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EXTENSION OF TWO MINIMAX THEOREMS OF S. PARK

Vesna Mišić and Ivan D. Aranđelović

Abstract. In this paper we prove two general minimax theorems which generalize famous classical saddle point theorems of M. Sion [**6**] and J. von Neumann. Our theorems also include some results of S. Park [**3**]-[**5**]. Results of this type have many applications in the Game theory, because they gives existence of solution of zero sums games.

1. Introduction

Today KKM theory, which investigated generalizations of KKM principle and theirs applications, is an important branch of nonlinear functional analysis (see [**2**], [**7**] and [**8**]).

Many definitions of convexity on topological spaces without algebraic structures exists. For convex fixed point theory and KKM theory, the most important of them are definition of Abstract convex spaces with KKM structure (shortly KKM spaces) introduced by S. Park (for details and references see [**4**], [**5**]). KKM spaces include earlier considered classes of topological spaces defined by of H. Komiya (Convex topological spaces), M. Lassonde (Convex spaces), C. Horvath (Pseudo convex spaces and C - spaces), C. Bardaro, R. Ceppitelli (H - spaces), S. Park and H. Kim (Generalized convex spaces), H. Ben-El-Mechaiekh, S. Chebbi, M. Florenzano and J.-V. Linares (L - spaces), J. Huang (spaces which satisfies property H) and X. P. Ding (FC - spaces).

In this paper we prove two general minimax theorems which generalize famous classical saddle point theorems of M. Sion [**6**] and J. von Neumann. Our theorems

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also include some results of S. Park [**3**]-[**5**]. Results of this type have many applications in the Game theory, because they give existence of solution of zero sums games.

2. Preliminary notes

Let X and Y be non-empty sets; we denote by 2^X family of all non-empty subsets of X, $\mathcal{F}(X)$ a family of all non-empty finite subsets of X and $\mathcal{P}(X)$ a family of all subsets of X. A *set-valued function* G from X into Y is a map $G : X \longrightarrow \mathcal{P}(Y)$. We denote such set-valued function by $G : X \multimap Y$. A *multifunction* G from X into Y is a map $G : X \longrightarrow 2^Y$ i.e. multifunction is a set-valued function with non-empty values. Let $G : X \multimap Y$ be a set-valued function. We define $G^{-1} : U_{x\in X}G(x) \multimap X$ by $G^{-1}(y) = \{x \in X : y \in G(x)\}$. If $B \subseteq Y$ then set $\{x \in X : G(x) \cap B \neq \emptyset\}$ is denoted by $G^{-1}(B)$. Let X be a non-empty set and let $F : X \multimap X$ be a set-valued function. $x_0 \in X$ is a *fixed point* of set-valued function F if and only if $x_0 \in F(x_0)$. Since statements $x \in F(x)$ and $x \in F^{-1}(x)$ are equivalent, set-valued functions F and F^{-1} have the same fixed points.

Let $\mathcal{S}$ be a linearly ordered set. If its every subset has a least upper bound, then S is *complete linearly ordered*.

Let $\mathcal{S}$ be a complete linearly ordered set. $f : X \longrightarrow \mathcal{S}$ is upper (lower) semi-continuous function on X if and only if set $\{x \in X : f(x) < r\}(\{x \in X : f(x) > r\})$ is open for each $r \in \mathcal{S}$. If X is the compact topological space and $f : X \longrightarrow \mathcal{S}$ upper (lower) semi-continuous function on X, then f has maximum (minimum) on X.

Let $\mathcal{A}$ be an arbitrary family of upper (lower) semi-continuous function defined on compact topological space X and let g be a function defined by $g(x) = \inf_{f\in A} f(x)(g(x) = \sup_{f\in A} f(x))$ for any $x \in X$. If g is finite on X then g is lower (upper) semi-continuous.

A family of sets has the *finite intersection property* if and only if the intersection of each its finite subfamily is non-empty.

An *abstract convex space* $(E, D; \Gamma)$ consists of a non-empty set E, a non-empty set D and a multifunction $\Gamma : \mathcal{F}(D) \multimap X$.

Let $(E, D; \Gamma)$ be an abstract convex space. For any $A \subseteq D$, the Γ-convex hull of A is denoted and defined by

$$\mathrm{conv}_\Gamma(A) = \bigcup_{B\subseteq A} \Gamma(B).$$

A subset $X \subseteq E$ is called a Γ-convex subset of $(E, D; \Gamma)$ relative to $D' \subseteq D$, if for any $A \subseteq D'$, we have $\Gamma(A) \subseteq X$, that is $\mathrm{conv}_\Gamma(D') \subseteq X$.

When $D \subseteq E$, the space is denoted by $(E \supseteq D; \Gamma)$. In such case, a subset X of E is said to be Γ-convex if

$$\mathrm{conv}_\Gamma(D \cap X) \subseteq X;$$

in other words, X is Γ-convex relative to $D \cap X$.

In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Let $(E, D; \Gamma)$ be an abstract convex space and Z be an arbitrary set, $F : E \multimap Z$ and $G : D \multimap Z$ be multifunctions which satisfies

$$F(\Gamma(A)) \subset G(A) := \bigcup_{y \in A} G(y).$$

Then G is called a KKM map with respect to F. A KKM map $G : D \multimap E$ is a KKM map with respect to identity map I_E.

A multifunction $F : E \multimap Z$ is said to have KKM property and called a $\mathcal{K}$-map if, for any KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}$ has the finite intersection property. We denote

$$\mathcal{K}(E, Z) := \{F \multimap Z : F \text{ is a } \mathcal{K} - \text{map}\}.$$

When Z is a topological space then $F : E \multimap Z$ is $\mathcal{KC}$-map ($\mathcal{KO}$-map) if, for any close-valued (open-valued) KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}$ has the finite intersection property. Sets $\mathcal{KC}(E, Z)$ and $\mathcal{KO}(E, Z)$ are defined similarly as $\mathcal{K}(E, Z)$. We have:

$$\mathcal{K}(E, Z) \subseteq \mathcal{KC}(E, Z) \bigcap \mathcal{KO}(E, Z).$$

Let $(E, D; \Gamma)$ be an abstract convex topological space (this mean that E is topological space and $(E, D; \Gamma)$ is abstract convex space) . The KKM principle is statement

$$\mathbb{I} \in \mathcal{KC}(E, E) \bigcap \mathcal{KO}(E, E).$$

KKM space is an abstract convex topological space satisfying the KKM principle.

Let $\mathcal{S}$ be a complete linearly ordered set, $(E, D; \Gamma)$ an abstract convex space and $f : X \longrightarrow \mathcal{S}$. f is said to be quasi-concave (quasi-convex) if for every $t \in \mathcal{S}$ set $\{x : f(x) > t\}$ ($\{x : f(x) < t\}$) is convex.

In [**4**],[**5**] S. Park proved the following fixed point theorem which generalizes famous Fan-Browder theorem.

Theorem 2.1 ([**4**],[**5**]). *Let $(E, D; \Gamma)$ be a KKM space, and $F : E \multimap E, G : E \multimap D$ set-valued functions which satisfies:*

(1) *for each $x \in E, conv_\Gamma(G(x)) \subseteq F(x)$ i.e. $F(x)$ is Γ-convex relative to $G(x)$;*

(2) *$E = G^{-1}(A)$ for some $A \subseteq \mathcal{F}(D)$;*

(3) *G has open values.*

Then F has a fixed point.

Next result was also proved by S. Park [**4**],[**5**].

Theorem 2.2 ([**4**],[**5**]). *Let $\{(E_i, D_i; \Gamma_i i)\}_{i \in I}$ be any family of abstract convex spaces. Let $E := \prod_{i \in I} E_i$ and $D := \prod_{i \in I} D_i$. For each $i \in I$ let $\pi_i : D \longrightarrow D_i$ be the projection. For each $A \in \mathcal{D}$ define $\Gamma(A) = \prod_{i \in I} \Gamma_i(\pi_i(A))$. Then $(E, D; \Gamma)$ is an abstract convex space.*

3. Results

Our first result is the following coincidence theorem which generalize earlier results of J. von Neumann and Ky Fan.

THEOREM 3.1. *Let $X \subseteq E$ and $Y \subseteq F$ be non-empty convex compact sets in abstract convex spaces (E, Γ_E) and (F, Γ_F), and $A, B : X \multimap Y$ set-valued functions such that:*

a) *$A(x)$ is open and $B(x)$ non-empty Γ_F-convex set for any $x \in X$;*

b) *$B^{-1}(y)$ open and $A^{-1}(y)$ non-empty Γ_E-convex set for any $y \in Y$.*

If $(E \times F, \Gamma_{E\times F})$ is a KKM space, where $\Gamma_{E\times F}$ is product convexity defined as above (Theorem(2.2)), then there exists $x_0 \in X$ such that

$$A(x_0) \bigcap B(x_0)B \neq \emptyset.$$

PROOF. From Theorem (2.2) it follows that set $Z = X \times Y$ in KKM space $E \times F$ is non-empty, Γ-convex and compact. Set-valued multifunction $T : Z \longrightarrow \mathcal{P}(Z)$ defined by

$$T(x, y) = A^{-1}(y) \times B(x)$$

satisfy:

: 1) $T(x, y)$ is non-empty Γ-convex set for each $(x, y) \in Z$;

: 2) $T^{-1}(x, y) = B^{-1}(y) \times A(x)$ is open set for each $(x, y) \in Z$.

By Theorem (2.1) we get that there exists fixed point of set-valued function T^{-1}. So there exists $(x_0, y_0) \in Z$ such that $(x_0, y_0) \in T(x_0, y_0)$. From $(x_0, y_0) \in A^{-1}(y_0) \times B(x_0)$ it follows that $y_0 \in A(x_0)$ and $y_0 \in B(x_0)$ which implies $y_0 \in A(x_0) \bigcap B(x_0)$. □

Now we shall prove our main result.

THEOREM 3.2. *Let (E, Γ_E) and (F, Γ_F) be abstract convex spaces, $\mathcal{S}$ a complete linearly ordered set, $X \subseteq E$ and $Y \subseteq F$ compact Γ-convex sets and $f, g : X \times Y \longrightarrow \mathcal{S}$. Suppose that:*

1) *$f(x, y) \leqslant g(x, y)$ for each $x \in X$ and $y \in Y$;*

2) *there exists a subset $T \subseteq \mathcal{S}$, such that $a \in g(X, Y)$ $b \in f(X, Y)$ with $a < b$ implies $T \cap (a, b) \neq \emptyset$;*

3) *for each $x \in X$ function $y \longrightarrow f(x, y)$ is lower semi-continuous on Y and set $\{y \in Y | g(x, y) < t\}$ is Γ_F-convex for any $t \in T$;*

4) *for each $y \in Y$ function $x \longrightarrow g(x, y)$ is upper semi-continuous on X and set $\{x \in X | f(x, y) > t\}$ is Γ_E-convex for any $t \in T$.*

If $(E \times F, \Gamma_{E\times F})$ is a KKM space, where $\Gamma_{E\times F}$ is product convexity defined as above (Theorem (2.2)), then

$$\min_{y\in Y} \sup_{x\in X} f(x, y) \leqslant \max_{x\in X} \inf_{y\in Y} g(x, y).$$

PROOF. Not that $y \longrightarrow \sup_{x\in X} f(x, y)$ is lower semi-continuous on Y and $x \longrightarrow \inf_{y\in Y} g(x, y)$ is upper semi-continuous on X. Therefore, the both sides of inequality exists. If

$$\max_{x\in X}\inf_{y\in Y} g(x,y) < \min_{y\in Y}\sup_{x\in X} f(x,y),$$

then by 2) it follows that there exists $t \in T$ such that

$$\max_{x\in X}\inf_{y\in Y} g(x,y) < t < \min_{y\in Y}\sup_{x\in X} f(x,y).$$

Let $A, B : X \multimap Y$ be set valued functions defined by:

$$A(x) = \{y \in Y : f(x,y) > r\} \text{ and } B(x) = y \in Y : g(x,y) < r\}.$$

By using Theorem (3.1) there exists $x_0 \in X$ such that $A(x_0)\bigcap B(x_0) \neq \emptyset$. This implies that there exists $y_0 \in Y$ such that $r < f(x_0, y_0) \leqslant f(x_0, y_0) < r$ which is a contradiction. □

If $x \longrightarrow f(x,y)$ is quasi-concave on X, for each $y \in Y, y \longrightarrow g(x,y)$ is quasi-convex on Y and $\mathcal{S} = \mathcal{R}$ then Theorem (3.2) reduce to S. Parks theorem ([**4**] -Theorem 21; [**5**] -Theorem 8.2.) If $f = g$ by using Lema 3.2 we obtained generalization of famous theorems on existence of saddle points of J. von Neumann and M. Sion [**6**]. It also include result of first author [**1**] -Theorem 3.

Now we present one result which extends one another minimax result of S. Park [**3**] - Theorem 3.

Theorem 3.3. *Let (E, Γ_E) and (F, Γ_F) be KKM spaces, $\mathcal{S}$ a complete linearly ordered set, $X \subseteq E$ be a Γ-convex set, $Y \subseteq F$ a compact Γ-convex set, $f : X \times Y \longrightarrow \mathcal{S}$. Suppose that:*

1) *there exists a subset $T \subseteq \mathcal{S}$, such that $a, b \in f(X,Y)$ with $a < b$ implies $T \cap\ (a,b) \neq \emptyset$;*

2) *for each $x \in X$ function $y \longrightarrow f(x,y)$ is lower semi-continuous on Y and set $\{y \in Y | f(x,y) < t\}$ is Γ_F-convex for any $t \in T$;*

3) *for each $y \in Y$ function $x \longrightarrow f(x,y)$ is upper semi-continuous on X and set $\{x \in X | f(x,y) > t\}$ is Γ_E-convex for any $t \in T$.*

If $(E \times F, \Gamma_{E\times F})$ is a KKM space, where $\Gamma_{E\times F}$ is product convexity defined as above (Theorem (2.2)), then

$$\sup_{x\in X}\min_{y\in Y} f(x,y) = \min_{y\in Y}\sup_{x\in X} f(x,y).$$

Proof. Since $f(x,y)$ is lower semi-continuous on Y, $p(x) = \min_{y\in Y} f(x,y)$ exists for each $x \in X$. Since $q(y) = \sup_{x\in X} f(x,y)$ is lower semi-continuous for each $y \in Y$, there exists $q(y_0) = \min_{x\in X} q(y)$. Note that

$$p(x) \leqslant \min y \in Y f(x,y) \leqslant f(x,y) \leqslant \max_{x\in X} f(x,y) = q(y),$$

for all $x \in X$ and $y \in Y$. Suppose that the equality does not hold. So

$$\sup_{x\in X} p(x) < \min y \in Y g(y).$$

This implies that there exists $r \in T$ such that:

$$\sup_{x\in X} p(x) < r < \min y \in Y g(y),$$

for some $r \in R$.

Let $A, B : X \multimap Y$ be set-valued functions defined by:

$$A(x) = \{y \in Y : f(x, y) > r\} \text{ and } B(x) = \{y \in Y : f(x, y) < r\}.$$

By using Theorem (3.1) there exists $x_0 \in X$ such that $A(x_0) \bigcap B(x_0) \neq \emptyset$. Hence there exists $y_0 \in Y$ such that $r < f(x_0, y_0) < r$ which is a contradiction. □

If E and F are convex spaces then Theorem (3.3) reduce to minimax result of S. Park [**3**]. It also include earlier result of first author [**1**] -Theorem 3.

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V. Mišić: University of East Sarajevo - Faculty of Transport and Traffic Engineering, Vojvode Mišića 52, 53000 Doboj, Bosnia and Herzegovina

E-mail address: vesnasmisic@yahoo.com

I. D. Arandelović: Univrsity of Belgrade, Faculty of Mechanical Engineering, 11000 Beograd, Kraljice Marije 16, Serbia

E-mail address: iarandjelovic@mas.bg.ac.rs