Supplementary Materials for 'Joint Spatial Modeling of Recurrent Infection and Growth with Processes under Intermittent Observation' by F.S. Nathoo

Web Appendix

Details on Computational Implementation

Details of the sampling algorithm for the joint spatial model are given below. Estimation of the conditional predictive ordinate using Monte Carlo samples from the posterior distribution is then discussed. Software for carrying out the computation has been written in the R programming language and is available from the author upon request.

Reparameterizations

We have found two simple reparameterizations particularly useful in improving mixing of the sampler.

1. Replacing the regression specification $\lambda_i\{t|x_{\boldsymbol{s_i}}^{(H)}(t)\} = \lambda_0(t)\omega_i \exp\{\beta x_{\boldsymbol{s_i}}(t)\}$ with a centered version $\lambda_i\{t|x_{\boldsymbol{s_i}}^{(H)}(t)\} = \lambda_0(t)\omega_i \exp\{\beta(x_{\boldsymbol{s_i}}(t) - c)\}$ where taking $c = \bar{Y}$ improves mixing of the sampler; in particular w.r.t. β and $\{\lambda_j\}$.

2. We have found it useful to hierarchically scale each kernel $K_l(\boldsymbol{u})$ so that the corresponding scale parameter σ_l, $l = 1, 2, 3$, is 'pushed back' into the distribution of the latent variables $\boldsymbol{X_j}$, $j = 1, \cdots, J$. In doing so, the distribution of the increments used in defining the discrete process convolution changes from $\mathbf{X_j} \overset{ind}{\sim} MVN_3(\mathbf{0}, |A_j|\mathbf{TT'})$ to $\mathbf{X_j} \overset{ind}{\sim} MVN_3(\mathbf{0}, |A_j| \operatorname{diag}\{\sigma_1, \sigma_2, \sigma_3\}\mathbf{TT'} \operatorname{diag}\{\sigma_1, \sigma_2, \sigma_3\})$ and the computation of each $K_l(\boldsymbol{u})$ no longer involves σ_l.

MCMC Algorithm

Denoting by $\{\mathbf{X}, \{\sigma_l^2\}, \{\psi_{lx}, \psi_{ly}, \nu_l\}, \boldsymbol{\rho}, \eta, \beta, \{\omega_i\}, \{\lambda_j\}, \sigma_\epsilon^2, \boldsymbol{\mu_z}\}$ the current state of the chain, we follow steps 1 to 10 below. One iteration of the sampler consists of a complete sweep through the ten steps, at the end of which the new state is recorded.

1. Update latent variables $\mathbf{X} = (\mathbf{X_1}, \ldots, \mathbf{X_J})'$: the full conditional density for $\mathbf{X}$ is denoted $\pi_{\mathbf{X}}(\cdot)$ and given by

$$\pi_{\mathbf{X}}(\mathbf{X}) \propto L(\boldsymbol{\theta}|\boldsymbol{N}, \boldsymbol{Y}) \times \left[\prod_{j=1}^{J} \pi(\mathbf{X_j}|\boldsymbol{\rho}, \sigma_1, \sigma_2, \sigma_3)\right].$$

An update based on the hybrid algorithm requires evaluation of $\nabla \log \pi_{\mathbf{X}}(\mathbf{X})$, a vector of length $3J$, which is easily obtained analytically using the chain rule. Numerical evaluation based on finite differences is also possible and gave identical results at the cost of slower computation. The analytic form for the components of $\nabla \log \pi_{\mathbf{X}}(\mathbf{X})$ are given by

$$\frac{\partial \log \pi_{\mathbf{X}}(\mathbf{X})}{\partial \mathbf{X_{j}}_1} = \sum_{i=1}^{n}\sum_{j=1}^{M_1} \frac{\partial \log P_N(N_{ij}|\mu_{N_{ij}})}{\partial \mu_{N_{ij}}} \frac{\partial \mu_{N_{ij}}}{\partial b_1(\boldsymbol{s_i})} \frac{\partial b_1(\boldsymbol{s_i})}{\partial \mathbf{X_{j}}_1}$$

$$+ \sum_{i=1}^{n}\sum_{j=1}^{M_2} \frac{\log P_Y(Y_{ij}|\mu_{Y_{ij}}, \sigma_\epsilon^2)}{\partial \mu_{Y_{ij}}} \frac{\partial \mu_{Y_{ij}}}{\partial b_1(\boldsymbol{s_i})} \frac{\partial b_1(\boldsymbol{s_i})}{\partial \mathbf{X_{j}}_1} + \frac{\partial \log \pi(\mathbf{X_j}|\boldsymbol{\rho}, \sigma_1, \sigma_2, \sigma_3)}{\partial \mathbf{X_{j}}_1}$$

$$j = 1, \ldots, J$$

$$\frac{\partial \log \pi_{\mathbf{X}}(\mathbf{X})}{\partial \mathbf{X_{j}}_2} = \sum_{i=1}^{n}\sum_{j=1}^{M_1} \frac{\partial \log P_N(N_{ij}|\mu_{N_{ij}})}{\partial \mu_{N_{ij}}} \frac{\partial \mu_{N_{ij}}}{\partial b_2(\boldsymbol{s_i})} \frac{\partial b_2(\boldsymbol{s_i})}{\partial \mathbf{X_{j}}_2}$$

$$+ \sum_{i=1}^{n}\sum_{j=1}^{M_2} \frac{\log P_Y(Y_{ij}|\mu_{Y_{ij}}, \sigma_\epsilon^2)}{\partial \mu_{Y_{ij}}} \frac{\partial \mu_{Y_{ij}}}{\partial b_2(\boldsymbol{s_i})} \frac{\partial b_2(\boldsymbol{s_i})}{\partial \mathbf{X_{j}}_2} + \frac{\partial \log \pi(\mathbf{X_j}|\boldsymbol{\rho}, \sigma_1, \sigma_2, \sigma_3)}{\partial \mathbf{X_{j}}_2}$$

$$j = 1, \ldots, J$$

$$\frac{\partial \log \pi_{\mathbf{X}}(\mathbf{X})}{\partial \mathbf{X_{j3}}} = \sum_{i=1}^{n} \sum_{j=1}^{M_1} \frac{\partial \log P_N(N_{ij}|\mu_{N_{ij}})}{\partial \mu_{N_{ij}}} \frac{\partial \mu_{N_{ij}}}{\partial b_3(\boldsymbol{s_i})} \frac{\partial b_3(\boldsymbol{s_i})}{\partial \mathbf{X_{j3}}}$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{M_2} \frac{\log P_Y(Y_{ij}|\mu_{Y_{ij}}, \sigma_\epsilon^2)}{\partial \mu_{Y_{ij}}} \frac{\partial \mu_{Y_{ij}}}{\partial b_3(\boldsymbol{s_i})} \frac{\partial b_3(\boldsymbol{s_i})}{\partial \mathbf{X_{j3}}} + \frac{\partial \log \pi(\mathbf{X_j}|\boldsymbol{\rho}, \sigma_1, \sigma_2, \sigma_3)}{\partial \mathbf{X_{j3}}}$$

$$j = 1, \ldots, J$$

Given these forms, and letting $\mathbf{X}^*$ denote the current value in the Markov chain for $\mathbf{X}$, the hybrid update, based on a step size $\delta > 0$, then proceeds as described in Section 3.2:

(a) *Simulate auxiliary variables* $\mathbf{U}^* \sim MVN_{3J}(\mathbf{0}, \mathbf{I})$.

Let $\mathbf{X^{(0)}} = \mathbf{X}^*$ *and* $\mathbf{U^{(0)}} = \mathbf{U}^* + \frac{\delta}{2} \nabla \log \pi_{\mathbf{X}}(\mathbf{X}^*)$

(b) *For* $l = 1, \ldots, L$, *let*

$$\mathbf{X^{(l)}} = \mathbf{X^{(l-1)}} + \delta\, \mathbf{U^{(l-1)}}$$

$$\mathbf{U^{(l)}} = \mathbf{U^{(l-1)}} + \delta_l \nabla \log \pi_{\mathbf{X}}(\mathbf{X^{(l)}})$$

where $\delta_l = \delta$ *for* $l < L$ *and* $\delta_L = \frac{\delta}{2}$.

(c) *Accept* $\mathbf{X^{(L)}}$ *as the new state for* $\mathbf{X}$ *with probability*

$$p = min\left(\frac{\pi_{\mathbf{X}}(\mathbf{X^{(L)}})}{\pi_{\mathbf{X}}(\mathbf{X}^*)} \exp\left\{ -\frac{1}{2} \left(\mathbf{U^{(L)}}' \mathbf{U^{(L)}} - \mathbf{U}^{*\prime} \mathbf{U}^* \right) \right\}, 1 \right)$$

else remain in the current state $\mathbf{X}^*$ *with probability* $1 - p$.

2. Block update $(\sigma_1^2, \sigma_2^2, \sigma_3^2)'$: The density of the full conditional distribution is proportional to

$$\left[\prod_{j=1}^{J} \pi(\mathbf{X_j}|\boldsymbol{\rho}, \sigma_1, \sigma_2, \sigma_3)\right] \times \left[\prod_{l=1}^{3} \pi(\sigma_l)\right].$$

We use a Metropolis-Hastings step with candidate generated from a multivariate log-normal distribution.

3. For each $l = 1, 2, 3$, block update $\{\psi_{lx}, \psi_{ly}, \nu_l\}$: The density of the full conditional distribution is proportional to

$$L(\boldsymbol{\theta}|\boldsymbol{N}, \boldsymbol{Y}) \times \pi(\psi_{lx})\pi(\psi_{ly})\pi(\nu_l).$$

We use a Metropolis-Hastings step where the candidate is generated from a *transformed* multivariate normal distribution.

4. For each $l = 1, 2, 3$, update cross-correlation parameter ρ_l. The full conditional distribution has density proportional to

$$\left[\prod_{j=1}^{J} \pi(\mathbf{X_j}|\boldsymbol{\rho}, \sigma_1, \sigma_2, \sigma_3)\right] \pi(\rho_l).$$

We discretize this density onto a fine grid which facilitates a Gibbs update.

5. Update frailty precision η: the full conditional distribution has density proportional to

$$\left[\prod_{i=1}^{n} \pi(\omega_i|\eta)\right] \pi(\eta).$$

We use a Metropolis-Hastings step with candidate generated from a log-normal distribution.

6. Update regression coefficient β: the full conditional distribution has density proportional to
$$\prod_{i=1}^{n}\left[\prod_{j=1}^{M_1} \mathrm{P}_N(N_{ij}|\mu_{N_{ij}})\right]\pi(\beta).$$
We use a random walk Metropolis step based on a Gaussian proposal distribution.

7. Update frailties ω_i: the full conditional distribution for ω_i is
$$\mathrm{Gamma}(\eta + \sum_{j=1}^{M_1} N_{ij}, \eta + \sum_{j=1}^{M_1} \lambda_j I_{ij})$$
where $I_{ij} = \int_{t_{j-1}^{(N)}}^{t_j^{(N)}} \exp\{\beta x_{\boldsymbol{s_i}}(t)\}dt$. We sample from the conditional distribution directly in a Gibbs update.

8. Update baseline hazard parameters λ_j: Upon adopting a conjugate Gamma(ϵ, ϵ) prior for λ_j, the full conditional distribution for λ_j is
$$\mathrm{Gamma}(\epsilon + \sum_{i=1}^{n} N_{ij}, \epsilon + \sum_{i=1}^{n} \omega_i I_{ij})$$
where $I_{ij} = \int_{t_{j-1}^{(N)}}^{t_j^{(N)}} \exp\{\beta x_{\boldsymbol{s_i}}(t)\}dt$. We sample from the conditional distribution directly in a Gibbs update.

9. Update the error precision σ_ϵ^2: Upon adopting a conjugate Inverse-Gamma (α_1, α_2) prior for σ_ϵ^2, the full conditional distribution for σ_ϵ^2 is

$$\text{Inverse-Gamma}(\alpha_1 + nM_2/2, \alpha_2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{M_2}(Y_{ij} - x_{\boldsymbol{s_i}}(t_j^{(H)}))^2).$$

We sample from the conditional distribution directly in a Gibbs update.

10. Update $\boldsymbol{\mu_z}$: the density of the full conditional distribution is proportional to

$$L(\boldsymbol{\theta}|\boldsymbol{N}, \boldsymbol{Y}) \times \left[\prod_{l=1}^{3} \pi(\mu_{Z_l})\right].$$

We use a random walk Metropolis step with candidate generated from a Multivariate normal distribution.

Monte Carlo Estimation of Conditional Predictive Ordinate

Having generated L samples from the posterior $\boldsymbol{\theta}^{(1)}, \ldots, \boldsymbol{\theta}^{(L)}$, the conditional predictive ordinates defined in Section 3.1 are computed using estimators based on a harmonic mean (see Gelfand and Dey, 1994)

$$CPO_{N_{ij}} \approx \left[\frac{1}{L}\sum_{l=1}^{L}\frac{1}{P_N(N_{ij}|\mu_{N_{ij}}^{(l)})}\right]^{-1}$$

$$CPO_{Y_{ij}} \approx \left[\frac{1}{L}\sum_{l=1}^{L}\frac{1}{P_Y(Y_{ij}|\mu_{Y_{ij}}^{(l)}, \sigma_\epsilon^{2(l)})}\right]^{-1}$$

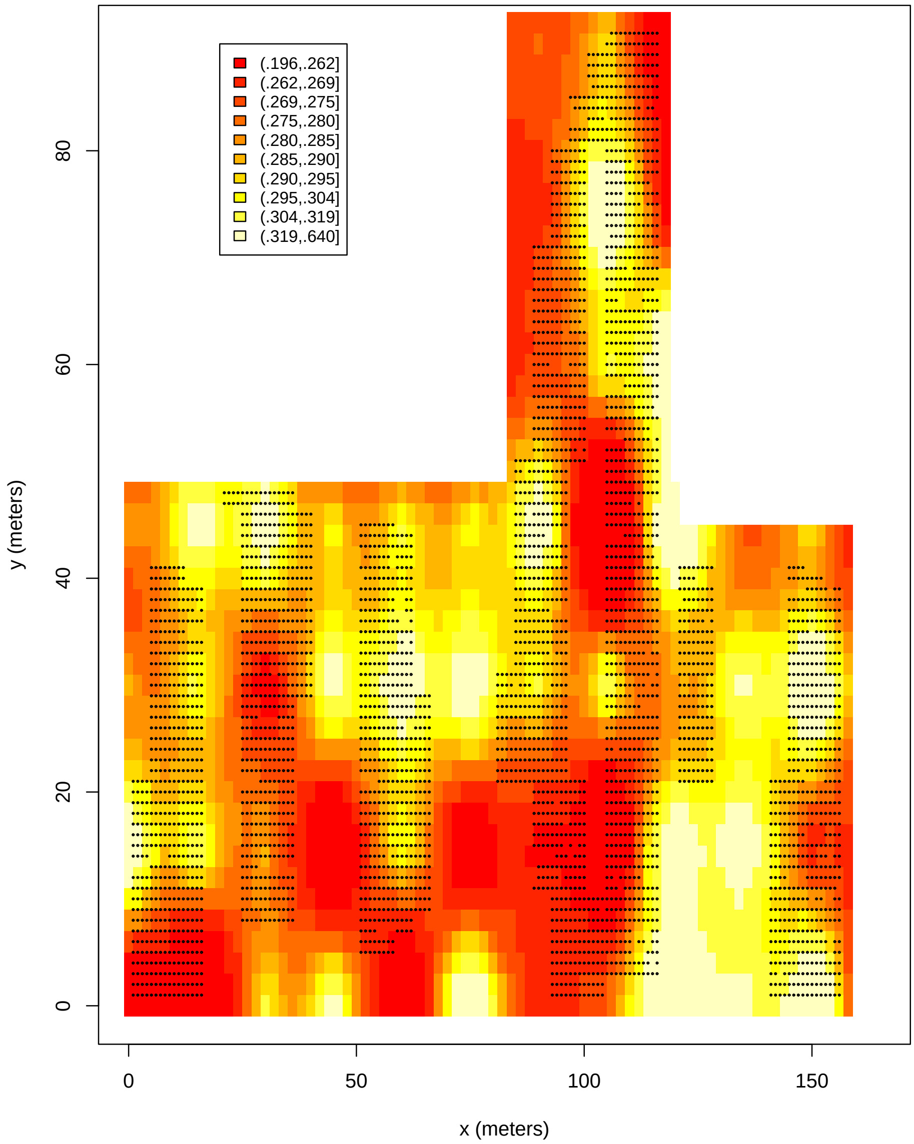

Figure 1. (Web) Image plot of the posterior mean interpolated surface for $b_3(\boldsymbol{s})$.