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Structure of the Small Amplitude Motion on Transversely Sheared Mean Flows

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Abstract

This paper considers the small amplitude unsteady motion of an inviscid non-heat conducting compressible fluid on a transversely sheared mean flow. It extends a previous result given in Goldstein (1978(b) and 1979(a)) which shows that the hydrodynamic component of the motion is determined by two arbitrary convected quantities in the absence of solid surfaces or other external sources. The result is important because it can be used to specify appropriate boundary conditions for unsteady surface interaction problems on transversely sheared mean flows in the same way that the vortical component of the Kovasznay (1953) decomposition is used to specify these conditions for surface interaction problems on uniform mean flows. But unlike the Kovasznay (1953) case the arbitrary convected quantities no longer bear a simple relation to the physical variables. One purpose of this paper is to derive a formula that relates these quantities to the (physically measurable) vorticity and pressure fluctuations in the flow.

1.0   Introduction

The small amplitude motion of an inviscid non-heat conducting compressible fluid is governed by the Linearized Euler Equations, i.e., the Euler equations linearized about an arbitrary, usually steady, solution to those equations— customarily referred to as the base flow. The simplest case occurs when the base flow is completely uniform. In a now classical paper, Kovasznay (1953) showed that the unsteady isentropic motion on this flow can be decomposed into the sum of a purely vortical disturbance that has no pressure fluctuations and an irrotational disturbance that carries the pressure fluctuations. The latter satisfies a second-order wave equation when the flow is compressible and should, as argued by Möhring (1976), either decay or propagate relative to the base flow. It can, therefore be associated with the acoustic component of the motion on these flows. The former, which moves downstream with the mean flow, i.e., it is a purely convected quantity, can be associated with the remaining, hydrodynamic, component of the motion. Any convected velocity field will satisfy the linearized momentum equation for this flow, but continuity only allows two of its components to be arbitrary. These two quantities can then be independently specified as steady state boundary conditions for unsteady surface interaction problems. This makes the Kovasznay decomposition particularly useful for analyzing the interaction of turbulence (which corresponds to the hydrodynamic component of the motion) with surfaces embedded in uniform mean flows (Sears, 1941) or in flows that become uniform in the upstream region (Hunt, 1973; Goldstein, 1978(a), 1979(b)). It is worth noting, however, that the Kovasznay decomposition is not unique because there are irrotational (homogeneous) solutions that carry no pressure fluctuations. There have been many attempts to extend these ideas to non-uniform base flows, but the situation is considerably more complicated when the entire base flow is non-uniform. The simplest case occurs when the base flow shear is uniform (i.e., constant).

In 1907, Orr (1907, pp. 26-29, see Drazin and Reid, 1981, pp. 147-151) pointed out that the linearized incompressible vorticity equation

$$\left(\frac{\partial}{\partial \tau} + U \frac{\partial}{\partial y_1} \right) \omega_3 = 0, \tag{1.1}$$

where τ denotes the time, y_1, y_2, y_3 are Cartesian coordinates, with y_1 being in the mean flow direction or equivalently the two-dimensional Rayleigh equation

$$\frac{\partial}{\partial y_1} \left(\frac{\partial}{\partial \tau} + U \frac{\partial}{\partial y_1} \right) \omega_3 = \left(\frac{\partial}{\partial \tau} + U \frac{\partial}{\partial y_1} \right) \left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} \right) v_2' = 0 \tag{1.2}$$

that determine the two-dimensional vorticity perturbation ω_3 and the unsteady transverse velocity perturbation $v_2'(y_2, \tau)$ for the two-dimensional small amplitude motion on a sheared mean flow with linear velocity profile

$$U = \lambda y_2 \tag{1.3}$$

can be integrated to obtain

$$\left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} \right) v_2' = \frac{\partial}{\partial y_1} \omega_c \left(\tau - \frac{y_1}{\lambda y_2}, y_2 \right) \tag{1.4}$$

where the transverse vorticity perturbation ω_3 is an arbitrary function of its arguments. Orr obtained an analytic solution to an initial value problem associated with this equation and used it to study the development of the velocity and pressure fluctuations starting from some initial state. But (1.4) can also be formulated as a steady-state (i.e., time-stationary) boundary value problem whose formal solution is given by

$$v_2'(\boldsymbol{x}, t) = \frac{\partial}{\partial x_1} \int_{-T}^{T} \int g_0(\boldsymbol{x}, t \mid \boldsymbol{y}, \tau) \omega_c \left(\tau - \frac{y_1}{\lambda y_2}, y_2 \right) d\boldsymbol{y} d\tau \tag{1.5}$$

where $\boldsymbol{x} = \{x_1, x_2\}$, $\boldsymbol{y} = \{y_1, y_2\}$ denote the two-dimensional Cartesian coordinates, the inner integration is over the entire $y_1 - y_2$ plane, T denotes a large time interval and g_0 denotes a Green's function, which is determined by

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) g_0(\boldsymbol{x}, t \mid \boldsymbol{y}, \tau) = \delta(t - \tau) \delta(\boldsymbol{y} - \boldsymbol{x}) \tag{1.6}$$

together with appropriate outgoing wave boundary conditions.

It can be shown that $v_2'(\boldsymbol{x}, t)$ will decay very slowly with x_1 when the arbitrary convected quantity ω_c $(\tau - y_1 / \lambda y_2, y_2)$ decays sufficiently slowly as $y_2 \to \pm\infty$. The latter can then be specified as an upstream boundary condition since (1.5) satisfies (1.2) for any choice of this quantity. Note that the statistics of any time-stationary turbulent shear flow can be represented by a sum of such time harmonic functions (see for example, Goldstein and Durbin 1980).

Equation (1.4), which bears some relation to the so-called continuous spectrum, was extended to three-dimensional compressible motions on base flows with constant mean shear by Möhring (1976) and to general planar compressible shear flows by Goldstein (1978(b), 1979(a)): hereafter referred to as G78 and G79, respectively) who showed how this more general (non-uniform mean flow) result can used to formulate surface interaction problems that are relevant to aircraft noise problems in which the surfaces

are typically embedded in transversely sheared mean flows. But, unlike the uniform flow and incompressible constant mean shear cases, the arbitrary convected quantity ω_c ($\tau - y_1 /U(y_2)$, y_2)—or its equivalent—can no longer be associated with a single component of the vorticity in these more general flows.

G78 and G79 obtained a result that is the natural generalization of the vortical solution in the Kovasznay (1953) decomposition by extending the single-convected-quantity, Orr-type, solutions to include an additional arbitrary convected quantity, but neither of these can be associated with a single component of the vorticity or, for that matter, any other specific physical (measurable) variable.

Much of modern Rapid Distortion Theory (RDT) is concerned with the small amplitude vortical motion on potential flows that originate from a completely uniform upstream flow (Sagaut and Cambon, 2008; Hunt, 1973, 1977; Goldstein, 1978(a), 1979(b)) or small scale motion on irrotational mean flows (e.g., Gartshore, Durbin and Hunt 1983 and Nazerenko, Kevlahan and Drubulle, 1999). An important application of existing rapid distortion theory is the specification of appropriate upstream boundary conditions, which can only be accomplished by using the Kovasznay decomposition (see, for example, Lele 1998, p. 432 and ff. and Colonius 2004, p.316). A natural extension of these ideas would be to develop a theory based on the small amplitude vortical motion on steady vortical base flows that originate from a parallel, or more generally, a transversely sheared mean flow in the upstream region (Colonius 2004, pp. 323-324).

In this paper, we consider the small-amplitude motion of an inviscid non-heat-conducting compressible fluid on a general transversely sheared mean flow with the aim of understanding its underlying structure and providing the necessary formalism to extend the class of mean flows for which applications of RDT can be considered.

The paper begins by using a newly uncovered connection to the adjoint Rayleigh operator to extend the G78 and G79 results to completely general (variable density, non-planar base flow, etc.) transversely sheared mean flows. As in G78 and G79 the unsteady motion, which can be calculated by solving an inhomogeneous equation, is determined by two arbitrary convected quantities in the absence of solid surfaces, external sources and Kelvin-Helmholtz instabilities. The relevant solution is the sum of two parts: a part which produces no pressure fluctuations and is driven by one of the two arbitrary convected quantities and a part that produces pressure fluctuations and is driven by the other arbitrary convected quantity.

A relation between the convected quantities and the physical variables is worked out in Section 3.0. It is shown that there is a specific linear combination of these quantities that is related to a linear combination of the physical variables (pressure, vorticity and their derivatives) by a single algebraic equation, which generalizes an equation obtained by Möhring (1976) for the linear base flow profile (1.3) with constant sound speed. But Möhring (1976) result, which is derived from the Beltrami equation (Emanuel,1993, p. 90), does not include a second convected quantity which, as shown in G79, must be included in order to completely represent small amplitude vortical motion on steady vortical base flows. It cannot, therefore, be used to represent general three-dimensional vortical disturbances. Our results generalize the classical Kovasznay (1953) decomposition to arbitrary transversely sheared mean flows and provide the formalism needed to apply RDT to these mean flows.

The paper is concluded with Section 4.0, which discusses the implication of these results and how they can be used to formulate RDT problems, i.e., problems that involve the interaction of turbulence with solid surfaces.

2.0   The Basic Result

We suppose that the flow is inviscid and non-heat conducting and assume an ideal gas so that the entropy is proportional to $\ln(p/\rho^\gamma)$, and the squared sound speed is $\gamma p/\rho$, where p denotes the pressure, ρ the density and γ the specific heat ratio. Then the inviscid pressure $p' = p - p_0$ and momentum flux

$$u_i \equiv \rho v_i', \tag{2.1}$$

perturbations (where v_i' denotes the velocity perturbation) on a transversely sheared mean flow with pressure p_0 = constant, velocity $U(y_T)$ and mean sound speed squared $c^2(y_T)$ are governed by the linearized momentum and energy equations

$$\frac{D_0 u_i}{D\tau} + \delta_{1i} u_j \frac{\partial U}{\partial y_j} + \frac{\partial}{\partial y_i} p' = 0 \tag{2.2}$$

$$\frac{D_0 p'}{D\tau} + \frac{\partial}{\partial y_j} c^2 u_j = 0, \tag{2.3}$$

where $y_T = \{y_2, y_3\}$, $y = \{y_1, y_2, y_3\} = \{y_1, y_T\}$,

$$\frac{D_o}{D\tau} \equiv \frac{\partial}{\partial \tau} + U \frac{\partial}{\partial y_1} \tag{2.4}$$

is the convective derivative.

G79 points out that the momentum Equation (2.2) will be identically satisfied for any function ϕ and any purely convected function $\vartheta(\tau - (y_1/U), y_T)$ when u_i and p' are determined by

$$p' = -\frac{D_0^3 \phi}{D\tau^3} \tag{2.5}$$

$$u_i = \left(\delta_{ij} \frac{D_0}{D\tau} - \delta_{i1} \frac{\partial U}{\partial y_j} \right) \Lambda_j + \varepsilon_{ijk} \frac{1}{c^2} \frac{\partial U}{\partial y_j} \frac{\partial}{\partial y_k} \vartheta\left(\tau - \frac{y_1}{U}, y_T \right) \tag{2.6}$$

where δ_{ij} denotes the Kronecker delta, ε_{ijk} denotes the alternating tensor and

$$\Lambda_j \equiv \frac{\partial}{\partial y_j} \frac{D_0 \phi}{D\tau} + 2 \frac{\partial U}{\partial y_j} \frac{\partial \phi}{\partial y_1} \tag{2.7}$$

denotes a kind of generalized particle displacement.
The arbitrary function ϕ can then be adjusted to ensure that the energy Equation (2.3) is also satisfied by substituting (2.5)--(2.7) into (2.3) to obtain

$$\frac{D_0}{D\tau} \left[\frac{\partial}{\partial y_i} c^2 \left(\frac{\partial}{\partial y_i} \frac{D_0 \phi}{D\tau} + 2 \frac{\partial U}{\partial y_i} \frac{\partial \phi}{\partial y_1} \right) - \frac{D_0^3 \phi}{D\tau^3} \right] = 0, \tag{2.8}$$

which can then be integrated to obtain

$$L_a \phi = -\tilde{\omega}_c \left(\tau - \frac{y_1}{U}, y_T \right) \tag{2.9}$$

where

$$L_a \equiv \frac{D_0^3}{D\tau^3} - \frac{\partial}{\partial y_i} c^2 \left(\frac{\partial}{\partial y_i} \frac{D_0}{D\tau} + 2\frac{\partial U}{\partial y_i} \frac{\partial}{\partial y_1} \right) \tag{2.10}$$

It is well known that the momentum flux perturbation, u_i can be eliminated between (2.2) and (2.3) to show that the pressure fluctuation p' also satisfies Rayleigh's equation

$$L\, p' = 0 \tag{2.11}$$

where

$$L \equiv \frac{D_o}{D\tau}\left(\frac{\partial}{\partial y_i} c^2 \frac{\partial}{\partial y_i} - \frac{D_o^2}{D\tau^2} \right) - 2\frac{\partial U}{\partial y_j}\frac{\partial}{\partial y_1} c^2 \frac{\partial}{\partial y_j} \tag{2.12}$$

denotes the usual Rayleigh operator. But it is readily verified that

$$
\begin{aligned}
vLu - uL_a v &= \frac{\partial}{\partial y_i} c^2 \left(u\frac{\partial}{\partial y_i}\frac{D_0 v}{D\tau} + 2u\frac{\partial U}{\partial y_i}\frac{\partial v}{\partial y_1} - \frac{\partial u}{\partial y_i}\frac{D_0 v}{D\tau} \right) \\
&+ \frac{D_o}{D\tau}\left(v\frac{\partial}{\partial y_i} c^2 \frac{\partial u}{\partial y_i} + \frac{D_0 u}{D\tau}\frac{D_o v}{D\tau} - u\frac{D_0^2 v}{D\tau^2} - v\frac{D_o^2 u}{D\tau^2} \right) - 2\frac{\partial}{\partial y_1}\left(c^2 \frac{\partial U}{\partial y_j} v\frac{\partial u}{\partial y_j} \right)
\end{aligned}
\tag{2.13}
$$

for any functions u, v, which shows that the Rayleigh operator L is adjoint to L_a (Morse and Feshbach, 1953, p. 870).

Let $g(\mathbf{y},\tau\,|\,\mathbf{x},t)$

$$L\, g(\mathbf{y},\tau\,|\,\mathbf{x},t) = \delta(\mathbf{y}-\mathbf{x})\delta(\tau-t) \tag{2.14}$$

be a Green's function for the Rayleigh operator L which exhibits outgoing wave behaviour as $|\mathbf{y}| \to \infty$, where, as usual, the first two arguments denote the dependent variables and the second two denote the source variables. We expect $g(\mathbf{y},\tau\,|\,\mathbf{x},t)$ to vanish as $\tau \to \infty$ for all finite $\mathbf{y}$ and as $y_1 \to \pm\infty$ for all finite τ when the base flow U is globally stable, and causality should ensure that it vanishes as $\tau \to -\infty$. We therefore suppose that

$$\lim_{\tau\to\pm\infty} g(\mathbf{y},\tau\,|\,\mathbf{x},t),\ \lim_{y_1\to\pm\infty} g(\mathbf{y},\tau\,|\,\mathbf{x},t) = 0, \tag{2.15}$$

Setting u equal to $g(\mathbf{y},\tau\,|\,\mathbf{x},t)$ in (2.13), letting v be a solution to (2.9) and using the divergence theorem shows that

$$
\begin{aligned}
\phi(\mathbf{x},t) &= -\int_{-T}^{T}\int_{V} g(\mathbf{y},\tau\,|\,\mathbf{x},t)\,\tilde{\omega}_c\left(\tau - \frac{y_1}{U(\mathbf{y}_T)}, \mathbf{y}_T \right) d\mathbf{y}\, d\tau \\
&+ \int_{-T}^{T}\int_{S} \hat{n}_i c^2 \left[g(\mathbf{y},\tau\,|\,\mathbf{x},t)\Lambda_i - \frac{\partial g(\mathbf{y},\tau\,|\,\mathbf{x},t)}{\partial y_i}\frac{D_0\phi}{D\tau} \right] dS(\mathbf{y})\, d\tau
\end{aligned}
\tag{2.16}
$$

where T denotes a very large but finite time interval, V is a region of space bounded by cylindrical surface(s) S, $\hat{\mathbf{n}} = \{\hat{n}_i\}$ is the unit outward-drawn normal to S, where we have omitted terms that are

negligibly small as $T \to \pm\infty$ and assumed that the contribution from the end caps can be neglected (Tam and Auriault, 1998). This formula expresses the solution to Equation (2.9) in terms of the volume source distribution $\tilde{\omega}_c (\tau - y_1/U(\boldsymbol{y}_T), \boldsymbol{y}_T)$ and the values of ϕ on some arbitrary cylindrical surfaces S. The Green's function g is not uniquely determined by (2.14) and can be required to satisfy certain boundary conditions on a portion of the surface S. The present analysis is somewhat unconventional in that the direct Green's function g now plays the role of an adjoint Green's function for the solution ϕ.

When the surface S is at infinity, i.e., V is all space

$$\phi(\boldsymbol{x},t) = -\int_{-T}^{T} \int_{V} g(\boldsymbol{y},\tau \mid \boldsymbol{x},t)\tilde{\omega}_c\left(\tau - \frac{y_1}{U(\boldsymbol{y}_T)}, \boldsymbol{y}_T\right) d\boldsymbol{y}d\tau \tag{2.17}$$

(which generalizes Equation (2.36) in G79) and Equation (2.6) then shows that the transverse velocity perturbation $v'_\perp$ is given by

$$\rho v'_\perp \equiv u_i(\boldsymbol{x},t)\frac{\partial U}{\partial x_i} \Big/ |\nabla U| = -\frac{\partial U/\partial x_i}{|\nabla U|}\int_{-T}^{T} \int_{V} g_i(\boldsymbol{y},\tau \mid \boldsymbol{x},t)\tilde{\omega}_c\left(\tau - \frac{y_1}{U(\boldsymbol{y}_T)}, \boldsymbol{y}_T\right) d\boldsymbol{y}d\tau \tag{2.18}$$

with

$$g_i(\boldsymbol{y},\tau \mid \boldsymbol{x},t) \equiv \frac{D_0}{Dt}\left(\frac{\partial}{\partial x_i}\frac{D_0}{Dt} + 2\frac{\partial U}{\partial x_i}\frac{\partial}{\partial x_1}\right)g(\boldsymbol{y},\tau \mid \boldsymbol{x},t), \tag{2.19}$$

In the more general case, where the surface integral does not vanish, Equation (2.5) shows that the pressure perturbation $p'(\boldsymbol{x},t)$ is given by

$$\begin{aligned}p'(\boldsymbol{x},t) = &\int_{-T}^{T} \int_{V} \frac{D_0^3 g(\boldsymbol{y},\tau \mid \boldsymbol{x},t)}{Dt^3}\tilde{\omega}_c\left(\tau - \frac{y_1}{U(\boldsymbol{y}_T)}, \boldsymbol{y}_T\right) d\boldsymbol{y}d\tau \\ &-\int_{-T}^{T} \int_{S} \hat{n}_i c^2 \left[\frac{D_0^3 g(\boldsymbol{y},\tau \mid \boldsymbol{x},t)}{Dt^3}\Lambda_i - \frac{\partial}{\partial y_i}\left(\frac{D_0^3 g(\boldsymbol{y},\tau \mid \boldsymbol{x},t)}{Dt^3}\right)\frac{D_0\phi}{D\tau}\right] dS(\boldsymbol{y})d\tau\end{aligned} \tag{2.20}$$

which reduces to

$$p'(\boldsymbol{x},t) = \int_{-T}^{T} \int_{V} \frac{D_0^3 g(\boldsymbol{y},\tau \mid \boldsymbol{x},t)}{Dt^3}\tilde{\omega}_c\left(\tau - \frac{y_1}{U(\boldsymbol{y}_T)}, \boldsymbol{y}_T\right) d\boldsymbol{y}d\tau \tag{2.21}$$

when the surface integral vanishes, i.e., when g and ϕ satisfy appropriate boundary conditions on any cylindrical surfaces that may be present in the flow.

2.1   Interpretation of Results

Equation (2.18) can be shown to generalize the Orr result (1.5) by inserting Equation (B.12) into that result, noting that the integral over the second term vanishes and that the Green's function (1.6) is self-adjoint (i.e., $g_0(\boldsymbol{y},\tau \mid \boldsymbol{x},t) = g_0(\boldsymbol{y},\tau \mid \boldsymbol{x},\tau)$) to show that it reduces to (1.5) for two dimensional incompressible

flows with constant mean shear when the arbitrary convected quantity $\tilde{\omega}_c\,(\tau - y_1/U(\boldsymbol{y}_T), \boldsymbol{y}_T)$ is replace by the renormalized quantity

$$\omega_c\left(\tau - \frac{y_1}{U(\boldsymbol{y}_T)}, \boldsymbol{y}_T\right) \equiv \tilde{\omega}_c\left(\tau - \frac{y_1}{U(\boldsymbol{y}_T)}, \boldsymbol{y}_T\right)|\nabla U|\,/\,c^2 \qquad (2.22)$$

which has dimensions of vorticity (based on the rescaled velocity u_i). The most significant difference between these results is that the convected quantity ω_c is no longer equal to the vorticity.

Möhring (1976) pointed out that the velocity fluctuations on a uniform base flow satisfy a third order wave equation which can be integrated to obtain an inhomogeneous second order equation whose homogeneous solutions can then be identified with the acoustic motion. But in the present case the acoustic motion must be identified with the homogeneous solutions of the third-order wave Equation (2.9), which cannot, in general, be reduced to second order because these solutions have both acoustic and hydrodynamic components. It can, however, be reduced to second order when the mean flow does not support hydrodynamic motion. This can be done for uniform mean flows by introducing the dependent variable $\Phi \equiv D_0\phi/D\tau$ (which can then be used in (2.5) and (2.6) to calculate the pressure and velocity). A similar reduction can also be done for two-dimensional flows with constant c^2 and mean shear (see Möhring, 1976). This can be seen by differentiating (2.9) with respect to y_2 and using (2.7) and (2.10) to obtain

$$\frac{D_0^2}{D\tau^2}\Lambda_2 - \frac{\partial^2}{\partial y_2^2}c^2\Lambda_2 - \frac{\partial^3}{\partial y_1^2\partial y_2}\left(c^2\frac{D_0\phi}{D\tau}\right) = -\frac{\partial}{\partial y_2}\tilde{\omega}_c\left(\tau - \frac{y_1}{U}, y_2\right) \qquad (2.23)$$

which becomes upon using (2.6) to eliminate Λ_2

$$\frac{1}{\lambda}\frac{\partial}{\partial y_1}\left[\frac{D_0^4\phi}{D\tau^4} - \frac{\partial^2}{\partial y_2^2}\left(c^2\frac{D_0^2\phi}{D\tau^2}\right)\right] - \frac{\partial^3}{\partial y_1^2\partial y_2}\left(c^2\frac{D_0\phi}{D\tau}\right) - \frac{1}{\lambda}\left(\frac{D_0^2 u_1}{D\tau^2} - \frac{\partial^2}{\partial y_2^2}c^2 u_1\right) = -\frac{\partial}{\partial y_2}\tilde{\omega}_c\left(\tau - \frac{y_1}{U}, y_2\right) \qquad (2.24)$$

And using (2.8) to eliminate $D_0^4\phi\,/\,D\tau^4$ leads to Möhring's (1976) result

$$\left[\left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}\right) - \frac{1}{c^2}\frac{D_0^2}{D\tau^2}\right]u_1 = -\frac{\lambda}{c^2}\frac{\partial}{\partial y_2}\tilde{\omega}_c\left(\tau - \frac{y_1}{U}, y_2\right) = -\frac{\partial}{\partial y_2}\omega_c\left(\tau - \frac{y_1}{U}, y_2\right) \qquad (2.25)$$

Möhring (1976) also showed that a similar result can be obtained for the three dimensional motion on two-dimensional base flows with constant c^2 and mean shear.

The unsteady motion is given by (2.5) to (2.7) with ϕ determined by the *inhomogeneous* solution (2.17) of (2.9) for the general transversely sheared mean flow considered in this paper. In the absence of solid surfaces and other external sources the resulting unsteady flow consists entirely of subsonically propagating disturbances when the flow is subsonic and, therefore, cannot radiate to the far field when the flow is unbounded (Goldstein, 2005 and 2009). This can easily be verified in any particular case by working out the relevant far field expansion. It is therefore appropriate to identify it with the hydrodynamic component of the motion. As in the Kovasznay result this flow is itself the sum of two parts, but in this case one of the parts (the part that is driven by the arbitrary convected quantity $\tilde{\omega}_c\,(\tau - y_1/U, \boldsymbol{y}_T)$) produces pressure fluctuations while the other part, which is driven by the arbitrary convected quantity $\vartheta(\tau - y_1/U, \boldsymbol{y}_T)$, does not. But unlike the Kovasznay result, the two arbitrary convected quantities are not experimentally observable variables.

3.0 Relation Between the Physical Variables and the Convected Quantities

Formulations that represent upstream boundary conditions in terms of the arbitrary convected quantities, $\tilde{\omega}_c\,(\tau - y_1/U, \boldsymbol{y}_T)$ and $\vartheta(\tau - y_1/U, \boldsymbol{y}_T)$, are most useful when these quantities can be related to the physical (preferably measurable) flow variables. This section shows that there is a local (i.e., non-integral) equation that relates a linear combination of $\tilde{\omega}_c\,(\tau - y_1/U, \boldsymbol{y}_T)$ and $\vartheta(\tau - y_1/U, \boldsymbol{y}_T)$ to these variables. The results are obtained by using relations derived in Appendix A. Relations between the vorticity and pressure fluctuations for the combined acoustic-like and discrete hydrodynamic component of the unsteady motion are also derived.

Equations (A.8) and (A.12) show that

$$\frac{\partial}{\partial y_1}\left(\tilde{\omega}_c - p' - \frac{\partial N_i}{\partial y_i}\eta_\perp\right) = -\frac{\partial}{\partial y_i}\left[\varepsilon_{ijk}\left(\omega_j - \omega_j^{(\infty)}\right)N_k\right] \tag{3.1}$$

and

$$N_i\left[\left(\omega_i - \omega_i^{(\infty)}\right) + \varepsilon_{ij1}\frac{\partial\eta_\perp}{\partial y_j}\right] = 0 \tag{3.2}$$

where $\tilde{\omega}_c$ is related to the rescaled vortical like quantity ω_c by (2.22),

$$N_i \equiv \frac{c^2}{|\nabla U|^2}\frac{\partial U}{\partial y_i}, \tag{3.3}$$

$$\omega_i \equiv \varepsilon_{ijk}\frac{\partial u_k}{\partial y_j} \tag{3.4}$$

is a density weighted vorticity based on u_i,

$$\omega_i^{(\infty)} \equiv \varepsilon_{knm}\varepsilon_{ijk}\frac{\partial}{\partial y_j}\frac{1}{c^2}\frac{\partial U}{\partial y_n}\frac{\partial\vartheta}{\partial y_m} = \left(\delta_{in}\delta_{jm} - \delta_{im}\delta_{jn}\right)\frac{\partial}{\partial y_j}\frac{1}{c^2}\frac{\partial U}{\partial y_n}\frac{\partial\vartheta}{\partial y_m}$$
$$= \frac{\partial}{\partial y_j}\frac{1}{c^2}\left(\frac{\partial U}{\partial y_i}\frac{\partial\vartheta}{\partial y_j} - \frac{\partial U}{\partial y_j}\frac{\partial\vartheta}{\partial y_i}\right) = \frac{\partial}{\partial y_j}\left[\frac{1}{c^2}\frac{\partial(U,\vartheta)}{\partial(y_i,y_j)}\right] \tag{3.5}$$

which has zero divergence, is the vorticity component generated by the second convected quantity ϑ,

$$\eta_\perp \equiv \frac{\partial U}{\partial y_i}\Lambda_i \tag{3.6}$$

can be interpreted as a transverse particle displacement since it follows from (2.6) that

$$u_i\frac{\partial U}{\partial y_i} = \frac{D_0}{D\tau}\eta_\perp \tag{3.7}$$

and

$$\frac{\partial(U,V)}{\partial(y_i,y_j)} \equiv \frac{\partial U}{\partial y_i}\frac{\partial V}{\partial y_j} - \frac{\partial U}{\partial y_j}\frac{\partial V}{\partial y_i} \qquad (3.8)$$

denotes the Jacobian. Notice that

$$\frac{D_0}{D\tau}\frac{\partial(U,\vartheta)}{\partial(y_i,y_j)} = 0 \qquad (3.9)$$

i.e., that $\partial(U,\vartheta)/\partial(y_i,y_j)$ is also a convected quantity. And it follows from (3.5) that

$$\frac{D_0}{D\tau}\omega_i^{(\infty)} = -\frac{1}{c^2}\frac{\partial U}{\partial y_j}\left(\frac{\partial U}{\partial y_i}\frac{\partial}{\partial y_j} - \frac{\partial U}{\partial y_j}\frac{\partial}{\partial y_i}\right)\frac{\partial \vartheta}{\partial y_1} \qquad (3.10)$$

and, therefore that

$$\frac{D_0}{D\tau}\left(\frac{\partial U}{\partial y_i}\omega_i^{(\infty)}\right) = 0 \qquad (3.11)$$

Equations (3.1) and (3.5) relate the arbitrary convected quantities $\tilde{\omega}_c\,(\tau - y_1/U(\boldsymbol{y}_T), \boldsymbol{y}_T)$ and $\vartheta(\tau - y_1/U(\boldsymbol{y}_T), \boldsymbol{y}_T)$ to the pressure p', the transverse vorticity components $\varepsilon_{ijk}\omega_j N_k$ and the particle displacement $\eta_\perp$, while Equations (3.2) and (3.5) relate the arbitrary convected quantity $\vartheta(\tau - y_1/U(\boldsymbol{y}_T), \boldsymbol{y}_T)$ to the pressure p', the normal vorticity component $\omega_j N_j$ and the particle displacement $\eta_\perp$. But the particle displacement $\eta_\perp$ (which, unlike the pressure and vorticity components, is not a physical variable in the usual sense) can be eliminated between Equations (3.1) and (3.2) when $\partial N_i/\partial y_i \neq 0$ to obtain the following equation

$$\chi_c\left(\tau - y_1/U\left(\boldsymbol{y}_T\right), \boldsymbol{y}_T\right) = \frac{\partial U}{\partial y_i}\left\{\varepsilon_{ij1}\frac{\partial}{\partial y_j}\left[\left(\frac{\partial N_k}{\partial y_k}\right)^{-1}\left(\frac{\partial p'}{\partial y_1} - \varepsilon_{lnm}\frac{\partial N_m\omega_n}{\partial y_l}\right)\right] - \frac{\partial \omega_i}{\partial y_1}\right\} \qquad (3.12)$$

that relates the linear combination

$$\chi_c\left(\tau - y_1/U\left(\boldsymbol{y}_T\right), \boldsymbol{y}_T\right) \equiv \frac{\partial U}{\partial y_i}\left\{\varepsilon_{ij1}\frac{\partial}{\partial y_j}\left[\left(\frac{\partial N_k}{\partial y_k}\right)^{-1}\left(\frac{\partial \tilde{\omega}_c}{\partial y_1} - \varepsilon_{lnm}\frac{\partial N_m\omega_n^{(\infty)}}{\partial y_l}\right)\right] - \frac{\partial \omega_i^{(\infty)}}{\partial y_1}\right\} \qquad (3.13)$$

of the two arbitrary convected quantities $\tilde{\omega}_c\,(\tau - y_1/U(\boldsymbol{y}_T), \boldsymbol{y}_T)$ and $\vartheta(\tau - y_1/U(\boldsymbol{y}_T), \boldsymbol{y}_T)$ to the physical variables p' and ω_i. Equations (3.3), (3.10), (3.11) and (A.7) show that $\chi_c(\tau - y_1/U(\boldsymbol{y}_T), \boldsymbol{y}_T)$ is itself an arbitrary convected quantity.

3.1 Application to Planar Mean Flows

These results are fairly complex, but most of the relevant literature is concerned with the small amplitude motion on planar base flows, where c^2 and U depend on a single Cartesian coordinate (say y_2). Equations (3.1), (3.2), and (3.5) then become much simpler and reduce to

$$\frac{\partial}{\partial y_1}\left(\tilde{\omega}_c - p' - \frac{dN_2}{dy_2}\eta_\perp\right) = N_2\left(\frac{\partial \omega_3}{\partial y_1} - \frac{\partial \omega_1}{\partial y_3}\right) \tag{3.14}$$

$$\frac{1}{N_2}\nabla_\parallel^2 \vartheta = \omega_2 + \frac{\partial \eta_\perp}{\partial y_3} \tag{3.15}$$

where

$$\nabla_\parallel^2 \equiv \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_3^2} \tag{3.16}$$

These equations show that the arbitrary convected quantities $\tilde{\omega}_c (\tau - y_1/U(\boldsymbol{y}_T), \boldsymbol{y}_T)$ and $\vartheta(\tau - y_1/U(\boldsymbol{y}_T), \boldsymbol{y}_T)$ are now related to p' the vorticity components ω_1, ω_3 and the particle displacement $\eta_\perp$ by fairly simple relations. Equations (3.12) and (3.13) now simplify to

$$\tilde{\chi}_c (\tau - y_1 / U(y_2), \boldsymbol{y}_T) = \frac{\partial}{\partial y_1}\left[\frac{\partial p'}{\partial y_3} - \frac{\partial}{\partial y_2}(N_2\omega_2)\right] - N_2\nabla_\parallel^2 \omega_1 \tag{3.17}$$

where

$$\tilde{\chi}_c (\tau - y_1 / U(y_2), \boldsymbol{y}_T) \equiv \frac{\partial}{\partial y_1}\left(\frac{\partial \tilde{\omega}_c}{\partial y_3} - \frac{1}{N_2}\frac{dN_2}{dy_2}\nabla_\parallel^2 \vartheta\right), \tag{3.18}$$

and we have used the fact that the solenoidal vectors ω_i and $\omega_i^{(\infty)}$ have zero divergence.

Notice that Equation (3.14) is now independent of $\omega_i^{(\infty)}$ and, therefore of the second convected quantity ϑ. But the divergence $\partial N_i/\partial y_i$ is equal to zero for the constant shear-constant c^2 parallel mean flow (1.3), since N_i is a is constant in that case and it follows from (2.22) that Equation (3.14) then reduces to Möhring's (1976) result

$$\frac{\partial}{\partial y_1}\left(\omega_c - \frac{p'\lambda}{c^2}\right) = \frac{\partial \omega_3}{\partial y_1} - \frac{\partial \omega_1}{\partial y_3} \tag{3.19}$$

that relates the single convected variable ω_c to the physical variables. Equation (3.17) relates the combined convected quantity $\tilde{\chi}_c (\tau - y_1/U(y_2), \boldsymbol{y}_T)$ to the physical variables p' and ω_i.

4.0   Concluding Remarks

This paper is based on a formal solution to the linearized Euler equations for a transversely sheared mean flow which shows that the general solution to these equations is given by an *inhomogeneous* solution to the adjoint Equation (2.9) and, like the Kovasznay decomposition for the unsteady motion on uniform flows, involves two arbitrary convected quantities $\vartheta(\tau - y_1/U, \boldsymbol{y}_T)$ and $\tilde{\omega}_c (\tau - y_1/U, \boldsymbol{y}_T)$ that can be used to specify appropriate boundary conditions for unsteady surface interaction problems and, thereby, extend the class of flows for which this can be done to arbitrary transversely sheared mean flows. It is shown that this equation reduces to the second order wave equation derived by Möhring (1976) for two dimensional motions on the linear base flow (1.3) with constant sound speed once an appropriate dependent variable is introduced. But unlike the Kovasznay result, the hydrodynamic component carries pressure fluctuations and the acoustic component is not irrotational. Section 3.0 shows that there is a

linear combination of the convected quantities that is equal to a linear combination of the pressure and vorticity

The results developed in this paper are expected to be particularly useful for representing incident vortical disturbances in Rapid Distortion Theory problems involving the interaction of turbulence with surfaces embedded in transversely sheared base flows or, more generally, in vortical base flows that asymptote to transversely sheared mean flows in the upstream region. Equations (2.18), (2.21), and similar equations that can be derived for the remaining velocity components, determine the velocity and pressure fluctuations in terms of the arbitrary convected quantities $\tilde{\omega}_c\,(\tau - y_1/U, \boldsymbol{y}_T)$ and $\vartheta(\tau - y_1/U, \boldsymbol{y}_T)$ in these flows.

One such application currently under investigation is to the noise generated by the interaction of a high Reynolds number rectangular jet with a nearby flat plate (Afsar, Goldstein and Leib 2013). This work applies the formalism developed in the present paper to predict the acoustic spectrum of the noise scattered from the trailing edge of the plate and compares it with experimental data. The predictions require a model for the autocovariance of the convected quantity $\tilde{\omega}_c\,(\tau - y_1/U, \boldsymbol{y}_T)$, whose validity can be tested by using the relations developed in Section 3.0 to relate it to physical quantities that can be measured in the laboratory.

Appendix A.—Relation Between Convected Quantities and Physical Variables

Equations (2.5) to (2.7), (2.9) and (2.10) imply that the convected quantity $\tilde{\omega}_c$ is related to the pressure perturbation p' and the generalized particle displacement Λ_i by

$$\tilde{\omega}_c = p' + \frac{\partial}{\partial y_i} c^2 \Lambda_i \tag{A.1}$$

$$u_i = \frac{D_0}{D\tau} \Lambda_i - \delta_{i1} \eta_\perp + \varepsilon_{ijk} \frac{1}{c^2} \frac{\partial U}{\partial y_j} \frac{\partial}{\partial y_k} \vartheta\left(\tau - \frac{y_1}{U}, \mathbf{y}_T\right) \tag{A.2}$$

where is $\eta_\perp$ given by (3.6).

The density-weighted vorticity ω_i is therefore, given by

$$\omega_i \equiv \varepsilon_{ijk} \frac{\partial u_k}{\partial y_j} = \frac{D_0}{D\tau} \varepsilon_{ijk} \frac{\partial \Lambda_k}{\partial y_j} + \varepsilon_{ijk} \frac{\partial U}{\partial y_j} \frac{\partial \Lambda_k}{\partial y_1} - \varepsilon_{ij1} \frac{\partial \eta_\perp}{\partial y_j} + \varepsilon_{knm} \varepsilon_{ijk} \frac{\partial}{\partial y_j} \frac{1}{c^2} \frac{\partial U}{\partial y_n} \frac{\partial \vartheta}{\partial y_m} \tag{A.3}$$

But since Equation (2.7) and the identities

$$\varepsilon_{ijk} \frac{\partial^2 U}{\partial y_j \partial y_k} = \varepsilon_{ijk} \frac{\partial^2 \phi}{\partial y_j \partial y_k} = 0 \tag{A.4}$$

show that

$$\varepsilon_{ijk} \frac{\partial \Lambda_k}{\partial y_j} = 2\varepsilon_{ijk} \frac{\partial U}{\partial y_k} \frac{\partial^2 \phi}{\partial y_1 \partial y_j} \tag{A.5}$$

this can be written as

$$\omega_i = -\varepsilon_{ij1} \frac{\partial \eta_\perp}{\partial y_j} + \varepsilon_{ijk} \frac{\partial U}{\partial y_j} \frac{\partial}{\partial y_1}\left(\Lambda_k - 2\frac{D_0}{D\tau} \frac{\partial \phi}{\partial y_k}\right) + \omega_i^{(\infty)} \tag{A.6}$$

where $\omega_i^{(\infty)}$ is given by (3.5). The definition (2.7) and the identity

$$\varepsilon_{ijk} \frac{\partial U}{\partial y_j} \frac{\partial U}{\partial y_k} = 0 \tag{A.7}$$

can now be used to show that (A.6) can be written as

$$\omega_i = -\varepsilon_{ij1} \frac{\partial \eta_\perp}{\partial y_j} - \varepsilon_{ijk} \frac{\partial U}{\partial y_j} \frac{\partial \Lambda_k}{\partial y_1} + \omega_i^{(\infty)} \tag{A.8}$$

And it, therefore, follows from (3.6) that the cross product $\varepsilon_{ijk} \omega_j \partial U / \partial y_k$ can be written as

$$\varepsilon_{ijk}\omega_j\frac{\partial U}{\partial y_k} = -\varepsilon_{ijk}\varepsilon_{1jn}\frac{\partial \eta_\perp}{\partial y_n}\frac{\partial U}{\partial y_k} - \varepsilon_{ijk}\varepsilon_{jmn}\frac{\partial U}{\partial y_m}\frac{\partial \Lambda_n}{\partial y_1}\frac{\partial U}{\partial y_k} + \varepsilon_{ijk}\omega_j^{(\infty)}\frac{\partial U}{\partial y_k}$$

$$= -\left(\delta_{i1}\delta_{kn} - \delta_{in}\delta_{k1}\right)\frac{\partial \eta_\perp}{\partial y_n}\frac{\partial U}{\partial y_k} + \left(\delta_{im}\delta_{kn} - \delta_{in}\delta_{km}\right)\frac{\partial U}{\partial y_m}\frac{\partial \Lambda_n}{\partial y_1}\frac{\partial U}{\partial y_k} + \varepsilon_{ijk}\omega_j^{(\infty)}\frac{\partial U}{\partial y_k} \qquad (A.9)$$

$$= -\delta_{i1}\frac{\partial \eta_\perp}{\partial y_k}\frac{\partial U}{\partial y_k} + \frac{\partial U}{\partial y_i}\frac{\partial \eta_\perp}{\partial y_1} - \frac{\partial U}{\partial y_k}\frac{\partial U}{\partial y_k}\frac{\partial \Lambda_i}{\partial y_1} + \varepsilon_{ijk}\omega_j^{(\infty)}\frac{\partial U}{\partial y_k}$$

which can be solved for $\partial \Lambda_i / \partial y_1$ to obtain

$$\frac{\partial \Lambda_i}{\partial y_1} = \left[\frac{\partial U}{\partial y_i}\frac{\partial \eta_\perp}{\partial y_1} - \delta_{1i}\frac{\partial \eta_\perp}{\partial y_k}\frac{\partial U}{\partial y_k} - \varepsilon_{ijk}\left(\omega_j - \omega_j^{(\infty)}\right)\frac{\partial U}{\partial y_k}\right]\bigg/|\nabla U|^2 \qquad (A.10)$$

Inserting this into (A.1) shows that

$$\frac{\partial \tilde{\omega}_c}{\partial y_1} = \frac{\partial}{\partial y_1}p' + \frac{\partial}{\partial y_i}\left\{\frac{c^2}{|\nabla U|^2}\left[\frac{\partial U}{\partial y_i}\frac{\partial \eta_\perp}{\partial y_1} - \delta_{1i}\frac{\partial \eta_\perp}{\partial y_k}\frac{\partial U}{\partial y_k} - \varepsilon_{ijk}\left(\omega_j - \omega_j^{(\infty)}\right)\frac{\partial U}{\partial y_k}\right]\right\} \qquad (A.11)$$

and, therefore, that

$$\frac{\partial \tilde{\omega}_c}{\partial y_1} = \frac{\partial}{\partial y_1}p' + \left[\frac{\partial}{\partial y_i}\left(\frac{\partial U}{\partial y_i}\frac{c^2}{|\nabla U|^2}\right)\right]\frac{\partial \eta_\perp}{\partial y_1} - \varepsilon_{ijk}\frac{\partial}{\partial y_i}\left(\omega_j\frac{c^2}{|\nabla U|^2}\right)\frac{\partial U}{\partial y_k} + \varepsilon_{ijk}\frac{\partial}{\partial y_i}\left(\omega_j^{(\infty)}\frac{c^2}{|\nabla U|^2}\right)\frac{\partial U}{\partial y_k} \qquad (A.12)$$

or

$$\frac{\partial \tilde{\omega}_c}{\partial y_1} = \frac{\partial}{\partial y_1}\left[p' + \eta_\perp\frac{\partial}{\partial y_i}\left(\frac{\partial U}{\partial y_i}\frac{c^2}{|\nabla U|^2}\right)\right] - \varepsilon_{ijk}\frac{\partial}{\partial y_i}\left[\frac{c^2}{|\nabla U|^2}\left(\omega_j - \omega_j^{(\infty)}\right)\frac{\partial U}{\partial y_k}\right] \qquad (A.13)$$

where the identity (A.7) has again been used.

Appendix B.—Limiting Form of Equation (2.18) for Linear Base Flows

The easiest way to show that Equation (2.18) reduces to Equation (1.5) is by noting that the Fourier transform

$$\bar{G}_i\left(\boldsymbol{y}_T \mid \boldsymbol{x}_T : \omega, k_1\right) \equiv \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\left[\omega(t-\tau)-k_1(x_1-y_1)\right]} g_i\left(\boldsymbol{y},\tau \mid \boldsymbol{x},t\right) d\left(x_1 - y_1\right) d\left(t - \tau\right) \tag{B.1}$$

of $g_i(\boldsymbol{y},\tau \mid \boldsymbol{x},t)$ is related to the reduced Green's function $\bar{G}\left(\boldsymbol{y}_T \mid \boldsymbol{x}_T : \omega, k_1\right)$

$$\bar{G}\left(\boldsymbol{y}_T \mid \boldsymbol{x}_T : \omega, k_1\right) \equiv \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\left[\omega(t-\tau)-k_1(x_1-y_1)\right]} \frac{D_0^3 g\left(\boldsymbol{y},\tau \mid \boldsymbol{x},t\right)}{Dt^3} d\left(x_1 - y_1\right) d\left(t - \tau\right) \tag{B.2}$$

which satisfies the reduced Rayleigh equation

$$L_R \bar{G} \equiv \nabla_T \cdot \left[\frac{c^2 \nabla_T \bar{G}}{\left(\omega - Uk_1\right)^2} \right] + \left[1 - \frac{c^2 k_1^2}{\left(\omega - Uk_1\right)^2} \right] \bar{G} = \frac{1}{(2\pi)^2} \delta\left(\boldsymbol{x}_T - \boldsymbol{y}_T\right) \tag{B.3}$$

with $U = U(y_T)$ and $\nabla_T \equiv \left\{ \partial / \partial y_2, \partial / \partial y_3 \right\}$ by

$$\bar{G}_i\left(\boldsymbol{y}_T \mid \boldsymbol{x}_T : \omega, k_1\right) \equiv \frac{1}{\left[ik_1 U\left(\boldsymbol{x}_T\right) - i\omega\right]} \frac{\partial}{\partial x_i} \bar{G}\left(\boldsymbol{y}_T \mid \boldsymbol{x}_T : \omega, k_1\right), \tag{B.4}$$

since $g(\boldsymbol{y},\tau \mid \boldsymbol{x},t)$ depends on τ, t and y_1, x_1 only in the combinations $t - \tau$ and $x_1 - y_1$, respectively.

It, therefore, follows that

$$\bar{G}_2\left(\boldsymbol{y}_T \mid \boldsymbol{x}_T : \omega, k_1\right) = \frac{1}{\left[ik_1 U\left(x_2\right) - i\omega\right]} \frac{\partial}{\partial x_2} \bar{G}\left(\boldsymbol{y}_T \mid \boldsymbol{x}_T : \omega, k_1\right)$$

$$= \frac{1}{\left[ik_1 U\left(x_2\right) - i\omega\right]} \frac{\partial}{\partial x_2} \bar{G}\left(\boldsymbol{x}_T \mid \boldsymbol{y}_T : \omega, k_1\right) \tag{B.5}$$

and

$$\left(\omega - Uk_1\right)^2 \frac{\partial}{\partial x_2} \left[\frac{1}{\left(\omega - Uk_1\right)^2} \frac{\partial \bar{G}}{\partial x_2} \right] + \frac{\partial^2 \bar{G}}{\partial x_3^2} - k_1^2 \bar{G} = \frac{\left(\omega - Uk_1\right)^2}{(2\pi c)^2} \delta\left(\boldsymbol{x}_T - \boldsymbol{y}_T\right) \tag{B.6}$$

for planar base flows with constant sound speed—which means that

$$\frac{\partial}{\partial x_2}\left\{(\omega-Uk_1)^2\frac{\partial}{\partial x_2}\left[\frac{i\bar{G}_2}{(\omega-Uk_1)}\right]\right\}+i(\omega-Uk_1)\left(\frac{\partial^2\bar{G}_2}{\partial x_3^2}-k_1^2\bar{G}_2\right)$$

$$=\frac{-1}{(2\pi c)^2}\frac{\partial}{\partial x_2}(\omega-Uk_1)^2\delta(\boldsymbol{x}_T-\boldsymbol{y}_T) \tag{B.7}$$

$$=i\frac{\partial}{\partial x_2}\left\{(\omega-Uk_1)\frac{\partial\bar{G}_2}{\partial x_2}+\lambda k_1\bar{G}_2\right\}+i(\omega-Uk_1)\left(\frac{\partial^2\bar{G}_2}{\partial x_3^2}-k_1^2\bar{G}_2\right)$$

and, therefore that

$$\left(\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}-k_1^2\right)\bar{G}_2=\frac{i}{(2\pi c)^2(\omega-Uk_1)}\frac{\partial}{\partial x_2}(\omega-Uk_1)^2\delta(\boldsymbol{x}_T-\boldsymbol{y}_T)$$

$$=\frac{i}{(2\pi c)^2}\left[\frac{\partial}{\partial x_2}(\omega-Uk_1)\delta(\boldsymbol{x}_T-\boldsymbol{y}_T)-\lambda k_1\delta(\boldsymbol{x}_T-\boldsymbol{y}_T)\right] \tag{B.8}$$

for the linear profile (1.3). And it follows from (B.8) that

$$\bar{G}_2=-\lambda ik_1\bar{G}_0/c^2+\frac{\partial}{\partial x_2}\left[\frac{i(\omega-U(y_2)k_1)\bar{G}_0}{c^2}\right] \tag{B.9}$$

where

$$\bar{G}_0(\boldsymbol{y}_T\mid\boldsymbol{x}_T:\omega,k_1)\equiv\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{i\left[\omega(t-\tau)-k_1(x_1-y_1)\right]}g_0(\boldsymbol{y},\tau\mid\boldsymbol{x},t)d(x_1-y_1)d(t-\tau) \tag{B.10}$$

satisfies

$$\left(\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}-k_1^2\right)\bar{G}_0=\frac{1}{(2\pi)^2}\delta(\boldsymbol{x}_T-\boldsymbol{y}_T) \tag{B.11}$$

Taking the inverse Fourier transform of (B.9) shows that

$$g_2(\boldsymbol{y},\tau\mid\boldsymbol{x},t)=-\frac{\lambda}{c^2}\frac{\partial}{\partial x_1}g_0(\boldsymbol{y},\tau\mid\boldsymbol{x},t)+\frac{\partial}{\partial x_2}\frac{1}{c^2}\frac{D_0}{D\tau}g_0(\boldsymbol{y},\tau\mid\boldsymbol{x},t), \tag{B.12}$$

while the limiting form of Equation(B.11) for a completely two dimension flow is just the Fourier transform of Equation (1.6).

References

Afsar, M., Goldstein, M.E. and Leib. S. J., 2013, Prediction of low-frequency trailing edge noise using rapid distortion theory, submitted to 14th European Turbulence Conference, Lyon, France, Sept 1-4, 2013.

Colonius, T, 2004, Modeling artificial boundary conditions for compressible flow, Annual Rev. Fluid Mech. 36, pp. 315–45

Drazin, P. G., Reid, W. H., 1981, Hydrodynamic Stability, Cambridge University Press, Cambridge, UK.

Emanuel, G., 1993, Analytical Fluid Dynamics, CRC Press Ann Arbor.

Gartshore, I.S., Durbin, P.A. and Hunt, J.C.R., 1983. The production of turbulent stress in a shear flow by irrotational fluctuations. J. Fluid Mech., 137, 307-329.

Goldstein, M.E., 1978(a), Unsteady vortical and entropic distortions of potential flows round arbitrary obstacles, J. Fluid Mech., Vol. 89, pp. 433-468

Goldstein, M.E., 1978(b), Characteristics of the unsteady motion on transversely sheared mean flows, J. Fluid Mech., Vol. 84, Part 2, pp. 305-329

Goldstein, M.E., 1979(a), Scattering and distortion of the unsteady motion on transversely sheared mean flows, J. Fluid Mech. Vol. 91, pp. 601-632

Goldstein, M.E., 1979(b), turbulence generated by entropy fluctuation with non-uniform mean flows , J. Fluid Mech., Vol. 93, Part 2, pp. 209-224

Goldstein, M.E., 2005, On Identifying the true sources of aerodynamic sound , J. Fluid Mech. Vol. 526, pp. 337-347

Goldstein, M.E., 2009, A theoretical basis for identifying the sound sources in a turbulent flow, International Journal of Aeroacoustics, Vol. 8, No. 4, pp. 283-300

Goldstein, M.E. and Durbin, P. A. 1980, The effect of finite turbulence spatial scale on the amplification of turbulence by a contracting stream, J. Fluid Mech., Vol. 98, pp. 473-508

Hunt, J. C. R., 1973, A theory of turbulent flow around two dimensional bluff bodies J. Fluid Mech. Vol. 61, pp. 625-706

Hunt, J. C. R., 1977, A review of the theory of rapidly distorted flows and its application s. 13th Biennial Fluid Dyn. Symp., Poland.

Kovasznay, L. S. G., 1953, Turbulence in supersonic flow, J. Aero Sci. 20, No. 10, pp. 657-674.

Lele, S. K, 1998, Direct Numerical Simulations of compressible turbulent flows: fundamentals and applications. Book chapter in Transition, Turbulence and combustion modeling lecture notes, 2nd Ercoftac summer school, Stockholm, Sweden.

Möhring, W., 1976 Über Schallwellen in Scherströmungen, Fortschritte der Akustik. DAGA 76 VDI, pp. 543-546

Nazarenko, S.V., Kevlahan, N. and Dubrulle, B., A WKB theory for rapid distortion of inhomogeneous turbulence, Journal of Fluid Mech., **390** (1999) 325.

Morse, P. M. and Feshbach, H., 1953, Methods of Theoretical Physics, McGraw-Hill Book Company

Orr, W. (1907), The stability and instability of the steady motions of a perfect liquid and of a viscous liquid, Proc. Roy. Irish Acad., A 27, 9-68 and 69-138

Sagaut, P. and Cambon, C., 2008, Homogeneous turbulence Dynamics, Cambridge University Press, Cambridge U.K

Sears, W.R., 1941, Some aspects of non-stationary airfoil theory and its practical application, J. Aero. Sci. Vol. 83, p. 104

Tam, C. K.W. and Auriault, L. 1998, mean flow refraction effects on sound radiated from localized sources in a jet, J. Fluid Mech. Vol. 370, pp. 149-174

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13. SUPPLEMENTARY NOTES

14. ABSTRACT
This paper considers the small amplitude unsteady motion of an inviscid non-heat conducting compressible fluid on a transversely sheared mean flow. It extends a previous result which shows that the hydrodynamic component of the motion is determined by two arbitrary convected quantities in the absence of solid surfaces or other external sources. The result is important because it can be used to specify appropriate boundary conditions for unsteady surface interaction problems on transversely sheared mean flows in the same way that the vortical component of the Kovasznay decomposition is used to specify these conditions for surface interaction problems on uniform mean flows. But unlike the Kovasznay case the arbitrary convected quantities no longer bear a simple relation to the physical variables. One purpose of this paper is to derive a formula that relates these quantities to the (physically measurable) vorticity and pressure fluctuations in the flow.

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