# Localized wave structures: Solitons and beyond

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The review is concerned with solitary waves and other localized structures in the systems described by a variety of generalizations of the Korteweg–de Vries (KdV) equation. Among the topics we focus upon are "radiating solitons", the generic structures made of a soliton-like pulses and oscillating tails. We also review properties of solitary waves in the generalized KdV equations with the modular and "sublinear" nonlinearities. Such equations have an interesting class of solutions, called *compactons*, solitary waves defined on a finite spatial interval. Both the properties of single solitons and the interactions between them are discussed. We show that even minor non-elastic effects in the soliton-soliton collisions can accumulate and result in a qualitatively different asymptotic behavior. A statistical description of soliton ensembles ("soliton gas") which emerges as a major theme has been discussed for several models. We briefly outline the recent progress in studies of ring solitons and lumps within the framework of the cylindrical KdV equation and its two-dimensional extension. Ring solitons and lumps (2D solitons) are of particular interest since they have many features in common with classical solitons and yet are qualitatively different. Particular attention is paid to interactions between the objects of different geometries, such as interaction of ring solitons and shear flows, ring solitons and lumps, lumps and line solitons. We conclude our review with views of the future developments of the selected lines of studies of localized wave structures in the theory of weakly nonlinear, weakly dispersive waves.

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**We review results and trends in studies of localized wave structures, which is an a vast area of research. Our review of 2015 was confined to a selection of particular models of classical one-dimensional solitary waves in weakly dispersive media. Here we focus upon results concerned with a broader class of localized structures, not confining our consideration to solitary waves understood as steady localized solutions. In particular, we discuss *radiating solitons*, the structures which are neither stationary nor localized, their tails are localized only at finite times. We discuss how the radiation can destroy solitary waves, how the structures with radiating tails can be born and how they might be destroyed by a mild inhomogeneity. We pick up examples for the radiating soliton discussion primarily in the context of internal gravity waves in a rotating stratified ocean, but the conclusions are relevant for a wide class of nonlinear systems.**

**The second major topic of consideration is the rapidly growing area of studies of solitary waves in the systems where nonlinearity cannot be approximated by commonly used power-like dependences; we overview recent results on solitons in systems with a variety of 'non-traditional' nonlinearities, including non-analytic, modular, and with the degree of nonlinearity less than one. Such 'non-traditional' nonlinearities lead to a plethora of exotic solitary structures, such as, for example, a class of '*pyramidal*' solitons. When the nonlinearity is less than one, the solitary waves are confined to a finite spatial interval and, therefore, are called *compactons*. Although the systems with such nonlinearities are non-integrable, interactions between compactons are qualitatively similar to those in the familiar integrable systems like the Korteweg–de Vries equation. Particular attention is paid to interactions of ensembles of solitons in the systems allowing for different polarities of solitons; in such systems, their interactions might lead to rogue wave-type solutions. The kinetics of various solitons in ensembles is viewed primarily from the perspective of rogue wave formation.**

**We also review the substantial recent progress in studies of cylindrical solitons. These structures are of particular interest since they have many features in common with the classical solitons and yet are qualitatively different. We also outline our views of the future development of the selected threads of studies of localized wave structures in the theory of weakly nonlinear weakly dispersive waves.**

Contents



## I. Introduction

In 2015, the authors of this paper attempted to sketch the progress in a few chosen directions of nonlinear wave theory in the aftermath of its revolutionary development in the 1960s–1970s. As a perhaps most characteristic example, we described various generalizations of the Korteweg-de Vries (KdV) equation which played an outstanding role in pioneering development of exact and asymptotic methods for many wave equations closely focusing on the concept of soliton and its particle-like properties.

Here, we revisit the issues aiming at a broader view of growing number of studies of localized structures, sometimes related but different from the KdV-type solitons. Broader mathematical and physical studies further developed the "soliton science" and its applications but, perhaps most importantly, involved other localized objects as well. And, as it often happens with new areas of mathematical physics, we can see a trend to apply mathematical achievements to physical problems, such as, for example, those related to oceanic waves and currents. In the decade after the publication of our previous review [Ostrovsky et al., 2015], a notable progress has been achieved in several branches of the soliton theory. Understandably, here we cannot cover all the variety and breadth of relevant new developments of the last decade and concentrate only on a few characteristic problems, which were closer to our interests. They include but not limited to:

- Complex behavior of solitons including their emergence under different competing factors such as low frequency dispersion, caused by, e.g., the Earth rotation, inhomogeneity and interaction with a long wave, and a subsequent vanishing due to radiation.

- Two- dimensional localized objects, solitary waves, and two-dimensional structures.

- Rather unusual structures such as solitons in systems with the KdV-type dispersion and various nonlinearities. Among them are "compactons" in systems with modular nonlinearity that resemble those introduced by Rosenau [1997] in equations with nonlinear dispersion.

It is also appropriate to state at the very beginning what major issues related to localized structures, we do not touch in our review. Localized structures caused by wave collapses could look very similar to solitary waves, however, since there are comprehensive reviews on solitons and collapses [Zakharov & Kuznetsov, 2012; Malomed, 2022], we left this topic aside. Solitons which in 2015 we viewed as solitary patterns can interact; rapidly advancing studies of their multiple collisions ("soliton gas" or "soliton kinetics") are reviewed in, e.g., Ref. [El, 2021]. Here we singled out only one aspect of such kinetics related to emergence of rogue waves.

Another area that is well covered in the literature and therefore not considered here is the so-called *dispersive shock waves* (also dubbed *solibores*). In such formations, solitons gradually emerge from stepwise initial perturbation. Albeit the fundamentals of dispersive shocks were developed fifty years ago [Gurevich & Pitaevskii, 1974; Whitham, 1974], the area is experiencing a renaissance, see the reviews [El & Hoefer, 2016; Kamchatnov, 2021].

One more area that is left outside of the scope of this review relates to solitary waves in nonconservative media with both energy pumping and dissipation being essential (the so-called *autosolitons*). This is a separate, very broad area related to many aspects of biophysics, chemistry, lasers, etc, (see, e.g., Kerner & Osipov [1994] and references therein).

Whereas considerable attention is paid here to the KdV-type equations with a variety of nonlinearities, we set aside the active research of evolution equations with generalizations of the dispersion terms. An overview of a particularly rapidly developing area concerned with fractional differential equations can be found in recent publications [Malomed, 2024; Kevrekids & Guevas-Maraver, 2024]. Dispersion in such systems is described by integral operators. At present, the evolution equations with integral dispersion are not well understood except for the Benjamin–Ono equation and its hierarchy (see, e.g., [Saut, 2019] and references therein).

A plethora of publications on the forced KdV and related forced evolution equations requires a separate dedicated review, and it is not discussed here.

Although we focus primarily on theoretical models of localized patterns, whenever possible, we try to relate the mathematical results with physical applications.

## II. Radiating solitons: their birth, life and destruction

### A. Radiation from attenuating solitons

We begin with some history. After the analogy between solitons and elastic material particles was established by considering the interaction of solitons, which gave the name to solitons, the most fundamental question to understand is what happens beyond the realm of exact solutions of integrable equations. A natural way towards this understanding is to add a small perturbation to such equations and consider how this perturbation affects the wave evolution. We begin with the perturbed Korteweg–de Vries (KdV) equation in the dimensionless form:

$$u_t + uu_x + u_{xxx} = \mu R(u), \qquad (1.1)$$

where $\mu$ is a small parameter and $R$ is an operator that can be responsible for a variety of perturbations. The effects of various perturbing factors such as different mechanisms of dissipation, front curvature, medium rotation, etc., on soliton evolution have been considered beginning from the 1970s; here we show how these studies developed more recently focusing on the non-localized field component generated by a perturbed soliton. This component can be dubbed soliton radiation, while the whole pattern, i.e. the localized pulse plus the nonlocalized component is natural to refer to as a *radiating soliton*. First, we briefly outline an asymptotic perturbation scheme for solitons known since the 1970s (see, e.g., [Gorshkov et al., 1974; Kaup &

Newell, 1978; Grimshaw, 1979; Gorshkov & Ostrovsky, 1981; Kivshar & Malomed, 1989]). The radiation from solitons is a truly universal phenomenon emerging in a large variety of physical contexts. Here we shall briefly outline the basics of the theory in its more recent form (see the books by Ostrovsky [2015; 2022] and references therein). The solution of Eq. (1.1) with a single soliton as a basic approximation is represented as:

$$u(x,t) = A(T)\,\mathrm{sech}^2\frac{\zeta}{\Delta(T)} + \sum_{n=1}^{J}\mu^n u^{(n)}(\zeta,T), \tag{1.2}$$

where $\Delta(T) = (12/A(T))^{1/2}$, $\mu \ll 1$ is a small parameter, $T = \mu t$ is a "slow time" and $\zeta = x - \int V(T)dt$, where $V = A/3$ is soliton velocity. Substitution of (1.2) into (1.1) yields in each order of $\mu$ the linear equation:

$$Gu^{(n)} = \frac{d}{d\zeta}\left[-V + U(\zeta) + \frac{d^2}{d\zeta^2}\right]u^{(n)} = H^{(n)},$$

where $H^{(n)}$ contains derivatives of the previous-order perturbations and the corresponding terms in expansion of $R(u)$. To keep all perturbations limited, the following "compatibility conditions" must be met: [Gorshkov & Ostrovsky, 1981]:

$$\int_{-\infty}^{\infty} UH^{(n)}d\zeta = 0, \quad \lim_{\zeta\to\pm\infty} H^{(n)} = 0. \tag{1.3}$$

Here $U(\zeta,T)$ is the unperturbed solution, in this case a localized solitary wave, and $H$ is a non-localized component ("radiation"); in the first approximation, $H^{(1)} = R(U) - U_T$. It is also assumed that in the first order, $R(U)$ is also localized.

The basic, first-order approximation for the soliton amplitude satisfies the equation:

$$\frac{dA}{dT} = \frac{4}{3}\frac{\int_{-\infty}^{\infty}\varphi(\theta)R[\varphi(\theta)]d\theta}{\int_{-\infty}^{\infty}\varphi^2(\theta)d\theta} = \int_{-\infty}^{\infty}\mathrm{sech}^2\theta\cdot R\left[\mathrm{sech}^2\theta\right]d\theta, \tag{1.4}$$

where $\theta = \zeta/\Delta$ and $\varphi(\theta) = U(\theta)/A = \mathrm{sech}^2\theta$. Many solutions of this or similar equations for different forms of the operator $R$ are known. For example, if $R = -qu$ (Rayleigh dissipation), soliton amplitude decreases exponentially: $A = A_0\exp(-4qT/3)$ (that is faster than $\exp(-qT)$ known for a linear wave). If $R = \gamma u_{xx}$ that corresponds to the Burgers type dissipation which mimics the effect of viscosity in fluids (in this case, Eq. (1.1) is the KdV–Burgers equation), then the soliton amplitude varies as [Gorshkov & Papko, 1977]:

$$A(T) = \frac{A_0}{1 + 4\gamma A_0 T / 45}. \tag{1.5}$$

It is noteworthy that at large times when $A_0T >> 45/4\gamma$, we have $A \approx 45/4\gamma T$, i.e., the amplitude ceases to depend on its initial value. Note that the nonlinear Chezy law, $R \sim u|u|$ (obtained, in particular, by Miles (1983), for water waves with bottom friction), yields the same damping law for a soliton as given by Eq. (1.5). A similar decay law was found to occur for weak shock waves in nonlinear gas dynamics and acoustics (see, e.g., [Naugolnykh & Ostrovsky, 1998]). However, this similarity is superficial, it is due to the specific relation between the KdV soliton amplitude and width; in general, these two types of dissipation produce quite different solutions. Various examples of attenuating solitons in physics and hydrodynamics can be found in the books and reviews of different years (see, e.g., [Ostrovsky, 2022] and numerous publications referred there).

A comprehensive analysis of the decay laws within the framework of perturbed most common evolution equations such as Korteweg–de Vries (KdV), Benjamin–Ono (BO), Kadomtsev–Petviashvili (KP), and rotation modified KdV (rKdV) equation with different dissipative terms, was presented in some early works within the perturbed KdV equation and in the recent papers within the perturbed KP and BO equations [Clarke et al., 2018; Grimshaw et al., 2018]. All these equations can be written in the form similar to Eq. (1.1):

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \alpha u\frac{\partial u}{\partial x} + \beta \hat{L}[u] + \delta \hat{D}[u] = \mu\left(\gamma \int u dx - \frac{c}{2}\int \frac{\partial^2 u}{\partial y^2} dx\right), \tag{1.6}$$

where $\mu$ is a small parameter, while the coefficients $c > 0$, $\alpha$, $\beta$, and $\gamma > 0$ depend on the environmental parameters of the particular medium (e.g., in the context of water waves they depend on the depth, stratification, shear flow, etc.), $\hat{L}[u]$ is a linear dispersion operator whose Fourier image is $k^3$ in the case of the KdV or KP equations and it is $|k|k$ in the case of the BO equation. In particular, for the KdV and KP equations, $\hat{L}[u] \equiv \frac{\partial^3 u}{\partial x^3}$, whereas for the BO equation, $\hat{L}[u] \equiv \frac{1}{\pi}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{+\infty} \frac{u(\xi,t)}{\xi - x} d\xi$. Here $\hat{D}[\upsilon]$ is the dissipative linear operator which can be expressed in the rather general form [Grimshaw, 2001]:

$$D[\upsilon] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (-ik)^m \tilde{\upsilon}(k,t) e^{ikx} dk\,, \tag{1.7}$$

where $\tilde{\upsilon}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \upsilon(x,t) e^{-ikx} dx$ is the Fourier transform of $u(x, t)$, and the parameter $m$ depends on the specific type of dissipation. In particular, $m = 0$ (with $\delta > 0$) corresponds to the linear Rayleigh damping when the dissipative term $\delta\hat{D}[u]$ in Eq. (1.6) reduces simply to $\delta u$. In the widely used model of dissipation with $m = 2$ and $\delta < 0$, $\hat{D}[u] = u_{xx}$ and Eq. (1.6) reduces to the KdV–Burgers equation; $m = 1/2$ is customarily used for modeling dissipation in bottom boundary layers. Note that in general both operators $\hat{L}[u]$ and $\hat{D}[u]$ can be fractional; in particular, as aforementioned, the BO equation can be considered as the fractional equation. In some cases, operators $\hat{L}[u]$ and $\hat{D}[u]$ can be nonlinear (see, for example, [Rosenau, 1977]). A well-known example of nonlinear operator $\hat{D}[u] \sim |u|u$ was used to model shallow water wave decay over a rugged bottom - the Chezy law of dissipation (see, e.g. [Grimshaw, 2001]).

Recall, that in the absence of dissipation ($\delta = 0$, $\gamma = 0$), the KdV, BO, and KP equations have soliton solutions. For the former two equations solitons are described by the following expressions:

$$u_{KdV}(x,t) = A\,\mathrm{sech}^2 \frac{x - Vt}{\Delta}; \qquad u_{BO}(x,t) = \frac{A}{1 + (x - Vt)^2/\Delta^2}, \tag{1.8}$$

where all parameters of solitons (width $\Delta$ and speed $V$) are here presented in terms of the amplitude $A$ and parameters of nonlinearity and dispersion $\alpha$ and $\beta$. For the KdV soliton $\Delta = (12\beta/\alpha A)^{1/2}$, $V = \alpha A/3$, whereas for the BO soliton $\Delta = 4\beta/\alpha A$, $V = \alpha A/4$

The KP equation has qualitatively different properties depending on the sign of the dispersion coefficient. When the coefficient $\beta > 0$, plane KdV-type solitons are stable with respect to small transverse perturbations and can propagate at a small but arbitrary angle to the $x$-axis. This situation occurs for shallow-water waves and for numerous other types of waves in media with negative dispersion; the corresponding equation is known as the KP2 equation. In the case of positive dispersion, $\beta < 0$, more common for waves in plasmas and solids, the basic equation is dubbed the KP1 equation. In such cases plane KdV-type solitons are unstable with respect to small transverse perturbations; however, another type of stable fully localized 2D solitons dubbed *lumps*, can exist (see, e.g., [Ablowitz & Segur, 1981]):

$$u(\xi, y, t) = 8\frac{V}{\alpha}\frac{1+\left(y/\Delta_y\right)^2+\left(\xi+|V|t\right)^2/\Delta_x^2}{\left[1+\left(y/\Delta_y\right)^2+\left(\xi+|V|t\right)^2/\Delta_x^2\right]^2}, \tag{1.9}$$

where $\xi = x - c\,t$, $V < 0$ is the speed of a lump in the Galilean coordinate frame moving with the speed $c$ with respect to immovable observer, $\Delta_x = \sqrt{24|\beta/\alpha A|}$, $\Delta_y = \sqrt{96c|\beta|/(\alpha A)^2}$ .

The decay laws for solitary waves derived within three basic models, KdV, BO, and KP1 are summarized in Table 1. In the case of the Rayleigh dissipation with $m = 0$, the amplitude of a solitary wave decays exponentially with time, $A(t) = A_0 e^{-t/\tau}$ In all other cases of dissipation, the decay has a power-type character: $A(t) = A_0\left(1+t/\tau\right)^{-n}$ with different powers $n$ and characteristic time $\tau$. For small dissipation, this asymptotic formula agrees well with direct numerical modelling of Eq. (1.6), KdV equation, BO equation, and KP1 equation (see, e.g., [Clarke et al., 2018; Grimshaw et al., 2018]).

It is interesting to note that the approximate formula for the adiabatic decay of BO solitons due to the Landau damping derived within the asymptotic theory, proved to be the exact solution to the BO equation with the Landau damping term [Grimshaw et al., 2018]. This is, apparently, the only known example of an exact nonstationary solution of nonlinear equation with the decaying solitary wave due to dissipation. It is noteworthy that all these mechanisms of dissipation lead to power laws of attenuation with the exception of the Rayleigh case, where the decay is exponential. Here $\Delta_0$ is the initial width of a soliton.

Clarke et al. [2018] also studied the adiabatic decay of lumps moving at an angle to the main *x*-axis within the KP1 equation and discovered that under the influence of dissipation a rectilinear lump motion is no longer possible: lump trajectories become curved, i.e. if a lump starts moving at a small angle to the *x*-axis, the angle increases in the course of lump motion. Two typical examples of lump trajectories under the influence of dissipation are shown in Fig. 1.

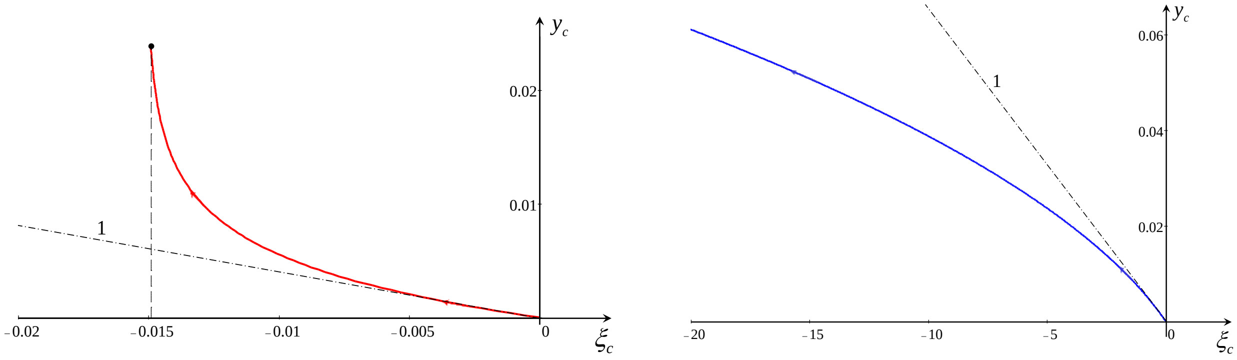
Table 1 (in the last two rows the index *m* is undetermined)

| Type of dissipation | Characteristic decay time of a KdV soliton [Grimshaw, 2001] | Characteristic decay time of a BO soliton [Grimshaw et al., 2018] | Characteristic decay time of a KP1 lump [Clarke et al., 2018] |
|---|---|---|---|
| Rayleigh dissipation, $m = 0$, $A = A_0 e^{-t/\tau}$ | $\tau = \frac{3}{4\delta}$ | $\tau = \frac{1}{2\delta}$ | $\tau = \frac{1}{4\delta}$ |
| Burgers dissipation, $m = 2$ | $A(t) = A_0(1 + t/\tau)^{-1}$; $\tau = \frac{15}{16}\frac{\Delta_0^2}{\lvert\delta\rvert}$ | $A(t) = A_0(1 + t/\tau)^{-1/2}$; $\tau = \frac{1}{2}\frac{\Delta_0^2}{\lvert\delta\rvert}$ | $A(t) = A_0(1 + t/\tau)^{-1}$; $\tau = \frac{\Delta_0^2}{8\lvert\delta\rvert}$ |
| Landau damping, $m = 1$ | $A(t) = A_0(1 + t/\tau)^{-1}$; $\tau = 48\sqrt{2\pi}\left\lvert\frac{\beta A_0}{\delta}\right\rvert\Delta_0^2$ | $A(t) = A_0(1 + t/\tau)^{-1}$; $\tau = \frac{\Delta_0}{\lvert\delta\rvert}$ | $A(t) = A_0(1 + t/\tau)^{-1}$; $\tau = 48\sqrt{2\pi}\left\lvert\frac{\beta A_0}{\delta}\right\rvert\Delta_0^4$ |
| Decay in a laminar boundary layer, $m = 1/2$ | $A(t) = A_0(1 + t/\tau)^{-4}$; $\tau = \frac{97}{2}\frac{A_0^2}{\lvert\delta\rvert}\Delta_0^{9/2}$ | $A(t) = A_0(1 + t/\tau)^{-2}$; $\tau = \frac{2}{\lvert\delta\rvert}\sqrt{\frac{\Delta_0}{\pi}}$ | $A(t) = A_0(1 + t/\tau)^{-4}$; $\tau \approx 10.2\frac{A_0^2}{\lvert\delta\rvert}\Delta_0^{9/2}$ |
| Nonlinear Chezy dissipation $R \sim \lvert u\rvert u$ | $A(t) = A_0(1 + t/\tau)^{-1}$; $\tau = \frac{5}{64}\left\lvert\frac{\alpha}{\beta\delta}\right\rvert\Delta_0^2$ | $A(t) = A_0(1 + t/\tau)^{-1}$; $\tau = \frac{\alpha}{6\beta\lvert\delta\rvert}\Delta_0$ | $A(t) = A_0(1 + t/\tau)^{-1}$; $\tau = \left\lvert\frac{\alpha}{\beta\delta}\right\rvert\frac{\Delta_0^2}{24}$ |
| Radiative dissipation in rotating media – see below | $A(t) = A_0(1 - t/\tau)^2$; $\tau = \frac{1}{\gamma\Delta_0}$ | $A(t) = A_0(1 - t/\tau)$; $\tau = \frac{1}{2\pi\gamma\Delta_0}$ | $A(t) = A_0(1 - t/\tau)^2$; $\tau = \frac{3\sqrt{2}}{4\gamma\Delta_0}$ |

Let us return to the problem of soliton radiation, that is the non-localized part of the perturbation. As mentioned, it is described by the second equation (1.3). Although at large $\lvert\zeta\rvert$, both $U$ and $R$ are exponentially small, the perturbation field $u^{(1)}$ can be non-vanishing. The expression for $u^{(1)}$ in this external area is:

$$u^{(1)}(\zeta, T, X) \approx C^{(1)}(T, X) - \frac{1}{V}\int_0^{\zeta}\left(R(U) - U_T\right)d\xi'. \qquad (1.10)$$

Here $C^{(1)}$ is an integration constant and $X = \mu x$ is the “slow” coordinate.




a) b)

Fig. 1. An example of lump trajectory within the KP1 equation with the Rayleigh dissipation (a) and with the Burgers dissipation (b). The vertical dashed line in (a) shows the `extinction distance' at which a lump completely vanishes as it approaches the black dot. Dashed-dotted lines 1 in both frames show the unperturbed trajectories. The coordinate subscript '$c$' stands for the lump centre. From [Clarke et al., 2018].

Note that for the soliton (1.2) and $|\zeta| >> \Delta$, we have:

$$\int_0^{\zeta} U_T \, d\zeta' = \sqrt{\frac{3}{A}}\frac{dA}{dT}\tanh\frac{\zeta}{\Delta}. \tag{1.11}$$

Here $\Delta = \sqrt{12/A}$ is the characteristic length of the soliton (as in Eq. (1.2)). In general, such perturbations can exist on each side of the localized solitary wave from each side of the latter. They can be matched on the trajectory of the soliton, $X = X_s(T)$ as

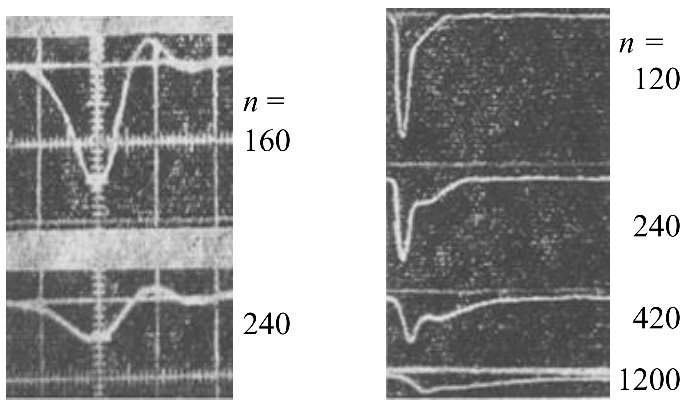
$$\left(u_+^{(1)} - u_-^{(1)}\right)\Big|_{X=X_s} = \frac{2}{V^{3/2}}\int_{-\infty}^{\infty} R(U)\left(\operatorname{sech}^2\theta - 1\right)d\theta. \tag{1.12}$$

Here the subscripts + and – refer to the regions ahead and behind the soliton, respectively. If, in particular, the initial condition defines a non-perturbed soliton at $X = 0$, i.e., $u_\pm^{(1)}(X,0) = 0$, we have $u_+^{(1)} \equiv 0$ (no radiation ahead of the soliton), and the radiation field behind the soliton has the form of a slowly varying "shelf" expanding into the region $0 < X < X_s$:

$$u_-^{(1)}(X,T) = u_-^{(1)}(T_s(X))\exp[-\delta(T - T_s(X))], \;\; 0 \le X \le X_s. \tag{1.13}$$

Here $\delta$ is a constant depending on the form of the functional $R$ in Eq. (1.1), and $T_s(X)$ is the time when the soliton center is located at $X = X_s$, , that is the function inverse to $X_s(T) = \int_0^T V(T')dT'$. Note also that the parameter $C^{(1)}$ in Eq. (1.10) is equal to $u_-^{(1)}/2$ taken at the soliton trajectory $X = X_s(T)$.

On the slow scale $X$, the solution (1.13) has a “jump” at $X = X_s(T)$. In reality, the “jump” is a transition area having a scale of the order of the soliton width $\Delta$. In their early work, Gorshkov and Papko [1977a] studied the cases of the Rayleigh and Burgers’ dissipation. In the former case the radiation field oscillates (close to Airy function) whereas in the latter case, the formation of a “shelf” behind a soliton eventually leads to into a triangle-shaped impulse attenuating according to the Burgers equation. They also experimentally observed this process in a nonlinear electric line (Fig. 2).




a) b)

Fig. 2. Oscillograms illustrating evolution of a soliton in a nonlinear electromagnetic line (a) with the Rayleigh-type dissipation, (b) with the Burgers – type dissipation. The numbers next to the photos show the numbers of line cells equivalent to the distances along the line. An impulse close to a KdV soliton was generated at $n = 1$. The time dependence of the voltage across the line is shown. Adapted from [Gorshkov & Papko, 1977a]

### B. The low-frequency dispersion due to the effect of fluid rotation

Another widely discussed version of Eq. (1.6) which takes into account the low-frequency dispersion caused by rotation is the rKdV equation [Ostrovsky, 1978]. In the one-dimensional case it can be rescaled to the form used in e.g. [Grimshaw et al., 1998a]:

$$\left(u_t + 3uu_x + u_{xxx}/4\right)_x = \mu u/2. \tag{1.14}$$

Specific features of this quite universal evolution equation were discussed in many publications, including our previous reviews [Ostrovsky et al, 2015; Stepanyants, 2020]. However, here we focus on the effect of radiation caused by rotation. First, note the "antisoliton theorem" [Leonov, 1981; Galkin & Stepanyants, 1991] stating that there are no stationary solitary waves in Eq. (1.14) due to a synchronously radiated "tail" carrying soliton's energy away. There is also a "zero-mass" constraint: an integral over $x$ of any localized or periodic condition is zero. As shown in [Grimshaw et al., 1998a; 1998b], the soliton amplitude decays as:

$$A(T) = A_0\left(1 - t/\tau\right)^2, \tag{1.15}$$

where $A_0 = A(t = 0)$, and $\tau = 1/\Delta_0$ (see the last row in Table 1) is the extinction time. Here the soliton disappears in a finite time $t = \tau$ being transformed into radiation (whereas the total wave energy is conserved). In [Grimshaw et al., 1998a] this effect was called "*terminal damping*."

The radiation field was also considered in [Grimshaw et al., 1998a]. This field consists of two components. One is a "nearfield" perturbation following synchronously the soliton until it disappears. It has the form]:

$$u = \sqrt{2}\sin\kappa\zeta, \quad w = -2\sqrt{A}\cos\kappa\zeta, \quad \kappa = \sqrt{\mu/2A}. \tag{1.16}$$

This quasi-stationary field spreads behind the soliton up to a distance $|\zeta| << \Delta/\mu$, where $\Delta$ is the characteristic soliton width. Behind that area there exists a non-stationary wave depending on slow variables of $X$ and $T$. In this area the wave is quasi-harmonic, with varying frequency and wave number. Note that, as follows from the dispersion relation of the linearized rKdV equation, $\omega = -k^3 + 1/k$, in the long-wave limit (neglecting the term with $k^3$), the wave group velocity $c_g = -1/k^2 < 0$. This means that in the laboratory frame of reference, the linear "tail" propagates more slowly than the soliton and carries its energy away. At a fixed wave group, the wave amplitude decreases as $T^{-1/2}$. The total structure of the field generated by a soliton is schematically shown in Fig. 3a). For details, see [Grimshaw et al., 1998a].

Another relevant characteristic example is the tail structure behind the adiabatically varying KdV soliton due to the cylindrical divergence that was studied by Johnson [1999] and Sidorovas et al. [2024]. These asymptotic solutions describing both the near-field and far-field tails were later found to be in a good agreement with the exact and numerical solutions presented in [Hu et al., 2023; Hu et al., 2024].

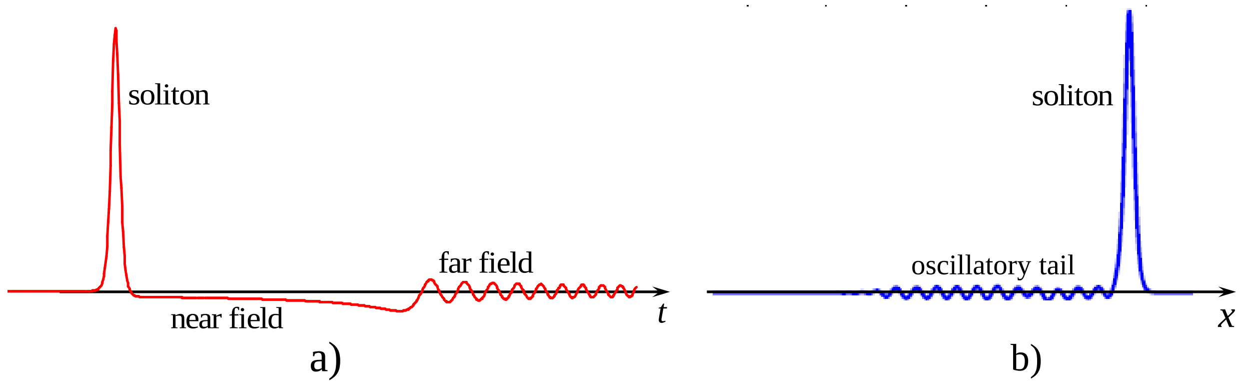



Fig. 3. Schematic of a radiating solitary wave in different systems: a) time dependence of $u$ at a fixed point (from [Hu et al., 2024]); b) snapshot of $u$ at a fixed time (from [Khusnutdinova et al., 2009]; used with permission).

Note that the tail structure can be different depending on the character of energy losses. In some cases, the tail can consist of only a shelf (that can be of either negative or positive polarity) without oscillations in the far field (see, for example, Fig. 2b). In other cases, the shelf can be absent, and small-amplitude oscillations are attached directly to the leading soliton as shown in Fig. 2a or even clearly in Fig. 3b from [Khusnutdinova et al., 2009].

### C. Solitons on a long wave with rotation

Now we shall describe more recent results related to soliton radiation. Note first that whereas the anti-soliton theorem is valid for the zero background, stable impulses close to solitons can exist if there is a source of energy compensating radiation losses. An early numerical observation of that was made in [Gilman et al., 1996] (see also [Chen & Boyd, 2001]), whereas the theory and detailed numerical study of this effect refers to [Ostrovsky and Stepanyants, 2016]. Starting from the same rKdV equation as above (i.e., Eq. (1.1) with $R_x = u$), the solution is represented as $u(t, x) = u_1(t, x) + u_2(t, x)$, where $u_1$ is a long background wave with the wavelength $\Lambda$, and $u_2$ is a KdV soliton with slowly varying amplitude and width corresponding to the first term in Eq. (1.2). If the soliton width is much smaller than $\Lambda$, the equations for $u_1$ and $u_2$ can be separated.

First, we assume that the function $u_1$ is given and it represents a particular stationary solution to rKdV in which the third-order derivative responsible for the small-scale dispersion is omitted (in that instance, it is often called the reduced rKdV). This stationary wave $u_2 = u_2(s = x-ct)$, where $c$ is a constant wave speed, satisfies the equation:

$$\frac{d^2}{ds^2}\left(\frac{1}{2}u_1^2 - cu_1\right) = u_1. \tag{1.17}$$

The shape of this wave can vary from the small- amplitude sinusoidal wave to the limiting periodic wave in the form of a sequence of parabolic arcs; all these waves have zero mean value. After separating, the small-scale wave (the soliton) and the long background wave, the equations describing evolution of a soliton with the amplitude $A(T)$ can be written in the form [Ostrovsky & Stepanyants, 2016]:

$$\begin{aligned}\frac{dS}{dT} &= \frac{A}{3} + u_1(S) - c,\\ \frac{dA}{dT} &= -\frac{4}{3}\frac{du_1}{dS} - 4\sqrt{3A},\end{aligned} \tag{1.18}$$

where phase $S$ is the soliton peak coordinate with respect to the minimum of the background wave profile in the reference frame moving with the background wave. Here the soliton amplitude is assumed to be much larger than that of the long wave. For any given $u_1(S)$, the analysis of this system is straightforward.

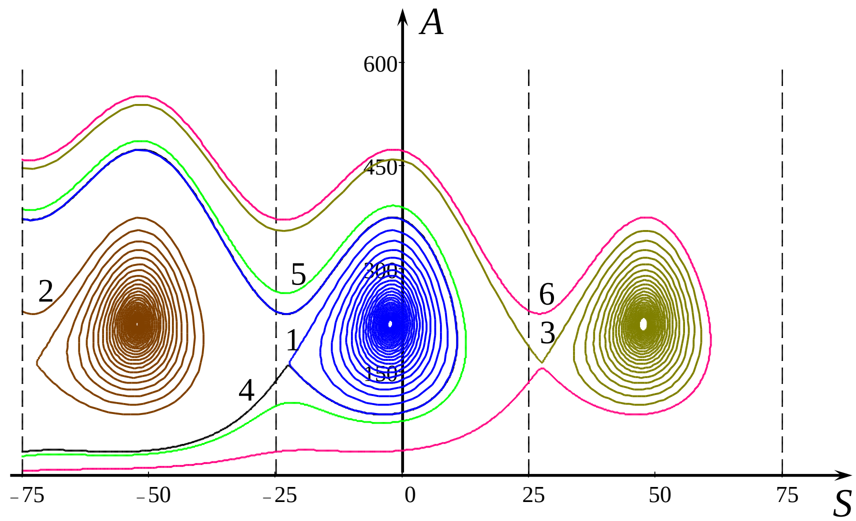



Fig. 4. The phase portrait of the dynamical system for three periods of a sinusoidal wave. Vertical dashed lines separate each wavelength. From [Ostrovsky & Stepanyants, 2016].

Figure 4 shows the phase plane of Eq. (1.18) for the case of a sinusoidal long wave with non-dimensional amplitude $U_0 = 10$ and length $\Lambda = 60$. There are two regimes of soliton dynamics, depending on the initial condition. The trajectories external with respect to the separatrices lead to a total damping of the soliton, whereas for those inside separatrices, the soliton is trapped within one wavelength and tends to a non-zero equilibrium. Similar results were obtained for a soliton riding on the long wave of maximal amplitude with a parabolic profile. An asymptotic position of the soliton is close (but not exactly equal) to the minimum of the parabola. The similar approach was used to describe a BO soliton riding on a long periodic wave in the deep two-layer fluid [Grimshaw et al., 2021].

In [Ostrovsky & Stepanyants, 2022] a more complex problem was considered when there are two solitons interacting with each other and with a long wave. In this case, four equations (two for soliton phases and two for their amplitudes ought to be solved instead of Eq. (1.18). One of the interesting results is that even if the initial amplitudes of solitons strongly differ, eventually they become close to each other.

### D. Joint action of rotation and inhomogeneity

A variety of scenarios emerges when a nonlinear wave propagates on an inhomogeneous background. This problem arises in wide variety of contexts, recall that the "rotation-type dispersion" does not need the true rotation to occur, it might be, for example, due to wave propagation through a media with a small scale random inhomogeneities [Benilov & Pelinovsky, 1988], magnetic field in quark-gluon plasma [Fogaça et al., 2020], or wave-guide dispersion when a wave is confined in the lateral direction like e.g. topographically trapped waves or acoustic waves in rods [Ostrovsky & Sutin, 1975], etc. Here we discuss this problem using as an example propagation of internal waves over a bottom topography on the rotating Earth. The rotation effect on nonlinear waves was observed by Farmer et al. [2009] in South China Sea. A consistent theoretical consideration with the account for topography was performed by Grimshaw et al. [2014] with the application to South China Sea too. They used an extension of the rKdV equation with the account of the term responsible for the bottom topography. In the physical variables the governing equation has the form:

$$\left(\eta_t + c\eta_x + \alpha\eta\eta_x + \beta\eta_{xxx} + \eta c Q_x / 2Q\right)_x = \gamma\eta. \qquad (1.19)$$

Here $c$ is the velocity of a long linear wave in the absence of rotation and topography, while $Q$ is proportional to the wave action. The parameters $\alpha(x), \beta(x), c(x), Q(x)$ depend on the fluid stratification, whereas $\gamma(x) = f^2/2c$, where $f$ is the Coriolis parameter. A similar equation without rotation was considered in many papers. Grimshaw et al. [2004, 2014] obtained an adiabatic solution of rKdV (1.19) for a soliton under the action of both factors: rotation and the horizontally inhomogeneous environment. In that work, Eq. (1.19) includes a horizontal flow with vertical shear $u_0(x, z)$. For $u_0 = 0$, we have $Q \sim c$, and at the adiabatic stage, soliton amplitude varies as [Stepanyants, 2019; Ostrovsky & Helfrich, 2019]:

$$A(x) = A_0\left[\frac{\alpha(x)\beta_0}{\alpha_0\beta(x)}\right]^{1/3}\left\{1-\Delta_0\int_{x_0}^{x}\frac{\gamma(x')}{c(x')}\left[\frac{\alpha(x)\beta_0}{\alpha_0\beta(x)}\right]^{2/3}dx'\right\}, \tag{1.20}$$

where $\Delta = \left(12\beta/\alpha A\right)^{1/2}$ is the characteristic soliton width and subscript 0 refers to initial values at $x = x_0$. The soliton "mass" varies as (Stepanyants, 2019):

$$M_s(x) = \int_{-\infty}^{\infty}\eta(x,t)dt = \frac{4}{c(x)}\sqrt{\frac{3A(x)\beta(x)}{\alpha(x)}}\,. \tag{1.21}$$

Since, as mentioned, the total field "mass" is zero, the "mass" of the radiated part is $M_r(x) = M_{s0} - M_s(x)$, and at the extinction moment $M_r = M_s(0)$.

Grimshaw et al. [2014] modified Eq. (1.19) by adding a cubic nonlinear term and employed to numerically simulate wave propagation in two directions from a source in Luzon Strait in the South China Sea. In parallel, they also simulated the full system of the Euler equations. The prime effect of rotation is the formation of a secondary wave train due to soliton radiation. Figure 5 illustrates this effect.

The above discussion of the effect of rotation on internal waves solitons illustrates a generic phenomenon: only strongly idealized equations admit soliton solutions; when perturbed they no longer support stationary soliton solutions; instead, the perturbed systems have solutions of the type of radiating solitons, i.e. a soliton-like pulse plus a radiating tail (with the only exception of decaying soliton within the BO equation with the Landau damping). The example we discussed at length demonstrates that the effect of such radiation can be quite profound. Although, indeed, at short time scales it results in a merely weak leakage of soliton energy and momentum, the losses are nonlinear and they destroy the soliton in finite time. The account of media inhomogeneity even in the absence of rotation causes a soliton to radiate and leads to solutions of the type of radiating solitons.

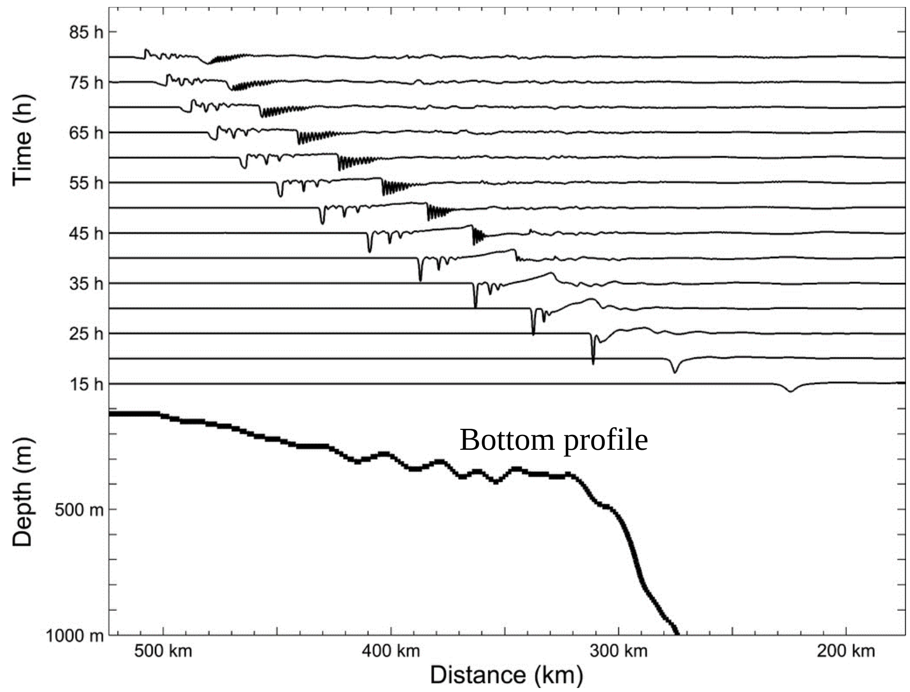



Fig. 5. Evolution of an internal soliton with initial amplitude of 90 m. The isopycnal of $\rho$ = 1024 kg/m$^3$, which is located at 100 m when at rest, is shown at time intervals of 5 h. The vertical axis for the topography also measures the displacement of the plotted isopycnals. From [Grimshaw et al., 2014] (© American Meteorological Society. Used with permission).

Here we focus on a different aspect of the effect of inhomogeneity - an alternative mechanism of destroying a soliton. The role of radiation is inessential here. In the process of wave evolution when the sign of quadratic nonlinearity coefficient $\alpha(x)$ changes, the wave undergoes a transformation changing its polarity with a possible formation of a new soliton. Naturally, near the point where quadratic nonlinearity coefficient changes its sign the cubic (or higher-order) nonlinearity becomes important. In the two-layer model, it happens when the thicknesses of the two layers, upper and lower, are equal, i.e. $h_1(x) = h_2(x)$. For the non-rotating fluid, the latter was considered by Grimshaw et al. [2010]. Ostrovsky & Helfrich [2019] considered this competition based on the "competition parameter" $G_s = X_{ex}/L$, where $X_{ex} = (c/\gamma)\sqrt{\alpha A_0/12\beta}$ is the extinction distance in the homogeneous case, corresponding to the dimensionless time $\tau$ in Eq. (1.15), and $L$ is the scale of bottom depth variation; for a two-layer fluid with a constant $h_1$ and constantly sloped $h_2$, $L$ is the distance at which $h_2$ turns to zero. Evidently, at small $G_s$ rotation effects dominate, and at large $G_s$ the bottom slope determines the wave evolution. An example of calculations is shown in Fig. 6. Here we illustrate both scenarios of soliton extinction outlined above. For a small bottom slope ($L$ = 178 km), the soliton amplitude goes to zero, due to radiation, before the point $h_2 = h_1$, similarly to the terminal damping model described above for a horizontally homogeneous layer. For steeper bottom slopes (smaller $L$), adiabatic theory predicts disappearance of the soliton at $h_2 = h_1$. Numerical calculations show some residual field at this point because they measure the field amplitude even when its profile strongly deviated from a soliton. As shown by [Grimshaw et al., 2004], after the destruction of the soliton near that point, a new, positive polarity soliton can be formed upon further propagation. The case of cubic nonlinearity is briefly discussed below.

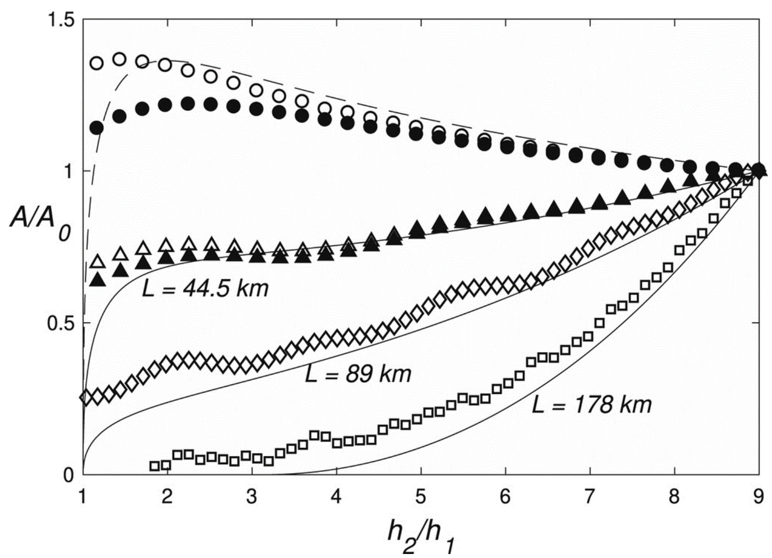



Fig. 6. Variation of soliton amplitude with the decrease of lower layer thickness (onshore propagation). Here $h_1$ = 50 m, $h_2$ = 450 m, $A_0$ = –10 m, $f = 10^{-5}$ s$^{-1}$, $\Delta\rho/\rho_1 = 5.1\cdot 10^{-4}$. The dashed line is adiabatic theory for the nonrotating case (valid for all slopes). Thin lines-adiabatic evolution with rotation for different bottom slopes (values of $L$). Open symbols show the corresponding numerical results. The solid symbols are for calculations from the nonrotating and rotating Gardner equations at $L$ = 44.5 km. From [Ostrovsky & Helfrich, 2019].

Figure 7 shows an example of wave propagation in one of these cases in which the terminal damping takes place. Before vanishing, the initial soliton generates the secondary one which propagates farther and eventually changes its polarity at around $x = 350h_0$ where $h_2 = h_1$.

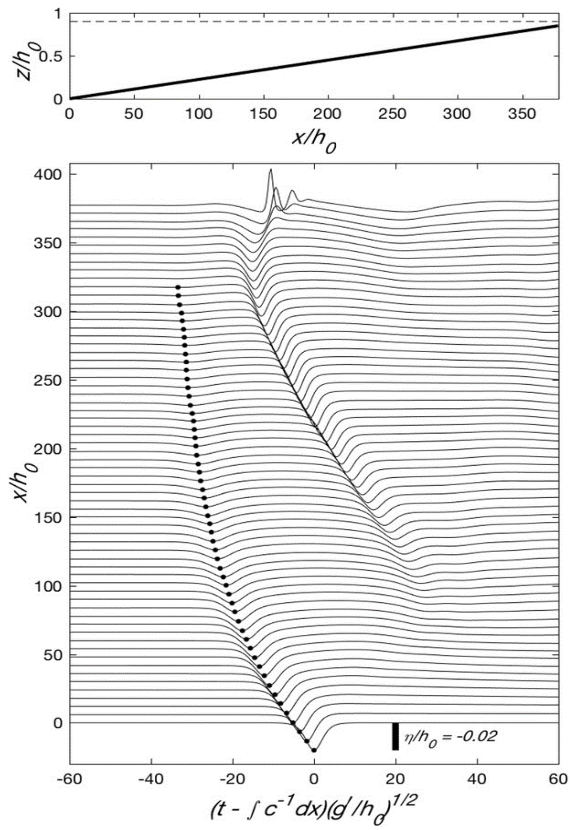



Fig. 7. A rKdV solution for the parameters shown in Fig. 5 and $L$ = 178 km. Top: The topography (heavy solid line) and the mean interface depth (dashed). Bottom: The numerical solution $\eta$ as a function of the "traveling" time $t - \int c^{-1}dx$, centered on the initial solitary wave. Here $x$, $z$, and $\eta$ are scaled with depth $h_0$ and $\tau$ with $(h_0/g')^{1/2}$. The $x$ locations of each time series of $\eta$ are indicated by the vertical axis. The dots show the local minima used to define the amplitude of the evolving solitary wave. From [Ostrovsky & Helfrich, 2019].

### E. The rKdV–Gardner equation

Accounting for the cubic nonlinearity in rKdV equation extends the variety of scenarios of soliton evolution. We already mentioned here some numerical solutions of this equation in application to internal waves. Its general form is [Holloway et al., 1999]:

$$\left(\eta_t + c\eta_x + \alpha\eta\eta_x + \alpha_1\eta^2\eta_x + \beta\eta_{xxx} + \eta cQ_x/2Q\right)_x = \gamma\eta. \quad (1.22)$$

A family of solitary solutions of this equation without rotation and inhomogeneity was studied in [Slunyaev & Pelinovskii, 1999; Slyunyaev, 2001, Grimshaw et al., 2010]. As an important particular case they include table-top structures with a limiting amplitude. Adiabatic evolution of a Gardner soliton in a homogeneous medium with rotation was considered in [Obregon et al., 2018]. Figure 8 shows the result of numerical solution for an initially table-top Gardner soliton. The formation of a "tail" destroying the soliton is well expressed there. As noted in [Obregon et al., 2018], further evolution transforms it into a bell-shaped impulse. A similar result was obtained in [Polukhina & Samarina, 2007] for the initially table-top Gardner soliton decaying in non-rotating medium due to the cylindrical divergence which we discuss below.

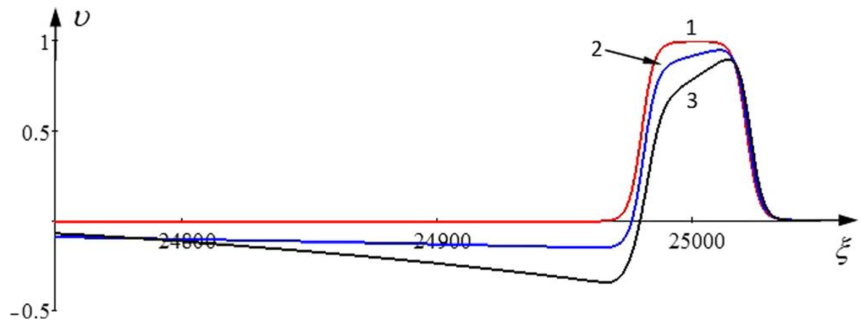



Fig. 8. Initial stage of degradation of a table-top soliton (line 1) under the action of rotation. Lines 2 and 3 show the variation of the wave profile in the course of solitary wave evolution. From [Obregon et al., 2018].

Joint contribution of rotation and inhomogeneity in the rKdV-Gardner equation was considered in [Helfrich & Ostrovsky, 2022]. First, an adiabatic solution for soliton evolution was obtained. It was noted that without rotation the soliton energy is conserved but its mass varies due to the inhomogeneity, since a trailing shelf is formed. In the typical case $\alpha_1 < 0$, when a soliton moves toward a point of polarity reversal, the trailing "tail" is negative so that the soliton mass increases due to its broadening [Grimshaw et al., 1998a]. In a homogeneous rotating environment both mass and energy decrease [Grimshaw et al., 1999]. With both homogeneity and rotation, the energy decreases but the wave mass can increase or decrease, depending on the interplay between these two effects. In [Helfrich & Ostrovsky, 2022], a numerical study of the above particular cases, as well as of the full rKdV equation with rotation and inhomogeneity was carried out. As for the rKdV equation, at small slopes a soliton attenuates to zero (terminal damping), but before that it radiates another soliton-like impulse (see Fig. 7). At steeper slopes which affect the wave stronger than rotation, the soliton reaches the point $h_2 = h_1$ where it is destroyed forming a complex oscillating train. Figure 9 shows wave shapes after passing that level ($x$ =1.2$L$ where, as above, $L$ is the distance to the point $h_2 = h_1$) at different values of $L$, i.e., bottom slopes.

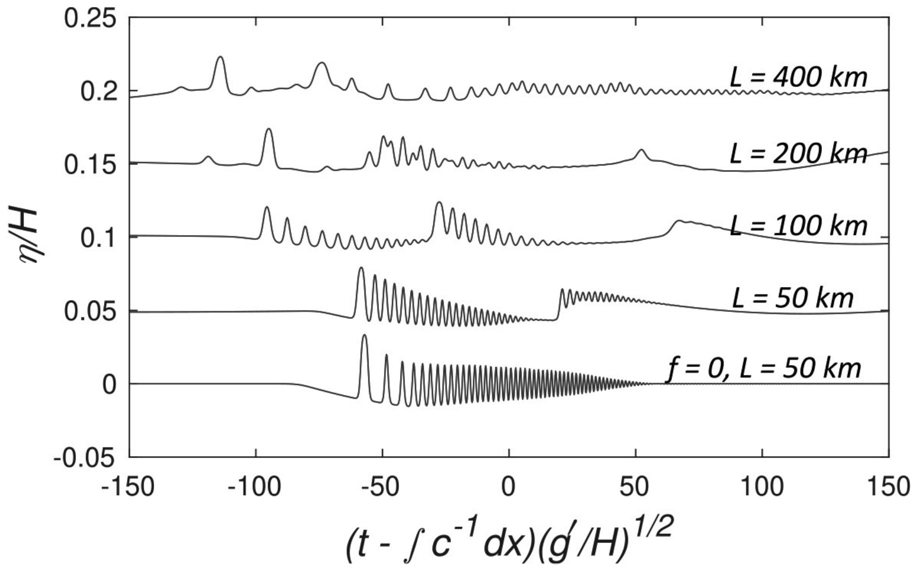



Fig. 9. The time dependence of interface displacement normalized by the total initial depth $H$ at $x = 1.2L$ and initial values $h_1$ = 50 m, $h_{20}$ = 450 m, $f = 10^{-4}$ s$^{-1}$. Initial amplitude $A_0$ = –25 m (so that the initial ratio $A_0/H$ = 0.05). From [Helfrich & Ostrovsky, 2022].

The massive difference between the effects of rotation and polarization change caused by inhomogeneity is clearly seen here. Note here one effect: when inhomogeneity prevails (small $L$), at $x > L$ the soliton generated a long oscillating tail similar to that in the non-rotation case (see Fig. 6), but with generation of a secondary train. This is specific of Gardner equation: for the rKdV equation at sufficiently mild slopes, the primary soliton completely decays, whereas the secondary train leaves a much shorter wave train at $x > L$. Here we outlined only a few scenarios of soliton decay, including some numerical situations for a two-layer fluid model.

Similar evolution equations occur in a variety physical context, such as, e.g., magneto-acoustic waves [Ruderman et al., 2023] and ultrasound waves in solid plates (see below). The universality of these models and the attention they attracted in the last decade suggest that the scenarios we outlined can be viewed as an important post-soliton development of nonlinear wave theory.

### F. Radiating elastic solitons

As another example of physically important radiating solitons, we consider waves in solids. Elastic solitons are known for nearly fifty years. Solitons in rods with both physical and geometric sources of nonlinearity were considered by Ostrovsky and Sutin [1977] and Samsonov [1984] (see also

[Samsonov, 2001; Porubov, 2003; Dai & Fan, 2004] and references therein), with new theoretical developments related to the systematic asymptotic derivation of Boussinesq-type equations in the recent papers by Garbuzov et al. [2019; 2020]. Samsonov with co-authors observed and modelled soliton-like structures in rods and plates [Samsonov, 2001]. Transmission and reflection of elastic solitons in layered structures with delamination was actively studied theoretically starting with the paper by Khusnutdinova and Samsonov [2008], where fission of a single incident soliton into several solitons in the delaminated area of the layered bar was predicted using the weakly-nonlinear analysis. This prediction was later confirmed by experimental observations by Dreiden et al. [2010]. Solitary structures in layered bars with other type of bonding (with and without delamination) were registered in [Dreiden et al., 2012]. Most recent related experimental and theoretical studies were concerned with the generation of undular bores following tensile fracture [Hooper et al., 2021; Hooper et al., 2022].

Solitary waves radiating a co-propagating one-sided oscillatory tail emerge in layered elastic bars with a thin and soft bonding between the layers. They can be modelled with a system of coupled Boussinesq-type equations derived from a complex lattice model by Khusnutdinova et al. [2009]:

$$\begin{aligned} u_{tt} - u_{xx} &= \varepsilon\left[\frac{1}{2}(u^2)_{xx} + u_{ttxx} - \delta(u-w)\right], \\ w_{tt} - w_{xx} &= \varepsilon\left[\frac{\alpha}{2}(w^2)_{xx} + \beta w_{ttxx} - \gamma(u-w)\right], \end{aligned} \qquad (1.23)$$

Here, $u$ and $w$ denote longitudinal strains in the layers, and the coefficients are defined by the properties of the layers. If the linear wave speeds of the layers are close (i.e. $c - 1 = O(\varepsilon)$, where $\varepsilon$ is a small amplitude parameter), then for the unidirectional propagation one can derive coupled rKdV equations, whereas in the opposite case, $c - 1 = O(1)$, the leading order rKdV equations uncouple. These regimes were studied by Khusnutdinova & Moore [2011]. In the coupled system, there exists a second branch of the linear dispersion relation which can be in synchronism (resonance) with a soliton, resulting in the appearance of a co-propagating radiating tail. Transmission and reflection of radiating solitons in a layered structure with a partially delaminated thin, soft bonding was modelled by Khusnutdinova and Tranter [2017]. Several structures supporting radiating solitons were examined; Fig. 10 shows an example. As one can see, although some radiation exists in a fully bonded structure, the delamination significantly increases it, so that the soliton attenuates. Emergence of a second soliton can also be seen here. Further theoretical study of soliton "tails" was carried out by Tamber & Tranter [2022].

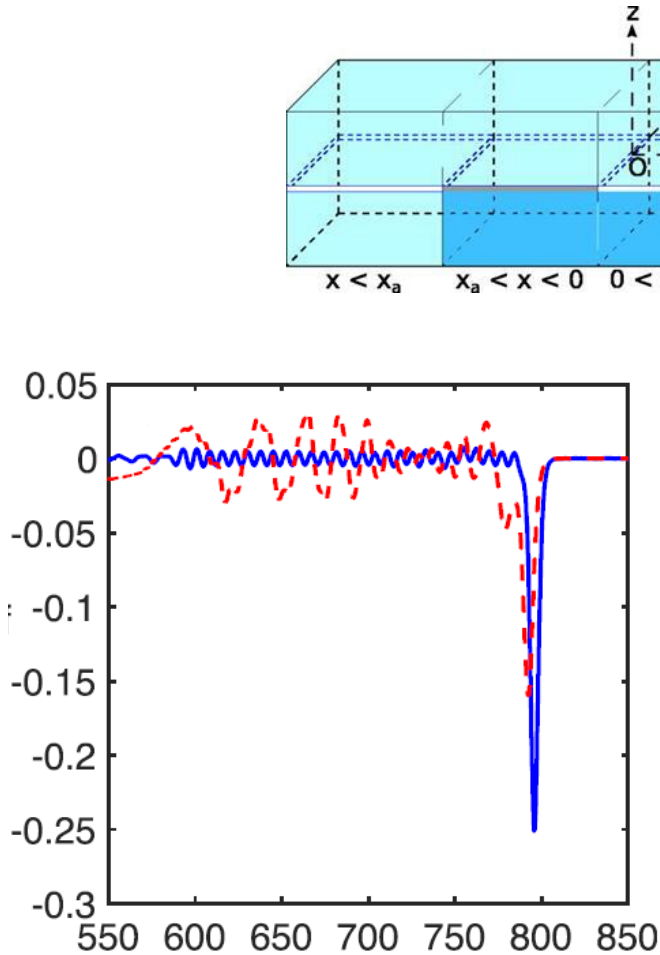



Fig. 10. Top: A sketch of a structure with a delamination and bonded layers. Bottom: A simulation of propagation of an initially solitary wave at a fixed time in the upper layer. Solid line: without delamination, dashed – with a long delamination section. From [Khusnutdinova & Tranter, 2017]. See details in that paper.

### G. Radiating solitons as intermediate asymptotic of the initial problem

Here, we first outline a broad class of situations with radiating solitons we focus upon and then illustrate our main point of this section which is the observation that often, when "genuine" solitons do not exist at all, or, for a particular branch of dispersion relation, the radiating solitons emerge as asymptotic solutions of evolution of an initial data. To this end, we employ a model describing wave-current interaction [Voronovich et al., 2006]. The advantage of this model, apart from its simplicity and relevance for some real situations, is that it supports various classes of solitary waves.

The particular class of radiating solitons we consider here occurs when a system supporting usual solitary waves is singularly perturbed by a high frequency dispersion term in such a way that the existence of usual solitary waves existing in the unperturbed system becomes prohibited, since there is a resonance between the unperturbed solution and much shorter high-frequency waves having the same phase velocities to leading order. The most popular example of such a system is the intensely studied singularly perturbed KdV equation: the KdV with a fifth order dispersion perturbation and many others within the frames of higher-order (e.g., 5th-order KdV) evolution equations [Akylas & Yang, 1995; Khusnutdinova et al., 2018] (we do not touch here "embedded solitons" which can exist as localized structures despite being in resonance with short linear waves. Such embedded solitons are often unstable, e.g., [Champneys et al., 2001; Yang, 2010; Khusnutdinova et al., 2018].

Here, however, as an example, we choose the system describing resonant interaction between internal gravity waves and a surface current in two-layer deep fluid with a thin upper layer with a boundary-layer shear current; the system was derived and examined in (Voronovich et al., 2006). The set of coupled evolution equations constituting the model reads:

$$a_t + 2aa_x - b_x = 0, \quad b_t + \delta b_x - \hat{H}[b_x] - a_x = 0, \tag{1.24}$$

where $a(x, t)$ and $b(x, t)$ are scaled amplitudes of the 'vorticity' wave and internal gravity wave, respectively; while $x$ and $t$ are slow horizontal coordinate and time, $\hat{H}[f]$ is the Hilbert transform of $f$. Amplitude $b$ of the internal gravity wave mode can be viewed as the deflection of the density interface; $a$ is normalized amplitude of the vorticity mode in the uppermost layer and can be viewed as the normalized perturbation of the surface velocity. The variables are scaled in such a way that the only parameter left is $\delta$, the mismatch in the phase velocities of the interacting waves. The vorticity mode appears owing to the presence of a boundary-layer current in the upper layer.

For small-amplitude harmonic solutions $\sim \exp[i\,k\,(x - v\,t)]$ the linear dispersion relation

$$v(k) = (1/2)\{\delta - |\mathrm{k}| \pm [(\delta - |\mathrm{k}|)^2 + 4)]^{1/2}\} \tag{1.25}$$

specifying the dependence of phase velocity $V$ on wavenumber $k$ is illustrated in Fig. 11. Linear waves belong to two different branches:

$$-\infty < V \le c- < 0 \quad \text{and} \quad 0 \le V \le c+, \tag{1.26}$$

which correspond to the vorticity and internal waves modified by their interaction in the vicinity of the resonance.

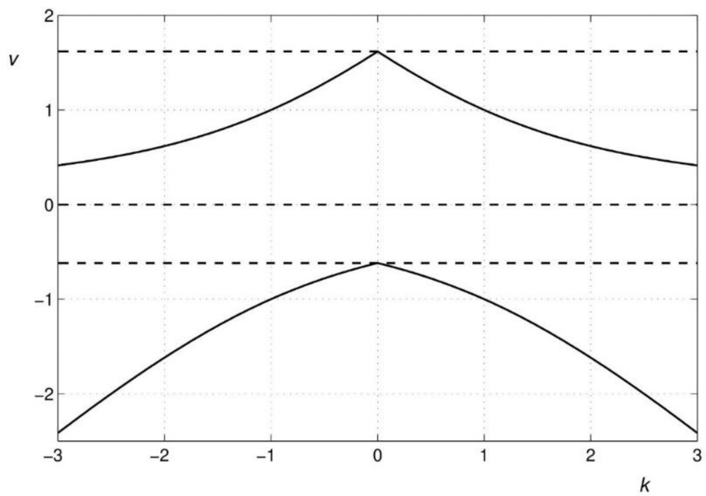



Fig. 11. An example of dispersion curve for linear waves in resonance with a shear current for $\delta$=1. The two branches correspond to the vorticity and internal waves modified by their interaction. From [Voronovich et al., 2006].

For any value of $\delta$ there exist two gaps in the spectrum of the linear wave speeds. Therefore, one may expect nonlinear solitary waves (*gap solitons*) to travel with the velocities lying inside the forbidden zones (see, e.g., [de Sterke & Sipe, 1994]). With this in mind, we look for stationary solutions advancing with a constant speed $V$:

$$a = a_s(\xi), \quad b = b_s(\xi), \quad \text{where } \xi = x - Vt. \tag{1.27}$$

Apart from the expected two classes of steady soliton solutions corresponding to each of the gaps, which we refer to as "fast" and "slow" solitons, there are also soliton-like solutions we call "*delocalized solitons*" with small amplitude tails not decaying at infinities, as illustrated by Fig. 12. The fast solitons have the opposite polarities of $a$ and $b$, while the slow and delocalized solitons pulses have the same polarities of $a$ and $b$. If we drop the assumption of stationarity, the delocalized solitons turn into usual radiating solitons, the symmetry disappears leaving the tail only on one side. In contrast to the earlier sections where the radiation plays a major role in the soliton evolution, the main point of this section is to highlight the minimal role the existence of tails plays in the evolution at relatively short timescales we focus upon here. Radiating solitary waves are expected to radiate linear harmonics and decay due to the Landau damping mechanism and this, indeed, happens. Yet, the rate of energy loss being asymptotically small, the radiating solitary waves prove to be quasi-stationary, i.e. long-lived patterns. Thus, these radiating solitary waves are effectively not so different from the classical ones and thus represent *intermediate asymptotic* in the temporal evolution of the localized pulses. Moreover, for a wide class of initial perturbations, they emerge as by far the highest solitary waves as illustrated by Fig. 13 obtained by direct numerical integration of the system (1.24) with the initial pulses having shape of the solitary wave with the width increased by the factor of four and the amplitude by the factor of two.

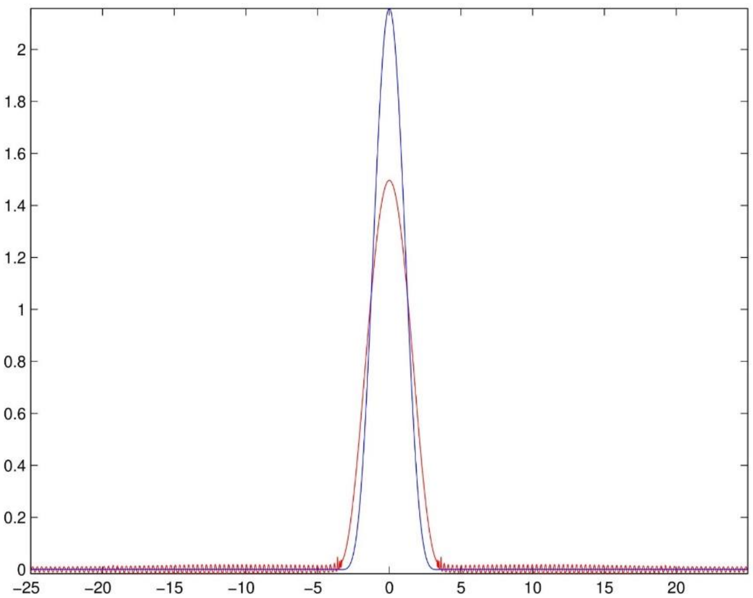



Fig. 12. Delocalized solitary wave profile $a(\xi)$ (in blue), $b(\xi)$ (in red). From [Voronovich et al., 2006].

The initial pulses do not represent a solution to Eq. (1.24) and have to evolve in the course of propagation. We reiterate that here we focus upon the timescales long enough to for the soliton-like pulses to emerge out of a generic localized initial condition, while not long enough for the radiation to have any sizeable effect. There are four solitary waves emerging from the initial pulse, two moving right and two others – left. The wave situated at $x$ = 90 when $t$ = 50 has field components of opposite signs, which distinguishes it as the fast solitary wave. Two waves moving left have negative velocities and obviously are the gap solitons. Yet the highest wave at $x$ = 20 and $t$ = 100 also has positive velocity, but its field components have the same polarity. Although no radiation is discernible by the naked eye, it is a clear manifestation of a radiating delocalized soliton in the evolutionary problem. Therefore, not only such objects should be taken into account in all studies of wave evolution, but for some aspects of the evolution they often represent the dominant feature. In our example the radiating soliton has the largest amplitude. The phenomenon is generic and thus, we expect that in all systems admitting radiating solitons with asymptotically small tails, they represent intermediate asymptotic in the temporal evolution. Indirect but ample evidence supporting our conjecture is provided by numerous observations of internal wave radiating solitary waves in the ocean. Indeed, all existing observations of internal wave solitons in the ocean are examples of radiating solitons, since the true solitons are prohibited.

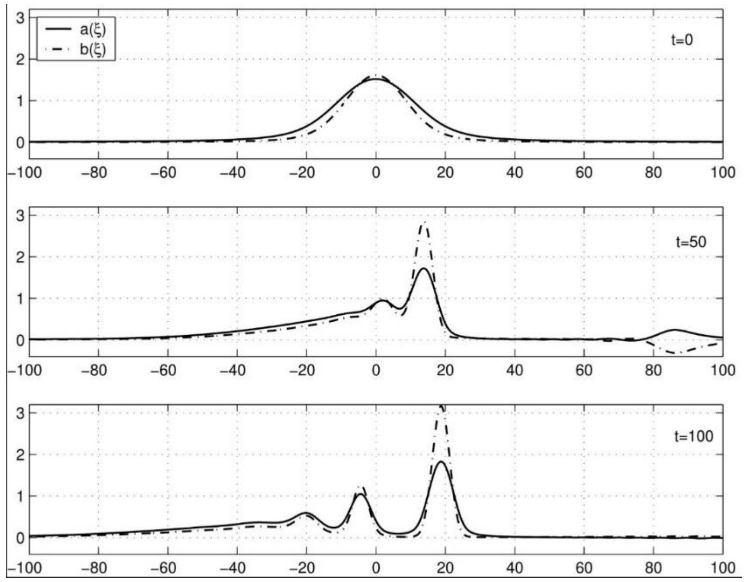



Fig. 13. Evolution of a sub-critical initial pulse in deep water, $\delta$ = 1. From [Voronovich et al., 2006].

## III. Solitons in 'non-traditional' evolution equations of the KdV-type

The KdV equation and the modified Korteweg–de Vries (mKdV) equation (as well as the Gardner equation that unites them) are now the etalon equations in the nonlinear wave physics. They appear in many branches of physics as the first approximation for weakly nonlinear and weakly dispersive waves when in the asymptotic derivation, the lower-order terms of the Taylor series are used for analytical functions describing nonlinearity. Within the framework of these etalon equations the properties of solitons and their interactions were thoroughly studied and at present are well understood. Below we examine a fundamental question whether, and if yes, to what extent the properties established for these equations can be extrapolated to the KdV-type equations with more sophisticated nonlinearities.

Naturally, there was an immediate move to investigate the generalized KdV equation:

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}+\frac{\partial^3 u}{\partial x^3}=0 \tag{3.1}$$

with nonlinear terms of a higher order: $f(u) \sim u^n$ $(n > 3)$. If the sign in front of the nonlinear term is positive, equation (3.1) has a solitary wave solution (here only the solitons on a zero pedestal are considered) for all $n$; however, the soliton's integral characteristics exhibit some unusual properties (for example, for $n > 3$ the soliton mass decreases with the increasing amplitude, and for $n > 4$ energy (momentum) also decreases), so that the solitons turn out to be unstable, which is easily shown by using the Kuznetsov criterion [Kuznetsov, 1984]. As a result, solutions to equation (3.1) explode, and their nature has been studied in detail; see, for example [Klein & Peter, 2015; Amodio et al., 2020; Bona & Hong, 2022] and references therein. From the physical point of view, it is necessary to take into account the mechanisms for limiting or arresting such an instability, which can be done by adding a higher-order term to Eq. (3.1) with a minus sign. One of these models, where $f(u) \sim u^3 - u^5$ arises in the dynamics of three-layer flows with a certain ratio on the layer thicknesses [Kurkina et al., 2011]. Note that many KdV-type equations arise in studies of stratified flows, and if the stratification is weak enough, the nonlinear function $f(u)$ can be represented by a high-order polynomial [Derzho, 2022]. In this case, the so-called pyramidal solitons can arise [Pelinovsky et al., 2021; 2022]. The simplest example of such a pyramidal soliton, shown in Fig. 14, follows from Eq. (3.1) with the nonlinear function

$$f(u)=d\Pi/du\,, \qquad \Pi=u^3\left[\left(u-u_1\right)^2+\varepsilon^2\right]\left(u-A\right)\,, \tag{3.2}$$

where $u_1$ is the intermediate soliton height, and $A$ is its amplitude. By controlling the small parameter $\varepsilon$, it is possible to obtain different base widths and the shapes of the soliton. By adding terms similar to those presented in square brackets, one can obtain a soliton with any number of humps or the *pyramidal soliton*. Such solitons are stable and appear in numerical simulations of, for example, oceanic internal waves.

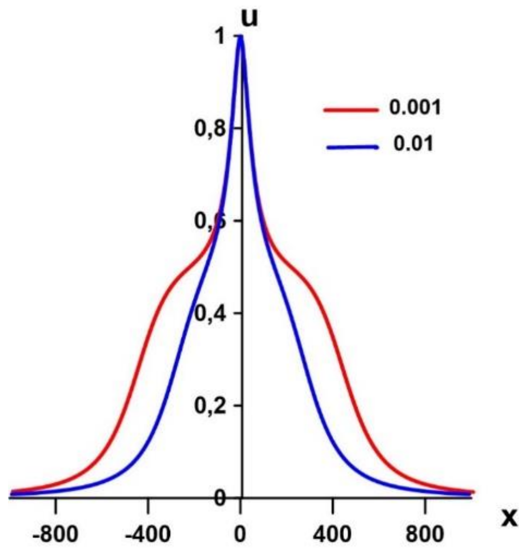



Fig. 14. Shape of the pyramidal soliton for various values of $\varepsilon$. From [Pelinovsky et al., 2022].

Different non-analytical functions, describing nonlinearity, arise in a number of applications. A striking example here is the *Shamel equation*, derived back in 1973, when in Eq. (3.1) the nonlinear function is represented by $f(u) \sim |u|^{3/2}$ [Schamel, 1973]. Originally obtained in plasma physics, in recent years the Schamel equation has been actively used to describe waves in metamaterials [Zemlyanukhin et al., 2019; 2021; Mogilevich & Popova, 2023] and electric circuits [Kengne et al., 2020; Aziz et al., 2020]. Although in contrast to the KdV and mKdV equations, the Schamel equation is not integrable, stable solitons of both polarities exist within its framework.

The so-called *logarithmic KdV equation* appeared in the context of the Fermi–Pasta–Ulam chains under a certain law of particle interaction [James & Pelinovsky, 2014; Carles & Pelinovsky, 2014; Wazwaz, 2016; Zhang & Li, 2020], in which the nonlinear function is $f(u) \sim u \log[uH(u)]$, where $H(u)$ is the Heaviside step function. In this equation, the solitons are also stable, and they have the form of Gaussian pulses.

The number of equations of the KdV-type with a non-analytic nonlinear function grew rapidly. In particular, the modular equation with $f(u) \sim |u|$ appeared in the bi-modular theory of elasticity [Rudenko, 2016], and then, as the *canonical modular equation* $f(u) \sim u|u|$ [Slunyaev et al., 2023], which differs from the canonical KdV equation only by the presence of modulus. For the equation with $f(u) \sim |u|$ there are no solitons on the zero pedestal (all unipolar solutions are linear). Within the framework of the canonical modular KdV equation, it is easy to construct families of multi-soliton solutions of the same polarity, since in this case we are dealing with the classical KdV equation. New effects here arise from the interaction of solitons of different polarities (see below). Finally, a class of sublinear equations appeared, when $f(u) \sim u|u|^b$ with $b < 0$ [Pelinovsky et al., 2021; Friedman et al., 2022]. In this case, solitary waves with exponential tails do not exist; however solitary waves in the form of '*compactons*' (structures defined only in a finite space interval) can exist. Compactons themselves are stable, but their dynamics is much more complex than the dynamics of solitary waves with exponential or algebraic tails. As mentioned, such structures, as well as their name, were first introduced by Rosenau (1997) for a class of equations with a nonlinear operator $\hat{L}[u]$.

Of course, it is also possible to combine various nonlinear functions, so in the literature one can find equations with names like the Schamel–KdV equation or the Schamel-logarithmic KdV equation, etc. Sometimes such equations are referred to as the generalized Gardner equation, since the terminology has not settled, we will not dwell upon it here. The presence of non-analytic nonlinear functions greatly hinders the mathematical proof of existence and uniqueness theorems, and here, in essence, there are only a few separate publications [Friedman et al., 2022].

It is relatively easy to study the class of stationary solitary waves within the framework of the generalized equation (3.1) regardless of the analyticity of the nonlinear function $f(u)$, since such a problem reduces to the ordinary second-order differential equation:

$$\frac{d^2u}{dy^2} - Vu + f(u) = 0 \tag{3.3}$$

($y = x - Vt$, $V$ is the soliton velocity), the solution of which is reduced to quadratures. If $f(u)$ is nonlinear with a power exceeding one, it already follows from Eq. (3.3) that the solitons (if they exist) have exponential asymptotic and move to the right, i.e faster than the longest linear waves ("supersonic" solitons), while the shape of the soliton is not important (including both "fat" and pyramidal soliton) [Pelinovsky et al., 2021; 2022] (the jargon term "fat soliton" comes from one of the solutions to the Gardner equations – see, e.g., line 2 in Fig. 1 in [Ostrovsky et al., 2015]). If $V = 0$, algebraic solitons with power tails are realized. Here we present the analytical expression for an algebraic soliton of the Schamel–KdV equation:

$$\frac{\partial u}{\partial t} - \left(\sqrt{|u|} - u\right)\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{3.4}$$

This equation has a particular explicit solution in the form:

$$u(x) = \left( \frac{12}{5} \frac{1}{1 + 4x^2/19} \right)^2 . \tag{3.5}$$

This solution is similar to the algebraic soliton in the Gardner equation (see, e.g., line 5 in Fig. 1 in [Ostrovsky et al., 2015]); it is structurally unstable: being perturbed, it transforms either into a breather or a soliton with exponential tails.

If the power of nonlinearity is less than one, the soliton is confined to a compact area (therefore it is called a compacton), at the ends of which an algebraic approximation is valid. These solutions are no longer described by analytical functions, so they must be carefully combined with the zero pedestal outside the compacton [Pelinovsky et al., 2021]. To prove the uniqueness of solutions in the form of compactons is not straightforward. As an example, we present a solution of Eq. (3.1) with the nonlinear function $f(u) \sim |u|^{1/2}$ in the form of a compacton [Pelinovsky D. et al., 2021]:

$$u = A \begin{cases} \sin^4 \left[ \frac{\pi}{2} - \sqrt{-V} \frac{|x|}{4} \right] & |x| < \frac{2\pi}{\sqrt{-V}}, \\ 0 & |x| > \frac{2\pi}{\sqrt{-V}}, \end{cases} \tag{3.6}$$

where $V = 4q/3\sqrt{A}$. Note that the compactons propagate to the left ("subsonic" solitons), and their speed decreases with increasing amplitude, so that small-amplitude compactons run faster than the large-amplitude ones. Due to the non-integrability of Eq. (3.1) with non-analytic nonlinear functions, the problem of generating compactons and their interaction with each other has to be studied numerically. A large number of such problems with various nonlinear functions was considered in [Garcia-Alvarado & Omel'yanov, 2014; D. Pelinovsky et al., 2021; Slunyaev et al., 2023; Flamarion et al., 2023; Didenkulova et al., 2023]. Note that the interaction of unipolar solitons and compactons within the framework of various versions of the KdV-type equations occurs, in essence, according to the same scenarios as in the classical KdV equation: overtaking a slower soliton by a faster one (if the velocities are very different) and the exchange in the case of close wave velocities. As an example, Fig. 15 shows the interaction of two solitons within the framework of the Shamel equation on the $x,t$-plane [Flamarion et al., 2023]. In this case, some energy is emitted, which, however, is very small (~$10^{-4}$), so that the solitons remain very robust structures, although the Shamel equation is not integrable.

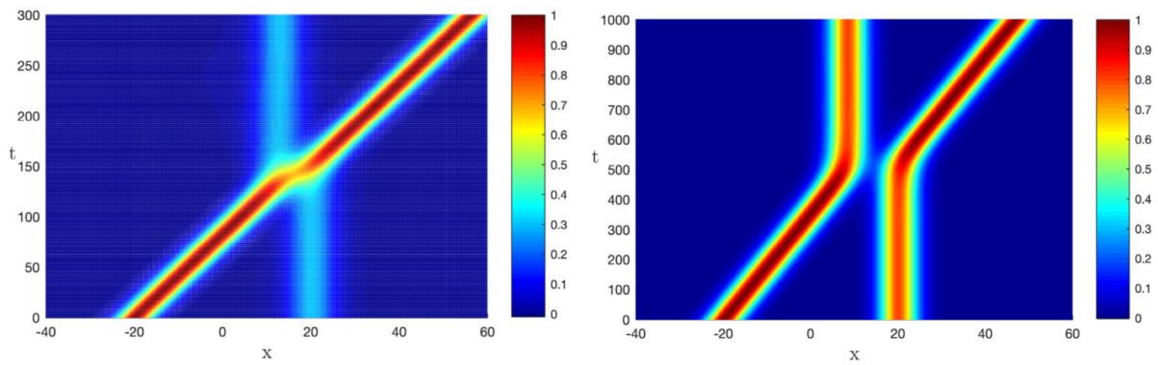



Fig. 15. Overtaking (left) and exchange (right) soliton interaction within the framework of the Schamel equation on the *x*,*t*-plane. From [Flamarion et al., 2023].

Of course, the quantitative characteristics of the interaction process (phase shift, amplitude ratio when changing the mode) depend on the specific type of equations, but qualitatively the process of interaction between two solitons of the same polarity occurs in the same way as in the integrable systems. It is really an amazing property, since in non-analytical equations there is no small parameter that characterizes their difference from the integrable ones, and yet, the processes associated with solitons and compactons occur according to the same scenarios.

If the system supports solitons of different polarities, then, as in the mKdV equation, a qualitatively different ‘absorbing-emitting’ interaction scenario becomes possible, in which the smaller of the two solitons is first absorbed into the larger one for a short time, and then is restored. In contrast to the integrable mKdV equation, in the non-integrable system there is no complete restoration of amplitudes of the interacting solitons, and the large soliton takes away part of the energy from the small one. This process within the modular equation $f(u) \sim u|u|$ is illustrated in Fig. 16 [Slunyaev et al., 2023], which shows the dependence of soliton amplitudes on time for the periodic boundary conditions, so that the trend towards the increase in the amplitude of a large soliton (the red line) and the decrease in the amplitude of a small amplitude soliton (the black line) is clearly visible.

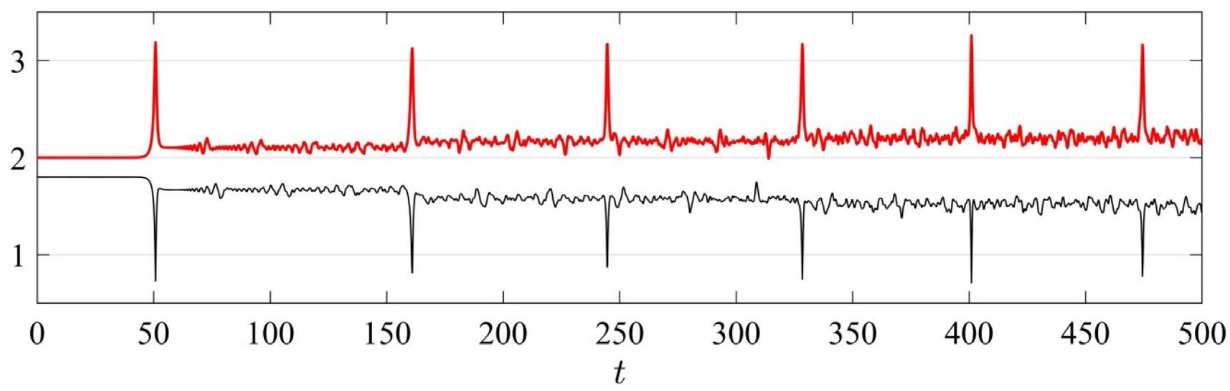



Fig. 16. Interaction of solitons of different polarities in the modular KdV equation. The larger soliton is plotted in red, the smaller one in black. The non-elastic character of interaction and energy transfer from small soliton to bigger one is clearly seen. From [Slunyaev et al., 2023].

The graph also shows small-scale oscillations associated with radiation during the interaction. Similar results were obtained within the framework of the Shamel equation [Didenkulova et al., 2023]. This difference in the behavior of the unipolar and bi-polar solitons is obviously associated with the zero level crossing, where the non-analyticity of the nonlinear function manifests itself. Note that the specific feature of solitary wave interaction in non-integrable systems when the larger amplitude soliton after interaction with small amplitude solitons becomes bigger is well-known [Krylov & Yankov, 1980; Zakharov et al., 1988; D'yachenko et al., 1989; Zakharov & Kuznetsov, 2012]. Although at a qualitative level, two-soliton collisions look very similar to those in the integrable models, even the slight differences (a small gain of energy by the larger soliton in the result of a collision and weak radiation during the interaction) can be highly consequential after many interactions, for example, in the bounded confinement, as a result of multiple collisions, eventually only one soliton survives (a "champion").

When solving the initial-value problem, the process is qualitatively similar to that known for the KdV equation: a wide pulse disintegrates into a sequence of solitons and a small amplitude dispersive train, while a narrow one spreads out and transforms into a large amplitude dispersion train of the Airy-type wave, whereas a soliton, if generated, has a small amplitude. Figure 17 illustrates the process of the wide pulse evolution within the framework of the sub-linear KdV equation, $f(u) \sim |u|^{-1/4}u$, in which solitary waves have the form of trigonometric compactons like in Eq. (3.6). The initial impulse (the green line) breaks up into approximately five compactons, which move to the left, with the small amplitude compactons having a higher speed. If these compactons left the source without interaction, they would move further apart (their positions and amplitudes are indicated by the dashed line in Fig. 17a). If during the decay of the initial perturbation the compactons do interact with each other, the radiation is quite strong and relatively large oscillations are formed, as shown in Fig. 17b in the semi-logarithmic scale.

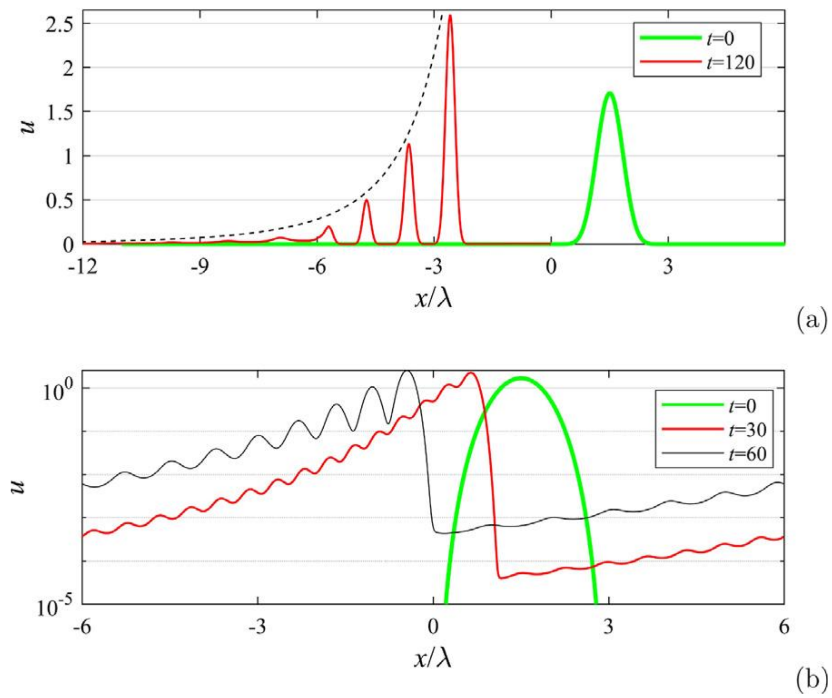



Fig. 17. Evolution of a wide pulse-like initial perturbation: (a) the long-time wave evolution and (b)the initial stage shown in the semi-logarithmic scale. From [Slunyaev et al., 2023].

A narrow pulse evolution within the framework of the same sub-linear KdV equation is shown in Fig. 18. Over time, the dispersive train disintegrates into a sequence of compactons of different signs, since quasi-sinusoidal oscillations are impossible due to strong nonlinearity when crossing the zero level (it is clearly visible in Fig. 18b). It is also clear that small amplitude compactons acquired a higher speed at the initial stage than without interaction (in this case, their amplitudes would be on the dashed line in Fig. 18a).

Compactons play a twofold role in the wave evolution within the framework of the sub-linear KdV equation. On the one hand, compactons behave similarly to the classical solitary waves: they survive collisions with other waves and represent the long-term asymptotic of the evolution problem. On the other hand, small-amplitude compactons play the role of dispersive waves in the linear KdV equation, since they quickly spread the residual energy of the initial perturbation which has not been taken by large-amplitude compactons. In the latter case, either a slowly decaying smooth tail appears first from the left of the perturbation, which splits later into small-amplitude compactons or new small-amplitude compactons are emitted by inelastically interacting compactons.

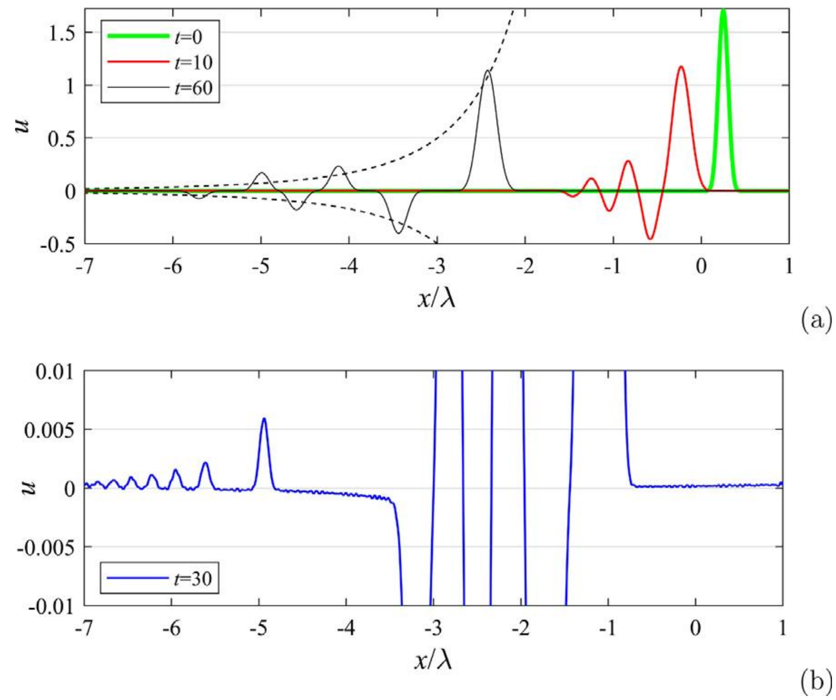



Fig. 18. Evolution of an initially narrow pulse: (a) the long-time wave evolution; (b) the structure of the small-amplitude tail. From [Slunyaev et al., 2023].

As it is known, an effective method for generating solitons is via interaction with external fields, the most effective transfer of energy occurs at velocities close to the velocities of long waves, i.e. at resonance. Such processes were actively studied within the framework of the forced equation of the KdV type:

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}+\frac{\partial^3 u}{\partial x^3}=W(x,t) \tag{3.7}$$

with a given function $W(x, t)$ being deterministic or random. In this case, Eq. (3.7) is not integrable (except for trivial representations for the external force), even if $f(u)$ is proportional to $u^2$ or $u^3$. Analytical results within the framework of the forced KdV equation with $f(u) \sim u^2$ were obtained for the case of a weak external force moving at a constant speed, since then it is possible to use asymptotic methods and obtain, in the first approximation, the system of ordinary differential equations for the amplitude and phase (i.e., position) of the soliton:

$$\frac{d}{dt}\int_{-\infty}^{+\infty}\frac{u^2(\zeta)}{2}d\zeta=\int_{-\infty}^{+\infty}u(\zeta)\frac{dW(\zeta+x)}{d\zeta}d\zeta\,, \tag{3.8}$$

$$\frac{dx}{dt}=\Delta+V(A)\,, \tag{3.9}$$

where $u(x)$ is the solution of the KdV equation, $V(A)$ is its speed, and $\Delta$ is the detuning from the external force resonance. This system of equations can be easily studied qualitatively on the phase plane assuming a constant $\Delta$, and with a variable detuning – by using a simple numerical integration. Within the framework of this system, the processes of soliton capturing by external fields and pushing it to the periphery (depending on the relationship between the soliton polarity and the external force) have been thoroughly studied. The literature on the analytical and numerical solutions of the forced KdV equation is enormous, and here we refer to the latest work [Ermakov & Stepanyants, 2019], where references to earlier works can be found.

For the case of arbitrary nonlinearity, in essence, Eqs. (3.8) and (3.9) do not change, and the specificity of particular equations lies in the soliton shape and its velocity dependence on the amplitude. Qualitatively, the same regimes of the soliton capture and repulsion by an external field are observed here. Such problems have recently been solved for the mKdV equation [Flamarion & Pelinovsky, 2022a; 2022b] and the Schamel equation [Flamarion & Pelinovsky, 2023].

In the general case with arbitrary initial conditions, with or without the presence of external forces, a large number of solitons can be excited in the system, so we can speak about soliton turbulence. Already in the classical work by Zakharov [1971], the kinetic equation for the KdV solitons was derived, and it was shown that in the rarefied soliton gas only the pair soliton interactions can occur. Consequently, the two-soliton interaction can be considered to be an elementary interaction within the random soliton ensemble. As usual in the turbulence theory, it is necessary to study the distribution functions and statistical field moments:

$$M_n(t)=\int_{-\infty}^{+\infty}<u^n(x,t)>dx\,, \tag{3.10}$$

where <…> means statistical averaging. The first two moments are invariants of Eq. (3.1). Therefore, the third and fourth moments are of interest, which allows us to calculate the *skewness* and *kurtosis* of a random field. Since the interaction of two solitons is an elementary interaction of soliton turbulence, as a first step it is worth calculating moments (3.10) only for two solitons (naturally, in this case we are not speaking about statistical averaging). These integrals cannot be evaluated analytically even in integrable systems, so they have to be calculated numerically. Within the framework of Eq. (3.1) with the nonlinear function $f(u) \sim u^m$ ($m$ = 3/2, 2, 3), qualitatively the same results were obtained for the same polarity solitons: the fourth moment decreases due to the interaction [Pelinovsky et al, 2013; Pelinovsky & Shurgalina, 2015; Flamarion et al., 2023], and in the case of solitons of different polarities it increases [Pelinovsky & Shurgalina, 2015; Didenkulova et al., 2023]. It is consistent with the decrease in the field at the moment of interaction of the same polarity solitons (under any overtaking or exchange scenario) and its strengthening during the interaction of solitons of different polarities (absorb-emit scenario), what we have already discussed above. When a larger number of solitons are interacting, qualitatively the same effects should be expected.

In the case of an ensemble of random solitons (soliton gas), some conclusions can be drawn by analyzing the moments in the limit of widely separated solitons (rarefied gas). At least, such an ensemble is easy to create at the initial time moment. For calculations it is convenient to solve a periodic problem and calculate all the moments on a finite but sufficiently large interval $L$, dividing the integrals in Eq. (3.10) by $L$. As an example, let us consider a soliton gas in the framework of the classical KdV equation with $f(u) = 6u$. Then, substituting the soliton solution in the form of the sum of $N$ solitons with different amplitudes and phases, the first moment (the average value) turns out to be equal to:

$$M_1 = 2\sqrt{2}\rho < A^{1/2} >, \tag{3.11}$$

where the gas density is $\rho = N/L$ and A is the soliton amplitude. The dispersion of the random field is:

$$\sigma^2 = <\left(u - < u >\right)^2> = \frac{8}{3\sqrt{2}}\rho < A^{3/2} > - 8\rho^2 < A >. \tag{3.12}$$

Since the dispersion of the random field must be positive, the limitation on the density of the soliton gas follows from Eq. (3.12) [El, 2016; Pelinovsky & Shurgalina, 2017]:

$$\rho < \rho_{cr} = \frac{< A^{3/2} >}{3\sqrt{2} < A >}, \tag{3.13}$$

that is, the soliton gas cannot be very dense. In essence, it follows that it is impossible to focus many KdV solitons into a single very large pulse (the "rogue wave"). It can be shown rigorously for the integrable KdV equation using the known exact $N$-soliton solutions [Tarasova & Slunyaev, 2023]. Similar results for any soliton gas consisting of the same polarity solitons can be obtained in the same way, although the quantitative expressions for the analogues of Eq. (3.13) will differ.

A qualitatively different situation occurs for the gas consisting of solitons of opposite polarities. Obviously, in this case the average field value will be small or equal to zero; but then the dispersion of the random gas is always positive, and there is no critical value for the gas density. This means that such a gas may contain areas of high density where rogue waves can arise [Pelinovsky & Shurgalina, 2016]. The specific magnitude of rogue waves depends on many factors, especially on the phase relationships between them at the initial time, and in the general case, it is impossible to predict the possible height of a rogue wave. If we take as an example the integrable mKdV equation, which allows for the existence of solitons of opposite polarities, the analysis here is again possible using the known $N$-soliton solution. In particular, it is possible to find the optimal expression for the amplitudes of solitons alternating in sign with certain phases, when the strongest focusing of solitons into a rogue wave occurs, the amplitude of which is equal to the sum of the amplitudes of all solitons [Slunyaev & Pelinovsky, 2016; Slunyaev & Tarasova, 2022]. This scenario is illustrated in Fig. 19. The initial group of six different-polar solitons (the black line at the top of Fig. 19) is grouped into an anomalously large wave (plotted in red). The short formation time of such a rogue wave is illustrated in the lower part of Fig. 19.

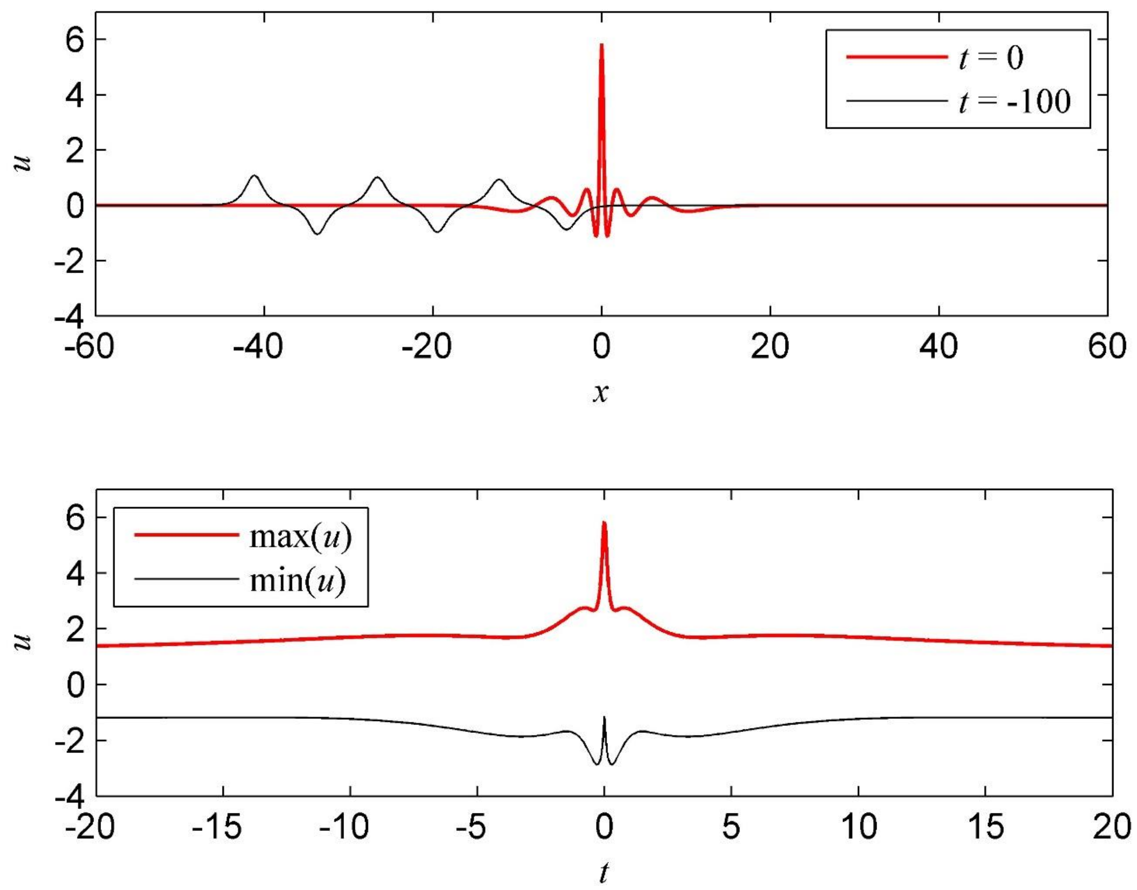



Fig. 19. A rogue wave formation from mKdV solitons under optimal focusing conditions. From [Slunyaev & Pelinovsky, 2016].

Similar results were also obtained for other integrable KdV equations, in particular for the Gardner equation, which combines quadratic and cubic nonlinearity [Slunyaev, 2019; Didenkulova, 2019]. For non-integrable versions of the KdV equations, it is impossible to predict the conditions for optimal soliton focusing, but in numerical experiments with the soliton gas the formation of rogue waves is observed in agreement with the general concept [Zakharov & Kuznetsov, 2012]. In particular, such simulations have been recently carried out within the framework of the Schamel equation, where solitons could have both polarities [Flamarion et al., 2024]. Let us note, however, that if the soliton gas comes to an equilibrium after a short transition period within the framework of the integrable KdV models, in non-integrable systems, due to radiation during the soliton interactions, the statistical field characteristics evolve with time, and the soliton gas is supplemented by dispersive wave packets.

Summarizing the discussion above, we conclude that the qualitative properties of solitons and their interactions for all the variety of examined nonlinearities are similar to those of the etalon systems: as a rule, the solitons are stable, while their soliton-soliton interaction develops qualitatively as in the KdV equation. However, for many solitons interacting, the mentioned small differences during and after each collision due to inelasticity of the collisions can accumulate, which changes the long term asymptotic qualitatively, resulting, for example, in the case of a confined domain in the disappearance of all solitons with initially smaller amplitudes and emergence of a single soliton “champion”.

The next natural direction of study is the evolution equations in which, along with modified nonlinearity, a more general integral dispersion is used. As mentioned, the most known example of such equations is the Benjamin–Ono equation, which is completely integrable. The properties of its solitons (although they have power-type rather than exponential tails) are in many ways similar to the properties of solitons in the KdV equation (see, for example, the review by Saut, 2019). However, the evolution equations combining non-quadratic nonlinearity and integral dispersion are the subject of future studies. At present, most actively studied are nonlinear waves in fractional dispersive models; the latest results there can be found in very recent publication [Malomed, 2024; Kevrekids & Guevas-Maraver, 2024].

## IV. Solitons and lumps in the Kadomtsev–Petviashvili equation

For obvious reasons in the soliton theory the lion share of attention was devoted to stable robust structures, however, recently the overlooked unstable or potentially unstable have attracted more attention. The motivation to study such structures is not just interesting mathematics, but realization that the instabilities are often not too strong, and under appropriate circumstances such ephemeral structures might emerge as intermediate asymptotic. We also note that often the instabilities in question have large spatial scales and, hence, cannot develop in confined spaces, which turns unstable object into stable ones in confined geometry. Below we include in our overview recent works concerned with such objects, without discussion of specific physical contexts where they might be important.

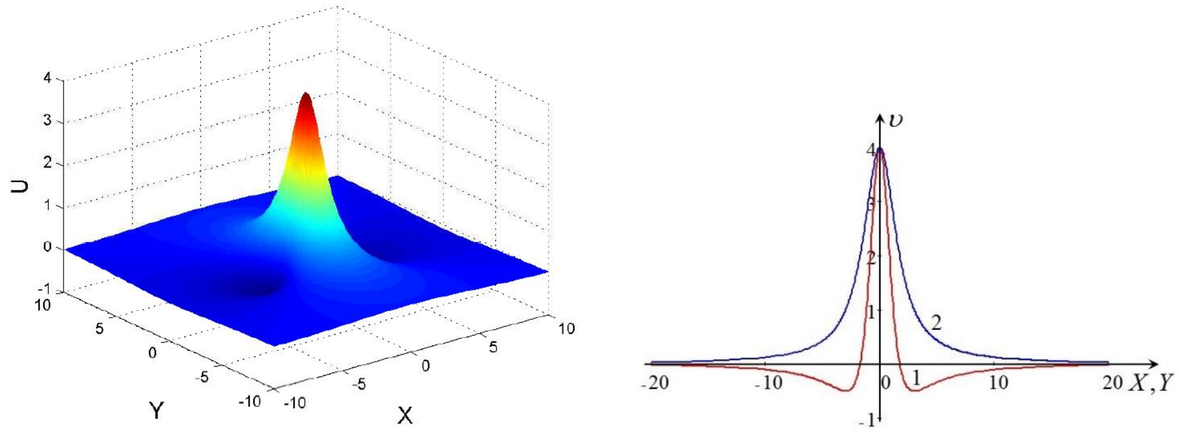
First, we consider two-dimensional structures described by the KP equation that can be presented in the form [Kadomtsev & Petviashvili, 1970]:

$$\frac{\partial}{\partial x}\left(\frac{\partial \upsilon}{\partial t}+c\frac{\partial \upsilon}{\partial x}+\alpha\upsilon\frac{\partial \upsilon}{\partial x}+\beta\frac{\partial^3 \upsilon}{\partial x^3}\right)=-\frac{c}{2}\frac{\partial^2 \upsilon}{\partial y^2}. \qquad (4.1)$$

This equation is completely integrable but has qualitatively different properties depending on the dispersion coefficient $\beta$. In the case of negative dispersion media when the coefficient $\beta$ is positive (surface and internal gravity waves, magnetosonic waves in plasma, etc.), this equation (dubbed the KP2) has solutions in the form of plane KdV-type solitary waves with line fronts propagating under an angle to the $x$-axis (see, for example, [Ablowitz & Segur, 1981]). Such solitons can interact with each other creating various nice patterns, see, for example photos at the websites of M.J. Ablowitz (Photographs, https://sites.google.com/site/ablowitz/linesolitons/x-type-interactions), D.E. Baldwin (Nonlinear waves, http://www.douglasbaldwin.com/nlwaves.html), and the book by Eremenko [2019]. Plane solitons are stable objects with respect to small perturbations; they can emerge from rather arbitrary initial perturbations and propagate on long distances [Apel et al., 2007].

In the case of positive dispersion media when $\beta < 0$ (for example, waves in magnetized plasma, waves in fluid layers with strong surface tension on the interfaces, waves in solids, etc.), the KP equation is dubbed KP1. It possesses solutions in the form of plane KdV-type solitons too propagating at different angles and interacting with each other. Plane solitons are known to be unstable with respect to front modulations of sufficiently long wavelengths $\Lambda > \Lambda_{cr} \equiv 24\pi\beta/(\alpha A\sqrt{3})$, where $A$ is the soliton amplitude [Kadomtsev & Petviashvili, 1970; Zakharov, 1975; Ostrovskii & Shrira, 1976; Pesenson, 1991]. The nonlinear development of the instability leads to the formation of completely localized 2D solitons dubbed *lumps* [Pelinovsky & Stepanyants, 1993], described by the analytical solution (1.9) (see, for example, [Ablowitz & Segur, 1981]). The lump shape is shown in Fig. 20. As follows from Eq. (1.9), lump field slowly decays in space as $u \sim r^{-2}$ where $r=\sqrt{x^2+y^2}$ when $r \to \infty$.

Lumps are stable with respect to small perturbations; their interactions with each other are somewhat unusual: after an interaction, their original shapes completely are restored but they do not experience phase shifts in the regular case. In the exceptional *resonant* case, phase shifts become undetermined [Gorshkov et al., 1993]. Lumps can form bound states consisting of several coupled lumps stationary moving in various directions resembling "*lump molecules*". Various multi-lump formations were obtained in [Hu et al., 2018; Zhang et al., 2023a] by different analytical methods; two simplest examples of a bi-lump and triple lump are shown in Fig. 21. Even more complex multi-lump formations were obtained in numerous publications – see, for example, [Zhang et al., 2022a,b; 2023a,b; Yang B. & Yang J., 2022; Chakravarty & Zowada, 2023; Han et al., 2023; He et al., 2023] and references therein.




(a) (b)

Fig. 20. Surface plot of a single symmetric lump (a) and its main cross-sections (b). Line 1 in frame (b) – cross-section along the *x*-axis; line 2 – cross-section along the *y*-axis.

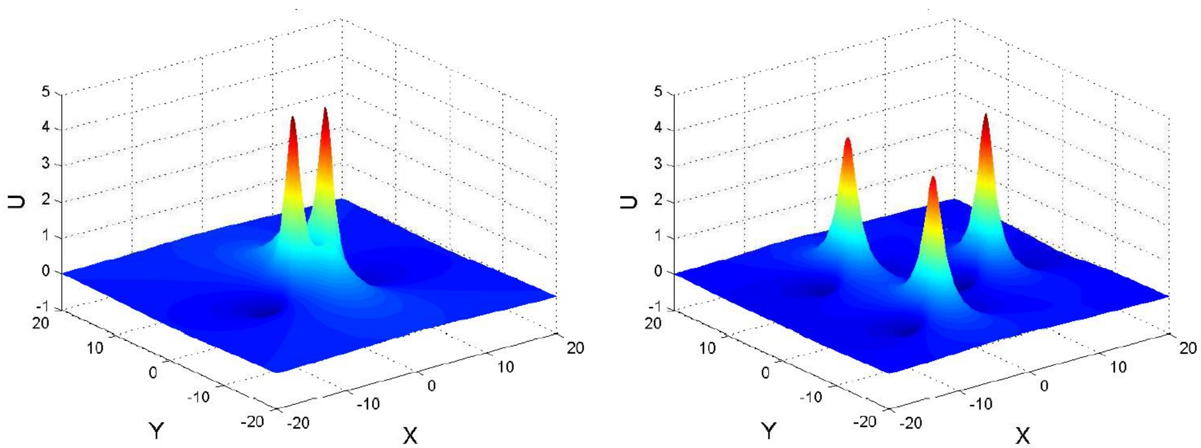



Fig. 21. Simplest multi-lump formations: left panel – a bi-lump; right panel – a triple lump. From [Zhang et al., 2023b].

A binding energy of all such bound states is zero which means that they are unstable concerning small perturbations. However, in the result of such instability, lump molecules do not disappear but experience a fission into several separate lumps. Disintegrations of multi-lumps in the course of their interactions under an angle to each other were studied in [Hu et al., 2018] and illustrated by movies (see the websites [Websites]). One of the examples is shown in Fig. 22.

As follows from this study, in general, lump molecules disintegrate in the course of interactions, but in some cases, two rather complex multi-lump formations can pass through each other preserving their entities. Such examples demonstrate the fundamental properties of lumps and their interactions in the elementary acts. A more general problem arises regarding the behavior of an ensemble of lumps with different amplitudes and phases (lump gas), their statistical properties, and their possible role in the description of soliton turbulence reminiscent of intensely studied soliton gas in the KdV and KdV-like systems [El, 2016; 2021].

To explain the difference between the regular and anomalous interaction of lumps, consider first the simplest regular case when two lumps move one after another along the $x$-axis with different speeds so that the faster moving lump is initially behind the slower moving. When their relative speed is not too big, the faster moving lump splits the slower moving one so that they form two lumps moving under an angle to each other. After a while these lumps start attracting each other and approaching. As the result, they form again two lumps moving along the $x$-axis one after another but now the faster moving lump is in front the slower moving. The distance between the lumps increases linearly with time.

The time of lump interaction depends on their amplitudes; it increases up to infinity when soliton amplitudes become equal. In the limiting case, after splitting, the lumps continue slowly moving away from each other and never come back. Such a degenerate solution is called *resonance* and gives rise to *anomalous scattering* [Gorshkov et al., 1993]. When three or more lumps enter the resonant interaction, the distance between them can vary as $d \sim t^p$ where $p$ is a fraction number [Chakravarty & Zowada, 2022a; 2022b; Dong et al., 2022; Yang B. & Yang J., 2022; Zhang et al., 2023b]. An example of such interaction is shown in Fig. 23. Figure 24 shows the divergence of lumps after interaction when the distances between them increase as $d \sim t^{1/3}$ [Zhang et al., 2023b].

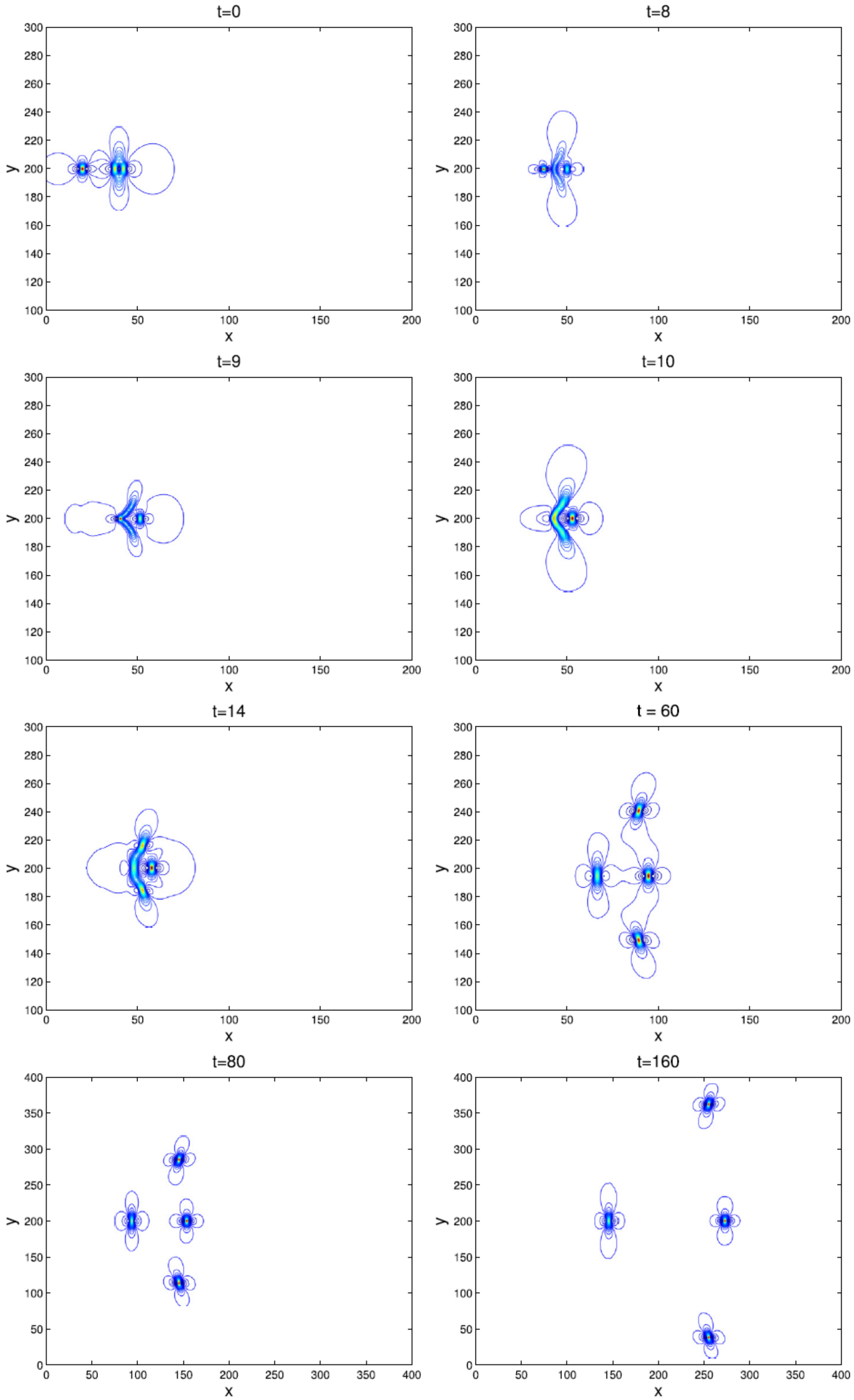



Fig. 22. Interaction of a single lump with a symmetrical bi-lump when they move initially along the *x*-axis with different speeds. The single lump approaches the bi-lump and after collision four single lumps appear [Hu et al., 2018].

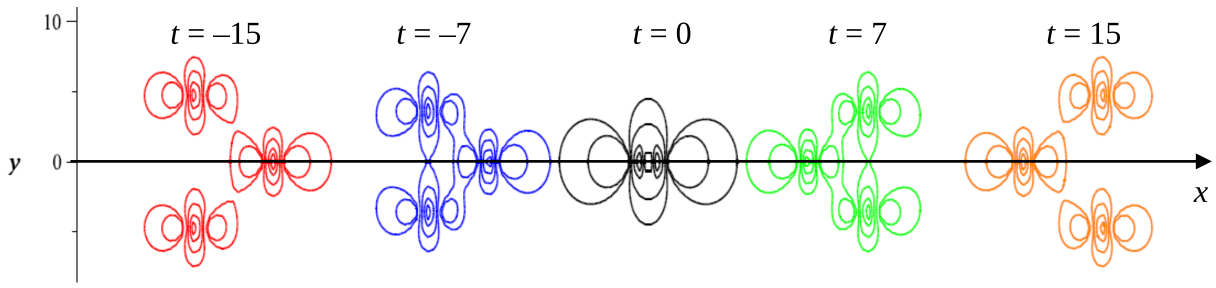



Fig. 23. Resonant interaction of three equal-amplitude lumps at different time moments. (The distance between the patterns shown in different times are not in scale.) [Zhang et al., 2023b].

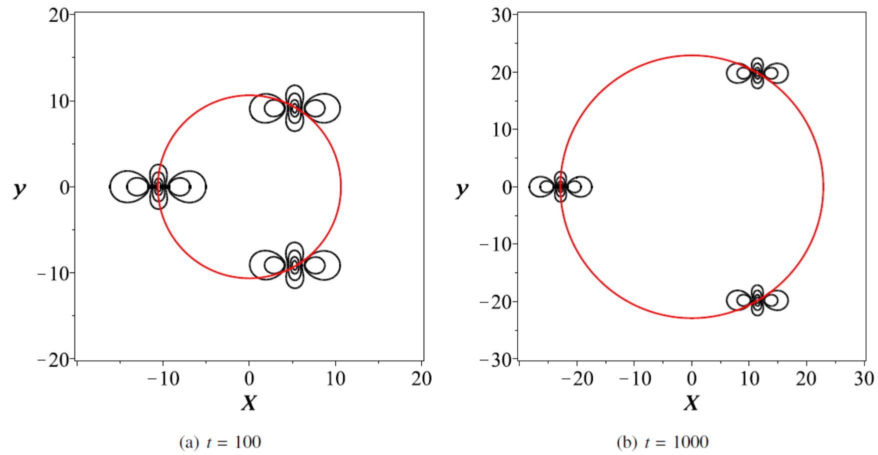



Fig. 24. Resonant interaction of three lumps shown in Fig. 23 in two time moments after the collision. Lumps are equally spaced on the circle whose radius increases with time as $R \sim t^{1/3}$ [Zhang et al., 2023b].

A resonant interaction can occur between lump chains too. As well known, there are solutions to the KP1 equation that describe infinite lump chains moving at different angles on the *x*,*y*-plane [Zaitsev, 1983; Gdanov & Trubnikov, 1984; Abramyan & Stepanyants, 1985]. Such chains moving at an angle to each other can interact in a regular way or resonantly depending on their parameters. These interactions are very similar to interactions of plane solitons in the KP2 equation. In the regular case, the interaction of two plane solitons results in the bending of their fronts on the *x*,*y*-plane and subsequent spatial phase shift. However, as was discovered by Newell & Redekopp [1977] and Miles [1977], at a certain relationship between soliton parameters, the resonant interaction can occur when two crossing plane solitons give birth to a third plane soliton; the phase shift in such a case becomes infinite. Two types of soliton interactions, regular and resonant, are shown in Fig. 25. In frame (a), the bridge between two pairs of wave fronts is of a finite length, and spatial phase shifts of solitons is finite too – cf. two red lines in frame (a). At a special relationship between soliton amplitudes and the angle between them, the bridge becomes infinite and reduces to the third soliton that is shown by red line in frame (b). A very similar situation occurs when two lump chains intersect each other at some angle. Figure 26 illustrates a regular and resonant interactions of two lump chains.

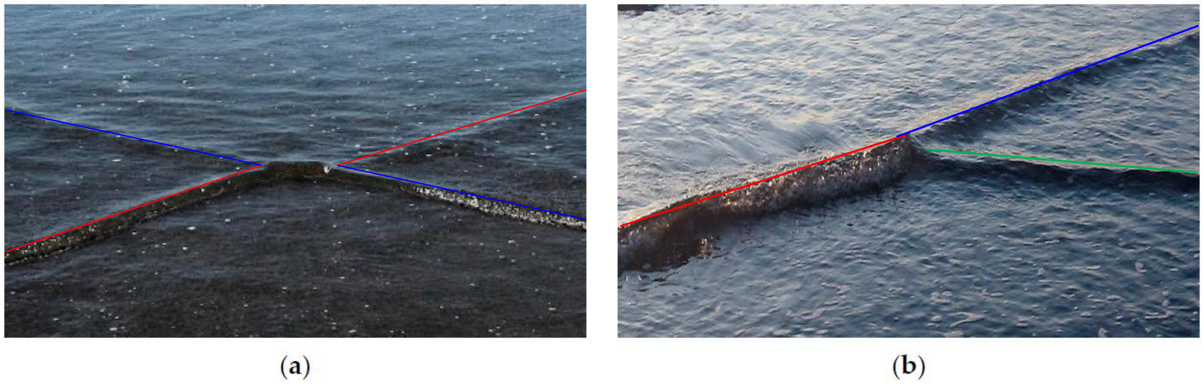



Fig. 25. Photographs of the observed wave patterns on shallow water which demonstrate a regular soliton interaction (a) and a resonant interaction (b) [Ablowitz & Baldwin, 2012]. Photos are taken with the kind permission of M. Ablowitz from his website: https://sites.google.com/site/ablowitz/linesolitons/x-type-interactions. The website was accessed on 23 April 2024. Colour lines were added by us to clearly demonstrate solitons fronts and spatial shifts due to the nonlinear interactions.

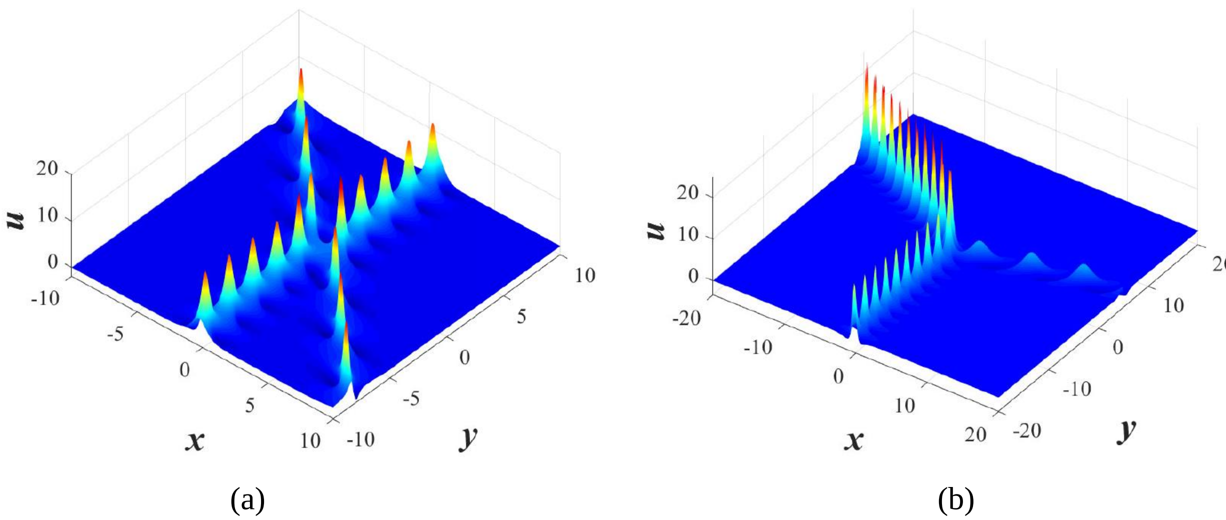



Fig. 26. Regular (a) and resonant (b) interactions of two lump chains [Zhang et al., 2022b].

Other interesting phenomena studied in recent years were concerned with lump and lump chains interacting with a plane soliton or with each other. In particular, [Stepanyants et al., 2022] studied an interaction of a lump and plane soliton. It was shown that such an interaction is elastic so that, both the lump and plane soliton restore their shapes as shown in Fig. 27. There is also a solution which represents a line soliton with a lump riding on it, and they stationary move together as shown in Fig. 28 [Stepanyants et al., 2022].

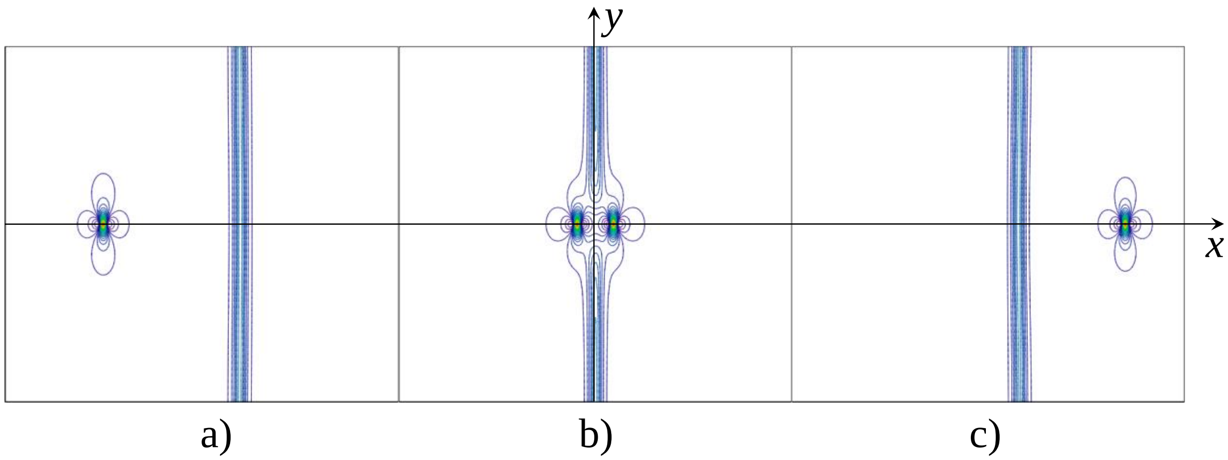



Fig. 27. Line soliton overtaking by a lump. Frame (a), $t$ = –200; frame (b), $t$ = –85; frame (c), $t$ = 0 [Stepanyants et al., 2022].

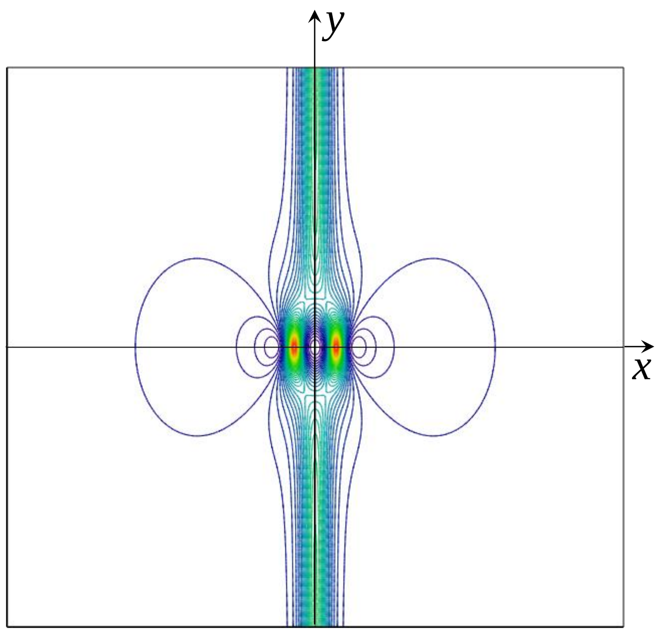



Fig. 28. Line soliton stationary moving together with a lump riding on it [Stepanyants et al., 2022].

Note that all such solutions containing plane solitons or lump chains are unstable with respect to small perturbations. As a result of instability, a number of new lumps can arise as shown, for example in Fig. 22. Nevertheless, solutions with plane solitons or lump chains beside their interest from the mathematical viewpoint can be of physical interest too because upon being somehow created somehow, they can exist for a long time, provided the instability growth rate is relatively small. Moreover, as aforementioned plane solitons (and lump chains) are stable with respect to perturbations of sufficiently small wavelengths with $\Lambda < \Lambda_{cr}$. This can occur, for example, in waveguides of a width $l < \Lambda_{cr}$. Meanwhile, we have to confess that even a single lump long known to be stable has not been observed thus far in any physical medium.

In the past decade, various nonstationary resonant interactions between plane solitons, lumps, and lump chains were discovered [Lester et al., 2021; Rao et al., 2022]. One of such interactions that is worth mentioning is an interaction of two parallel moving plane solitons of equal amplitudes [Stepanyants et al., 2022]. In the case of KdV equation or KP2 equation, such solitons located at a big distance from each other experience an "exchange-type" interaction [Ostrovsky, 2022] when some portion of energy from the rear soliton is transferred to the leading soliton. In the result of this, the amplitude and speed of a leading soliton slightly increase, whereas they decrease in the rear soliton. After that, the amplitudes and speeds of solitons become slightly different, and the distance between them *linearly increases* with time. Even in the case of the amplitude of one of the solitons being weakly disturbed, the general picture described above remains the same.

The situation can be different in the case of the KP1 equation. As was shown in Ref. [Stepanyants et al., 2022], there are such perturbations of infinitesimal amplitude on one of the plane solutions that lead to the emission of a lump that absorbs by another plane soliton, and then both solitons having equal amplitudes at the infinity diverge from each other logarithmically with time, i.e. the distance between them increases as $d \sim \ln t$. This is illustrated by Fig. 29. There is also an analogous solution that describes emission and absorption of a lump chain by two equal amplitude solitons when one of them is slightly disturbed [Stepanyants et al., 2022].

There are many other examples of resonant solutions not only in the KP equation but in several other equations of physical interest (cylindrical KP equation, Davey–Stewartson equation [Gilson, 1992], nonlinear Schrodinger equation [Chabchoub et al., 2021] et al.).

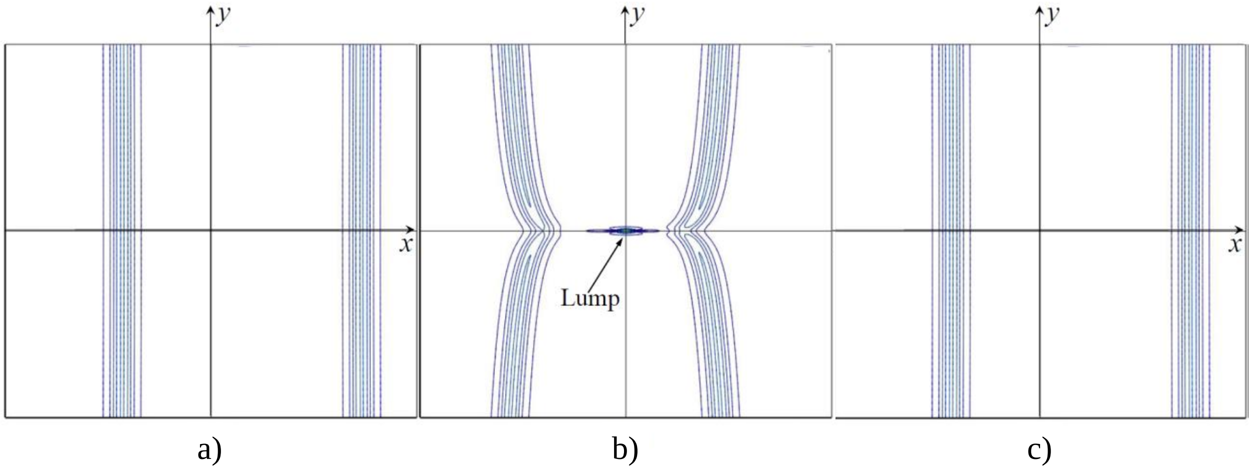



Fig. 29. Two line solitons of equal amplitudes; left of them is slightly disturbed (a); the disturbance growths with time and emits a lump (b); the lump is absorbed by the right soliton (c) [Xu et al., 2019; Stepanyants et al., 2022].

### V. Solitons and lumps in the cylindrical geometry

One of the directions of the soliton theory that has particularly advanced in the past decade is the study of solutions to the cylindrical KdV (cKdV) and KP (cKP) equations. The cKdV equation was first derived by Iordansky in 1959 [Iordansky, 1959] for the description of diverging water waves in a shallow fluid. It was independently derived in the similar context by Lugovtsovs A.A. and B.A. in 1969 [Lugovtsovs A.A. and B.A., 1969]. In 1974 the similar equation was derived for plasma waves by Maxon & Viecceli [Maxon & Viecelli, 1974]. Later, a generalized cKdV equation which also includes a lateral dependence of a wave field on the azimuthal variable $\varphi$ was derived for shallow-water waves by Johnson [Johnson, 1980]. This equation is very similar to the KP equation in the plane case and is called the cylindrical KP (cKP) equation; for the diverging waves it has a form:

$$\frac{\partial}{\partial t}\left(\frac{\partial \upsilon}{\partial r}+\frac{1}{c}\frac{\partial \upsilon}{\partial t}-\frac{\alpha}{c}\upsilon\frac{\partial \upsilon}{\partial t}-\frac{\beta}{2c^5}\frac{\partial^3 \upsilon}{\partial t^3}+\frac{\upsilon}{2r}\right)=-\frac{c}{2r^2}\frac{\partial^2 \upsilon}{\partial \varphi^2}, \tag{5.1}$$

where $r$ is a radial variable, $c$ is a long-wave speed in the linear approximation, and $\alpha$ and $\beta$ are the nonlinear and dispersive coefficients respectively. For axisymmetric waves when function $\upsilon$ does not depend on $\varphi$, this equation reduces to the cKdV equation. As was shown by Dryuma [Dryuma, 1976, 1983], both cKdV and cKP equations are integrable.

Due to the importance of these equations for physical applications, there were several attempts to obtain its approximate analytical solutions in the axisymmetric case [Ostrovsky & Pelinovsky 1977; Cumberbatch, 1978; Ko & Kuel, 1979], solve it numerically [Maxon & Viecceli, 1974; Ko & Kuel, 1979; Ramirez et al., 2002; Fraunie & Stepanyants, 2002], and study experimentally for converging and diverging nonlinear waves of solitary wave type [Hershkowitz, Romesser 1974; Stepanyants, 1981; Weidman, 1988; Ramirez et al., 2002]. In all these studies, the universal decay character was obtained for nonlinear solitary waves and self-similar solutions $A \sim r^{-2/3}$, where $A$ is a solitary wave amplitude.

Exact solutions to the cKdV equation were first obtained by Calogero and Degasperis [Calogero & Degasperis, 1978; 1982] and by Nakamura and Chen [Nakamura & Chen, 1981].

$$\upsilon(r,t) = 2\frac{\partial^2}{\partial t^2}\ln\left\{1+\frac{q}{(12r)^{1/3}}\left[zW^2-\left(\frac{dW}{dz}\right)^2\right]\right\}, \qquad z=\frac{t-t_0}{(12r)^{1/3}}, \tag{5.2}$$

where $q$ is an arbitrary real parameter and $W(z)$ is one of the Airy functions, either of the first kind Ai($z$) [Calogero & Degasperis, 1978; 1982] or of the second kind Bi($z$) [Nakamura & Chen, 1981]. Within these solutions, wave amplitudes also decay with the distance as $A \sim r^{-2/3}$. A recent analysis [Hu et al., 2023, 2024] revealed that solutions described by the Bi($z$) function are singular and, therefore, are not interesting from the physical point of view; whereas solutions based on the Ai($z$) function can be nonsingular at certain values of the parameter $q$ and very close to numerically obtained solutions for solitary waves developed from the KdV soliton. Left panel in Figure 30 shows wave shapes of cylindrical outgoing waves for different values of parameter $q$. As one can see from this figure, the leading part of the wave resembles KdV soliton for a very big negative value of $q$. The detailed comparison confirms that it is indeed indistinguishable from the KdV soliton.

As was shown in [Calogero & Degasperis, 1978; 1982; Nakamura & Chen, 1981], there are solutions that are analogous to N-soliton solutions in the KdV equation. Moreover, a general pulse-type initial perturbation experiences disintegration onto several cylindrical solitons which can experience elastic-type interactions (for details see) [Hu et al., 2024].

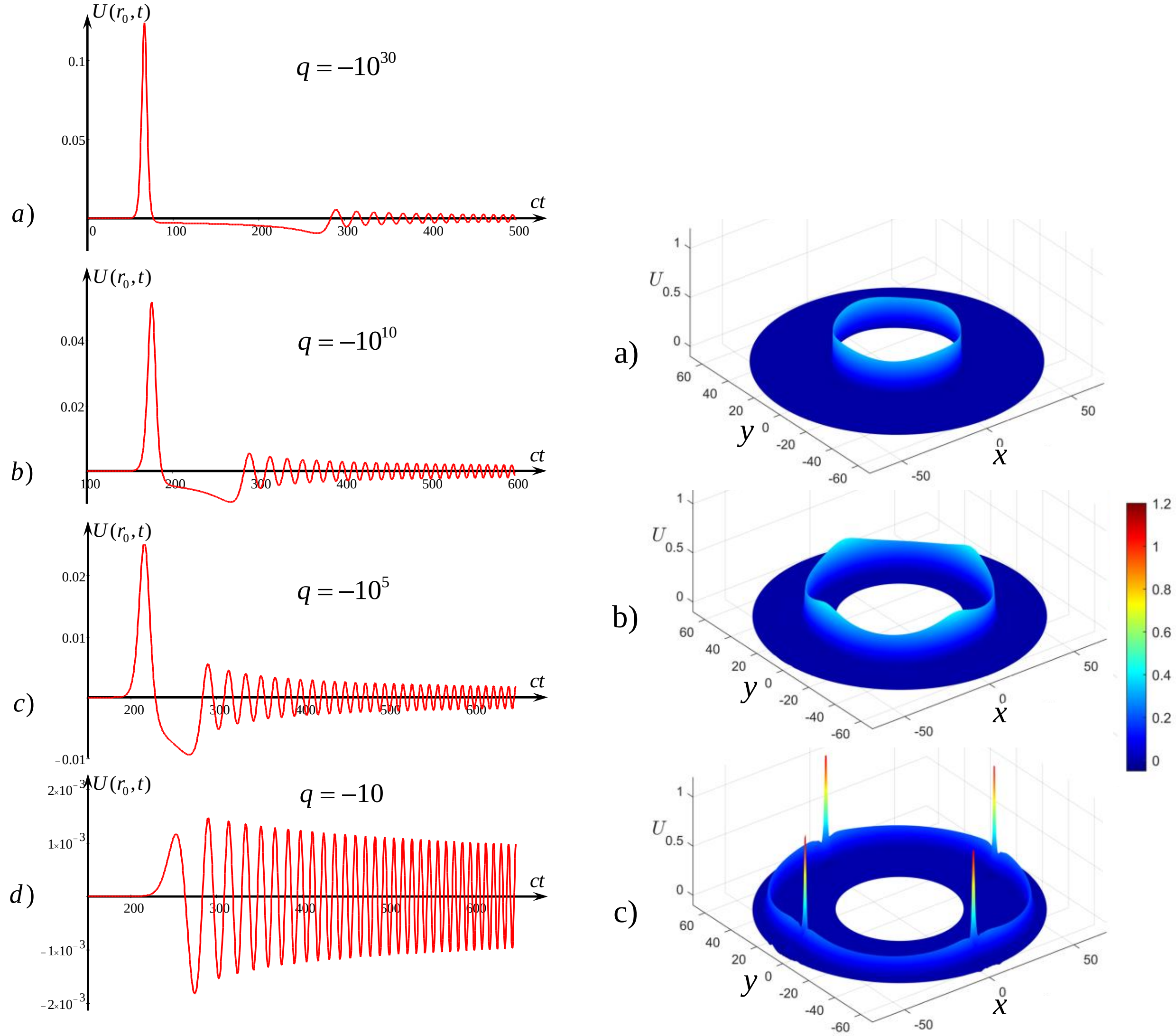



Fig. 30. Left: Shapes of solution (5.1) with the first kind of Airy function $W(z) \equiv \mathrm{Ai}(z)$ for different values of parameter $q$. Right: Development of a modulation instability in the cKP equation with positive dispersion. a) $t$ = 120; b) $t$ = 170; c) $t$ = 200. From [Hu et al., 2024].

Ring solitons, like plane solitons, can be unstable with respect to the front modulation in media with positive and negative dispersion. For negative dispersion ($\beta > 0$) the instability for converging solitary waves was first found in [Ostrovsky & Shrira 1976]. Only small perturbations were examined, while the nonlinear stage of the perturbation evolution still remains to be investigated. In contrast, in media with positive dispersion, when $\beta < 0$ in Eq. (5.1)., a qualitative description of both linear and nonlinear stages of such instability was presented in [Hu et al., 2024]. Numerical calculations performed there have demonstrated that in the course of instability development, a ring soliton can experience fission onto several lumps propagating in different directions as shown in the right panel of Fig. 30. Initially, it was a small azimuthal modulation of a mode number 4 on a ring Ai-soliton (right panel a) which gives rise to four outgoing lumps. A corresponding analytical solution was found in. [Zhang et al., 2024]. The analytical approach based on the Darboux–Matveev transform was used to derive exact solutions that describe regular and resonant interactions of ring waves with lump chains [Zhang et al., 2024]. One of examples of such regular interactions is shown in Fig. 31. Similar solutions were derived in [Klein et al., 2007], but they were presented in inappropriate variables which make them difficult for physical interpretations. Note also that lump solutions were obtained in [Khusnutdinova et al., 2013; Yang et al., 2024] for the KP-type equation in the elliptic coordinates.

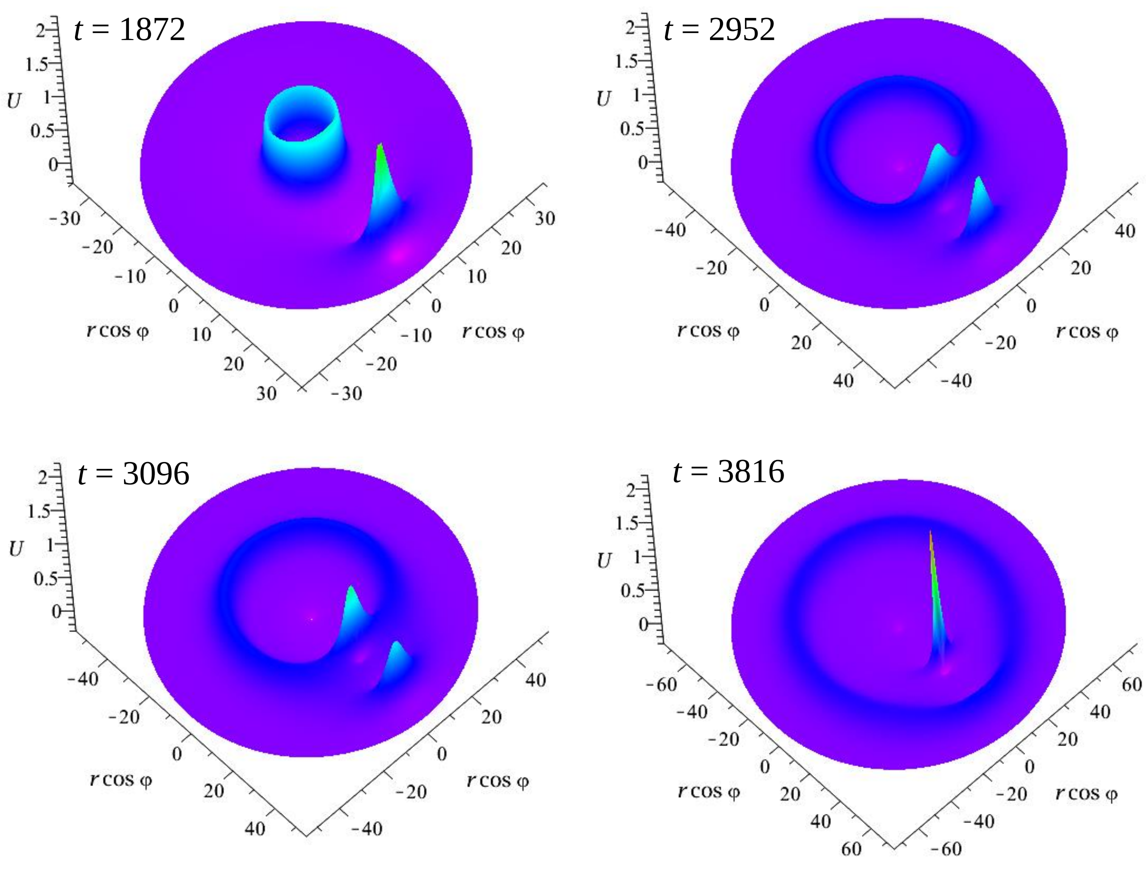



Fig. 31. An example of a regular interaction when a ring soliton overtakes a single lump. From [Zhang et al., 2024].

As an example of a resonant interaction in the case of positive dispersion, we mention an absorption of a lump chain by a ring soliton (of course, there is a reverse process when a ring soliton emits a lump chain); an example of such an interaction is shown in Fig. 32. Similar examples of regular and resonant interactions of two lump chains were obtained in the cited paper [Zhang et al., 2024].

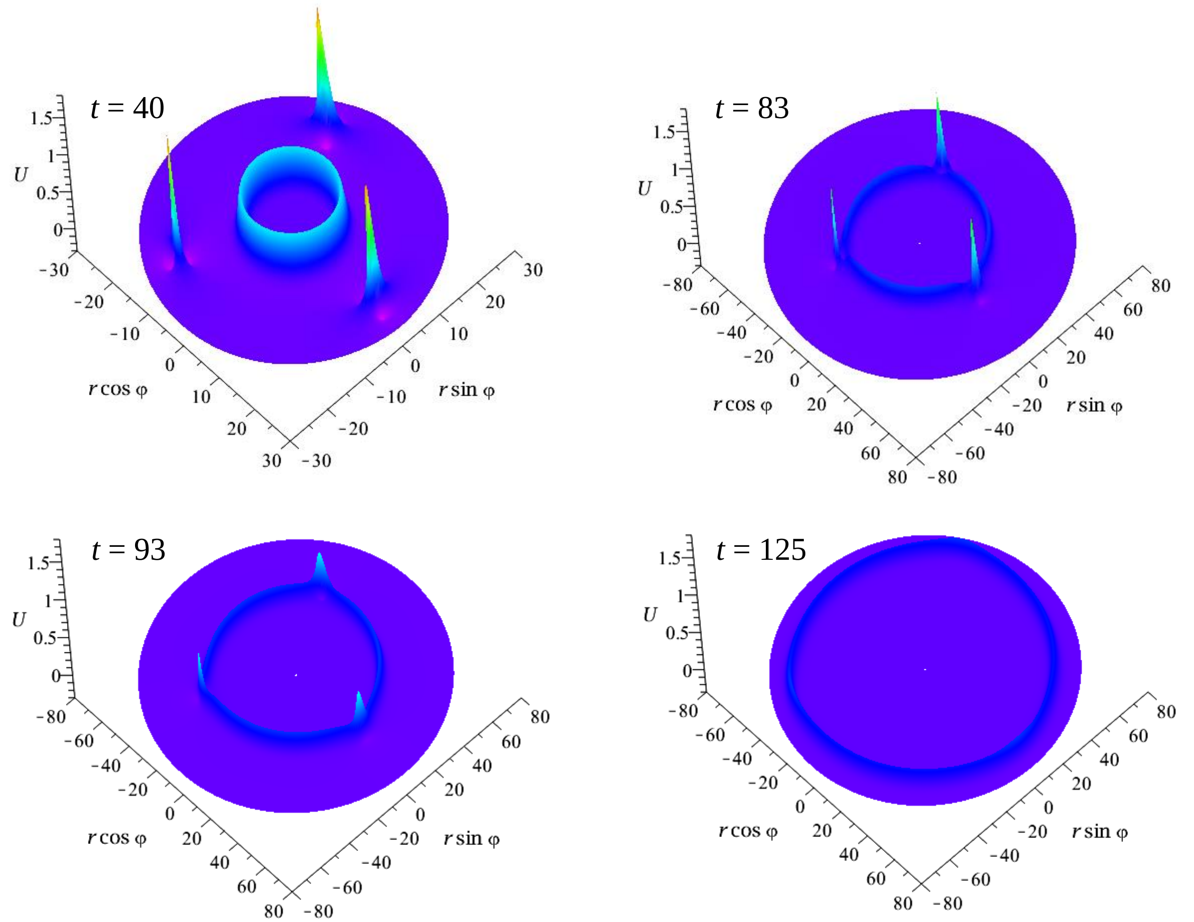
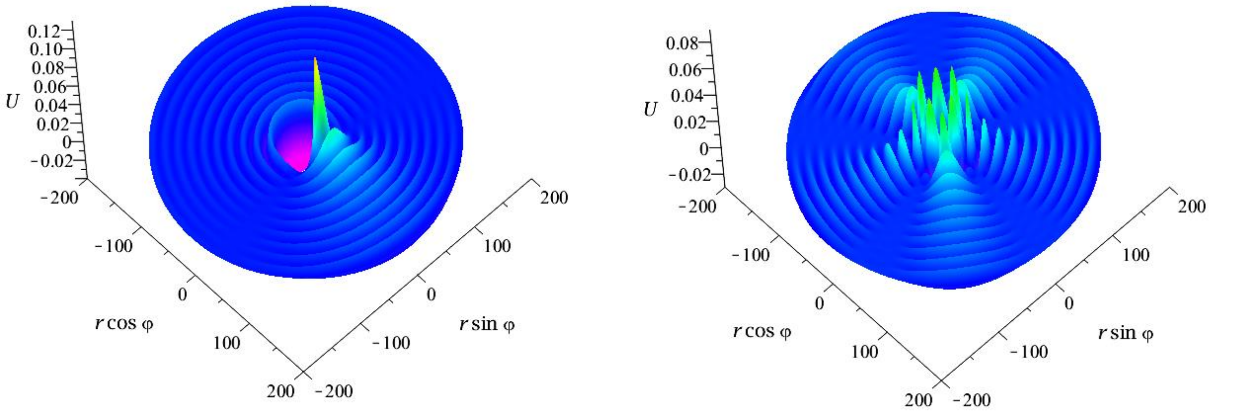



Fig. 32. Another example of a resonant interaction in case of positive dispersion – absorption of a lump chain by a circular soliton. From [Zhang et al., 2024b].

It is worth mentioning also the existence of solutions in the cKP1 equation that describe ripplons, compact formation with oscillatory structure in space; as well as ripplon chains shown in Fig. 33. In the course of propagation, ripplon amplitudes decrease due to the radial spreading as $A \sim r^{-2/3}$. Similar ripplon formations with horseshoe fronts were found within the plane KP1 equation (see [Zhang et al., 2024] and references therein). Apparently, all these entities, solitons, lumps, and ripplons can play a certain role in the theory of strong turbulence which is considered some time as an ensemble of stable particle-like solitary waves.




a) $m = 1$ b) $m = 5$

Fig. 33. A single outgoing ripplon with the azimuthal number $m = 1$ (a) and a chain of ripplons with the azimuthal number $m = 5$ (b). From [Zhang et al., 2024b].

### A. Generalized approaches to the description of nonlinear cylindrical waves

Important limitations of the cKdV equation are that it describes concentric waves only far from the centre of a cylindrical coordinate system and it is applicable only to either outgoing or ingoing waves, but not to their interaction. To overcome these limitations, one can use a set of Boussinesq equations; however, it would be more convenient to deal with only one equation capable to describe waves propagating bot outward and inward. Such an equation was obtained in Ref. [Arkhipov et al, 2015] where the authors derived a single nonlinear equation for the axisymmetric waves that describes wave traveling in both directions and applicable not only far from the centre, but even in its vicinity. In the context of shallow-water waves, the derived equation has the form:

$$\frac{\partial^2\psi}{\partial t^2} - c^2 r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) - \frac{g}{2}r\frac{\partial}{\partial r}\left[\frac{1}{r^2}\left(\frac{\partial\psi}{\partial r}\right)^2\right] - \frac{1}{h}r\frac{\partial}{\partial r}\left[\frac{1}{r^2}\left(\frac{\partial\psi}{\partial t}\right)^2\right] - \beta r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^3\psi}{\partial r\partial t^2}\right) = 0, \qquad (5.3)$$

where $c^2 = gh$, $g$ is the acceleration due to gravity, $h$ is the water depth, $\beta = h^3/3 - \sigma/\rho g$, $\sigma$ is the surface tension between the air and water, $\rho$ is the water density, and $\psi$ is the auxiliary function that is related to the water-surface disturbance by the equation $\eta = (1/r)(\partial\psi/\partial r)$.

On the basis of this new equation a number of numerical experiments were carried out for the particular problems on the evolution of surface waves originated from the localised perturbations. It was demonstrated that Eq. (5.3) can indeed describe the evolution of a perturbation given at the centre $r = 0$. It was also observed that an axisymmetric pulse-type initial perturbation given on a ring at $r = r_0$ splits into two parts one of which travels outward experiencing disintegration into solitons, whereas another one travels toward the centre, increases in amplitude but remains finite at $r = 0$, then it reflects from the centre and travels outward; this is illustrated by Fig. 34.

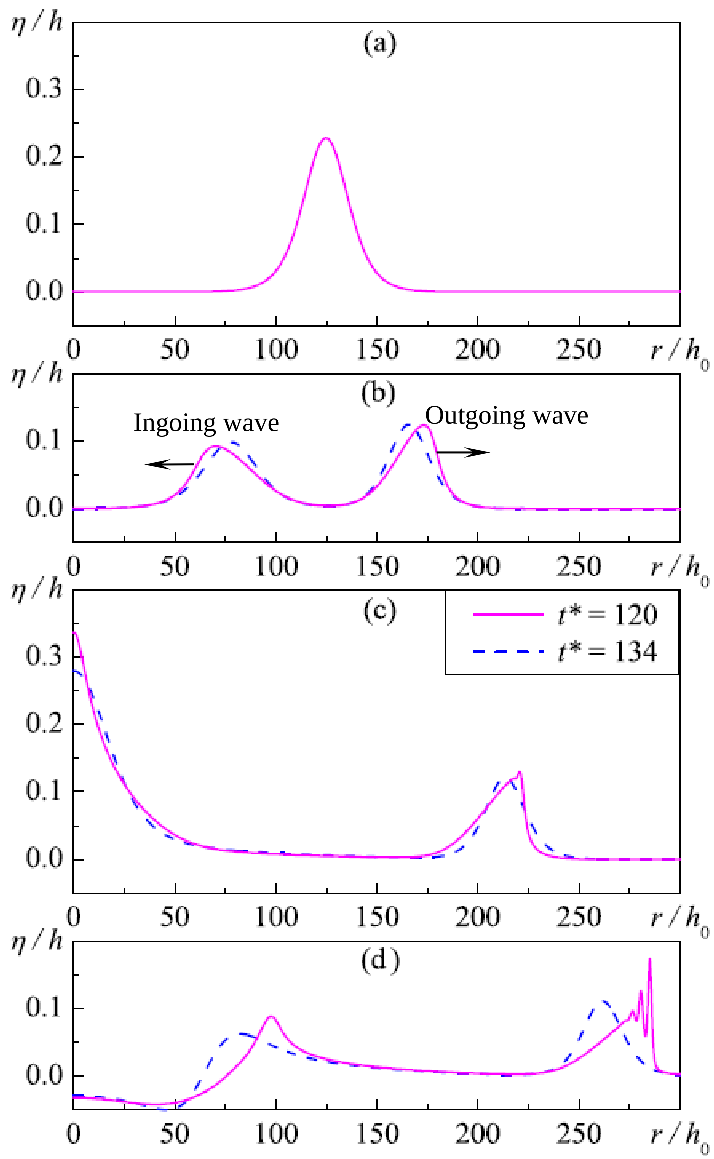



Fig. 34. Evolution of a ring initial perturbation centred around $r = r_0 = 125$ at $t = 0$ (panel (a)). Solid lines – numerical solutions of Eq. (5.3); dashed lines – numerical solutions to the linearised Eq. (5.3) (in both cases the surface tension effect was neglected, $\sigma = 0$); $t^*$ is the dimensionless

time when the incoming waves reaches the centre. From [Arkhipov et al, 2015]; used with permission.

In [Khusnutdinova & Zhang, 2016b] the authors exploited a step-by-step approach to describe concentric outgoing waves propagating from the origin of a cylindric coordinate system in a two-layer fluid. At the initial stage, the authors used an exact solution of a linear 2D long-wave equation derived by Dobrokhotov & Sekerzh-Zen'kovich [2010]. Then, at a big distance from the center, the solution provides a pulse-type outgoing perturbation that can be used as the initial condition for the cKdV equation. With this initial condition, the authors studied numerically the dynamics of surface and interfacial internal waves for initial perturbation of an opposite polarity (for the surface elevation and depression). Figure 35 illustrates surface and internal waveforms at different times.

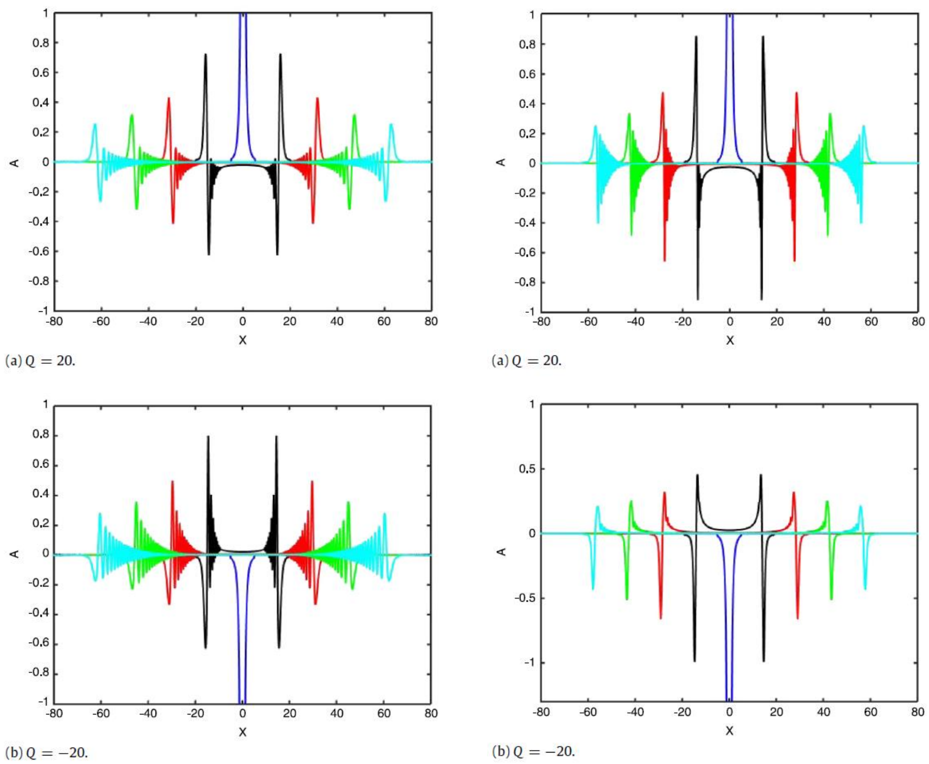



Fig. 35. Waveforms of surface (left panels (a) and (b)) and internal (right panels (a) and (b)) ring gravity waves propagating in the opposite directions (at the angles $\theta = 0$ and $\theta = \pi$) from the center for the initial elevation (a) and depression (b). The parameter $Q$ characterizes the amplitude and polarity of the initial perturbation. Waveforms are shown in the the dimensionless units for the dimensionless times $t = 0$, 16, 32, 48, and 64 in the left panels and $t = 0$, 70, 140, 210, and 280 in the right panels. From [Khusnutdinova & Zhang, 2016b]; used with permission.

The authors presented also numerical solutions for the evolution of a table-top circular initial perturbation and formation of dissipationless shock waves (circular undular bores). This problem represents the circular analog of the well-known dam-breaking problem. It was demonstrated a big difference in wave structures of linear dispersionless problem when there are no ondulations and nonlinear dispersive problem with front disintegration onto a number of solitary waves. The effect of a piecewise-constant shear flow on ring waves generated from a localised initial condition was also studied. This will be described below from a more general viewpoint.

To conclude this section, we mention the recent publication by Sidorovas et al. [2024] where the higher-order cKdV equation was derived both for outward- and inward-propagating water waves within the scope of the 2D Boussinesq, Serre-Green-Naghdi, and Matsuno systems. The Matsuno system contains all relevant nonlinear and dispersive terms of the full Euler equation. Solutions to this equation were studied numerically and compared with numerical solutions within the 2D Boussinesq system and analytical solutions of the cKdV equation. The main conclusion of the paper is that the high-order cKdV model provides a significantly more accurate description of water waves and extends the range of validity of the weakly-nonlinear modelling to waves of moderate amplitudes.

### B. Solitons and lumps in the cylindrical Gardner equation

As well-known, in some case, for the adequate description of wave process, the quadratic nonlinearity is insufficient and cubic nonlinear effects should be taken into consideration. Such a situation occurs, for example, in the description of internal waves in two-layer fluid (see, e.g., [Apel et al., 2007; Ostrovsky et al., 2015] and references therein). Then, the adequate model equation that contains both the quadratic and cubic nonlinear terms in the plane case is the Gardner equation [Ostrovsky et al., 2015]. In the cylindrical case, this equation reads [Polukhina, Samarina, 2007; Gorshkov et al., 2021]:

$$\frac{\partial \upsilon}{\partial r} + \frac{1}{c}\frac{\partial \upsilon}{\partial t} - \frac{\alpha}{c}\upsilon\frac{\partial \upsilon}{\partial t} + \frac{\alpha_1}{c}\upsilon^2\frac{\partial \upsilon}{\partial t} - \frac{\beta}{2c^5}\frac{\partial^3 \upsilon}{\partial t^3} + \frac{\upsilon}{2r} = 0 \,. \tag{5.4}$$

The Gardner equation is integrable and has different types of soliton solutions (KdV type bell-shaped solitons, “fat solitons”, table-top solitons) depending on the coefficient $a_1$. The adiabatic decay of solitons was studied in Ref. [Polukhina, Samarina, 2007] by the asymptotic method and validated by numerical modelling. Unfortunately, there is no simple formula to describe soliton amplitude decay with the distance like in the cKdV case when $A \sim r^{-2/3}$. In this study, it was shown, in particular, that for the negative coefficient $a_1$, a bell-shaped soliton of a positive polarity transforms into a breather when its amplitude becomes less than some critical value.

In the paper [Gorshkov et al., 2021], the authors studied the evolution of cylindrical table-top soliton beyond the adiabatic approximation; such solitons can exist when $\alpha_1 > 0$. It was shown that in the course of propagation, the initial pulse becomes essentially nonstationary; however, its description can be achieved through the matching of two kinks representing the front and rear slopes of a wide soliton (see Fig. 36b) with weakly dispersive wave fields inside and outside the pulse.

In the case of cylindrically diverging solitons (see Fig. 36), the non-stationarity of the process is less pronounced compared to the case of converging waves. In particular, the difference between the magnitudes of fields and front and rear slope speeds does not exceed 1.2. The duration of a solitary wave decreases both in the nonstationary and quasi-stationary cases; however, the regular character of the evolution turns out to be possible only for relatively short initial solitons. For solitary waves with a long duration, a singularity appears on their top. The singularity generates field oscillations, the growth of which, in turn, leads to soliton decay into relatively short solitary waves.

For cylindrically converging solitary waves (see Fig. 37), their evolution occurs with the increase in their durations. However, their shapes notably differ from the rectangular shape and the difference in field slopes and velocities of the front and rear parts are so big that do not allow one to characterize the process as quasi-stationary. The approach used made it possible to determine for the initial solitary wave the critical value of the duration $\Delta_{cr}$ starting from which it decays $\Delta_{cr}(r_0) = 0.07r_0$, where $r_0$ is the radius where the initial soliton was set up. The corresponding critical width of the soliton $L_{cr} = 0.42r_0$.

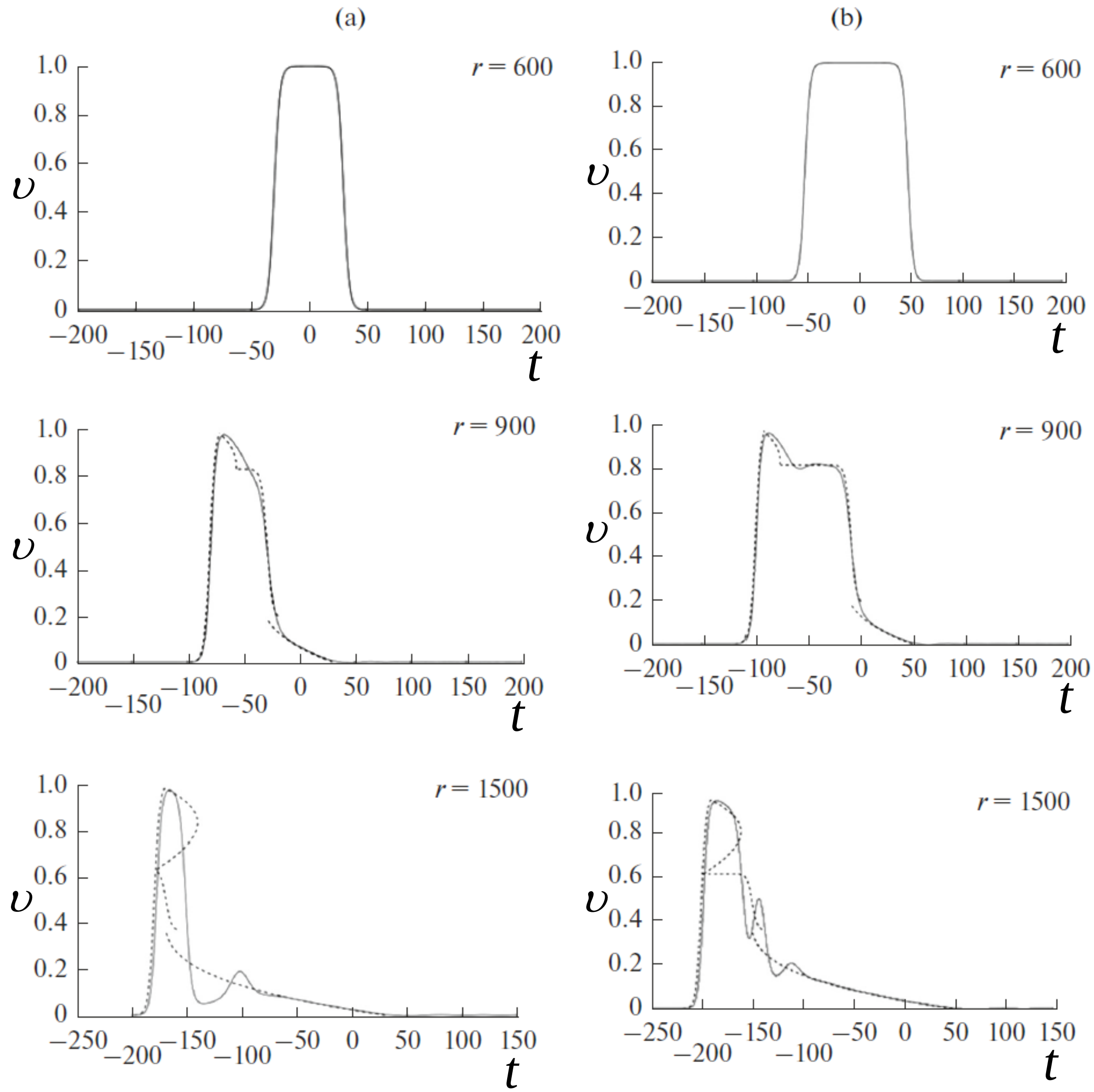



Fig. 36. Evolution of a relatively narrow diverging table-top soliton at different distances *r* (left panel (a)) and a wide table-top soliton (right panel (b)). Solid lines are numerical solutions of the cylindrical Gardner equation; dashed lines – theoretical results obtained within the approximate model. From [Gorshkov et al., 2021].

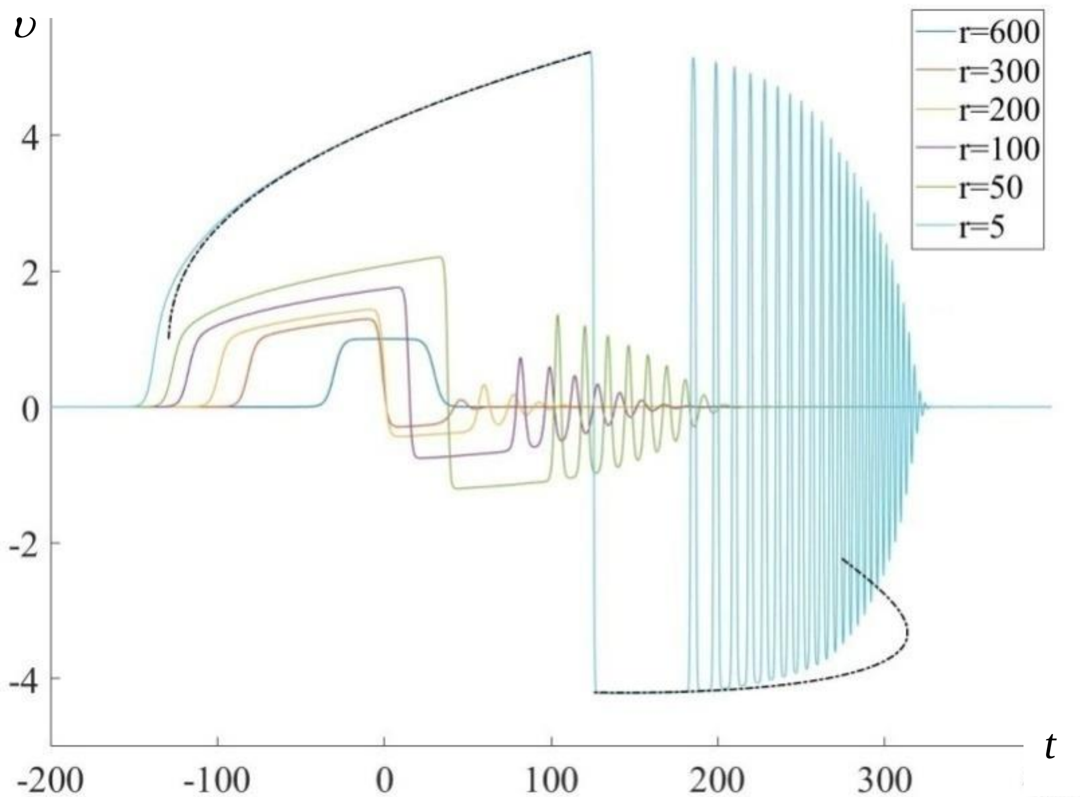



Fig. 37. Evol ution of a cylindrically converging soliton with an initial duration $\Delta(r_0)$ = 60 at different values of $r$ ($r = r_0$ = 600, $r$ = 300, 200, 100, 50, and 5). Solid lines represent numerical calculation, and the dotted-dashed line represents a theoretical dependence within the approximate model. From [Gorshkov et al., 2021].

### A. Internal ring solitons on a shear flow

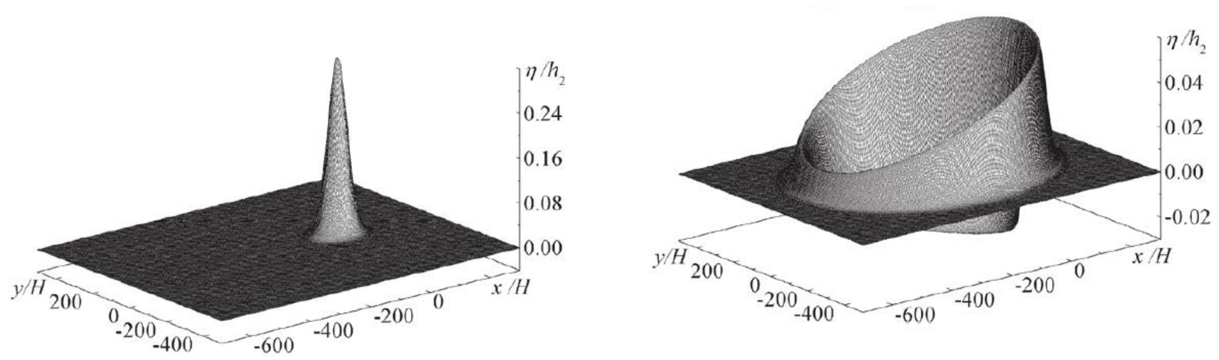
Taking into account the effect of shear flow on ring solitons is challenging since the geometries of the shear flow and ring solitons are different. Such an incompatibility of geometries for different factors is not uncommon in numerous other physical contexts, but here we confine our consideration to the context of surface or internal waves on a shear flow. In the series of papers [Khusnutdinova, Zhang, 2016a; 2016b; Khusnutdinova, 2020; Tseluiko et al., 2023] the authors studied nonlinear quasi-circular surface and internal waves propagating on depth-dependent shear flows in density stratified fluid. Shear flows effect led to a distortion of wavefronts of surface and internal waves but the result for internal waves was rather unexpected. A first in the long wave approximation a model 2+1-dimensional cKdV equation was derived for the description of weakly nonlinear waves propagating at different directions with respect to the basic flow direction; the equation has the following form:

$$\mu_1 \frac{\partial \upsilon}{\partial r} + \mu_2 \upsilon \frac{\partial \upsilon}{\partial \xi} + \mu_3 \frac{\partial^3 \upsilon}{\partial \xi^3} + \frac{\mu_4}{r} \upsilon + \frac{\mu_5}{r} \frac{\partial \upsilon}{\partial \theta} = 0, \quad (5.5)$$

where function $\upsilon(r, \theta, t)$ describes a perturbation of a water or isopycnal surface, $\mu_i$ are some coefficients which depend on the stratification and shear-flow structure, and $\theta$ is an angular variable in the horizontal plane, $\xi = rk(\theta) - ct$ with $c$ being a wave speed of long linear waves when a shear flow is absent and $k(\theta) = 1$, whereas in the presence of a shear flow function $k(\theta)$ describes the shape of a wavefront. A similar equation was derived earlier by Johnson [Johnson, 1990] for surface quasi-cylindrical waves on a shear flow in a fluid of a constant density, but as was shown in [Khusnutdinova, Zhang, 2016a], in such a case, the coefficient $\mu_5$ is identically zero, and Eq. (5.4) reduces to the ordinary cKdV equation, whereas in the general case, $\mu_5 \neq 0$.

The developed theory was illustrated by the case of wave propagation in a two-layer fluid of different densities moving with different speeds. It was discovered a striking difference in the shapes of wavefronts of surface and interfacial (internal) waves propagating in the same shear flow. While wavefronts of surface waves elongate along the current, wavefronts of internal waves squeeze in the direction of the current (see Figs. 8 and 9 in [Khusnutdinova, Zhang, 2016a]). The difference between the waveforms of surface waves and internal waves for several values of dimensionless speed differences $\Delta U$ between two layers was illustrated by numerical solutions of Eq. (5.4).

Some more interesting details of ring wave propagation in a two-layer fluid under the action of a piece-wise-constant shear flow were presented in [Khusnutdinova, Zhang, 2016b]. The most striking effect was observed when the wave heights of surface and internal waves decrease faster upstream than downstream, although the effect of a shear flow on surface waves is weaker than its effect on interfacial waves. This result obtained from the model equation (5.4) agrees with the result of numerical solution of the Boussinesq-type set of equations for internal waves in a two-layer fluid with the rigid-lid boundary condition on the water surface [Arkhipov et al., 2013]. However, in [Arkhipov et al., 2013], the authors were able to simulate wave development from the axisymmetric Gaussian initial perturbation given at the origin, $r = 0$ (see Fig. 38a), whereas the authors of Ref. [Khusnutdinova, Zhang, 2016b] used a model initial condition that mimics the result obtained in [Arkhipov et al., 2013] when a wavefront developed from the initial perturbation was far enough from the centre, so that equation Eq. (5.4) can be used (see Fig. 38b). Note, that the squeezing of the wavefront of internal wave was not noted in [Arkhipov et al., 2013]; whereas in [Khusnutdinova, Zhang, 2016b], the authors observed a front squeezing. Influence of dissipation on the internal wave formation from the initial pulse-type perturbation was studied numerically in Ref. [Arkhipov et al., 2007].




(a) (b)

Fig. 38. The initial bell-shaped perturbation (a) and a wave shape formed at some distance during its evolution. The vertical scale in Fig. 38b is four times smaller than that in Fig. 38a. (Reproduced from [Arkhipov et al., 2013]. Copyright © year Elsevier Masson SAS. All rights reserved.)

Further generalisation of the problem of nonlinear circular wave propagation was made in the papers [Khusnutdinova, 2020; Hooper et al., 2021b; Tseluiko et al., 2023]. In the first two papers, the authors have shown that the squeezing in the flow direction of the initially circular front of an interfacial wave is a rather general phenomenon which also occurs in a smooth velocity profile. Specifically, in those papers, the velocity was assumed to be zero in the lower layer, and then, it gradually increased up to the surface in the upper layer (the rigid-lid approximation was used in the paper). Unfortunately, there was a calculation error in the paper [Khusnutdinova, 2020] but it had a minor effect on the final results. The error has been rectified in the follow-up article by Hooper et al., [2021b] (private communication by K. Khusnutdinova).

In the latter paper [Tseluiko et al., 2023], the authors studied wavefronts deformation in the three-layer fluid with the linear velocity profile under the rigid-lid approximation. It was shown that in such a model the wavefront of the faster baroclinic mode is elongated in the direction of the current like a surface mode in the previous study [Khusnutdinova, Zhang, 2016a], whereas the wavefront of the slower mode is squeezed. Moreover, depending on the vorticity strength, several different regimes have been identified. When the vertical shear is relatively weak, a part of the wavefront is capable of propagating upstream, but when the shear is strong enough, the whole wavefront propagates downstream. A richer behaviour was observed for the slower mode. As the shear increases, singularities of the swallowtail-type can arise and, eventually, solutions with compact wavefronts crossing the downstream axis cease to exist. The authors showed that the latter effect is related to the long-wave instability of the basic flow. The cKdV-type equation (5.4) was derived for each mode and the evolution of wave modes was studied numerically. A soliton creation in the upstream direction was revealed when the wavefronts expanded, and nonlinearity and dispersion effects became stronger.

Thus, in all these studies it was demonstrated that a shear flow in a stratified flued can provide nontrivial wave fronts of outgoing surface and internal waves originated from circular of pulse-type initial perturbations.

## VI. Concluding remarks

The selected topics we discussed here are in no way closed; the review provides just a snapshot of a few lines of research chosen at a somewhat arbitrary moment from the viewpoint of their intrinsic evolution. Here, we attempt to outline our views on the likely continuations and perspectives of the threads and trends that we discussed.

Let us start with the radiating solitons. First, we reiterate that the radiating solitons that we discussed represent a generic phenomenon in real physical systems, much more general than the classical fully localized stationary solitons. It would not be an exaggeration to say that the classical solitons can be viewed as a limiting case of radiating solitons occurring in the strongly idealized systems. We didn't aim at surveying a huge variety of physical causes of radiation by solitons since our focus is on elucidating its main implications. Considering radiation caused by the low-frequency dispersion in the context of long waves in rotating fluids as a representative example, we outlined how it results in a finite life span of initially soliton-like pulses, while in a model of wave-current resonance, we showed an example of the generation of radiating solitons because of fission of a large initial pulse. The radiating solitons are intermediate asymptotic for a large class of systems and initial conditions. We expect a qualitatively similar behaviour for weakly dispersive nonlinear waves in most physical contexts.

As regards physical applications, along with the oceanic waves, we outlined a less thoroughly investigated but equally interesting and potentially important class of acoustic solitons in solids, including specific effects associated with radiating and/or coupled solitons.

We overviewed a wide class of solitary wave solutions in the KdV-like systems with various nonlinearities. In such systems, the single hump solitary wave solutions are robust, while the interaction of two solitons occurs roughly as in the integrable systems. One of the differences between interactions in the generic non-integrable and idealized exceptional integrable systems is that in the generic case interactions are inelastic: there is always radiation that accompanies an interaction. As a result of such an inelastic interaction, usually the larger soliton gets even bigger, whereas a smaller one gets smaller. There are examples where, as a result of multiple repeating of such interactions in a confined environment, the larger soliton grows by sucking all the energy out of smaller ones, whereas the smaller one disappears. How general is such a scenario, currently we do not know. An alternative scenario is that the radiation somehow restores the smaller solitary waves; there are examples where solitons interact through the radiation [Gorshkov & Ostrovsky (1984)]. At the moment, we lack understanding which would enable us to say *a priori* what scenario should be expected for a particular system. Certainly, this challenge warrants further efforts.

We note that here we confined our review only to the KdV-type systems. There is also an infinite variety of weakly dispersive nonlinear evolution equations supporting soliton solutions with dispersion described by pseudo-differential operators (see, e.g., [Shrira & Voronovich, 1996; Oloo & Shrira, 2023] and references therein). Note that nonlocal dispersion might also include non-local generation and dissipation. In contrast to the familiar situation where a broad range of physical problems is funnelled into a small number of "canonical" equations, in case of nonlocal dispersion one must deal with numerous non-universal (including fractional) equations. Novel approaches may be needed in this area.

Systems of coupled weakly dispersive nonlinear evolution equations occurring in physical contexts where there are different modes or polarizations also possess stable solitary wave solutions (see, e.g., [Alias et al., (2014]). Therefore, there is a huge untapped potential for extending this line of research to include more general systems. At the end, we want to know what the solitary solutions of such systems are, how their solitons interact and radiate and what respects they differ from the multitude of the systems already studied. We can expect a steady progress in clarifying these issues.

The recent advances in the description of statistical properties of soliton gas – the kinetics of solitons, which we briefly mentioned in our review, were mostly confined to solitons in integrable models, whereas much less is known about the inelastic kinetics of solitons (see, however, [Gorshkov & Papko, 1977b; Dyachenko et al., 1989]). Worth mentioning here is also the work on soliton turbulence in the systems with external pumping [Gorshkov et al., 1977]. In view of a sharp rise of interest in statistical properties of soliton gas and focussing of significant efforts in this direction, we expect substantial progress there.

Another area where we are expecting further progress is in describing kinetics of two-dimensional solitons. The recent results on lump interactions can be viewed as a foundation for this challenge within the KP1 model. At a first glance there is no interaction within such an equation: the collisions affect neither amplitude nor phase of the colliding lumps (with just one caveat regarding the resonance interactions mentioned in Section IV) so that we have a "super-noble gas" of lumps. However, beyond the paraxial approximation central for the derivation of the KP equation, similar lumps are exact solutions of the Boussinesq equations; their interaction is no longer elastic which is expected to lead to non-trivial but tractable kinetics. Also, as discussed in this review, within the KP1 equation there also exist complicated structures comprised of lumps.

How their ephemeral existence might affect lump statistics we do not know yet and, since the system is integrable, we might hope for the answer in not-too-distant future. Note, that the two-dimensional Zakharov–Kuznetsov equation [Zakharov & Kuznetsov, 1974] supports stable 2D solitons which interact inelastically. To our knowledge, so far there were no attempts of studying "turbulence" described by such models.

It is worth mentioning also about an open question of nonlinear stage of instability of converging ring solitons. To our knowledge, this issue is now being investigated by two independent groups employing different numerical models. One can expect the formation of shocks propagating in opposite azimuthal directions on a soliton front as was predicted in [Pesenson, 1991]. Hopefully, with this issue, we will get clarity in a not-too-distant future.

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