A variational approach to frame-indifferent quasistatic viscoelasticity of rate type

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June 6, 2024

Abstract

The three-dimensional dynamical model for nonlinear viscoelasticity of strain-rate type is investigated in a quasistatic setting under the assumption of higher order regularity of the deformation, which, in the literature is referred to as the case of non-simple materials. Existence of weak solutions is proven using a time-discretization technique while respecting the condition of dynamical frame-indifference. Some observations on frame-indifference for strain-rate type stresses are made and corrections are proposed for some related work in the literature. Finally, a counterexample is given to show that the assumed higher order regularity is necessary in order to obtain required compactness.

1 Introduction

The requirement for a well-posed qualitative mathematical theory for properly formulated dynamics, based on fundamental physical principles, has been recognized for a long time. In order to realize this purpose one needs to have answers to some questions that can be stated generally for any evolution equation associated with a nonelliptic variational integral. This paper aims to address the question of existence of solutions for frame-indifferent viscoelastic models of rate type. An extensive overview about such models can be found in [23]. These dynamical models have been successfully studied in the literature, including the one-dimensional case ([11], [22], [1], [15]), the general three-dimensional case ([12], [26], [13]), and the thermodynamical case when temperature dependance is also taken into account ([27]).

In [24], for the first time quasistatic approximation was considered in the context of nonlinear viscoelasticity of rate type and a variational approach was introduced in a three-dimensional setting for the existence of solutions while handling the dynamical frame-indifference of the stress. Following [24], in [7] the one-dimensional problem was studied which was proved to be equivalent to a gradient flow due to the quasistatic nature of the governing equations resulting from neglecting the inertia term. Similarly, in [20], the quasistatic case was considered and a variational approach was adopted via metric gradient flows. After [24], approximating dynamical models by the corresponding quasistatic equations have been adopted by many studies in various contexts. However, none of the studies for such rate-type viscoelastic models has been successful in obtaining well-posedness while dealing with the requirement of frame-indifference.

The main difference of the current work from the model investigated in [24] is that here a higher order gradient term is included in the equation of linear momentum balance in order to obtain necessary compactness while passing to the limit after the application of a time-discretization method. More precisely, a term including the second gradient of the deformation is added into the system representing *the strain-gradient*. Materials with such models are called *non-simple materials*. These models have been successfully studied in the literature in various contexts, the most relevant to the current work

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being [14], [3], [19] and [21]. More precisely, in [14] and [19] the same model considered here is investigated. However, the focus of the first work is more on the linearized case rather than large strains while the latter one focuses on possible self-contact in deformed configuration. In [21], the same model is coupled with a suitable heat equation so that thermal effects are also taken into account. While this makes the system to be studied more general, the analysis is more complicated and some restrictions are put into place, including regularization of the mechanical equation with a strain-rate term in order to handle necessary compactness. More recently, [3] studied thermoviscoelasticity in the quasistatic setting by refining the results obtained in [21].

The present work contributes to the above mentioned literature in three ways. Firstly, new investigations are proven for frame-indifferent stresses of rate type as well as the corresponding dissipation potentials. Also, some expressions on frame-indifference in the literature are proven to be incorrect. Secondly, weak solutions to frame-indifferent quasistatic viscoelasticity in the context of non-simple materials is proven to exist using a variational approach and by applying a time-discretization method. While this can be seen to be less general as a special case of the systems investigated in [21] and [3] due to being isothermal, the analysis is rather clear focusing solely on the mechanical equation, in particular, on a specific viscoelastic part of the stress tensor. Thirdly, a counterexample is given in the static case, which can be viewed as the state at each fixed time-step, showing that without the inclusion of the strain-gradient term in the model, it is not possible to obtain compactness leading to the existence of solutions.

The content of the paper is as follows. In Section 2 we describe some preliminary notions that will be used throughout the paper. Then, in Section 3 we introduce the governing equations to be studied. Section 4 is devoted to frame-indifference with some observations about rate type stresses and dissipation potentials, and corrections to some statements in the literature. In Section 5 we state the assumptions, introduce the time-discretization scheme and prove the main result. Finally, in Section 6 we give our counterexample before the concluding remarks in the last part.

2 Preliminaries

For a homogeneous elastic body with a reference configuration $\Omega \subset \mathbb{R}^3$ and with unit reference density, a motion is an evolution of diffeomorphisms $y(\cdot, t) : \Omega \to \mathbb{R}^3$, where $t \in [0, T] \subset \mathbb{R}$. The *deformation gradient* at time t is written as $\nabla y(x, t)$, or equivalently Dy or F can be identified with the $n \times n$ matrix of partial derivatives given as

$$(Dy)_{i\alpha} = y_{i,\alpha} = \frac{\partial y_i}{\partial x_\alpha}.$$

We would like our model to be physically realistic and hence it is necessary to avoid interpenetration of matter so that two distinct material points cannot simultaneously occupy the same position in space. In order to ensure this, it is required that for (almost) every t, the actual position field $y(\cdot, t)$ is injective. This is equivalent to say that the deformation y is invertible in Ω. We can still allow some cases where, for example, self-contact occurs on the boundary (see [5] for more information). Therefore, it is enough to assume that the admissible deformations satisfy the constraint

$$\det \nabla y(x, t) > 0, \tag{2.1}$$

ensuring that the admissible deformations are *orientation-preserving* and *locally invertible*. As discussed in [5] by examples, however, local invertibility does not imply global invertibility.

An elastic material is *hyperelastic* if there exists a function $W : \Omega \times \mathrm{GL}^+(3) \to \mathbb{R}$ differentiable with respect to the variable $F \in \mathrm{GL}^+(3)$ for each $x \in \Omega$ such that the *first Piola-Kirchhoff stress tensor* is given by

$$T_R(x, F) = \frac{\partial W}{\partial F}(x, F). \tag{2.2}$$

Here $\mathrm{GL}^+(3)$ denotes the set of matrices in $\mathbb{R}^{3\times 3}$ with positive determinant. The function W is called the *stored-energy function*. Naturally, if the material is *homogeneous*, it is a function of F only (cf. [8], [9]), which is the case we consider in this work. As noted by Ball [5], this is more restrictive than saying that Ω is occupied by the same material at each point, since it is possible to have some pre-existing stresses. We can also define the *second Piola-Kirchhoff stress tensor* as

$$\hat{T}(x,F) = F^{-1}T_R(x,F), \tag{2.3}$$

and the *Cauchy stress tensor* as

$$T(x,F) = (\det F)^{-1}T_R(x,F)F^{\mathrm{T}}. \tag{2.4}$$

The elastic energy corresponding to the deformation y is defined as

$$I(y) = \int_\Omega W(\nabla y(x,t))dx. \tag{2.5}$$

Unless stated otherwise, we will make the following convention that the initial free energy is finite,

$$\int_\Omega W(\nabla y(x,0))dx < \infty.$$

The matrix

$$C = \nabla y^{\mathrm{T}}\nabla y \tag{2.6}$$

is called *the right Cauchy-Green strain tensor*. It is symmetric and is positive-definite where ∇y is nonsingular.

3 Modelling

We use the Lagrangian formulation in the domain $\Omega \subset \mathbb{R}^3$ bounded with a smooth boundary Γ. We consider the time variable $t \in [0,T]$. A generalization of Kelvin-Voigt type viscoelasticity can be modelled as

$$\ddot{y} - \operatorname{Div} DW(\nabla y) - \operatorname{Div} S(\nabla y, \nabla\dot{y}) - f(t) = 0, \tag{3.1}$$

where the constitutive equation for the first Piola-Kirchhoff stress tensor reads $T_R = W(\nabla y) + S(\nabla y, \nabla\dot{y})$ with S being the viscoelastic part, and $f(t)$ is the external mechanical loading which might consist of a dead force and boundary traction.

In this paper, we have two main postulates; firstly, we consider the quasistatic approximation for (3.1) meaning that the inertial effects are neglected; secondly, in order to gain enough regularity to handle the physical nonlinearities, we make the assumption that there is an energy contribution coming from the strain gradient. As a result the total free energy (2.5) becomes

$$E(y(t)) := \int_\Omega \big(W(\nabla y) + \mathcal{H}(\nabla^2 y)\big)dx, \tag{3.2}$$

where $\mathcal{H} = \mathcal{H}(\nabla F)$ is the potential corresponding to the elastic hyperstress. As mentioned in the introduction, materials with such elastic energy are referred to as *second-grade* or *non-simple materials* in the literature. Unfortunately, the problem of existence of solutions for simple materials, that is, without having the higher order gradient term, is still open both in the quasistatic and dynamical cases. We can introducing the dissipation potential Ψ as

$$D_H\Psi(F,H) = S(F,H). \tag{3.3}$$

As a result the balance of linear momentum in the quasistatic case implies

$$\operatorname{Div}\big(DW(F) - \operatorname{Div}\mathcal{H}'(\nabla F)\big) + \operatorname{Div} S(F,\dot{F}) = f(t), \tag{3.4}$$

In this paper, we are interested in existence of solutions of (3.4) which can also be viewed as an abstract gradient flow. In the case of simple materials in one space dimension, it is explicitly shown by Ball and Şengül [7] that this model is equivalent to the equation of gradient flows.

We impose the following initial conditions

$$y(x,0) = y_0 \;\text{ on }\; \Omega. \tag{3.5}$$

For boundary conditions, for simplicity, and without loss of generality, we only consider $f(t)$ being a time-dependant dead force given as

$$f : [0,T] \times \Omega \to \mathbb{R}^3. \tag{3.6}$$

The reader is referred to a recent paper by Mielke and Roubicek [21] for the treatment of boundary conditions including traction.

4 Frame-indifference

4.1 Definition and mathematical expression

The mechanical behaviour of materials is governed by some general principles one of which is the principle of frame-indifference. As a general axiom in physics, it states that the response of a material must be independent of the observer (see e.g. [25]). In particular, it restricts the form of the constitutive functions and thus plays an important role in nonlinear continuum mechanics. We state it as follows:

> *The Principle of Frame-Indifference (Objectivity)*: Constitutive functions are invariant under rigid motions.

In order to express this principle as a mathematical condition, we first note that a change of observer can be seen as application of rigid-body motions on the current configuration. Since a rigid-body motion consists of a translation and a rotation, in each of these motions, the relative positions of the points of the material remain the same. As the deformation gradient is not effected by the translations of the origin, the corresponding expression for the stress becomes

$$\tilde{T}_R(x,\tilde{t}) \;=\; R(t)\, T_R(x,t). \tag{4.1}$$

A formal mathematical statement can be given by the following result.

Lemma 4.1. *Any frame-indifferent stress tensor $S(F,\dot{F})$ can be written as*

$$S(F,\dot{F}) \;=\; R\, S(U,\dot{U}), \tag{4.2}$$

where $R \in \mathrm{SO}(3)$ and U is the right Cauchy-Green stretch tensor. Here SO(3) *denotes the set of rotations in $\mathbb{R}^3$.*

Proof. By polar decomposition theorem, we have

$$\begin{aligned} R^T S(F,\dot{F}) &= S(R^T F, \dot{\widehat{R^T F}}) = S(R^T R\, U, \dot{R}^T F + R^T \dot{F}) \\ &= S(U, \dot{R}^T R\, U + R^T(\dot{R}\, U + R\dot{U})) \\ &= S(U, (\dot{R}^T R + R^T \dot{R})\, U + \dot{U})) \;=\; S(U,\dot{U}) \end{aligned}$$

as required. □

We can obtain a more convenient form of (4.2) by using the second Piola-Kirchhoff stress tensor as follows.

$$T(F,\dot{F}) \;=\; F^{-1} S(F,\dot{F}) = U^{-1} S(U,\dot{U}) =: G(C,\dot{C}).$$

Thus we have

$$S(F,\dot{F}) \;=\; F\, G(C,\dot{C}). \tag{4.3}$$

It is also worth mentioning that rotations are involved in both material symmetry and frame-indifference, but they act differently. More precisely, in material symmetry, the rotation acts in the reference configuration and in frame-indifference, the rotation acts in the deformed configuration. Therefore, it is not possible to obtain one variant by rotating another. In other words, given symmetric matrices U_1 and U_2, it is not possible to find a rotation R such that $RU_1 = U_2$, since this would be inconsistent with the uniqueness property stated in the polar decomposition theorem.

4.2 Further observations on frame-indifference

This section is devoted to some trivial but crucial observations we make on frame-indifference.

Lemma 4.2. *Any frame-indifferent stress* $S(F,\dot F)$ *should satisfy*

$$S(F,\dot F):\dot F=\frac{1}{2}\,G(C,\dot C):\dot C.$$

Proof. We know by (4.3) that any frame-indifferent $S(F,\dot F)$ takes the form $S(F,\dot F)\,=\,F\,G(C,\dot C)$. Therefore, using the fact that $G(C,\dot C)$ is symmetric, we get

$$\begin{aligned} S(F,\dot F)\colon\dot F &= F\,G(C,\dot C)\colon\dot F=G(C,\dot C)\colon F^T\dot F=G(C,\dot C)\colon\dot F^TF\\ &= G(C,\dot C)\colon\frac{1}{2}\left(\dot F^TF+F^T\dot F\right)=\frac{1}{2}\,G(C,\dot C)\colon\dot C \end{aligned}$$

as required. □

Lemma 4.3. *The condition*

$$G(C,\dot C):\dot C\;\geq\;\gamma\,|\dot F|^2,\;\;\gamma>0\;\;\textit{a constant,}\tag{4.4}$$

contradicts frame-indifference.

Proof. Assume for contradiction that there exists a frame-indifferent S satisfying (4.4). By Lemma 4.2, we have

$$S(F,\dot F):\dot F\;\geq\;\gamma\,|\dot F|^2\quad\Leftrightarrow\quad G(C,\dot C):\dot C\;\geq\;2\,\gamma\,|\dot F|^2.$$

Choosing $F=R(t)=\exp(Kt)\,\in\,\mathrm{SO}(3)$, where K is skew, we get

$$\dot F\;=\;K\,\exp(Kt)\;\;\text{and}\;\;|\dot F|^2\;=\;|K|^2\;\neq\;0.$$

However, $C\;=\;F^TF\;=\;R^TR\;=\;1$ implies $\dot C\;=\;0$, giving a contradiction. □

Remark 4.1. *The condition*

$$G(C,\dot C):\dot C\;\geq\;\gamma\,|\dot C|^2,\;\;\gamma>0\;\;\textit{a constant,}$$

does not contradict frame-indifference as can be seen easily by choosing $G(C,\dot C)\;=\;\dot C$ *in (4.4).*

In contradiction to the claim of Tvedt [26] we have that

Lemma 4.4. *The assumption*

$$(S(F,\dot F)-S(F,\dot{\hat F})):(\dot F-\dot{\hat F})\geq\gamma\,|\dot F-\dot{\hat F}|^2\,,\;\gamma>0\tag{4.5}$$

is incompatible with frame-indifference.

Proof. If, for contradiction, the claim was true, then there would exist a frame-indifferent S satisfying (4.3) so that (4.5) would give

$$(F\,G(C,\dot C)-F\,G(C,\dot{\hat F}^TF+F^T\dot{\hat F})):(\dot F-\dot{\hat F})\;\geq\;\gamma\,|\dot F-\dot{\hat F}|^2.\tag{4.6}$$

Let us define $A\;:=\;(G(C,\dot C)-G(C,\dot{\hat F}^TF+F^T\dot{\hat F}))$ so that we get

$$\begin{aligned} &(F\,G(C,\dot C)-F\,G(C,\dot{\hat F}^TF+F^T\dot{\hat F})):(\dot F-\dot{\hat F})=\;A\;:\;F^T(\dot F-\dot{\hat F})=\\ &\quad=\frac{1}{2}\left[A+A^T\right]\,:\,F^T(\dot F-\dot{\hat F})=\frac{1}{2}\left[A:F^T(\dot F-\dot{\hat F})+A^T:F^T(\dot F-\dot{\hat F})\right]\\ &\quad=\frac{1}{2}\left[A:F^T(\dot F-\dot{\hat F})+A:(\dot F-\dot{\hat F})^T\,F\right]=\frac{1}{2}\,A\,:\,\left[F^T(\dot F-\dot{\hat F})+(\dot F-\dot{\hat F})^T\,F\right]\\ &\quad=\frac{1}{2}\,A\,:\,\left[F^T\dot F-F^T\dot{\hat F}+\dot F^TF-\dot{\hat F}^TF\right]=\frac{1}{2}\,A\,:\,\left[\dot C-(F^T\dot{\hat F}+\dot{\hat F}^TF)\right]. \end{aligned}$$

Therefore, (4.6) is now equivalent to

$$\frac{1}{2}\left(G(C,\dot{C}) - G(C,\dot{\tilde{F}}^T F + F^T\dot{\tilde{F}})\right) : \left(\dot{C} - (F^T\dot{\tilde{F}} + \dot{\tilde{F}}^T F)\right) \,\geq\, \gamma\,|\dot{F} - \dot{\tilde{F}}|^2.$$

However, for any given G, we can choose $F = I$ in this inequality and obtain

$$\frac{1}{2}\left(G(1,\dot{F}^T + \dot{F}) - G(1,\dot{\tilde{F}}^T + \dot{\tilde{F}})\right) : \left(\dot{F}^T + \dot{F} - \dot{\tilde{F}} - \dot{\tilde{F}}^T)\right) \,\geq\, \gamma\,|\dot{F} - \dot{\tilde{F}}|^2.$$

Choosing $\dot{F} = 0$ now gives

$$\frac{1}{2}\left(G(1,0) - G(1,\dot{\tilde{F}}^T + \dot{\tilde{F}})\right) : \left(-\dot{\tilde{F}} - \dot{\tilde{F}}^T)\right) \,\geq\, \gamma\,| - \dot{\tilde{F}}|^2.$$

Finally, choosing $\dot{\tilde{F}}$ to be a nonzero and skew matrix makes the left-hand side vanish whereas the right hand side remains positive. This gives a contradiction proving the claim. □

Contradicting a claim of Antman [2] we have that

Lemma 4.5. *The following statement is incompatible with frame-indifference.*

$$\big(S(F,\dot{F}+\dot{H}) \,-\, S(F,\dot{F})\big) \,:\, \dot{H} \,>\, 0\,, \quad \forall\,\dot{H}\neq 0, \quad \forall\,\dot{F}. \tag{4.7}$$

Proof. If, for contradiction, the claim was true, then there would exist a frame-indifferent stress tensor S satisfying (4.3) and (4.7) would be equivalent to

$$\left\{F[G(F^TF,(\dot{F}+\dot{H})^T F + F^T(\dot{F}+\dot{H})) - G(F^TF,\dot{F}^T F + F^T\dot{F})]\right\} : \dot{H} \,>\, 0,$$
$$\forall\;\dot{H}\neq 0 \;\text{ and }\; \forall\;\dot{F}.$$

Taking $F = I$ in this inequality gives

$$\left\{G(1,(\dot{F}+\dot{H})^T + (\dot{F}+\dot{H})) - G(1,\dot{F}^T + \dot{F})\right\} : \dot{H} \,>\, 0.$$

Letting $\dot{F}$ be skew reduces it further to

$$\left\{G(1,\dot{H}^T + \dot{H}) - G(1,0)\right\} : \dot{H} \,>\, 0.$$

We can choose $\dot{H}$ to be a nonzero and skew matrix which will make the left-hand side vanish, giving a contradiction. □

4.3 Dissipation Potentials

In this section, we aim to show that it is possible to have convex (in $\dot{F}$) potential functions $\Psi(F,\dot{F})$ leading to frame-indifferent viscoelastic stress $S(F,\dot{F})$ as a result of the relation (3.3).

Lemma 4.6. *Let*

$$\Psi(F,H) \,=\, \frac{1}{4}\,|F^T H + H^T F\,|^2. \tag{4.8}$$

Then, $\Psi(F,H)$ is convex in H, and

$$S(F,\dot{F}) \,:=\, \frac{\partial\Psi}{\partial\dot{F}}\,(F,\dot{F}) \,=\, F(\dot{F}^T F \,+\, F^T\dot{F}) \,=\, F\,\dot{C},$$

which is frame-indifferent.

Proof. The convexity of $\Psi(F,H)$ follows immediately from the fact that it is a nonnegative quadratic form in H. We now show that $S(F,\dot F) = F\,\dot C$. We have

$$\begin{aligned}
\frac{\partial \Psi}{\partial H}(F,H) &= \frac{1}{4}\,\frac{\partial}{\partial H_{k\beta}}\,|F^T H + H^T F\,|^2 = \\
&= \frac{1}{2}\,(F_{\alpha i}\,H_{\alpha j} + H_{\alpha i}\,F_{\alpha j})\,(F_{\alpha i}\,\delta_{k\alpha}\,\delta_{j\beta} + \delta_{k\alpha}\,\delta_{i\beta}\,F_{\alpha j}) \\
&= \frac{1}{2}\,\left(F^T H + H^T F\right)_{ij}\left(F_{ki}\,\delta_{j\beta} + \delta_{i\beta}\,F_{kj}\right) \\
&= \frac{1}{2}\,\left((F^T H + H^T F)_{i\beta}\,F_{ki} + (F^T H + H^T F)_{\beta j}\,F_{kj}\right) \\
&= \frac{1}{2}\,\left((F^T H + H^T F)^T_{\beta i}\,F^T_{ik} + (F^T H + H^T F)_{\beta j} F^T_{jk}\right) \\
&= \frac{1}{2}\,\left(((F^T H + H^T F)^T\,F^T)_{\beta k} \,+\, ((F^T H + H^T F)\,F^T)_{\beta k}\right) \\
&= \frac{1}{2}\,\left[\left((F^T H + H^T F)^T + (F^T H + H^T F)\right)F^T\right]^T_{k\beta} \\
&= \frac{1}{2}\,\left(\left(H^T F + F^T H + F^T H + H^T F\right)F^T\right)^T \\
&= \left((F^T H + H^T F)\,F^T\right)^T \,=\, F\left(F^T H + H^T F\right)^T \\
&= F\,(H^T F + F^T H).
\end{aligned}$$

As $\dot C \,=\, \dot F^T F + F^T \dot F$, putting $H = \dot F$ above proves the claim. Frame-indifference of $S(F,\dot F)$ follows immediately from (4.3). ☐

Lemma 4.7. *Let*

$$\Psi(F,H) \,=\, \frac{1}{4}\,|H\,F^{-1} \,+\, F^{-T}\,H^T\,|^2. \tag{4.9}$$

Then, $\Psi(F,H)$ is convex in H, and

$$S(F,\dot F) := \frac{\partial \Psi}{\partial \dot F}(F,\dot F) \,=\, (\dot F\,F^{-1} + (\dot F\,F^{-1})^T)\,F^{-T} \,=\, F\,C^{-1}\,\dot C\,C^{-1},$$

which is frame-indifferent.

Proof. Following the proof in Lemma 4.6, one can easily show that $\Psi(F,H)$ is convex with respect to H. Therefore we skip it here and prove only that $S(F,\dot F) = \,F\,C^{-1}\,\dot C\,C^{-1}$, whose frame-indifference follows from (4.3) again. In order to calculate S we adopt a different approach from that of the previous result[1]. We have

$$\Psi(F, A+\varepsilon B) = \frac{1}{4}\big|(A+\varepsilon B)\,F^{-1} + F^{-T}\,(A+\varepsilon B)^T\big|^2.$$

Therefore, we obtain

$$\begin{aligned}
&\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\,\Psi(F, A+\varepsilon B) = \\
&\quad= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\,\frac{1}{4}\big|(AF^{-1} + F^{-T}A^T) + \varepsilon(BF^{-1} + F^{-T}B^T)\big|^2 \\
&\quad= \frac{1}{2}\big(AF^{-1} + F^{-T}A^T\big) : \big(BF^{-1} + F^{-T}B^T\big) \\
&\quad= \frac{1}{2}\big(AF^{-1} + F^{-T}A^T\big) : BF^{-1} + \frac{1}{2}\big(AF^{-1} + F^{-T}A^T\big) : F^{-T}B^T \\
&\quad= \frac{1}{2}\big(AF^{-1}F^{-T} + F^{-T}A^TF^{-T}\big) : B + \frac{1}{2}\big(F^{-1}AF^{-1} + F^{-1}F^{-T}A^T\big) : B^T \\
&\quad= \frac{1}{2}\big(AF^{-1}F^{-T} + F^{-T}A^TF^{-T}\big) : B + \frac{1}{2}\big(F^{-T}A^TF^{-T} + AF^{-1}F^{-T}\big) : B \\
&\quad= \big(AF^{-1}F^{-T} + F^{-T}A^TF^{-T}\big) : B.
\end{aligned}$$

[1]We thank Gero Friesecke for this proof, which is much simpler than the original one.

This gives

$$\frac{\partial \Psi}{\partial A}(F, A) = F^{-T}\, A^T\, F^{-T} + A\, F^{-1}\, F^{-T}$$

and hence

$$\begin{aligned}
\frac{\partial \Psi}{\partial \dot F}(F, \dot F) &= F^{-T}\, \dot F^T\, F^{-T} + \dot F\, F^{-1}\, F^{-T} \\
&= (FF^{-1})\, F^{-T}\, \dot F^T\, (FF^{-1})\, F^{-T} + (FF^{-1})\, (F^{-T}F^T)\, \dot F\, F^{-1}\, F^{-T} \\
&= F\, (F^{-1}F^{-T})\, (\dot F^T F)\, (F^{-1}F^{-T}) + F\, (F^{-1}F^{-T})\, (F^T \dot F)\, (F^{-1}F^{-T}) \\
&= F\, C^{-1}\, \dot C\, C^{-1}
\end{aligned}$$

as required. □

Remark 4.2. *Strict convexity of* $\Psi(F, \cdot)$ *is incompatible with frame-indifference (see [12]).*

As a result, we conclude that there exist dissipation potentials Ψ which not only satisfy convexity, but also give frame-indifferent S in (3.3).

5 Existence of solutions

In this section we prove existence of weak solutions to (3.4) complemented with initial and boundary conditions using a time-discretization method as first introduced in [24] and [20], and more recently followed in [21].

5.1 Assumptions and auxiliary results

We make the following assumptions on the free energy density, the strain-gradient function and the dissipation potential. It is important to note that, as indicated in [14], as a result of the inclusion of the higher order gradient term, it is not necessary to have any kind of convexity condition on the stored-energy density function W.

(i) $W : \mathrm{GL}^+(3) \to \mathbb{R}^+$ is C^2, frame-indifferent, and satisfies

$$W(F) \geq \alpha |F|^s + \frac{\alpha}{(\det F)^q},\ \forall F \in \mathrm{GL}^+(3), \tag{5.1}$$

where $\alpha > 0$ is a constant, $s > 2$, $q \geq 3p/(p-3)$ (with $p > 3$) and $\mathrm{GL}^+(3)$ is as before.

(ii) $\mathcal{H} : \mathbb{R}^{3\times3\times3} \to \mathbb{R}^+$ is convex, frame-indifferent, and C^1 with

$$\alpha |G|^p \leq \mathcal{H}(G) \leq K(1 + |G|^p),\ \forall G \in \mathbb{R}^{3\times3\times3}, \tag{5.2}$$

where K is a constant, possibly very large, and $p > 3$.

(iii) $\Psi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^+$ is continuous and given by (4.8) so that $S(F, \dot F) = F\dot C$.

(iv) $f(t) \in L^2([0,T]; \mathbb{R}^3)$ and $y_0 \in W^{1,2}(\Omega)$.

We will make use of the following auxiliary results proven in [21] (see also [18]). We include them here for the convenience of the reader.

Proposition 5.1. *Assume that the components* W *and* $\mathcal{H}$ *of the energy function* $E : W^{2,p}(\Omega; \mathbb{R}^3) \to \mathbb{R}$ *satisfy assumptions* (5.1) *and* (5.2)*. Then, for each constant* $C > 0$ *there is a constant* $\hat C > 0$ *such that all* y *with* $E(y(t)) \leq C$ *satisfy*

$$\|y\|_{W^{2,p}} \leq \hat C, \quad \|y\|_{C^{1,1-3/p}} \leq \hat C, \det \nabla y(x) \geq 1/\hat C, \|(\nabla y)^{-1}\|_{C^{1,1-3/p}} \leq \hat C.$$

Proposition 5.2. *For fixed $\lambda \in (0,1)$ and positive constant $K > 1$ define the set*

$$S_K := \{F \in C^\lambda(\Omega;\mathbb{R}^{3\times 3}); \|F\|_{C^\lambda} \le K, \min_{x\in\Omega} \det F(x) \ge 1/K\}.$$

Then, for all $K > 1$, there exists a constant $c_K > 0$ such that for all $F \in S_K$ we have

$$\forall v \in W^{1,2}(\Omega;\mathbb{R}^3) \, : \, \int_\Omega |F^{\mathrm{T}}\nabla v + (\nabla v)^{\mathrm{T}} F|^2 dx \ge c_K \|v\|_{W^{1,2}}.$$

Combining Propositions (5.1) and (5.2) one can obtain the following corollary.

Corollary 5.1. *Given $C > 0$, there exists a $C_K > 0$ such that for all y with $E(y(t)) \le C$ we have*

$$\forall v \in W^{1,2}(\Omega;\mathbb{R}^3) \, : \, \int_\Omega |(\nabla y)^{\mathrm{T}}\nabla v + (\nabla v)^{\mathrm{T}}\nabla y|^2 dx \ge C_K \|v\|_{W^{1,2}}.$$

We give the definition of a weak solution as follows:

Definition 5.1. *A function $y : [0,T]\times\Omega \to \mathbb{R}^3$ is called a weak solution of* (3.4) *with initial conditions* (3.5) *and boundary conditions* (3.6) *if $y \in L^2([0,T]; W^{2,p}(\Omega,\mathbb{R}^3))$ satisfying* $\det \nabla y > 0$ *as well as the identity*

$$\int_0^T \int_\Omega \Big(S(F,\dot{F}) + DW(F)\Big) \colon \nabla z + \mathcal{H}(\nabla^2 y) \vdots \nabla^2 z\Big) dx dt = \int_0^T \int_\Omega f \cdot z dx dt, \tag{5.3}$$

for all smooth $z : [0,T]\times\Omega \to \mathbb{R}^3$.

As a result of assumption (iii), testing equation (3.4) by $\dot{y}$ and integrating over Ω, we obtain the energy equality as

$$\int_\Omega DW(\nabla y) : \nabla\dot{y} dx + \int_\Omega |\dot{C}|^2 dx + \frac{d}{dt}\int_\Omega \mathcal{H}(\nabla^2 y) dx = \int_\Omega f(t)\cdot\dot{y} dx.$$

Integrating this equality with respect to time and using (3.2) we obtain

$$E(y(T)) + \int_0^T \int_\Omega |\dot{C}|^2 dx dt = E(y(0)) + \int_0^T \int_\Omega f(t)\cdot\dot{y} dx dt. \tag{5.4}$$

This relation is vital for the apriori estimates we obtain for the solutions in the following sections.

Throughout the paper, we use the standard notation for L^p as Lebesgue spaces, and similarly $W^{k,p}$ for Sobolev spaces of functions whose k-th order weak derivatives are in L^p. Spaces $W^{k,p}([0,T];X)$ denote Banach spaces of mappings from $L^p([0,T];X)$ whose k-th order weak derivatives with respect to time variable are also in $L^p([0,T];X)$. Also, $H^k = W^{k,2}$.

5.2 Time-discretization

In this section we introduce the variational approach we adopt for the existence of weak solutions of the equation

$$\mathrm{Div}\big(DW(\nabla y) + S(\nabla y,\nabla\dot{y}) - \mathrm{Div}\,\mathcal{H}'(\nabla^2 y)\big) - f = 0, \tag{5.5}$$

and define approximate solutions to (5.5) by means of the following implicit time-discretization scheme. For a fixed time stepsize $\tau > 0$ and initial data $y_0 \in W^{1,2}(\Omega)$ we inductively define

$$\begin{aligned} y_\tau^0 &:= y_0 \\ y_\tau^k &:= \text{ a minimizer of the functional } J_\tau^k(y) \;\; (k \in \mathbb{N}), \end{aligned}$$

where

$$J_\tau^k(y) = \int_\Omega \left(W(\nabla y) + \tau\,\Psi\left(\nabla y_\tau^{k-1}, \frac{\nabla y - \nabla y_\tau^{k-1}}{\tau}\right) - \mathrm{Div}\,\mathcal{H}(\nabla^2 y)\right) dx - \int_\Omega f_\tau^k \cdot y \, dx, \tag{5.6}$$

where $f_\tau^k := \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau} f(t)dt$. Note that for $k \geq 1$, the minimizers y_τ^k satisfy the Euler-Lagrange equations

$$\int_\Omega \left(DW(\nabla y_\tau^k) \colon \nabla z + \Psi_q \left(\nabla y_\tau^{k-1}, \frac{\nabla y_\tau^k - \nabla y_\tau^{k-1}}{\tau} \right) : \nabla z + D\mathcal{H}(\nabla^2 y_\tau^k) \vdots \nabla^2 z - f_\tau^k \cdot z \right) dx = 0, \quad (5.7)$$

for all $z \in C_0^\infty(\Omega)$, which represent a weak, time-discretized version of (5.5) for $\Psi_q(p,q) = S(p,q)$. For the last term in the integrand, we used Gateaux differentiability of the strain-gradient density (see [21]).

5.3 Existence result

We can now state the result on the existence of minimizers for the discrete scheme.

Theorem 5.1. *Let the assumptions (i), (ii) and (iii) be satisfied. Let $y_\tau^0 = y_0$ be as in (iv), $N \in \mathbb{N}$ and $\tau = T/N$. Then for $k = 1, 2, \ldots, N$, y_τ^k can be found by solving the minimization problem* (5.6) *whose minimizers satisfy the time-discretized version of problem* (5.5) *in the weak sense as in* (5.7).

Proof. By assumptions (5.1) and (5.2), the minimization problem (5.6) is coercive on $W^{2,p}(\Omega, \mathbb{R}^3)$. By (iii) we know that Ψ is given as in (4.8). By Lemma 4.6 we obtain that

$$\Psi \left(\nabla y_\tau^{k-1}, \frac{\nabla y - \nabla y_\tau^{k-1}}{\tau} \right)$$

is convex in y. This implies lower semicontinuity in $W^{1,2}(\Omega, \mathbb{R}^3)$. As a result we obtain a minimizer y_τ^k with $E(y_\tau^k) < \infty$. Hence, by Proposition 5.1, we know that the minimizer satisfies $\det \nabla y(x) \geq \delta > 0$. So, y_τ^k satisfies the Euler-Lagrange equation (5.7), which completes the proof. □

We now define piecewise constant interpolants using the discrete approximations y_τ^k for $k = 0, \ldots, N$, which is standard in the convergence of discrete schemes. They are given as

$$y^\tau(x,t) := y_\tau^k(x) \quad \text{for} \quad t \in ((k-1)\tau, k\tau). \quad (5.8)$$

Now, we prove the apriori estimates for the solutions.

Proposition 5.3. *Let assumptions (i) - (iv) satisfied. Also, assume that $\Psi(\cdot, 0) = 0$. Then, there exists a constant $C > 0$ such that the piecewise constant interpolants $y_\tau \in W^{2,p}$ given in* (5.8) *satisfy the following apriori estimates:*

$$\begin{aligned} &\|y_\tau\|_{L^\infty([0,T];W^{2,p}(\Omega;\mathbb{R}^3)) \cap H^1([0,T];H^1(\Omega;\mathbb{R}^3))} \leq C \\ &\det \nabla y_\tau \geq 1/C. \end{aligned} \quad (5.9)$$

Proof. Substituting $y = y_\tau^{k-1}$ in (5.6) and using both the assumption that $\Psi(\cdot, 0) = 0$ and the fact that $y = y_\tau^k$ is a global minimum we obtain

$$\begin{aligned} &\int_\Omega \big(W(\nabla y_\tau^k) - W(\nabla y_\tau^{k-1}) \big) dx + \int_\Omega \tau \, \Psi \left(\nabla y_\tau^{k-1}, \frac{\nabla y_\tau^k - \nabla y_\tau^{k-1}}{\tau} \right) dx \\ - &\int_\Omega \big(\mathrm{Div}\, \mathcal{H}(\nabla^2 y_\tau^k) - \mathrm{Div}\, \mathcal{H}(\nabla^2 y_\tau^{k-1}) \big) dx \leq \tau \int_\Omega f_\tau^k \cdot \left(\frac{y_\tau^k - y_\tau^{k-1}}{\tau} \right) dx. \end{aligned}$$

Using $\Psi \geq 0$ and

$$f_\tau^k \cdot \left(\frac{y_\tau^k - y_\tau^{k-1}}{\tau} \right) \leq \|f_\tau^k\|_{H^{-1}} \left\| \frac{y_\tau^k - y_\tau^{k-1}}{\tau} \right\|_{H^1} \leq \hat{C} \|f_\tau^k\|_{H^{-1}} \left\| \frac{\nabla y_\tau^k - \nabla y_\tau^{k-1}}{\tau} \right\|_{L^2},$$

we obtain the recursive estimate

$$E(y_\tau^k) - E(y_\tau^{k-1}) \leq \tau C_1 \|f_\tau^k\|_{H^{-1}}^2 + \tau C_2 \left\| \frac{\nabla y_\tau^k - \nabla y_\tau^{k-1}}{\tau} \right\|_{L^2}^2,$$

where $\hat{C}, C_1$ and C_2 are generic constants. Using a discrete Grönwall-type estimate for the total energy together with its definition (3.2), we obtain the desired estimate in $L^\infty([0,T];W^{2,p}(\Omega;\mathbb{R}^3))$. Moreover, by assumption (iii) we know that Ψ is given as in (4.8). By invoking Corollary 5.1 we obtain the desired estimate in $H^1([0,T];H^1(\Omega;\mathbb{R}^3))$. □

By this proposition, one can conclude that the constant interpolants defined in (5.8) satisfy a suitable discretized version of (5.5). Now we can prove convergence as $\tau \to 0$.

Theorem 5.2. *Let the assumptions (i)-(iv) hold. Then, as $\tau \to 0$, there exists a limit function y such that*

$$y_\tau \to y \quad \text{weakly}^* \quad \text{in } L^\infty([0,T];W^{2,p}(\Omega;\mathbb{R}^3)) \cap H^1([0,T];H^1(\Omega;\mathbb{R}^3)). \tag{5.10}$$

Also,

$$\nabla y_\tau \to \nabla y \ \text{ strongly } \ \text{in } L^\infty([0,T];\mathbb{R}^{3\times 3}), \tag{5.11}$$

and any such y is a weak solution to (5.5) *complemented with the initial and boundary values as in* (3.5) *and* (3.6). *Moreover, it satisfies the energy balance* (5.4).

Proof. By the apriori estimates obtained in Proposition 5.3, we can extract a subsequence (which we do not relabel for simplicity) such that (5.10) holds. To obtain (5.11) we argue as follows (see also [21]). By the continuous embedding $W^{1,p}(\Omega) \subset C^\gamma(\Omega)$ with $\gamma = 1 - 3/p$, we have $\|\nabla y_\tau\|_{C^\gamma} \le C$. Moreover, (5.9) gives the Hölder estimate

$$\|\nabla y_\tau(t_1) - \nabla y_\tau(t_2)\|_{L^2(\Omega;\mathbb{R}^3)} \le C_1 |t_1 - t_2|^{1/2},$$

for all $t_1, t_2 \in [0,T]$. Using the interpolation $\|\cdot\|_{C^\beta} \le C\|\cdot\|_{C^\gamma}^{1-\alpha}\|\cdot\|_{L^2}^{\alpha}$ and the apriori estimated in Proposition 5.3, we can conclude that ∇y_τ is uniformly bounded in C^α leading to the desired convergence (5.11) by an application of Arzelá-Ascoli theorem. For the convergence in the energy balance, we use the from of S given in assumption (iii), and apply Minty's trick to the strain-gradient part as a result of (5.11). □

We can state the following corollary.

Corollary 5.2. *Considering time-discretization for $\dot{C} = (\nabla \dot{y})^{\mathrm{T}} \nabla y + \nabla \dot{y} (\nabla y)^{\mathrm{T}}$, where $C = F^{\mathrm{T}} F$ is the right Cauchy-Green stretch tensor, and defining the appropriate constant interpolant C_τ, we obtain*

$$C_\tau \to C \ \text{ strongly in } H^1([0,T];L^2(\Omega)),$$

where $C = (\nabla y)^{\mathrm{T}} \nabla y$.

6 A counterexample

We prove the following result and the counterexample showing that the assumption of strong convergence is necessary for compactness (see also [24]).

Theorem 6.1. *Consider the sequence $\{y^{(j)}\}_{j=1}^\infty$ and assume that the following convergences hold:*

(i) $y^{(j)} \overset{*}{\rightharpoonup} y$ *in* $W^{1,\infty}(\Omega;\mathbb{R}^{3\times 3})$,
(ii) $\det \nabla y, \det \nabla y^{(j)} > 0$ *for* a.e. $x \in \Omega$ *and for all* j,
(iii) $U^{(j)} = \sqrt{\nabla y^{(j)\mathrm{T}} \nabla y^{(j)}} \to U$ *for* a.e. $x \in \Omega$.

Then, $\nabla y^{\mathrm{T}} \nabla y = U^2$ holds.

Proof. Assumption (i) immediately gives, by Theorem 3.4 in [4], that

$$\det \nabla y^{(j)} \overset{*}{\rightharpoonup} \det \nabla y \ \text{ in } L^\infty(\Omega) \tag{6.1a}$$

$$\operatorname{cof} \nabla y^{(j)} \overset{*}{\rightharpoonup} \operatorname{cof} \nabla y \ \text{ in } L^\infty(\Omega). \tag{6.1b}$$

By assumption (ii) and polar decomposition theorem, we have $\nabla y^{(j)} = R^{(j)}\, U^{(j)}$, where $R^{(j)} \in \mathrm{SO}(3)$ and $U^{(j)}$ is the right stretch tensor. This gives

$$\det \nabla y^{(j)} \;=\; \det R^{(j)}\, \det U^{(j)} \;=\; \det U^{(j)}.$$

Therefore, by assumption (iii) we obtain

$$\det \nabla y^{(j)} \;\to\; \det U \;\text{ for a.e. } x \in \Omega. \tag{6.2}$$

Convergences (6.1a) and (6.2) immediately give

$$\det \nabla y \;=\; \det U. \tag{6.3}$$

Similarly, we have

$$\operatorname{cof} \nabla y^{(j)} \;=\; \operatorname{cof} R^{(j)} \operatorname{cof} U^{(j)} \;=\; R^{(j)} \operatorname{cof} U^{(j)}$$

and hence, by (6.1b), we obtain

$$R^{(j)} \operatorname{cof} U^{(j)} \;\overset{*}{\rightharpoonup}\; \operatorname{cof} \nabla y \;\text{ in }\; L^{\infty}(\Omega).$$

This shows that $\operatorname{cof} U^{(j)}$ is uniformly bounded, which together with assumption (iii) gives

$$\operatorname{cof} U^{(j)} \;\to\; \operatorname{cof} U \;\text{ in }\; L^{q}(\Omega),\, 1 \leq\, q \,<\, \infty. \tag{6.4}$$

Without loss of generality, we can say that $R^{(j)} \overset{*}{\rightharpoonup} R$ in $L^{\infty}(\Omega)$, which implies

$$R^{(j)} \;\rightharpoonup\; R \;\text{ in }\; L^{p}(\Omega),\, 1 \,\leq\, p \,<\, \infty. \tag{6.5}$$

Choosing $q = p'$ in (6.4) thus gives

$$R^{(j)} \operatorname{cof} U^{(j)} \;\rightharpoonup\; R \operatorname{cof} U \;\text{ in }\; L^{1}(\Omega). \tag{6.6}$$

Convergences (6.1b) and (6.6) imply that

$$\operatorname{cof} Dy \;=\; R \operatorname{cof} U. \tag{6.7}$$

By (6.3), (6.7) and the fact that $\operatorname{cof} F = (\det F)\, F^{-\mathrm{T}}$, for any $F \in \mathrm{GL}^{+}(3)$, we obtain

$$(\det U)\, \nabla y^{-T} \;=\; (\det Dy)\, Dy^{-\mathrm{T}} \;=\; \operatorname{cof} \nabla y \;=\; R \operatorname{cof} U \;=\; R\, (\det U)\, U^{-\mathrm{T}},$$

giving

$$\nabla y^{-\mathrm{T}} \;=\; R\, U^{-\mathrm{T}}. \tag{6.8}$$

As we do not know whether R is a rotation and U is symmetric or not, (6.8) is still not enough. However, by boundedness of $\nabla y^{(j)}$ and $R^{(j)}$, and assumption (iii), we deduce that

$$U^{(j)} \;\to\; U \;\text{ in }\; L^{r}(\Omega),\, 1 \leq r < \infty.$$

Choosing $p = r'$ in (6.5) thus gives

$$R^{(j)}\, U^{(j)} \;\rightharpoonup\; R\, U \;\text{ in }\; L^{1}(\Omega). \tag{6.9}$$

Assumption (i) and (6.9) imply that

$$\nabla y \;=\; R\, U. \tag{6.10}$$

Therefore, by (6.8) and (6.10) we have

$$\nabla y^{\mathrm{T}}\, \nabla y \;=\; U^{\mathrm{T}}\, R^{-1}\, R\, U \;=\; U^{\mathrm{T}}\, U. \tag{6.11}$$

Equations (6.10) and (6.11) prove that $R \in \mathrm{SO}(3)$. Hence by assumption (ii) and polar decomposition theorem we can conclude that U is symmetric which, by (6.11), immediately gives the result. □

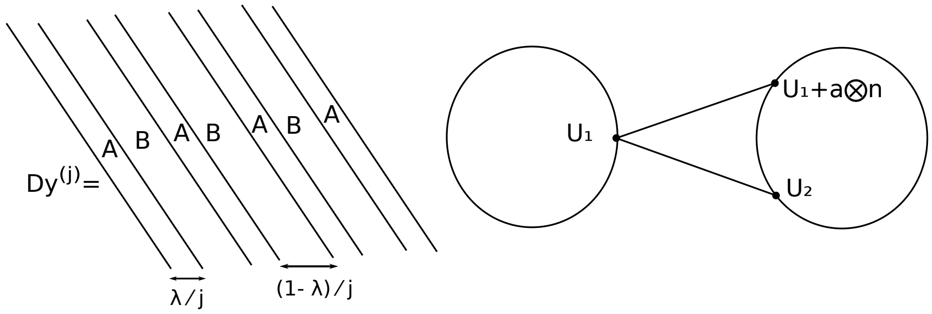

Figure 1: Laminate in a two-well problem

We now state the following crucial remark which shows that without the inclusion of strain-gradient term in the model, we would not be able to obtain Theorem 5.2.

Proposition 6.1. *If assumption* (iii) *is not satisfied, then the conclusion of Theorem 6.1 does not hold.*

Proof. We give the following counterexample in order to prove the claim. Consider the simple laminate in a two-well problem as shown in Figure 1 formed from gradients A, B satisfying $A - B = a \otimes n$ with separating interfaces with normal n, the A layers having thickness λ/j and B layers $(1-\lambda)/j$ for $0 < \lambda < 1$.

Let $A \in \mathrm{SO}(3)\, U_1, B \in \mathrm{SO}(3)\, U_2$ and choose

$$A = U_1 \quad \text{and} \quad B = U_1 + \, a \otimes n.$$

Then, $\nabla y^{(j)}$ satisfies (see e.g. [5], [6])

$$\nabla y^{(j)} \overset{*}{\rightharpoonup} \lambda U_1 + (1-\lambda)(U_1 + a \otimes n) = U_1 + (1-\lambda)\, a \otimes n =: \nabla y.$$

Therefore,

$$U \;=\; \sqrt{\nabla y^{\mathrm{T}} \nabla y} \;=\; \sqrt{(U_1 + (1-\lambda)\, n \otimes a)(U_1 + (1-\lambda)\, a \otimes n)}.$$

On the other hand, we could also choose

$$U_A^{(j)} = U_1 \quad \text{and} \quad U_B^{(j)} = U_2$$

where $U_A^{(j)} = U^{(j)}(x)\big|_{x \in A}$ and similarly for B. In this case we would get

$$U^{(j)} \overset{*}{\rightharpoonup} \lambda U_1 + (1-\lambda) U_2 =: U.$$

However,

$$\lambda U_1 + (1-\lambda) U_2$$

and

$$\sqrt{(U_1 + (1-\lambda)\, n \otimes a)(U_1 + (1-\lambda)\, a \otimes n)}$$

are not necessarily equal, contradicting the conclusion of Theorem 6.1. □

7 Conclusion

In this contribution, a very long-standing open problem of well-posedness of nonlinear viscoelasticity of strain-rate type in high space dimensions while obeying the conditions of frame-indifference is revisited. As a result of the adopted modelling postulates, including neglecting the inertia term as well as adding a higher order regularity term for deformation, existence of weak solutions are obtained as a result of the application of a time-discretization method together with a minimization argument. However, the problem of existence of solutions for the fully dynamical case without assuming additional regularity of the deformation is still open. We hope that the observations made, however small, and the counterexample given will be able to shed some light onto the discovery of new methods to be developed or new approaches to be adopted in order to tackle this problem.

Acknowledgements. YŞ thanks John M. Ball for his guidance and valuable contributions to the results obtained in this work.

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