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Published in:
New Journal of Physics

DOI:
10.1088/1367-2630/abfdf6

Publication date:
2021

New Journal of Physics
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To cite this article: Luiz C B da Silva and Efi Efrati 2021 *New J. Phys.* **23** 063016

New Journal of Physics
The open access journal at the forefront of physics
Deutsche Physikalische Gesellschaft DPG
IOP Institute of Physics
Published in partnership with: Deutsche Physikalische Gesellschaft and the Institute of Physics

CrossMark
OPEN ACCESS

RECEIVED
28 February 2021

REVISED
27 April 2021

ACCEPTED FOR PUBLICATION
4 May 2021

PUBLISHED
8 June 2021

PAPER

Moving frames and compatibility conditions for three-dimensional director fields

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Keywords: liquid crystal, director field, compatibility, geometric frustration, moving frame

Abstract

The geometry and topology of the region in which a director field is embedded impose limitations on the kind of supported orientational order. These limitations manifest as compatibility conditions that relate the quantities describing the director field to the geometry of the embedding space. For example, in two dimensions the splay and bend fields suffice to determine a director uniquely (up to rigid motions) and must comply with one relation linear in the Gaussian curvature of the embedding manifold. In 3D there are additional local fields describing the director, i.e. fields available to a local observer residing within the material, and a number of distinct ways to yield geometric frustration. So far it was unknown how many such local fields are required to uniquely describe a 3D director field, nor what are the compatibility relations they must satisfy. In this work, we address these questions directly. We employ the method of moving frames to show that a director field is fully determined by five local fields. These fields are shown to be related to each other and to the curvature of the embedding space through six differential relations. As an application of our method, we characterize all uniform distortion director fields, i.e., directors for which all the local characterizing fields are constant in space, in manifolds of constant curvature. The classification of such phases has been recently provided for directors in Euclidean space, where the textures correspond to foliations of space by parallel congruent helices. For non-vanishing curvature, we show that the pure twist phase is the only solution in positively curved space, while in the hyperbolic space uniform distortion fields correspond to foliations of space by (non-necessarily parallel) congruent helices. Further analysis of the obtained compatibility fields is expected to allow to also construct new non-uniform director fields.

1. Introduction

Liquid crystals are a state of matter characterized by the presence of an orientational order but no, or only partial, positional order. In many cases, the ordering can be described in terms of a unit vector field $\mathbf{n}$, called the director [1, 2]. Liquid crystals pervade our daily lives, from computer and smart-phone displays to optical switches enabling fast and efficient communication. In recent years, liquid crystals also found applications as controllable and responsive materials [3–6], and similar phases were identified outside of soft matter systems, for example in the nematic order observed in iron based superconductors [7, 8].

The liquid crystalline orientational 'texture' often manifests the shape and interactions between its constituents. Elongated and straight constituents with steric interactions favor the nematic phase in which the director's orientation is uniform in space. In contrast, chiral constituents may favor a twisted director field, while elongated and curved constituents may favor a bent director. However, not all such locally preferred tendencies can be globally realized by a director field in a finite domain. For example, the two-dimensional straight nematic texture with vanishing splay and bend cannot be realized on any open region on the surface of a sphere [9]. Here, the splay and bend of a director field $\mathbf{n}$ are given by $s = \nabla \cdot \mathbf{n}$ and $b = \|(\mathbf{n} \cdot \nabla)\mathbf{n}\|$, respectively, and constitute the basic distortion modes of any two-dimensional director field. Similarly, the phase of constant non-vanishing bend and vanishing splay cannot be realized in the plane [10]. It is thus natural to ask what local tendencies could be realized by a director texture, and conversely how many such local descriptors are required to uniquely determine a texture.

Recently, it was shown that any two-dimensional director field is fully described by its bend and splay fields, and that the values these scalar fields obtain for any realizable texture satisfy $K = -b^2 - s^2 - \mathbf{n}\cdot\nabla s + \mathbf{n}_\perp\cdot\nabla b$ [9], where K is the Gaussian curvature of the surface S in which the field is embedded and $\mathbf{n}_\perp$ is the field in S normal to $\mathbf{n}$. The identification of the class of all admissible textures also allowed addressing the notion of optimal compromise for unrealizable frustrated states. These results are, however, presently limited to two-dimensional systems. For three-dimensional liquid crystals there are additional distortion fields, such as the twist and saddle-splay that do not have corresponding fields in two-dimensional systems. Moreover, the three-dimensional geometry is associated with additional compatibility conditions; while for two-dimensional Riemannian geometry there is only one local geometric charge, in three dimensions there are three independent scalar Riemannian charges. Thus, the three-dimensional case is expected to lead to a larger set of relations involving a greater number of fields. Presently, it is unknown how many fields are required to uniquely determine a director field in three dimensions, nor how many relations these fields must satisfy to correspond to a realizable texture.

Many frustrated assemblies, in which the constituents locally favor an arrangement that cannot be globally realized, exhibit a super-extensive ground state energy for isotropic domains; i.e. the energetic cost of the optimal compromise in these systems increases faster than linearly with their mass. Recently, it was shown that the exact order and structure of the compatibility conditions completely determine this super-extensive behavior and can be used to predict the exponent related to the super-extensive growth of the ground state energy [11]. The purpose of the work presented here is to further advance recent efforts aimed at understanding and quantifying frustration in three-dimensional liquid crystals. We provide a definitive answer to the above questions by writing explicitly the six differential relations that form the compatibility conditions relating the five fields that describe a director field in three dimensions. These six equations relate the fields and their derivatives to each other and to the curvature tensor of the 3D manifold where the director field lives. As an application of our results, we also characterize all uniform distortion fields in the three-sphere, $\mathbb{S}^3$, and hyperbolic space, $\mathbb{H}^3$, showing in particular that in hyperbolic space uniform distortion fields also correspond to a foliation of space by (non-necessarily parallel) helices[1]. Thus, together with the results of reference [12], we complete the characterization of uniform distortion fields for all the three homogeneous and isotropic geometries.

For flat space, satisfaction of the compatibility conditions constitutes a necessary and sufficient condition for the existence of a corresponding director field. We thus conclude that knowledge of the five scalar fields that describe the director; namely the twist, t, splay s, bend b, biaxial splay Δ, and relative orientation between the principal biaxial splay direction and the bend direction ϕ, suffice for defining a texture, unique up to rigid motions, provided that they satisfy the compatibility conditions.

2. Background: geometric frustration in three-dimensional director fields

The present work joins ongoing efforts to better understand the underlying geometry of three-dimensional director fields. Recent insightful interpretations of the basic distortion modes of unit director fields in three dimensions identified these distortion modes with distinct components of the director gradient, $J = \nabla\mathbf{n}$ [13, 14]. The splay corresponds to the trace of J, while the bend is a vector in the space perpendicular to $\mathbf{n}$ and thus contributes two degrees of freedom. The remaining modes contribute to the components of J in the two-dimensional space normal to $\mathbf{n}$, and are traceless. The twist $t = \mathbf{n}\cdot(\nabla\times\mathbf{n})$ corresponds to the anti-symmetric component, while the biaxial splay is identified with the remaining traceless symmetric structure and thus contributes two degrees of freedom as well. This yields a total count of six independent contributions to J [13, 14]. However, the freedom in assigning a base to the space perpendicular to $\mathbf{n}$ eliminates one of these to yield five total intrinsic fields that describe a director. We identify these as the splay, bend, twist, saddle-splay and the relative orientation between the direction of the bend vector and the principal direction of the biaxial splay.

These local descriptors of the liquid crystalline order may be associated with non trivial reference values induced by the structure and relative interactions of their constituents. Considering phases composed of identical constituents, it is natural to assume that these reference values will be uniform in space and manifest the underling symmetry of their constituent. However, as was recently shown [12], the space of phases associated with such constant descriptors, termed 'uniform distortions', is very limited, necessitating

[1] By a helix we mean a curve of constant curvature and constant torsion.

more complex textures. For example, chiral constituents favoring the unrealizable uniform double twist produce the blue phase in which defect lines, separating biaxially twisted columns, are periodically arranged [1]. Similarly, achiral bent core liquid crystals form chiral meso-phases displaying giant optical activity [15, 16] and heliconical ordering [17]. The constituents in this case locally favor a phase of vanishing twist, splay and saddle-splay and a constant non-vanishing bend. Such a phase cannot be realized in Euclidean space and instead the system incorporates a twist in order to accommodate the uniform bend resulting in the observed heliconical phase [10].

Focusing on uniform distortions Virga showed that all such textures correspond to foliations of the three-dimensional Euclidean space by parallel helices [12]. His results relied on vector calculus, where the motion of the frame $\{\mathbf{n}=\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3\}$ is described in terms of the so-called connectors vector fields and the compatibility conditions for the deformation modes associated with a director then follow from the symmetry of the tensors $\mathbf{n}_i\cdot\nabla^2\mathbf{n}_j$ [12]. In particular, it was shown that the pure bend phase favored by bent core liquid crystals is indeed frustrated, and predicted the heliconical phase with uniform twist as a plausible compromise. For small enough domains, however, one might expect other non-uniform distortions to yield the optimal compromise [11].

Similar arguments show that the attempted pure double twist phase resulting in the blue phase is also frustrated in Euclidean space [12]. This attempted phase, however, can be accommodated in a three-dimensional spherical geometry of an appropriate radius [18]. Other examples of uniform distortion fields have been recently provided for all the eight Thurston geometries [19], where it is shown that each pure mode of director deformation can fill space without frustration for at least one type of geometry.

In this work we seek to obtain the full compatibility conditions for three-dimensional director fields. Naturally, one may seek to exploit the same reasoning that was exploited to yield the compatibility conditions in two dimensions [9]. However, the method employed there relies heavily on the existence of a natural orthogonal frame of coordinates such that the parametric curves are tangent to $\mathbf{n}$ and to the perpendicular unit vector $\mathbf{n}_\perp$. This, however, could not be generalized to three dimensions. A general field of an orthonormal triad in 3D cannot be associated with the tangents of parametric curves. Instead, one needs to study the properties of the orthonormal triad field without resorting to coordinates; the mathematical formalism which achieves this is called *the method of moving frames* [20], also known as vielbein formalism in the context of relativity [21]. Given a 3D director field $\mathbf{n}_1=\mathbf{n}$ and its two normals $\mathbf{n}_2$ and $\mathbf{n}_3=\mathbf{n}_1\times\mathbf{n}_2$ one can build the corresponding dual frame of differential forms which together with the so-called connection forms describe the geometry of 3D space using the differential form structure equations. This formalism also allows for an invariant formulation of vector calculus operators, which means that quantities and energy functionals used in the description of 3D liquid crystals can be rewritten as exterior differential systems, i.e., differential equations in terms of differential forms and operations defined on them.

Though more abstract than the vector calculus method [12], the approach based on differential forms allows us to obtain manageable equations and to investigate director fields in both Euclidean and curved Riemannian spaces in an equal foot. This helps in better understanding how the Euclidean space frustrates the existence of certain phases.

When concluding the writing of this manuscript a parallel effort to obtain the compatibility conditions using moving frames by Pollard and Alexander came to our attention [22]. We briefly relate to the similarities and differences between these works in the discussion section.

3. Differential forms and moving frames

Given a coordinate system $(x^1,\ldots,x^m)$ on an open and connected set $U\subseteq\mathbb{R}^m$, the corresponding vector fields tangent to the coordinate curves are denoted by $\{\frac{\partial}{\partial x^i}\}$, while their dual fields (or *covectors*) are denoted by $\mathrm{d}x^i$, i.e., when applied to a vector $v=v^i\frac{\partial}{\partial x^i}$ (sum on repeated indices), we have $\mathrm{d}x^i(v)=v^i$.

The differential of a scalar function f is defined as $\mathrm{d}f=\frac{\partial f}{\partial x^i}\,\mathrm{d}x^i$ and, consequently, $\mathrm{d}x^i$ can be alternatively seem as the differential of the ith coordinate function. From now on, a field of covectors $p\in U\mapsto\eta_p\in(T_pU)^*$ is called a *differential one-form*, while a function is a *zero-form*. Notice that we can write any one-form as $\eta=a_i\mathrm{d}x^i$ for some scalar fields a_i and that there is an isomorphism between vector fields and one-forms: $a_i\,\mathrm{d}x^i\leftrightarrow a_i\frac{\partial}{\partial x^i}$. Given two differential one-forms η and ω, we define the *exterior product* $\eta\wedge\omega$ as the anti-symmetric bilinear map $(\eta\wedge\omega)(u,v)=\eta(u)\omega(v)-\eta(v)\omega(u)$. We shall refer to $\eta\wedge\omega$ as a *differential two-form*. We can define the *exterior derivative* of a one-form $\eta=a_i\mathrm{d}x^i$ as the two-form $\mathrm{d}\eta=\mathrm{d}a_i\wedge\mathrm{d}x^i=\frac{\partial a_i}{\partial x^j}\mathrm{d}x^j\wedge\mathrm{d}x^i$. The vector space of two-forms are generated by $\{\mathrm{d}x^i\wedge\mathrm{d}x^j\}_{1\leqslant i<j\leqslant m}$ and, therefore, it has dimension $m(m-1)/2$. More generally, a *differential k-form* is an anti-symmetric k-linear map and the corresponding vector space is generated by the basis

$\{\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_k}\}_{1\leqslant i_1<\ldots<i_k\leqslant m}$, where $(\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_k})(v_1,\ldots,v_m)=\det(\mathrm{d}x^{i_r}(v_s))_{rs}$. The *exterior derivative* of a k-form $\eta=a_{i_1\ldots i_k}\mathrm{d}x^{i_1}\wedge\ldots\mathrm{d}x^{i_k}$ is the $(k+1)$-form $\mathrm{d}\eta=\mathrm{d}a_{i_1\ldots i_k}\wedge\mathrm{d}x^{i_1}\wedge\ldots\mathrm{d}x^{i_k}$. In addition, d is linear and satisfies the product rule $\mathrm{d}(\eta\wedge\omega)=\mathrm{d}\eta\wedge\omega+(-1)^k\eta\wedge\mathrm{d}\omega$, where η is a k-form and ω is a ℓ-form. A remarkable property of the exterior derivative d is that $\mathrm{d}^2=0$, i.e., the differential of the k-form $\mathrm{d}\eta$ always vanishes. (As an exercise, the reader can easily verify this property for zero- and one-forms.)

Instead of using coordinate fields, we may consider in U any set of orthonormal vector fields $\{\mathbf{n}_1,\ldots,\mathbf{n}_m\}$ along with its set of dual fields $\{\eta^1,\ldots,\eta^m\}$, i.e., $\eta^i(\mathbf{n}_j)=\delta^i_j$, where δ^i_j is the Kronecker delta. Since each $\mathbf{n}_i$ is a smooth map from U to $\mathbb{R}^m$, if we write it in coordinates $\mathbf{n}_i=n_i^j\frac{\partial}{\partial x^j}$, its differential[2] is $\mathrm{d}\mathbf{n}_i=(\mathrm{d}n_i^1,\ldots,\mathrm{d}n_i^m)=\frac{\partial\mathbf{n}_i}{\partial\mathbf{x}}\mathrm{d}\mathbf{x}$, where $\frac{\partial\mathbf{n}_i}{\partial\mathbf{x}}$ is the Jacobian matrix acting by matrix multiplication on $\mathrm{d}\mathbf{x}=(\mathrm{d}x^1,\ldots,\mathrm{d}x^m)$. Alternatively, the differential $\mathrm{d}\mathbf{n}_i$ acting on a tangent vector $v\in T_pU$ can be written as a linear combination

$$(\mathrm{d}\mathbf{n}_i)_p(v)=\eta_i^j(p,v)\mathbf{n}_j(p). \tag{1}$$

In what follows, we shall omit the explicit dependence on p and v and simply write $\mathrm{d}\mathbf{n}_i=\eta_i^j\mathbf{n}_j$. For a fixed point p, the functions η_i^j are linear and, therefore, each $p\mapsto\eta_i^j(p,\cdot)$ defines a one-form. From the orthonormality of $\{\mathbf{n}_i\}$ it follows that $\eta_i^j=-\eta_j^i$.

If $\mathbf{r}:U\to\mathbb{R}^m$ denotes the inclusion map, its differential can be written as $\mathrm{d}\mathbf{r}=\eta^i\mathbf{n}_i$. Geometrically, given a moving frame $\{\mathbf{n}_i\}$, the set of one-forms $\{\eta^i\}$ describes infinitesimal translations of the moving frame while the one-forms $\{\eta_i^j\}$ describes infinitesimal rotations. Now, using that $\mathrm{d}^2\mathbf{r}=0$ and $\mathrm{d}^2\mathbf{n}_i=0$, we have the so-called *structure equations*

$$\begin{cases}\mathrm{d}\eta^i=\eta^k\wedge\eta_k^i\\ \mathrm{d}\eta_j^i=\eta_j^k\wedge\eta_k^i\end{cases},\quad i,j\in\{1,\ldots,m\}. \tag{2}$$

For the Euclidean case, these constitute the integrability conditions for the existence of a moving frame with dual frame $\{\eta^i\}$ and connection forms $\{\eta_i^j\}$ [23]. See [24], lemma 2 with $k=0$, for an elementary proof.

As an example of these ideas, consider in $\mathbb{R}^3$ the moving frame given by the vector fields $\mathbf{n}_1=(\cos\phi\cos\theta,\cos\phi\sin\theta,\sin\phi)$, $\mathbf{n}_2=(-\sin\theta,\cos\theta,0)$ and $\mathbf{n}_3=(-\sin\phi\cos\theta,-\sin\phi\sin\theta,\cos\phi)$, where $\theta=\theta(x,y,z)$ and $\phi=\phi(x,y,z)$ are smooth functions on $U\subseteq\mathbb{R}^3$. Computing their differential gives

$$\begin{aligned}\mathrm{d}\mathbf{n}_1&=\mathrm{d}\phi(-\sin\phi\cos\theta,-\sin\phi\sin\theta,\cos\phi)+\mathrm{d}\theta(-\cos\phi\sin\theta,\cos\phi\cos\theta,0)\\&=\cos\phi\,\mathrm{d}\theta\,\mathbf{n}_2+\mathrm{d}\phi\,\mathbf{n}_3,\\ \mathrm{d}\mathbf{n}_2&=\mathrm{d}\theta(-\cos\theta,-\sin\theta,0)=-\cos\phi\,\mathrm{d}\theta\,\mathbf{n}_1+\sin\phi\,\mathrm{d}\theta\,\mathbf{n}_3,\\ \mathrm{d}\mathbf{n}_3&=-\mathrm{d}\phi(\cos\phi\cos\theta,\cos\phi\sin\theta,\sin\phi)+\mathrm{d}\theta(\sin\phi\sin\theta,-\sin\phi\cos\theta,0)\\&=-\mathrm{d}\phi\,\mathbf{n}_1-\sin\phi\,\mathrm{d}\theta\,\mathbf{n}_2.\end{aligned}$$

Therefore, the one-forms η_i^j associated with $\{\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3\}$ are $\eta_1^2=\cos\phi\,\mathrm{d}\theta$, $\eta_1^3=\mathrm{d}\phi$, and $\eta_2^3=\sin\phi\,\mathrm{d}\theta$. We leave as an exercise checking the validity of the structure equations $\mathrm{d}\eta_j^i=\eta_j^k\wedge\eta_k^i$.

The one-forms η_i^k are also known as *connection forms* since they determine the connection coefficients of the covariant derivative. Indeed, given two vector fields $u=u^i\mathbf{n}_i$ and $v=v^i\mathbf{n}_i$ in U, the covariant derivative of u in the direction of v, $\nabla_v u$, can be written using moving frames as

$$\nabla_v u=(\mathrm{d}u)(v)=\mathrm{d}(u^k\mathbf{n}_k)(v)=\left[\mathrm{d}u^k(v)+u^j\eta_j^k(v)\right]\mathbf{n}_k. \tag{3}$$

Therefore, the connection forms can be alternatively computed from the Levi-Civita connection ∇ by using the relation $\eta_j^k(v)=\langle\nabla_v\mathbf{n}_j,\mathbf{n}_k\rangle$. In addition, given two tangent vectors $u,v\in T_pU$, the inner product between them is $g(u^i\mathbf{n}_i,v^j\mathbf{n}_j)=u^iv^j\delta_{ij}=u^iv^i=\eta^i(u)\eta^i(v)$. The metric g in U is then written as $g=(\eta^1)^2+\cdots+(\eta^m)^2$. It follows that the geometry of $U\subseteq\mathbb{R}^m$ is entirely contained in the sets of one-forms $\{\eta^i\}$ and $\{\eta_i^j\}$.

To accomplish the goal of doing differential geometry using moving frames, we should be able to compute differential operators using differential forms. To do that, we need the Hodge star operator $\star$, which takes k-forms to $(m-k)$-forms. Geometrically, we proceed as follows. Given a k-form $\omega=\omega^1\wedge\cdots\wedge\omega^k$, where $\{\omega^i\}$ is linearly independent, consider the k-dimensional vector subspace V of

[2] The use of the same symbol for both the differential of a map between manifolds and the exterior derivative of a differential form is justified by the possibility of seeing the differential as a vector-valued one-form, see, e.g. subsection 2.8 of reference [20].

$\mathbb{R}^m$ generated by the vectors $\{v_1, \ldots, v_k\}$ associated with $\{\omega^1, \ldots, \omega^k\}$. We then pick a basis $\{v_{k+1}, \ldots, v_m\}$ of the vector space $V^\perp$ orthogonal to V and consider $\omega^{k+1}, \ldots, \omega^m$, the one-forms associated with the vectors of this basis. Then, we define $\star\omega = \pm\lambda\, \omega^{k+1} \wedge \cdots \wedge \omega^m$, where λ is the k-volume of the solid generated by $\{v_i\}_{i=1}^k$ and the sign corresponds to the orientation of $\mathcal{B} = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_m\}$, i.e., plus if $\mathcal{B}$ has the same orientation as the canonical basis of $\mathbb{R}^m$ and minus if otherwise. Finally, we compute $\star$ for a generic linear form by demanding linearity. As an example, in $\mathbb{R}^3$ the Hodge star operator acting on one-forms gives $\star\mathrm{d}x^1 = \mathrm{d}x^2 \wedge \mathrm{d}x^3$, $\star\mathrm{d}x^2 = -\mathrm{d}x^1 \wedge \mathrm{d}x^3$, and $\star\mathrm{d}x^3 = \mathrm{d}x^1 \wedge \mathrm{d}x^2$. Finally, the curl and divergence of $\mathbf{n}$ are associated with differential forms according to

$$\nabla \times \hat{\mathbf{n}} \leftrightarrow \star(\mathrm{d}\eta) \quad \text{and} \quad \nabla \cdot \hat{\mathbf{n}} = \star[\mathrm{d}(\star\eta)], \tag{4}$$

where η is the one-form dual to $\hat{\mathbf{n}}$.

4. Compatibility condition for two-dimensional director fields

Director fields $\mathbf{n}$ in 2D are fully described by their bend $b = \|\mathbf{n} \times \nabla \times \mathbf{n}\|$ and splay $s = \nabla \cdot \mathbf{n}$. However, the splay and bend are not independent functions and they are related to the curvature of the ambient surface by [9]

$$-K = s^2 + b^2 + \mathbf{n} \cdot \nabla s - \mathbf{n}^\perp \cdot \nabla b.$$

In this section we provide an alternative proof for the 2D compatibility equation via moving frames. But, first, we shall illustrate how the moving frame method can be used to describe the geometry of surfaces.

Let $\mathbf{r} : U \to S \subset \mathbb{R}^3$ be a surface and $\mathbf{N}$ its unit normal. If $\{\mathbf{n}_1, \mathbf{n}_2\}$ is a field of orthonormal bases for the tangent planes, we then define a moving frame in 3D as $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3 := \mathbf{N}\}$ along with its dual frame $\{\eta^1, \eta^2, \eta^3\}$. Since we are interested on the surface geometry, we shall restrict our attention to η^i, η_i^j when applied to tangent vectors. Then, in this restricted setting it follows that

$$\forall \quad v = v^1\mathbf{n}_1 + v^2\mathbf{n}_2 \in T_\mathrm{p}S, \eta^3(v) = 0.$$

Therefore, seeing η^3 as a 2D differential form on S implies $\eta^3 = 0$. Thus, the one-forms $\eta_1^2, \eta_1^3, \eta_2^3$ can be written as a linear combination of η^1 and η^2 only, i.e., they can also be seen as differential forms on the surface. This process of seeing η^i and η_j^i as 2D differential forms can be rigorously justified by using $\mathbf{r}$ to pullback the one-forms η^i and η_i^j to U: the pullback of a k-form η is the k-form $\omega = \mathbf{r}^*\eta$ defined by $\omega_\mathrm{p}(v_1, \ldots, v_k) = \eta_{\mathbf{r}(\mathrm{p})}(\mathrm{d}\mathbf{r}(v_1), \ldots, \mathrm{d}\mathbf{r}(v_k))$. Now, since the pullback operation * commutes with d and $\wedge$ [25], the one-forms $\mathbf{r}^*\eta^i$ and $\mathbf{r}^*\eta_j^i$ satisfy the same structure equations as η^i and η_j^i. Thus, with some abuse of notation, we simply write $\eta^i = \mathbf{r}^*\eta^i$ and $\eta_j^i = \mathbf{r}^*\eta_j^i$, which finally justifies seeing η^i and η_i^j as one-forms over $S = \mathbf{r}(U)$[3].

From the fact that $\eta^3 = 0$ on S, it follows that $\mathrm{d}\eta^3 = 0$ on S. Then, the structure equations in (2) imply that $\eta^1 \wedge \eta_1^3 + \eta^2 \wedge \eta_2^3 = 0$. An important result for differential forms is the Cartan lemma [20, 26], which says that if $\omega^1, \ldots, \omega^k$ are linearly independent one-forms and if there exist one-forms $\theta^1, \ldots, \theta^k$ such that $\sum_{i=1}^k \omega^i \wedge \theta^i = 0$, then $\theta^i = a_j^i\omega^j$ with $a_i^j = a_j^i$. Therefore, since the set $\{\eta^1, \eta^2\}$ is linearly independent, from the Cartan lemma we may write

$$\eta_1^3 = a_1^1\eta^1 + a_2^1\eta^2 \quad \text{and} \quad \eta_2^3 = a_1^2\eta^1 + a_2^2\eta^2, \qquad a_i^j = a_j^i. \tag{5}$$

From $\mathrm{d}\mathbf{n}_3 = \eta_3^1\mathbf{n}_1 + \eta_3^2\mathbf{n}_2 = -(\eta_1^3\mathbf{n}_1 + \eta_2^3\mathbf{n}_2)$, it follows that the coefficients a_j^i precisely describe the shape operator of S. Then, the mean (H) and Gaussian (K) curvatures can be written as

$$H = \frac{1}{2}\,\mathrm{tr}(a) = \frac{a_1^1 + a_2^2}{2} \quad \text{and} \quad K = \det(a) = a_1^1 a_2^2 - (a_1^2)^2. \tag{6}$$

It remains to find the interpretation of η_1^2. From $\eta_1^2(v) = \langle \nabla_v \mathbf{n}_1, \mathbf{n}_2 \rangle$, we see that we can write $\eta_1^2 = \eta_1^2(\mathbf{n}_1)\eta^1 + \eta_1^2(\mathbf{n}_1)\eta^2 = \kappa_\mathrm{g}\eta^1 + \kappa_\mathrm{g}^\perp\eta^2$, where κ_g and $\kappa_\mathrm{g}^\perp$ are the geodesic curvatures of the integral curves of $\mathbf{n}_1$ and $\mathbf{n}_2$, respectively. In addition, taking the exterior derivative provides the important relation $\mathrm{d}\eta_1^2 = \eta_1^3 \wedge \eta_1^2 = -K\eta^1 \wedge \eta^2$. This relation will be the key to finding the compatibility equation for director fields in 2D.

[3] As an alternative to using pullbacks, we could consider a foliation of space by surfaces parallel to S spanning a region parametrized as $\mathbf{R}(x^1, x^2, x^3) = \mathbf{r}(x^1, x^2) + x^3\mathbf{N}(x^1, x^2)$. Since we are only interested on tangent directions, any dependence of η_i^j on η^3 does not contribute to the final result. In addition, following this idea, η^3 is nothing but the differential of the x^3-coordinate, which implies that $\mathrm{d}\eta^3 = 0$. As shown in the main text, this is the key property allowing us to use the moving frame method to study the differential geometry of surfaces in space.

We have just seen that for a surface in 3D the intrinsic geometry is encoded in η^1, η^2, and η_1^2, while the extrinsic geometry comes from η_1^3 and η_2^3. (The second fundamental form II can be written as $\mathrm{II} = \eta^i\eta_i^3$.) The equation $\mathrm{d}\eta_1^2 = -K\eta^1 \wedge \eta^2$ in 2D indicates that for moving frames in a Riemannian manifold the second set of structure equations, equation (2), must be modified to account for the curvature of the ambient manifold: for a 2D manifold with Gaussian curvature $K = R_{1212}$, the structure equations associated with the one-forms $\{\eta^1, \eta^2\}$ and η_1^2 are $\mathrm{d}\eta^1 = \eta^2 \wedge \eta_2^1$, $\mathrm{d}\eta^2 = \eta^1 \wedge \eta_1^2$, and $\mathrm{d}\eta_1^2 - \eta_1^k \wedge \eta_k^2 = \mathrm{d}\eta_1^2 = -K\eta^1 \wedge \eta^2$.

In general, for a moving frame $\{\mathbf{n}_i\}_{i=1}^m$ in a Riemannian manifold M^m with curvature tensor $R_{ijk\ell} = R^i_{jk\ell}$, the structure equations are [20, 26]

$$\mathrm{d}\eta^i = \eta^k \wedge \eta_k^i \quad \text{and} \quad \mathrm{d}\eta_j^i - \eta_j^k \wedge \eta_k^i = -\frac{1}{2}R^i_{jk\ell}\eta^k \wedge \eta^\ell = -\sum_{k<\ell} R_{ijk\ell}\eta^k \wedge \eta^\ell, \tag{7}$$

where we used that the operation of raising and lowering indices is trivial since the metric coefficients associated with the moving frame are $\delta_{ij} = \langle \mathbf{n}_i, \mathbf{n}_j \rangle$.

Now, let $\hat{\mathbf{n}}$ be a director field on a 2D Riemannian manifold $(M^2, \langle\cdot,\cdot\rangle)$. We may introduce a moving frame $\{\mathbf{n}_1 := \hat{\boldsymbol{n}}, \mathbf{n}_2 := \hat{\boldsymbol{n}}^\perp\}$ along with its coframe $\{\eta^1, \eta^2\}$. As we have seen, we can write $\eta_1^2 = \kappa_g\eta^1 + \kappa_g^\perp\eta^2$.

On the one hand, the splay $s = \nabla \cdot \hat{\mathbf{n}}$ is computed as

$$s = \star\, \mathrm{d} \star \eta^1 = \star\, \mathrm{d}\eta^2 = \star(\eta^1 \wedge \eta_1^2) = \kappa_g^\perp. \tag{8}$$

On the other hand, the bend $b = \|\hat{\mathbf{n}} \times \nabla \times \hat{\mathbf{n}}\|$ is

$$b = \left\|\star(\eta^1 \wedge \star\, \mathrm{d}\eta^1)\right\| = \left\|\star[\eta^1 \wedge \star(\eta^2 \wedge \eta_2^1)]\right\| = \left\|\star[\eta^1 \wedge \star(\kappa_g\eta^1 \wedge \eta^2)]\right\| = \left\|\kappa_g\eta^2\right\| = \kappa_g. \tag{9}$$

This last equation also shows that, in 2D, we may write $b = -\nabla \cdot \mathbf{n}_2 = \star \mathrm{d} \star \eta^2$. In short, we have the following relation

$$\eta_1^2 = b\eta^1 + s\eta^2. \tag{10}$$

Now we shall apply the findings above in order to write the compatibility equation for 2D director fields as found in [9], but using moving frames.

Theorem 1. (Compatibility condition in 2D). *Let $\hat{\mathbf{n}}$ be a director field with splay s and bend b on a 2D manifold M^2 with Gaussian curvature K. Then,*

$$-K = s^2 + b^2 + (\hat{\mathbf{n}} \cdot \nabla)s - (\hat{\boldsymbol{n}}^\perp \cdot \nabla)b, \tag{11}$$

where $(v \cdot \nabla)$ is the directional derivative in the direction of v and $\langle \hat{\mathbf{n}}, \hat{\boldsymbol{n}}^\perp \rangle = 0$.

Proof. The exterior derivative of η_1^2 is

$$\begin{aligned}\mathrm{d}\eta_1^2 &= \mathrm{d}(b\eta^1 + s\eta^2) = \mathrm{d}b \wedge \eta^1 + b\,\mathrm{d}\eta^1 + \mathrm{d}s \wedge \eta^2 + s\,\mathrm{d}\eta^2 \\ &= [(\hat{\boldsymbol{n}}^\perp \cdot \nabla)b]\eta^2 \wedge \eta^1 + b\eta^2 \wedge \eta_2^1 + [(\hat{\boldsymbol{n}} \cdot \nabla)s]\eta^1 \wedge \eta^2 + s\eta^1 \wedge \eta_1^2 \\ &= \left[(\hat{\boldsymbol{n}} \cdot \nabla)s - (\hat{\boldsymbol{n}}^\perp \cdot \nabla)b + b^2 + s^2\right]\eta^1 \wedge \eta^2.\end{aligned}$$

Now, using that $\mathrm{d}\eta_1^2 = -K\eta^1 \wedge \eta^2$ we deduce the desired equality.

5. Three-dimensional director fields

Inspired by the study of 2D director fields, the strategy in 3D will consist of writing the one-forms η_i^j in terms of the deformation modes of a director field $\mathbf{n}$ and then from the structure equations associated with $\mathrm{d}\eta_i^j$ we will obtain the compatibility equations.

In 2D, there are two deformation modes (bend and splay), while in 3D there are 6 modes, which can be further reduced to 5. Indeed, as discussed in section 2, taking into account rotations that preserve the director $\mathbf{n}$, the gradient $\nabla\mathbf{n}$ decomposes as [13, 14]

$$\nabla_\alpha n_\beta = -n_\alpha b_\beta + \frac{s}{2}(\delta_{\alpha\beta} - n_\alpha n_\beta) + \frac{t}{2}\epsilon_{\alpha\beta\gamma}n_\gamma + \Delta_{\alpha\beta}, \tag{12}$$

where Greek indices indicate Cartesian coordinates and $\mathbf{b} = -\mathbf{n} \cdot \nabla\mathbf{n}$ is the *bend vector*, whose norm b (the *bend*) gives the curvature of the integral lines of $\mathbf{n}$, $s = \nabla \cdot \mathbf{n}$ is the *splay*, $t = \mathbf{n} \cdot \nabla \times \mathbf{n}$ is the *twist*, and Δ_{ij} are the coefficients of the *biaxial-splay* [14].

In 2D, the coefficients of the one-form η_1^2 are related to the geometry of the integral curves of the director and its orthogonal field. Given an integral curve of $\mathbf{n}_i$ in 3D, we can consider $\{\mathbf{n}_i, \mathbf{n}_{i+1}, \mathbf{n}_{i+2}\}$ as a positive orthonormal moving trihedron along it, e.g., for $\mathbf{n}_2$ we have $\{\mathbf{n}_2, \mathbf{n}_3, \mathbf{n}_1\}$. The equations of motion of such a moving trihedron along the $\mathbf{n}_i$-integral curves are

$$\nabla_{\mathbf{n}_i} \begin{pmatrix} \mathbf{n}_i \\ \mathbf{n}_{i+1} \\ \mathbf{n}_{i+2} \end{pmatrix} = \begin{pmatrix} 0 & \kappa_i^1 & \kappa_i^2 \\ -\kappa_i^1 & 0 & \omega_i \\ -\kappa_i^2 & -\omega_i & 0 \end{pmatrix} \begin{pmatrix} \mathbf{n}_i \\ \mathbf{n}_{i+1} \\ \mathbf{n}_{i+2} \end{pmatrix}, \tag{13}$$

where κ_i^1 and κ_i^2 relate to the (geodesic) curvature function κ_i as $\kappa_i(s) = \sqrt{[\kappa_i^1(s)]^2 + [\kappa_i^2(s)]^2}$ and ω_i relates to the torsion τ_i as $\omega_i(s) = \tau_i(s) - \theta'(s)$, where θ is the angle between the (Frenet) principal normal and $\mathbf{n}_{i+1}$ [27]. Thus, using the property $\eta_i^j(v) = \langle \nabla_v \mathbf{n}_i, \mathbf{n}_j \rangle$, the one-forms η_i^j when written in the basis $\{\eta^1, \eta^2, \eta^3\}$ are

$$\begin{cases} \eta_1^2 = \kappa_1^1 \eta^1 - \kappa_2^2 \eta^2 + \omega_3 \eta^3 \\ \eta_1^3 = \kappa_1^2 \eta^1 - \omega_2 \eta^2 - \kappa_3^1 \eta^3 \\ \eta_2^3 = \omega_1 \eta^1 + \kappa_2^1 \eta^2 - \kappa_3^2 \eta^3. \end{cases} \tag{14}$$

The one-forms η_1^2 and η_1^3 provide information about the gradient of the director $\mathbf{n}_1$, $\mathrm{d}\mathbf{n}_1 = \eta_1^2 \mathbf{n}_2 + \eta_1^3 \mathbf{n}_3$. The components dual to the director $\mathbf{n}$ then contains information about the bend vector $\mathbf{b} = -\nabla_{\mathbf{n}} \mathbf{n} = b\mathbf{n} \times \nabla \times \mathbf{n}$, $b = \sqrt{(\kappa_1^1)^2 + (\kappa_1^2)^2}$. The remaining components of $\mathrm{d}\mathbf{n}_1$ can be decomposed into an antisymmetric and a symmetric part, where the symmetric part can be further decomposed into a trace and traceless operator. This decomposition provides the twist t, splay s, and biaxial splay coefficients Δ_{ij}, respectively. Thus, from

$$\begin{pmatrix} -\kappa_2^2 & \omega_3 \\ -\omega_2 & -\kappa_3^1 \end{pmatrix} = \frac{t}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{s}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \Delta_1 & \Delta_2 \\ \Delta_2 & -\Delta_1 \end{pmatrix}, \tag{15}$$

we can write

$$t = -(\omega_2 + \omega_3), \qquad s = -(\kappa_2^2 + \kappa_3^1), \qquad \Delta_1 = \frac{\kappa_3^1 - \kappa_2^2}{2} \quad \text{and} \quad \Delta_2 = \frac{\omega_3 - \omega_2}{2}. \tag{16}$$

By inverting these relations, we can finally rewrite η_1^2 and η_1^3 in equation (14) as

$$\eta_1^2 = -b_\perp \eta^1 + \left(\frac{s}{2} + \Delta_1 \right) \eta^2 + \left(-\frac{t}{2} + \Delta_2 \right) \eta^3, \tag{17}$$

and

$$\eta_1^3 = -b_\times \eta^1 + \left(\frac{t}{2} + \Delta_2 \right) \eta^2 + \left(\frac{s}{2} - \Delta_1 \right) \eta^3, \tag{18}$$

where we write the bend vector as $\mathbf{b} = b_\perp \mathbf{n}_2 + b_\times \mathbf{n}_3$.

The compatibility conditions then come from the structure equations

$$\Omega_1^2 = \mathrm{d}\eta_1^2 - \eta_1^3 \wedge \eta_3^2, \qquad \Omega_1^3 = \mathrm{d}\eta_1^3 - \eta_1^2 \wedge \eta_2^3, \quad \text{and} \quad \Omega_2^3 = \mathrm{d}\eta_2^3 - \eta_2^1 \wedge \eta_1^3,$$

where Ω_i^j are the curvature forms whose coordinates in the basis $\{\eta^i \wedge \eta^j\}$ provide the coefficients of the curvature tensor as defined in equation (7). Together, the three structure equations provide the $3^4 = 81$ coefficients of the curvature tensor $R_{ijk\ell}$. From $\Omega_i^j = -\Omega_j^i$ and the fact that Ω_i^j is a differential form, it follows that $R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k}$ reducing these to only 9 independent entries. However, the first Bianchi identity (which is required to further reduce these to only 6 independent components) cannot be proved directly from the above definition. Proving this identity requires differentiating $\mathrm{d}\eta^i = \eta^k \wedge \eta_k^i$ to obtain $\eta^k \wedge \Omega_k^i = 0$, from which follows that $R_{ijk\ell} = R_{k\ell ij}$ (see [26] or [20], p 376). While these relations hold for connection forms that are obtained from a moving frame, there could be one-forms η_i^j that would fail to satisfy these relations. Such forms could not be the connection forms of a moving frame in *any* Riemannian geometry. Thus, while for any compatible set of moving frames the Riemann curvature tensor contains only six independent entries, requiring the satisfaction of the first Bianchi identity yields three additional non-trivial compatibility conditions resulting in the following nine equations:

$$\begin{cases} R_{1212} &= -\left(\frac{s}{2}+\Delta_1\right)_{,1} - b_{\perp,2} - b_\perp^2 - \frac{s^2}{4} + \frac{t^2}{4} - s\Delta_1 - (\Delta)^2 + 2\omega_1\Delta_2 + \kappa_2^1 b_\times, \\ R_{1213} &= -\left(-\frac{t}{2}+\Delta_2\right)_{,1} - b_{\perp,3} - b_\perp b_\times - s\left(-\frac{t}{2}+\Delta_2\right) - 2\omega_1\Delta_1 - \kappa_3^2 b_\times, \\ R_{1223} &= -\left(-\frac{t}{2}+\Delta_2\right)_{,2} + \left(\frac{s}{2}+\Delta_1\right)_{,3} + tb_\perp - 2\kappa_2^1\Delta_1 + 2\kappa_3^2\Delta_2, \\ R_{1312} &= -\left(\frac{t}{2}+\Delta_2\right)_{,1} - b_{\times,2} - b_\perp b_\times - s\left(\frac{t}{2}+\Delta_2\right) - 2\omega_1\Delta_1 - \kappa_2^1 b_\perp, \\ R_{1313} &= -\left(\frac{s}{2}-\Delta_1\right)_{,1} - b_{\times,3} - b_\times^2 - \frac{s^2}{4} + \frac{t^2}{4} + s\Delta_1 - (\Delta)^2 - 2\omega_1\Delta_2 + \kappa_3^2 b_\perp, \\ R_{1323} &= -\left(\frac{s}{2}-\Delta_1\right)_{,2} + \left(\frac{t}{2}+\Delta_2\right)_{,3} + tb_\times - 2\kappa_2^1\Delta_2 - 2\kappa_3^2\Delta_1, \\ R_{2312} &= -\kappa_{2,1}^1 + \omega_{1,2} - (b_\times + \kappa_2^1)\left(\frac{s}{2}+\Delta_1\right) + (b_\perp + \kappa_3^2)\left(\frac{t}{2}+\Delta_2\right) + b_\perp\omega_1 - \kappa_3^2\omega_1, \\ R_{2313} &= \kappa_{3,1}^2 + \omega_{1,3} + (b_\perp + \kappa_3^2)\left(\frac{s}{2}-\Delta_1\right) - (b_\times + \kappa_2^1)\left(-\frac{t}{2}+\Delta_2\right) + b_\times\omega_1 - \kappa_2^1\omega_1, \\ R_{2323} &= \kappa_{3,2}^2 + \kappa_{2,3}^1 - (\kappa_2^1)^2 - (\kappa_3^2)^2 - t\omega_1 - \frac{s^2}{4} - \frac{t^2}{4} + (\Delta)^2, \end{cases} \tag{19}$$

where $f_{,i} = \mathbf{n}_i \cdot \nabla f$ denotes the derivative of f in the direction of $\mathbf{n}_i$ and we denote $\Delta = \sqrt{(\Delta_1)^2 + (\Delta_2)^2}$.

The gradient of the director field can be written in terms of the deformations modes $b_\perp, b_\times, s, t, \Delta_1$, and Δ_2. However, notice that by choosing $\mathbf{n}_2$ to be either the normalized bend vector or an eigenvector of the biaxial splay implies we have a Gauge freedom allowing us to set either $b_\times = 0$ or $\Delta_2 = 0$. Therefore, this reduces the number of degrees of freedom from 6 to 5. In addition, the equations for the curvature tensor were written in terms of 9 functions, six of which can be written in terms of the deformation modes. Thus, the remaining three, κ_2^1, κ_3^2, and ω_1 must be superfluous. We will prove this last assertion in the next two subsections, where we divide the study into director fields with either $\Delta^2 = (\Delta_1)^2 + (\Delta_2)^2 > 0$ or $\Delta^2 = (\Delta_1)^2 + (\Delta_2)^2 = 0$ on all points. In short, we will have six 6 compatibility equations in 5 functions.

5.1. Director fields with non-vanishing biaxial splay

Let us assume non-vanishing biaxial splay $(\Delta)^2 = (\Delta_1)^2 + (\Delta_2)^2 > 0$. Then, using the equations for $R_{ijk\ell}$ we can compute the sum $\Delta_1 R_{1223} + \Delta_2 R_{1323}$, which allows us to write κ_2^1 as

$$\kappa_2^1 = -\frac{\Delta_1 R_{1223} + \Delta_2 R_{1323}}{2\Delta^2} + \frac{\Delta_1 t_{,2} - \Delta_2 s_{,2} + \Delta_1 s_{,3} + \Delta_2 t_{,3}}{2\Delta^2} - \frac{\Delta_1\Delta_{2,2} - \Delta_2\Delta_{1,2}}{2\Delta^2} + \frac{(\Delta^2)_{,3}}{4\Delta^2} + t\frac{b_\perp\Delta_1 + b_\times\Delta_2}{2\Delta^2}. \tag{20}$$

Using the equations for $R_{ijk\ell}$ we can find $\Delta_2 R_{1223} - \Delta_1 R_{1323}$, which allows us to write κ_3^2 as

$$\kappa_3^2 = -\frac{\Delta_1 R_{1323} - \Delta_2 R_{1223}}{2\Delta^2} - \frac{\Delta_2 t_{,2} + \Delta_1 s_{,2} - \Delta_1 t_{,3} + \Delta_2 s_{,3}}{2\Delta^2} + \frac{\Delta_1\Delta_{2,3} - \Delta_2\Delta_{1,3}}{2\Delta^2} + \frac{(\Delta^2)_{,2}}{4\Delta^2} - t\frac{b_\perp\Delta_2 - b_\times\Delta_1}{2\Delta^2}. \tag{21}$$

Analogously, computing $-\Delta_2 R_{1212} + \Delta_2 R_{1313} + \Delta_1 R_{1213} + \Delta_1 R_{1312}$ allows us to write ω_1 as

$$\omega_1 = \frac{\Delta_2 R_{1212} - \Delta_2 R_{1313} - \Delta_1 R_{1213} - \Delta_1 R_{1312}}{4\Delta^2} + \frac{\Delta_2\Delta_{1,1} - \Delta_1\Delta_{2,1}}{2\Delta^2} - \frac{\Delta_1 b_{\times,2} + \Delta_1 b_{\perp,3} - \Delta_2 b_{\perp,2} + \Delta_2 b_{\times,3}}{4\Delta^2} + \frac{(b_\perp^2 - b_\times^2)\Delta_2 - 2b_\perp b_\times\Delta_1}{4\Delta^2} - \kappa_2^1\frac{b_\perp\Delta_1 + b_\times\Delta_2}{4\Delta^2} + \kappa_3^2\frac{b_\perp\Delta_2 - b_\times\Delta_1}{4\Delta^2}. \tag{22}$$

Now, substituting the expressions for κ_2^1 and κ_3^2 in the equation above, we finally have

$$\omega_1 = \frac{\Delta_2 R_{1212} - \Delta_2 R_{1313} - \Delta_1 R_{1213} - \Delta_1 R_{1312}}{4\Delta^2} + \frac{b_\perp R_{1223} + b_\times R_{1323}}{8\Delta^2} + \frac{\Delta_2\Delta_{1,1} - \Delta_1\Delta_{2,1}}{2\Delta^2} - \frac{\Delta_1 b_{\times,2} + \Delta_1 b_{\perp,3} - \Delta_2 b_{\perp,2} + \Delta_2 b_{\times,3}}{4\Delta^2} - \frac{tb^2}{8\Delta^2} - \frac{b_\times\Delta_{1,2} - b_\perp\Delta_{2,2} + b_\perp\Delta_{1,3} + b_\times\Delta_{2,3}}{8\Delta^2} + \frac{(b_\perp^2 - b_\times^2)\Delta_2 - 2b_\perp b_\times\Delta_1}{4\Delta^2}. \tag{23}$$

There are three other linearly independent combinations we can construct with the equations for R_{12ij} and R_{13ij}. Indeed, using the equations for $R_{ijk\ell}$ we can compute $R_{1212}+R_{1313}$, $R_{1312}-R_{1213}=0$, and $\Delta_1R_{1212}+\Delta_2R_{1213}+\Delta_2R_{1312}-\Delta_1R_{1313}$, which give

$$R_{1212}+R_{1313}=-s_{,1}-b_{\perp,2}-b_{\times,3}-b^2-\frac{s^2}{2}+\frac{t^2}{2}-2\Delta^2+\kappa_2^1b_\times+\kappa_3^2b_\perp, \tag{24}$$

$$0=R_{1312}-R_{1213}=-t_{,1}-b_{\times,2}+b_{\perp,3}-st-\kappa_2^1b_\perp+\kappa_3^2b_\times, \tag{25}$$

and

$$s=\frac{\Delta_1[R_{1313}-R_{1212}]-\Delta_2[R_{1213}+R_{1312}]}{2\Delta^2}-\frac{[(b_\perp^2-b_\times^2)\Delta_1+2b_\perp b_\times\Delta_2]}{2\Delta^2}$$
$$-\frac{\Delta_1[b_{\perp,2}-b_{\times,3}]+\Delta_2[b_{\times,2}+b_{\perp,3}]}{2\Delta^2}-\kappa_2^1\frac{b_\perp\Delta_2-b_\times\Delta_1}{2\Delta^2}-\kappa_3^2\frac{b_\perp\Delta_1+b_\times\Delta_2}{2\Delta^2}. \tag{26}$$

Note we can set $0=R_{1312}-R_{1213}$ since this is required by the symmetries of the curvature tensor $R_{ijk\ell}$.

Now, substituting κ_2^1, κ_3^2, and ω_1 from equations (20), (21), and (23) in the three equations we just obtained, we obtain three differential equations of first order involving the deformations modes. In addition, if we also substitute κ_2^1, κ_3^2, and ω_1 in the equations for R_{23ij}, we will obtain another set of three differential equations involving the deformations modes and their first and second derivatives.

5.2. Director fields with vanishing biaxial splay

Now, let us assume a vanishing biaxial splay $(\Delta)^2=(\Delta_1)^2+(\Delta_2)^2=0$. If we assume that $\mathbf{b}\neq 0$, then from the equations of $R_{ijk\ell}$ we can compute $b_\times R_{1212}-b_\perp R_{1312}$, which allows us to write κ_2^1 as

$$\kappa_2^1=\frac{b_\times R_{1212}-b_\perp R_{1312}}{b^2}+\frac{b_\times s_{,1}-b_\perp t_{,1}}{2b^2}+\frac{b_\perp b_{\times,2}-b_\times b_{\perp,2}}{b^2}-\frac{(t^2-s^2)b_\times+2stb_\perp}{4b^2}. \tag{27}$$

In addition, from the equations of $R_{ijk\ell}$ we can compute $b_\perp R_{1313}-b_\times R_{1213}$, which allows us to write κ_3^2 as

$$\kappa_3^2=\frac{b_\perp R_{1313}-b_\times R_{1213}}{b^2}+\frac{b_\times t_{,1}+b_\perp s_{,1}}{2b^2}+\frac{b_\perp b_{\times,3}-b_\times b_{\perp,3}}{b^2}+\frac{(s^2-t^2)b_\perp+2stb_\times}{4b^2}. \tag{28}$$

Note that when $\Delta=0$, we can write ω_1 as a function of the deformation modes $\{s,t,b_\perp,b_\times\}$ by substituting for κ_2^1 and κ_3^2 in the equation for R_{2323}. Alternatively, we can get rid of ω_1 by choosing $\mathbf{n}_2$ and $\mathbf{n}_3$ such that $\omega_1=0$.

On the other hand, if $\mathbf{b}=0$, then there are some restrictions on the geometry of the ambient manifold. Indeed, we straightforwardly conclude that $R_{1212}=R_{1313}$ and $R_{1213}=-R_{1312}$, which by using the symmetry $R_{1213}=R_{1312}$ allows us to deduce that $R_{1213}=R_{1312}=0$.

6. Uniform distortion director fields on manifolds of constant curvature

In this section, we provide a characterization of uniform distortion fields, i.e., director fields $\mathbf{n}$ for which the deformation modes $\{s,t,b,\Delta_1,\Delta_2\}$ are all constant, in manifolds of constant sectional curvature. As a consequence, it will follow that no combination of values other than the pure twist phase exist in positive curvature. For negative curvature, the examples of uniform distortion fields with $s^2+4b^2=4$ and $t=0$ provided in [19] are in fact the most general case under the assumption that the biaxial splay vanishes. However, our results will also imply that it is possible to have uniform distortion fields in negative curvature with non-vanishing biaxial splay and, as in the Euclidean space, these phases correspond to foliations of space by helices.

From now on, let us assume that we have a director field in a curved space M^3 of constant curvature R_0. This means that the curvature tensor is given by $R_{ijk\ell}=R_0(\delta_{ik}\delta_{j\ell}-\delta_{i\ell}\delta_{jk})$ [28]: $R_0<0$ if M^3 is locally isometric to a hyperbolic space, $R_0=0$ if M^3 is locally isometric to the Euclidean space, and $R_0>0$ if M^3 is locally isometric to a three-sphere. Let us also introduce the shorthand notation $\langle D\mathbf{b},\mathbf{b}\rangle=(b_\perp^2-b_\times^2)\Delta_1+2b_\perp b_\times\Delta_2=b^2\Delta\,\cos(2\phi)$ and $\langle JD\mathbf{b},\mathbf{b}\rangle=(b_\times^2-b_\perp^2)\Delta_2+2b_\perp b_\times\Delta_1=b^2\Delta\,\sin(2\phi)$, where ϕ is the angle formed by the bend vector and the principal direction of the biaxial splay. (In other words, D and J denote the biaxial splay and the counterclockwise $\frac{\pi}{2}$-rotation acting as linear operators on the plane normal to the director field, respectively.) The results of this section can be summarized as follows

Theorem 2. *Let M^3 be a manifold of constant sectional curvature R_0 and let $\mathbf{n}$ be a director field in it with constant deformation modes $\{s,t,b_\perp,b_\times,\Delta_1,\Delta_2\}$.*

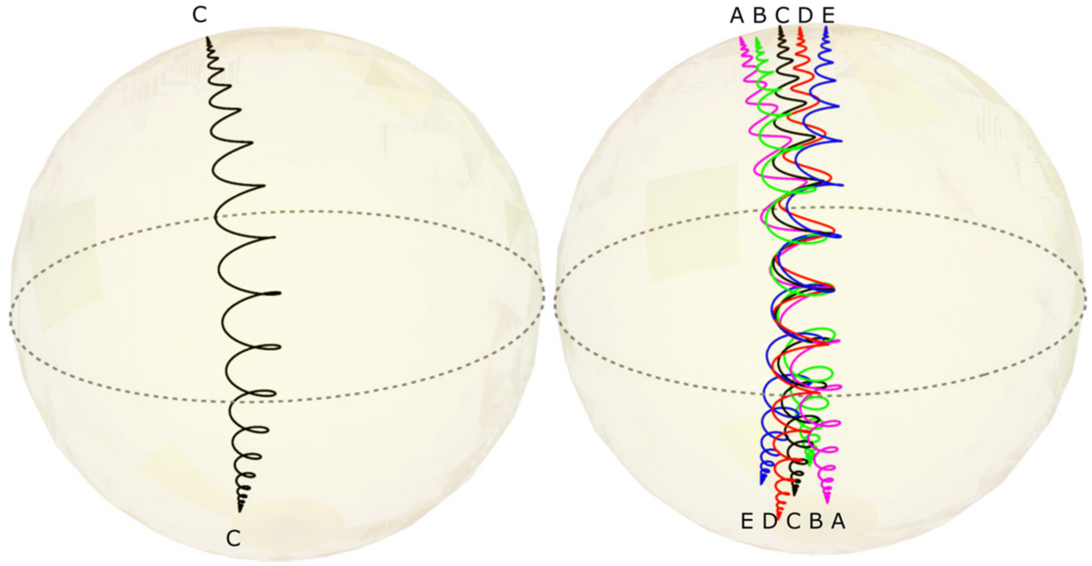

Figure 1. (Left) A single integral curve of a uniform distortion field **n** in hyperbolic space $\mathbb{H}^3$ using the Poincaré ball model. (Right) A collection of five integral curves of a uniform distortion field in hyperbolic space $\mathbb{H}^3$ forming congruent helices of similar curvature $\kappa = b$ and torsion $\tau = \omega_1$. The collection of all integral curves constitute a foliation of $\mathbb{H}^3$ by congruent helices. The dashed gray line indicates the points (x, y, z) in the ideal boundary with $z = 0$. The different helices are assigned five distinct letters (colors) to help guide the eye. Note that the distance between the integral curves increases as we move away from the point $(0, 0, 0)$ indicating a non-zero splay. This is expected in hyperbolic space since the failure of the parallel postulate is equivalent to the non-existence of equidistant geodesics and, therefore, we expect to not have equidistant congruent curves in $\mathbb{H}^3$. (In the figure, the sectional curvature is $R_0 = -1$, $b = 4.0$ (with $b_\perp = -b$ and $b_\times = 0$), $\Delta_1 = \Delta_2 = \frac{\sqrt{2}}{2}$, $t = 2\Delta = 2$, and $s = -\frac{1}{2\Delta^2}[(b_\perp^2 - b_\times^2)\Delta_1 + 2b_\perp b_\times \Delta_2] = -4\sqrt{2}$.)

(*a*) *If* $R_0 > 0$, *then* $s = b = \Delta = 0$ *and* $t = \pm 2\sqrt{R_0}$ *is the only solution.*

(*b*) *If* $R_0 = 0$, *then* $b = s = t = 0$ *when* $\Delta = 0$. *On the other hand, when* $\Delta \neq 0$, *then* $s = 0$, $t = \pm 2\Delta$, *and* $\phi = \frac{(2k+1)\pi}{4}$, $k \in \{0, 1, 2, 3\}$, *i.e., the bend vector* b *bisects the principal directions of the biaxial splay, where* $k = 0$ *or* $k = 2$ *if* $t = 2\Delta$ *and* $k = 1, 3$ *if* $t = -2\Delta$.

(*c*) *If* $R_0 < 0$, *then* $t = 0$ *and* $s^2 + 4b^2 = -4R_0$ *when* $\Delta = 0$. *On the other hand, when* $\Delta \neq 0$, *then* $t = \pm 2\Delta$, $s = -\frac{1}{2\Delta^2}\langle D\mathbf{b}, \mathbf{b}\rangle$, *and the deformation modes are subjected to the restriction* $1 + \frac{b^2}{4\Delta^2} \geqslant \sqrt{-\frac{R_0}{\Delta^2}}$.

In (b) and (c), the bend and biaxial splay are the free parameters describing the families of solutions.

Illustrations of uniform distortion fields in the Euclidean space and three-sphere can be found in references [12, 19], respectively. In our figure 1 we illustrate a uniform distortion field in hyperbolic space. As we show below, similarly to the case of Euclidean space, in $\mathbb{H}^3$ uniform distortions also give rise to foliation of space by congruent helices.

In the next subsections we are going to provide a proof for this theorem by analyzing the restrictions imposed by the compatibility equations on the values of the deformation modes. But, before that, let us discuss the implications on the geometry of the integral curves of the director field.

For $R_0 > 0$, it is known that the vector field tangent to the fibers of the Hopf fibration provides an example of a uniform distortion field [18, 19]. It turns out that this is the only possibility. Indeed, given any uniform distortion field **n** on a manifold of constant positive curvature, the integral curves of **n** are geodesics. In addition, from the fact that s, Δ_1, and Δ_2 all vanish, we deduce that any two integral curves are parallel to each other, from which follows that the fibration provided by the integral curves of **n** is locally a Hopf fibration [29].

For $R_0 \leqslant 0$, the integral curves of a uniform distortion field do not have to be geodesics. In general, they are helices, i.e., curves with constant curvature and torsion. Indeed, the equations of motion of the $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ are

$$\nabla_{\mathbf{n}_1}\begin{pmatrix}\mathbf{n}_1\\ \mathbf{n}_2\\ \mathbf{n}_3\end{pmatrix} = \begin{pmatrix}0 & -b_\perp & -b_\times\\ b_\perp & 0 & \omega_1\\ b_\times & -\omega_1 & 0\end{pmatrix}\begin{pmatrix}\mathbf{n}_1\\ \mathbf{n}_2\\ \mathbf{n}_3\end{pmatrix},$$

where ω_1 is constant and given by equation (23). We can obtain the Frenet frame $\{\mathbf{T} = \mathbf{n}, \mathbf{N}, \mathbf{B}\}$ from $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ by a rotation of an angle θ on the normal plane. Then, we can write $\kappa_1^1 = \kappa\cos\theta$, $\kappa_1^2 = \kappa\sin\theta$, and $\theta' = \tau - \omega_1$ [27]. Since κ_1^i and ω_1 are all constant, we deduce that κ and τ are also

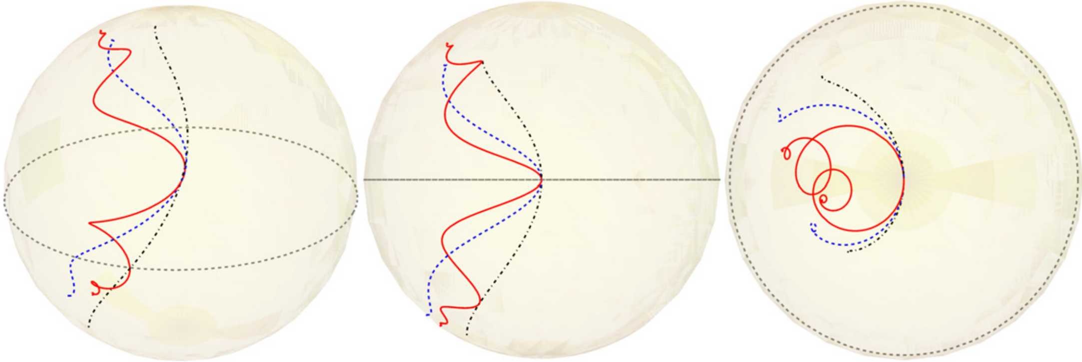

Figure 2. Helices in $\mathbb{H}^3$ in the Poincaré ball model shown, from left to right in perspective-view, side-view (xz-plane projection), and top-view (xy-plane projection), respectively. The dashed gray line indicates the points (x, y, z) in the ideal boundary with $z = 0$. Up to rigid motions in $\mathbb{H}^3$, every hyperbolic helix γ is parametrized by $\gamma(t) = \frac{1}{1+\cosh(\theta)\cosh(\omega_1 t)}(\cosh(\theta)\sinh(\omega_1 t),$ $\sinh(\theta)\cos(\omega_2 t), \sinh(\theta)\sin(\omega_2 t))$, where $\theta = \theta(\omega_1, \omega_2)$ and ω_1, ω_2 are constants depending on κ and τ. In the figure, the sectional curvature is $R_0 = -1$, the torsion is $\tau = \frac{1}{2}$ while the curvature is $\kappa = \frac{1}{2}$ for the dot-dashed black curve, $\kappa = 1$ for the dashed blue curve, and $\kappa = \frac{3}{2}$ for the full red curve. Note that the oscillatory behavior of γ is more prominent for large values of κ.

constant. As a consequence, the integral curves of $\mathbf{n}$ form helices and make a constant angle with the Darboux vector field $\mathbf{w} = \omega_1\mathbf{n}_1 - \kappa_1^2\mathbf{n}_2 + \kappa_1^1\mathbf{n}_3$ or $\mathbf{w} = \tau\mathbf{T} + \kappa\mathbf{B}$ if we use the Frenet frame: if s denotes the arc-length of the integral curves of $\mathbf{n}$, then $\frac{d}{ds}\langle\mathbf{w},\mathbf{n}\rangle = \langle\nabla_{\mathbf{n}}\mathbf{w},\mathbf{n}\rangle + \langle\mathbf{w},\nabla_{\mathbf{n}}\mathbf{n}\rangle = \langle\tau'\mathbf{n} + b'\mathbf{B} + \tau b\mathbf{N} - \tau b\mathbf{N},\mathbf{n}\rangle + \langle\mathbf{w}, b\mathbf{N}\rangle = 0$.

Virga's strategy to characterize the helicoidal phases in Euclidean space [12] consisted in investigating the behavior of the frame along a generic curve in space (not necessarily an integral curve). This gives rise to an operator whose eigenvector can be shown to be constant and, in addition, the integral lines of the director field precess around this fixed direction. We can provide an alternative proof by showing that all integral curves of a uniform distortion field $\mathbf{n}$ in a flat manifold have the same axis, i.e., we may show that $\mathbf{v}\cdot\nabla\mathbf{w} = 0$ for every direction $\mathbf{v}$.

Using equations (13) and (16) to write some of the κ_i^j's and ω_i's as functions of the deformation modes, we conclude that

$$\mathbf{w}_{,1} = (b_\times b_\perp - b_\perp b_\times)\mathbf{n}_1 + (b_\perp\omega_1 - \omega_1 b_\perp)\mathbf{n}_2 + (b_\times\omega_1 - \omega_1 b_\times)\mathbf{n}_3 = 0,$$

$$\mathbf{w}_{,2} = \left[b_\perp\left(\frac{t}{2} + \Delta_2\right) - b_\times\Delta_1\right]\mathbf{n}_1 + (\omega_1\Delta_1 + b_\perp\kappa_2^1)\mathbf{n}_2 + \left[\omega_1\left(\frac{t}{2} + \Delta_2\right) + b_\times\kappa_2^1\right]\mathbf{n}_3,$$

and

$$\mathbf{w}_{,3} = \left[b_\times\left(\frac{t}{2} - \Delta_2\right) + b_\perp\left(\frac{s}{2} - \Delta_1\right)\right]\mathbf{n}_1 - \left[\omega_1\left(\frac{t}{2} - \Delta_2\right) + b_\perp\kappa_3^2\right]\mathbf{n}_2 + \left[\omega_1\left(\frac{s}{2} - \Delta_1\right) - b_\times\kappa_3^2\right]\mathbf{n}_3.$$

On the one hand, for $R_0 = 0$, substitution of the values of the deformation modes of a uniform distortion field allows us to deduce that $\mathbf{n}_i\cdot\nabla\mathbf{w} = 0$, $i = 1, 2, 3$. Therefore, $\mathbf{n}$ provides a foliation of space by parallel helices. On the other hand, for $R_0 < 0$, we still have that $\mathbf{w}_{,1} = 0$ and, therefore, $\mathbf{w}$ is parallel transported along the integral curves of $\mathbf{n}$. However, in general $\mathbf{w}_{,2}$ and $\mathbf{w}_{,3}$ do not vanish, implying that $\mathbf{n}$ provides a foliation of hyperbolic space by helices which are not necessarily parallel. In a hyperbolic space, we need to distinguish between three types of helices. First notice that a curve with zero torsion is necessarily contained in a totally geodesic surface, i.e., locally the surface is a copy of a hyperbolic plane of curvature R_0. There are three types of planes curves with constant curvature $b > 0$: circles if $b \in (\sqrt{-R_0}, \infty)$, horocycles if $b = \sqrt{-R_0}$, and hypercycles if $b \in (0, \sqrt{-R_0})$ [30]. Therefore, depending on the values of the bend b, we expect three families of helices in hyperbolic geometry. Representative members of each of the families of hyperbolic helices are illustrated in figure 2.

6.1. Uniform distortion fields with vanishing biaxial splay $(\Delta)^2 = (\Delta_1)^2 + (\Delta_2)^2 = 0$

First, assume we have $b_\perp = b_\times = 0$. Then, from the equations for R_{1212} and R_{1213}, it follows that $R_0 = \frac{1}{4}(t^2 - s^2)$ and $st = 0$. Consequently, $s = 0$ or $t = 0$ and we finally conclude

$$\Delta = 0, \quad b = 0 \Rightarrow \begin{cases} s = 0 \text{ and } t = \pm 2\sqrt{R_0} & \text{if } \quad R_0 \geqslant 0 \\ t = 0 \text{ and } s = \pm 2\sqrt{-R_0} & \text{if } \quad R_0 \leqslant 0 \end{cases}. \tag{29}$$

In particular, in Euclidean space, $\Delta = 0$ and $b = 0$ imply that the director field is constant: $\mathrm{d}\mathbf{n} \equiv 0$.

Now, assume that $\Delta = 0$ but $b \neq 0$. From $R_{1223} = 0$ and $R_{1323} = 0$, we necessarily have $t = 0$. From equations (27) and (28) it follows

$$\kappa_2^1 = \frac{b_\times}{b^2}\left(R_0 + \frac{s^2}{4}\right) \quad \text{and} \quad \kappa_3^2 = \frac{b_\perp}{b^2}\left(R_0 + \frac{s^2}{4}\right). \tag{30}$$

Substituting the expressions for κ_2^1, κ_3^2 in the equation for R_{2323} gives

$$R_0 = -\frac{s^2}{4} - \left(R_0 + \frac{s^2}{4}\right)^2 \frac{b_\perp^2 + b_\times^2}{b^4} \Rightarrow \frac{1}{b^2}\left(R_0 + \frac{s^2}{4}\right)\left(b^2 + R_0 + \frac{s^2}{4}\right) = 0. \tag{31}$$

Therefore, $s^2 = -4R_0$ or $s^2 = -4b^2 - 4R_0$.

On the one hand, we see that if $R_0 \geqslant 0$, then there exists no solution with $b \neq 0$. On the other hand, if $R_0 < 0$, we could equally have either $s^2 = -4R_0$ or $s^2 = -4b^2 - 4R_0$ (there is no sign obstruction for $R_0 < 0$). However, only $s^2 = -4b^2 - 4R_0$ is allowed. Indeed, if it were $s^2 = -R_0$, then substituting the expressions for κ_2^1 and κ_3^2 above in the equations for R_{1212}, R_{1313} and summing them would give

$$2R_0 = -b^2 - \frac{s^2}{2} + R_0 + \frac{s^2}{4} \Rightarrow R_0 = -b^2 - \frac{s^2}{4} = -b^2 - \frac{(-4R_0)}{4} \Rightarrow b^2 = 0. \tag{32}$$

This contradicts the assumption that $b \neq 0$. Finally, we conclude that

$$\Delta = 0, \quad b \neq 0 \Rightarrow \begin{cases} \nexists\,\text{solution} & \text{if} \quad R_0 > 0 \\ t = 0 \text{ and } s^2 + 4b^2 = -4R_0 & \text{if} \quad R_0 \leqslant 0 \end{cases}. \tag{33}$$

Notice that in the case $R_0 < 0$, such as in hyperbolic space, the configuration with $\Delta = 0$ and $t = 0$ becomes the trivial director field in Euclidean space in the limit $R_0 \to 0^-$.

6.2. Uniform distortion fields with non-vanishing biaxial splay $(\Delta)^2 = (\Delta_1)^2 + (\Delta_2)^2 > 0$

If all deformation modes are constant, it follows from equations (20), (21), and (23) that κ_2^1, κ_3^2, and ω_1 are also constant and equal to

$$\kappa_2^1 = t\frac{b_\perp\Delta_1 + b_\times\Delta_2}{2\Delta^2}, \qquad \kappa_3^2 = -t\frac{b_\perp\Delta_2 - b_\times\Delta_1}{2\Delta^2}, \qquad \omega_1 = -\frac{tb^2}{8\Delta^2} - \frac{\langle JD\mathbf{b}, \mathbf{b}\rangle}{4\Delta^2}. \tag{34}$$

Now, substituting κ_2^1 and κ_3^2 in equation (26), implies that the splay is given by

$$s = -\frac{\langle D\mathbf{b}, \mathbf{b}\rangle}{2\Delta^2}. \tag{35}$$

In addition, substituting κ_2^1, κ_3^2, and ω_1 in equations (24), (25), and R_{2323} from equation (19), gives

$$2R_0 = -b^2 - \frac{s^2}{2} + \frac{t^2}{2} - 2\Delta^2 + t\frac{(b_\times^2 - b_\perp^2)\Delta_2 + 2b_\perp b_\times\Delta_1}{2\Delta^2}, \tag{36}$$

$$0 = -st - t\frac{(b_\perp^2 - b_\times^2)\Delta_1 + 2b_\perp b_\times\Delta_2}{2\Delta^2}, \tag{37}$$

and

$$R_0 = -\frac{s^2}{4} - \frac{t^2}{4} + \Delta^2 - \frac{t^2b^2}{8\Delta^2} + t\frac{(b_\times^2 - b_\perp^2)\Delta_2 + 2b_\perp b_\times\Delta_1}{4\Delta^2}. \tag{38}$$

Notice that from the expression we got for the splay in equation (35), it follows that equation (37) is redundant. Subtracting equation (36) from twice the last equation above allows us to conclude that

$$0 = \left(1 + \frac{b^2}{4\Delta^2}\right)(t^2 - 4\Delta^2) \Rightarrow t = \pm 2\Delta. \tag{39}$$

Remark 1. The equations for R_{2312} and R_{2313} in (19) provide no further constraints. In fact, substituting κ_2^1, κ_3^2, and ω_1 in R_{2312} and in R_{2313} respectively gives

$$\left(1 + \frac{b^2}{4\Delta^2}\right)\left(1 - \frac{t^2}{4\Delta^2}\right)(b_\times\Delta_1 - b_\perp\Delta_2) = 0,$$

and

$$\left(1 + \frac{b^2}{4\Delta^2}\right)\left(1 - \frac{t^2}{4\Delta^2}\right)(b_\perp\Delta_1 + b_\times\Delta_2) = 0.$$

Now, taking into account that $t = \pm 2\Delta$, which is obtained from R_{2323}, the two equations above vanish identically.

Let us write $t = 2\delta\Delta$, $\delta = \pm 1$, and substitute for t and s in equation (36). Thus,

$$2R_0 = -b^2 - \frac{\langle D\mathbf{b}, \mathbf{b}\rangle^2}{8\Delta^4} + \delta\frac{\langle JD\mathbf{b}, \mathbf{b}\rangle}{\Delta}, \tag{40}$$

from which we find that

$$2R_0\Delta + \frac{\langle D\mathbf{b}, \mathbf{b}\rangle^2}{8\Delta^3} + b^2\Delta + \delta\langle JD\mathbf{b}, \mathbf{b}\rangle = 0. \tag{41}$$

Now, from the Cauchy–Schwarz inequality, it follows that

$$|\langle JD\mathbf{b}, \mathbf{b}\rangle| \leqslant \|JD\mathbf{b}\|\|\mathbf{b}\| \leqslant b^2\Delta \Rightarrow 0 \leqslant b^2\Delta + \delta\langle JD\mathbf{b}, \mathbf{b}\rangle.$$

We immediately have the following conclusions:

(a) If $R_0 > 0$, then equation (41) is a sum of non-negative numbers. However, $R_0\Delta > 0$ and, consequently, there must be no uniform director field with $\Delta > 0$ on a space of constant positive curvature, such as the three-sphere.

(b) If $R_0 = 0$, then we must have $\langle D\mathbf{b}, \mathbf{b}\rangle = 0$ and also $b^2\Delta + \delta\langle JD\mathbf{b}, \mathbf{b}\rangle = 0$. It follows that in a space of vanishing curvature the splay s must vanish and the bend vector $\mathbf{b}$ bisects the principal directions of the biaxial splay, i.e., $\phi = \frac{(2k+1)\pi}{4}$, $k \in \{0, 1, 2, 3\}$, where $k = 0$ or $k = 2$ if $t = 2\Delta$ and $k = 1, 3$ if $t = -2\Delta$.

It remains to further analyze uniform distortion director fields in hyperbolic geometry, i.e., $R_0 < 0$. Seeing equation (36) as a quadratic polynomial in t, its discriminant is

$$\text{disc.} = 4\Delta^2\left(1 + \frac{b^2}{4\Delta^2}\right)^2 + 4R_0. \tag{42}$$

Thus, the requirement that $t \in \mathbb{R}$ demands disc. $\geqslant 0$, which implies

$$1 + \frac{b^2}{4\Delta^2} \geqslant \sqrt{-\frac{R_0}{\Delta^2}}. \tag{43}$$

Note that if we choose $\Delta \geqslant \sqrt{-R_0}$, then the above inequality imposes no restriction on the values of the bend b.

7. Discussion

In the intrinsic approach materials are described only through quantities available to an observer residing within the material [11]. These quantities may be associated with some non-trivial locally preferred reference values that manifest the constituents' shape and mutual interactions. We show that a collection of five such scalar (and pseudoscalar) fields suffice to characterize the director texture. These fields can be chosen to be the bend, splay, twist, saddle-splay and the relative orientation between the principal biaxial splay direction and the bend direction.

In 2D only two such fields suffice to uniquely prescribe a director field. The compatibility conditions in the 2D case amount to a single first order differential relation [9]. In three dimensions we obtained six differential relations. Three of first order, and three of second order. Thus the system is of at most second order; it is presently unknown whether the system can be further reduced to yield a purely first order system or not. Understanding the degree and structure of the compatibility conditions is important not for taxonomical reasons, but rather as these determine the super-extensive rate at which energy accumulates when a frustrated phase grows in size [11].

Though more abstract than an approach uniquely based on vector calculus, the method of moving frames allows us to obtain manageable equations and to investigate director fields in both Euclidean and curved Riemannian spaces in an equal foot. In particular it allows us to find all uniform distortion fields for all isotropic homogeneous Riemannian manifolds.

The exhaustive nature of the compatibility conditions presented here allows us to assert that the well known constant twist phase in $\mathbb{S}^3$ is, in fact, the only uniform distortion field $\mathbb{S}^3$ supports. For $\mathbb{H}^3$ we extend the result of Sadoc *et al* [19] who found particular solutions with vanishing twist and biaxial splay, to show that these constitute all possible textures foliated by planar curves in $\mathbb{H}^3$. Moreover, we find all the textures foliated by non-planar curves characterized by non-vanishing twist and biaxial splay that grow in proportion to each other. In general, we showed that uniform distortion fields in $\mathbb{H}^3$ yield textures foliated by congruent helices, encompassing the previous results.

The full compatibility conditions provided here allow us to extend our understanding of three-dimensional frustrated textures in Euclidean space to the realm of non-uniform distortion fields. Small enough domains of bent-core liquid crystals are expected to allow a non-uniform distortion field associated with an elastic energy that is lower than that of the uniform twist-bent phase [31]. Knowledge of the full compatibility conditions provides a path for constructing such low energy solutions for small enough domains: starting with a state of pure bend at some point in the domain, satisfaction of the compatibility conditions necessitates certain gradients to assume a non-vanishing value. Incorporating the constitutive law at this point may help select which of these gradients will be chosen to balance the attempted constant pure bend. Such 'propagation' of solutions may also find use in solving the inverse design of three-dimensional responsive material, analogously to the procedure carried out for 2D in [4].

Considering the five characterizing fields as given quantities and solving for the corresponding director field may also be carried out as long as the compatibility conditions are satisfied. The compatibility conditions, in turn, can be interpreted both in terms of a material frame, where the fields gradients are given in terms of their projections on the director orientations, and a lab frame in which the field gradients are given explicitly in terms of the embedding space coordinates. It is important to note that the information contained in the two viewpoints is not equivalent. The material intrinsic description, which is more natural, may be used to integrate the director field from knowledge about its local behavior. Such an approach is particularly useful for solving inverse design problems [4], and for constructing new optimal textures. The lab frame approach is somewhat less natural as it assumes that the fields are given in terms of the stationary embedding space coordinates, yet the director is unknown. For the two-dimensional case the lab-frame approach allowed obtaining the director field directly from the gradients of the bend and splay functions, provided they were compatible [9]. For general fields, b and s, one may apply the reconstruction formula provided that the gradients of these fields are large enough. However, this does not assure that these fields were indeed compatible. In this approach the compatibility condition is replaced by self-consistency conditions equating the splay and bend of the resulting director field with those used to generate it. One may expect that these conditions will produce two second order differential equations for the fields b and s that are independent of the director orientation, however to the best of our knowledge these relations have not yet been obtained.

When finishing this manuscript, it came to our attention that a similar approach to the one presented here was recently pursued by Pollard and Alexander [22]. There, the authors also develop the idea of using the moving frame method to obtain the compatibility conditions for the deformation modes of a director field expressed in terms of the curvature tensor of the ambient manifold. In addition, they present the lab frame reconstruction formulae for the director in terms of the fixed frame spatial gradients of the deformation modes. These, much like their two-dimensional analogs are assured to satisfy the self-consistency conditions if the scalar deformations modes were indeed compatible, yet for general fields do not have to be self-consistent. Finally, they discuss the examples constructed by Sadoc *et al* [19] in the language of moving frames by exploiting the fact that there exists an underlying Lie algebra structure associated with uniform distortion fields. While the choice of applications differs between our work and that of Pollard and Alexander, the main guiding principles and calculations of the components of the Riemann curvature tensor are similar. In particular the nine equations (19)–(25) and (31)–(33) in their manuscript can be directly translated to the nine equations derived here (19). While we further reduce these nine equations in eight unknown fields to six equations in the five deformation modes, the compatibility conditions remain equivalent, and any set of deformation modes determined compatible by one of the methods would be compatible with respect to the other.

Acknowledgments

This work was funded by the Israel Science Foundation Grant No. 1479/16. EE thanks the Ascher Foundation for their support. LCBdS acknowledges the support provided by the Morá Miriam Rozen Gerber fellowship for Brazilian postdocs.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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References

[1] de Gennes P G and Prost J 1995 *The Physics of Liquid Crystals* 2 edn (Oxford: Oxford University Press)
[2] Virga E G 1995 *Variational Theories for Liquid Crystals* (Boca Raton, FL: CRC Press)
[3] Aharoni H, Sharon E and Kupferman R 2014 Geometry of thin nematic elastomer sheets *Phys. Rev. Lett.* **113** 257801
[4] Griniasty I, Aharoni H and Efrati E 2019 Curved geometries from planar director fields: solving the two-dimensional inverse problem *Phys. Rev. Lett.* **123** 127801
[5] Warner M and Mostajeran C 2018 Nematic director fields and topographies of solid shells of revolution *Proc. R. Soc.* A **474** 20170566
[6] Sydney Gladman A, Matsumoto E A, Nuzzo R G, Mahadevan L and Lewis J A 2016 Biomimetic 4D printing *Nat. Mater.* **15** 413
[7] Fernandes R M, Chubukov A V and Schmalian J 2014 What drives nematic order in iron-based superconductors? *Nat. Phys.* **10** 97
[8] Wang F, Kivelson S A and Lee D-H 2015 Nematicity and quantum paramagnetism in FeSe *Nat. Phys.* **11** 959
[9] Niv I and Efrati E 2018 Geometric frustration and compatibility conditions for two-dimensional director fields *Soft Matter* **14** 424
Niv I and Efrati E 2018 Correction: geometric frustration and compatibility conditions for two-dimensional director fields *Soft Matter* **14** 1068
[10] Meyer R B 1976 *Structural Problems in Liquid Crystal Physics* (*Les Houches Summer School in Theoretical Physics* vol 25) (London: Gordon and Breach) pp 273–373
[11] Meiri S and Efrati E 2021 Cumulative geometric frustration in physical assemblies (arXiv:2101.09816)
[12] Virga E G 2019 Uniform distortions and generalized elasticity of liquid crystals *Phys. Rev.* E **100** 052701
[13] Machon T and Alexander G P 2016 Umbilic lines in orientational order *Phys. Rev.* X **6** 011033
[14] Selinger J V 2018 Interpretation of saddle-splay and the Oseen–Frank free energy in liquid crystals *Liq. Cryst. Rev.* **6** 129
[15] Borshch V *et al* 2013 Nematic twist-bend phase with nanoscale modulation of molecular orientation *Nat. Commun.* **4** 2635
[16] Hough L E *et al* 2009 Chiral isotropic liquids from achiral molecules *Science* **325** 452
[17] Chen D *et al* 2014 Twist-bend heliconical chiral nematic liquid crystal phase of an achiral rigid bent-core mesogen *Phys. Rev.* E **89** 022506
[18] Sethna J P, Wright D C and Mermin N D 1983 Relieving cholesteric frustration: the blue phase in a curved space *Phys. Rev. Lett.* **51** 467
[19] Sadoc J-F, Mosseri R and Selinger J V 2020 Liquid crystal director fields in three-dimensional non-Euclidean geometries *New J. Phys.* **22** 093036
[20] Clelland J N 2017 *From Frenet to Cartan: The Method of Moving Frames* (Providence, RI: American Mathematical Society)
[21] Carroll S 2019 *Spacetime and Geometry: An Introduction to General Relativity* (Cambridge: Cambridge University Press)
[22] Pollard J and Alexander G P 2021 Intrinsic geometry and director reconstruction for three-dimensional liquid crystals *New J. Phys.* https://doi.org/10.1088/1367-2630/abfdf4
[23] Griffiths P 1974 On Cartan's method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry *Duke Math. J.* **41** 775
[24] Tenenblat K 1971 On isometric immersions of Riemannian manifolds *Bull. Braz. Math. Soc.* **2** 23
[25] do Carmo M P 1994 *Differential Forms and Applications* (Berlin: Springer)
[26] do Carmo M P 2012 *O Método do Referencial Móvel* (Rio de Janeiro: Publicações Matemáticas, Instituto de Matemática Pura e Aplicada)
[27] Takagi S and Tanzawa T 1992 Quantum mechanics of a particle confined to a twisted ring *Prog. Theor. Phys.* **87** 561
[28] Spivak M 1979 *A Comprehensive Introduction to Differential Geometry* vol 4 (Boston, MA: Publish or Perish)
[29] Nuchi H 2015 Fiberwise homogeneous fibrations of the three-dimensional space forms by geodesics *Algebr. Geom. Topol.* **15** 3059
[30] Ramsay A and Richtmyer R D 1994 *Introduction to Hyperbolic Geometry* (New York: Springer)
[31] Virga E G 2014 Double-well elastic theory for twist-bend nematic phases *Phys. Rev.* E **89** 052502