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Tuning the Total Displacement of Membranes

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Abstract

In this paper we study a design problem to tune the robustness of a membrane by changing its vulnerability. Consider an energy functional corresponding to solutions of Poisson's equation with Robin boundary conditions. The aim is to find functions in a rearrangement class such that their energies would be a given specific value. We prove that this design problem has a solution and also we propose a way to find it. Furthermore, we derive some topological and geometrical properties of the configuration of the vulnerability. In addition, some explicit solutions are found analytically when the domain is an N-ball. For general domain we develop a numerical algorithm based on rearrangements to find the solution. The algorithm evolves both minimization and maximization processes over two different rearrangement classes. Our algorithm works efficiently for various domains and the numerical results obtained coincide with our analytical findings.

Keywords: Laplacian Operator, Robin Boundary Condition, Rearrangement, Design Problem
2010 MSC: 49A20, 49J20, 35J05, 35J20, 74E30

1. Introduction

Rearrangement design problems arise naturally in many applications such as fluid mechanics and mechanical vibrations, just to name a few [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. In many of these problems we consider an energy functional depending on solutions of a partial differential equation where its design coefficients are in a rearrangement class of functions. For example, in the design of mechanical vibration that explore the possibility to control the total displacement we have an energy functional which depends on the solution of a Poisson's equation where the right-hand side function is in a rearrangement class. Moreover, to find a stationary and stable flow in the planar motion under an irrotational body force of an incompressible, inviscid fluid contained in an infinite cylinder of uniform cross-section we should find the extremizer of an energy functional corresponding the solutions of a Poisson's equation [1, 2, 3, 16].

In this article, we study an intermediate problem arising in the design of mechanical vibration that explore the possibility to control the total displacement. The governing elliptic partial differential equation (PDE) is

$$\begin{cases} -\Delta u(\mathbf{x}) = f(\mathbf{x}) & \text{in} \quad \Omega, \\ \frac{\partial u(\mathbf{x})}{\partial n} + \beta u(\mathbf{x}) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Δ is the Laplace operator acting on the function $u(\mathbf{x})$ defined on a bounded smooth domain $\Omega \subset \mathbb{R}^N$, $f \in L^2(\Omega)$, β is a given positive constant, and $\frac{\partial}{\partial n}$ is the outward normal derivative along the

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boundary $\partial\Omega$. As the solution u depends on f, we use the notation u_f to emphasize this dependence. For a given $f \in L^2(\Omega)$, $u = u_f \in H^1(\Omega)$ is a (weak) solution of (1.1) if and only if we have

$$\int_\Omega \nabla u_f \cdot \nabla \phi d\mathbf{x} + \beta \int_{\partial\Omega} u_f \phi dS = \int_\Omega f \phi d\mathbf{x}, \quad \forall \phi \in H^1(\Omega). \tag{1.2}$$

It is noteworthy that the solution u_f of (1.1) corresponding to f is the unique maximizer of the following problem

$$\sup_{u \in H^1(\Omega)} \mathcal{G}(f, u, \beta), \tag{1.3}$$

where

$$\mathcal{G}(f, u, \beta) = 2\int_\Omega f u d\mathbf{x} - \int_\Omega |\nabla u|^2 d\mathbf{x} - \beta \int_{\partial\Omega} u^2 dS,$$

and the maximum value is $J(f) = \int_\Omega f u_f d\mathbf{x}$.

In mechanical vibration, (1.1) models the steady state of a vibrating membrane with a constant force, e.g. the gravity force of earth, applied to it. Moreover, let the magnitude of the tension be a constant. Accordingly, the function $f(\mathbf{x})$ in (1.1) can be considered as an external force such that the quantity

$$\int_\Omega f(\mathbf{x}) d\mathbf{x},$$

measures the mass of the membrane. The Robin condition would be considered to imagine that the membrane at its boundary points are free to move along a track but are attached to a coiled spring or rubber band obeying Hooke's law which tends to pull it back to the equilibrium position. In that case the membrane would exchange some of its energy with the coiled spring [19].

The objective function

$$J(f) = \int_\Omega f u_f d\mathbf{x}, \tag{1.4}$$

which is called the total displacement, measures the robustness of the membrane from the physical point of view. The following shape optimization problem has been considered by several authors [1, 2, 3, 16, 20, 21]: Assume that we want to build a membrane with a prescribed total mass $\gamma > 0$ and consists of two given materials with densities M and m ($M > m > 0$). The aim is to distribute these materials in such a way that the total displacement of the resulting membrane is optimal or desired. This means that the function $f(\mathbf{x})$ should be considered in the following admissible set

$$\mathcal{A}_{m,M,\gamma} = \{f \mid f(\mathbf{x}) = M\chi_D + m\chi_{D^c} \text{ where } D \subset \Omega, |D| = A < |\Omega|\},$$

where A is a prescribed constant. The set $\mathcal{A}_{m,M,\gamma}$ which is called a rearrangement class of function is a subset of $L^2(\Omega)$ and its weak closure is

$$\mathcal{B}_{m,M,\gamma} = \{f \mid m \le f(\mathbf{x}) \le M, \int_\Omega f(\mathbf{x}) d\mathbf{x} = \gamma\},$$

where $\gamma := MA + m(|\Omega| - A)$, see [1, 2, 22]. The difference between functions in two sets $\mathcal{A}_{m,M,\gamma}$ and $\mathcal{B}_{m,M,\gamma}$ is whether f takes values between m and M.

The following shape optimization problems have been studied

$$\check{J}_\mathcal{A} := \min\{J(f) : f \in \mathcal{A}_{m,M,\gamma}\}. \tag{1.5}$$

$$\hat{J}_\mathcal{A} := \max\{J(f) : f \in \mathcal{A}_{m,M,\gamma}\}, \tag{1.6}$$

previously in [20, 21] and the optimal forces are found from all possible external forces in $\mathcal{A}_{m,M,\gamma}$ which minimize or maximize the total displacement, correspondingly. The distribution of materials in the membrane which leads to external forces with minimum and maximum vulnerability are obtained. For the minimization problem Liu and Emamizadeh have established the existence and uniqueness of the solution and showed that the solution is a non-decreasing function when the domain is an N-ball [20]. Kao and Mohammadi studied both minimization and maximization problems and obtained analytical solutions for N-balls. In addition, the properties of the extremizers on general domains including topology and geometry of the optimizers have been derived. Moreover, efficient algorithms based on finite element methods and rearrangement techniques are proposed to determine the extremizers in just a few iterations on general domains [21].

In this paper we consider the following design problem

$$J_c := \{J(f) = c \,:\, f \in \mathcal{A}_{m,M,\gamma}\}, \tag{1.7}$$

where $c \in (\check{J}_{\mathcal{A}}, \hat{J}_{\mathcal{A}})$. Instead of searching for external forces with minimum and maximum vulnerability, we address the question whether it is possible to tune the robustness of the membrane by changing its vulnerability in $c \in (\check{J}_{\mathcal{A}}, \hat{J}_{\mathcal{A}})$. It is noteworthy that an intermediate value problem has been considered in [20] for a linear energy functional.

In this paper, we first prove an existence result for (1.7). After that we provide a formula to derive a solution of (1.7) by dividing it into two separate rearrangement optimization problems. This formula allows us to determine an analytical solution to (1.7) when the domain is an N-ball. Moreover, we prove that the solution inherits some kind of Steiner symmetry while our domain is Steiner symmetric with respect to hyperplanes. For general domains, a numerical algorithm based on rearrangement techniques and the formula is developed to determine the solution of (1.7). The algorithm is capable to obtain the solution efficiently for domains with different geometries. The numerical results coincide with our analytical findings.

The paper is organized in the following way. In section 2, we report the analytical results including existence of a solution (1.7) and some geometrical and topological properties of our solutions. Moreover, the explicit solutions of N-balls are provided. Section 3 is devoted to our numerical method and we illustrate several numerical examples.

2. Analytical Results

Considering $f \in \mathcal{B}_{m,M,\gamma}$, we have $u_f(\mathbf{x}) > 0$ and $u_f \in H^2_{\text{loc}}(\Omega) \cap C^{1,\theta}$ where $\theta \in (0,1)$. Moreover, function u_f attains its minimum only on $\partial\Omega$ and its maximum at an interior point of Ω.

As we will see later, Problem (1.7) does not have a unique solution in general. Here, we address the question of existence for Problem (1.7). Without loss of generality in this section we can assume that $m = 1$ and set $\epsilon = (M - 1)$. Then, a function f in $\mathcal{A}_{m,M,\gamma}$ is of the form $f = 1 + \epsilon\chi_D$ such that $|D| = A$.

Theorem 1. *Problem* (1.7) *has a solution* $f_c \in \mathcal{A}_{m,M,\gamma}$.

Proof. In view of Theorem 5.1 in [20] and Theorem 3 in [21], we know that Problems (1.5) and (1.6) have solutions $\check{f}$ and $\hat{f}$ in $\mathcal{A}_{m,M,\gamma}$ respectively. It is known that the set $\mathcal{A}_{m,M,\gamma}$ is path connected using Lemma 2.11 in [2]. Hence, there is a continuous function $\eta \in C([0,1],\, \mathcal{A}_{m,M,\gamma})$ such that

$$\eta(0) = \check{f}, \quad \eta(1) = \hat{f}.$$

Define function $\xi : [0,1] \to [\check{J}_{\mathcal{A}}, \hat{J}_{\mathcal{A}}]$ where

$$\xi(t) = J(\eta(t)) = \int_\Omega \eta(t) u_{\eta(t)} d\mathbf{x}.$$

Recall that functional $J(\cdot)$ is continuous, see [20, Lemma 5.2], and so ξ is a continuous function. Employing the intermediate value theorem one can find $\bar{t} \in [0,1]$ such that $\xi(\bar{t}) = c$ and then $f_c = \eta(\bar{t})$ is a solution for (1.7). $\square$

In what follows, we determine a solution of (1.7) by using solutions of two rearrangement optimization problems. To do so, we need the following lemma.

Lemma 2.1. *Let D_1 be a measurable subset of Ω. Then maximization problem*

$$\max_{D\subset D_1^c,\ |D|=t} J(1+\epsilon\chi_{D_1}+\epsilon\chi_D),$$

has a solution D_2. Moreover, this solution is uniquely defined by

$$D_2=\{\mathbf{x}\in D_1^c:\ u_{1+\epsilon\chi_{D_1}+\epsilon\chi_{D_2}}(\mathbf{x})\geq\theta\},\quad \textit{with}\quad \theta=\sup\{s:\ |\{\mathbf{x}\in D_1^c:\ u_{1+\epsilon\chi_{D_1}+\epsilon\chi_{D_2}}(\mathbf{x})\geq s\}|\geq t\}. \tag{2.1}$$

Proof. Let us recall here that

$$\int_\Omega f u_g d\mathbf{x}=\int_\Omega g u_f d\mathbf{x}, \tag{2.2}$$

for all $f,g\in L^2(\Omega)$, [20]. Setting $f=1+\epsilon\chi_{D_1}$, this symmetry property and the fact that D is a subset of D_1^c yields

$$\begin{aligned}J(f+\epsilon\chi_D)&=\int_\Omega f u_f d\mathbf{x}+\epsilon\int_\Omega f u_{\chi_D} d\mathbf{x}+\epsilon\int_{D_1^c}\chi_D u_f d\mathbf{x}+\epsilon^2\int_{D_1^c}\chi_D u_{\chi_D} d\mathbf{x}\\&=\int_\Omega f u_f d\mathbf{x}+2\epsilon\int_{D_1^c}\chi_D u_f d\mathbf{x}+\epsilon^2\int_{D_1^c}\chi_D u_{\chi_D} d\mathbf{x}.\end{aligned} \tag{2.3}$$

Hence, $J(f+\epsilon\chi_D)$ can be considered as a functional where χ_D belongs to the rearrangement set $\mathcal{A}_{0,1,t}\subset L^2(D_1^c)$ and our maximization problem is an optimization of this functional over $\mathcal{A}_{0,1,t}$. In order to prove the existence of a maximizer let consider the maximization problem over the weak closure set $\mathcal{B}_{0,1,t}$. It is well-known that $\mathcal{B}_{0,1,t}\subset L^2(D_1^c)$ is a convex weakly sequentially compact set with $\mathcal{A}_{0,1,t}$ as its extreme points [1, 2]. Due to the weak continuity of the functional $J(f+\epsilon\chi_D)$, see [20, Lemma 5.2], we deduce that there is a maximum for the functional over the weakly compact set $\mathcal{B}_{0,1,t}$. Moreover, the maximum have to be in $\mathcal{A}_{0,1,t}$ in view of the convexity of the functional, see [20, Lemma 5.2]. So far, we have shown that the maximization problem has a solution D_2.

In order to establish the next assertion in the theorem, we claim that

$$\int_{D_1^c}\chi_{D_2}\,u_{f+\epsilon\chi_{D_2}}d\mathbf{x}\geq\int_{D_1^c}\chi_D\,u_{f+\epsilon\chi_{D_2}}d\mathbf{x},\quad \text{for every}\quad D\subset D_1^c,\quad \text{with}\quad |D|=t. \tag{2.4}$$

To prove the claim, we argue by contradiction. Assume there is a set $\mathcal{D}\subset D_1^c$ with $|\mathcal{D}|=t$ such that

$$\int_{D_1^c}\chi_{D_2}\,u_{f+\epsilon\chi_{D_2}}d\mathbf{x}<\int_{D_1^c}\chi_{\mathcal{D}}\,u_{f+\epsilon\chi_{D_2}}d\mathbf{x}. \tag{2.5}$$

This inequality reveals that

$$\int_\Omega(f+\epsilon\chi_{D_2})u_{f+\epsilon\chi_{D_2}}d\mathbf{x}<\int_\Omega(f+\epsilon\chi_{\mathcal{D}})u_{f+\epsilon\chi_{D_2}}d\mathbf{x}, \tag{2.6}$$

in view of (2.3). Then, using (2.6) we have

$$\begin{aligned}\int_\Omega(f+\epsilon\chi_{D_2})u_{f+\epsilon\chi_{D_2}}d\mathbf{x}&=\mathcal{G}(f+\epsilon\chi_{D_2},u_{f+\epsilon\chi_{D_2}},\beta)<\mathcal{G}(f+\epsilon\chi_{\mathcal{D}},u_{f+\epsilon\chi_{D_2}},\beta)\leq\sup_{u\in H^1(\Omega)}\mathcal{G}(f+\epsilon\chi_{\mathcal{D}},u,\beta)\\&=\int_\Omega(f+\epsilon\chi_{\mathcal{D}})u_{f+\epsilon\chi_{\mathcal{D}}}d\mathbf{x},\end{aligned}$$

which contradicts the maximality of $f+\epsilon\chi_{D_2}$ and the claim is proven.

From (2.4) we deduce that $\int_{D_1^c}\chi_{D_2}u_{f+\epsilon\chi_{D_2}}d\mathbf{x}$ is a maximizer for the functional $L(\chi_D):=\int_{D_1^c}\chi_D u_{f+\epsilon\chi_{D_2}}d\mathbf{x}$ over the rearrangement class $\mathcal{A}_{0,1,t}\subset L^2(D_1^c)$. On the other hand, due to Lemma 2.1 in [21], we infer that $u_{f+\epsilon\chi_{D_2}}$ satisfies $-\Delta u=f+\epsilon\chi_{D_2}$ almost everywhere in Ω and so employing Lemma 7.7 in [23] we observe that its level sets have measure zero. Now, Lemma 2.9 in [2] yields that there is a non-decreasing function $\eta : \mathbb{R} \to \mathbb{R}$ such that $\eta(u_{f+\epsilon\chi_{D_2}})$ is in the rearrangement class $\mathcal{A}_{0,1,t}$. Moreover, Lemma 2.4 in [2] reveals that $\eta(u_{f+\epsilon\chi_{D_2}})$ is the unique maximizer of the functional $L(\chi_D) = \int_{D_1^c} \chi_D \, u_{f+\epsilon\chi_{D_2}} d\mathbf{x}$ over $\mathcal{A}_{0,1,t} \subset L^2(D_1^c)$ and so $\chi_{D_2} = \eta(u_{f+\epsilon\chi_{D_2}})$. This yields the second assertion of the theorem and (2.1). □

Theorem 2. *There is a solution $f = 1 + \epsilon\chi_{D_1} + \epsilon\chi_{D_2}$ for (1.7) such that*

$$1 + \epsilon\chi_{D_1} = \underset{D\subset\Omega,\ |D|=A-t}{\arg\min} J(1 + \epsilon\chi_D), \quad 1 + \epsilon\chi_{D_1} + \epsilon\chi_{D_2} = \underset{D\subset D_1^c,\ |D|=t}{\arg\max} J(1 + \epsilon\chi_{D_1} + \epsilon\chi_D), \tag{2.7}$$

where t is a number in $(0, A)$.

Proof. Recall that the minimization problem in (2.7) is a rearrangement optimization problem on rearrangement classes $\mathcal{A}_{1,1+\epsilon,\gamma_t}$ with $\gamma_t = |\Omega| + \epsilon(A - t)$. Then applying Theorem 5.1 in [20], we observe that minimization problem in (2.7) is uniquely solvable. Moreover, using Lemma 2.1, we know that the maximization problem in (2.7) has a solution. Let us define $\xi(t) = J(1 + \epsilon\chi_{D_1} + \epsilon\chi_{D_2})$ for $t \in (0, A)$ then it is inferred that $\xi(t)$ is well-defined. Moreover, it is easy to check that

$$\xi(0) = \check{J}_{\mathcal{A}} \quad \xi(A) = \hat{J}_{\mathcal{A}}.$$

In order to employ the intermediate value theorem, we show that ξ is continuous.

Consider $\{t_n\}_1^\infty$ in $(0, A)$ such that $t_n \to \bar{t}$ as $n \to \infty$. We establish that $\xi(t_n) \to \xi(\bar{t})$ when $n \to \infty$. At first, we show that the solution of the minimization problem in (2.7) corresponding to t_n converge to the solution corresponding to $\bar{t}$. It is noteworthy to mention here that a similar question has been investigated in [24].

For each t_n, problems in (2.7) have solutions $1+\epsilon\chi_{D_1^n}$ and $1+\epsilon\chi_{D_1^n}+\epsilon\chi_{D_2^n}$ respectively. Furthermore, $1 + \epsilon\chi_{\bar{D}_1}$ and $1 + \epsilon\chi_{\bar{D}_1} + \epsilon\chi_{\bar{D}_2}$ are the solutions of the problems in (2.7) corresponding to $\bar{t}$. There are sub-sequences (still denoted by $\{\chi_{D_1^n}\}_1^\infty$, $\{\chi_{D_2^n}\}_1^\infty$) such that

$$\chi_{D_1^n} \rightharpoonup \eta_1, \quad \chi_{D_2^n} \rightharpoonup \eta_2, \quad \text{in } L^2(\Omega), \tag{2.8}$$

as $n \to \infty$. It is easy to check that $1 + \epsilon\eta_1$ belongs to $\mathcal{B}_{1,1+\epsilon,\gamma_{\bar{t}}}$. Consider an arbitrary function $1 + \epsilon\chi_D$ in the rearrangement class $\mathcal{A}_{1,1+\epsilon,\gamma_{\bar{t}}}$. We claim that there is a sequence of functions $\{\chi_{E_n}\}_1^\infty$ where $E_n \subset \Omega$ with $|E_n| = A - t_n$ and $\chi_{E_n} \to \chi_D$ in $L^2(\Omega)$. The sets $\{E_n\}_1^\infty$ can be constructed in the following way. If $A - t_n > A - \bar{t}$, then set $E_n = D \cup F_n$ where $F_n \subset D^c$ with $|F_n| = \bar{t} - t_n$. If $A - t_n <= A - \bar{t}$, then we set $E_n = D \setminus F_n$ where $F_n \subset D$ with $|F_n| = t_n - \bar{t}$. Now in view of weak continuity of J and (2.7), we observe that

$$J(1 + \epsilon\chi_D) = \lim_{n\to\infty} J(1 + \epsilon\chi_{E_n}) \geq \lim_{n\to\infty} J(1 + \epsilon\chi_{D_1^n}) = J(1 + \epsilon\eta_1). \tag{2.9}$$

Let us recall here that due to the weak continuity and strict convexity of the functional J, see [20, Lemma 5.2] , the minimizer in (2.7) is uniquely solvable even considering the minimization problem over the weak closure of the rearrangement class, $\mathcal{B}_{1,1+\epsilon,\gamma_t}$. This fact and (2.9) reveal that $1 + \epsilon\eta_1$ is the unique solution of the minimization problem in (2.7) when $t = \bar{t}$ and so we have $\eta_1 = \chi_{\bar{D}_1}$.

So far, we have shown that the solution of the minimization problems in (2.7) corresponding to t_n converge to $1+\epsilon\chi_{\bar{D}_1}$, the solution of the minimization corresponding to $\bar{t}$. Since $\|\chi_{D_1^n}\|_{L^2(\Omega)} \to \|\chi_{\bar{D}_1}\|_{L^2(\Omega)}$ and in view of (2.8), we have

$$\chi_{D_1^n} \to \chi_{\bar{D}_1}, \quad \text{in } L^2(\Omega), \tag{2.10}$$

invoking a special case of the Radon-Riesz theorem. This yields that

$$\chi_{D_1^n}(\mathbf{x}) \to \chi_{\bar{D}_1}(\mathbf{x}), \quad \text{a.e.} \tag{2.11}$$

Next, we show that the support of function η_2 is a subset of $\bar{D}_1^c$. Consider an entire point $\mathbf{x}_0 \in \bar{D}_1$. Then applying (2.11), we observe that $\mathbf{x}_0 \in D_1^n$ for large n and so $\chi_{D_2^n}(\mathbf{x}_0) = 0$ for such n since $D_2^n \subset (D_1^n)^c$. Hence, we obtain

$$\chi_{D_2^n}(\mathbf{x})\chi_{\bar{D}_1}(\mathbf{x}) \to 0, \quad \text{a.e.} \tag{2.12}$$

Then we see that

$$\int_\Omega \eta_2 \chi_{\bar{D}_1} d\mathbf{x} = \lim_{n\to\infty} \int_\Omega \chi_{D_2^n} \chi_{\bar{D}_1} d\mathbf{x} = 0, \tag{2.13}$$

employing (2.8), (2.12). Therefore, we observe that the support of η_2 is a subset of $\bar{D}_1^c$.

Consider an arbitrary set $D \subset \bar{D}_1^c$ with $|D| = \bar{t}$. One can find a sequence of sets $\{D_n\}_1^\infty$ where $D_n \subset (D_1^n)^c$ with $|D_n| = t_n$ and also $\chi_{D_n} \to \chi_D$ in $L^2(\Omega)$. The sequence of sets $\{D_n\}_1^\infty$ is constructed in the following way. For each n, set $E_n = (D_1^n)^c \cap D$. Remember that $\chi_{(D_1^n)^c} \to \chi_{(\bar{D}_1)^c}$ a.e. applying (2.11) and so $\chi_{E_n} \to \chi_D$ in $L^2(\Omega)$. But, the problem is that may be $|E_n| = \theta_n \neq t_n$. If $\theta_n > t_n$ then we consider a set $F_n \subset E_n$ with $|F_n| = \theta_n - t_n$ and $D_n = E_n \setminus F_n$. If $\theta_n < t_n$ then set $D_n = E_n \cup F_n$ with $|F_n| = t_n - \theta_n$ such that $F_n \subset (D_1^n)^c$ and $|F_n \cap D| = 0$. Now it is easy to check that $\chi_{D_n} \to \chi_D$ in $L^2(\Omega)$. Using this, the weak continuity of J, (2.7) and (2.10), we have

$$J(1+\epsilon\chi_{\bar{D}_1}+\epsilon\chi_D) = \lim_{n\to\infty} J(1+\epsilon\chi_{D_1^n}+\epsilon\chi_{D_n}) \le \lim_{n\to\infty} J(1+\epsilon\chi_{D_1^n}+\epsilon\chi_{D_2^n}) = J(1+\epsilon\chi_{\bar{D}_1}+\epsilon\eta_2). \tag{2.14}$$

Let us recall here that the maximizer in the maximization problem of (2.7) is also a solution when considering the problem over the weak closure of the rearrangement class due to the weak continuity of J. We have shown that the support of η_2 is a subset of $\bar{D}_1^c$ and so it belongs to $\mathcal{B}_{0,1,\bar{t}} \subset L^2(\bar{D}_1^c)$. Expression (2.14) reveals that indeed η_2 is a solution of the maximization problem in (2.7) considering it over the weak closure of the corresponding rearrangement class and so we have

$$J(1+\epsilon\chi_{\bar{D}_1}+\epsilon\eta_2) = J(1+\epsilon\chi_{\bar{D}_1}+\epsilon\chi_{\bar{D}_2}).$$

In summary, we have shown

$$\lim_{n\to\infty} \xi(t_n) = \lim_{n\to\infty} J(1+\epsilon\chi_{D_1^n}+\epsilon\chi_{D_2^n}) = J(1+\epsilon\chi_{\bar{D}_1}+\epsilon\eta_2) = J(1+\epsilon\chi_{\bar{D}_1}+\epsilon\chi_{\bar{D}_2}) = \xi(\bar{t}).$$

Consequently, we obtain that $\xi : [0, A] \to [\check{J}_{\mathcal{A}}, \hat{J}_{\mathcal{A}}]$ is a continuous function and the intermediate value theorem yields that (1.7) has a solution in the form asserted in this theorem. □

Remark 2.2. *In view of Theorem 1 in [21], it is noteworthy that the minimizer in* (2.7) *has the following form*

$$D_1 = \{\mathbf{x} \in \Omega : \ u_{1+\epsilon\chi_{D_1}}(\mathbf{x}) \le \tau\}, \quad \tau = \inf\{s : \ |\{\mathbf{x} \in \Omega : \ u_{1+\epsilon\chi_{D_1}}(\mathbf{x}) \le s\}| \ge A - t\}. \tag{2.15}$$

Also, there is a connected component $\mathcal{D}_0$ of the interior of D_1 hits the boundary, i.e., $\bar{\mathcal{D}}_0 \cap \partial\Omega \neq \emptyset$. Moreover, if $A - t$ is large enough then D_1 contains a tubular neighborhood of the boundary $\partial\Omega$ and $\partial\Omega \subset \partial D_1$. In particular, if $\Omega \subset \mathbb{R}^2$ is simply connected, then D_1 is connected when $A - t$ is large enough.

The next two theorems determine a geometrical property of a solution of (1.7).

Theorem 3. *Let $f = 1 + \epsilon\chi_{D_1} + \epsilon\chi_{D_2}$ be a solution of* (1.7) *derived from Theorem 2 and $u_{1+\epsilon\chi_{D_1}+\epsilon\chi_{D_2}}$ is not constant on $\partial\Omega$. If t is large enough, then both D_1 and D_2 touch the boundary $\partial\Omega$.*

Proof. In view of Remark 2.2 we know that D_1 touches the boundary. When t is large enough then due to (2.1) we infer that D_2 touches the boundary using a method similar to that for proof of Theorem 4-(iv) in [21]. □

Theorem 4. *Let $f = 1 + \epsilon\chi_{D_1} + \epsilon\chi_{D_2}$ be a solution of* (1.7) *obtained in Theorem 2 when $\beta = \infty$, the Dirichlet boundary conditions. If Ω is a simply connected subset of $\mathbb{R}^2$, then D_1 is a connected tubular neighborhood of the boundary $\partial\Omega$.*

Proof. In view of Remark 2.2 and the Dirichlet boundary conditions we see that $\partial\Omega \subset \partial D_1$. This shows that D_1 contains a tubular neighborhood of the boundary $\partial\Omega$. In order to establish that D_1 is connected we argue by contradiction. Assume there is an open subset $\mathcal{D}_0$ of $D_1 = \{\mathbf{x} \in \Omega : \ u_{1+\epsilon\chi_{D_1}}(\mathbf{x}) \le \tau\}$ such that $\partial \mathcal{D}_0 \subset \{\mathbf{x} \in \Omega : \ u_{1+\epsilon\chi_{D_1}}(\mathbf{x}) \geq \tau\}$. Then we see $\partial \mathcal{D}_0 \subset \{\mathbf{x} \in \Omega : \ u_{1+\epsilon\chi_{D_1}}(\mathbf{x}) = \tau\}$. Consequently, $u_{1+\epsilon\chi_{D_1}}$ has a minimum in $\mathcal{D}_0$ and also

$$\begin{cases} -\Delta u_{1+\epsilon\chi_{D_1}}(\mathbf{x}) = 1+\epsilon & \text{in} \quad \mathcal{D}_0, \\ \quad u_{1+\epsilon\chi_{D_1}}(\mathbf{x}) = \tau & \text{on} \quad \partial\mathcal{D}_0, \end{cases} \tag{2.16}$$

which contradicts the maximum principle [25]. □

Next theorem reveals a symmetry property for solutions of problems in (2.7).

Theorem 5. *Let Ω be a Steiner symmetric domain with respect to a hyperplane T and $\beta = \infty$, the Dirichlet boundary conditions. Assume $1+\epsilon\chi_{D_1}$ is the unique solution of the minimization problem in (2.7). Then, D_1^c is a Steiner symmetric domain with respect to hyperplane T. Moreover, the maximization problem in (2.7) has a solution $1+\epsilon\chi_{D_1}+\epsilon\chi_{D_2}$ where D_2 is Steiner symmetric with respect to T.*

Proof. Let $w = u_{1+\epsilon\chi_{D_1}}$. Then it is known that w is the unique minimizer of the following functional over $H_0^1(\Omega)$

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 d\mathbf{x} + \int_\Omega \zeta(u) d\mathbf{x}, \tag{2.17}$$

where $\zeta : \mathbb{R} \to \mathbb{R}$ is a convex and so continuous function [20]. Assume that w^* is a Steiner symmetrization of function w with respect to the hyperplane T. It is known that $w^* \in H_0^1(\Omega)$, and also

$$\int_\Omega |\nabla w|^2 d\mathbf{x} \geq \int_\Omega |\nabla w^*|^2 d\mathbf{x}, \quad \int_\Omega \zeta(w) d\mathbf{x} = \int_\Omega \zeta(w^*) d\mathbf{x}, \tag{2.18}$$

see [2, 26, 27]. Consequently, we observe that

$$\begin{aligned} I(w) &= \frac{1}{2}\int_\Omega |\nabla w|^2 d\mathbf{x} + \int_\Omega \zeta(w) d\mathbf{x} \geq \frac{1}{2}\int_\Omega |\nabla w^*|^2 d\mathbf{x} + \int_\Omega \zeta(w^*) d\mathbf{x} \\ &= I(w^*), \end{aligned}$$

using (2.18). This shows that $w = w^*$ since w is the unique minimizer of $I(u)$. Employing (2.15), we know that there is $\tau > 0$ such that $D_1^c = \{\mathbf{x} \in \Omega : \ w(\mathbf{x}) > \tau\}$. This yields that D_1^c is Steiner symmetric since w is Steiner symmetric with respect to T.

Now we turn to the second assertion of the theorem. Using (2.3), we see that

$$J(f+\epsilon\chi_{D_2}) = \int_\Omega f u_f d\mathbf{x} + 2\epsilon \int_{D_1^c} \chi_{D_2} u_f d\mathbf{x} + \epsilon^2 \int_{D_1^c} \chi_{D_2} u_{\chi_{D_2}} d\mathbf{x}, \tag{2.19}$$

where indeed $f = 1+\epsilon\chi_{D_1}$ and u_f is w. Then, u_f is Steiner symmetric. In the second integral of (2.19), invoking (2.2) and Hardy-Littlewood inequality we have

$$\int_{D_1^c} \chi_{D_2} u_f d\mathbf{x} = \int_\Omega \chi_{D_2} u_f d\mathbf{x} \leq \int_\Omega \chi_{D_2^*} u_f d\mathbf{x} = \int_{D_1^c} \chi_{D_2^*} u_f d\mathbf{x}, \tag{2.20}$$

where D_2^* is the Steiner symmetrization of the set D_2. In the last equality we have used the fact that $D_2^* \subset D_1^c$ since D_1^c is Steiner symmetric with respect to T.

For the third integral of (2.19), let us recall that we have

$$\begin{cases} -\Delta u_{\chi_{D_2}} = \chi_{D_2} & \text{in} \quad \Omega, \\ \quad u_{\chi_{D_2}} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

and by using a method similar to that in the proof of Theorem 5 in [21] one can say

$$\int_{D_1^c} \chi_{D_2} u_{\chi_{D_2}} d\mathbf{x} = \int_\Omega \chi_{D_2} u_{\chi_{D_2}} d\mathbf{x} \leq \int_\Omega \chi_{D_2^*} u_2^* d\mathbf{x}, \tag{2.21}$$

where u_2^* is the solution of (1.1) corresponding to $\chi_{D_2^*}$. Applying (2.20) and (2.21), we deduce that

$$J(f+\epsilon\chi_{D_2}) \leq \int_\Omega f u_f d\mathbf{x} + 2\epsilon \int_{D_1^c} \chi_{D_2^*} u_f d\mathbf{x} + \int_\Omega \chi_{D_2^*} u_2^* d\mathbf{x} = J(f+\epsilon\chi_{D_2^*}),$$

and hence D_2^* is a solution of the maximization problem in (2.7). □

Due to the Steiner symmetry property of D_1^c and its subset D_2 we obtain the following theorem.

Theorem 6. *Let $\Omega \subset \mathbb{R}^N$ be Steiner symmetric with respect to a family of N mutually perpendicular hyperplanes $\{T_i\}_1^N$, $\beta = \infty$ and $f = 1 + \epsilon\chi_{D_1} + \epsilon\chi_{D_2}$ be a solution of (1.7) derived by (2.7) such that D_1^c and D_2 are Steiner symmetric with respect to those hyperplanes. Then,*
i) both functions $u_{1+\epsilon\chi_{D_1}}(\mathbf{x})$ and $u_{1+\epsilon\chi_{D_1}+\chi_{D_2}}(\mathbf{x})$ have a unique maximum point which is the intersection point of $\{T_i\}_1^N$.
ii) The sets D_1^c and D_2 are star-shaped domains.

Proof. The proof of (i)-(ii) can be done using a method similar to that for the proof of Theorem 6 in [21]. □

One can determine a solution for (1.7) when Ω is an N-ball. Define $\mathcal{B}(0,a)$ as a ball in $\mathbb{R}^N$ centered at the origin with radius a. Set $D_2^t = \mathcal{B}(0,r_2)$ such that $|D_2^t| = t$ and $D_1^t = \mathcal{B}(0,a) \setminus \mathcal{B}(0,r_1)$ where the radius r_1 is chosen such that $|D_1^t| = A - t$. It is easy to check that

$$r_2 = (\frac{t}{\sigma_N})^{\frac{1}{N}}, \quad r_1 = (\frac{\sigma_N a^N - A + t}{\sigma_N})^{\frac{1}{N}}, \tag{2.22}$$

where σ_N is the volume of unit ball $\mathcal{B}(0,1)$.

Theorem 7. *Let $\Omega = \mathcal{B}(0,a)$ and $\beta = \infty$. Then there is $\bar{t}$ in $(0,A)$ such that $f_c = 1 + \epsilon\chi_{D_1^{\bar{t}}} + \epsilon\chi_{D_2^{\bar{t}}}$ is a solution for (1.7).*

Proof. We know that Ω is symmetric with respect to all hyper-planes T which pass through the origin. Employing Theorem 5, $D_1^{\bar{t}}$ which is a ring around the boundary is the unique solution of the minimization problem in (2.7). Moreover, $D_2^{\bar{t}}$ is a solution of the maximization problem in (2.7).

□

Remark 2.3. *Indeed, the solution provided by Theorem 2 is one of solutions for (1.7). Although the minimization problem in (2.7) has a unique solution, the maximization problem may have different solutions. Even for the case that $\Omega = \mathcal{B}(0,a)$, we do not have a proof that the maximization problem in (2.7) has a unique solution. However, it has been established that a ball is the only radial maximizer for the maximization problem in (2.7) [21].*

2.1. Explicit Solutions for (1.7)

The explicit solutions for design problem like (1.7) are rare due the fact that we do not have so much information on the topology or geometry of the solution. This section is devoted for explicit solution of (1.7) when the domain is a ball.

First we consider the one-dimensional case $\Omega = (0,1)$. In this case we can find a solution for (1.7) for general β.

Theorem 8. *Let $\Omega = (0,1)$. There is $t \in (0,A)$ such that*

$$f = 1 + \epsilon\chi_{[0,\frac{A-t}{2}]} + \epsilon\chi_{[\frac{1-t}{2},\frac{1+t}{2}]} + \epsilon\chi_{[1-\frac{A-t}{2},1]},$$

is a solution of (1.7).

Proof. Recall that there is $t \in (0, A)$ where (1.7) has a solution in the form (2.7). For one-dimensional domain $\Omega = (0, 1)$, the solution of the minimization problem in (2.7) is available explicitly [21]. Indeed in the minimization problem we have $D_1 = [0, \frac{A-t}{2}] \cup [1 - \frac{A-t}{2}, 1]$ and $u_{1+\epsilon\chi_{D_1}}$ is symmetric around $x = \frac{1}{2}$ and increasing in $[0, \frac{1}{2}]$.

Now setting $f = 1 + \epsilon\chi_{D_1}$, recall that

$$J(f + \epsilon\chi_D) = \int_\Omega f u_f d\mathbf{x} + 2\epsilon \int_{D_1^c} \chi_D u_f dx + \epsilon^2 \int_{D_1^c} \chi_D u_{\chi_D} dx,$$

where

$$\begin{cases} -\Delta u_{\chi_D} = \chi_D & \text{in} \quad \Omega, \\ \frac{\partial u_{\chi_D}}{\partial n} + \beta u_{\chi_D} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

For $D \subset \Omega$ with $|D| = t$, it is inferred by using a method similar to that in the proof of Theorem 5 in [21] that

$$\int_{D_1^c} \chi_D u_{\chi_D} dx \leq \int_{D_1^c} \chi_{D_2} u_{\chi_{D_2}} dx,$$

where $D_2 = [\frac{1-t}{2}, \frac{1+t}{2}]$. Invoking Hardy-Littlewood inequality and the symmetry of u_f, it is observed that

$$\int_{D_1^c} \chi_D u_f dx \leq \int_{D_1^c} (\chi_D)^* u_f^* dx = \int_{D_1^c} \chi_{D_2} u_f dx.$$

Therefore, we can conclude that D_2 is a maximizer for the maximization problem in (2.7). $\square$

In view of Theorem 8, let

$$f = 1 + \epsilon\chi_{[0, \frac{A-t}{2}]} + \epsilon\chi_{[\frac{1-t}{2}, \frac{1+t}{2}]} + \epsilon\chi_{[1-\frac{A-t}{2}, 1]},$$

and it is easy to check that $f \in \mathcal{A}_{m,M,\gamma}$. Inserting this f into (1.1), we obtain

$$u_f(x) = \begin{cases} -(\frac{1+\epsilon}{2})x^2 + (\frac{A\epsilon+1}{2})x + \frac{A\epsilon+1}{2\beta}, & 0 \leq x \leq \frac{A-t}{2}, \\ -\frac{x^2}{2} + \frac{x(1+\epsilon t)}{2} + \frac{\beta\,\epsilon(A-t)^2+4(A\epsilon+1)}{8\beta}, & \frac{A-t}{2} \leq x \leq \frac{1-t}{2}, \\ -(\frac{1+\epsilon}{2})x^2 + (\frac{1+\epsilon}{2})x + \frac{\beta\epsilon(A-1)(A-2t+1)+4(A\epsilon+1)}{8\beta}, & \frac{1-t}{2} \leq x \leq \frac{1+t}{2}, \\ -\frac{x^2}{2} + \frac{x(1-\epsilon t)}{2} + \frac{\beta\,\epsilon(A-t)^2+4\beta\,\epsilon\,t+4(A\epsilon+1)}{8\beta}, & \frac{1+t}{2} \leq x \leq 1 - \frac{A-t}{2}, \\ -(\frac{1+\epsilon}{2})x^2 + (\frac{-A\epsilon+2\epsilon+1}{2})x + \frac{\beta\epsilon(A-1)+A\epsilon+1}{2\beta}, & 1 - \frac{A-t}{2} \leq x \leq 1. \end{cases} \tag{2.23}$$

Using this formula, we derive

$$J(f) = \frac{1}{12\beta}(-3\beta\epsilon(\epsilon+1)(A-1)t^2 + 3\beta\epsilon(A-1)^2 t + A^2(A\beta+6)\epsilon^2 + A\left(-A^2\beta + 3A\beta + 12\right)\epsilon + \beta + 6). \tag{2.24}$$

In order to find a solution for (1.7), we should solve quadratic equation $J(f) = c$ with respect to t. Applying Theorem 8 we know that this equation has a solution $\bar{t}$ in $(0, A)$. It is noteworthy that (2.24) is increasing with respect to t in $(0, A)$ since $A < 1$ and the minimum point of this quadratic equation is $\frac{A-1}{2(\epsilon+1)}$. Therefore, $\bar{t}$ is unique and we have just one solution for (1.7) in the form mentioned in Theorem 8.

Remark 2.4. *We do not have the uniqueness property for the solutions of* (1.7)*. Here, we calculate two other solutions for* (1.7) *when* $\Omega = (0, 1)$*.*

Consider the following function

$$g_t = \chi_{[0,t]} + (1+\epsilon)\chi_{[t, t+\frac{A}{2}]} + \chi_{[t+\frac{A}{2}, 1-(t+\frac{A}{2})]} + (1+\epsilon)\chi_{[1-(t+\frac{A}{2}), 1-t]} + \chi_{[1-t, 1]}, \quad t \in [0, \frac{1}{2} - \frac{A}{2}], \tag{2.25}$$

in the rearrangement class $\mathcal{A}_{m,M,\gamma}$. Theorem 5.1 in [20] and Theorem 4 in [21] tell us that

$$\check{f} = 1 + \epsilon\chi_{[0,\frac{A}{2}]} + \epsilon\chi_{[1-\frac{A}{2},1]}, \quad \hat{f} = 1 + \epsilon\chi_{[\frac{1}{2}-\frac{A}{2},\frac{1}{2}+\frac{A}{2}]},$$

are the unique minimizer and maximizer of $J(f)$ over the set $\mathcal{A}_{m,M,\gamma}$ corresponding to the cases g_0 and $g_{\frac{1}{2}-\frac{A}{2}}$ respectively. Hence, g_t is a continuous path in $\mathcal{A}_{m,M,\gamma}$ connecting $\check{f}$ to $\hat{f}$. Consider

$$J(t) = \int_\Omega g_t u_{g_t} d\mathbf{x},$$

then, we have

$$J(t) = \frac{1}{12\beta}(6A\beta\epsilon t\,(-2t + 2 - A + A\epsilon)\,t + A^2\,(A\beta + 6)\,\epsilon^2 + A\left(-A^2\beta + 3A\beta + 12\right)\epsilon + \beta + 6). \quad (2.26)$$

Since $t \in [0, \frac{1}{2} - \frac{A}{2}]$, we have $(-2t + 2 - A + A\epsilon) > 0$. Then $J(t)$ is a monotone function in t. This shows that the equation $J(t) = c$ has a unique solution $\bar{t} \in [0, \frac{1}{2} - \frac{A}{2}]$. This yields that problem (1.7) has a solution in form (2.25).

The solution in form (2.25) has a symmetry around point $x = \frac{1}{2}$ and one can find an asymmetric solution as well. Consider

$$g_t^1 = (1+\epsilon)\chi_{[0,\frac{A}{2}]} + \chi_{[\frac{A}{2},t]} + (1+\epsilon)\chi_{[t,t+\frac{A}{2}]} + \chi_{[t+\frac{A}{2},1]}, \quad t \in [\frac{A}{2}, 1 - \frac{A}{2}], \quad (2.27)$$

this is a path connecting the decreasing rearrangement of $\check{f}$ which is $f^\Delta = (1+\epsilon)\chi_{[0,A]} + \chi_{[A,1]}$ and $\check{f}$. On the other hand, set

$$g_t^2 = \chi_{[0,t-\frac{A}{2}]} + (1+\epsilon)\chi_{[t-\frac{A}{2},t+\frac{A}{2}]} + \chi_{[t+\frac{A}{2},1]}, \quad t \in [\frac{A}{2}, \frac{1}{2}], \quad (2.28)$$

which defines a path connecting f^Δ and $\hat{f}$. Consequently, if $c \in (\check{J}_{\mathcal{A}}, \hat{J}_{\mathcal{A}})$ then (1.7) has a solution which is in the form g_t^1 or g_t^2 where none of them are symmetric.

Now we derive a solution for (1.7) when $\Omega = \mathcal{B}(0, a)$ and $\beta = \infty$. In view of Theorem 7, we know that f_c is a radial function such that $f_c(r) = 1 + \epsilon\chi_{[0,r_1]}(r) + \epsilon\chi_{[r_2,a]}(r), \quad 0 \le r \le a$ with $r_1 < r_2$. This explicit formula allows to determine the value of J_c and the solution of (1.1) corresponding to $f_c(r)$. Although Theorem 7 is valid when $\beta = \infty$, we derive u_{f_c} and $J(f_c)$ for the general Robin boundary condition. This is because of the fact that our numerical experiments suggest that $f_c(r)$ is also the solution of (1.7) when $\beta < \infty$.

Since f_c is radial, u_{f_c} should be a radial function and indeed it satisfies the following boundary value problem

$$-\frac{1}{r^{N-1}}(r^{N-1}u')' = f_c(r), \quad u'(0) = 0, \quad u'(a) + \beta u(a) = 0. \quad (2.29)$$

Now, integrating this equation we obtain

$$u_{f_c}(r) = \frac{1}{\beta a^{N-1}}\int_0^a s^{N-1} f_c(s)ds + \int_r^a \frac{1}{t^{N-1}}\int_0^t s^{N-1} f_c(s)dsdt. \quad (2.30)$$

Then, one can calculate $u_{f_c}(r)$ explicitly for different N. Using integration in polar coordinates, we have

$$J_c = \int_\Omega f_c u_{f_c} d\mathbf{x} = N\sigma_N \int_0^a r^{N-1} f_c(r) u_{f_c}(r)dr. \quad (2.31)$$

Let us set $N = 2$ and derive the explicit formula for f_c when Ω is a circle. Indeed, we only should calculate the parameter $\bar{t}$ in the formula of f_c mentioned in Theorem 7. Consider $f_c(r) = 1 + \epsilon\chi_{[0,r_1]}(r) +$

$\epsilon\chi_{[r_2,a]}(r)$ where t in formula (2.22) is an arbitrary number in $(0, A)$. Now employing formulas (2.22) and (2.30) for this f_c, we obtain

$$u_f(r) = \begin{cases} \frac{u_1}{4\pi\beta a}, & 0 \le r \le r_1, \\ \frac{u_2}{4\pi\beta a}, & r_1 \le r \le r_2, \\ \frac{u_3}{4\pi\beta a}, & r_2 \le r \le a, \end{cases} \tag{2.32}$$

where

$$u_1 = \epsilon\left((\pi a^2 - A + t)\ln(\pi a^2 - A + t) - (\pi a^2 - A)\ln(\pi a^2) - \ln(t)t\right)a\beta + \pi a^2\left(a\beta + 2\right) + (-r^2(\epsilon+1)\pi + \epsilon A)a\beta + 2\epsilon A,$$
$$u_2 = \beta((\pi a^2 - A + t)\ln((\pi a^2 - A + t)/\pi) - 2(\pi a^2 - A)\ln(a) - 2\ln(r)t)\epsilon a + \pi a^2\left(a\beta + 2\right) + ((A - t)\epsilon - \pi r^2)a\beta + 2\epsilon A,$$
$$u_3 = 2a\beta\epsilon\ln\left(\frac{r}{a}\right)\left(\pi a^2 - A\right) + a\left(\beta\left(\epsilon + 1\right)\left(a^2 - r^2\right) + 2a\right)\pi + 2\epsilon A.$$

Now, employing formula (2.31), it is obtained that

$$J(t) = (\frac{1}{8\pi a\beta})(-2a\beta\epsilon^2\left((A - \pi a^2)^2 - t^2\right)\ln\left(\pi a^2 - A + t\right) + 2a\beta\epsilon^2\left(A - \pi a^2\right)^2\ln\left(\pi a^2\right) - 2\ln\left(t\right)a\beta\epsilon^2 t^2$$
$$+ \pi^2 a^4\left(a\beta + 4\right) - 2\left(\left(A - t\right)\epsilon - 2t\right)\pi\beta\epsilon a^3 + 8\epsilon A\pi a^2 + \left(\left(3A - 2t\right)\epsilon + 2A - 4t\right)\epsilon A a\beta + 4A^2\epsilon^2). \tag{2.33}$$

Solving non-linear equation $J(t) = c$ with respect to t, we derive $\bar{t}$.

Now we assume $N = 3$ and derive the explicit formula for f_c when Ω is a sphere. Again, we only should calculate the parameter $\bar{t}$ in the formula of f_c mentioned in Theorem 7. Similar to that of $N = 2$, we consider $f_c(r) = 1 + \epsilon\chi_{[0,r_1]}(r) + \epsilon\chi_{[r_2,a]}(r)$ where t in formula (2.22) is an arbitrary number in $(0, A)$. Then using formulas (2.22) and (2.30), we have

$$u_f(r) = \begin{cases} \frac{v_1}{6\beta a^2}, & 0 \le r \le r_1, \\ \\ \frac{v_2}{6a^2\beta r}, & r_1 \le r \le r_2, \\ \\ \frac{\left(a^3 + {r_1}^3 - {r_2}^3\right)\epsilon + a^3}{3\beta a^2} + \frac{v_3}{6ar}, & r_2 \le r \le 1, \end{cases} \tag{2.34}$$

where

$$v_1 = \beta\left(1 + \epsilon\right)a^4 + 2\left(\epsilon + 1\right)a^3 - \beta\left(\left(r^2 - 3{r_1}^2 + 3{r_2}^2\right)\epsilon + r^2\right)a^2 + 2\epsilon\left({r_2}^3 - {r_1}^3\right)\left(a\beta - 1\right),$$
$$v_2 = a^3 r\left(1 + \epsilon\right)\left(a\beta + 2\right) - a^2\beta\left(r^3 + 3\epsilon r r_2^2 - 2\epsilon r_1^3\right) + 2\epsilon\left({r_2}^3 - {r_1}^3\right)\left(a\beta - 1\right)r,$$
$$v_3 = \left(a - r\right)\left(ar\left(a + r\right)\left(1 + \epsilon\right) + 2\epsilon\left({r_1}^3 - {r_2}^3\right)\right).$$

Now, applying formula (2.31), it is obtained that

$$J = \frac{4\pi}{45a^2\beta}(a^6\left(a\beta + 5\right)\left(1 + \epsilon\right)^2 + 5\epsilon\left(1 + \epsilon\right)\left(a\beta + 2\right)\left({r_1}^3 - {r_2}^3\right)a^3 + 3\epsilon^2\left(2{r_1}^5 - 5{r_1}^3{r_2}^2 + 3{r_2}^5\right)\beta a^2$$
$$+ 3a^2\epsilon\beta\left({r_2}^5 - {r_1}^5\right) - 5\epsilon^2\left({r_2}^3 - {r_1}^3\right)^2\left(a\beta - 1\right)).$$

Substituting the values of r_1 and r_2 according to formula (2.22) into J we arrive at $J(t)$ which is non-linear with respect to t. Solving $J(t) = c$ we obtain $\bar{t}$.

3. Numerical Discretization, Rearrangement Algorithms, and Numerical Results

As analytical solutions of the intermediate value problem (1.7) can be found only on domains with very simple geometry such as an interval, a disk, and a sphere, we propose an iterative algorithm to compute

Algorithm 1 A pseudo code to find the solution f of Problem (1.7)

Given Ω, m, M, A and c.
(1) Find $\check{J}_{\mathcal{A}} := \min\{J(f) \,:\, f \in \mathcal{A}_{m,M,\gamma}\}$ and $\hat{J}_{\mathcal{A}} := \max\{J(f) \,:\, f \in \mathcal{A}_{m,M,\gamma}\}$.
(2) If $c < \check{J}_{\mathcal{A}}$ or $c > \hat{J}_{\mathcal{A}}$ then stop and report that no such f exists;
else
 use bisection algorithm to find $t \in [0, A]$ which satisfies $J(f) - c = 0$
 where $f = 1 + \epsilon\chi_{D_1} + \epsilon\chi_{D_2}$ and
 $1 + \epsilon\chi_{D_1} = \underset{D\subset\Omega,\ |D|=A-t}{\arg\min} J(1 + \epsilon\chi_D), \quad 1 + \epsilon\chi_{D_1} + \epsilon\chi_{D_2} = \underset{D\subset D_1^c,\ |D|=t}{\arg\max} J(1 + \epsilon\chi_{D_1} + \epsilon\chi_D).$
end

solutions on general domains. The algorithm consists of several essential calculations: the forward solver, the minimum solver, and the maximum solver. Here we will discuss each one in detail.

The forward solver is to find the solution u of Poisson's equation (1.1) when f, β, and Ω are specified. We use a finite element approach which is based on the variational form of (1.1) and approximate u by a piecewise polynomial function. For simplicity, we use polynomial of degree one which leads to a second order convergence for the solution u. Our calculation is implemented by MATLAB partial differential equation toolbox.

We have shown in Theorem 2 that there exists a $t \in (0, A)$ such that f can be found to satisfies the conditions (2.7). We use a bisection algorithm to find this particular t. Thus, we just need to focus on how to determine D_1 and D_2 such that conditions (2.7) are satisfied for a given t.

The minimum solver is to determined D_1 such that

$$1 + \epsilon\chi_{D_1} = \underset{D\subset\Omega,\ |D|=A-t}{\arg\min} J(1 + \epsilon\chi_D), \tag{3.1}$$

for given Ω, t, and A. We use the rearrangement approach, Algorithm 2, proposed in [21] to find the optimal set D_1. The maximum solver is to determined D_2 such that

$$1 + \epsilon\chi_{D_1} + \epsilon\chi_{D_2} = \underset{D\subset D_1^c,\ |D|=t}{\arg\max} J(1 + \epsilon\chi_{D_1} + \epsilon\chi_D), \tag{3.2}$$

for given Ω, D_1, and t. Similarly, we use the rearrangement approach, Algorithm 1, proposed in [21] to find the optimal set D_2. A pseudo code is given in Algorithm 1.

In the following numerical simulations, we choose $m = 1$ and $M = 2$ for all examples. The mesh size will be reported for each individual cases. The stopping criterion is that the absolute value of the difference between the numerical value of J and c is less that 10^{-6}.

In Figure 1, we show the results on a circle with $2,097,152$ triangular elements. The radius of the circle is $a = 2$, $|D| = \pi$ and $\beta = 1$. The theoretical minimal and maximal values are $\check{J}_{\mathcal{A}} \approx 26.7735$ and $\hat{J}_{\mathcal{A}} \approx 32.8974$ as provided by Formula (2.33) when $t \to 0$ and $t = A$. The minimizer is achieved when $t = 0$ and the set D is a ring which attaches to the boundary while the maximizer is achieved when $t = |A|$ and the set D is a disk in the center of the domain, as shown in Figure 1 (a) and (b). We then solve the intermediate value problem (1.7) and choose J to be the mean value of $\check{J}_{\mathcal{A}}$ and $\hat{J}_{\mathcal{A}}$, i.e. $J \approx 29.8355$. A solution f_c of (1.7) is a radial function such that D consists of two regions. One region is a ring attached to the boundary of the circle while the other region is a disk in the center of the domain. In Figure 2, we show the f_c which achieves $(1-c)\check{J}_{\mathcal{A}} + c\hat{J}_{\mathcal{A}}$ for $c = 0.25, 0.5$, and $c = 0.75$, respectively, with the parameter $\beta = 1$ in the Robin boundary condition. When c increases, we observe that the area of the light gray disk gets larger while the ring becomes thinner. This results match the analytical radial solution given in Formula (2.32) with r_1 and r_2 given in Formula (2.22) where t needs to be determined numerically. Similar results are obtained for $\beta = 10$ and β approaching to the infinity (Dirichlet boundary condition) in Figures 3 and 4, respectively.

In Figures 5, 6, and 7, the f_c which achieves $(1-c)\check{J}_{\mathcal{A}} + c\hat{J}_{\mathcal{A}}$ for $c = 0.25, 0.5$, and $c = 0.75$ with $\beta = 1$, $\beta = 10$, and Dirichlet boundary conditions are shown on a unit square, respectively. These calculations

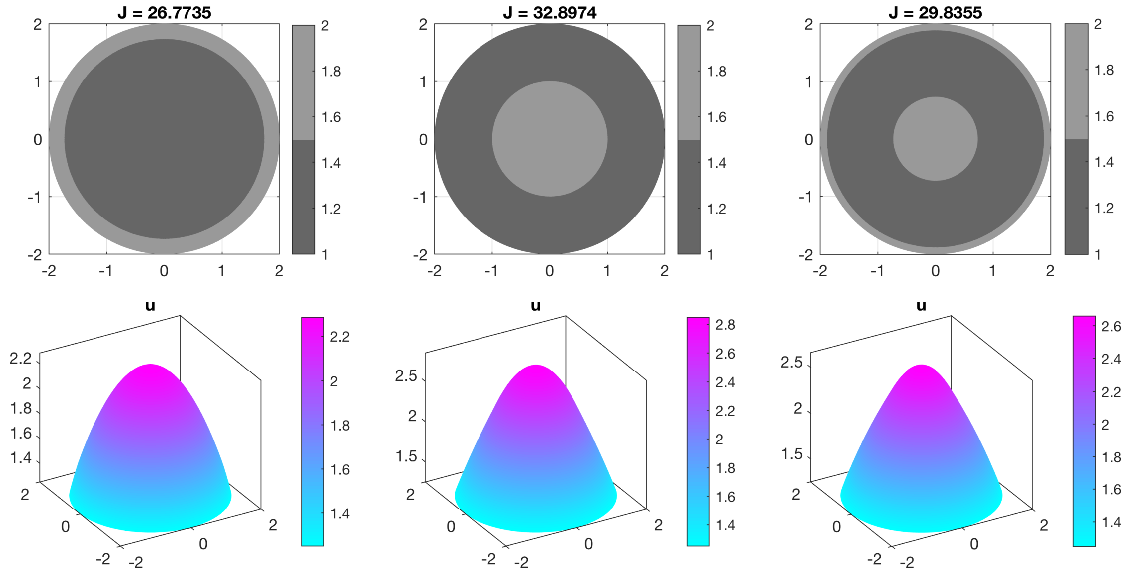

Figure 1: The solutions f and their corresponding u. $\beta = 1$ (a) $\check{J}_{\mathcal{A}}$ (b)$\hat{J}_{\mathcal{A}}$ (c) $J = 0.5(\check{J}_{\mathcal{A}} + \hat{J}_{\mathcal{A}})$

are performed on a triangular mesh with $1,048,576$ elements and $|D| = 0.25$. The set D of the minimizer $\check{f}$ contains a tubular neighborhood of the boundary and is connected while the set D of the maximizer $\hat{f}$ is a disk in the center [21]. When one seeks for f_c which takes an intermediate value between $\check{J}_{\mathcal{A}}$ and $\hat{J}_{\mathcal{A}}$, one can achieve this by interleaving D and D^c. We see that the solutions have D^c (dark gray region) which looks like a ring. This ring may or may not touch the boundary of the domain. When the boundary condition is Dirichlet and $J \neq \hat{J}_{\mathcal{A}}$, this ring does not touch the boundary as shown in Figure 7.

In Figures 8, 9, and 10, the f_c which achieves $(1-c)\check{J}_{\mathcal{A}} + c\hat{J}_{\mathcal{A}}$ for $c = 0.25, 0.5$, and $c = 0.75$ with $\beta = 1$, $\beta = 10$, and Dirichlet boundary conditions are shown on a cross-shaped domain which is Steiner symmetric with respect to x- and y-axis, respectively. These calculations are performed on a triangular mesh with $1,835,008$ elements and $|D| = 0.3|\Omega|$. As discussed in Theorem 5, assuming $1 + \epsilon\chi_{D_1}$ is the unique solution of the minimization problem in (2.7), D_1^c is a Steiner symmetric domain with respect to x- and y-axis. Moreover, the maximization problem in (2.7) has a solution $1 + \epsilon\chi_{D_1} + \epsilon\chi_{D_2}$ where D_2 is Steiner symmetric with respect to x- and y-axis. These D_1 and D_2 are the light gray regions in the figures. Furthermore, as discussed in Theorem 6, the sets D_1^c and D_2 are star-shaped domains.

In Figures 11, 12, and 13, the f_c which achieves $(1-c)\check{J}_{\mathcal{A}} + c\hat{J}_{\mathcal{A}}$ for $c = 0.25, 0.5$, and $c = 0.75$ with $\beta = 1$, $\beta = 10$, and Dirichlet boundary conditions are shown on an ellipse with two circular holes, respectively. These calculations are performed on a triangular mesh with $2,506,752$ elements and $|D| = 0.37|\Omega|$. One can see how the topology of dark gray region changes with respect to different β and c. It is interesting to see that the dark gray region could have one or two holes in these simulation results. It is likely to expect that the dark gray region could even have three holes when J is chosen to be very close to $\check{J}_{\mathcal{A}}$.

The results on an annulus with $2,752,512$ elements and $|D| = 0.5|\Omega|$ are shown in Figures 14, 15, and 16. These results are interesting as they demonstrate that it is possible to have D_1 and D_2 being connected with each other as shown in Figures 14. On an annulus, the set D of the minimizer $\check{f}$ contains two concentric rings with one attached the inner boundary and the other attached the outer boundary while the set D of the maximizer $\hat{f}$ is either an interior ring for large β or forms a connected region which stays on one side of the domain for small β [21]. We see that, when β is small, D_1 and D_2 are connected as shown in Figures 14 although they are disjoint sets. When β is large enough, D_1 and D_2 are disconnected from each other. The set D_1 consists of two concentric rings with one attached the inner

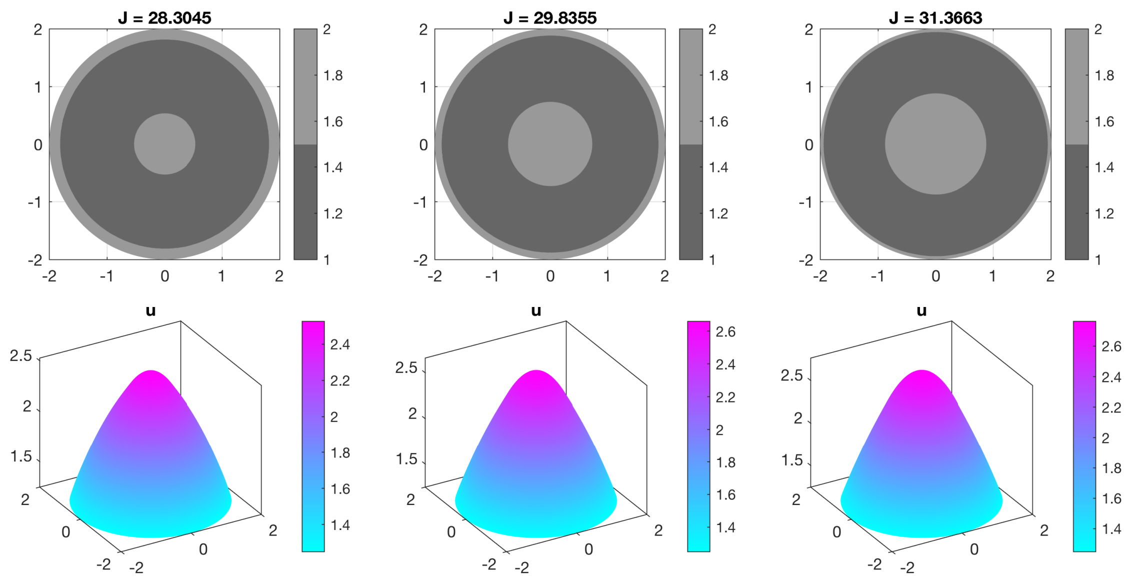

Figure 2: The solutions f and their corresponding u. $\beta = 1$ (a) $J = 0.75\check{J}_{\mathcal{A}} + 0.25\hat{J}_{\mathcal{A}}$ (b) $J = 0.5(\check{J}_{\mathcal{A}} + \hat{J}_{\mathcal{A}})$ (c) $J = 0.25\check{J}_{\mathcal{A}} + 0.75\hat{J}_{\mathcal{A}}$.

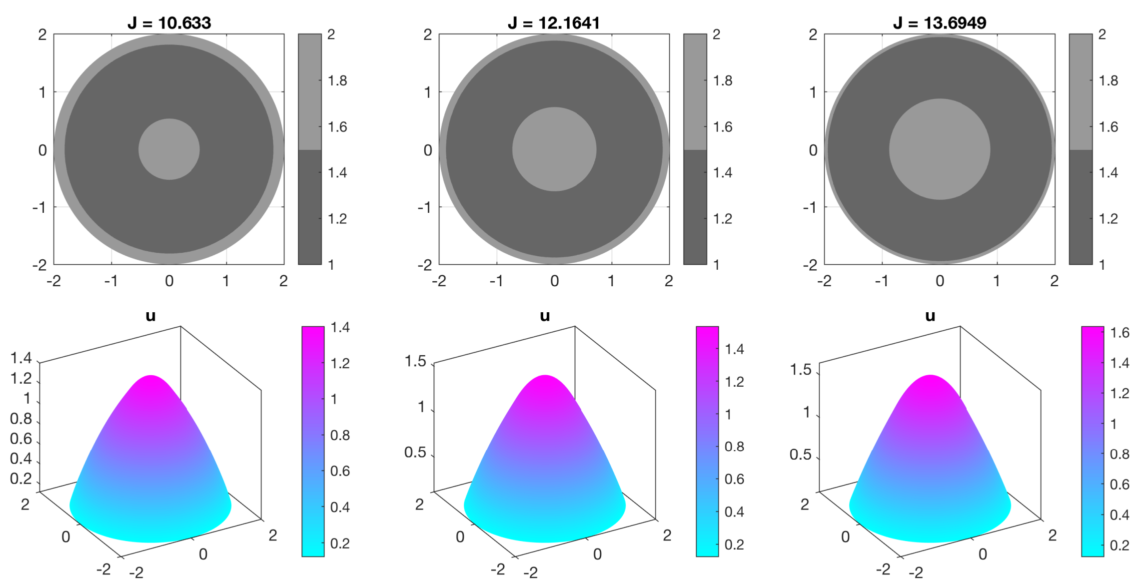

Figure 3: The solutions f and their corresponding u. $\beta = 10$ (a) $J = 0.75\check{J}_{\mathcal{A}} + 0.25\hat{J}_{\mathcal{A}}$ (b) $J = 0.5(\check{J}_{\mathcal{A}} + \hat{J}_{\mathcal{A}})$ (c) $J = 0.25\check{J}_{\mathcal{A}} + 0.75\hat{J}_{\mathcal{A}}$.

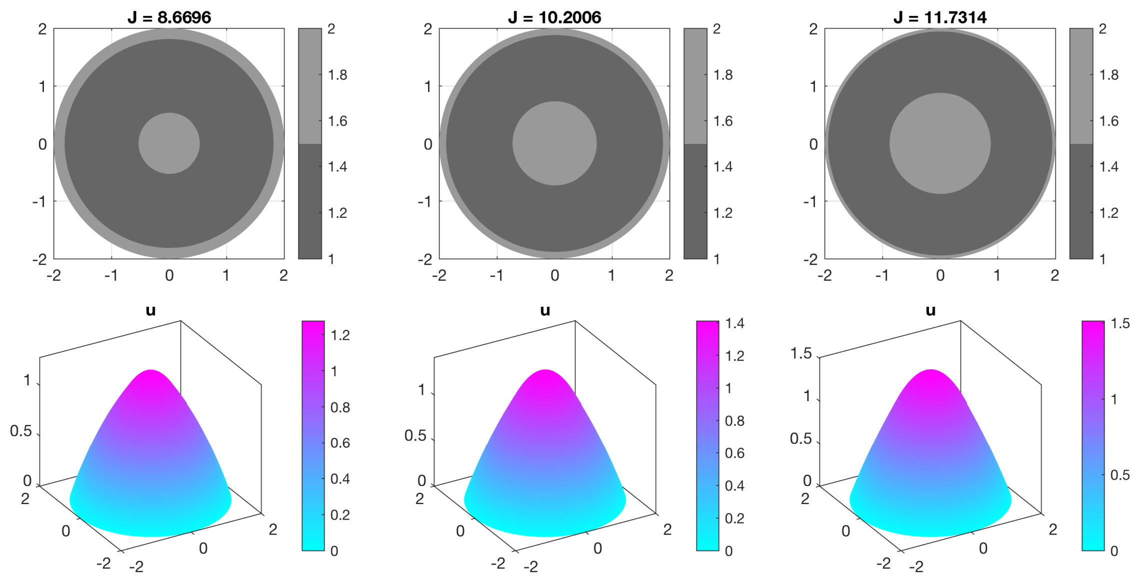

Figure 4: The solutions f and their corresponding u. Dirichlet boundary condition (a) $J = 0.75\check{J}_{\mathcal{A}} + 0.25\hat{J}_{\mathcal{A}}$ (b) $J = 0.5(\check{J}_{\mathcal{A}} + \hat{J}_{\mathcal{A}})$ (c) $J = 0.25\check{J}_{\mathcal{A}} + 0.75\hat{J}_{\mathcal{A}}$.

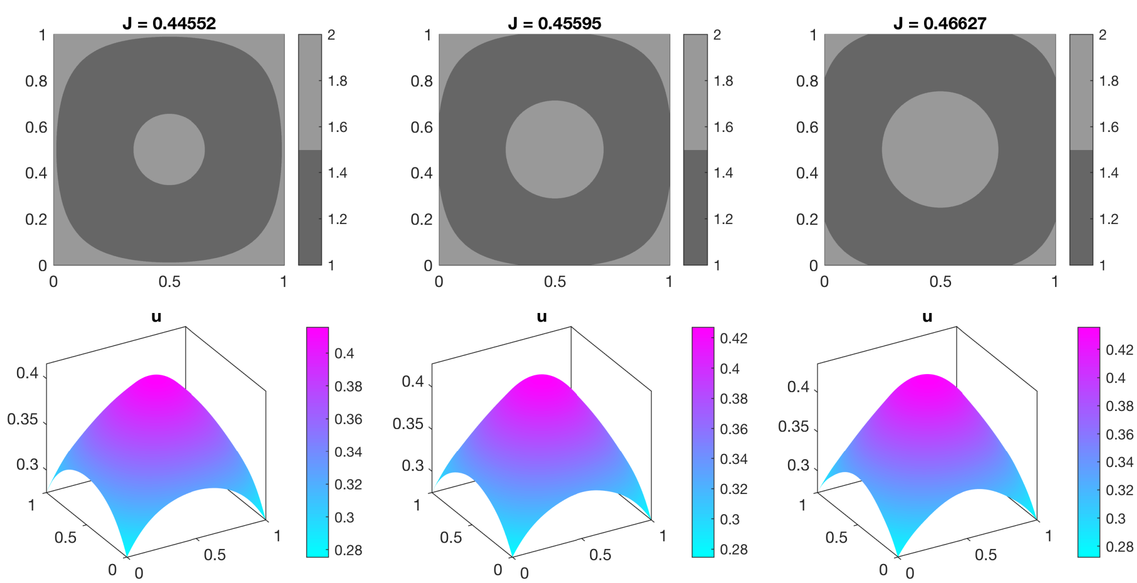

Figure 5: The solutions f and their corresponding u. $\beta = 1$ (a) $J = 0.75\check{J}_{\mathcal{A}} + 0.25\hat{J}_{\mathcal{A}}$ (b) $J = 0.5(\check{J}_{\mathcal{A}} + \hat{J}_{\mathcal{A}})$ (c) $J = 0.25\check{J}_{\mathcal{A}} + 0.75\hat{J}_{\mathcal{A}}$.

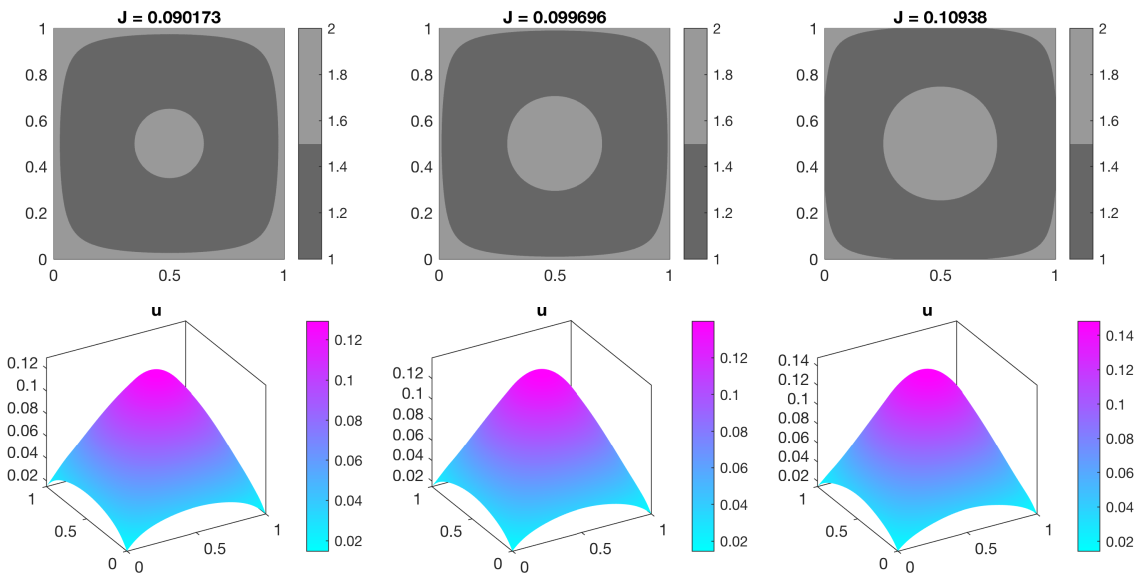

Figure 6: The solutions f and their corresponding u. $\beta = 10$ (a) $J = 0.75\check{J}_{\mathcal{A}} + 0.25\hat{J}_{\mathcal{A}}$ (b) $J = 0.5(\check{J}_{\mathcal{A}} + \hat{J}_{\mathcal{A}})$ (c) $J = 0.25\check{J}_{\mathcal{A}} + 0.75\hat{J}_{\mathcal{A}}$.

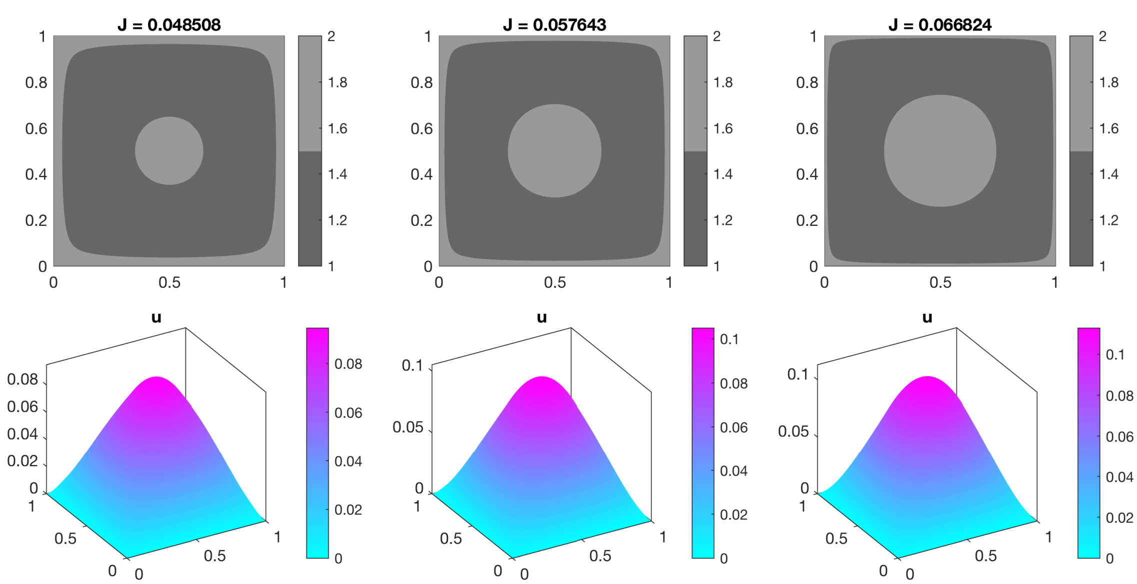

Figure 7: The solutions f and their corresponding u. Dirichlet boundary condition. (a) $J = 0.75\check{J}_{\mathcal{A}} + 0.25\hat{J}_{\mathcal{A}}$ (b) $J = 0.5(\check{J}_{\mathcal{A}} + \hat{J}_{\mathcal{A}})$ (c) $J = 0.25\check{J}_{\mathcal{A}} + 0.75\hat{J}_{\mathcal{A}}$.

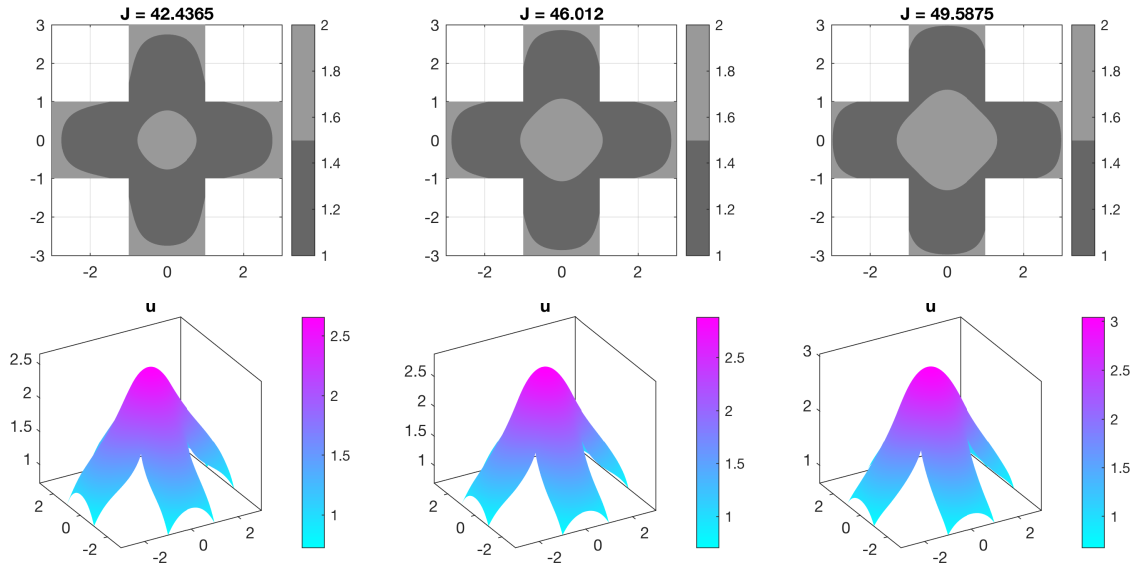

Figure 8: The solutions f and their corresponding u. $\beta = 1$ (a) $J = 0.75\check{J}_{\mathcal{A}} + 0.25\hat{J}_{\mathcal{A}}$ (b) $J = 0.5(\check{J}_{\mathcal{A}} + \hat{J}_{\mathcal{A}})$ (c) $J = 0.25\check{J}_{\mathcal{A}} + 0.75\hat{J}_{\mathcal{A}}$.

boundary and the other attached the outer boundary. The set D_2 could be a simply connected domain (Figures 15 (a-c) and 16 (a)) or an interior ring (Figures 16 (b) and (c)) depending on the choice of β and c. It is noteworthy that for the annulus we do not have uniqueness obviously in view of our numerical results. This is due to the fact that in Figures 14, 15, and 16 (a) a rotation of the light gray domain about the origin by any degree yields another solution.

Acknowledgement

Authors would like to thank Mathematics division, National Center of Theoretical Sciences, Taipei, Taiwan for hosting a research pair program during June 15-June 30, 2019 so they can accomplish this project and initiate new projects on shape optimizations.

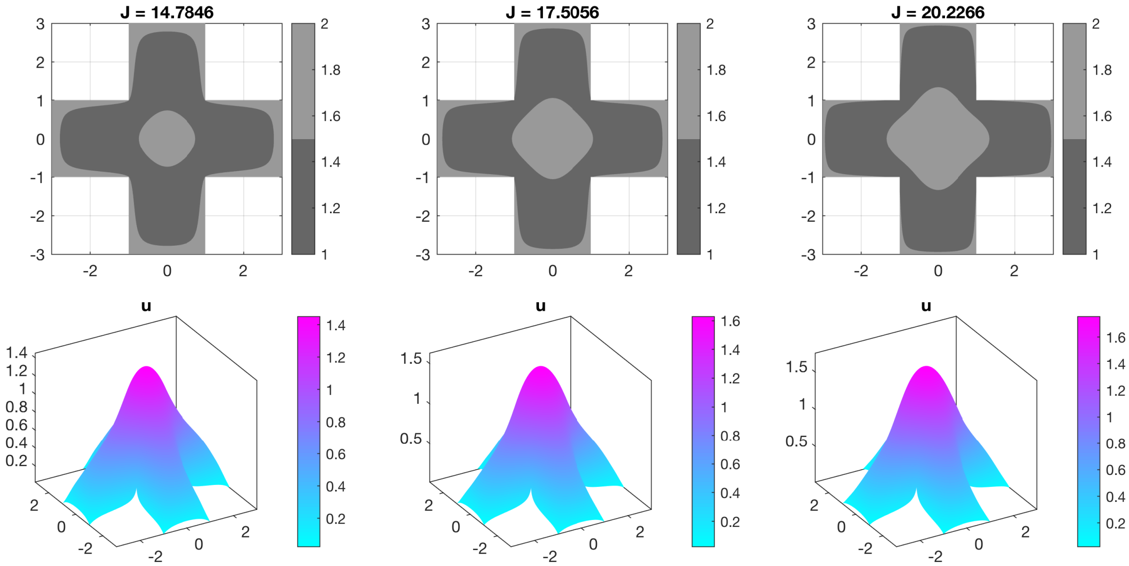

Figure 9: The solutions f and their corresponding u. $\beta = 10$ (a) $J = 0.75\check{J}_{\mathcal{A}} + 0.25\hat{J}_{\mathcal{A}}$ (b) $J = 0.5(\check{J}_{\mathcal{A}} + \hat{J}_{\mathcal{A}})$ (c) $J = 0.25\check{J}_{\mathcal{A}} + 0.75\hat{J}_{\mathcal{A}}$.

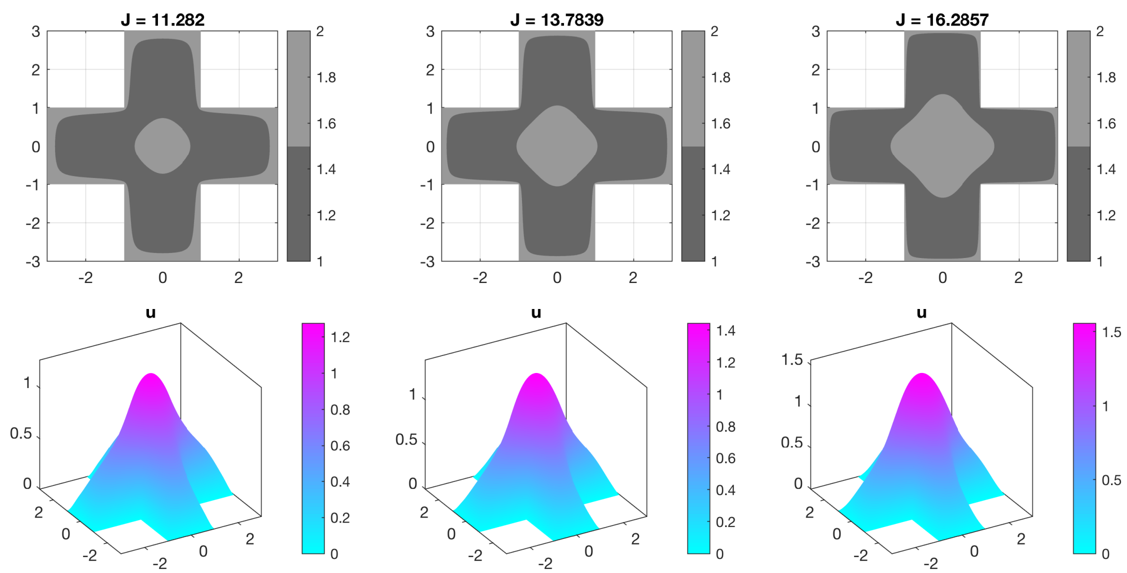

Figure 10: The solutions f and their corresponding u. Dirichlet boundary condition (a) $J = 0.75\check{J}_{\mathcal{A}} + 0.25\hat{J}_{\mathcal{A}}$ (b) $J = 0.5(\check{J}_{\mathcal{A}} + \hat{J}_{\mathcal{A}})$ (c) $J = 0.25\check{J}_{\mathcal{A}} + 0.75\hat{J}_{\mathcal{A}}$.

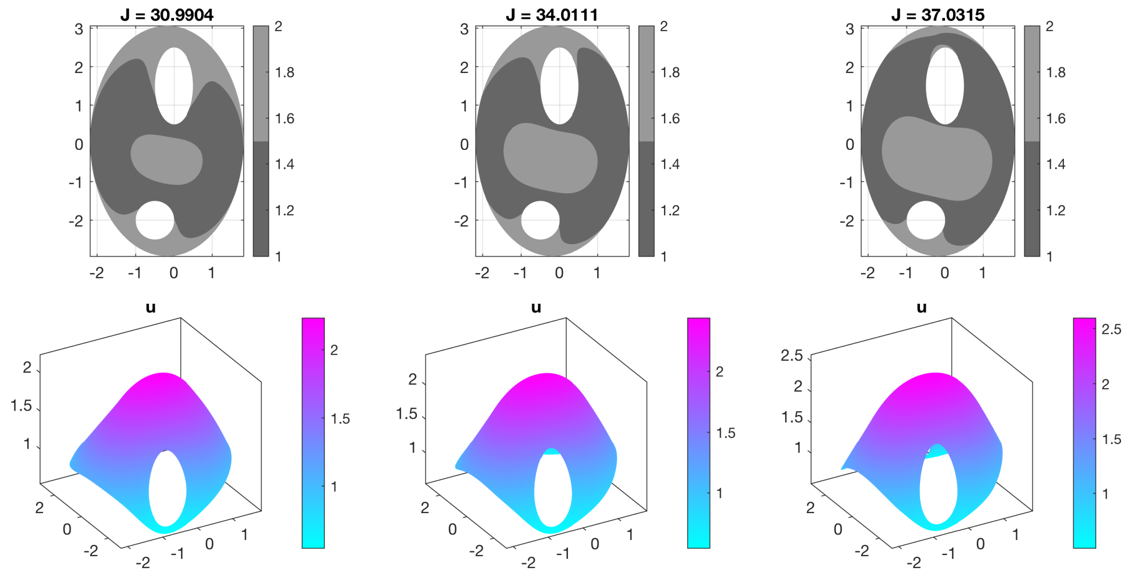

Figure 11: The solutions f and their corresponding u. $\beta = 1$ (a) $J = 0.75\check{J}_{\mathcal{A}} + 0.25\hat{J}_{\mathcal{A}}$ (b) $J = 0.5(\check{J}_{\mathcal{A}} + \hat{J}_{\mathcal{A}})$ (c) $J = 0.25\check{J}_{\mathcal{A}} + 0.75\hat{J}_{\mathcal{A}}$.

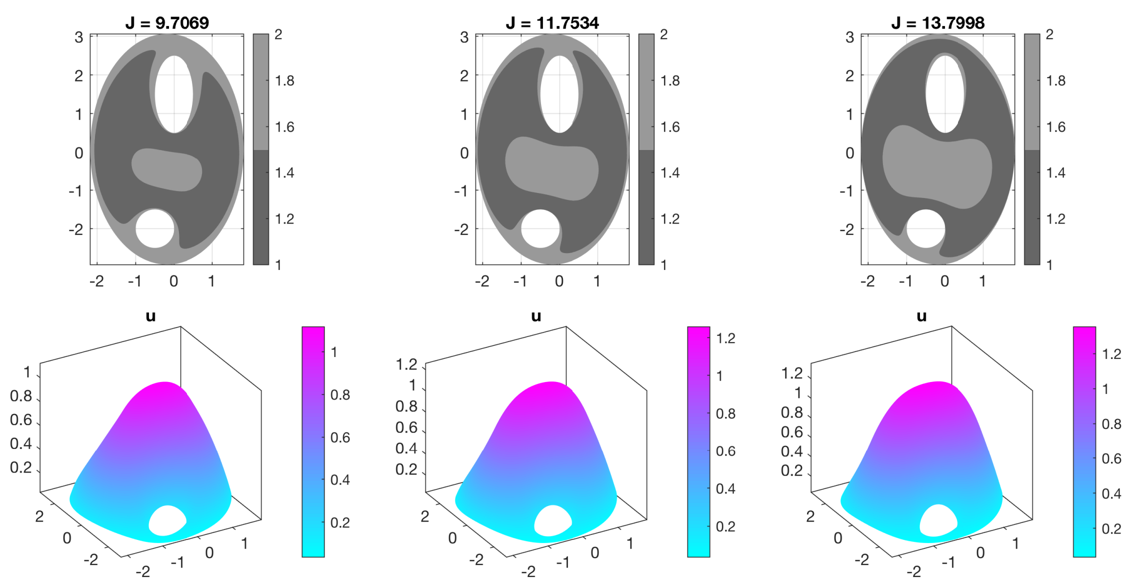

Figure 12: The solutions f and their corresponding u. $\beta = 10$ (a) $J = 0.75\check{J}_{\mathcal{A}} + 0.25\hat{J}_{\mathcal{A}}$ (b) $J = 0.5(\check{J}_{\mathcal{A}} + \hat{J}_{\mathcal{A}})$ (c) $J = 0.25\check{J}_{\mathcal{A}} + 0.75\hat{J}_{\mathcal{A}}$.

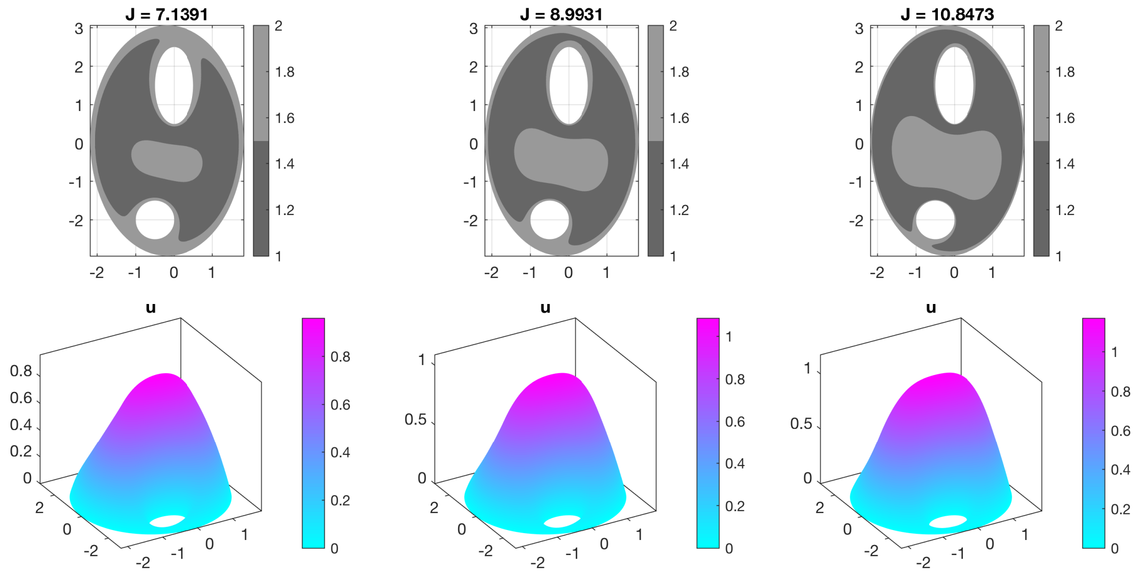

Figure 13: The solutions f and their corresponding u. Dirichlet boundary condition (a) $J = 0.75\check{J}_{\mathcal{A}} + 0.25\hat{J}_{\mathcal{A}}$ (b) $J = 0.5(\check{J}_{\mathcal{A}} + \hat{J}_{\mathcal{A}})$ (c) $J = 0.25\check{J}_{\mathcal{A}} + 0.75\hat{J}_{\mathcal{A}}$.

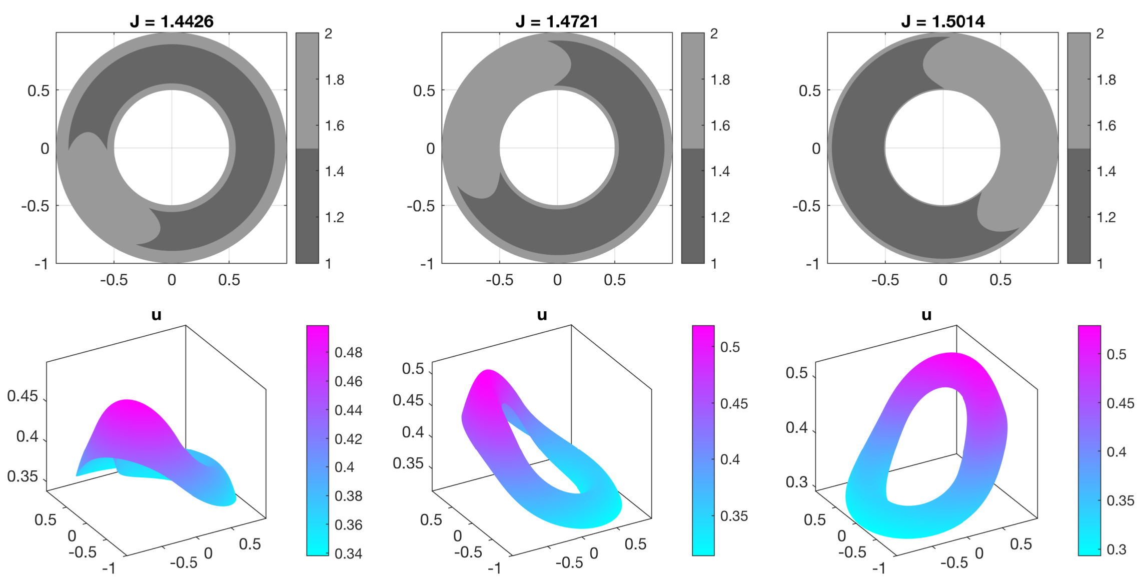

Figure 14: The solutions f and their corresponding u. $\beta = 1$ (a) $J = 0.75\check{J}_{\mathcal{A}} + 0.25\hat{J}_{\mathcal{A}}$ (b) $J = 0.5(\check{J}_{\mathcal{A}} + \hat{J}_{\mathcal{A}})$ (c) $J = 0.25\check{J}_{\mathcal{A}} + 0.75\hat{J}_{\mathcal{A}}$.

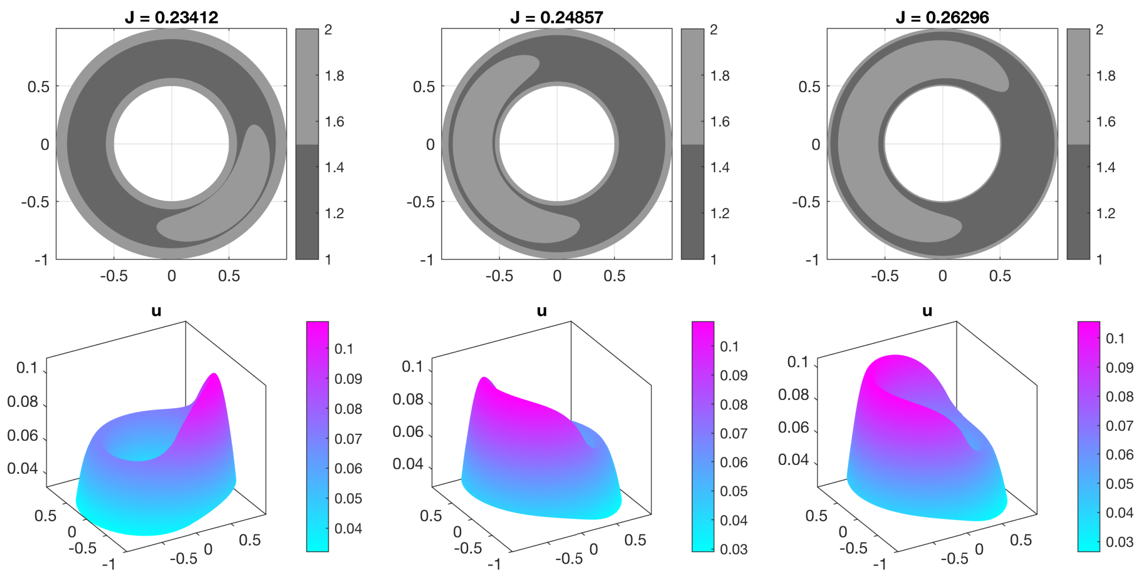

Figure 15: The solutions f and their corresponding u. $\beta = 10$ (a) $J = 0.75\check{J}_{\mathcal{A}} + 0.25\hat{J}_{\mathcal{A}}$ (b) $J = 0.5(\check{J}_{\mathcal{A}} + \hat{J}_{\mathcal{A}})$ (c) $J = 0.25\check{J}_{\mathcal{A}} + 0.75\hat{J}_{\mathcal{A}}$.

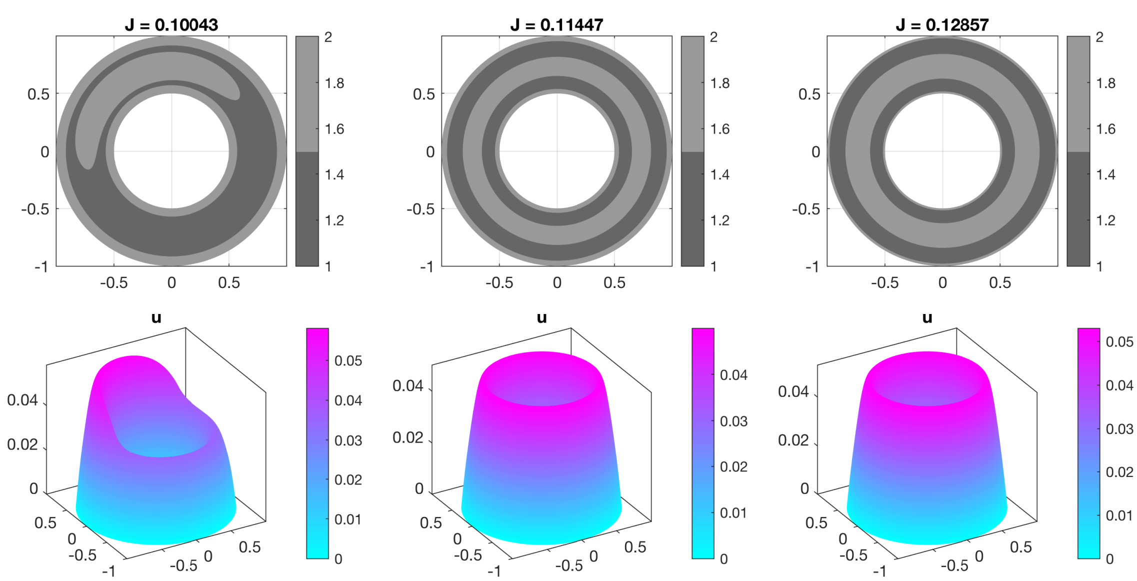

Figure 16: The solutions f and their corresponding u. Dirichlet boundary condition (a) $J = 0.75\check{J}_{\mathcal{A}} + 0.25\hat{J}_{\mathcal{A}}$ (b) $J = 0.5(\check{J}_{\mathcal{A}} + \hat{J}_{\mathcal{A}})$ (c) $J = 0.25\check{J}_{\mathcal{A}} + 0.75\hat{J}_{\mathcal{A}}$.

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