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# ON $K$-STABILITY OF $\mathbb{P}^3$ BLOWN UP ALONG THE DISJOINT UNION OF A TWISTED CUBIC CURVE AND A LINE

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ABSTRACT. We find all $K$-polystable smooth Fano threefolds that can be obtained as blowup of $\mathbb{P}^3$ along the disjoint union of a twisted cubic curve and a line.

CONTENTS



## 1. INTRODUCTION

Smooth Fano threefolds defined over the field $\mathbb{C}$ have been classified in [11, 12, 14, 15] into 105 families. The detailed description of these families can be found in [2]. In [2] the following problem was posed:

**Calabi Problem.** *Find all $K$-polystable smooth Fano threefolds in each family.*

This problem was solved for 72 families. A great contribution to solving this problem was made by the authors of [2]. After their work only 34 families were left. The same year I. Cheltsov and J. Park obtained the result for one more family [3]. Suppose $X$ is a general member of the family №$\mathscr{N}$, then [2, Main Theorem] tells us that

$$X \text{ is } K\text{-polystable} \iff \mathscr{N} \notin \left\{\begin{array}{l} 2.23, 2.26, 2.28, 2.30, 2.31, 2.33, 2.35, 2.36, \\ 3.14, 3.16, 3.18, 3.21, 3.22, 3.23, \\ 3.24, 3.26, 3.28, 3.29, 3.30, 3.31, \\ 4.5, 4.8, 4.9, 4.10, 4.11, 4.12, \\ 5.2 \end{array}\right\}.$$

Suppose $X$ is a member of Family 3.12. Then we describe $X$ as the blowup $\pi : X \to \mathbb{P}^3$ of $\mathbb{P}^3$ at a twisted cubic $C$ and line $L$ that is disjoint from $C$ (see Section 3 for explicit description of all members of this family). These threefolds form a one-dimensional family. Groups of automorphisms of such threefolds are finite except for one threefold which has automorphism group $\mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$. It was shown in [2, §5.18] that this threefold is $K$-polystable. Moreover, it was used to show that the general member of this family is $K$-polystable. Furthermore, in [2, §7.7] it was shown that there exists a non $K$-polystable member in this family and it was conjectured that all other smooth Fano threefolds in Family 3.12 are $K$-polystable. The goal of this work is to prove this conjecture and complete the description of all $K$-polystable smooth Fano threefolds of Picard rank 3 and degree 28 started in [2].

**Main Theorem.** *All the smooth threefolds except one in Family 3.12 are $K$-polystable.*

Hence, all smooth Fano threefolds in Family 2.12 except one described in [2, §7.7] admit a Kähler-Einstein metric.

1.1. **Plan of the paper.** In Section 2 we state the results which will use to prove Main Theorem. In Section 3 we will discuss the equivariant geometry of $\mathbb{P}^3$ which will help us to understand the equivariant geometry of $X$ in Family 3.12. We will focus our attention on the members in Family 3.12 for which the $K$-polystability has not been proved yet. In this section we show that $\mathrm{Aut}(X) \cong \mathrm{Aut}(\mathbb{P}^3, L+C)$ contains a subgroup $G \cong \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. We will show that there are no $G$-fixed points on $\mathbb{P}^3$, describe $G$-invariant quadrics containing $C$ on $\mathbb{P}^3$ and $G$-invariant lines on $\mathbb{P}^3$. At the end of this section we give description of the Mori cone and the cone of effective divisors on $X$. Finally, in Section 4 we prove our Main Theorem.

1.2. **Plan of the proof.** If $X$ is not $K$-polystable then it follows from [20, Corollary 4.14] that there exists a $G$-invariant prime divisor $F$ over $X$ such that $\beta(F) \leqslant 0$ where $\beta(F)$ was defined in [10], see also [2, Definition 1.2.1] and Section 2. Let $Z$ be the center of $F$ on $X$. Then $Z$ is not a point since $X$ has no $G$-fixed points, and $Z$ is not a surface by [9, Theorem 10.1], so that $Z$ is a $G$-invariant irreducible curve. Then we derive a contradiction as follows:

① We use Abban-Zhuang theory (see [1]) and its corollary [2, Collolary 1.7.26] to exclude the case when $\pi(Z)$ is a line such that $\pi(Z) \neq L$ and $\pi(Z) \cap C = \emptyset$. This is done in Lemma 4.1.
② We use Abban-Zhuang theory ([1]) and its corollary [2, Collolary 1.7.26] to exclude the case when $\pi(Z) \subset L$. This is done in Lemma 4.2.
③ We use Abban-Zhuang theory(see [1]) and its corollary [2, Collolary 1.7.26] to show that $\pi(Z)$ is not contained in a $G$-invariant quadric passing through $C$. This is done in Lemma 4.3.
④ Using ①, ②, ③ we deduce that $\pi(Z)$ is not a line.
⑤ It follows from $\beta(F) \leq 0$ that $Z$ is contained in $\mathrm{Nklt}(X, \lambda D)$ for some $G$-invariant effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ and $\lambda \in \mathbb{Q}$ such that $\lambda < \frac{3}{4}$. Moreover it follows from ②, ③ and description of the cone of effective divisors on $X$ that $Z$ is not contained in the surface in $\mathrm{Nklt}(X, \lambda D)$. See Corollary 4.6.
⑥ Using ⑤ we derive that $\pi(Z) \not\subset C$. This is done in Corollary 4.7.
⑦ Finally, we use ⑥ to show that $\pi(Z)$ is a line in $\mathbb{P}^3$, which contradicts ④.

## 2. Preliminary results

Let $X$ be a Fano variety with Kawamata log terminal singularities, let $G$ be a reductive subgroup in $\mathrm{Aut}(X)$, let $f\colon \widetilde{X} \to X$ be a $G$-equivariant birational morphism, let $F$ be a $G$-invariant prime divisor in $\widetilde{X}$, and let $n = \dim(X)$.

**Definition 2.1.** We say that $F$ is a $G$-invariant prime divisor *over* the Fano variety $X$. If $F$ is $f$-exceptional, we say that $F$ is an exceptional $G$-invariant prime divisor *over* $X$. We will denotes the subvariety $f(F)$ by $C_X(F)$.

Let

$$S_X(F) = \frac{1}{(-K_X)^n} \int_0^{\tau} \mathrm{vol}(f^*(-K_X) - xF)dx,$$

where $\tau = \tau(F)$ is the pseudo-effective threshold of $F$ with respect to $-K_X$, i.e. we have

$$\tau(F) = \sup\Big\{u \in \mathbb{Q}_{>0} \ \Big|\ f^*(-K_X) - uF \text{ is big}\Big\}. \tag{2.0.1}$$

Let $\beta(F) = A_X(F) - S_X(F)$, where $A_X(F)$ is the log discrepancy of the divisor $F$.

**Theorem 2.2** ([20, Corollary 4.14])**.** *Suppose that $\beta(F) > 0$ for every $G$-invariant prime divisor $F$ over $X$. Then $X$ is K-polystable.*

**Theorem 2.3** ([9, Theorem 10.1])**.** *Let $X$ be any smooth Fano threefold that is not contained in the following 41 deformation families:*

*№1.17, №2.23, №2.26, №2.28, №2.30, №2.31, №2.33, №2.34, №2.35, №2.36, №3.9, №3.14, №3.16, №3.18, №3.19, №3.21, №3.22, №3.23, №3.24, №3.25, №3.26, №3.28, №3.29, №3.30, №3.31, №4.2, №4.4, №4.5, №4.7, №4.8, №4.9, №4.10, №4.11, №4.12, №5.2, №5.3, №6.1, №7.1, №8.1, №9.1, №10.1.*

*Then $S_X(Y) < 1$ for every irreducible surface $Y \subset X$, i.e. $X$ is divisorially stable.*

**Theorem 2.4** ([2, Collolary 1.7.26])**.** *Let $X$ be a smooth Fano threefold, let $Y$ be an irreducible normal surface in the threefold $X$, let $Z$ be an irreducible curve in $Y$, and let $F$ be a prime divisor over the threefold $X$ such that $C_X(F) = Z$. Then*

$$\frac{A_X(F)}{S_X(F)} \geqslant \min\left\{\frac{1}{S_X(Y)}, \frac{1}{S\big(W^Y_{\bullet,\bullet}; Z\big)}\right\} \tag{2.0.2}$$

*and*

$$S\big(W^Y_{\bullet,\bullet}; Z\big) = \frac{3}{(-K_X)^3}\int_0^{\tau} \big(P(u)^2\cdot Y\big)\cdot \mathrm{ord}_Z\Big(N(u)\big|_Y\Big)du + \frac{3}{(-K_X)^3}\int_0^{\tau}\int_0^{\infty} \mathrm{vol}\big(P(u)\big|_Y - vZ\big)dvdu,$$

*where $P(u)$ is the positive part of the Zariski decomposition of the divisor $-K_X - uY$, and $N(u)$ is its negative part.*

**Lemma 2.5** ([2, Lemma 1.4.4])**.** *Let $X$ be a Fano variety with at most Kawamata log terminal singularities of dimension $n \geqslant 2$, $Z$ be a proper irreducible subvariety in $X$. Let $f\colon \widetilde{X} \to X$ be an arbitrary $G$-equivariant birational morphism, let $F$ be a $G$-invariant prime divisor in $\widetilde{X}$ such that $Z \subseteq f(F)$, and let $\tau(F)$ satisfy 2.0.1. Suppose in addition that $X$ is smooth and $\dim(Z) \geqslant 1$. Then*

$$\frac{A_X(F)}{S_X(F)} > \frac{n+1}{n}\alpha_{G,Z}(X),$$

*where*

$$\alpha_{G,Z}(X) = \sup\left\{\lambda\in\mathbb{Q}\ \middle|\ \begin{array}{l}\textit{the pair } (X,\lambda D) \textit{ is log canonical at general point of } Z \textit{ for any}\\ \textit{effective } G\textit{-invariant } \mathbb{Q}\textit{-divisor } D \textit{ on } X \textit{ such that } D\sim_{\mathbb{Q}} -K_X\end{array}\right\}.$$

**Lemma 2.6** ([2, Corollary A.13]). *Suppose $X=\mathbb{P}^3$ and $B_X\sim_{\mathbb{Q}}-\lambda K_X$ for some rational number $\lambda<\frac{3}{4}$. Let $Z$ be the union of one-dimensional components of* $\mathrm{Nklt}(X,B_X)$. *Then $\mathcal{O}_{\mathbb{P}^3}(1)\cdot Z\leqslant 1$.*

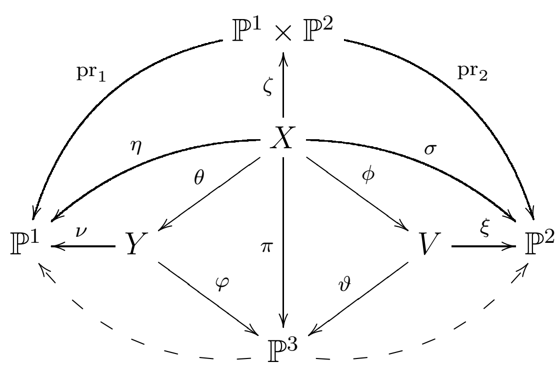
**Lemma 2.7** ([2, Corollary A.15]). *Suppose that $X$ is a smooth Fano threefold, $-K_X$ is nef and big, $B_X\sim_{\mathbb{Q}}-\lambda K_X$ for some rational number $\lambda<1$, and there exists a surjective morphism with connected fibers $\phi\colon X\to\mathbb{P}^1$. Set $H=\phi^*(\mathcal{O}_{\mathbb{P}^1}(1))$. Let $Z$ be the union of one-dimensional components of* $\mathrm{Nklt}(X,\lambda B_X)$. *Then $H\cdot Z\leqslant 1$.*

## 3. Geometry of Fano Threefolds in Family №3.12

3.1. **Basic properties of Fano Threefolds in Family №3.12.** Let $C$ be the smooth twisted cubic curve in $\mathbb{P}^3$ that is the image of the map $\mathbb{P}^1\hookrightarrow\mathbb{P}^3$ given by

$$[x:y]\to[x^3:x^2y:xy^2:y^3]$$

let $L$ be a line in $\mathbb{P}^3$ that is disjoint from $C$, and let $\pi:X\to\mathbb{P}^3$ be the blow up of $\mathbb{P}^3$ along $C$ and $L$. Then $X$ is *a Fano threefold in family 3.12* and all threefolds in this family can be obtained this way. Note that there exists the following commutative diagram:




Where:

- $\varphi$ is the blowup of a line $L$,
- $\vartheta$ is the blowup of a curve $C$,
- $\phi$ is the blowup of a curve $\vartheta^*L$,
- $\theta$ is the blowup of a curve $\varphi^*C$,
- the left dashed arrow is the linear projection from the line $L$,
- the right dashed arrow is given by the linear system of quadrics that contain $C$,
- $\xi$ is a $\mathbb{P}^1$-bundle,
- $\nu$ is a $\mathbb{P}^2$-bundle,
- $\sigma$ is a non-standard conic bundle,
- $\eta$ is a fibration into the del Pezzo surfaces of degree 6,
- $\zeta$ is the contraction of the proper transforms of the quartic surface in $\mathbb{P}^3$ that is spanned by the secants of the curve $C$ that intersect $L$,
- pr$_1$ and pr$_2$ are projections to the first and the second factors, respectively.

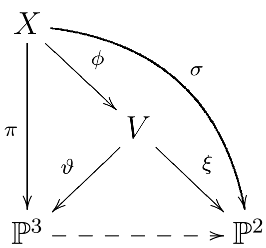
Let $H$ be a plane in $\mathbb{P}^3$, $E_L$ be the exceptional surface of $\pi$ that is mapped to $L$, $E_C$ be the exceptional surface of $\pi$ that is mapped to $C$, $R$ be $\zeta$-exceptional surface. Then

$$R \sim_{\mathbb{Q}} \pi^*(4H) - 2E_C - E_L,$$

and

$$-K_X \sim_{\mathbb{Q}} \pi^*(4H) - E_C - E_L.$$

3.2. **Construction of $R$.** Consider the commutative diagram:




Where $\xi$ is a $\mathbb{P}^1$-bundle given by the linear system $|\vartheta^*(2H) - E_C|$, $\vartheta$ is the blowup of $C$ and the dashed arrow is given by the linear system of quadrics containing $C$, $\phi$ is the blowup of $\vartheta^* L$. Denote $\tilde{L} = \vartheta^* L$. What is the image of $\tilde{L}$ in $\mathbb{P}^2$? We have that

$$\tilde{L} \cdot (\vartheta^*(2H) - E_C) = 2$$

which means that $\xi(\tilde{L})$ is a conic. The preimage of this conic are all the secants of $C$ which intersect $\tilde{L}$. Therefore, $\pi(R)$ is spanned by secants of $C$ that intersect $L$. Note that the class of the preimage is

$$\xi^*(\mathcal{O}_{\mathbb{P}^2}(2)) = 2(\vartheta^*(2H) - E_C) = \vartheta^*(4H) - 2E_C.$$

Note that moreover that $\xi(\tilde{L})$ is a smooth conic and $\xi$ is a $\mathbb{P}^1$-bundle thus the preimage of $\xi(\tilde{L})$ is a smooth surface so it is smooth along $\tilde{L}$ thus the class of $R$ in $\mathbb{P}^3$ is given by

$$R \sim_{\mathbb{Q}} \pi^*(4H) - 2E_C - E_L.$$

3.3. **$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-action on $\mathbb{P}^3$ and Fano threefolds in Family 3.12.** Note that $\mathrm{Aut}(X) \cong \mathrm{Aut}(X, C + L)$. On the other hand, we have

$$\mathrm{Aut}(\mathbb{P}^3, C) = \mathrm{PGL}_2(\mathbb{C}),$$

where $\mathrm{Aut}(\mathbb{P}^3, C)$ is the group of automorphisms of $\mathbb{P}^3$ which fix $C$ as a set.

3.3.1. *Types of threefolds in Family 3.12.* We look at the projection from the line $L$ which is disjoint from $C$ to $\mathbb{P}^1$:

$$\phi_L : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1,$$

which gives a 3-cover of $\mathbb{P}^1$:

$$\phi_L|_C : C \xrightarrow{3:1} \mathbb{P}^1.$$

Then by Riemann-Hurwitz we have that the degree of the ramification divisor is 4. The multiplicity in each ramification point is either 2 or 3. So we have 3 options for ramification points:

- there are two ramification points both of multiplicity 3,

- there is one ramification point of multiplicity 3 and two ramification points of multiplicity 2,
- there are four ramification points of multiplicity 2.

We see that there are at least two ramification points on $C$. By acting on $C$ by the $\mathrm{PGL}(2,\mathbb{C})$ we can make these points to be $p_1 = [1:0]$, $p_2 = [0:1]$ on $C$. Now we look at the line $L$. It is the intersection of 2 planes which are tangent to $C$ at points $p_1$ and $p_2$ (note that these planes are different since the plane intersects the cubic $C$ in three points, so the same plane cannot be tangent to $C$ at two points) so it is given by the equations:

$$L : \begin{cases} x_0 = r_1 x_1, \\ x_3 = r_2 x_2. \end{cases}$$

Now we have 3 cases:
(1) $r_1 = r_2 = 0$ so $L$ is given by the equations:

$$L : \begin{cases} x_0 = 0, \\ x_3 = 0. \end{cases}$$

Here we have two ramification points of multiplicity 3. This case was described in [2]. The corresponding threefold $X$ is $K$-polystable in this case.
(2) $r_1 = 0$, $r_2 \neq 0$ (which is symmetric to the case $r_1 \neq 0$, $r_2 = 0$) so $L$ is given by the equations:

$$L : \begin{cases} x_0 = 0, \\ x_3 = r_2 x_2. \end{cases}$$

Using the action of $\mathbb{C}^*$ by the matrix which fixes $C$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_2^2 & 0 \\ 0 & 0 & 0 & r_2^3 \end{pmatrix}$$

We can assume that $L$ is given by

$$L : \begin{cases} x_0 = 0, \\ x_3 = x_2. \end{cases}$$

Here we have one ramification point of multiplicity 3 and two ramification points of multiplicity 2. This case was described in [2] where it was proved in that $X$ is not $K$-polystable.
(3) $r_1 \neq 0$, $r_2 \neq 0$ so $L$ is given by the equations

$$L : \begin{cases} x_0 = r_1 x_1, \\ x_3 = r_2 x_2. \end{cases}$$

Using the action of $\mathbb{C}^*$ by the matrix which fixes $C$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 \\ 0 & 0 & 0 & \lambda^3 \end{pmatrix},$$

where $\lambda$ satisfies $\lambda^2 = \frac{r_2}{r_1}$. We can assume that $L$ is given by

$$L : \begin{cases} x_0 = rx_1, \\ x_3 = rx_2. \end{cases}$$

Note that:

- $r \neq 0$ since otherwise we are in case (1),
- $r \neq \pm 1$ since otherwise $L$ intersects $C$ which is prohibited,
- $r \neq \pm 3$ since otherwise there exists a plane containing $L$ which is tangent to $C$ with multiplicity 3 (it is a plane given by $x_3 + 3x_3 + 3x_1 + x_0 = 0$ in case $r = -3$ and a plane $-x_3 + 3x_3 - 3x_1 + x_0 = 0$ in case $r = 3$) so this case is projectively isomorphic to the case (2).

Now the involution on $\mathbb{P}^3$ given by $[x_0 : x_1 : x_2 : x_3] \to [x_3 : x_2 : x_1 : x_0]$ fixes $C$ an $L$. We can do it for any pair of four ramification points on $C \cong \mathbb{P}^1$. This gives the action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. More precisely this group is generated by the involutions viewed on $\mathbb{P}^1$:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -\frac{r(r^2-5+(r^4-10r^2+9)^{1/2})}{2(r^2-3+(r^4-10r^2+9)^{1/2})} \\ \frac{r^2+3+(r^4-10r^2+9)^{1/2}}{4r} & -1 \end{pmatrix}$$

The action on $\mathbb{P}^3$ is given by the map induced by $[x : y] \to [x^3 : x^2y : xy^2 : y^2]$.

3.3.2. *$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-fixed points on $X$.* From now on we assume until the end of this section that we are in case (3) of the previous part and $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In particular, $\mathrm{Aut}(X)$ is finite.

**Lemma 3.1.** *There are no $G$-invariant planes in $\mathbb{P}^3$*

*Proof.* Note that $G \hookrightarrow \mathrm{Aut}(C)$ since $C$ is a spatial curve. If there exists a $G$-invariant plane $\Pi$ consider the intersection of $\Pi$ with $C$. There are three points in $\Pi \cap C$ counted with multiplicities. Thus, since the order of $G$ is 4 then there is a $G$-fixed point on $C \cong \mathbb{P}^1$, which is a contradiction. □

**Corollary 3.2.** *There are no $G$-fixed points in $\mathbb{P}^3$.*

**Corollary 3.3.** *The threefold $X$ does not contain $G$-invariant points.*

3.3.3. *$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-invariant Quadrics Containing $C$.* Let $\mathcal{M}$ be the linear system of quadrics in $\mathbb{P}^3$ that contain the curve $C$.

**Lemma 3.4.** *The linear system $\mathcal{M}$ is 3-dimensional, it contains exactly 3 $G$-invariant surfaces, and these surfaces are smooth.*

*Proof.* Note that all this statement does not depend on the equivariant choice of coordinates. We know that groups isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ are conjugate in $\mathrm{PGL}_2(\mathbb{C})$ so we can choose coordinates such that the generators of our group will look like:

$$\tau_1 : [x : y] \to [y : x],$$

$$\tau_2 : [x : y] \to [x : -y].$$

Which gives us the action on $\mathbb{P}^3$ by:

$$\tau_1 : [x_0 : x_1 : x_2 : x_3] \to [x_3 : x_2 : x_1 : x_0],$$

$$\tau_2 : [x_0 : x_1 : x_2 : x_3] \to [x_0 : -x_1 : x_2 : -x_3].$$

The linear system $\mathcal{M}$ is clearly 3-dimensional. We can provide the equations for 3 $G$-invariant quadrics containing $C$:

$$Q_1 : x_0x_3 = x_1x_2, \quad Q_2 : x_1^2 + x_2^2 = x_0x_2 + x_1x_3, \quad Q_3 : x_1^2 - x_2^2 = x_0x_2 - x_1x_3.$$

Note that $(\tau_1, \tau_2)$ acts on the equation of:

- $Q_1$ by multiplying it by $(1, -1)$,
- $Q_2$ by multiplying it by $(1,\ 1)$,
- $Q_3$ by multiplying it by $(-1, 1)$.

Thus since all the pairs are different and $\mathcal{M}$ is 3-dimensional there are exactly 3 $G$-invariant quadrics which we listed above. Note that these quadrics are smooth. □

Now take a $G$-invariant quadric $Q \in \mathcal{M}$ and look at the intersection of it with $L$. Note that $L \not\subset Q$ since $L$ does not intersect $C$. $Q \cap L$ cannot be one point since we do not have $G$-fixed points thus The intersection $Q \cap L$ consists of two distinct points. This pair of points does not lie on curves of bidegree $(1, 0)$ or $(0, 1)$ (since these are the lines on $Q$ and we know that $L \not\subset Q$). Now we see that the blowup $\tilde{Q} \to Q$ at these points is a del Pezzo surface of degree 6.

3.3.4. *$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-invariant lines.* Let us describe $G$-invariant lines in $\mathbb{P}^3$. As in proof of Lemma 3.4, we may assume that $G$ is generated by

$$\tau_1 : [x : y] \to [y : x],$$

$$\tau_2 : [x : y] \to [x : -y].$$

Which gives us the action on $\mathbb{P}^3$ by:

$$\tau_1 : [x_0 : x_1 : x_2 : x_3] \to [x_3 : x_2 : x_1 : x_0],$$

$$\tau_2 : [x_0 : x_1 : x_2 : x_3] \to [x_0 : -x_1 : x_2 : -x_3].$$

In this case all $G$-invariant lines are of the form:

$$\begin{cases} \lambda x_0 + \mu x_2 = 0, \\ \lambda x_3 + \mu x_1 = 0, \end{cases}$$

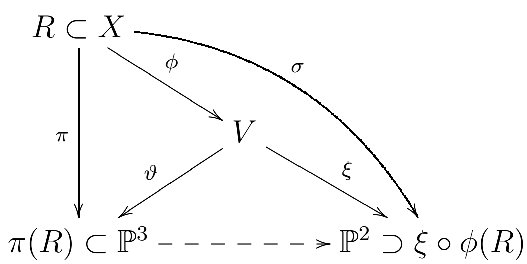
where $[\lambda : \mu] \in \mathbb{P}^1$. All such lines do not intersect each other and lie on the quadric $Q_4$ given by $x_1x_0 = x_2x_3$. We see that $\mathbb{P}^3$ contains infinitely may $G$-invariant lines and all of them are contained in $Q_4$. Among them there are 3 lines that intersect $C$. We can describe them explicitly. Let's now look at the intersection of this quadric with $C$. There are exactly 6 such points:

$$\begin{aligned} P_1 &= [0:1] = [0:0:0:1], \\ P_2 &= [1:0] = [1:0:0:0], \\ P_3 &= [1:1] = [1:1:1:1], \\ P_4 &= [1:-1] = [1:-1:1:-1], \\ P_5 &= [1:i] = [1:i:-1:-i], \\ P_6 &= [1:-i] = [1:-i:-1:i], \end{aligned}$$

here in the third column are given the corresponding coordinates on $C \subset \mathbb{P}^3$. Note that $\tau_1$ exchanges $P_1$ and $P_2$, $\tau_2$ exchanges $P_3$ and $P_4$, $\tau_2$ exchanges $P_5$ and $P_6$ and we obtain three pairs of points which belong to the same line (different one for each pair). We denote these lines $L_{12}$, $L_{34}$, $L_{56}$, where $L_{ij}$ is the line connecting points $P_i$ and $P_j$.

**Lemma 3.5.** *Suppose that $Z$ is an irreducible curve on $X$, $\pi(Z)$ is its image on $\mathbb{P}^3$ and $\pi(Z)$ is a line different from $L$, $L_{12}$, $L_{34}$, $L_{56}$ then $Z \not\subset R$.*

*Proof.* Suppose $Z$ is contained in $R$. Consider the following commutative diagram from section 3.2:




Where the bottom dashed arrow is given by the linear system of quadrics containing $C$. Using the equations of quadrics which form the basis of the linear system $\mathcal{M}$ defined in Section 3.3.3 we get the explicit map:

$$\mathbb{P}^3 \dashrightarrow \mathbb{P}^2 \text{ where } [x_0 : x_1 : x_2 : x_3] \dashrightarrow [x_0x_3 - x_1x_2 : x_1^2 - x_0x_2 : x_2^2 - x_1x_3].$$

We know that $\xi \circ \phi(R)$ is a conic. Let's write its equation in $\mathbb{P}^2$ with coordinates $[x : y : z]$:

$$a_1x^2 + a_2xy + a_3xz + a_4y^2 + a_5yz + a_6z^2 = 0.$$

We want to look at the preimage of this equation in $\mathbb{P}^3$ which will give the equation for $\pi(R)$. Substituting $[x_0x_3 - x_1x_2 : x_1^2 - x_0x_2 : x_2^2 - x_1x_3]$ into the defining equation of $\xi \circ \phi(R)$ we get:

$$\pi(R) : a_1x_3^2x_0^2 - a_2x_2x_3x_0^2 + a_4x_2^2x_0^2 + a_2x_3x_0x_1^2 - 2a_4x_2x_0x_1^2 + a_2x_2^2x_0x_1 + (-2a_1 + a_5)x_2x_3x_0x_1 -$$

$$-a_3x_3^2x_0x_1 + a_3x_3x_2^2x_0 - a_5x_2^3x_0 + a_4x_1^4 - a_2x_1^3x_2 - a_5x_3x_1^3 +$$

$$+(a_1 + a_5)x_2^2x_1^2 + a_3x_2x_3x_1^2 + a_6x_3^2x_1^2 - a_3x_2^3x_1 - 2a_6x_2^2x_3x_1 + a_6x_2^4 = 0.$$

Recall from section 3.3.4 that all $G$-invariant lines are of the form

$$\begin{cases} \lambda x_0 + \mu x_2 = 0, \\ \lambda x_3 + \mu x_1 = 0, \end{cases}$$

where $[\lambda : \mu] \in \mathbb{P}^1$. Now $L$ is given by

$$L = L_s : \begin{cases} x_0 + sx_2 = 0, \\ x_3 + sx_1 = 0, \end{cases}$$

for $s \in \mathbb{C}$. Note that $s \neq 0$ since otherwise $X$ would have an infinite group of automorphisms. Similarly $\pi(Z)$ is given by

$$\pi(Z) = L_t : \begin{cases} x_0 + tx_2 = 0, \\ x_3 + tx_1 = 0, \end{cases}$$

for $t \in \mathbb{C}$. By our assumption $L_s$ is contained in $\pi(R)$. This gives

$$\{a_1 = -1/s,\ a_2 = 0,\ a_3 = 0,\ a_4 = 1,\ a_5 = (s^2+1)/s, a_6 = 1\}.$$

So that $\pi(R)$ is given by

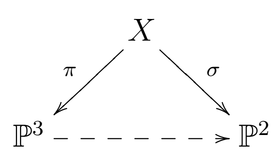
$$\pi(R) : x_2^2x_0^2s - x_3^2x_0^2 - 2x_2x_0x_1^2s + (s^2+3)x_2x_3x_0x_1 + (-s^2-1)x_2^3x_0 + sx_1^4 + \\ +(-s^2-1)x_3x_1^3 + s^2x_1^2x_2^2 + x_3^2x_1^2s - 2x_2^2x_3x_1s + sx_2^4 = 0.$$

Similarly since $L_t$ is contained in $\pi(R)$ we get

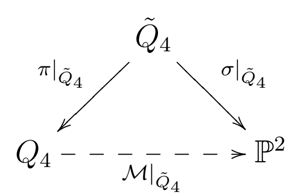
$$\begin{cases} -t^4 - 4ts + (s^2+3)t^2 + s^2 = 0, \\ s + (-s^2-1)t + t^2s = 0. \end{cases}$$

and the solution to this system is $s = t$ which means that $L_s = L$ and $L_t = \pi(Z)$ coincide contradicting the assumption on $Z$. $\square$

*Remark* 3.6. Let us use the assumptions and the notations from the proof of Lemma 3.5. There is another way to show that the points in $\{b_1, b_2, b_3, b_4\}$ are in general position which as discussed above is equivalent to showing that $Z \not\subset R$. Suppose the opposite, i.e. that $Z \subset R$. Now as in the proof of lemma above we take suitable $t, s \in \mathbb{C}$ such that $L_s = L$ and $L_t = \pi(Z)$. Consider the following commutative diagram:




Where the bottom dashed arrow is given by the linear system $\mathcal{M}$ of quadrics containing $C$ and a morphism $\sigma|_{\tilde{Q}_4}$ is given by the linear system $|\pi^*(2H) - E_C|$. Now we restrict this diagram to the quadric $Q_4$ which is the quadric which consists of $G$-invariant curves, $\tilde{Q}_4$ is the strict transform of $Q_4$ on $X$ :




The restriction $\pi|_{\tilde{Q}_4}$ is the blow up of the intersection points $C \cap Q_4 = \{P_1, ..., P_6\}$, which we described in 3.3.4. Observe that $\tilde{Q}_4$ is not a del Pezzo surface. Indeed, let $\tilde{L}_{ij}$ be in $\{\tilde{L}_{12}, \tilde{L}_{34}, \tilde{L}_{56}\}$ (here $L_{ij}$ is line connecting $P_i$ and $P_j$) then $\tilde{L}_{12}$, $\tilde{L}_{34}$, $\tilde{L}_{56}$ where $\tilde{L}_{ij}$ is the strict transform of the line $L_{ij}$ are (-2)-curves in $\tilde{Q}_4$ since the lines $L_{12}$, $L_{34}$, $L_{56}$ lie in $Q_4$. So that they has trivial intersection with $-K_{\tilde{Q}_4}$. In fact, using the coordinates of the points $P_1, ..., P_6$ and the equation of $Q_4$ one can show that the lines $L_{ij}$ are the only secants of the curve $C$ that are contained in $Q_4$. On the other hand,

$$-K_{\tilde{Q}_4} \sim (\pi^*(2H) - E_C)|_{\tilde{Q}_4},$$

so that $-K_{\tilde{Q}_4}$ is nef and big, and the only curves in $\tilde{Q}_4$ that has trivial intersection with $-K_{\tilde{Q}_4}$ are the three curves $\tilde{L}_{ij}$. Note that these curves do not intersect the curve $Z$. Taking the Stein factorization of the morphism $\sigma|_{\tilde{Q}_4} : \tilde{Q}_4 \to \mathbb{P}^2$, we obtain the following commutative diagram:

$$\begin{array}{ccc} \tilde{Q}_4 & \xrightarrow{\text{contraction of (-2) curves}} & \bar{Q}_4 \\ {\scriptstyle \pi|_{\tilde{Q}_4}}\downarrow & \searrow^{\sigma|_{\tilde{Q}_4}} & \downarrow{\scriptstyle \beta} \\ Q_4 & \dashrightarrow & \mathbb{P}^2 \end{array}$$

where $\bar{Q}_4$ is a singular del Pezzo surface of degree 2 with three singular points of type $A_1$, $\beta$ is the double cover given by $|-K_{\bar{Q}_4}|$, and the dasharrow is the rational map given by the restriction of the linear system $\mathcal{M}$ to $Q_4$. Suppose $\bar{L}_s$, $\bar{L}_t$ are the images of $\tilde{L}_s$, $\tilde{L}_t$ which are the strict transforms of $L_s$ and $L_t$ on $\tilde{Q}_4$. $\bar{L}_s$ and $\bar{L}_t$ do not pass through singular points, because $L_s$ and $L_t$ are disjoint from the lines $L_{ij}$. Since both $L_t, L_s \subset \pi(R)$ by assumption then by construction of $R$ given in Section 3.2 we get that $\beta(\bar{L}_s) = \beta(\bar{L}_t)$ is the same conic in $\mathbb{P}^2$. So we see that:

$$\bar{L}_s + \bar{L}_t = \beta^*(\mathcal{O}_{\mathbb{P}^2}(2)) = -2K_{\bar{Q}_4}.$$

By the adjunction formula we have:

$$K_{\bar{Q}_4} \cdot \bar{L}_s + \bar{L}_s^2 = -2 \Rightarrow -K_{\bar{Q}_4} \cdot \bar{L}_s = 2.$$

So we get

$$0 = \bar{L}_s \cdot \bar{L}_s + \bar{L}_t \cdot \bar{L}_s = -2K_{\bar{Q}_4} \cdot \bar{L}_s = 4.$$

Which gives us a contradiction. Thus $Z \not\subset R$.

3.4. **Mori Cone.** Let $l_L$, $l_C$, $l_R$ be the general fibers of the natural projections $E_L \to L$, $E_C \to C$, $R \to \sigma(R)$. Observe that we can contract any of two rays $\mathbb{R}_{\geq 0}[l_L]$, $\mathbb{R}_{\geq 0}[l_C]$, $\mathbb{R}_{\geq 0}[l_R]$. Indeed $l_C$ and $l_L$ are contracted by $\pi : X \to \mathbb{P}^3$, $l_R$ and $l_L$ are contracted by $\sigma : X \to \mathbb{P}^2$, $l_R$ and $l_C$ are contracted by $\eta : X \to \mathbb{P}^1$. Thus, these curves generate 3 extreme rays $\mathbb{R}_{\geq 0}[l_L]$, $\mathbb{R}_{\geq 0}[l_C]$, $\mathbb{R}_{\geq 0}[l_R]$ of the Mori cone $\overline{\mathrm{NE}(X)}$.

3.5. **Cone of Effective Divisors.**

**Lemma 3.7.** *Suppose $S$ is a surface in $X$ then*

$$S \sim a(\pi^*(H) - E_L) + b(2\pi^*(H) - E_C) + cR + eE_L + fE_C,$$

*for $a, b, c, e, f \in \mathbb{Z}_{\geq 0}$.*

*Proof.* Suppose $\pi(S) \subset \mathbb{P}^3$ is the surface of degree $d$ in $\mathbb{P}^3$. Then we have

$$S \sim d\pi^*(H) - m_L E_L - m_C E_C,$$

where $m_L$ is the multiplicity of $\pi(S)$ in $L$, $m_C$ is the multiplicity of $\pi(S)$ in $C$. Suppose that $S \neq E_C$, $S \neq E_L$ and $S \neq R$ for all $n$. Now let's intersect $S$ with three extreme rays

$l_L$, $l_C$, $l_R$ corresponding to $L$, $C$, $R$:

- $\pi^*(H) \cdot l_C = 0$,
- $\pi^*(H) \cdot l_L = 0$,
- $\pi^*(H) \cdot l_R = 1$,
- $E_L \cdot l_C = 0$,
- $E_L \cdot l_L = -1$,
- $E_L \cdot l_R = 1$,
- $E_C \cdot l_C = -1$,
- $E_C \cdot l_L = 0$,
- $E_C \cdot l_R = 2$.

So we have that:

$$S \cdot l_C = m_C \geq 0, \quad S \cdot l_L = m_L \geq 0, \quad S \cdot l_R = d - m_L - 2m_C \geq 0.$$

Moreover if $l_1$ is the general line intersecting $L$, $l_2$ is the general secant of $C$ then we get strict inequalities:

$$S \cdot l_1 = d - m_L > 0, \quad S \cdot l_2 = d - 2m_C > 0$$

Now we want to find the integer positive solutions for:

$$d\pi^*(H) - m_L E_L - m_C E_C = a(\pi^*(H) - E_L) + b(2\pi^*(H) - E_C) + cR + eE_L + fE_C.$$

Comparing the coefficients we get:

$$d = a + 2b + 4c, \quad m_C = b + 2c - f, \quad m_L = a + c - e.$$

The non-negative solution to this system can be given by

$$\begin{cases} a = -2m_C + d, \\ b = m_C, \\ c = 0, \\ e = -m_L - 2m_C + d, \\ f = 0. \end{cases}$$

Thus, the cone of effective divisors over $\mathbb{Z}$ is generated by $\pi^*(H) - E_L$, $2\pi^*(H) - E_C$, $R$, $E_L$, $E_C$. $\square$

**Corollary 3.8.** *Cone of effective divisors of $X$ is generated over $\mathbb{Q}$ by $\pi^*(H) - E_L$, $R$, $E_L$, $E_C$. More precisely, suppose $S$ is a surface in $X$ then*

$$S \sim_{\mathbb{Q}} a(\pi^*(H) - E_L) + cR + eE_L + fE_C,$$

*for unique $a, c, e, f \in \mathbb{Q}_{\geq 0}$.*

## 4. Proof of the Main Theorem

Let $C$ be a twisted cubic in $\mathbb{P}^3$ that is the image of the map $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ given by

$$[x : y] \to [x^3 : x^2y : xy^2 : y^3],$$

and $L$ be a line in $\mathbb{P}^3$ that is disjoint from $C$ given by

$$L : \begin{cases} x_0 = rx_1, \\ x_3 = rx_2 \end{cases}$$

where $r \neq 0$, $r \neq \pm 1$, $r \neq \pm 3$ in the coordinates presented in section 3.3.1. $X$ is a smooth Fano 3-folds in Family 3.12 obtained by blowing up $\mathbb{P}^3$ in $C$ and $L$, and $G$ is the subgroup in $\mathrm{Aut}(X)$ such that $G \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ described in Section 3.1.

Suppose $X$ is not $K$-polystable. By Theorem 2.2, there exists a $G$-invariant prime divisor $F$ over $X$ such that $\beta(F) = A_X(F) - S_X(F) \leqslant 0$. Let us seek for a contradiction. Let $Z = C_X(F)$. Then $Z$ is not a point by Corollary 3.1, and $Z$ is not a surface by Theorem 2.3, so that $Z$ is a $G$-invariant irreducible curve.

**Lemma 4.1.** *Suppose that $\pi(Z) \neq L$ then $\pi(Z)$ is not one of the $G$-invariant lines which does not intersect $C$.*

*Proof.* Let's take a $G$-invariant line $\pi(Z)$ that does not intersect $C$ and consider a plane $H$ which contains this line. It intersects a line $L$ in one point and a twisted cubic $C$ at three points. Let $S$ be the proper transform of $H$ on $X$. In this case we have that the induced map $\pi|_S : S \to H$ is the blowup of a plane $H$ in 4 points $b_1 = H \cap L$, $b_2, b_3, b_4 = H \cap C$. We now need to check that these points are in general position to conclude that $S$ is a del Pezzo surface of degree 5.

To prove that we need to show that the points in $\{b_1, b_2, b_3, b_4\}$ are in general position which means that no three of them belong to the same line. Note that $b_2, b_3, b_4 = H \cap C$ does not belong to the same line, because $C$ is an intersection of quadrics. So the only option is that $b_1$ and two points from the set $\{b_2, b_3, b_4\}$ belongs to the same line. Suppose $H$ is a general plane and $b_1$ and 2 points among $\{b_2, b_3, b_4\}$ are contained in one line $\ell$. From Section 3.2 we know that $\pi(R)$ is spanned by secants of $C$ that intersect $L$, so $H$ contains such secant $\ell$. Moreover $\pi(Z)$ intersects $\ell$, so we see that $\pi(Z)$ intersects a general secant of $C$ that is contained in $\pi(R)$. Then $Z \subset R$ which contradicts Lemma 3.5. So we can choose the hyperplane $H$ in such a way that the points in $\{b_1, b_2, b_3, b_4\}$ are in general position.

Thus, $S$ is a del Pezzo surface of degree 5 with the exceptional divisors $E_1$, $E_2$, $E_3$, $E_4$ corresponding to points $b_1, b_2, b_3, b_4$, $L_{ij}$ are the preimages of lines connecting $b_i$ and $b_j$ for $i \in \{1, ..., 4\}$. Recall that $E_1$, $E_2$, $E_3$, $E_4$ and $L_{ij}$ generate the Mori Cone $\overline{\mathrm{NE}(S)}$. We have that

$$-K_X \sim \pi^*(4H) - E_C - E_L,$$

$$R \sim \pi^*(4H) - 2E_C - E_L,$$

and moreover

$$\pi^*(H)|_S \sim S|_S \sim Z, \quad E_L|_S \sim E_1, \quad E_C|_S \sim E_2 + E_3 + E_4.$$

By Theorem 2.3, we have $S_X(S) < 1$. Thus, we conclude that $S(W^S_{\bullet,\bullet}; Z) \geqslant 1$ by Corollary 2.4. Let us compute $S(W^S_{\bullet,\bullet}; Z)$. Take $u \in \mathbb{R}_{\geqslant 0}$. Observe that

$$-K_X - uS \sim (1 - u/3)R + u/3(H - E_L) + (1 - 2u/3)E_C,$$

which implies that $-K_X - uS$ is pseudo-effective if and only if $u \leqslant \frac{3}{2}$. Let $P(u) = P(-K_X - uS)$ be a positive part of Zariski decomposition and $N(u) = N(-K_X - uS)$ be a negative part of Zariski decomposition. Here we use the notations introduced in Theorem 2.4.

$$P(u) = \begin{cases} -K_X - uS \text{ if } 0 \leqslant u \leqslant 1, \\ -K_X - uS - (u-1)R \text{ if } 1 \leqslant u \leqslant \dfrac{3}{2}, \end{cases}$$

and

$$N(u) = \begin{cases} 0 \text{ if } 0 \leqslant u \leqslant 1, \\ (u-1)R \text{ if } 1 \leqslant u \leqslant \dfrac{3}{2}. \end{cases}$$

Then take any $v \in \mathbb{R}_{\geqslant 0}$. Suppose $P(u, v)$ is a positive part of the Zariski decomposition of $(-K_X - uS)|_S - vZ$, $N(u, v)$ is a negative part of the Zariski decomposition of $(-K_X -$

$uS)|_S - vZ$.
If $u \in [0,1]$, we have

$$P(u)|_S - vZ \sim (-K_X - uS)|_S - vZ = (4H - E_C - E_L - uS)|_S - vZ =$$

$$= 4Z - (E_2 + E_3 + E_4) - E_1 - uZ - vZ = (4-u-v)Z - (E_1 + E_2 + E_3 + E_4).$$

For $u \in [0,1]$ we find $v$ such that the divisor $P(u)|_S - vZ$ is nef. We have that:

- $P(u,v) \cdot Z = 4 - u - v,$
- $P(u,v) \cdot E_i = 1,$ for $i \in \{1, ..., 4\},$
- $P(u,v) \cdot L_{ij} = 2 - u - v,$ for $i, j \in \{1, ..., 4\}.$

To check when the divisor is nef we should choose the strongest inequality from the following system:

$$\begin{cases} 4 - u - v \geq 0, \\ 2 - u - v \geq 0, \end{cases} \Rightarrow v \leq 2 - u.$$

Thus for $v \leq 2-u$ we have that the divisor $P(u)|_S - vZ$ is nef. Note that for $v = 2-u$ we have that $P(u, 2-u)^2 = 0$ so $P(u)|_S - vZ$ is pseudo-effective until $v$ satisfies $v \leq 2-u$. We see that the Zariski decomposition for $u \in [0,1]$, $v \leq 2-u$ is given by

$$P(u,v) = (4-u-v)Z - (E_1 + E_2 + E_3 + E_4), \quad N(u,v) = 0.$$

For $u \in [1, 3/2]$ and $v \in \mathbb{R}_{\geqslant 0}$ we have that:

$$P(u)|_S - vZ \sim (-K_X - uS - (u-1)R)|_S - vZ =$$

$$= (4H - E_C - E_L - uS - (u-1)(4H - 2E_C - E_L))|_S - vZ =$$

$$= (8 - 5u - v)Z + (2u-3)(E_2 + E_3 + E_4) + (u-2)E_1.$$

We have that:

- $P(u,v) \cdot Z = 8 - 5u - v,$
- $P(u,v) \cdot E_1 = 2 - u,$
- $P(u,v) \cdot E_i = 3 - 2u$ for $i \in \{2,3,4\},$
- $P(u,v) \cdot L_{1i} = 3 - 2u - v$ for $L_{1i} \in \{L_{12}, L_{13}, L_{14}\},$
- $P(u,v) \cdot L_{ij} = 3 - 2u - v$ for $L_{ij} \in \{L_{23}, L_{34}, L_{24}\}.$

To check when the divisor is nef we should choose the strongest inequality from the following system:

$$\begin{cases} 8 - 5u - v \geq 0, \\ 3 - 2u - v \geq 0, \\ 2 - u - v \geq 0, \end{cases} \Rightarrow v \leq 3 - 2u.$$

Thus, for $v \leq 3-2u$ we have that the divisor $P(u)|_S - vZ$ is nef. We see that the Zariski decomposition for $u \in [1, 3/2]$ $v \leq 3 - 2u$ is given by

$$P(u,v) = (8 - 5u - v)Z + (2u-3)(E_2 + E_3 + E_4) + (u-2)E_1, \quad N(u,v) = 0.$$

Suppose $v \geq 3 - 2u$. Let us find the Zariski decomposition of $P(u)|_S - vZ$. $P(u,v)$ is given by

$$P(u,v) = (8 - 5u - v)Z + (2u-3)(E_2 + E_3 + E_4) + (u-2)E_1 - aL_{12} - bL_{13} - cL_{14},$$

for some $a, b, c$. Note that we should have $P(u,v)\cdot L_{1i}=0$ for $i\in\{2,3,4\}$ thus $a=b=c=-(3-2u-v)$. So we obtain that:

$$P(u,v)=(8-5u-v)Z+(2u-3)(E_2+E_3+E_4)+(u-2)E_1+(3-2u-v)(L_{12}+L_{13}+L_{14}),$$

$$N(u,v)=-(3-2u-v)(L_{12}+L_{13}+L_{14}).$$

We have that:

- $P(u,v)\cdot Z=-11u-4v+17$,
- $P(u,v)\cdot E_1=-7u-3v+11$,
- $P(u,v)\cdot E_i=-v-4u+6$ for $i\in\{2,3,4\}$,
- $P(u,v)\cdot L_{1i}=0$ for $L_{1i}\in\{L_{12},L_{13},L_{14}\}$,
- $P(u,v)\cdot L_{ij}=5-3u-2v$ for $L_{ij}\in\{L_{23},L_{24},L_{34}\}$.

To check when the divisor is pseudo-effective we should choose the strongest inequality from the following system:

$$\begin{cases}-11u-4v+17\geq 0,\\ -7u-3v+11\geq 0,\\ -v-4u+6\geq 0,\\ 5-3u-2v\geq 0,\end{cases}\Rightarrow\begin{cases}v\leq\frac{-11u+17}{4},\\ v\leq\frac{-7u+11}{3},\\ v\leq 6-4u,\\ v\leq\frac{-3u+5}{2}.\end{cases}$$

So for $u\in[1,7/5]$ we get $v\leq\frac{-3u+5}{2}$ and for $u\in[7/5,3/2]$ we get $v\leq 6-4u$. Note that

$$P(u,v)^2=2(3u-5+2v)(4u-6+v).$$

Thus, for $v=\frac{-3u+5}{2}$ or $v=6-4u$ we have that $P(u,v)^2=0$. Note that for $v=2-u$ we have that $P(u,v)^2=0$ so $P(u)|_S-vZ$ is pseudo-effective until $v$ is in these intervals. The Corollary 2.4 gives us

$$S\big(W^S_{\bullet,\bullet};Z\big)=\frac{3}{(-K_X)^3}\int_0^{3/2}\big(P(u)^2\cdot S\big)\cdot\mathrm{ord}_Z\Big(N(u)\big|_S\Big)du+\frac{3}{(-K_X)^3}\int_0^{3/2}\int_0^{\infty}\mathrm{vol}\big(P(u)\big|_S-vZ\big)dvdu.$$

Note that $\mathrm{ord}_Z\Big(N(u)\big|_S\Big)=0$ because $Z\not\subset R$. So we are only left with the second part of the integral which equals:

$$S\big(W^S_{\bullet,\bullet};Z\big)=\frac{3}{28}\int_0^1\int_0^{2-u}\big((4-u-v)Z-(E_1+E_2+E_3+E_4)\big)^2dvdu+$$

$$+\frac{3}{28}\int_1^{3/2}\int_0^{2u-3}\big((8-5u-v)Z+(2u-3)(E_2+E_3+E_4)+(u-2)E_1\big)^2dvdu+$$

$$+\frac{3}{28}\int_1^{7/5}\int_{3-2u}^{\frac{5-3u}{2}}\big((8-5u-v)Z+(2u-3)(E_2+E_3+E_4)+(u-2)E_1)+(3-2u-v)(L_{12}+L_{13}+L_{14})\big)^2dvdu+$$

$$+\frac{3}{28}\int_{7/5}^{3/2}\int_{3-2u}^{6-4u}\big((8-5u-v)Z+(2u-3)(E_2+E_3+E_4)+(u-2)E_1)+(3-2u-v)(L_{12}+L_{13}+L_{14})\big)^2dvdu=$$

$$=\frac{3}{28}\int_0^1\int_0^{2-u}\big((4-u-v)^2-4\big)dvdu+\frac{3}{28}\int_1^{3/2}\int_0^{2u-3}\big(12u^2+10uv+v^2-40u-16v+33\big)dvdu+$$

$$+\frac{3}{28}\int_1^{7/5}\int_{3-2u}^{\frac{5-3u}{2}}\big(24u^2+22uv+4v^2-76u-34v+60\big)dvdu+$$

$$+\frac{3}{28}\int_1^{7/5}\int_{3-2u}^{6-4u}\big(24u^2+22uv+4v^2-76u-34v+60\big)dvdu=\frac{753}{1120}<1.$$

The obtained contradiction completes the proof of the lemma. □

**Lemma 4.2.** *One has $Z\not\subset E_L$.*

*Proof.* Suppose that $Z\subset E_L$. Observe that $E_L\cong\mathbb{P}^1\times\mathbb{P}^1$. Let $\mathbf{s}$ be the section of the natural projection $E_L\to L$ such that $\mathbf{s}^2=0$, and $\mathbf{l}$ be a fiber of this projection. Then

$$E_L|_{E_L}\sim-\mathbf{s}+\mathbf{l},\ \pi^*(H)|_{E_L}\sim\mathbf{l},\ R|_{E_L}\sim\mathbf{s}+3\mathbf{l},$$

and $E_C$ and $E_L$ are disjoint. By Theorem 2.3, we have $S_X(E_L)<1$. Thus, we conclude that $S(W^{E_L}_{\bullet,\bullet};Z)\geqslant 1$ by Corollary 2.4. Let us compute $S(W^{E_L}_{\bullet,\bullet};Z)$. Take $u\in\mathbb{R}_{\geqslant 0}$. Observe that

$$-K_X-uE_L\sim_{\mathbb{R}}\frac{1}{2}R+2(\pi^*(H)-E_L)+\big(\frac{3}{2}-u\big)E_L,$$

which implies that $-K_X-uE_L$ is pseudo-effective if and only if $u\leqslant\frac{3}{2}$. Let $P(u)=P(-K_X-uE_L)$ and $N(u)=N(-K_X-uE_L)$. Then

$$P(u)=\begin{cases}-K_X-uE_L\text{ for }0\leqslant u\leqslant 1,\\ (8-4u)\pi^*(H)-(3-2u)E_C-2E_L\text{ for }1\leqslant u\leqslant\frac{3}{2},\end{cases}$$

and

$$N(u)=\begin{cases}0\text{ for }0\leqslant u\leqslant 1,\\ (u-1)R\text{ for }1\leqslant u\leqslant\frac{3}{2}.\end{cases}$$

1 Suppose that $Z\neq R|_{E_L}$ and $Z\sim a\mathbf{s}+b\mathbf{l}$. Note that $a\geq 1$ since $\mathbb{P}^3$ does not contain $G$-fixed points. Then using Corollary 2.4 we obtain

$$S\big(W^{E_L}_{\bullet,\bullet};Z\big)=\frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty}\operatorname{vol}\Big(P(u)\big|_Q-v(a\mathbf{s}+b\mathbf{l})\Big)dvdu\leqslant\frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty}\operatorname{vol}\Big(P(u)\big|_Q-v\mathbf{s}\Big)dvdu.$$

so it is enough to show that the last integral is less than 1 to deduce a contradiction. So suppose $Z\sim\mathbf{s}$. We have that:

$$P(u)|_{E_L}-v\mathbf{s}\sim_{\mathbb{R}}\begin{cases}(1+u-v)\mathbf{s}+(3-u)\mathbf{l}\text{ for }0\leq u\leq 1,\\ (2-v)\mathbf{s}+(6-4u)\mathbf{l}\text{ for }1\leq u\leq 3/2.\end{cases}$$

Then Corollary 2.4 gives

$$S\big(W^{E_L}_{\bullet,\bullet};\mathbf{s}\big)=\frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty}\operatorname{vol}\Big(P(u)\big|_Q-v\mathbf{s}\Big)dvdu=$$

$$\frac{3}{28}\int_0^1\int_0^{1+u}2(3-u)(1+u-v)dvdu+\frac{3}{28}\int_1^{3/2}\int_0^2 4(v-2)(-3+2u)dvdu=\frac{13}{16}<1.$$

So we obtained a contradiction with Corollary 2.4.

Thus, for any $G$-invariant curve $Z\subset E_L$ such that $Z\neq R|_{E_L}$ we have $S\big(W^{Q}_{\bullet,\bullet};Z\big)<1$-contradiction with Corollary 2.4.

$\boxed{2}$ Suppose $Z = R|_{E_L} \sim \mathbf{s} + 3\mathbf{l}$. Take any $v \in \mathbb{R}_{\geqslant 0}$ then we have:

$$P(u)|_{E_L} - vZ \sim_{\mathbb{R}} \begin{cases} (1+u-v)\mathbf{s} + (3-u-3v)\mathbf{l}, \text{ for } 0 \le u \le 1, \\ (2-v)\mathbf{s} + (6-4u-3v)\mathbf{l}, \text{ for } 1 \le u \le 3/2. \end{cases}$$

Hence, if $Z = R|_{E_L}$, then Corollary 2.4 gives

$$S\big(W^{E_L}_{\bullet,\bullet}; Z\big) = \frac{3}{28}\int_1^{\frac{3}{2}} (u-1)E_L \cdot \big((8-4u)\pi^*(H) - (3-2u)E_C - 2E_L\big)^2 du +$$

$$+ \frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty} \mathrm{vol}\Big(P(u)\big|_{E_L} - vZ\Big) dv du =$$

$$= \frac{3}{28}\int_1^{\frac{3}{2}} 4(u-1)(6-4u)du + \frac{3}{28}\int_0^1\int_0^{\frac{3-u}{3}} 2(1+u-v)(3-u-3v)dvdu +$$

$$+ \frac{3}{28}\int_1^{\frac{3}{2}}\int_0^{\frac{6-4u}{3}} 2(2-v)(6-4u-3v)dvdu = \frac{19}{56} < 1.$$

The obtained contradiction with Corollary 2.4 and completes the proof of the lemma. □

**Lemma 4.3.** *Let $Q$ be a $G$-invariant quadric surface in $\mathbb{P}^3$ that contains $C$. Then $Z \not\subset Q$.*

*Proof.* Suppose that $Z \subset Q$. Let us seek for a contradiction. Recall that $\pi(Q)$ is a smooth quadric surface in $\mathbb{P}^3$ that contains the twisted cubic curve $C$, and it does not contain line $L$. Let us identify $\pi(Q) = \mathbb{P}^1 \times \mathbb{P}^1$ such that $C$ is a curve in $\pi(Q)$ of degree $(1,2)$. Then $\pi$ induces a birational morphism $\varphi\colon Q \to \mathbb{P}^1 \times \mathbb{P}^1$ that is a blow up of two intersection points $\pi(Q) \cap L$, which are not contained in the curve $C$. Moreover, the surface $Q$ is a smooth del Pezzo surface of degree 6, because the points of the intersection $\pi(Q) \cap L$ are not contained in one line in $\pi(Q)$ since otherwise this line would be $L$. But $L$ is not contained in $\pi(Q)$ which is a contradiction.
By Theorem 2.3, we have $S_X(Q) < 1$. Then $S(W^Q_{\bullet,\bullet}; Z) \geqslant 1$ by Corollary 2.4. Let us show that $S(W^Q_{\bullet,\bullet}; Z) < 1$, which would give us the desired contradiction.
Take $u \in \mathbb{R}_{\geqslant 0}$. Then

$$-K_X - uQ \sim_{\mathbb{R}} 2\pi^*(H) - E_L + (1-u)(2\pi^*(H) - E_C),$$

which implies that $-K_X - uQ$ is nef for every $u \in [0,1]$. On the other hand, we have

$$-K_X - uQ \sim_{\mathbb{R}} (4-2u)\big(\pi^*(H) - E_L\big) + (3-2u)E_L + (u-1)E_C,$$

so that the divisor $-K_X - uS$ is pseudo-effective $\iff$ $u \in [0, \frac{3}{2}]$. Let $P(u) = P(-K_X - uQ)$ and $N(u) = N(-K_X - uQ)$. Then we have

$$P\big(-K_X - uQ\big) = \begin{cases} -K_X - uQ \text{ for } 0 \leqslant u \leqslant 1, \\ (4-2u)\pi^*(H) - E_L \text{ for } 1 \leqslant u \leqslant \dfrac{3}{2}, \end{cases}$$

and

$$N(-K_X - uQ) = \begin{cases} 0 \text{ for } 0 \leqslant u \leqslant 1, \\ (u-1)E_C \text{ for } 1 \leqslant u \leqslant \frac{3}{2}. \end{cases}$$

Let us introduce some notation on $Q$. Suppose $\varphi : Q \to \mathbb{P}^1 \times \mathbb{P}^1$ is the blowup at points $A_1$, $A_2$ First, we denote by $\ell_1$ and $\ell_2$ the proper transforms on $Q$ of general curves $k_1$ and

$k_2$ in $\mathbb{P}^1 \times \mathbb{P}^1$ of degrees $(1,0)$ and $(0,1)$, respectively. Second, we denote by $e_1$ and $e_1$ the exceptional curves of $\varphi$ which correspond to points $A_1$, $A_2$ respectively. Third, we let $F_{11}$, $F_{12}$, $F_{21}$, $F_{22}$ be the $(-1)$-curves on $Q$ such that

$$F_{11} \sim \ell_1 - e_1,\ F_{12} \sim \ell_1 - e_2,\ F_{21} \sim \ell_2 - e_1,\ F_{22} \sim \ell_2 - e_2.$$

Then

$$\pi^*(H)|_Q \sim \ell_1 + \ell_2,\ E_L|_Q \sim e_1 + e_2,\ E_C|_Q \sim \ell_1 + 2\ell_2.$$

$\boxed{1}$ Suppose $Z \neq E_C|_Q$, then $\varphi(Z)$ is a curve since since $Z \neq e_1$ and $Z \neq e_2$, because neither $e_1$ nor $e_2$ is $G$-invariant. Now we have that $\varphi(Z) \sim ak_1 + bk_2$ and so

$$Z \sim a\ell_1 + b\ell_2 - m_1e_1 - m_2e_2,$$

where $m_1$ is a multiplicity of $\varphi(Z)$ at point $A_1$, $m_2$ is a multiplicity of $\varphi(Z)$ at point $A_2$. Note that $G$ exchanges $A_1$ and $A_2$ and $Z$ is a $G$-invariant curve thus $m_1 = m_2 =: m$. We know that $Z \notin \{F_{11}, F_{12}, F_{21}, F_{22}\}$ since $F_{ij}$-s ($i, j \in \{1,2\}$) are not $G$-invariant. Thus:

$$0 \leq Z \cdot F_{11} = (a\ell_1 + b\ell_2 - m_1e_1 - m_2e_2)(l_1 - e_1) = b - m \Rightarrow b \geq m,$$

$$0 \leq Z \cdot F_{22} = (a\ell_1 + b\ell_2 - m_1e_1 - m_2e_2)(l_2 - e_2) = a - m \Rightarrow a \geq m.$$

Now we have that

$$Z \sim a\ell_1 + b\ell_2 - m(e_1 + e_2) =$$

$$= \begin{cases} \underbrace{(a-b)}_{\geq 0}\ell_1 + \underbrace{m}_{\geq 0}(\ell_1 + \ell_2 - e_1 - e_2) + \underbrace{(b-m)}_{\geq 0}(\ell_1 + \ell_2), & \text{for } a \geq b, \\ \underbrace{(b-a)}_{\geq 0}\ell_2 + \underbrace{m}_{\geq 0}(\ell_1 + \ell_2 - e_1 - e_2) + \underbrace{(a-m)}_{\geq 0}(\ell_1 + \ell_2), & \text{for } b \geq a. \end{cases}$$

So we can decompose each curve $Z$ as the sum of $\ell_1$, $\ell_2$, $\ell_1 + \ell_2 - e_1 - e_2$ with non-negative coefficients, i.e. $Z \sim c_1\ell_1 + c_2\ell_2 + c_3(\ell_1 + \ell_2 - e_1 - e_2)$ for some non-negative integers $c_1$, $c_2$, $c_3$. Note if for example $c_1 \geq 1$ then:

$$S\big(W^Q_{\bullet,\bullet}; Z\big) = \frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty} \mathrm{vol}\Big(P(u)\big|_Q - v(c_1\ell_1 + c_2\ell_2 + c_3(\ell_1 + \ell_2 - e_1 - e_2))\Big)dvdu \leqslant$$

$$\leqslant \frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty} \mathrm{vol}\Big(P(u)\big|_Q - v\ell_1\Big)dvdu.$$

and similarly for $c_2$ and $c_3$. So it is enough to get $S\big(W^Q_{\bullet,\bullet}; Z\big) < 1$ for $Z \sim \ell_1$, $Z \sim \ell_2$, $Z \sim \ell_1 + \ell_2 - e_1 - e_2$ to deduce a contradiction.

1). Suppose $Z \sim \ell_1$.

Take any $v \in \mathbb{R}_{\geq 0}$. Suppose $P(u,v)$ is a positive part of the Zariski decomposition of $(-K_X - uS)|_S - vZ$, $N(u,v)$ is a negative part of the Zariski decomposition of $(-K_X - uS)|_S - vZ$.

If $u \in [0,1]$ then

$$P(u)|_Q - vZ \sim (-K_X - uQ)|_Q - vZ = -e_1 - e_2 + (3 - u - v)\ell_1 + 2\ell_2.$$

Now we find $v$ such that the divisor $P(u)|_Q - vZ$ is nef. We have that:

- $P(u,v) \cdot e_1 = 1$,
- $P(u,v) \cdot e_2 = 1$,
- $P(u,v) \cdot F_{11} = 1$,
- $P(u,v) \cdot F_{12} = 1$,
- $P(u,v) \cdot F_{21} = 2 - u - v$,
- $P(u,v) \cdot F_{22} = 2 - u - v$.

Thus, for $v \leq 2-u$ we have that the divisor $P(u)|_Q - vZ$ is nef.
Suppose $v \geq 2-u$. We find the Zariski decomposition of $P(u)|_Q - vZ$.

$$P(u,v) = (-e_1 - e_2 + (3-u-v)\ell_1 + 2\ell_2) + (2-u-v)(F_{21} + F_{22})$$

$$= (-3+u+v)e_1 + (-3+u+v)e_2 + (3-u-v)\ell_1 + (6-2u-2v)\ell_2,$$

$$N(u,v) = -(2-u-v)(F_{21}+F_{22}) = -(2-u-v)(2\ell_2 - e_1 - e_2).$$

We have that:

- $P(u,v) \cdot e_1 = 3-u-v$,
- $P(u,v) \cdot e_2 = 3-u-v$,
- $P(u,v) \cdot F_{11} = 3-u-v$,
- $P(u,v) \cdot F_{12} = 3-u-v$,
- $P(u,v) \cdot F_{21} = 0$,
- $P(u,v) \cdot F_{22} = 0$.

We get $v \leq 3-u$. Note that for $v = 3-u$ we have that $P_2(u, 3-u)^2 = 0$ so $P(u)|_S - vZ$ is pseudo-effective if and only if $v$ satisfies $v \leq 3-u$.
For $u \in [1, 3/2]$ and $v \in \mathbb{R}_{\geqslant 0}$ we have that:

$$P(u)|_Q - vZ \sim (-K_X - uQ)|_Q - vZ = -e_1 - e_2 + (4-2u-v)\ell_1 + (4-2u)\ell_2.$$

Now we find $v$ such that the divisor $P(u)|_Q - vZ$ is nef. We have that:

- $P(u,v) \cdot e_1 = 1$,
- $P(u,v) \cdot e_2 = 1$,
- $P(u,v) \cdot F_{11} = 3-2u$,
- $P(u,v) \cdot F_{12} = 3-2u$,
- $P(u,v) \cdot F_{21} = 3-2u-v$,
- $P(u,v) \cdot F_{22} = 3-2u-v$.

Thus for $v \leq 3-2u$ we have that the divisor $P(u)|_Q - vZ$ is nef.
Suppose $v \geq 3-2u$. We find the Zariski decomposition of $P(u)|_S - vZ$.

$$P(u,v) = (-e_1 - e_2 + (4-2u-v)\ell_1 + (4-2u)\ell_2) + (3-2u-v)(F_{21}+F_{22}) =$$

$$= (-4+2u+v)e_1 + (-4+2u+v)e_2 + (4-2u-v)\ell_1 + (10-6u-2v)\ell_2,$$

$$N(u,v) = -(3-2u-v)(F_{21}+F_{22}) = (3-2u-v)(2\ell_2 - e_1 - e_2).$$

We have that:

- $P(u,v) \cdot e_1 = 4-2u-v$,
- $P(u,v) \cdot e_2 = 4-2u-v$,
- $P(u,v) \cdot F_{11} = 6-4u-v$,
- $P(u,v) \cdot F_{12} = 6-4u-v$,
- $P(u,v) \cdot F_{21} = 0$,
- $P(u,v) \cdot F_{22} = 0$.

To check when the divisor is pseudo-effective we should choose the strongest inequality from the following system:

$$\begin{cases} v \leq 4-2u, \\ v \leq 6-4u, \end{cases} \Rightarrow v \leq 6-4u.$$

We get $v \leq 6-4u$. Note that for $v = 6-4u$ we have that $P_4(u, 6-4u)^2 = 0$ so $P(u)|_S - vZ$ is pseudo-effective if and only if $v \leq 6-4u$. The Corollary 2.4 gives us:

$$S\big(W^Q_{\bullet,\bullet}; Z\big) = \frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty} \mathrm{vol}\Big(P(u)\big|_Q - v\ell_1\Big)dvdu = \frac{3}{28}\int_0^1\int_0^{2-u}\Big(10-4u-4v\Big)dvdv+$$
$$+\frac{3}{28}\int_0^1\int_{2-u}^{3-u} 2(3-u-v)^2 dvdu + \frac{3}{28}\int_1^{\frac{3}{2}}\int_0^{3-2u}\Big(8u^2+4uv-32u-8v+30\Big)dvdu+$$
$$+\frac{3}{28}\int_1^{\frac{3}{2}}\int_{3-2u}^{6-4u} 2(4-2u-v)(6-4u-v)dvdu = \frac{109}{112}.$$

So we obtained a contradiction with Corollary 2.4.

2). Suppose $Z \sim \ell_2$.

Take any $v \in \mathbb{R}_{\geq 0}$. Abusing the notations suppose $P(u,v)$ is a positive part of the Zariski decomposition of $(-K_X - uS)|_S - vZ$, $N(u,v)$ is a negative part of the Zariski decomposition of $(-K_X - uS)|_S - vZ$.

If $u \in [0,1]$ then

$$P(u)|_Q - vZ \sim (-K_X - uQ)|_Q - vZ = -e_1 - e_2 + (3-u)\ell_1 + (2-v)\ell_2.$$

Now we find $v$ such that the divisor $P(u)|_Q - vZ$ is nef. We have that:

- $P(u,v)\cdot e_1 = 1$,
- $P(u,v)\cdot e_2 = 1$,
- $P(u,v)\cdot F_{11} = 1-v$,
- $P(u,v)\cdot F_{12} = 1-v$,
- $P(u,v)\cdot F_{21} = 2-u$,
- $P(u,v)\cdot F_{22} = 2-u$.

Thus for $v \leq 1$ we have that the divisor $P(u)|_Q - vZ$ is nef.

Suppose $v \geq 1$. We find the Zariski decomposition of $P(u)|_Q - vZ$.

$$P(u,v) = (-e_1 - e_2 + (3-u)\ell_1 + (2-v)\ell_2) + (1-v)(F_{11}+F_{12}) =$$
$$= (v-2)e_1 + (v-2)e_2 + (5-u-2v)\ell_1 + (2-v)\ell_2,$$
$$N(u,v) = -(v-1)(F_{11}+F_{12}) = -(v-1)(2\ell_1 - e_1 - e_2).$$

We have that:

- $P(u,v)\cdot e_1 = 2-v$,
- $P(u,v)\cdot e_2 = 2-v$,
- $P(u,v)\cdot F_{11} = 0$,
- $P(u,v)\cdot F_{12} = 0$,
- $P(u,v)\cdot F_{21} = -v+3-u$,
- $P(u,v)\cdot F_{22} = -v+3-u$.

To check when the divisor is pseudo-effective we should choose the strongest inequality from the following system:

$$\begin{cases} v \leq 2, \\ v \leq 3-u, \end{cases} \Rightarrow v \leq 2.$$

We get $v \leq 2$. Note that for $v = 2$ we have that $P_2(u,2)^2 = 0$ so $P(u)|_S - vZ$ is pseudo-effective if and only if $v \leq 2$.

For $u \in [1, 3/2]$ and $v \in \mathbb{R}_{\geqslant 0}$ we have that:

$$P(u)|_Q - vZ \sim (-K_X - uQ)|_Q - vZ = -e_1 - e_2 + (4-2u)\ell_1 + (4-2u-v)\ell_2.$$

Now we find $v$ such that the divisor $P(u)|_Q - vZ$ is nef. We have that:

- $P(u,v)\cdot e_1=1$,
- $P(u,v)\cdot e_2=1$,
- $P(u,v)\cdot F_{11}=3-2u-v$,
- $P(u,v)\cdot F_{12}=3-2u-v$,
- $P(u,v)\cdot F_{21}=3-2u$,
- $P(u,v)\cdot F_{22}=3-2u$.

Thus for $v\le 3-2u$ we have that the divisor $P(u)|_Q-vZ$ is nef.
Suppose $v\ge 3-2u$. We find the Zariski decomposition of $P(u)|_S-vZ$.

$$P(u,v)=(-e_1-e_2+(4-2u)\ell_1+(4-2u-v)\ell_2)+(3-2u-v)(F_{11}+F_{12})=$$
$$=(-4+2u+v)e_1+(-4+2u+v)e_2+(10-6u-2v)\ell_1+(4-2u-v)l_2,$$
$$N(u,v)=-(3-2u-v)(F_{11}+F_{12})=(3-2u-v)(2\ell_1-e_1-e_2).$$

We have that:

- $P(u,v)\cdot e_1=4-2u-v$,
- $P(u,v)\cdot e_2=4-2u-v$,
- $P(u,v)\cdot F_{11}=0$,
- $P(u,v)\cdot F_{12}=0$,
- $P(u,v)\cdot F_{21}=6-4u-v$,
- $P(u,v)\cdot F_{22}=6-4u-v$.

We get $v\le 6-4u$. Note that for $v=6-4u$ we have that $P_4(u,6-4u)^2=0$ so $P(u)|_S-vZ$ is pseudo-effective if and only if $v\le 6-4u$. The Corollary 2.4 gives us:

$$S\big(W^Q_{\bullet,\bullet};Z\big)=\frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty}\mathrm{vol}\Big(P(u)\big|_Q-v\ell_2\Big)dvdu=\frac{3}{28}\int_0^1\int_0^1\Big(2uv-4u-6v+10\Big)dvdv+$$
$$+\frac{3}{28}\int_0^1\int_1^2 2(v-2)(v-3+u)dvdu+\frac{3}{28}\int_1^{\frac{3}{2}}\int_0^{3-2u}\Big(8u^2+4uv-32u-8v+30\Big)dvdu+$$
$$+\frac{3}{28}\int_1^{\frac{3}{2}}\int_{3-2u}^{6-4u}2(4-2u-v)(6-4u-v)dvdu=\frac{89}{112}<1.$$

So we obtained a contradiction with Corollary 2.4.
3). Suppose $Z\sim\ell_1+\ell_2-e_1-e_2$.
Take any $v\in\mathbb{R}_{\ge 0}$. Abusing the notations suppose $P(u,v)$ is a positive part of the Zariski decomposition of $(-K_X-uS)|_S-vZ$, $N(u,v)$ is a negative part of the Zariski decomposition of $(-K_X-uS)|_S-vZ$.
If $u\in[0,1]$ then:

$$P(u)|_Q-vZ\sim(-K_X-uQ)|_Q-vZ=(-1+v)e_1+(-1+v)e_2+(3-u-v)\ell_1+(2-v)\ell_2.$$

Now we find $v$ such that the divisor $P(u)|_Q-vZ$ is nef. We have that:

- $P(u,v)\cdot e_1=-v+1$,
- $P(u,v)\cdot e_2=-v+1$,
- $P(u,v)\cdot F_{11}=1$,
- $P(u,v)\cdot F_{12}=1$,
- $P(u,v)\cdot F_{21}=2-u$,
- $P(u,v)\cdot F_{22}=2-u$.

Thus for $v\le 1$ we have that the divisor $P(u)|_Q-vZ$ is nef.
Suppose $v\ge 1$. We find the Zariski decomposition of $P(u)|_Q-vZ$.

$$P(u,v)=((-1+v)e_1+(-1+v)e_2+(3-u-v)\ell_1+(2-v)\ell_2)+(1-v)(e_1+e_2)=$$
$$=(3-u-v)\ell_1+(2-v)\ell_2,$$
$$N(u,v)=-(v-1)(e_1+e_2).$$

We have that:

- $P(u,v)\cdot e_1 = 0,$
- $P(u,v)\cdot e_2 = 0,$
- $P(u,v)\cdot F_{11} = 2-v,$
- $P(u,v)\cdot F_{12} = 2-v,$
- $P(u,v)\cdot F_{21} = -v+3-u,$
- $P(u,v)\cdot F_{22} = -v+3-u.$

To check when the divisor is pseudo-effective we should choose the strongest inequality from the following system:

$$\begin{cases} v \le 2, \\ v \le 3-u, \end{cases} \quad \Rightarrow v \le 2.$$

We get $v \le 2$. Note that for $v=2$ we have that $P_2(u,2)^2 = 0$ so $P(u)|_S - vZ$ is pseudo-effective until $v$ satisfies $v \le 2$.
For $u \in [1, 3/2]$ and $v \in \mathbb{R}_{\geqslant 0}$ we have that:

$$P(u)|_Q - vZ \sim (-K_X - uQ)|_Q - vZ = (v-1)e_1 + (v-1)e_2 + (4-2u-v)\ell_1 + (4-2u-v)\ell_2.$$

Now we find $v$ such that the divisor $P(u)|_Q - vZ$ is nef. We have that:

- $P(u,v)\cdot e_1 = -v+1,$
- $P(u,v)\cdot e_2 = -v+1,$
- $P(u,v)\cdot F_{11} = 3-2u,$
- $P(u,v)\cdot F_{12} = 3-2u,$
- $P(u,v)\cdot F_{21} = 3-2u,$
- $P(u,v)\cdot F_{22} = 3-2u.$

Thus, for $v \le 1$ we have that the divisor $P(u)|_Q - vZ$ is nef.
Suppose $v \ge 3-2u$. We find the Zariski decomposition of $P(u)|_S - vZ$.

$$P(u,v) = ((v-1)e_1 + (v-1)e_2 + (4-2u-v)\ell_1 + (4-2u-v)\ell_2) + (1-v)(e_1+e_2) =$$

$$= (4-2u-v)\ell_1 + (4-2u-v)\ell_2,$$

$$N(u,v) = -(3-2u-v)(e_1+e_2).$$

We have that:

- $P(u,v)\cdot e_1 = 0,$
- $P(u,v)\cdot e_2 = 0,$
- $P(u,v)\cdot F_{11} = 4-2u-v,$
- $P(u,v)\cdot F_{12} = 4-2u-v,$
- $P(u,v)\cdot F_{21} = 4-2u-v,$
- $P(u,v)\cdot F_{22} = 4-2u-v.$

We get $v \le 4-2u$. Note that for $v = 4-2u$ we have that $P(u, 4-2u)^2 = 0$ so $P(u)|_S - vZ$ is pseudo-effective until $v$ satisfies $v \le 4-2u$. The Corollary 2.4 gives us:

$$S\big(W^Q_{\bullet,\bullet}; Z\big) = \frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty} \mathrm{vol}\Big(P(u)\big|_Q - v\ell_2\Big)dvdu = \frac{3}{28}\int_0^1\int_0^1 \Big(2uv-4u-6v+10\Big)dvdv +$$
$$+\frac{3}{28}\int_0^1\int_1^2 2(v-2)(-3+u+v)dvdu + \frac{3}{28}\int_1^{\frac{3}{2}}\int_0^1 \Big(2(2u-3)(2u+2v-5)\Big)dvdu +$$
$$+\frac{3}{28}\int_1^{\frac{3}{2}}\int_1^{4-2u} 2(-4+2u+v)^2 dvdu = \frac{47}{56} < 1.$$

So we obtained a contradiction with Corollary 2.4.

Thus, for any $G$-invariant curve $Z \subset Q$ such that $Z \neq E_C|_Q$ we have $S\big(W^Q_{\bullet,\bullet}; Z\big) < 1$ which is impossible by Corollary 2.4.

$\boxed{2}$ Suppose $Z = E_C|_Q$, then

$$S\big(W^Q_{\bullet,\bullet}; Z\big) = \frac{3}{28}\int_0^{\frac{3}{2}} \Big(P(u)\cdot P(u)\cdot Q\Big)\mathrm{ord}_Z\Big(N(u)\big|_Q\Big)du + \frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty} \mathrm{vol}\Big(P(u)\big|_Q - vZ\Big)dvdu =$$

$$= \frac{3}{28}\int_1^{\frac{3}{2}} (u-1)\big((4-2u)\pi^*(H)-E_L\big)^2\cdot\big(2\pi^*(H)-E_C\big)du + \frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty} \mathrm{vol}\Big(P(u)\big|_Q - vZ\Big)dvdu =$$

$$= \frac{3}{28}\int_1^{\frac{3}{2}} (u-1)\big(2(4-2u)^2-2\big)du + \frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty} \mathrm{vol}\Big(P(u)\big|_Q - vZ\Big)dvdu =$$

$$= \frac{5}{224} + \frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty} \mathrm{vol}\Big(P(u)\big|_Q - v(\ell_1+2\ell_2)\Big)dvdu \leqslant$$

$$\leqslant \frac{5}{224} + \frac{3}{28}\int_0^{\frac{3}{2}}\int_0^{\infty} \mathrm{vol}\Big(P(u)\big|_Q - v\ell_1\Big)dvdu = \frac{5}{224} + \frac{109}{112} = \frac{223}{224} < 1.$$

So we obtained a contradiction with Corollary 2.4. This completes the proof of the lemma. □

**Corollary 4.4.** *The curve $\pi(Z)$ is not a line that intersect $C$.*

*Proof.* Recall from Section 3.3.4 that there are exactly 3 $G$-invariant lines in $\mathbb{P}^3$ that intersect the curve $C$. These are the lines $L_{12}$, $L_{34}$, $L_{56}$. We have that $L_{12} \subset Q_2 \cap Q_3$, $L_{34} \subset Q_1 \cap Q_2$, $L_{34} \subset Q_1 \cap Q_3$. Thus, the lemma above gives us the result. □

By Lemma 2.5, one has $\alpha_{G,Z}(X) < \frac{3}{4}$. Thus, by lemma [2, Lemma 1.4.1] and it's proof, there is a $G$-invariant effective $\mathbb{Q}$-divisor $D$ on the threefold $X$ such that $D \sim_{\mathbb{Q}} -K_X$ and $Z \subseteq \mathrm{Nklt}(X, \lambda D)$ for some positive rational number $\lambda < \frac{3}{4}$.

**Lemma 4.5.** *Let $S$ be an irreducible surface in $X$. Suppose that $S \subset \mathrm{Nklt}(X, \lambda D)$. Then either $S \in |\pi^*(2H) - E_C|$ and $S$ is $G$-invariant or $S = E_L$.*

*Proof.* We have $D \sim_{\mathbb{Q}} 4\pi^*(H) - E_C - E_L$ and $\lambda < \frac{3}{4}$. By assumption we have $D = aS + \Delta$ where $a \in \mathbb{Q}$ such that $a \geq \frac{1}{\lambda} > \frac{4}{3}$ and $\Delta$ is an effective $\mathbb{Q}$ divisor on $X$ whose support does not contain $S$.
Assume $S = E_C$. Then we get

$$\pi^*(4H) - E_C - E_L \sim_{\mathbb{Q}} aE_C + \Delta \Rightarrow \Delta \sim_{\mathbb{Q}} \pi^*(4H) - (1+a)E_C - E_L = R - (a-1)E_C$$

-contradiction.
Assume $S \neq E_L$, $S \neq E_C$ then $\pi(S) \subset \mathbb{P}^3$ is the surface of some degree $d$. We have that:

$$4H \sim_{\mathbb{Q}} a\pi(S) + \pi(\Delta) \Rightarrow 4 \geq ad \Rightarrow d = 1 \text{ or } d = 2.$$

The latter holds since $a > \frac{4}{3}$. Then $S$ is given by

$$S \sim_{\mathbb{Q}} d\pi^*(H) - m_L E_L - m_C E_C.$$

By Corollary 3.8 we know that the cone of effective divisors is generated by $E_L$, $E_C$, $R$, $H - E_L$ so we have that:

$$\Delta \sim_{\mathbb{Q}} \pi^*(4H) - E_C - E_L - a(\pi^*(4H) - (1+a)E_C - E_L = R - (a-1)E_C) \sim$$

$$\sim a_1E_L + a_2E_C + a_3(\pi^*(4H) - 2E_C - E_L) + a_4(\pi^*(H) - E_L)$$

for $a_1 \geq 0,\ a_2 \geq 0,\ a_3 \geq 0,\ a_4 \geq 0$, which gives us a system of equations:

$$\begin{cases} -4a_3 - a_4 - d + 4 = 0, \\ -1 + m_L - a_1 + a_3 + a_4 = 0, \\ -1 + m_C - a_2 + 2a_3 = 0, \\ a_1 \geq 0,\ a_2 \geq 0,\ a_3 \geq 0,\ a_4 \geq 0, a > 4/3 \end{cases} \tag{4.0.1}$$

Thus, if $d = 2$ then $(m_L, m_C) = (0, 1)$ or $(m_L, m_C) = (1, 1)$ so we have the following options:

- $m_C = 1$ and $m_L = 1$. This gives us the linear system $|S| = |2\pi^*(H) - m_L E_L - m_C E_C|$ which on $\mathbb{P}^3$ corresponds to the linear system of quadrics which contain a line $L$ and a cubic $C$. But this is impossible since by assumption $L$ and $C$ so not intersect. Thus, this linear system does not contain effective divisors.
- $m_C = 1$, $m_L = 0$. Suppose $S \in |2\pi^*(H) - E_C|$ and $S$ is not $G$-invariant. We have that $S \subset \mathrm{Nklt}(X, \lambda D)$ where $D$ is a $G$-invariant $\mathbb{Q}$-divisor. We can write $D$ as $D = \sum_i a_i D_i$ where $D_i$-s are irreducible components of $D$, $a_i \in \mathbb{Q}_{>0}$ and we have $S = D_i$ for some $i$. We assumed that $S$ is not $G$-invariant thus if we take a non-trivial element $g \in G$ we will have that $S' = g(S)$ is $D_i$ which is one of the components of $D$ for $i \neq j$. Moreover $a_j = a_j = a$ since $D$ is $G$-invariant so we can write:

  $$\pi^*(4H) - E_C - E_L \sim_{\mathbb{Q}} 2a(2\pi^*(H) - E_C) + \Delta.$$

  Where $\Delta$ is an effective $\mathbb{Q}$-divisor. Thus:

  $$4(1 - a)\pi^*(H) + (-1 + 2a)E_C - E_L \sim_{\mathbb{Q}} \Delta.$$

  By Corollary 3.8 we know that the cone of effective divisors is generated by $E_L$, $E_C$, $R$, $H - E_L$ so we can write $\Delta$ as:

  $$\Delta = a_1 E_L + a_2 E_C + a_3(4\pi^*(H) - 2E_C - E_L) + a_4(H - E_L).$$

  For $a_1 \geq 0,\ a_2 \geq 0,\ a_3 \geq 0,\ a_4 \geq 0$. Solving the system of equations on coefficients we get that it has no solutions.
  Suppose $S \in |2\pi^*(H) - E_C|$ and $S$ is $G$-invariant. This case is possible.

Similarly, if $d = 1$ then $(m_L, m_C) = (0, 0)$ or $(m_L, m_C) = (1, 0)$ so we have the following options:

- $m_C = 0$, $m_L = 1$. Suppose $S \in |\pi^*(H) - E_L|$. We have that $S \subset \mathrm{Nklt}(X, \lambda D)$ where $D$ is a $G$-invariant $\mathbb{Q}$-divisor. We can write $D$ as $D = \sum_i a_i D_i$ where $D_i$-s are irreducible components of $D$, $a_i \in \mathbb{Q}_{>0}$ and we have $S = D_i$ for some $i$. Note that $S \in |\pi^*(H) - E_L|$ where $|\pi^*(H) - E_L|$ does not have $G$-invariant elements. Take a non-trivial element $g \in G$. We have that $S' = g(S)$ is $D_i$ which is one of the components of $D$ for $i \neq j$. Moreover $a_j = a_j = a$ since $D$ is $G$-invariant so we can write:

  $$\pi^*(4H) - E_C - E_L \sim_{\mathbb{Q}} 2a(\pi^*(H) - E_L) + \Delta.$$

  Where $\Delta$ is an effective $\mathbb{Q}$-divisor. Thus:

  $$(4 - 2a)\pi^*(H) - E_C - (1 - 2a)E_L \sim_{\mathbb{Q}} \Delta.$$

By Corollary 3.8 we know that the cone of effective divisors is generated by $E_L$, $E_C$, $R$, $H - E_L$ so we can write $\Delta$ as:

$$\Delta = a_1 E_L + a_2 E_C + a_3(4\pi^*(H) - 2E_C - E_L) + a_4(\pi^*(H) - E_L).$$

For $a_1 \geq 0$, $a_2 \geq 0$, $a_3 \geq 0$, $a_4 \geq 0$. Solving the system of equations on coefficients we get that it has no solutions.

- $m_C = 0$, $m_L = 0$. Suppose $S \in |\pi^*(H)|$. We have that $S \subset \mathrm{Nklt}(X, \lambda D)$ where $D$ is a $G$-invariant $\mathbb{Q}$-divisor. We can write $D$ as $D = \sum_i a_i D_i$ where $D_i$-s are irreducible components of $D$, $a_i \in \mathbb{Q}_{>0}$ and we have $S = D_i$ for some $i$. Note that $S \in |\pi^*(H)|$ where $|\pi^*(H)|$ does not have $G$-invariant elements. Take a non-trivial element $g \in G$. We have that $S' = g(S)$ is $D_i$ which is one of the components of $D$ for $i \neq j$. Moreover $a_j = a_j = a$ since $D$ is $G$-invariant so we can write:

$$\pi^*(4H) - E_C - E_L \sim_{\mathbb{Q}} 2a\pi^*(H) + \Delta.$$

Where $\Delta$ is an effective $\mathbb{Q}$-divisor. Thus:

$$(4 - 2a)\pi^*(H) - E_C - E_L \sim_{\mathbb{Q}} \Delta.$$

By Corollary 3.8 we know that the cone of effective divisors is generated by $E_L$, $E_C$, $R$, $H - E_L$ so we can write $\Delta$ as:

$$\Delta = a_1 E_L + a_2 E_C + a_3(4\pi^*(H) - 2E_C - E_L) + a_4(H - E_L).$$

For $a_1 \geq 0,\ a_2 \geq 0,\ a_3 \geq 0,\ a_4 \geq 0$. Solving the system of equations on coefficients we get that it has no solutions.

We see that we excluded all options except $S \in |\pi^*(2H) - E_C|$ and $S$ is $G$-invariant or $S = E_L$. □

**Corollary 4.6.** *$\pi(Z)$ is not a surface in* $\mathrm{Nklt}(X, \lambda D)$.

**Corollary 4.7.** *One has $Z \not\subset E_C$.*

*Proof.* Suppose that $Z \subset E_C$. Observe that $\pi(Z)$ is not a point, since $\mathbb{P}^3$ does not have $G$-fixed points by Lemma 3.1. Hence, we see that $\pi(Z)$ is the twisted cubic $C$.
Let $S$ be a general fiber of $\eta$. Then $S \cdot Z \geq 3$, which contradicts Lemma 2.7. □

**Lemma 4.8.** *The curve $\pi(Z)$ is the line.*

*Proof.* Let $\overline{D} = \pi(D)$, $\overline{Z} = \pi(Z)$. We see that $\overline{Z}$ is a $G$-invariant curve in $\mathbb{P}^3$ such that such that Z is not contained in a $G$-invariant surface in $|2\pi^*(H) - E_C|$ (by Lemma 4.3), $Z \not\subset E_L$ (by Lemma 4.2) and $Z \not\subset E_C$ (by Lemma 4.7). Then $\overline{Z} \subset (\mathbb{P}^3, \lambda\overline{D})$ and $\overline{Z}$ is not contained in any surface contained in $\mathrm{Nklt}(\mathbb{P}^3, \lambda\overline{D})$ by Lemma 4.5. Now we apply Lemma 2.6 and get that $\mathcal{O}_{\mathbb{P}^3}(1) \cdot \overline{Z} \leq 1$. Thus $\mathcal{O}_{\mathbb{P}^3}(1) \cdot \overline{Z} = 1$ so $\pi(Z)$ is a line. □

**Corollary 4.9.** *Such irreducible curve $Z$ does not exist.*

*Proof.* By Lemma 4.8 we know that $\pi(Z)$ is a line. We have that $\pi(Z) \neq L$ (by Lemma 4.2), $\pi(Z)$ is not one of the $G$-invariant lines which does not intersect $C$ and $\pi(Z) \neq L$ (by Lemma 4.1) and $\pi(Z)$ is not one of the $G$-invariant lines which intersect $C$ (by Corollary 4.4). So such irreducible curve $Z$ does not exist. □

This completes the proof of Main Theorem.

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