Polynomial worst-case iteration complexity of quasi-Newton primal-dual interior point algorithms for linear programming

Polynomial worst-case iteration complexity of quasi-Newton primal-dual interior point algorithms for linear programming

J. Gondzio* F. N. C. Sobral†

Technical Report ERGO, School of Mathematics, September 9, 2022

Abstract

Quasi-Newton methods are well known techniques for large-scale numerical optimization. They use an approximation of the Hessian in optimization problems or the Jacobian in system of nonlinear equations. In the Interior Point context, quasi-Newton algorithms compute low-rank updates of the matrix associated with the Newton systems, instead of computing it from scratch at every iteration. In this work, we show that a simplified quasi-Newton primal-dual interior point algorithm for linear programming enjoys polynomial worst-case iteration complexity. Feasible and infeasible cases of the algorithm are considered and the most common neighborhoods of the central path are analyzed. To the best of our knowledge, this is the first attempt to deliver polynomial worst-case iteration complexity bounds for these methods. Unsurprisingly, the worst-case complexity results obtained when quasi-Newton directions are used are worse than their counterparts when Newton directions are employed. However, quasi-Newton updates are very attractive for large-scale optimization problems where the cost of factorizing the matrices is much higher than the cost of solving linear systems.
Keywords: Quasi-Newton methods, Broyden update, Primal-dual Interior Point Methods, Polynomial worst-case iteration complexity
MSC codes: 90C05, 90C51, 90C53

1 Introduction

Let us consider the following general linear programming problem

$$\min \quad c^T x, \quad \text{s.t.} \quad Ax = b,\ x \geq 0, \tag{1}$$

where $x, c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$. We assume that (1) is feasible and the rows of A are linearly independent. Define function $F : \mathbb{R}^{2n+m} \to \mathbb{R}^{2n+m}$ by

$$F(x, \lambda, z) = \begin{bmatrix} A^T\lambda + z - c \\ Ax - b \\ XZe \end{bmatrix}, \tag{2}$$

where $X, Z \in \mathbb{R}^{n\times n}$ are diagonal matrices defined by $X = \operatorname{diag}(x)$ and $Z = \operatorname{diag}(z)$, respectively, and e is the vector of ones of appropriate size. First order necessary optimality conditions for (1) state that, if $x^* \geq 0$ is a minimizer, then there exist $z^* \in \mathbb{R}^n$, $z^* \geq 0$, and $\lambda^* \in \mathbb{R}^m$ such that $F(x^*, \lambda^*, z^*) = 0$ holds.

Interior point methods (IPMs) try to follow the so-called central-path of problem (1), defined by the solution of the perturbed KKT system $F(x, \lambda, z) = \begin{bmatrix} 0 & 0 & \mu e \end{bmatrix}^T$, as $\mu \to 0$. Instead of solving such a system exactly, primal-dual IPMs apply one iteration of Newton method for a given value of μ_k at iteration k. In order to calculate this step, the Jacobian of F is needed

$$J(x, \lambda, z) = \begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ Z & 0 & X \end{bmatrix}. \tag{3}$$

*School of Mathematics, University of Edinburgh, Edinburgh, EH9 3FD, Scotland, United Kingdom. Email: J.Gondzio@ed.ac.uk
† Corresponding author. Department of Mathematics, State University of Maringá, Avenida Colombo, 5790, Paraná, Brazil, 87020-900. Phone: +55 44 30116211. E-mail: fncsobral@uem.br

With an iterate (x^k, λ^k, z^k) at step k, the classical Newton direction is calculated by solving the following system

$$\begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ Z^k & 0 & X^k \end{bmatrix} \begin{bmatrix} \Delta x^k \\ \Delta \lambda^k \\ \Delta z^k \end{bmatrix} = \begin{bmatrix} c - z^k - A^T \lambda^k \\ b - Ax^k \\ \sigma_k \mu_k e - X^k Z^k e \end{bmatrix}, \tag{4}$$

where μ_k is set to be the average complementarity gap ${x^k}^T s^k/n$ and $\sigma_k \in (0,1)$ determines its target reduction.

While the coefficient matrix in (4) can be efficiently evaluated and stored, it changes at each iteration. The solution of (4) is usually accomplished by direct methods, using suitable matrix factorizations [9, 15], or by iterative methods [4], computing preconditioners to improve their convergence properties. In this paper we are concerned with classes of problems for which it is advantageous to approximate $J(x^k, \lambda^k, z^k)$ in order to reduce the cost of solving (4).

Usually, IPMs do not deal explicitly with the unreduced system (4), but rather consider its reduced form as augmented system (which is symmetric) or as normal equations (whose coefficient matrix is positive definite) [8]. The interest in working directly with unreduced systems has attracted more attention in the recent years, since they have good sparsity structure and also interesting spectral properties [14]. In [18], numerical experiments comparing preconditioners for unreduced and augmented systems were made. The appeal for using preconditioners for unreduced systems is their good conditioning close to the solution.

Gondzio and Sobral [10] considered unreduced systems in a way similar to [18]. They studied the Jacobian of F and asked the question whether it is possible to approximate it by classical quasi-Newton approaches for nonlinear systems. Although this might have seemed an obvious thing to attempt, the only previous use of quasi-Newton strategies in IPMs was to update the preconditioners [1, 2, 12, 13]. Broyden low-rank updates were used and the numerical experiments showed that this approach is effective for IPMs when the cost of solving linear systems is considerably lower than the cost of computing the factorization of the Jacobian (or its associated reduced form).

Recently, Ek and Forsgren [6] presented a theoretical background and numerical experiments regarding a different kind of low rank updates. The proposed update is based on the Eckart-Young-Mirsky theorem, rather than on the secant equation satisfied by the Broyden update, and affects only the "third row" of matrix $J(x^k, \lambda^k, z^k)$, related to the nonlinear part of F. Convex quadratic optimization problems were considered and local convergence was established for a simplified primal-dual interior point algorithm, but no complexity bound was provided. It is worth mentioning that the iteration worst-case complexity of $O(\sqrt{n})$ was shown for a short-step primal algorithm by Gonzaga [11], where low-rank updates were used to compute the projection matrix needed by such type of algorithms. Secant equations were also used in [5] for the same purpose, but without complexity results.

Polynomial worst-case iteration complexity is a key feature of IPMs for linear and convex-quadratic problems [19]. It is achieved by taking steps in the Newton direction (4) such that the new iterate belongs to some neighborhood of the central path. In case of linear programming, it is well known that the iteration worst-case complexity involves polynomials of orders between $\sqrt{n}$ and n^2, depending on the type of neighborhood of the central path used and whether feasible of infeasible iterates are allowed [21]. Those results have also been generalized to symmetric cone optimization problems [20].

This work is intended to provide the first steps towards the study of iteration worst-case complexity of quasi-Newton primal-dual interior point algorithms. We present non-trivial extensions of well known complexity results from [21] and properties that arise when Broyden "bad" quasi-Newton updates are used. Worst-case complexity is proven for both feasible and infeasible cases in the most commonly used neighborhoods. The theoretical study is motivated by the very promising results from [10] in quadratic programming problems. As expected, the degrees of polynomials in the complexity results are higher than those obtained when steps in Newton directions are made.

The paper is organized as follows. In Section 2 we review basic quasi-Newton concepts and the properties of quasi-Newton algorithms presented in [10]. Then, in Section 3 we analyze the worst-case complexity for the feasible case, considering two popular neighborhoods of the central path : $\mathcal{N}_2$ and $\mathcal{N}_s$. Section 4 is devoted to the infeasible case when the iterates are confined to the $\mathcal{N}_s$ neighborhood. Final comments, observations and possible directions of future work are discussed in Section 5.

Notation We define $\|\cdot\|$ as the Euclidean norm for vectors and the induced ℓ_2-norm for matrices. We will use the short versions F_k and J_k to describe $F(x^k, \lambda^k, z^k)$ and $J(x^k, \lambda^k, z^k)$, respectively. In addition, we will use both inline (x^k, λ^k, z^k) and matrix $\begin{bmatrix} x^k \\ \lambda^k \\ z^k \end{bmatrix}$ notations to address vectors in this work.

2 Background

Given a function $G : \mathbb{R}^N \to \mathbb{R}^N$, suppose that we want to solve the nonlinear system $G(\bar{x}) = 0$. Secant methods iteratively construct a linear model $M_k(\bar{x})$ of G which interpolates the last two computed iterates of the method. At each iteration, they need to compute an approximation to the Jacobian of G, which has to satisfy the secant equation

$$Bs_{k-1} = y_{k-1},$$

where $s_{k-1} = \bar{x}^k - \bar{x}^{k-1}$ and $y_{k-1} = G(\bar{x}^k) - G(\bar{x}^{k-1})$. There are infinitely many solutions to the secant equation for $N \geq 2$ and different approaches generate different secant methods [16]. Among them, the Broyden "bad" approach uses the already computed approximation to the inverse of G at $\bar{x}^{k-1}$, called H_{k-1}, to compute the current approximation H_k as

$$H_k = H_{k-1} + \frac{(s_{k-1} - H_{k-1}y_{k-1})y_{k-1}^T}{y_{k-1}^T y_{k-1}} = H_{k-1}V_{k-1} + \frac{s_{k-1}y_{k-1}^T}{\rho_{k-1}}, \tag{5}$$

where $V_{k-1} = \left(I - \frac{y_{k-1}y_{k-1}^T}{\rho_{k-1}}\right)$ and $\rho_{k-1} = y_{k-1}^T y_{k-1}$. The Broyden "bad" update is a rank-1 update where H_k is the matrix closest to H_{k-1} in the Frobenius norm which satisfies the secant equation. After ℓ updates of an approximation $H_{k-\ell}$, current approximation H_k is given by

$$\begin{aligned} H_k &= H_{k-1}V_{k-1} + \frac{s_{k-1}y_{k-1}^T}{\rho_{k-1}} \\ &= H_{k-\ell}\left(\prod_{j=k-\ell}^{k-1} V_j\right) + \sum_{i=1}^{\ell}\left(\frac{s_{k-i}y_{k-i}^T}{\rho_{k-i}} \prod_{j=k-i+1}^{k-1} V_j\right). \end{aligned} \tag{6}$$

For the specific case of this work, where G is given by F defined in (2), we have that $N = 2n + m$, $\bar{x} = (x, \lambda, z)$ and the vectors s_{k-1} and y_{k-1} from the secant equation assume a more specific description

$$s_{k-1} = \bar{\alpha}_{k-1}\begin{bmatrix} \Delta x^{k-1} \\ \Delta\lambda^{k-1} \\ \Delta z^{k-1} \end{bmatrix} \quad \text{and} \quad y_{k-1} = \begin{bmatrix} \bar{\alpha}_{k-1}(A^T\Delta\lambda^{k-1} + \Delta z^{k-1}) \\ \bar{\alpha}_{k-1}A\Delta x^{k-1} \\ X^kZ^ke - X^{k-1}Z^{k-1}e \end{bmatrix}, \tag{7}$$

where $\bar{\alpha}_{k-1} \in (0, 1]$ is the step-size taken at iteration $k - 1$ towards the solution of (4).

In [10], the authors described an interior point method based on low rank quasi-Newton approximations to the Jacobian of F. The Broyden updates were tested, and the computational experience revealed that the most efficient one was the Broyden "bad" update.

Since we are interested in finding an approximate solution of the linear system given by the Newton method (4), in the Broyden "bad" approach, given $\ell \geq 0$, the following direction is computed

$$\begin{bmatrix} \Delta x^k \\ \Delta\lambda^k \\ \Delta z^k \end{bmatrix} = H_k \begin{bmatrix} c - A^T\lambda^k - z^k \\ b - Ax^k \\ \sigma_k\mu_k e - X^kZ^ke \end{bmatrix}. \tag{8}$$

If $H_{k-\ell} = J_{k-\ell}^{-1}$ and $\ell = 0$, system (8) turns out to be exactly (4). Therefore, in the same way as discussed in [10], we assume that the initial approximation $H_{k-\ell}$ is given by the perfect approximation $J_{k-\ell}^{-1}$. When $\ell > 0$, the quasi-Newton procedure strongly uses the fact that the factorization of $J_{k-\ell}$ (or a good preconditioner) has already been computed. In addition, Lemma 1, taken from [10], shows that, with this choice of initial approximation, the Broyden "bad" update has an interesting alternative interpretation.

Lemma 1. *Assume that $k, \ell \geq 0$ and H_k is the approximation of J_k^{-1} constructed by ℓ updates (6) using initial approximation $H_{k-\ell} = J_{k-\ell}^{-1}$. Given $v \in \mathbb{R}^{2n+m}$, the computation of $r = H_kv$ is equivalent to the solution of*

$$J_{k-\ell}r = v + \begin{bmatrix} 0 \\ 0 \\ \sum_{i=1}^{\ell} \gamma_i \left[\bar{\alpha}_{k-i}\left(Z^{k-\ell}\Delta x^{k-i} + X^{k-\ell}\Delta z^{k-i}\right) - \left(X^{k-i+1}Z^{k-i+1} - X^{k-i}Z^{k-i}\right)e\right] \end{bmatrix},$$

where $\gamma_i = \dfrac{y_{k-i}^T \prod_{j=k-i+1}^{k-1} V_j}{\rho_{k-i}} v$, *for* $i = 1, \ldots, \ell$.

Lemma 1 is the basis of the analysis developed in this work. It states that we can study quasi-Newton steps using the Jacobian of the Newton step. The only difference is the right-hand side. Using this property of the Broyden "bad" update we are able to extend the well known complexity results described in [21]. The difficulty in the analysis will be mostly caused by the extra term, added to the usual right-hand side of (4). It is important to note that Lemma 1 does not assume that the iterates are feasible, hence it is useful in both feasible and infeasible cases. Although by (6), the sparsity structure of the third row of B_k (the inverse of H_k) is lost when $\ell \geq 1$, we can see that the structural sparsity of $J_{k-\ell}$ can still be used to solve the linear systems.

Let us define a skeleton primal-dual quasi-Newton interior point algorithm. It is given by Algorithm 1 and generates a sequence of alternating Newton and quasi-Newton steps. Clearly, by the nature of update (5), the first step needs to be a Newton step.

Algorithm 1 Conceptual Quasi-Newton Interior Point algorithm.

```
Input: F, J and (x^0, λ^0, z^0)
for k = 0, 1, ... do
    if k is odd then
        ℓ ← 1                                        ▷ Quasi-Newton iteration
    else
        ℓ ← 0                                        ▷ Newton iteration
    end if
    Calculate (Δx^k, Δλ^k, Δz^k) by solving (8)
    Calculate
        (x^{k+1}, λ^{k+1}, z^{k+1}) = (x^k, λ^k, z^k) + ᾱ_k(Δx^k, Δλ^k, Δz^k)
    for a suitable choice of ᾱ_k ∈ [0, 1], such that x^{k+1}, z^{k+1} > 0
end for
```

Let us analyze what happens when a sequence of two steps is performed: at iteration k the Newton step is made (with stepsize $\bar{\alpha}_k$) and then at iteration $k+1$ the quasi-Newton step is taken (with stepsize $\bar{\alpha}$). For Newton step at iteration k, we observe that

$$\begin{aligned} X^{k+1}Z^{k+1}e &= \left(X^k + \bar{\alpha}_k \Delta X^k\right)\left(Z^k + \bar{\alpha}_k \Delta Z^k\right)e \\ &= X^k Z^k e + \bar{\alpha}_k \left(Z^k \Delta x^k + X^k \Delta z^k\right) + \bar{\alpha}_k^2 \Delta X^k \Delta Z^k e \\ &= X^k Z^k e + \bar{\alpha}_k(\sigma_k \mu_k e - X^k Z^k e) + \bar{\alpha}_k^2 \Delta X^k \Delta Z^k e \\ &= (1-\bar{\alpha}_k) X^k Z^k e + \bar{\alpha}_k \sigma_k \mu_k e + \bar{\alpha}_k^2 \Delta X^k \Delta Z^k e. \end{aligned} \tag{9}$$

Later, in the proofs of several technical results, we will need to analyze the error produced when the quasi-Newton direction $(\Delta x^{k+1}, \Delta\lambda^{k+1}, \Delta z^{k+1})$ is multiplied by J_{k+1}:

$$\begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ Z^{k+1} & 0 & X^{k+1} \end{bmatrix} \begin{bmatrix} \Delta x^{k+1} \\ \Delta\lambda^{k+1} \\ \Delta z^{k+1} \end{bmatrix} = \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ Z^{k+1} - Z^k & 0 & X^{k+1} - X^k \end{bmatrix} + \begin{bmatrix} 0 & A^T & I \\ A & 0 & 0 \\ Z^k & 0 & X^k \end{bmatrix} \right) \begin{bmatrix} \Delta x^{k+1} \\ \Delta\lambda^{k+1} \\ \Delta z^{k+1} \end{bmatrix}.$$

Applying Lemma 1 for iteration $k+1$ with $\ell = 1$ and then observing that $(\Delta x^k, \Delta\lambda^k, \Delta z^k)$ solves the Newton system (4) and using (9), the third block equation in Lemma 1 gives

$$\begin{aligned} Z^k \Delta x^{k+1} + X^k \Delta z^{k+1} &= \sigma_{k+1}\mu_{k+1} e - X^{k+1} Z^{k+1} e + \gamma_1 \left[\bar{\alpha}_k \left(Z^k \Delta x^k + X^k \Delta z^k\right) - \left(X^{k+1} Z^{k+1} - X^k Z^k\right) e\right] \\ &= \sigma_{k+1}\mu_{k+1} e - X^{k+1} Z^{k+1} e + \gamma_1 (\bar{\alpha}_k \sigma_k \mu_k e + (1-\bar{\alpha}_k) X^k Z^k e - X^{k+1} Z^{k+1} e) \\ &= \sigma_{k+1}\mu_{k+1} e - X^{k+1} Z^{k+1} e - \gamma_1 \bar{\alpha}_k^2 \Delta X^k \Delta Z^k e. \end{aligned} \tag{10}$$

Hence, using (7) and (10)

$$\begin{aligned} Z^{k+1}\Delta x^{k+1} + X^{k+1}\Delta z^{k+1} &= (Z^{k+1} - Z^k)\Delta x^{k+1} + (X^{k+1} - X^k)\Delta z^{k+1} + Z^k \Delta x^{k+1} + X^k \Delta z^{k+1} \\ &= \bar{\alpha}_k \left(\Delta Z^k \Delta X^{k+1} e + \Delta X^k \Delta Z^{k+1} e\right) + \sigma_{k+1}\mu_{k+1} e - X^{k+1} Z^{k+1} e - \gamma_1 \bar{\alpha}_k^2 \Delta X^k \Delta Z^k e, \end{aligned} \tag{11}$$

where ΔZ^k, ΔX^k, ΔZ^{k+1} and ΔX^{k+1} are given by $\mathrm{diag}(\Delta z^k)$, $\mathrm{diag}(\Delta x^k)$, $\mathrm{diag}(\Delta z^{k+1})$ and $\mathrm{diag}(\Delta x^{k+1})$, respectively. Next we compute the new complementarity products obtained after a sequence of two steps, apply (11), and add and subtract the term $\bar{\alpha}_k^2 \Delta X^k \Delta Z^k e$ to derive

$$\begin{aligned}
X^{k+2}Z^{k+2}e &= (X^{k+1} + \bar{\alpha}\Delta X^{k+1})(Z^{k+1} + \bar{\alpha}\Delta Z^{k+1})e \\
&= X^{k+1}Z^{k+1}e + \bar{\alpha}(Z^{k+1}\Delta X^{k+1} + X^{k+1}\Delta Z^{k+1})e + \bar{\alpha}^2 \Delta X^{k+1}\Delta Z^{k+1} e \\
&= X^{k+1}Z^{k+1}e + \bar{\alpha}\bar{\alpha}_k(\Delta Z^k \Delta X^{k+1} e + \Delta X^k \Delta Z^{k+1} e) + \bar{\alpha}^2 \Delta X^{k+1}\Delta Z^{k+1} e + \bar{\alpha}_k^2 \Delta X^k \Delta Z^k e \\
&\quad - \bar{\alpha}_k^2 \Delta X^k \Delta Z^k e + \bar{\alpha}\sigma_{k+1}\mu_{k+1} e - \bar{\alpha} X^{k+1}Z^{k+1} e - \gamma_1 \bar{\alpha}\bar{\alpha}_k^2 \Delta X^k \Delta Z^k e \\
&= (1-\bar{\alpha})X^{k+1}Z^{k+1}e + (\bar{\alpha}_k \Delta X^k + \bar{\alpha}\Delta X^{k+1})(\bar{\alpha}_k \Delta Z^k + \bar{\alpha}\Delta Z^{k+1})e \\
&\quad + \bar{\alpha}\sigma_{k+1}\mu_{k+1} e - (1+\bar{\alpha}\gamma_1)\bar{\alpha}_k^2 \Delta X^k \Delta Z^k e.
\end{aligned} \tag{12}$$

By multiplying both sides of equation (12) with e^T we get the complementarity product at iteration $k+2$:

$$\begin{aligned}
&(x^{k+1} + \bar{\alpha}\Delta x^{k+1})^T (z^{k+1} + \bar{\alpha}\Delta z^{k+1}) = \\
&\quad = (1 - \bar{\alpha}(1-\sigma_{k+1}))x^{k+1^T} z^{k+1} + (\bar{\alpha}_k \Delta x^k + \bar{\alpha}\Delta x^{k+1})^T (\bar{\alpha}_k \Delta z^k + \bar{\alpha}\Delta z^{k+1}) - (1+\gamma_1\bar{\alpha})\bar{\alpha}_k^2 \Delta x^{k^T} \Delta z^k.
\end{aligned} \tag{13}$$

It is worth noting that the final expression in (12) involves a composite direction $(\bar{\alpha}_k \Delta x^k + \bar{\alpha}\Delta x^{k+1}, \bar{\alpha}_k \Delta z^k + \bar{\alpha}\Delta z^{k+1})$ which corresponds to an aggregate of two consecutive steps: in Newton direction at iteration k and in quasi-Newton direction at iteration $k+1$. Much of the effort of the analysis presented in this paper is focused on this composite direction. Let us mention that we will also use the component-wise versions of equations (9), (11) and (12). For example, in case of (12) this gives

$$\begin{aligned}
&\left[x^{k+1} + \bar{\alpha}\Delta x^{k+1}\right]_i \left[z^{k+1} + \bar{\alpha}\Delta z^{k+1}\right]_i = (1-\bar{\alpha})\left(x_i^{k+1} z_i^{k+1}\right) \\
&\quad + \left[\bar{\alpha}_k \Delta x^k + \bar{\alpha}\Delta x^{k+1}\right]_i \left[\bar{\alpha}_k \Delta z^k + \bar{\alpha}\Delta z^{k+1}\right]_i + \bar{\alpha}\sigma_{k+1}\mu_{k+1} - (1+\bar{\alpha}\gamma_1)\bar{\alpha}_k^2 [\Delta x^k]_i [\Delta z^k]_i.
\end{aligned} \tag{14}$$

Observe that equations (9)-(14) are valid for both feasible and infeasible algorithm. However, the analysis in Sections 3 and 4 will distinguish between these two cases because in the feasible one we are able to take advantage of the orthogonality of primal and dual directions and exploit it to deliver better final worst-case complexity results.

Before we conclude the brief background section and take the reader through a detailed analysis of different versions of the primal-dual quasi-Newton interior point algorithm, let us observe that equation (12) involves an important term γ_1. By Lemma 1, γ_1 can be seen as the scalar coefficient of the projection of vector v onto the subspace generated by vector y_k:

$$\mathcal{P}_{y_k}(v) = \frac{y_k^T v}{y_k^T y_k} y_k = \gamma_1 y_k.$$

Using the non-expansive property of projections we conclude that

$$\|\mathcal{P}_{y_k}(v)\| \leq \|v\| \iff \|\gamma_1 y_k\| \leq \|v\| \iff |\gamma_1| \leq \frac{\|v\|}{\|y_k\|}. \tag{15}$$

In the next lemma, a lower bound for $\|y_k\|$ is derived. It states that the denominator of (15) can be bounded away from zero if a sufficient decrease of $\mu = x^T z/n$ is ensured and non-null step-sizes are taken. The bound for $\|v\|$ involves the right-hand side in (8) and therefore depends on the feasibility of iterates and on the choice of the centering parameter σ.

Lemma 2. *Let $k+1$ be a quasi-Newton iteration of Algorithm 1 and y_k be the quasi-Newton vector defined by* (7) *to construct H_{k+1} by the Broyden "bad" update* (6)*. Suppose that $\mu_{k+1} \leq (1-\rho_k\bar{\alpha}_k)\mu_k$ holds, for $\bar{\alpha}_k, \rho_k \in [0,1]$. Then*

$$\|y_k\| \geq \frac{\rho_k}{2}\bar{\alpha}_k \mu_k.$$

Proof. If $\bar{\alpha}_k = 0$ or $\rho_k = 0$ the result trivially holds, so we can assume that $\bar{\alpha}_k, \rho_k \in (0,1]$. Suppose, by contradiction, that $\|y_k\| < \rho_k\bar{\alpha}_k\mu_k/2$. Therefore, by definition of y_k,

$$\begin{aligned}
\|X^{k+1}Z^{k+1}e - X^k Z^k e\| \leq \|y_k\| < \frac{\rho_k\bar{\alpha}_k}{2}\mu_k &\Rightarrow |x_i^{k+1} z_i^{k+1} - x_i^k z_i^k| < \frac{\rho_k\bar{\alpha}_k}{2}\mu_k, i = 1,\ldots,n \\
&\Rightarrow x_i^{k+1} z_i^{k+1} - x_i^k z_i^k > -\frac{\rho_k\bar{\alpha}_k}{2}\mu_k, i = 1,\ldots,n \\
&\Rightarrow \mu_{k+1} - \mu_k > -\frac{\rho_k\bar{\alpha}_k}{2}\mu_k,
\end{aligned}$$

where the last result was obtained by adding up all the n previous inequalities and dividing by n. By hypothesis we have that $\mu_{k+1} \leq (1 - \rho_k \bar{\alpha}_k)\mu_k$ and, therefore,

$$-\frac{\rho_k \bar{\alpha}_k}{2}\mu_k < \mu_{k+1} - \mu_k \leq -\rho_k \bar{\alpha}_k \mu_k,$$

which implies $\rho_k/2 > \rho_k$ and is a clear absurd. Thus, we conclude that $\|y_k\| \geq \rho_k \bar{\alpha}_k \mu_k/2$. □

3 Worst-case complexity in the feasible case

For all the results in this section, we suppose that (x^0, λ^0, z^0) is primal and dual feasible, given by Assumption 1.

Assumption 1. $(x^0, \lambda^0, z^0) \in \mathcal{F} \doteq \{(x, \lambda, z) \mid Ax = b, A^T\lambda + z = c, x > 0, z > 0\}$.

Our analysis follows closely the theory in [21]. Under Assumption 1 the primal and dual directions are orthogonal to each other [21, Lemma 5.1]. We show that the same holds for quasi-Newton directions.

Lemma 3. *If $k+1$ is a quasi-Newton iteration of Algorithm 1, then* ${\Delta x^{k+1}}^T \Delta z^{k+1} = 0$.

Proof. Using Lemma 1 with $r = Hv$ defined by their respective terms in (8) and by the primal and dual feasibility of (x^0, λ^0, z^0), we observe that the first two block rows of system (8) (at iteration $k+1$) given by

$$\left\{ \begin{array}{ccccc} & A^T\Delta\lambda^{k+1} & + & \Delta z^{k+1} & = 0 \\ A\Delta x^{k+1} & & & & = 0 \end{array} \right.$$

are the same as in the system solved by the usual Newton step. Therefore,

$${\Delta x^{k+1}}^T \Delta z^{k+1} = -{\Delta x^{k+1}}^T A^T \Delta\lambda^{k+1} = -\left(A\Delta x^{k+1}\right)^T \Delta\lambda^{k+1} = 0.$$

□

Corollary 1. *If k is a Newton iteration and $k+1$ is a quasi-Newton iteration, then*

$$\left(\bar{\alpha}_k \Delta x^k + \bar{\alpha}\Delta x^{k+1}\right)^T \left(\bar{\alpha}_k \Delta z^k + \bar{\alpha}\Delta z^{k+1}\right) = {\Delta x^k}^T \Delta z^k = {\Delta x^k}^T \Delta z^{k+1} = {\Delta z^k}^T \Delta x^{k+1} = 0.$$

Proof. Since, by equation (7), $(\Delta x^k, \Delta\lambda^k, \Delta z^k)$ was computed by the Newton step at iteration k we have that ${\Delta x^k}^T \Delta z^k = 0$ by the same arguments of Lemma 3. By Lemma 1 the first two equations do not change between iterations k and $k+1$, and we have the desired results. □

The second important result is that the quasi-Newton step can decrease the barrier parameter μ in exactly the same way as the Newton step (see [21, Lemma 5.1]). Recall that by definition $\mu = (x^T z)/n$.

Lemma 4. *Let $k+1$ be a quasi-Newton iteration of Algorithm 1. Then for any feasible step-size $\bar{\alpha} \in [0, 1]$*

$$\mu(\bar{\alpha}) = (1 - \bar{\alpha}(1 - \sigma_{k+1}))\mu_{k+1}. \tag{16}$$

Proof. By equation (13) and using Corollary 1, we obtain

$$\begin{aligned} n\mu(\bar{\alpha}) &= (x^{k+2})^T z^{k+2} = (x^{k+1} + \bar{\alpha}\Delta x^{k+1})^T (z^{k+1} + \bar{\alpha}\Delta z^{k+1}) \\ &= (1 - \bar{\alpha}){x^{k+1}}^T z^{k+1} + (\bar{\alpha}_k \Delta x^k + \bar{\alpha}\Delta x^{k+1})^T (\bar{\alpha}_k \Delta z^k + \bar{\alpha}\Delta z^{k+1}) + \bar{\alpha}\sigma_{k+1} n\mu_{k+1} - (1 + \bar{\alpha}\gamma_1)\bar{\alpha}_k^2 {\Delta x^k}^T \Delta z^k \\ &= (1 - \bar{\alpha}(1 - \sigma_{k+1}))n\mu_{k+1}. \end{aligned}$$

By dividing both sides of the last equation by n we get the desired result. □

Then, using Lemma 4, after Newton step at iteration k and quasi-Newton step at iteration $k+1$, we get

$$\begin{aligned} \|(\mu(\bar{\alpha}) - \mu_k)e\|^2 = (\mu(\bar{\alpha}) - \mu_k)^2 n \ &= \ [(1 - \bar{\alpha}(1 - \sigma_{k+1}))\mu_{k+1} - \mu_k]^2 n \\ &= [(1 - \bar{\alpha}(1 - \sigma_{k+1}))(1 - \bar{\alpha}_k(1 - \sigma_k))\mu_k - \mu_k]^2 n \\ &= [1 - (1 - \bar{\alpha}(1 - \sigma_{k+1}))(1 - \bar{\alpha}_k(1 - \sigma_k))]^2 \mu_k^2 n. \end{aligned} \tag{17}$$

It is worth noting that (in the feasible case) the term γ_1 originating from Lemma 1 does not have any influence on the value of $\mu(\bar{\alpha})$.

3.1 The $\mathcal{N}_2$ neighborhood

In this section we will consider a short-step interior point method and employ the notion of $\mathcal{N}_2$ neighborhood of the central path

$$\mathcal{N}_2(\theta) = \{(x, \lambda, z) \in \mathcal{F} \mid \|XZe - \mu e\| \leq \theta\mu\},$$

where $\mathcal{F}$ is the set of primal and dual feasible points such that $x, z > 0$, see Assumption 1. For all considerations in this subsection we add the following assumption.

Assumption 2. $(x^k, \lambda^k, z^k) \in \mathcal{N}_2(\theta_k)$ *and* $(x^{k+1}, \lambda^{k+1}, z^{k+1}) \in \mathcal{N}_2(\theta_{k+1})$, *for* $\theta_k, \theta_{k+1} \in (0, 1)$.

Our main goal is to show that the new iterate

$$(x^{k+2}, \lambda^{k+2}, z^{k+2}) = (x^{k+1}, \lambda^{k+1}, z^{k+1}) + \bar{\alpha}(\Delta x^{k+1}, \Delta\lambda^{k+1}, \Delta z^{k+1})$$

also belongs to $\mathcal{N}_2(\theta_{k+2})$, for suitable choices of $\theta_{k+2} \in (0, 1)$, $\bar{\alpha}$ and $\bar{\alpha}_k$. Therefore, we are interested in the analysis of the Euclidean norm of the vector $(X^{k+1} + \bar{\alpha}\Delta X^{k+1})(Z^{k+1} + \bar{\alpha}\Delta Z^{k+1})e - \mu(\bar{\alpha})e$ and to deliver it we will exploit several useful results stated earlier in equations (12), (14) and Lemma 4. Combining (14) and Lemma 4 we get

$$\begin{aligned} &\left[x^{k+1} + \bar{\alpha}\Delta x^{k+1}\right]_i \left[z^{k+1} + \bar{\alpha}\Delta z^{k+1}\right]_i - \mu(\bar{\alpha}) = \\ &\quad = (1 - \bar{\alpha})\left(x_i^{k+1} z_i^{k+1} - \mu_{k+1}\right) + \left[\bar{\alpha}_k \Delta x^k + \bar{\alpha}\Delta x^{k+1}\right]_i \left[\bar{\alpha}_k \Delta z^k + \bar{\alpha}\Delta z^{k+1}\right]_i - (1 + \bar{\alpha}\gamma_1)\bar{\alpha}_k^2 [\Delta x^k]_i [\Delta z^k]_i. \end{aligned} \tag{18}$$

By (18) and Assumption 2, we deliver the following bound on the proximity measure of the $\mathcal{N}_2$ neighborhood of the iterate after the quasi-Newton step

$$\begin{aligned} \|&\left(X^{k+1} + \bar{\alpha}\Delta X^{k+1}\right)\left(Z^{k+1} + \bar{\alpha}\Delta Z^{k+1}\right)e - \mu(\bar{\alpha})e\| = \\ &= \|\left\{(1 - \bar{\alpha})\left(x_i^{k+1} z_i^{k+1} - \mu_{k+1}\right) + \left[\bar{\alpha}_k \Delta x^k + \bar{\alpha}\Delta x^{k+1}\right]_i \left[\bar{\alpha}_k \Delta z^k + \bar{\alpha}\Delta z^{k+1}\right]_i \right. \\ &\qquad \left. -(1 + \bar{\alpha}\gamma_1)\bar{\alpha}_k^2 [\Delta x^k]_i [\Delta z^k]_i\right\}_{i=1}^n \| \\ &\leq (1 - \bar{\alpha})\|X^{k+1} Z^{k+1} e - \mu_{k+1} e\| + \|\left(\bar{\alpha}_k \Delta X^k + \bar{\alpha}\Delta X^{k+1}\right)\left(\bar{\alpha}_k \Delta Z^k + \bar{\alpha}\Delta Z^{k+1}\right)e\| \\ &\quad + |1 + \bar{\alpha}\gamma_1|\bar{\alpha}_k^2 \|\Delta X^k \Delta Z^k e\| \\ &\leq (1 - \bar{\alpha})\theta_{k+1}\mu_{k+1} + |1 + \bar{\alpha}\gamma_1|\bar{\alpha}_k^2 \frac{\theta_k^2 + n(1 - \sigma_k)^2}{2^{3/2}(1 - \theta_k)}\mu_k \\ &\quad + \|\left(\bar{\alpha}_k \Delta X^k + \bar{\alpha}\Delta X^{k+1}\right)\left(\bar{\alpha}_k \Delta Z^k + \bar{\alpha}\Delta Z^{k+1}\right)e\|. \end{aligned} \tag{19}$$

In the last inequality we used the bound on the error in the Newton step $\|\Delta X^k \Delta Z^k e\|$, see [21, Lemma 5.4].

To further exploit (19) we need bounds on two terms which appear in it: $1 + \bar{\alpha}\gamma_1$ and the second-order error contributed by the composite direction $\|\left(\bar{\alpha}_k \Delta X^k + \bar{\alpha}\Delta X^{k+1}\right)\left(\bar{\alpha}_k \Delta Z^k + \bar{\alpha}\Delta Z^{k+1}\right)e\|$. The following technical result delivers a bound for $|\gamma_1|$. (Observe that γ_1 is evaluated only when the quasi-Newton iteration is performed.)

Lemma 5. *Let* $k + 1$ *be a quasi-Newton iteration of Algorithm 1. Suppose that* v *in Lemma 1 is given by the right-hand side of* (4) *and Assumption 2 holds. If* $\bar{\alpha}_k \in (0, 1]$ *and* $\sigma_k \in [0, 1)$, *then*

$$|\gamma_1| \leq \frac{2(1 - \bar{\alpha}_k(1 - \sigma_k))\sqrt{\theta_{k+1}^2 + (1 - \sigma_{k+1})^2 n}}{\bar{\alpha}_k(1 - \sigma_k)},$$

where γ_1 *is defined in Lemma 1.*

Proof. We use the assumptions of the lemma, the fact that the iterates are primal and dual feasible, the property $e^T(\mu_{k+1}e - X^{k+1}Z^{k+1}e) = 0$ and the relation $\mu_{k+1} = (1 - \bar{\alpha}_k(1 - \sigma_k))\mu_k$ to derive the result

$$\begin{aligned} \|v\| &= \|\sigma_{k+1}\mu_{k+1}e - X^{k+1}Z^{k+1}e\| = \|(\mu_{k+1}e - X^{k+1}Z^{k+1}e) - (1 - \sigma_{k+1})\mu_{k+1}e\| \\ &= \sqrt{\|\mu_{k+1}e - X^{k+1}Z^{k+1}e\|^2 - 2(1 - \sigma_{k+1})\mu_{k+1}e^T(\mu_{k+1}e - X^{k+1}Z^{k+1}e) + \|(1 - \sigma_{k+1})\mu_{k+1}e\|^2} \\ &\leq \sqrt{\theta_{k+1}^2 \mu_{k+1}^2 + (1 - \sigma_{k+1})^2 \mu_{k+1}^2 n} = (1 - \bar{\alpha}_k(1 - \sigma_k))\mu_k \sqrt{\theta_{k+1}^2 + (1 - \sigma_{k+1})^2 n}\ . \end{aligned}$$

By defining $\rho_k = 1-\sigma_k$ we can see that $\mu_{k+1} = (1-(1-\sigma_k)\bar{\alpha}_k)\mu_k = (1-\rho_k\bar{\alpha}_k)\mu_k$ which ensures the sufficient decrease condition of Lemma 2. Since $\bar{\alpha}_k > 0$ and $\sigma_k < 1$, by the assumptions of the lemma, we have that $\|y_k\| > 0$. Then, by simple substitution of the previous equation and Lemma 2 in (15) we have

$$|\gamma_1| \le \frac{\|v_k\|}{\|y_k\|} \le \frac{(1-\bar{\alpha}_k(1-\sigma_k))\mu_k\sqrt{\theta_{k+1}^2+(1-\sigma_{k+1})^2 n}}{\frac{1-\sigma_k}{2}\bar{\alpha}_k\mu_k} = \frac{2(1-\bar{\alpha}_k(1-\sigma_k))\sqrt{\theta_{k+1}^2+(1-\sigma_{k+1})^2 n}}{\bar{\alpha}_k(1-\sigma_k)}.$$

□

Next we turn our attention to the error in the composite direction $\|\left(\bar{\alpha}_k\Delta X^k + \bar{\alpha}\Delta X^{k+1}\right)\left(\bar{\alpha}_k\Delta Z^k + \bar{\alpha}\Delta Z^{k+1}\right)e\|$ and start from a technical result.

Lemma 6. *If $k+1$ is a quasi-Newton iteration of Algorithm 1, then*

$$\begin{aligned} &Z^k\left(\bar{\alpha}_k\Delta X^k+\bar{\alpha}\Delta X^{k+1}\right)e + X^k\left(\bar{\alpha}_k\Delta Z^k+\bar{\alpha}\Delta Z^{k+1}\right)e = \\ &\quad = (\bar{\alpha}+\bar{\alpha}_k(1-\bar{\alpha}))\left(\mu_k e - X^k Z^k e\right) + (\mu(\bar{\alpha})-\mu_k)e - (1+\gamma_1)\bar{\alpha}\bar{\alpha}_k^2\Delta X^k\Delta Z^k e. \end{aligned} \tag{20}$$

Proof. We use equations (4), (10) and (9) and some simple manipulations to obtain

$$\begin{aligned} Z^k&\left(\bar{\alpha}_k\Delta X^k+\bar{\alpha}\Delta X^{k+1}\right)e + X^k\left(\bar{\alpha}_k\Delta Z^k+\bar{\alpha}\Delta Z^{k+1}\right)e = \bar{\alpha}_k\left(Z^k\Delta x^k + X^k\Delta z^k\right) + \bar{\alpha}\left(Z^k\Delta x^{k+1}+X^k\Delta z^{k+1}\right) \\ &= \bar{\alpha}_k(\sigma_k\mu_k e - X^kZ^ke) + \bar{\alpha}\left(\sigma_{k+1}\mu_{k+1}e - X^{k+1}Z^{k+1}e - \gamma_1\bar{\alpha}_k^2\Delta X^k\Delta Z^k e\right) \\ &= \bar{\alpha}_k(\sigma_k\mu_k e - X^kZ^ke) + \bar{\alpha}\left(\sigma_{k+1}\mu_{k+1}e - (1-\bar{\alpha}_k)X^kZ^ke - \bar{\alpha}_k\sigma_k\mu_k e - \bar{\alpha}_k^2\Delta X^k\Delta Z^k e - \gamma_1\bar{\alpha}_k^2\Delta X^k\Delta Z^k e\right) \\ &= (1-\bar{\alpha})\bar{\alpha}_k\sigma_k\mu_k e - (\bar{\alpha}+\bar{\alpha}_k(1-\bar{\alpha}))X^kZ^ke + \bar{\alpha}\sigma_{k+1}\mu_{k+1}e - (1+\gamma_1)\bar{\alpha}\bar{\alpha}_k^2\Delta X^k\Delta Z^k e. \end{aligned} \tag{21}$$

After adding and subtracting the term $(\bar{\alpha}+\bar{\alpha}_k(1-\bar{\alpha}))\mu_k e$ we further rearrange the previous equation

$$\begin{aligned} &Z^k\left(\bar{\alpha}_k\Delta X^k+\bar{\alpha}\Delta X^{k+1}\right)e + X^k\left(\bar{\alpha}_k\Delta Z^k+\bar{\alpha}\Delta Z^{k+1}\right)e = \\ &\quad = (\bar{\alpha}+\bar{\alpha}_k(1-\bar{\alpha}))\left(\mu_k e - X^kZ^ke\right) - (1+\gamma_1)\bar{\alpha}\bar{\alpha}_k^2\Delta X^k\Delta Z^k e \\ &\quad\quad + ((1-\bar{\alpha})\bar{\alpha}_k\sigma_k - (\bar{\alpha}+\bar{\alpha}_k(1-\bar{\alpha})))\mu_k e + \bar{\alpha}\sigma_{k+1}\mu_{k+1}e. \end{aligned} \tag{22}$$

Then using $\mu_{k+1} = (1-\bar{\alpha}_k(1-\sigma_k))\mu_k$ (which clearly holds for a step in Newton direction) and Lemma 4 which delivers a similar result for a step in quasi-Newton direction, we get:

$$\begin{aligned} ((1-\bar{\alpha})\bar{\alpha}_k\sigma_k - (\bar{\alpha}+\bar{\alpha}_k(1-\bar{\alpha})))\mu_k e + \bar{\alpha}\sigma_{k+1}\mu_{k+1}e &= (\bar{\alpha}_k\sigma_k - \bar{\alpha}\bar{\alpha}_k\sigma_k - \bar{\alpha} - \bar{\alpha}_k + \bar{\alpha}\bar{\alpha}_k)\mu_k e + \bar{\alpha}\sigma_{k+1}\mu_{k+1}e \\ &= \left[-\bar{\alpha}(1-\bar{\alpha}_k(1-\sigma_k)) - \bar{\alpha}_k(1-\sigma_k)\right]\mu_k e + \bar{\alpha}\sigma_{k+1}\mu_{k+1}e \\ &= \left[-1-\bar{\alpha}(1-\bar{\alpha}_k(1-\sigma_k)) + 1 - \bar{\alpha}_k(1-\sigma_k)\right]\mu_k e + \bar{\alpha}\sigma_{k+1}\mu_{k+1}e \\ &= -\mu_k e - \bar{\alpha}\mu_{k+1}e + \mu_{k+1}e + \bar{\alpha}\sigma_{k+1}\mu_{k+1}e \quad = \quad (1-\bar{\alpha}(1-\sigma_{k+1}))\mu_{k+1}e - \mu_k e \\ &= \mu(\bar{\alpha})e - \mu_k e. \end{aligned} \tag{23}$$

By substituting (23) in (22), the desired result is obtained. □

Lemma 7. *Let $k+1$ be a quasi-Newton iteration of Algorithm 1 and suppose that Assumption 2 holds. Then*

$$\begin{aligned} &\|\left(\bar{\alpha}_k\Delta X^k+\bar{\alpha}\Delta X^{k+1}\right)\left(\bar{\alpha}_k\Delta Z^k+\bar{\alpha}\Delta Z^{k+1}\right)e\| \le \\ &\le \frac{\mu_k}{2^{3/2}(1-\theta_k)}\left[[1-(1-\bar{\alpha}(1-\sigma_{k+1}))(1-\bar{\alpha}_k(1-\sigma_k))]^2 n + \left((\bar{\alpha}+\bar{\alpha}_k(1-\bar{\alpha}))\theta_k + |1+\gamma_1|\bar{\alpha}\bar{\alpha}_k^2\frac{\theta_k^2+n(1-\sigma_k)^2}{2^{3/2}(1-\theta_k)}\right)^2\right]. \end{aligned}$$

Proof. Let us first define $D^k = (X^k)^{1/2}(Z^k)^{-1/2}$ and the scaled vectors

$$u_k = \left(D^k\right)^{-1}\left(\bar{\alpha}_k\Delta X^k+\bar{\alpha}\Delta X^{k+1}\right)e \quad \text{and} \quad v_k = D^k\left(\bar{\alpha}_k\Delta Z^k+\bar{\alpha}\Delta Z^{k+1}\right)e.$$

Using Lemma 3, Corollary 1 and observing that all the involved matrices are diagonal, we get $u_k^T v_k = 0$. Hence vectors u_k and v_k satisfy the assumptions of Lemma 5.3 in [21]. With $U_k = \mathrm{diag}(u_k)$ and $V_k = \mathrm{diag}(v_k)$, we write

$$\begin{aligned} \|\left(\bar{\alpha}_k\Delta X^k+\bar{\alpha}\Delta X^{k+1}\right)\left(\bar{\alpha}_k\Delta Z^k+\bar{\alpha}\Delta Z^{k+1}\right)e\| &= \|\left(D^k\right)^{-1}\left(\bar{\alpha}_k\Delta X^k+\bar{\alpha}\Delta X^{k+1}\right)D^k\left(\bar{\alpha}_k\Delta Z^k+\bar{\alpha}\Delta Z^{k+1}\right)e\| \\ &= \|U_kV_ke\| \;\le\; 2^{-3/2}\|u_k+v_k\|^2 \\ &= 2^{-3/2}\|\left(D^k\right)^{-1}\left(\bar{\alpha}_k\Delta X^k+\bar{\alpha}\Delta X^{k+1}\right)e + D^k\left(\bar{\alpha}_k\Delta Z^k+\bar{\alpha}\Delta Z^{k+1}\right)e\|^2. \end{aligned} \tag{24}$$

After multiplying both sides of (20) by $(X^kZ^k)^{-1/2}$ and replacing it in (24) we obtain

$$
\begin{aligned}
&\| \left(\bar{\alpha}_k \Delta X^k + \bar{\alpha}\Delta X^{k+1}\right)\left(\bar{\alpha}_k \Delta Z^k + \bar{\alpha}\Delta Z^{k+1}\right) e\| \le \\
&\le 2^{-3/2} \|(X^kZ^k)^{-1/2}\left\{(\bar{\alpha} + \bar{\alpha}_k(1-\bar{\alpha}))\left(\mu_k e - X^kZ^ke\right) + (\mu(\bar{\alpha}) - \mu_k)e - (1+\gamma_1)\bar{\alpha}\bar{\alpha}_k^2 \Delta X^k \Delta Z^k e\right\}\|^2 \\
&= 2^{-3/2} \sum_{i=1}^n \frac{\left\{(\bar{\alpha} + \bar{\alpha}_k(1-\bar{\alpha}))\left(\mu_k - x_i^kz_i^k\right) + \mu(\bar{\alpha}) - \mu_k - (1+\gamma_1)\bar{\alpha}\bar{\alpha}_k^2[\Delta x^k]_i[\Delta z^k]_i\right\}^2}{x_i^kz_i^k} \qquad (25)\\
&\le \frac{2^{-3/2}}{(1-\theta_k)\mu_k} \|(\bar{\alpha} + \bar{\alpha}_k(1-\bar{\alpha}))\left(\mu_k e - X^kZ^ke\right) - (1+\gamma_1)\bar{\alpha}\bar{\alpha}_k^2 \Delta X^k\Delta Z^k e + (\mu(\bar{\alpha}) - \mu_k)e\|^2,
\end{aligned}
$$

where the last inequality comes from the fact that (x^k, λ^k, z^k) belongs to $\mathcal{N}_2(\theta_k)$, hence $(1-\theta_k)\mu_k \le x_i^kz_i^k \le (1+\theta_k)\mu_k$ for all i. Now we use Lemma 3 again and the definition of μ_k to observe that

$$
e^T\left((\bar{\alpha} + \bar{\alpha}_k(1-\bar{\alpha}))\left(\mu_k e - X^kZ^ke\right) - (1+\gamma_1)\bar{\alpha}\bar{\alpha}_k^2 \Delta X^k \Delta Z^k e\right) = 0 \qquad (26)
$$

and further rearrange (25):

$$
\begin{aligned}
&\| \left(\bar{\alpha}_k \Delta X^k + \bar{\alpha}\Delta X^{k+1}\right)\left(\bar{\alpha}_k \Delta Z^k + \bar{\alpha}\Delta Z^{k+1}\right) e\| \le \\
&\le \frac{2^{-3/2}}{(1-\theta_k)\mu_k}\left[\left\|(\bar{\alpha} + \bar{\alpha}_k(1-\bar{\alpha}))\left(\mu_k e - X^kZ^ke\right) - (1+\gamma_1)\bar{\alpha}\bar{\alpha}_k^2 \Delta X^k\Delta Z^k e\right\|^2 + \|(\mu(\bar{\alpha}) - \mu_k)e\|^2\right]. \qquad (27)
\end{aligned}
$$

The second norm on the right-hand side of (27) is given by (17). Using the definition of $\mathcal{N}_2(\theta_k)$ neighborhood and Lemma 5.4 from [21], for the first norm we get

$$
\begin{aligned}
\|(\bar{\alpha} + \bar{\alpha}_k(1-\bar{\alpha}))\;&\left(\mu_k e - X^kZ^ke\right) - (1+\gamma_1)\bar{\alpha}\bar{\alpha}_k^2 \Delta X^k\Delta Z^k e\right\|^2 \le \\
&\le \left(\left\|(\bar{\alpha} + \bar{\alpha}_k(1-\bar{\alpha}))\left(\mu_k e - X^kZ^ke\right)\right\| + \left\|(1+\gamma_1)\bar{\alpha}\bar{\alpha}_k^2 \Delta X^k\Delta Z^k e\right\|\right)^2 \qquad (28)\\
&\le \left((\bar{\alpha} + \bar{\alpha}_k(1-\bar{\alpha}))\theta_k + |1+\gamma_1|\bar{\alpha}\bar{\alpha}_k^2 \frac{\theta_k^2 + n(1-\sigma_k)^2}{2^{3/2}(1-\theta_k)}\right)^2 \mu_k^2.
\end{aligned}
$$

Finally, by substituting (17) and (28) in (27) we obtain the required result. ☐

We are ready to state the main result of this subsection. In Theorem 1 we show that it is possible to choose sufficiently small values for the step-sizes $\bar{\alpha}_k$ and $\bar{\alpha}$ such that $(x^{k+2}, \lambda^{k+2}, z^{k+2}) \in \mathcal{N}_2(\theta_k)$. Therefore, we ensure that the quasi-Newton step taken after a Newton step remains in the $\mathcal{N}_2$ neighborhood. This implies that all the iterates generated by the algorithm belong to $\mathcal{N}_2$. The upper bounds for the step-sizes as stated in the theorem are then used to determine the worst-case complexity of Algorithm 1 operating in $\mathcal{N}_2$. First, recall [21, Theorem 5.6] that it is possible to choose parameters $\theta_k, \theta_{k+1} \in (0, 1)$ and $\sigma_k, \sigma_{k+1} \in (0, 1)$ so that

$$
\frac{\theta_k^2 + n(1-\sigma_k)^2}{2^{3/2}(1-\theta_k)} \le \theta_k\sigma_k \quad \text{and} \quad \frac{\theta_{k+1}^2 + n(1-\sigma_{k+1})^2}{2^{3/2}(1-\theta_{k+1})} \le \theta_{k+1}\sigma_{k+1}. \qquad (29)
$$

Theorem 1. *Let $k+1$ be a quasi-Newton iteration of Algorithm 1. Suppose that Assumptions 1 and 2 hold and that $\theta_{k+1} = \theta_k$ and $\sigma_{k+1} = \sigma_k$. If the step-sizes in Newton and quasi-Newton iterations $\bar{\alpha}_k$ and $\bar{\alpha}$ satisfy*

$$
\bar{\alpha}_k \in \left(0, \min\left\{\frac{1-\sigma_k}{4\sigma_k}, \frac{\sigma_k(1-\sigma_k)}{10(1-\sigma_k)+4}\right\}\right] \quad \textit{and} \quad \bar{\alpha} \in \left[\bar{\alpha}_k, \frac{\sigma_k(1-\sigma_k)}{10(1-\sigma_k)+4}\right] \qquad (30)
$$

for $\sigma_k \in [0, 1)$ and $\theta_k \in (0, 16/25)$, then

$$
(x^{k+2}, \lambda^{k+2}, z^{k+2}) \doteq (x^{k+1}, \lambda^{k+1}, z^{k+1}) + \bar{\alpha}(\Delta x^{k+1}, \Delta\lambda^{k+1}, \Delta z^{k+1}) \in \mathcal{N}_2(\theta_k).
$$

Proof. We will first show that for all $\bar{\alpha}_k$ and $\bar{\alpha}$ satisfying the conditions of the theorem

$$
\| \left(X^{k+1} + \bar{\alpha}\Delta X^{k+1}\right)\left(Z^{k+1} + \bar{\alpha}\Delta Z^{k+1}\right) e - \mu(\bar{\alpha})e\| - \theta_k\mu(\bar{\alpha}) \le 0 \qquad (31)
$$

holds. This condition ensures that the Newton step at iteration $k+2$ will be successfully performed and the algorithm converges. By inequality (19), condition (31) is satisfied if

$$
\begin{aligned}
[(1-\bar{\alpha})\theta_{k+1}\mu_{k+1} - \theta_k\mu(\bar{\alpha})] + |1+\bar{\alpha}\gamma_1|\bar{\alpha}_k^2 &\frac{\theta_k^2 + n(1-\sigma_k)^2}{2^{3/2}(1-\theta_k)}\mu_k \\
&+ \| \left(\bar{\alpha}_k\Delta X^k + \bar{\alpha}\Delta X^{k+1}\right)\left(\bar{\alpha}_k\Delta Z^k + \bar{\alpha}\Delta Z^{k+1}\right) e\| \le 0. \qquad (32)
\end{aligned}
$$

We will derive bounds to each term on the left-hand side of this inequality in order to find an expression in a form $K_1\bar{\alpha}^2 - K_2\bar{\alpha}$, $K_1, K_2 > 0$, which will be nonpositive for small values of $\bar{\alpha}$.

For the first term, we use the fact that $\theta_{k+1} = \theta_k$, $\sigma_{k+1} = \sigma_k$ and $\bar{\alpha}_k \in (0, 1)$. In addition, we use Lemma 4 to expand $\mu(\bar{\alpha})$ and the fact that μ_{k+1} was calculated in the Newton step k to obtain

$$(1-\bar{\alpha})\theta_{k+1}\mu_{k+1} - \theta_k\mu(\bar{\alpha}) = [(1-\bar{\alpha})(\theta_{k+1}-\theta_k) - \bar{\alpha}\sigma_{k+1}\theta_k](1-\bar{\alpha}_k(1-\sigma_k))\mu_k \leq -\bar{\alpha}\sigma_k^2\theta_k\mu_k. \tag{33}$$

For the second and third terms, we first apply (29) in Lemma 5 to simplify the bound of γ_1:

$$\begin{aligned} |\gamma_1| &\leq \frac{2(1-\bar{\alpha}_k(1-\sigma_k))\sqrt{2^{3/2}(1-\theta_{k+1})}\sqrt{\frac{\theta_{k+1}^2+(1-\sigma_{k+1})^2 n}{2^{3/2}(1-\theta_{k+1})}}}{\bar{\alpha}_k(1-\sigma_k)} \\ &\leq \frac{2(1-\bar{\alpha}_k(1-\sigma_k))\sqrt{2^{3/2}(1-\theta_{k+1})}\sqrt{\theta_{k+1}\sigma_{k+1}}}{\bar{\alpha}_k(1-\sigma_k)} \;\leq\; \frac{4}{\bar{\alpha}_k(1-\sigma_k)} - 4. \end{aligned} \tag{34}$$

Therefore, using (29) again, we derive the following bound to the second term

$$\begin{aligned} |1+\bar{\alpha}\gamma_1|\bar{\alpha}_k^2\frac{\theta_k^2+n(1-\sigma_k)^2}{2^{3/2}(1-\theta_k)}\mu_k &\leq \left[1+\bar{\alpha}\left(\frac{4}{\bar{\alpha}_k(1-\sigma_k)}-4\right)\right]\bar{\alpha}_k^2\theta_k\sigma_k\mu_k \\ &= \bar{\alpha}_k^2\theta_k\sigma_k\mu_k + 4\bar{\alpha}\bar{\alpha}_k\left(\frac{1-\bar{\alpha}_k(1-\sigma_k)}{1-\sigma_k}\right)\theta_k\sigma_k\mu_k \;\leq\; \bar{\alpha}^2\left[1+\frac{4}{1-\sigma_k}\right]\theta_k\sigma_k\mu_k, \end{aligned} \tag{35}$$

where in the last inequality we used the condition $\bar{\alpha}_k \leq \bar{\alpha} \leq 1$ from (30). For the third term in (32), we use the bound obtained in Lemma 7 and analyze each part of it independently. First, since $\bar{\alpha}_k \leq \bar{\alpha}$ and $\sigma_{k+1} = \sigma_k$, we observe that

$$\begin{aligned} [1-(1-\bar{\alpha}(1-\sigma_{k+1}))(1-\bar{\alpha}_k(1-\sigma_k))]^2 n &= [\bar{\alpha}_k(1-\sigma_k)+\bar{\alpha}(1-\bar{\alpha}_k(1-\sigma_k))(1-\sigma_k))]^2 n \\ &\leq \bar{\alpha}^2(1-\sigma_k)^2[1+(1-\bar{\alpha}_k(1-\sigma_k))]^2 n \leq 4\bar{\alpha}^2(1-\sigma_k)^2 n. \end{aligned} \tag{36}$$

Using bound (34), assumption $\bar{\alpha}_k \leq (1-\sigma_k)/(4\sigma_k)$ in (30), and (29) again, we also obtain

$$\begin{aligned} \left[(\bar{\alpha}+\bar{\alpha}_k(1-\bar{\alpha}))\theta_k + |1+\gamma_1|\bar{\alpha}\bar{\alpha}_k^2\frac{\theta_k^2+n(1-\sigma_k)^2}{2^{3/2}(1-\theta_k)}\right]^2 &\leq \left[\bar{\alpha}+\bar{\alpha}_k(1-\bar{\alpha})+\left(\frac{4}{\bar{\alpha}_k(1-\sigma_k)}-3\right)\bar{\alpha}\bar{\alpha}_k^2\sigma_k\right]^2\theta_k^2 \\ &\leq \left[\bar{\alpha}+\bar{\alpha}_k(1-\bar{\alpha})+\frac{4}{\bar{\alpha}_k(1-\sigma_k)}\bar{\alpha}\bar{\alpha}_k^2\sigma_k\right]^2\theta_k^2 \\ &\leq \bar{\alpha}^2\left[1+(1-\bar{\alpha})+\frac{4\bar{\alpha}_k\sigma_k}{(1-\sigma_k)}\right]^2\theta_k^2 \\ &\leq 9\bar{\alpha}^2\theta_k^2. \end{aligned} \tag{37}$$

By combining (36) and (37) in the statement of Lemma 7 and applying (29) once more, we derive a bound to the third term of (32)

$$\|\left(\bar{\alpha}_k\Delta X^k+\bar{\alpha}\Delta X^{k+1}\right)\left(\bar{\alpha}_k\Delta Z^k+\bar{\alpha}\Delta Z^{k+1}\right)e\| \leq \frac{\mu_k}{2^{3/2}(1-\theta_k)}\left(4\bar{\alpha}^2(1-\sigma_k)^2 n+9\bar{\alpha}^2\theta_k^2\right) \leq 9\bar{\alpha}^2\theta_k\sigma_k\mu_k. \tag{38}$$

By (33), (35) and (38), the following bound on expression in (32) is obtained

$$\begin{aligned} (1-\bar{\alpha})\theta_{k+1}\mu_{k+1} - \theta_k\mu(\bar{\alpha}) + |1+\bar{\alpha}\gamma_1|\bar{\alpha}_k^2\frac{\theta_k^2+n(1-\sigma_k)^2}{2^{3/2}(1-\theta_k)}\mu_k& \\ +\|\left(\bar{\alpha}_k\Delta X^k+\bar{\alpha}\Delta X^{k+1}\right)\left(\bar{\alpha}_k\Delta Z^k+\bar{\alpha}\Delta Z^{k+1}\right)e\|& \\ \leq -\bar{\alpha}\sigma_k^2\theta_k\mu_k + \bar{\alpha}^2\left[1+\frac{4}{1-\sigma_k}\right]\theta_k\sigma_k\mu_k + 9\bar{\alpha}^2\theta_k\sigma_k\mu_k& \\ \leq \bar{\alpha}\left[10\bar{\alpha}+\frac{4\bar{\alpha}}{1-\sigma_k}-\sigma_k\right]\theta_k\sigma_k\mu_k,& \end{aligned} \tag{39}$$

which is negative only if $\bar{\alpha} \leq \sigma_k(1-\sigma_k)/(10(1-\sigma_k)+4)$, as requested by (30). This bound on $\bar{\alpha}$ implies a bound on $\bar{\alpha}_k$, due to condition $\bar{\alpha}_k \leq \bar{\alpha}$. Therefore, we arrive in the step-size conditions (30) of the theorem. Using (31) we also have that

$$\left(x_i^{k+1}+\bar{\alpha}[\Delta x^{k+1}]_i\right)\left(z_i^{k+1}+\bar{\alpha}[\Delta z^{k+1}]_i\right) \geq (1-\theta_k)\mu(\bar{\alpha}) > 0. \tag{40}$$

We now show that the new iterate belongs to $\mathcal{F}$. By Assumption 1 and Lemma 1 we know that all iterates remain primal and dual feasible. It remains to show that x^{k+2} and z^{k+2} are strictly positive. We follow the same arguments as [17] and adapt them to our case. Suppose by contradiction that $x_i^{k+2} \le 0$ or $z_i^{k+2} \le 0$ hold for some i. By (40), we have that $x_i^{k+2} < 0$ and $z_i^{k+2} < 0$ and that implies $x_i^k z_i^k < (\bar{\alpha}_k[\Delta x^k]_i + \bar{\alpha}[\Delta x^{k+1}]_i)(\bar{\alpha}_k[\Delta z^k]_i + \bar{\alpha}[\Delta z^{k+1}]_i) \le 9\bar{\alpha}^2\theta_k\sigma_k\mu_k$ by inequality (38). Since $(x^k, \lambda^k, z^k) \in \mathcal{N}_2(\theta_k)$ and $\bar{\alpha} \le 1/4$, by (30), we conclude that $(1-\theta_k)\mu_k < (9/16)\theta_k\mu_k$ and, therefore, $\theta_k > 16/25$, which contradicts the choice of θ_k. Hence, $(x^{k+2}, \lambda^{k+2}, z^{k+2})$ belongs to $\mathcal{N}_2(\theta_k)$ and the Newton step at iteration $k+2$ also falls in the $\mathcal{N}_2(\theta_k)$ neighborhood. □

For the polynomial convergence of Algorithm 1, we define $\sigma_k = \sigma_{k+1} = 1 - 0.4/\sqrt{n}$ and $\theta_k = \theta_{k+1} = \theta_{k+2} = 0.4$, which satisfy condition (29) [21] and maintain all the previous results. By Lemma 4, $\mu(\bar{\alpha}) \le \mu_{k+1} \le \mu_k$. So it is enough to look just at the Newton steps, which are easier to analyze. Using (30) it is not hard to see that

$$\bar{\alpha}_k \ge \min\left\{\frac{1-\sigma_k}{4\sigma_k}, \frac{\sigma_k(1-\sigma_k)}{10(1-\sigma_k)+4}\right\} \ge \min\left\{\frac{0.1}{\sqrt{n}}, \frac{0.03}{\sqrt{n}}\right\} = \frac{0.03}{\sqrt{n}}.$$

Therefore,

$$\mu_{k+1} \le \left(1 - \frac{0.03}{\sqrt{n}}\frac{0.4}{\sqrt{n}}\right)\mu_k = \left(1 - \frac{0.012}{n}\right)\mu_k, \quad k = 0, 2, 4, \ldots,$$

from which the convergence with the worst-case iteration complexity of $O(n)$ can be derived.

3.2 The $\mathcal{N}_s$ neighborhood

Colombo and Gondzio [3] used the symmetric neighborhood $\mathcal{N}_s(\gamma)$, defined by

$$\mathcal{N}_s(\gamma) \doteq \left\{(x, \lambda, z) \in \mathcal{F} \mid \gamma\mu \le x_i z_i \le \frac{1}{\gamma}\mu, i = 1, \ldots, n\right\},$$

for $\gamma \in (0,1)$, which is related with the $\mathcal{N}_{-\infty}(\gamma)$ neighborhood used in long-step primal-dual interior point algorithms. The idea of the symmetric neighborhood is to add an upper bound on the complementarity pairs, so that their products do not become too large with respect to the average. The authors showed that the worst-case iteration complexity for linear feasible primal-dual interior point methods remains $O(n)$ and the new neighborhood has a better practical interpretation. As `HOPDM` [7] implements the $\mathcal{N}_s$ neighborhood and it was used in the numerical experiments in [10] for quasi-Newton IPM, it is natural to ask about the iteration complexity of Algorithm 1 operating in the $\mathcal{N}_s$ neighborhood. The analysis presented below will follow closely that from Subsection 3.1. We start from an assumption, but it is worth observing that, from [3, 21], this assumption holds if the step-size in the Newton direction is sufficiently small: $\bar{\alpha}_k \in \left[0, \min\left\{2^{3/2}\frac{1-\gamma}{1+\gamma}\frac{\sigma_k}{n}, 2^{3/2}\gamma\frac{1-\gamma}{1+\gamma}\frac{\sigma_k}{n}\right\}\right]$.

Assumption 3. *Let $\gamma \in (0,1)$ and $(x^k, \lambda^k, z^k) \in \mathcal{N}_s(\gamma)$. Let the iterate after a step $\bar{\alpha}_k$ in Newton direction also satisfy $(x^{k+1}, \lambda^{k+1}, z^{k+1}) \in \mathcal{N}_s(\gamma)$.*

Our main goal is to show that the next iterate obtained after a step in the quasi-Newton direction

$$(x^{k+2}, \lambda^{k+2}, z^{k+2}) = (x^{k+1}, \lambda^{k+1}, z^{k+1}) + \bar{\alpha}(\Delta x^{k+1}, \Delta\lambda^{k+1}, \Delta z^{k+1})$$

also belongs to $\mathcal{N}_s(\gamma)$ if suitable step-sizes $\bar{\alpha}$ and $\bar{\alpha}_k$ are chosen. To demonstrate this, we will consider lower and upper bounds on the complementarity products in the $\mathcal{N}_s(\gamma)$ neighborhood using two possible values of $\zeta \in \{\gamma, \frac{1}{\gamma}\}$. We start the analysis from expanding the complementarity product at the quasi-Newton iteration

$$\left[(X^{k+1} + \bar{\alpha}\Delta X^{k+1})(Z^{k+1} + \bar{\alpha}\Delta Z^{k+1})e\right]_i - \zeta\mu(\bar{\alpha}), \tag{41}$$

where $\bar{\alpha} \in (0,1)$ is the step-size associated with the quasi-Newton direction. To guarantee that the new iterate belongs to $\mathcal{N}_s(\gamma)$, for $\zeta = \gamma$ the expression in (41) should be non-negative and for $\zeta = 1/\gamma$ it should be non-positive, for $i = 1, \ldots, n$. Using (14) and Lemma 4, we rewrite the expression in (41)

$$\begin{aligned}\left[x^{k+1} + \bar{\alpha}\Delta x^{k+1}\right]_i \left[z^{k+1} + \bar{\alpha}\Delta z^{k+1}\right]_i - \zeta\mu(\bar{\alpha}) \quad &= \quad (1-\bar{\alpha})(x_i^{k+1}z_i^{k+1} - \zeta\mu_{k+1}) + (1-\zeta)\bar{\alpha}\sigma_{k+1}\mu_{k+1} \\ &+ \left[\bar{\alpha}_k\Delta x^k + \bar{\alpha}\Delta x^{k+1}\right]_i \left[\bar{\alpha}_k\Delta z^k + \bar{\alpha}\Delta z^{k+1}\right]_i - (1+\gamma_1\bar{\alpha})\bar{\alpha}_k^2[\Delta x^k]_i[\Delta z^k]_i.\end{aligned} \tag{42}$$

To deliver the main result of this section we will need a bound for the quasi-Newton term γ_1 defined in Lemma 1 when the algorithm operates in the $\mathcal{N}_s(\gamma)$ neighborhood.

Lemma 8. *Let $k+1$ be a quasi-Newton iteration of Algorithm 1 operating in the $\mathcal{N}_s(\gamma)$ neighborhood. Suppose that v in Lemma 1 is given by the right-hand side of (4) and Assumption 3 holds. If $\bar{\alpha}_k \in (0,1]$ and $\sigma_k \in [0,1)$, then*

$$|\gamma_1| \leq \frac{2\sqrt{n}}{(1-\sigma_k)\bar{\alpha}_k \gamma}.$$

Proof. Using the conditions of the lemma, the definition of μ and the fact that $\sigma_{k+1} \in [0,1]$, we have that

$$\begin{aligned}
\|v\| &= \|\sigma_{k+1}\mu_{k+1}e - X^{k+1}Z^{k+1}e\| = \sqrt{\sigma_{k+1}^2\mu_{k+1}^2 n - 2\sigma_{k+1}\mu_{k+1}^2 n + \sum_{i=1}^n (x_i^{k+1}z_i^{k+1})^2} \\
&\leq \sqrt{(1/\gamma^2)\mu_{k+1}^2 n - (2-\sigma_{k+1})\sigma_{k+1}\mu_{k+1}^2 n} \leq \sqrt{(1/\gamma^2) - \sigma_{k+1}}\sqrt{n}\mu_{k+1} \\
&\leq \frac{\sqrt{n}}{\gamma}\mu_{k+1} = \frac{(1-\bar{\alpha}_k(1-\sigma_k))\sqrt{n}}{\gamma}\mu_k \leq \frac{\sqrt{n}}{\gamma}\mu_k.
\end{aligned}$$

Since $\bar{\alpha}_k \in (0,1]$ and $\sigma_k \in [0,1)$, by defining $\rho_k = 1-\sigma_k > 0$ we can again use $\mu_{k+1} = (1-\bar{\alpha}_k(1-\sigma_k))\mu_k$ and Lemma 2 to ensure that $\|y_k\| > 0$. Therefore, by (15), Lemma 2 and the previous result,

$$|\gamma_1| \leq \frac{\|v\|}{\|y_k\|} \leq \frac{(1-\bar{\alpha}_k(1-\sigma_k))\sqrt{n}\mu_k}{\gamma}\frac{2}{(1-\sigma_k)\bar{\alpha}_k\mu_k} = \frac{2(1-\bar{\alpha}_k(1-\sigma_k))\sqrt{n}}{(1-\sigma_k)\bar{\alpha}_k\gamma} \leq \frac{2\sqrt{n}}{(1-\sigma_k)\bar{\alpha}_k\gamma}.$$

□

The next lemma delivers a bound for term $\left[\bar{\alpha}_k \Delta x^k + \bar{\alpha}\Delta x^{k+1}\right]_i \left[\bar{\alpha}_k \Delta z^k + \bar{\alpha}\Delta z^{k+1}\right]_i$ under Assumption 3. Most of the calculations have already been made in the proof of Lemma 7. Let us mention that Lemmas 7 and 9 can be viewed as the quasi-Newton versions of [21, Lemma 5.10].

Lemma 9. *Let $k+1$ be a quasi-Newton iteration of Algorithm 1 and suppose that Assumption 3 holds. Then,*

$$\begin{aligned}
&\|\left(\bar{\alpha}_k\Delta X^k + \bar{\alpha}\Delta X^{k+1}\right)\left(\bar{\alpha}_k\Delta Z^k + \bar{\alpha}\Delta Z^{k+1}\right)e\| \leq \\
&\leq \frac{n\mu_k}{2^{3/2}\gamma}\left\{\left[(\bar{\alpha} + \bar{\alpha}_k(1-\bar{\alpha})) + \frac{|1+\gamma_1|\bar{\alpha}\bar{\alpha}_k^2}{2^{3/2}}\right]^2\left(\frac{1+\gamma}{\gamma}\right)^2 n + [1-(1-\bar{\alpha}(1-\sigma_{k+1}))(1-\bar{\alpha}_k(1-\sigma_k))]^2\right\}.
\end{aligned}$$

Proof. We observe that many arguments used at the beginning of the proof of Lemma 7 remain valid for the $\mathcal{N}_s$ neighborhood. (Only the bound $x_i^k z_i^k \geq (1-\theta_k)\mu_k$ needs to be replaced with $x_i^k z_i^k \geq \gamma\mu_k$.) Then inequalities (25) and (27) are replaced with the following

$$\begin{aligned}
&\|\left(\bar{\alpha}_k\Delta X^k + \bar{\alpha}\Delta X^{k+1}\right)\left(\bar{\alpha}_k\Delta Z^k + \bar{\alpha}\Delta Z^{k+1}\right)e\| \leq \\
&\quad\leq 2^{-3/2}\sum_{i=1}^n \frac{\left\{(\bar{\alpha}+\bar{\alpha}_k(1-\bar{\alpha}))\left(\mu_k - x_i^k z_i^k\right) + \mu(\bar{\alpha}) - \mu_k - (1+\gamma_1)\bar{\alpha}\bar{\alpha}_k^2[\Delta x^k]_i[\Delta z^k]_i\right\}^2}{x_i^k z_i^k} \\
&\quad\leq \frac{1}{2^{3/2}\gamma\mu_k}\|(\bar{\alpha}+\bar{\alpha}_k(1-\bar{\alpha}))\left(\mu_k e - X^k Z^k e\right) + (\mu(\bar{\alpha}) - \mu_k)e - (1+\gamma_1)\bar{\alpha}\bar{\alpha}_k^2\Delta X^k\Delta Z^k e\|^2 \\
&\quad= \frac{1}{2^{3/2}\gamma\mu_k}\left[\|(\bar{\alpha}+\bar{\alpha}_k(1-\bar{\alpha}))\left(\mu_k e - X^k Z^k e\right) - (1+\gamma_1)\bar{\alpha}\bar{\alpha}_k^2\Delta X^k\Delta Z^k e\|^2 + \|(\mu(\bar{\alpha})-\mu_k)e\|^2\right],
\end{aligned} \tag{43}$$

where in the last equality we have used equation (26). We already have the expression for $\|(\mu(\bar{\alpha})-\mu_k)e\|^2$ (see (17)), but we need a bound for the first norm in (43). We observe that

$$\|\mu_k e - X^k Z^k e\| = \sqrt{\sum_{i=1}^n (x_i^k z_i^k)^2 - 2\mu_k^2 n + \mu_k^2 n} \leq \sqrt{\frac{\mu_k^2 n}{\gamma^2} - \mu_k^2 n} \leq \frac{1}{\gamma}\sqrt{n}\mu_k \leq \frac{1+\gamma}{\gamma}\sqrt{n}\mu_k. \tag{44}$$

Therefore, by (44) and the bound on $\left\|\Delta X^k \Delta Z^k e\right\|$ obtained in [21, Lemma 5.10] (which also holds for $\mathcal{N}_s$ [3])

$$\begin{aligned}
&\left\|(\bar{\alpha}+\bar{\alpha}_k(1-\bar{\alpha}))\left(\mu_k e - X^k Z^k e\right) - (1+\gamma_1)\bar{\alpha}\bar{\alpha}_k^2\Delta X^k\Delta Z^k e\right\|^2 \leq \\
&\qquad\leq \left[(\bar{\alpha}+\bar{\alpha}_k(1-\bar{\alpha}))\left\|\mu_k e - X^k Z^k e\right\| + |1+\gamma_1|\bar{\alpha}\bar{\alpha}_k^2\left\|\Delta X^k\Delta Z^k e\right\|\right]^2 \\
&\qquad\leq \left[(\bar{\alpha}+\bar{\alpha}_k(1-\bar{\alpha}))\left(\frac{1+\gamma}{\gamma}\right)\sqrt{n}\mu_k + \frac{|1+\gamma_1|\bar{\alpha}\bar{\alpha}_k^2}{2^{3/2}}\left(\frac{1+\gamma}{\gamma}\right)\mu_k n\right]^2 \qquad (45) \\
&\qquad\leq \left[(\bar{\alpha}+\bar{\alpha}_k(1-\bar{\alpha})) + \frac{|1+\gamma_1|\bar{\alpha}\bar{\alpha}_k^2}{2^{3/2}}\right]^2\left(\frac{1+\gamma}{\gamma}\right)^2 n^2\mu_k^2,
\end{aligned}$$

since $n \geq 1$. Using (17) and (45) in (43), we obtain the desired result. $\square$

Using (42) and the bounds obtained so far, we now show that, for sufficiently small step-sizes $\bar{\alpha}_k$ and $\bar{\alpha}$, in the Newton and quasi-Newton iterations, respectively, the point $(x^{k+2}, \lambda^{k+2}, z^{k+2})$ also belongs to $\mathcal{N}_s(\gamma)$. The upper bounds for the step-sizes delivered by the theorem below are then used to determine the $O(n^3)$ iteration worst-case complexity of Algorithm 1 operating in $\mathcal{N}_s(\gamma)$.

Theorem 2. *Suppose that $k+1$ is a quasi-Newton iteration of Algorithm 1 and Assumption 3 holds. Define*

$$l = \frac{\frac{1}{2}\sigma_{min}}{\frac{3}{2^{3/2}\gamma}\left(2 + \frac{1}{\gamma(1-\sigma_{max})}\right)^2 \left(\frac{1+\gamma}{\gamma}\right)^2},$$

where $0 < \sigma_{min} \leq \sigma_k \leq \sigma_{max} < 1$ for all $k = 0, 1, 2, \ldots$ If

$$\bar{\alpha}_k \in \left(0, \frac{(1-\gamma)l}{2n^3}\right] \quad \text{and} \quad \bar{\alpha} \in \left[2\bar{\alpha}_k, \frac{(1-\gamma)l}{n^3}\right] \tag{46}$$

then $\gamma\mu(\bar{\alpha}) \leq x_i^{k+2} z_i^{k+2} \leq (1/\gamma)\mu(\bar{\alpha})$ for all $i = 1, \ldots, n$. If, in addition, $\gamma \geq \sigma_{min}/4$, then $(x^{k+2}, \lambda^{k+2}, z^{k+2}) \in \mathcal{N}_s(\gamma)$.

Proof. By construction we guarantee that $\bar{\alpha} \geq 2\bar{\alpha}_k$ as needed in (46). We start the proof by setting $\xi = \gamma$ in (42) and showing that $\left[x^{k+1} + \bar{\alpha}\Delta x^{k+1}\right]_i \left[z^{k+1} + \bar{\alpha}\Delta z^{k+1}\right]_i - \gamma\mu(\bar{\alpha}) \geq 0$ for sufficiently small step-sizes. By (42), using Assumption 3, Lemma 9 and [21, Lemma 5.10], we obtain

$$\begin{aligned}
&\left[x^{k+1} + \bar{\alpha}\Delta x^{k+1}\right]_i \left[z^{k+1} + \bar{\alpha}\Delta z^{k+1}\right]_i - \gamma\mu(\bar{\alpha}) = \\
&\quad = (1-\bar{\alpha})(x_i^{k+1} z_i^{k+1} - \gamma\mu_{k+1}) + (1-\gamma)\bar{\alpha}\sigma_{k+1}\mu_{k+1} \\
&\quad\quad + \left[\bar{\alpha}_k\Delta x^k + \bar{\alpha}\Delta x^{k+1}\right]_i \left[\bar{\alpha}_k\Delta z^k + \bar{\alpha}\Delta z^{k+1}\right]_i - (1+\gamma_1\bar{\alpha})\bar{\alpha}_k^2[\Delta x^k]_i[\Delta z^k]_i \\
&\quad \geq (1-\gamma)\bar{\alpha}\sigma_{k+1}(1-\bar{\alpha}_k(1-\sigma_k))\mu_k - (1+\gamma_1\bar{\alpha})\bar{\alpha}_k^2 \left(\frac{1+\gamma}{\gamma}\right) \frac{n}{2^{3/2}}\mu_k \\
&\quad\quad - \frac{n}{2^{3/2}\gamma}\left\{\left[\bar{\alpha} + \bar{\alpha}_k(1-\bar{\alpha}) + \frac{|1+\gamma_1|\bar{\alpha}\bar{\alpha}_k^2}{2^{3/2}}\right]^2 \left(\frac{1+\gamma}{\gamma}\right)^2 n + [1-(1-\bar{\alpha}(1-\sigma_{k+1}))(1-\bar{\alpha}_k(1-\sigma_k))]^2\right\}\mu_k \\
&\quad = \mu_k\Bigg\{(1-\gamma)\bar{\alpha}\sigma_{k+1}(1-\bar{\alpha}_k(1-\sigma_k)) - (1+\gamma_1\bar{\alpha})\bar{\alpha}_k^2\left(\frac{1+\gamma}{\gamma}\right)\frac{n}{2^{3/2}} \\
&\quad\quad - \left[\bar{\alpha} + \bar{\alpha}_k(1-\bar{\alpha}) + \frac{|1+\gamma_1|\bar{\alpha}\bar{\alpha}_k^2}{2^{3/2}}\right]^2\left(\frac{1+\gamma}{\gamma}\right)^2 \frac{n^2}{2^{3/2}\gamma} - [1-(1-\bar{\alpha}(1-\sigma_{k+1}))(1-\bar{\alpha}_k(1-\sigma_k))]^2 \frac{n}{2^{3/2}\gamma}\Bigg\}.
\end{aligned} \tag{47}$$

We will rearrange this expression and represent it in a form $K_1\bar{\alpha} - K_2\bar{\alpha}^2$ with $K_1, K_2 > 0$. Next, we will show that (47) is non-negative for sufficiently small values of $\bar{\alpha}_k$ and $\bar{\alpha}$. To deliver the desired results every term inside the curly brackets in (47) will be bounded. Since $\bar{\alpha}_k \leq \frac{1}{2}\bar{\alpha} \leq \frac{1}{2}$, we have $1 - \bar{\alpha}_k(1-\sigma_k) \geq 1 - \frac{1}{2}(1-\sigma_k) \geq \frac{1}{2}$ hence for the first term in (47) we obtain

$$(1-\gamma)\bar{\alpha}\sigma_{k+1}(1-\bar{\alpha}_k(1-\sigma_k)) \geq \frac{1}{2}[(1-\gamma)\sigma_{k+1}]\bar{\alpha}. \tag{48}$$

Using condition $\bar{\alpha} \geq 2\bar{\alpha}_k$ (guaranteed by (46)), $n \geq 1$ and Lemma 8, the second term becomes

$$\begin{aligned}
(1+\gamma_1\bar{\alpha})\bar{\alpha}_k^2\left(\frac{1+\gamma}{\gamma}\right)\frac{n}{2^{3/2}} &\leq (\bar{\alpha}_k^2 + |\gamma_1|\bar{\alpha}\bar{\alpha}_k^2)\left(\frac{1+\gamma}{\gamma}\right)\frac{n}{2^{3/2}} \leq \left(\bar{\alpha}_k^2 + \frac{2\sqrt{n}}{\gamma(1-\sigma_k)}\bar{\alpha}\bar{\alpha}_k\right)\left(\frac{1+\gamma}{\gamma}\right)\frac{n}{2^{3/2}} \\
&\leq \left(\frac{\bar{\alpha}^2}{4} + \frac{\sqrt{n}}{\gamma(1-\sigma_k)}\bar{\alpha}^2\right)\left(\frac{1+\gamma}{\gamma}\right)\frac{n}{2^{3/2}} = \left(\frac{1}{4} + \frac{\sqrt{n}}{\gamma(1-\sigma_k)}\right)\left(\frac{1+\gamma}{\gamma}\right)\frac{n}{2^{3/2}}\bar{\alpha}^2 \\
&\leq \left(2 + \frac{1}{\gamma(1-\sigma_k)}\right)\left(\frac{1+\gamma}{\gamma}\right)\frac{n\sqrt{n}}{2^{3/2}\gamma}\bar{\alpha}^2.
\end{aligned} \tag{49}$$

By applying condition $\bar{\alpha} \geq 2\bar{\alpha}_k$ again, we conclude that

$$\bar{\alpha} + \bar{\alpha}_k(1-\bar{\alpha}) = \bar{\alpha} + \bar{\alpha}_k - \bar{\alpha}\bar{\alpha}_k \leq \bar{\alpha} + \frac{1}{2}\bar{\alpha} - \bar{\alpha}\bar{\alpha}_k \leq \frac{3}{2}\bar{\alpha} \tag{50}$$

and, using (50) and the same arguments as before, for the third term in (47) we obtain

$$\begin{aligned}\left[\bar{\alpha}+\bar{\alpha}_k(1-\bar{\alpha})+\frac{|1+\gamma_1|\bar{\alpha}\bar{\alpha}_k^2}{2^{3/2}}\right]^2\left(\frac{1+\gamma}{\gamma}\right)^2\frac{n^2}{2^{3/2}\gamma} &\leq \left[\frac{3}{2}\bar{\alpha}+\left(1+\frac{2\sqrt{n}}{\gamma(1-\sigma_k)\bar{\alpha}_k}\right)\frac{\bar{\alpha}\bar{\alpha}_k^2}{2^{3/2}}\right]^2\left(\frac{1+\gamma}{\gamma}\right)^2\frac{n^2}{2^{3/2}\gamma}\\ &\leq \left(\frac{3}{2}+\frac{1}{2}+\frac{\sqrt{n}}{\gamma(1-\sigma_k)}\right)^2\left(\frac{1+\gamma}{\gamma}\right)^2\frac{n^2}{2^{3/2}\gamma}\bar{\alpha}^2\\ &\leq \left(2+\frac{1}{\gamma(1-\sigma_k)}\right)^2\left(\frac{1+\gamma}{\gamma}\right)^2\frac{n^3}{2^{3/2}\gamma}\bar{\alpha}^2.\end{aligned} \tag{51}$$

In a similar fashion, the last term of (47) can be bounded as

$$\begin{aligned}[1-(1-\bar{\alpha}(1-\sigma_{k+1}))(1-\bar{\alpha}_k(1-\sigma_k))]^2\frac{n}{2^{3/2}\gamma} &= [1-(1-\bar{\alpha}_k(1-\sigma_k))+\bar{\alpha}(1-\sigma_{k+1})(1-\bar{\alpha}_k(1-\sigma_k))]^2\frac{n}{2^{3/2}\gamma}\\ &= [\bar{\alpha}_k(1-\sigma_k)+\bar{\alpha}(1-\sigma_{k+1})(1-\bar{\alpha}_k(1-\sigma_k))]^2\frac{n}{2^{3/2}\gamma}\\ &\leq \left[\frac{(1-\sigma_k)}{2}+(1-\sigma_{k+1})(1-\bar{\alpha}_k(1-\sigma_k))\right]^2\frac{n}{2^{3/2}\gamma}\bar{\alpha}^2 \quad\leq\quad 4\frac{n}{2^{3/2}\gamma}\bar{\alpha}^2.\end{aligned} \tag{52}$$

Since $n \geq 1$ and assuming without loss of generality that $\left(2+\frac{1}{\gamma(1-\sigma_k)}\right)\left(\frac{1+\gamma}{\gamma}\right) \geq 4$, using (48), (49), (51) and (52) in (47) we have that

$$\begin{aligned}&\left[x^{k+1}+\bar{\alpha}\Delta x^{k+1}\right]_i\left[z^{k+1}+\bar{\alpha}\Delta z^{k+1}\right]_i-\gamma\mu(\bar{\alpha}) \geq\\ &\qquad\geq \left\{\frac{1}{2}[(1-\gamma)\sigma_{k+1}]\bar{\alpha}-3\left(2+\frac{1}{\gamma(1-\sigma_k)}\right)^2\left(\frac{1+\gamma}{\gamma}\right)^2\frac{n^3}{2^{3/2}\gamma}\bar{\alpha}^2\right\}\mu_k\\ &\qquad\geq \left\{\frac{1}{2}[(1-\gamma)\sigma_{min}]\bar{\alpha}-3\left(2+\frac{1}{\gamma(1-\sigma_{max})}\right)^2\left(\frac{1+\gamma}{\gamma}\right)^2\frac{n^3}{2^{3/2}\gamma}\bar{\alpha}^2\right\}\mu_k.\end{aligned}$$

Therefore, in order to guarantee that $\left[x^{k+1}+\bar{\alpha}\Delta x^{k+1}\right]_i\left[z^{k+1}+\bar{\alpha}\Delta z^{k+1}\right]_i \geq \gamma\mu(\bar{\alpha})$ for $i=1,\ldots,n$, it is sufficient that the quasi-Newton step-size $\bar{\alpha}$ satisfies

$$\bar{\alpha} \leq \frac{\frac{1}{2}(1-\gamma)\sigma_{min}}{\frac{3}{2^{3/2}\gamma}\left(2+\frac{1}{\gamma(1-\sigma_{max})}\right)^2\left(\frac{1+\gamma}{\gamma}\right)^2 n^3} = \frac{(1-\gamma)l}{n^3}. \tag{53}$$

We now set $\xi = 1/\gamma$ in (42). In order to show that the resulting expression is non-positive, we use the same arguments as before, that is, Lemma 9 and equations (48), (49), (51) and (52) to obtain

$$
\begin{aligned}
&\left[x^{k+1} + \bar{\alpha}\Delta x^{k+1}\right]_i \left[z^{k+1} + \bar{\alpha}\Delta z^{k+1}\right]_i - (1/\gamma)\mu(\bar{\alpha}) = \\
&\quad = (1-\bar{\alpha})(x_i^{k+1} z_i^{k+1} - (1/\gamma)\mu_{k+1}) + (1-(1/\gamma))\bar{\alpha}\sigma_{k+1}\mu_{k+1} \\
&\qquad + \left[\bar{\alpha}_k\Delta x^k + \bar{\alpha}\Delta x^{k+1}\right]_i \left[\bar{\alpha}_k \Delta z^k + \bar{\alpha}\Delta z^{k+1}\right]_i - (1+\gamma_1\bar{\alpha})\bar{\alpha}_k^2[\Delta x^k]_i[\Delta z^k]_i \\
&\quad \leq (1-(1/\gamma))\bar{\alpha}\sigma_{k+1}\mu_{k+1} + \left[\bar{\alpha}_k\Delta x^k + \bar{\alpha}\Delta x^{k+1}\right]_i \left[\bar{\alpha}_k \Delta z^k + \bar{\alpha}\Delta z^{k+1}\right]_i + |1+\gamma_1\bar{\alpha}|\bar{\alpha}_k^2 |[\Delta x^k]_i[\Delta z^k]_i| \\
&\quad \leq (1-(1/\gamma))\bar{\alpha}\sigma_{k+1}(1-\bar{\alpha}_k(1-\sigma_k))\mu_k + |1+\gamma_1\bar{\alpha}|\bar{\alpha}_k^2 \left(\frac{1+\gamma}{\gamma}\right)\frac{n}{2^{3/2}}\mu_k \\
&\qquad + \frac{n}{2^{3/2}\gamma}\left\{\left[\bar{\alpha}+\bar{\alpha}_k(1-\bar{\alpha})+\frac{|1+\gamma_1|\bar{\alpha}\bar{\alpha}_k^2}{2^{3/2}}\right]^2\left(\frac{1+\gamma}{\gamma}\right)^2 n + [1-(1-\bar{\alpha}(1-\sigma_{k+1}))(1-\bar{\alpha}_k(1-\sigma_k))]^2\right\}\mu_k \\
&\quad = \mu_k\left\{|1+\gamma_1\bar{\alpha}|\bar{\alpha}_k^2\left(\frac{1+\gamma}{\gamma}\right)\frac{n}{2^{3/2}} + \left[\bar{\alpha}+\bar{\alpha}_k(1-\bar{\alpha})+\frac{|1+\gamma_1|\bar{\alpha}\bar{\alpha}_k^2}{2^{3/2}}\right]^2\left(\frac{1+\gamma}{\gamma}\right)^2\frac{n^2}{2^{3/2}\gamma}\right. \\
&\qquad\qquad \left. + [1-(1-\bar{\alpha}(1-\sigma_{k+1}))(1-\bar{\alpha}_k(1-\sigma_k))]^2\frac{n}{2^{3/2}\gamma} - \left(\frac{1-\gamma}{\gamma}\right)\bar{\alpha}\sigma_{k+1}(1-\bar{\alpha}_k(1-\sigma_k))\right\} \\
&\quad \leq \mu_k\left\{3\left(2+\frac{1}{\gamma(1-\sigma_k)}\right)^2\left(\frac{1+\gamma}{\gamma}\right)^2\frac{n^3}{2^{3/2}\gamma}\bar{\alpha}^2 - \frac{1}{2}\sigma_{k+1}\left(\frac{1-\gamma}{\gamma}\right)\bar{\alpha}\right\} \\
&\quad \leq \mu_k\left\{3\left(2+\frac{1}{\gamma(1-\sigma_{max})}\right)^2\left(\frac{1+\gamma}{\gamma}\right)^2\frac{n^3}{2^{3/2}\gamma}\bar{\alpha}^2 - \frac{1}{2}\sigma_{min}\left(\frac{1-\gamma}{\gamma}\right)\bar{\alpha}\right\}.
\end{aligned}
\tag{54}
$$

Therefore, if

$$
\bar{\alpha} \leq \frac{\left(\frac{1-\gamma}{\gamma}\right) l}{n^3}, \tag{55}
$$

then by (54) we have that $\left[x^{k+1} + \bar{\alpha}\Delta x^{k+1}\right]_i \left[z^{k+1} + \bar{\alpha}\Delta z^{k+1}\right]_i \leq (1/\gamma)\mu(\bar{\alpha})$ for all $i = 1, \ldots, n$. Observe that bound (53) is tighter than (55) because $\gamma \in (0, 1)$ hence (53) appears as an upper bound on $\bar{\alpha}$ in (46). The remaining bounds in (46) are consistent with the need to satisfy $\bar{\alpha} \geq 2\bar{\alpha}_k$.

It remains to show that x^{k+2} and z^{k+2} are strictly positive. Similarly to Theorem 1, if, by contradiction, $x_i^{k+2} \leq 0$ or $z_i^{k+2} \leq 0$ for some i, then we must have $x_i^{k+2} < 0$ and $z_i^{k+2} < 0$. On one hand, we already know that $x_i^k z_i^k \geq \gamma\mu_k$, by Assumption 3. Hence, by Lemma 9, (51), (52) and similar arguments to those used in this proof

$$
\begin{aligned}
x_i^k z_i^k &< \left[\bar{\alpha}_k\Delta x^k + \bar{\alpha}\Delta x^{k+1}\right]_i \left[\bar{\alpha}_k\Delta z^k + \bar{\alpha}\Delta z^{k+1}\right]_i \leq \|\left(\bar{\alpha}_k\Delta X^k + \bar{\alpha}\Delta X^{k+1}\right)\left(\bar{\alpha}_k\Delta Z^k + \bar{\alpha}\Delta Z^{k+1}\right)e\| \\
&\leq \frac{3}{2^{3/2}\gamma}\left(2+\frac{1}{\gamma(1-\sigma_{max})}\right)^2\left(\frac{1+\gamma}{\gamma}\right)^2\bar{\alpha}^2\mu_k n^3 = \frac{\sigma_{min}\bar{\alpha}^2\mu_k n^3}{2l}.
\end{aligned}
$$

Hence we conclude that $\gamma < \sigma_{min} n^3\bar{\alpha}^2/(2l)$ which, together with condition $\gamma \geq \sigma_{min}/4$ and (46), implies the following absurd

$$
\frac{\sigma_{min}}{4} \leq \gamma < \frac{\sigma_{min} n^3\bar{\alpha}^2}{2l} \leq \frac{\sigma_{min} n^3\bar{\alpha}}{2l} \leq \frac{\sigma_{min}(1-\gamma)}{4} \leq \frac{\sigma_{min}}{4}.
$$

By Assumption 1, we have that $(x^{k+2}, \lambda^{k+2}, z^{k+2})$ belongs to $\mathcal{F}$, hence we conclude that it belongs to $\mathcal{N}_s(\gamma)$. □

To deliver the worst-case polynomial complexity, we follow the same arguments as those presented after the proof of Theorem 1 and consider only Newton iterations which are simpler to analyze. By (46), we have that

$$
\bar{\alpha}_k \geq \frac{(1-\gamma)l}{2n^3}
$$

and, by [21, Lemma 5.1],

$$
\mu_{k+1} = (1-\bar{\alpha}_k(1-\sigma_k))\mu_k \leq \left(1 - \frac{(1-\gamma)l}{2n^3}\right)\mu_k, \quad k = 0, 2, 4, \ldots,
$$

from which the convergence with the worst-case iteration complexity of $O(n^3)$ can be established easily.

Because of the close relation of the symmetric neighborhood and the $\mathcal{N}_{-\infty}(\gamma)$ neighborhood, one should expect similar complexity results to that of Theorem 2. However, if the $\mathcal{N}_s(\gamma)$ neighborhood is not applied in Lemma 8, then a naive approach is to bound $\sum_{i=1}^n \left(x_i^{k+1} z_i^{k+1}\right)^2$ by $\mu_{k+1}^2 n^2$. Such approach would increase the worst-case iteration complexity to $O(n^4)$, as the degree would be increased in equation (51). A different approach to reduce the worst-case polynomial degree when working with the $\mathcal{N}_{-\infty}(\gamma)$ neighborhood is subject to future developments.

4 Worst-case complexity in the infeasible case

In this section, we analyze worst-case iteration complexity for linear programming problems without assuming feasibility of the starting point. The analysis uses some ideas developed in Section 3 and follows [21, Chapter 6], by adding extra requirements on the computation of step-size $\bar{\alpha}_k$. The proofs are modified to consider the symmetric neighborhood and admit the quasi-Newton steps.

We start by defining r_b^k and r_c^k to be the primal and dual infeasibility vectors at point (x^k, λ^k, z^k) which appear in the right-hand side of equation (4):

$$r_b^k = Ax^k - b \quad \text{and} \quad r_c^k = A^T \lambda^k + z^k - c.$$

Then, given an initial guess (x^0, λ^0, z^0) and parameters $\beta \geq 1$ and $\gamma \in (0, 1)$, the infeasible version of the symmetric neighborhood is defined as follows

$$\mathcal{N}_s(\gamma, \beta) \doteq \left\{ (x, \lambda, z) \;\middle|\; \|(r_b, r_c)\| \leq \frac{\|(r_b^0, r_c^0)\|}{\mu_0} \beta \mu, x > 0, z > 0 \text{ and } \gamma \mu \leq x_i z_i \leq \frac{1}{\gamma} \mu, i = 1, \ldots, n \right\}.$$

In order to obtain complexity results, $\bar{\alpha}_k$ needs to be computed in such a way that the new iterate remains in $\mathcal{N}_s(\gamma, \beta)$ and a sufficient decrease condition for μ_k is satisfied. More precisely, given $\alpha_{\text{dec}} \in (0, 1)$, $\bar{\alpha}_k$ is the largest value in $[0, 1]$ (or a fixed fraction of it) such that

$$(x^k, \lambda^k, z^k) + \bar{\alpha}_k (\Delta x^k, \Delta \lambda^k, \Delta z^k) \in \mathcal{N}_s(\gamma, \beta) \quad \text{and} \quad \mu(\bar{\alpha}_k) \leq (1 - \alpha_{\text{dec}} \bar{\alpha}_k) \mu_k. \tag{56}$$

The goal of this section is to show that, assuming that one quasi-Newton step is performed from a point in $\mathcal{N}_s(\gamma, \beta)$ by Algorithm 1, then conditions (56) are satisfied by the new point. When only Newton steps are taken, it is shown in [21, Lemma 6.7] that there is an interval $[0, \hat{\alpha}]$ such that (56) holds for the $\mathcal{N}_{-\infty}(\gamma, \beta)$ neighborhood. This is the case if a special starting point is used and $\hat{\alpha} \geq \bar{\delta}/n^2$, where $\bar{\delta}$ is a constant independent of n. In Lemma 10, we extend those results in order to ensure that the iterates belong to $\mathcal{N}_s(\gamma, \beta)$ when only Newton steps are taken.

First, let us recall some results from [21] that will be used frequently. We define the scalar $\nu_k = \prod_{i=0}^{k-1} (1 - \bar{\alpha}_i)$, which allows us to write vectors r_b^k and r_c^k as $r_b^k = \nu_k r_b^0$ and $r_c^k = \nu_k r_c^0$, respectively. The parameter σ_k used in (8) satisfies $0 < \sigma_{min} \leq \sigma_k \leq \sigma_{max} < 1$ for all k. We also assume that a special initial point is used in Algorithm 1, given by

$$(x^0, \lambda^0, z^0) = \begin{bmatrix} \xi e & 0 & \xi e \end{bmatrix}^T, \tag{57}$$

where ξ is such that $\|(x^*, z^*)\|_\infty \leq \xi$, for some primal-dual solution (x^*, λ^*, z^*).

Let k be a Newton iteration, $(x^k, \lambda^k, z^k) \in \mathcal{N}_s(\gamma, \beta)$ and $D^k = (X^k)^{1/2} (Z^k)^{-1/2}$. Also, let $\omega = 9\beta/\gamma^{1/2}$ be a constant independent of n. When (57) is used as the starting point, the following bounds hold:

$$\nu_k \|(x^k, z^k)\|_1 \leq \frac{4\beta}{\xi} n \mu_k \tag{58}$$

$$\mu_0 = \xi^2 \tag{59}$$

$$\|(D^k)^{-1} \Delta x^k\| \leq \omega n \mu_k^{1/2} \quad \text{and} \quad \|D^k \Delta z^k\| \leq \omega n \mu_k^{1/2}. \tag{60}$$

The proofs for these bounds can be found in Lemmas 6.4 and 6.6 of [21], respectively.

We start the analysis from looking at Newton step in $\mathcal{N}_s(\gamma, \beta)$ neighborhood.

Lemma 10. *If $(x^k, \lambda^k, z^k) \in \mathcal{N}_s(\gamma, \beta)$ and the Newton step is taken at iteration k, then there exists a constant $\bar{\bar{\delta}}$ independent of n and a value $\hat{\alpha} \geq \bar{\bar{\delta}}/n^2$, independent of k, such that for all $\bar{\alpha}_k \in [0, \hat{\alpha}]$*

$$(1 - \bar{\alpha}_k) {x^k}^T z^k \leq (x^k + \bar{\alpha}_k \Delta x^k)^T (z^k + \bar{\alpha}_k \Delta z^k) \leq (1 - \alpha_{\text{dec}} \bar{\alpha}_k) {x^k}^T z^k \tag{61}$$

$$\gamma \mu(\bar{\alpha}_k) \leq [x^k + \bar{\alpha}_k \Delta x^k]_i [z^k + \bar{\alpha}_k \Delta z^k]_i \leq \frac{1}{\gamma} \mu(\bar{\alpha}_k) \tag{62}$$

Proof. We only need to address the right inequality of (62), since all the other results were detailed in [21, Lemma 6.7]. Since k is a Newton iteration, by (60), we obtain

$$|\Delta x_i^k \Delta z_i^k| \leq |[(D^k)^{-1}]_{ii} \Delta x_i^k||[D^k]_{ii} \Delta z_i^k| \leq \|(D^k)^{-1} \Delta x^k\| \|D^k \Delta z^k\| \leq \omega^2 n^2 \mu_k.$$

As $(\Delta x^k, \Delta\lambda^k, \Delta z^k)$ solves (4) and $(x^k, \lambda^k, z^k) \in \mathcal{N}_s(\gamma, \beta)$, using a component-wise version of (9) we get

$$\begin{aligned} [x^k + \bar\alpha_k \Delta x^k]_i [z^k + \bar\alpha_k \Delta z^k]_i &= (1 - \bar\alpha_k) x_i^k z_i^k + \bar\alpha_k \sigma_k \mu_k + \bar\alpha_k^2 \Delta x_i^k \Delta z_i^k \\ &\leq \frac{1 - \bar\alpha_k}{\gamma} \mu_k + \bar\alpha_k \sigma_k \mu_k + \bar\alpha_k^2 \omega^2 n^2 \mu_k. \end{aligned} \tag{63}$$

Using similar arguments and equation (9) again, we also obtain

$$\begin{aligned} \frac{1}{\gamma} \mu(\bar\alpha_k) = \frac{1}{\gamma} \frac{(x^k + \bar\alpha_k \Delta x^k)^T (z^k + \bar\alpha_k \Delta z^k)}{n} &\geq \frac{1 - \bar\alpha_k}{\gamma} \mu_k + \frac{\bar\alpha_k \sigma_k}{\gamma} \mu_k - \frac{\bar\alpha_k^2 |\Delta x^{k^T} \Delta z^k|}{\gamma n} \\ &\geq \frac{1 - \bar\alpha_k}{\gamma} \mu_k + \frac{\bar\alpha_k \sigma_k}{\gamma} \mu_k - \frac{\bar\alpha_k^2 \omega^2 n}{\gamma} \mu_k. \end{aligned} \tag{64}$$

Using (63), (64) and the fact that $n \geq 1$, we obtain

$$[x^k + \bar\alpha_k \Delta x^k]_i [z^k + \bar\alpha_k \Delta z^k]_i - \frac{1}{\gamma} \mu(\bar\alpha_k) \leq \bar\alpha_k \sigma_k \left(1 - \frac{1}{\gamma}\right) \mu_k + \bar\alpha_k^2 \omega^2 n^2 \left(1 + \frac{1}{\gamma}\right) \mu_k.$$

The right-hand side of this inequality is non-positive if $\bar\alpha_k \leq \dfrac{\sigma_{min}(1-\gamma)}{\omega^2 n^2 (1+\gamma)}$. By defining $\bar{\bar\delta}$ as the minimum of $\dfrac{\sigma_{min}(1-\gamma)}{\omega^2(1+\gamma)}$ and $\bar\delta$ defined in [21, Lemma 6.7], we obtain the desired result. □

By Lemma 10, if k is a Newton iteration of Algorithm 1 and $\bar\alpha_k \leq \bar{\bar\delta}/n^2$, then point $(x^{k+1}, \lambda^{k+1}, z^{k+1})$ satisfies (56). If only Newton steps are made, then $O(n^2)$ worst-case iteration complexity can be proved for $\mathcal{N}_s(\gamma, \beta)$ (see [21, Chapter 6]).

Based on the previous paragraphs, we set some assumptions that will be used in the remaining of this section. As usual, at iteration k the Newton step is taken and at iteration $k+1$ the quasi-Newton step is made. We assume that (x^0, λ^0, z^0) satisfies (57), both (x^k, λ^k, z^k) and $\bar\alpha_k$ satisfy (56), and $(x^{k+1}, \lambda^{k+1}, z^{k+1}) \in \mathcal{N}_s(\gamma, \beta)$. In our analysis, we will use extensively various well known results for Newton steps, such as properties (58)–(62). We observe that, while in the standard Newton approach one has to compute bounds for $\Delta x^{k^T} \Delta z^k$, in the quasi-Newton approach (see for example (13)), it is necessary to additionally get bounds for γ_1 and $(\bar\alpha_k \Delta x^k + \bar\alpha \Delta x^{k+1})^T (\bar\alpha_k \Delta z^k + \bar\alpha \Delta z^{k+1})$. In the next lemma, we give a bound for γ_1 without assuming feasibility of iterates.

Lemma 11. *Let $k+1$ be a quasi-Newton iteration of Algorithm 1. Suppose that v in Lemma 1 is given by the right-hand side of (4) at iteration $k+1$ and $\bar\alpha_k \in (0,1]$. Then, there exists a constant $C_3 \geq 1$, independent of n and k such that*

$$|\gamma_1| \leq C_3 \frac{\sqrt{n}}{\bar\alpha_k}. \tag{65}$$

Proof. Using scalar ν_{k+1} defined in the beginning of this section, vector v at iteration $k+1$ is given by

$$v = \begin{bmatrix} 0 \\ 0 \\ \sigma_{k+1}\mu_{k+1} e \end{bmatrix} - F(x^{k+1}, \lambda^{k+1}, z^{k+1}) = \begin{bmatrix} -r_c^{k+1} \\ -r_b^{k+1} \\ \sigma_{k+1}\mu_{k+1} e - X^{k+1} Z^{k+1} e \end{bmatrix} = \begin{bmatrix} -\nu_{k+1} r_c^0 \\ -\nu_{k+1} r_b^0 \\ \sigma_{k+1}\mu_{k+1} e - X^{k+1} Z^{k+1} e \end{bmatrix}. \tag{66}$$

We observe that, since $e^T(\mu_{k+1} e - X^{k+1} Z^{k+1} e) = 0$, by (56) we obtain

$$\begin{aligned} \|\sigma_{k+1}\mu_{k+1} e - X^{k+1} Z^{k+1} e\|^2 &= \|(\sigma_{k+1} - 1)\mu_{k+1} e + (\mu_{k+1} e - X^{k+1} Z^{k+1} e)\|^2 \\ &= (\sigma_{k+1} - 1)^2 \mu_{k+1}^2 n + \|\mu_{k+1} e - X^{k+1} Z^{k+1} e\|^2 \\ &= (\sigma_{k+1} - 1)^2 \mu_{k+1}^2 n + \|X^{k+1} Z^{k+1} e\|^2 - n\mu_{k+1}^2 \\ &\leq -(2 - \sigma_{k+1})\sigma_{k+1} n \mu_{k+1}^2 + \gamma^{-2} n \mu_{k+1}^2 \ \leq \ \gamma^{-2} n \mu_{k+1}^2. \end{aligned} \tag{67}$$

Taking the 2-norm in (66) and using (59) and (67), we get

$$\begin{aligned}\|v\|^2 = \|(r_c^{k+1}, r_b^{k+1})\|^2 + \|\sigma_{k+1}\mu_{k+1}e - X^{k+1}Z^{k+1}e\|^2 &\le \left(\frac{\|(r_c^0, r_b^0)\|\beta}{\mu_0}\right)^2 \mu_{k+1}^2 + \gamma^{-2} n \mu_{k+1}^2 \\ &\le \left[\left(\frac{\|(r_c^0, r_b^0)\|\beta}{\xi^2}\right)^2 + \gamma^{-2}\right] n\mu_{k+1}^2.\end{aligned} \tag{68}$$

By defining $\rho_k = \alpha_{\text{dec}} > 0$, we can see that condition (56) results in the sufficient decrease condition of Lemma 2. Since $\bar{\alpha}_k \in (0,1]$, this ensures that $\|y_k\| > 0$. Using Lemma 2, (56) and (68) in equation (15), we conclude that

$$|\gamma_1| \le \frac{\|v\|}{\|y_k\|} \le \frac{\left[\left(\frac{\|(r_c^0, r_b^0)\|\beta}{\xi^2}\right)^2 + \gamma^{-2}\right]^{1/2} \sqrt{n}\mu_{k+1}}{\frac{\alpha_{\text{dec}}\bar{\alpha}_k}{2}\mu_k} \le C_3 \frac{\sqrt{n}}{\bar{\alpha}_k},$$

with $C_3 = 2(\gamma\alpha_{\text{dec}})^{-1}\left[\left(\frac{\|(r_c^0, r_b^0)\|\beta\gamma}{\xi^2}\right)^2 + 1\right]^{1/2}$. In the last inequality we used (56) to yield $\mu_{k+1}/\mu_k \le 1$. □

Our goal now is to bound the term $(\bar{\alpha}_k \Delta x^k + \bar{\alpha}\Delta x^{k+1})^T(\bar{\alpha}_k \Delta z^k + \bar{\alpha}\Delta z^{k+1})$ in (13). We follow the same approach as [21] but compute, instead, bounds of $(D^k)^{-1}(\bar{\alpha}_k\Delta x^k + \bar{\alpha}\Delta x^{k+1})$ and $D^k(\bar{\alpha}_k \Delta z^k + \bar{\alpha}\Delta z^{k+1})$, where $D^k = (X^k)^{1/2}(Z^k)^{-1/2}$. It is important to note that matrix D^k is related to the Newton iteration, hence we can use Lemma 1 and the enjoyable properties of the true Jacobian.

First, we show what happens when matrix A multiplies the combined direction $\bar{\alpha}_k\Delta x^k + \bar{\alpha}\Delta x^{k+1}$:

$$\begin{aligned}A(\bar{\alpha}_k\Delta x^k + \bar{\alpha}\Delta x^{k+1}) &= \bar{\alpha}_k A\Delta x^k + \bar{\alpha}A\Delta x^{k+1} = -\bar{\alpha}_k(Ax^k - b) - \bar{\alpha}(Ax^{k+1} - b) \\ &= -\bar{\alpha}_k(Ax^k - b) - \bar{\alpha}(A(x^k + \bar{\alpha}_k\Delta x^k) - b) \;=\; -\bar{\alpha}_k(Ax^k - b) - \bar{\alpha}(Ax^k - b) + \bar{\alpha}_k\bar{\alpha}(Ax^k - b) \\ &= (\bar{\alpha}_k\bar{\alpha} - \bar{\alpha}_k - \bar{\alpha})(Ax^k - b) \;=\; (\bar{\alpha}_k\bar{\alpha} - \bar{\alpha}_k - \bar{\alpha})\nu_k(Ax^0 - b) \\ &= A\left[(\bar{\alpha}_k\bar{\alpha} - \bar{\alpha}_k - \bar{\alpha})\nu_k(x^0 - x^*)\right],\end{aligned}$$

where x^* is the primal solution of (1) used to define constant ξ in (57). By defining $\hat{\nu}_k = -(\bar{\alpha}_k\bar{\alpha} - \bar{\alpha}_k - \bar{\alpha})\nu_k = (1 - (1-\bar{\alpha}_k)(1-\bar{\alpha}))\nu_k$, we can see that the point $\hat{x} = \bar{\alpha}_k\Delta x^k + \bar{\alpha}\Delta x^{k+1} + \hat{\nu}_k(x^0 - x^*)$ is such that $A\hat{x} = 0$. Similar arguments can be used to show that $\hat{z} = \bar{\alpha}_k\Delta z^k + \bar{\alpha}\Delta z^{k+1} + \hat{\nu}_k(z^0 - z^*)$ and $\hat{\lambda} = \bar{\alpha}_k\Delta\lambda^k + \bar{\alpha}\Delta\lambda^{k+1} + \hat{\nu}_k(\lambda^0 - \lambda^*)$ satisfy $A^T\hat{\lambda} + \hat{z} = 0$. Therefore, it is not hard to see that

$$\hat{x}^T\hat{z} = \left(\bar{\alpha}_k\Delta x^k + \bar{\alpha}\Delta x^{k+1} + \hat{\nu}_k(x^0 - x^*)\right)^T\left(\bar{\alpha}_k\Delta z^k + \bar{\alpha}\Delta z^{k+1} + \hat{\nu}_k(z^0 - z^*)\right) = 0. \tag{69}$$

We now multiply the third row of the coefficient matrix in (4) by $\begin{bmatrix}\hat{x} & \hat{\lambda} & \hat{z}\end{bmatrix}$ to obtain

$$\begin{aligned}Z^k\hat{x} + X^k\hat{z} &= Z^k\left(\bar{\alpha}_k\Delta x^k + \bar{\alpha}\Delta x^{k+1} + \hat{\nu}_k(x^0 - x^*)\right) + X^k\left(\bar{\alpha}_k\Delta z^k + \bar{\alpha}\Delta z^{k+1} + \hat{\nu}_k(z^0 - z^*)\right) \\ &= Z^k(\bar{\alpha}_k\Delta x^k + \bar{\alpha}\Delta x^{k+1}) + X^k(\bar{\alpha}_k\Delta z^k + \bar{\alpha}\Delta z^{k+1}) + \hat{\nu}_k Z^k(x^0 - x^*) + \hat{\nu}_k X^k(z^0 - z^*).\end{aligned}$$

Multiplying this equation on both sides by $(X^kZ^k)^{-1/2}$, we get

$$\begin{aligned}(D^k)^{-1}\hat{x} + D^k\hat{z} = (X^kZ^k)^{-1/2}&\left[Z^k(\bar{\alpha}_k\Delta x^k + \bar{\alpha}\Delta x^{k+1}) + X^k(\bar{\alpha}_k\Delta z^k + \bar{\alpha}\Delta z^{k+1})\right] \\ &+ \hat{\nu}_k(D^k)^{-1}(x^0 - x^*) + \hat{\nu}_k D^k(z^0 - z^*).\end{aligned} \tag{70}$$

Using (69) and (70), we conclude that

$$\begin{aligned}\|(D^k)^{-1}\hat{x}\|^2 + \|D^k\hat{z}\|^2 &= \|(D^k)^{-1}\hat{x} + D^k\hat{z}\|^2 \\ &= \Big\|(X^kZ^k)^{-1/2}\left[Z^k(\bar{\alpha}_k\Delta x^k + \bar{\alpha}\Delta x^{k+1}) + X^k(\bar{\alpha}_k\Delta z^k + \bar{\alpha}\Delta z^{k+1})\right] \\ &\qquad + \hat{\nu}_k(D^k)^{-1}(x^0 - x^*) + \hat{\nu}_k D^k(z^0 - z^*)\Big\|^2.\end{aligned}$$

Using this relation we get

$$\begin{aligned}\|(D^k)^{-1}\hat{x}\| \le &\left\|(X^kZ^k)^{-1/2}\left[Z^k(\bar{\alpha}_k\Delta x^k + \bar{\alpha}\Delta x^{k+1}) + X^k(\bar{\alpha}_k\Delta z^k + \bar{\alpha}\Delta z^{k+1})\right]\right\| \\ &+ \hat{\nu}_k\|{D^k}^{-1}(x^0 - x^*)\| + \hat{\nu}_k\|D^k(z^0 - z^*)\|\end{aligned} \tag{71}$$

and observe that the same bound holds for $\|D^k\hat{z}\|$. In the next two lemmas, we compute the bounds for the terms in the right-hand side of (71).

Lemma 12. *Let $k+1$ be a quasi-Newton iteration of Algorithm 1. Let ω and C_3 be the constants (independent of k and n) defined in equation* (60) *and Lemma 11, respectively. Then*

$$\left\|(X^kZ^k)^{-1/2}\left[Z^k(\bar{\alpha}_k\Delta x^k+\bar{\alpha}\Delta x^{k+1})+X^k(\bar{\alpha}_k\Delta z^k+\bar{\alpha}\Delta z^{k+1})\right]\right\| \le$$
$$\le \gamma^{-1/2}\left[(\sigma_{max}+\gamma^{-1})(\bar{\alpha}_k+\bar{\alpha})+(\bar{\alpha}_k+C_3)\bar{\alpha}\bar{\alpha}_k\omega^2\right]n^{5/2}\mu_k^{1/2}.$$

Proof. First, we observe that the diagonal matrices ΔX^k and ΔZ^k were computed in the Newton iteration and use (60) to obtain

$$\|\Delta X^k\Delta Z^k e\| \le \|(D^k)^{-1}\Delta X^k\|\|D^k\Delta Z^k e\| \le \|(D^k)^{-1}\Delta X^k\|\omega n\mu_k^{1/2}.$$

Since both $(D^k)^{-1}$ and ΔX^k are diagonal matrices, using the property of the induced 2-norm of matrices, we get

$$\|(D^k)^{-1}\Delta X^k\| = \max_{i=1,\dots,n}\frac{|[\Delta x^k]_i|}{[D^k]_{ii}} = \|(D^k)^{-1}\Delta X^k e\|_\infty \le \|(D^k)^{-1}\Delta X^k e\| \le \omega n\mu_k^{1/2}.$$

Hence,

$$\|\Delta X^k\Delta Z^k e\| \le \omega^2 n^2\mu_k. \tag{72}$$

Now, we use equation (21) (which does not depend on the feasibility of the iterate or the type of the neighborhood) to expand the desired expression in the statement of this Lemma. Additionally, we use equation (56) to bound μ_{k+1} and X^kZ^ke as well as Lemma 11 and equation (72) to derive:

$$\begin{aligned}
&\left\|(X^k\ Z^k)^{-1/2}\left[Z^k(\bar{\alpha}_k\Delta x^k+\bar{\alpha}\Delta x^{k+1})+X^k(\bar{\alpha}_k\Delta z^k+\bar{\alpha}\Delta z^{k+1})\right]\right\| = \\
&= \left(\sum_{i=1}^n \frac{\left((1-\bar{\alpha})\bar{\alpha}_k\sigma_k\mu_k-(\bar{\alpha}+\bar{\alpha}_k(1-\bar{\alpha}))x_i^kz_i^k+\bar{\alpha}\sigma_{k+1}\mu_{k+1}-(1+\gamma_1)\bar{\alpha}\bar{\alpha}_k^2[\Delta x^k]_i[\Delta z^k]_i\right)^2}{x_i^kz_i^k}\right)^{1/2} \\
&\le (\gamma\mu_k)^{-1/2}\left\|(1-\bar{\alpha})\bar{\alpha}_k\sigma_k\mu_k e-(\bar{\alpha}+\bar{\alpha}_k(1-\bar{\alpha}))X^kZ^ke+\bar{\alpha}\sigma_{k+1}\mu_{k+1}e-(1+\gamma_1)\bar{\alpha}\bar{\alpha}_k^2\Delta X^k\Delta Z^k e\right\| \\
&\le (\gamma\mu_k)^{-1/2}\left[(1-\bar{\alpha})\bar{\alpha}_k\sigma_k\sqrt{n}\mu_k+(\bar{\alpha}+\bar{\alpha}_k(1-\bar{\alpha}))\gamma^{-1}\sqrt{n}\mu_k+\bar{\alpha}\sigma_{k+1}\sqrt{n}\mu_k+|1+\gamma_1|\bar{\alpha}\bar{\alpha}_k^2\omega^2n^2\mu_k\right] \\
&\le \gamma^{-1/2}\left[\left((\sigma_k+\gamma^{-1})(1-\bar{\alpha})\bar{\alpha}_k+(\sigma_{k+1}+\gamma^{-1})\bar{\alpha}\right)\sqrt{n}+\left(1+C_3\frac{\sqrt{n}}{\bar{\alpha}_k}\right)\bar{\alpha}\bar{\alpha}_k^2\omega^2n^2\right]\mu_k^{1/2} \\
&\le \gamma^{-1/2}\left[(\sigma_{max}+\gamma^{-1})(\bar{\alpha}_k+\bar{\alpha})\sqrt{n}+(\bar{\alpha}_k+C_3)\bar{\alpha}\bar{\alpha}_k\omega^2n^{5/2}\right]\mu_k^{1/2} \\
&\le \gamma^{-1/2}\left[(\sigma_{max}+\gamma^{-1})(\bar{\alpha}_k+\bar{\alpha})+(\bar{\alpha}_k+C_3)\bar{\alpha}\bar{\alpha}_k\omega^2\right]n^{5/2}\mu_k^{1/2}.
\end{aligned}$$

∎

Lemma 13. *Let $k+1$ be a quasi-Newton iteration of Algorithm 1. Let ω and C_3 be the constants (independent of k and n) defined before. Then,*

$$\|(D^k)^{-1}(\bar{\alpha}_k\Delta x^k+\bar{\alpha}\Delta x^{k+1})\| \le \left[(\sigma_{max}+\gamma^{-1}+8\beta)(\bar{\alpha}_k+\bar{\alpha})+(\bar{\alpha}_k+C_3)\omega^2\bar{\alpha}_k\bar{\alpha}\right]\gamma^{-1/2}n^{5/2}\mu_k^{1/2}$$
$$\|D^k(\bar{\alpha}_k\Delta z^k+\bar{\alpha}\Delta z^{k+1})\| \le \left[(\sigma_{max}+\gamma^{-1}+8\beta)(\bar{\alpha}_k+\bar{\alpha})+(\bar{\alpha}_k+C_3)\omega^2\bar{\alpha}_k\bar{\alpha}\right]\gamma^{-1/2}n^{5/2}\mu_k^{1/2}.$$

Proof. We will only consider $\|(D^k)^{-1}(\bar{\alpha}_k\Delta x^k+\bar{\alpha}\Delta x^{k+1})\|$, since getting a bound for $\|D^k(\bar{\alpha}_k\Delta z^k+\bar{\alpha}\Delta z^{k+1})\|$ follows the same arguments. We use the definition of $\hat{x}$ (see (69)), add and subtract $\hat{\nu}_k(D^k)^{-1}(x^0-x^*)$ inside the norm, and then use the triangle inequality and (71) to obtain

$$\begin{aligned}
\|(D^k)^{-1}(\bar{\alpha}_k\Delta x^k+\bar{\alpha}\Delta x^{k+1})\| &= \|(D^k)^{-1}\hat{x}-\hat{\nu}_k(D^k)^{-1}(x^0-x^*)\| \le \|(D^k)^{-1}\hat{x}\|+\hat{\nu}_k\|(D^k)^{-1}(x^0-x^*)\| \\
&\le \left\|(X^kZ^k)^{-1/2}\left[Z^k(\bar{\alpha}_k\Delta x^k+\bar{\alpha}\Delta x^{k+1})+X^k(\bar{\alpha}_k\Delta z^k+\bar{\alpha}\Delta z^{k+1})\right]\right\| \\
&\quad +2\hat{\nu}_k\|(D^k)^{-1}(x^0-x^*)\|+2\hat{\nu}_k\|D^k(z^0-z^*)\|,
\end{aligned} \tag{73}$$

where another term $\hat{\nu}_k\|D^k(z^0-z^*)\|$ was added in the last inequality. We already have bounds for the first term in the right-hand side of (73), by Lemma 12. From [21, Lemma 6.6] we know that

$$\begin{aligned}
\|(D^k)^{-1}(x^0-x^*)\| &\le \|(x^0-x^*)\|\|(D^k)^{-1}\| \le \xi\|(D^k)^{-1}\| = \xi\max_{i=1,\dots,n}\frac{1}{[D^k]_{ii}} = \xi\|(D^k)^{-1}e\|_\infty \\
&\le \xi\|(D^k)^{-1}e\| = \xi\|(X^kZ^k)^{-1/2}Z^ke\| \le \xi\|(X^kZ^k)^{-1/2}\|\|z^k\|_1
\end{aligned}$$

and, similarly, $\|D^k(z^0 - z^*)\| \le \xi\|(X^kZ^k)^{-1/2}\|\|x^k\|_1$. We recall that $\hat{\nu}_k = (1-(1-\bar{\alpha}_k)(1-\bar{\alpha}))\nu_k$ and apply all these inequalities together with (58) to obtain

$$\begin{aligned}2\hat{\nu}_k\|(D^k)^{-1}(x^0-x^*)\| + 2\hat{\nu}_k\|D^k(z^0-z^*)\| &\le 2\xi(1-(1-\bar{\alpha}_k)(1-\bar{\alpha}))\|(X^kZ^k)^{-1/2}\|\nu_k\|(x^k,z^k)\|_1 \\ &\le 2\xi(1-(1-\bar{\alpha}_k)(1-\bar{\alpha}))(\gamma\mu_k)^{-1/2}\frac{4\beta}{\xi}n\mu_k \\ &= (1-(1-\bar{\alpha}_k)(1-\bar{\alpha}))\frac{8\beta}{\gamma^{1/2}}n\mu_k^{1/2},\end{aligned}$$

which provides bounds for the last two terms in (73). Therefore, using Lemma 12 and the above inequality we write

$$\begin{aligned}\|D^{k^{-1}}(\bar{\alpha}_k\Delta x^k + \bar{\alpha}\Delta x^{k+1})\| &\le \left[\frac{(\sigma_{max}+\gamma^{-1})(\bar{\alpha}_k+\bar{\alpha}) + (\bar{\alpha}_k+C_3)\bar{\alpha}\bar{\alpha}_k\omega^2}{\gamma^{1/2}}n^{5/2} + \frac{8\beta(1-(1-\bar{\alpha}_k)(1-\bar{\alpha}))}{\gamma^{1/2}}n\right]\mu_k^{1/2} \\ &\le \frac{(\sigma_{max}+\gamma^{-1})(\bar{\alpha}_k+\bar{\alpha}) + (\bar{\alpha}_k+C_3)\bar{\alpha}\bar{\alpha}_k\omega^2 + 8\beta(1-(1-\bar{\alpha}_k)(1-\bar{\alpha}))}{\gamma^{1/2}}n^{5/2}\mu_k^{1/2} \\ &\le \left[(\sigma_{max}+\gamma^{-1}+8\beta)(\bar{\alpha}_k+\bar{\alpha}) + (\bar{\alpha}_k+C_3)\omega^2\bar{\alpha}_k\bar{\alpha}\right]\gamma^{-1/2}n^{5/2}\mu_k^{1/2}\end{aligned}$$

and the lemma is proved. □

If we restrict the choices of $\bar{\alpha}_k$ and $\bar{\alpha}$, then the bounds obtained in Lemma 13 can be significantly simplified as shown in the corollary below.

Corollary 2. *If $\bar{\alpha}_k \le (\sigma_{max}+\gamma^{-1}+8\beta)((1+C_3)\omega^2)^{-1}$ and $\bar{\alpha} \le C_3^{-1}\bar{\alpha}_k$, then there exists a constant C_6, independent of n, such that*

$$\|(D^k)^{-1}(\bar{\alpha}_k\Delta x^k + \bar{\alpha}\Delta x^{k+1})\| \le C_6\bar{\alpha}_k n^{5/2}\mu_k^{1/2} \quad \textit{and} \quad \|D^k(\bar{\alpha}_k\Delta z^k + \bar{\alpha}\Delta z^{k+1})\| \le C_6\bar{\alpha}_k n^{5/2}\mu_k^{1/2}.$$

Proof. Using the bounds on $\bar{\alpha}$ and $\bar{\alpha}_k$ assumed in the lemma we obtain $(\bar{\alpha}_k + C_3)\omega^2\bar{\alpha}_k \le (\sigma_{max}+\gamma^{-1}+8\beta)$, hence, by $\bar{\alpha} \le C_3^{-1}\bar{\alpha}_k$,

$$\begin{aligned}(\sigma_{max}+\gamma^{-1}+8\beta)(\bar{\alpha}_k+\bar{\alpha}) + (\bar{\alpha}_k+C_3)\omega^2\bar{\alpha}_k\bar{\alpha} &\le (\sigma_{max}+\gamma^{-1}+8\beta)(\bar{\alpha}_k+2\bar{\alpha}) \\ &\le (\sigma_{max}+\gamma^{-1}+8\beta)(1+2C_3^{-1})\bar{\alpha}_k,\end{aligned}$$

and the conclusion follows by defining $C_6 = (\sigma_{max}+\gamma^{-1}+8\beta)(1+2C_3^{-1})\gamma^{-1/2}$. □

We are now ready to prove the polynomial worst-case iteration complexity of Algorithm 1 in the infeasible case. Lemma 10 dealt with the Newton step of the method. Theorems 3 and 4 provide the results for the quasi-Newton step.

Theorem 3. *Suppose that $k+1$ is a quasi-Newton step of Algorithm 1 and all the hypotheses of this section hold. If the following condition holds*

$$\alpha_{\mathrm{dec}} + \sigma_{max} \le 1 - \sigma_{min} \tag{74}$$

then, there exists a constant C_5, independent of n, such that, if

$$\bar{\alpha}_k \in \left(0, \min\left\{1, \left((1+C_3)\omega^2\right)^{-1}, \frac{C_5}{n^5}\right\}\right] \quad \textit{and} \quad \bar{\alpha} \in \left[(C_3C_5)^{-1}n^5\bar{\alpha}_k^2,\ C_3^{-1}\bar{\alpha}_k\right], \tag{75}$$

then the iterate after the quasi-Newton step satisfies

$$(x^{k+1}+\bar{\alpha}\Delta x^{k+1})^T(z^{k+1}+\bar{\alpha}\Delta z^{k+1}) \ge (1-\bar{\alpha}){x^{k+1}}^T z^{k+1} \tag{76}$$

$$(x^{k+1}+\bar{\alpha}\Delta x^{k+1})^T(z^{k+1}+\bar{\alpha}\Delta z^{k+1}) \le (1-\alpha_{\mathrm{dec}}\bar{\alpha}){x^{k+1}}^T z^{k+1} \tag{77}$$

$$[x^k+\bar{\alpha}_k\Delta x^k]_i[z^k+\bar{\alpha}_k\Delta z^k]_i \ge \gamma\mu(\bar{\alpha}) \tag{78}$$

$$[x^k+\bar{\alpha}_k\Delta x^k]_i[z^k+\bar{\alpha}_k\Delta z^k]_i \le \frac{1}{\gamma}\mu(\bar{\alpha}). \tag{79}$$

Proof. By Lemma 11 we know that $C_3 \geq 1$. Hence, the given intervals for $\bar{\alpha}$ and $\bar{\alpha}_k$ are well defined. We begin the proof with inequality (76). Using (60), we get

$$\bar{\alpha}_k^2 \Delta x^{k^T} \Delta z^k \leq \bar{\alpha}_k^2 \|D^{k^{-1}} \Delta x^k\| \|D^k \Delta z^k\| \leq \bar{\alpha}_k^2 \omega^2 n^2 \mu_k. \tag{80}$$

By Corollary 2, we obtain

$$\begin{aligned}(\bar{\alpha}_k \Delta x^k + \bar{\alpha}\Delta x^{k+1})^T (\bar{\alpha}_k \Delta z^k + \bar{\alpha}\Delta z^{k+1}) &\leq \|(D^k)^{-1}(\bar{\alpha}_k \Delta x^k + \bar{\alpha}\Delta x^{k+1})\| \|D^k(\bar{\alpha}_k \Delta z^k + \bar{\alpha}\Delta z^{k+1})\| \\ &\leq C_6^2 n^5 \mu_k \bar{\alpha}_k^2.\end{aligned} \tag{81}$$

Finally, by using inequality (61) from Lemma 10, we get

$$\bar{\alpha}\sigma_{k+1} x^{k+1^T} z^{k+1} \geq \bar{\alpha}(1-\bar{\alpha}_k)\sigma_{k+1} x^{k^T} z^k \geq \sigma_{min}\bar{\alpha}(1-\bar{\alpha}_k) n\mu_k. \tag{82}$$

Starting from equation (13), using (80)–(82), and then applying the bounds for $|\gamma_1|$ from Lemma 11 together with the bound $\bar{\alpha} \leq C_3^{-1}\bar{\alpha}_k$ (assumed in (75)) to deliver $1 + \bar{\alpha}|\gamma_1| \leq 1 + C_3^{-1}\bar{\alpha}_k C_3 \frac{\sqrt{n}}{\bar{\alpha}_k} = 1 + \sqrt{n}$, we get

$$\begin{aligned}&(x^{k+1} + \bar{\alpha}\Delta x^{k+1})^T (z^{k+1} + \bar{\alpha}\Delta z^{k+1}) - (1-\bar{\alpha}) x^{k+1^T} z^{k+1} \geq \\ &\geq \sigma_{min}\bar{\alpha}(1-\bar{\alpha}_k) n\mu_k - C_6^2 n^5 \mu_k \bar{\alpha}_k^2 - (1+\gamma_1\bar{\alpha})\bar{\alpha}_k^2 \Delta x^{k^T} \Delta z^k \\ &\geq \sigma_{min}\bar{\alpha}(1-\bar{\alpha}_k) n\mu_k - (1+\bar{\alpha}|\gamma_1|)\bar{\alpha}_k^2 |\Delta x^{k^T} \Delta z^k| - C_6^2 n^5 \mu_k \bar{\alpha}_k^2 \\ &\geq \left(\sigma_{min}\bar{\alpha} - \sigma_{min}\bar{\alpha}\bar{\alpha}_k - (1+n^{1/2})\bar{\alpha}_k^2\omega^2 n - C_6^2 n^4 \bar{\alpha}_k^2\right) n\mu_k \\ &\geq \left[\sigma_{min}\bar{\alpha} - \left(C_3^{-1}\sigma_{min} + 2\omega^2 + C_6^2\right)\bar{\alpha}_k^2 n^4\right] n\mu_k.\end{aligned}$$

By defining

$$\kappa = 2\omega^2 + C_6^2 \quad \text{and} \quad C_4 = C_3^{-1}\sigma_{min}\left(C_3^{-1}\sigma_{min} + \kappa\right)^{-1},$$

the right-hand side of the previous inequality is non-negative if $\bar{\alpha} \geq (C_4 C_3)^{-1} n^4 \bar{\alpha}_k^2$ and $\bar{\alpha}_k \leq C_4/n^4$. The lower bound for $\bar{\alpha}$ is necessary to avoid too small step in the quasi-Newton direction. It is obtained by writing $\bar{\alpha}$ as a function of $\bar{\alpha}_k$, in order to guarantee its non-negativity. The upper bound for $\bar{\alpha}_k$ is obtained by requesting that $\bar{\alpha}_k$ has to be chosen such that $(C_4 C_3)^{-1} n^4 \bar{\alpha}_k^2 \leq C_3^{-1}\bar{\alpha}_k$ holds.

For inequality (77), we start by subtracting $(1-\alpha_{\text{dec}}\bar{\alpha}) x^{k+1^T} z^{k+1}$ from both sides of equation (13)

$$\begin{aligned}&(x^{k+1} + \bar{\alpha}\Delta x^{k+1})^T (z^{k+1} + \bar{\alpha}\Delta z^{k+1}) - (1-\alpha_{\text{dec}}\bar{\alpha}) x^{k+1^T} z^{k+1} = \\ &\quad = -(1-\alpha_{\text{dec}} - \sigma_{k+1})\bar{\alpha} x^{k+1^T} z^{k+1} + (\bar{\alpha}_k \Delta x^k + \bar{\alpha}\Delta x^{k+1})^T (\bar{\alpha}_k \Delta z^k + \bar{\alpha}\Delta z^{k+1}) - (1+\bar{\alpha}\gamma_1)\bar{\alpha}_k^2 \Delta x^{k^T} \Delta z^k.\end{aligned}$$

We observe that the first term is negative by the assumption (74) of the Theorem, since $\sigma_{k+1} \leq \sigma_{max}$. Next, from the left inequality in (61), we have $(1-\bar{\alpha}_k) n\mu_k \leq n\mu_{k+1}$ and then by using the condition $\alpha_{\text{dec}} + \sigma_{max} \leq 1 - \sigma_{min}$, (80) and (81), we conclude that

$$\begin{aligned}&(x^{k+1} + \bar{\alpha}\Delta x^{k+1})^T (z^{k+1} + \bar{\alpha}\Delta z^{k+1}) - (1-\alpha_{\text{dec}}\bar{\alpha}) x^{k+1^T} z^{k+1} \leq \\ &\quad \leq -\sigma_{min}\bar{\alpha}(1-\bar{\alpha}_k) n\mu_k + (1+\bar{\alpha}|\gamma_1|)\bar{\alpha}_k^2 |\Delta x^{k^T} \Delta z^k| + |(\bar{\alpha}_k \Delta x^k + \bar{\alpha}\Delta x^{k+1})^T (\bar{\alpha}_k \Delta z^k + \bar{\alpha}\Delta z^{k+1})| \\ &\quad \leq -\sigma_{min}\bar{\alpha}(1-\bar{\alpha}_k) n\mu_k + (1+n^{1/2})\bar{\alpha}_k^2\omega^2 n^2 \mu_k + C_6^2 \bar{\alpha}_k^2 n^5 \mu_k \\ &\quad \leq \left[-\sigma_{min}\bar{\alpha} + \left(\sigma_{min} C_3^{-1} + \kappa\right)\bar{\alpha}_k^2 n^4\right] n\mu_k,\end{aligned}$$

where in the last inequality we used the bound $\bar{\alpha} \leq C_3^{-1}\bar{\alpha}_k$ to deliver $\sigma_{min}\bar{\alpha}\bar{\alpha}_k n\mu_k \leq \sigma_{min} C_3^{-1}\bar{\alpha}_k^2 n\mu_k$. The right-hand side of the previous inequality will be non-positive if, as before, $\bar{\alpha} \geq (C_4 C_3)^{-1} n^4 \bar{\alpha}_k^2$ and $\bar{\alpha}_k \leq C_4/n^4$.

To prove inequalities (78) and (79), we first look at (14) which is a component-wise version of (12). We also need to derive component-wise versions of (81) and (80). For (81) we have

$$\begin{aligned}[\bar{\alpha}_k \Delta x^k + \bar{\alpha}\Delta x^{k+1}]_i [\bar{\alpha}_k \Delta z^k + \bar{\alpha}\Delta z^{k+1}]_i &\leq |[(D^k)^{-1}]_{ii} [\bar{\alpha}_k \Delta x^k + \bar{\alpha}\Delta x^{k+1}]_i| \; |[D^k]_{ii} [\bar{\alpha}_k \Delta z^k + \bar{\alpha}\Delta z^{k+1}]_i| \\ &\leq \|(D^k)^{-1}(\bar{\alpha}_k \Delta x^k + \bar{\alpha}\Delta x^{k+1})\| \|D^k(\bar{\alpha}_k \Delta z^k + \bar{\alpha}\Delta z^{k+1})\| \\ &\leq C_6^2 n^5 \mu_k \bar{\alpha}_k^2,\end{aligned} \tag{83}$$

and, by a similar approach, equation (80), Lemma 11 and the assumption of the theorem $\bar{\alpha} \le C_3^{-1}\bar{\alpha}_k$ we get

$$\begin{aligned}(1+\bar{\alpha}\gamma_1)\bar{\alpha}_k^2[\Delta x^k]_i[\Delta z^k]_i &\le (1+\bar{\alpha}|\gamma_1|)\bar{\alpha}_k^2|[\Delta x^k]_i[\Delta z^k]_i| \\ &\le (1+n^{1/2})\bar{\alpha}_k^2\|(D^k)^{-1}\Delta X^k\|\|D^k\Delta Z^k\| \\ &\le 2\bar{\alpha}_k^2\omega^2 n^{5/2}\mu_k.\end{aligned} \tag{84}$$

Now, taking (14) and applying (62) and (83)–(84) we get

$$\begin{aligned}&[x^{k+1}+\bar{\alpha}\Delta x^{k+1}]_i[z^{k+1}+\bar{\alpha}\Delta z^{k+1}]_i \ge \\ &\quad\ge (1-\bar{\alpha})\gamma\mu_{k+1} - |[\bar{\alpha}_k\Delta x^k+\bar{\alpha}\Delta x^{k+1}]_i[\bar{\alpha}_k\Delta x^k+\bar{\alpha}\Delta z^{k+1}]_i| - (1+\bar{\alpha}|\gamma_1|)\bar{\alpha}_k^2|[\Delta x^k]_i[\Delta z^k]_i| + \bar{\alpha}\sigma_{k+1}\mu_{k+1} \\ &\quad\ge (1-\bar{\alpha})\gamma\mu_{k+1} + \bar{\alpha}\sigma_{k+1}\mu_{k+1} - C_6^2 n^5\mu_k\bar{\alpha}_k^2 - 2\bar{\alpha}_k^2\omega^2 n^{5/2}\mu_k \\ &\quad\ge (1-\bar{\alpha})\gamma\mu_{k+1} + \bar{\alpha}\sigma_{k+1}\mu_{k+1} - \kappa n^5\bar{\alpha}_k^2\mu_k\end{aligned}$$

and from (13), using (80)–(81) we get

$$\begin{aligned}\gamma\mu(\bar{\alpha}) &= \gamma\frac{(x^{k+1}+\bar{\alpha}\Delta x^{k+1})^T(z^{k+1}+\bar{\alpha}\Delta z^{k+1})}{n} \le \\ &\le \gamma\left[(1-\bar{\alpha}(1-\sigma_{k+1}))\mu_{k+1} + \frac{1+\bar{\alpha}|\gamma_1|}{n}\bar{\alpha}_k^2|\Delta x^{k^T}\Delta z^k| + \frac{|(\bar{\alpha}_k\Delta x^k+\bar{\alpha}\Delta x^{k+1})^T(\bar{\alpha}_k\Delta z^k+\bar{\alpha}\Delta z^{k+1})|}{n}\right] \\ &\le (1-\bar{\alpha})\gamma\mu_{k+1} + \bar{\alpha}\sigma_{k+1}\gamma\mu_{k+1} + (1+n^{1/2})\gamma\omega^2 n\bar{\alpha}_k^2\mu_k + C_6^2\gamma n^4\bar{\alpha}_k^2\mu_k \\ &\le (1-\bar{\alpha})\gamma\mu_{k+1} + \bar{\alpha}\sigma_{k+1}\gamma\mu_{k+1} + \kappa\gamma n^4\bar{\alpha}_k^2\mu_k.\end{aligned}$$

By combining the above two inequalities, then using (82) and (61) and $\bar{\alpha} \le C_3^{-1}\bar{\alpha}_k$, we obtain

$$\begin{aligned}[x^{k+1}+\bar{\alpha}\Delta x^{k+1}]_i[z^{k+1}+\bar{\alpha}\Delta z^{k+1}]_i - \gamma\mu(\bar{\alpha}) &\ge (1-\gamma)\bar{\alpha}\sigma_{k+1}\mu_{k+1} - (1+\gamma)\kappa n^5\bar{\alpha}_k^2\mu_k \\ &\ge (1-\gamma)\sigma_{min}\bar{\alpha}(1-\bar{\alpha}_k)\mu_k - (1+\gamma)\kappa n^5\bar{\alpha}_k^2\mu_k \\ &\ge \left[\sigma_{min}\bar{\alpha} - \left(C_3^{-1}\sigma_{min} + \left(\frac{1+\gamma}{1-\gamma}\right)\kappa\right)n^5\bar{\alpha}_k^2\right](1-\gamma)\mu_k.\end{aligned}$$

The right-hand side of this inequality is non-negative, and hence (78) holds, if $\bar{\alpha} \ge (C_3C_5)^{-1}n^5\bar{\alpha}_k^2$ and $\bar{\alpha}_k \le C_5/n^5$, with

$$C_5 = C_3^{-1}\sigma_{min}\left(C_3^{-1}\sigma_{min} + \left(\frac{1+\gamma}{1-\gamma}\right)\kappa\right)^{-1}.$$

Finally, in order to show (79), we use the same ideas to obtain the following two inequalities

$$[x^{k+1}+\bar{\alpha}\Delta x^{k+1}]_i[z^{k+1}+\bar{\alpha}\Delta z^{k+1}]_i \le \frac{1-\bar{\alpha}}{\gamma}\mu_{k+1} + \bar{\alpha}\sigma_{k+1}\mu_{k+1} + \kappa n^5\bar{\alpha}_k^2\mu_k$$

and

$$\frac{1}{\gamma}\mu(\bar{\alpha}) \ge \frac{1-\bar{\alpha}}{\gamma}\mu_{k+1} + \frac{\bar{\alpha}\sigma_{k+1}}{\gamma}\mu_{k+1} - \frac{\kappa n^4\bar{\alpha}_k^2}{\gamma}\mu_k.$$

By combining these two inequalities and using (61) and $\bar{\alpha} \le C_3^{-1}\bar{\alpha}_k$ once more, we have that

$$\begin{aligned}[x^{k+1}+\bar{\alpha}\Delta x^{k+1}]_i[z^{k+1}+\bar{\alpha}\Delta z^{k+1}]_i - \frac{1}{\gamma}\mu(\bar{\alpha}) &\le \left(1-\frac{1}{\gamma}\right)\sigma_{k+1}\bar{\alpha}\mu_{k+1} + \left(1+\frac{1}{\gamma}\right)\kappa n^5\bar{\alpha}_k^2\mu_k \\ &= -\left(\frac{1-\gamma}{\gamma}\right)\sigma_{k+1}\bar{\alpha}\mu_{k+1} + \left(\frac{1+\gamma}{\gamma}\right)\kappa n^5\bar{\alpha}_k^2\mu_k \\ &\le -\left(\frac{1-\gamma}{\gamma}\right)\sigma_{min}\bar{\alpha}(1-\bar{\alpha}_k)\mu_k + \left(\frac{1+\gamma}{\gamma}\right)\kappa n^5\bar{\alpha}_k^2\mu_k \\ &\le \left[-\sigma_{min}\bar{\alpha} + \left(C_3^{-1}\sigma_{min} + \left(\frac{1+\gamma}{1-\gamma}\right)\kappa\right)n^5\bar{\alpha}_k^2\right]\left(\frac{1-\gamma}{\gamma}\right)\mu_k.\end{aligned}$$

Again, the right-hand side of this inequality is non-positive if $\bar{\alpha} \ge (C_3C_5)^{-1}n^5\bar{\alpha}_k^2$ and $\bar{\alpha}_k \le C_5/n^5$, with C_5 defined before. Since $C_5 \le C_4$, we observe that $(C_3C_5)^{-1}n^5 \ge (C_3C_4)^{-1}n^5$, and this explains the lower bound on $\bar{\alpha}$. We also observe that in (75), $\bar{\alpha}_k$ is bounded from above by C_5/n^5 hence $(C_3C_5)^{-1}n^5\bar{\alpha}_k^2 \le C_3^{-1}\bar{\alpha}_k$, which guarantees that the interval for $\bar{\alpha}$ is not empty. Similar observations can be made to justify the upper bound of $\bar{\alpha}_k$ in (75) and this concludes the proof. □

To further show that $(x^{k+2}, \lambda^{k+2}, z^{k+2})$ belongs to the $\mathcal{N}_s(\gamma, \beta)$ neighborhood, and therefore satisfies (56), we need to ensure that γ is not too close to 0.

Theorem 4. *Let the hypotheses of Theorem 3 hold. If, in addition, we request that*

$$\gamma \geq 2\left(-8\beta + \sqrt{(8\beta+2)^2 + \frac{4}{3\sigma_{min}}}\right)^{-1} \tag{85}$$

then $(x^{k+2}, \lambda^{k+2}, z^{k+2})$ *satisfies* (56).

Proof. Using inequality (76) in Theorem 3

$$\frac{\|(r_b^{k+2}, r_c^{k+2})\|}{\mu_{k+2}} \leq \frac{(1-\bar{\alpha})\|(r_b^{k+1}, r_c^{k+1})\|}{(1-\bar{\alpha})\mu_{k+1}} \leq \frac{\|(r_b^0, r_c^0)\|}{\mu_0}\beta$$

and, by (78) and (79)

$$0 < \gamma\mu_{k+2} \leq x_i^{k+2} z_i^{k+2} \leq \frac{1}{\gamma}\mu_{k+2}.$$

To show that $x^{k+2} > 0$ and $z^{k+2} > 0$, we suppose by contradiction that $x_i^{k+2} \leq 0$ or $z_i^{k+2} \leq 0$ occurs for some i and follow the same arguments as those used in Theorems 1 and 2 to conclude, using inequality (83), that $\gamma < C_6^2 n^5 \bar{\alpha}_k^2$ should hold.

By using the fact that $C_3 \geq 1$, defined in Lemma 11, and $\kappa \geq 1$ and $C_6 \geq 1$, defined in Theorem 3, and basic manipulation, we conclude that $C_6 \leq 3(1 + \gamma^{-1} + 8\beta)\gamma^{-1/2}$ and $C_5 \leq \sigma_{min}(1-\gamma)$, where C_5 was also defined in Theorem 3. Then, using (75) for $\bar{\alpha}_k$ we obtain

$$\gamma < C_6^2 n^5 \bar{\alpha}_k^2 \leq \frac{(C_5 C_6)^2}{n^5} \leq \frac{9(1+\gamma^{-1}+8\beta)^2(1-\gamma)^2}{\gamma}\sigma_{min}^2 \tag{86}$$

which implies that $\gamma < 2\left(-8\beta + \sqrt{(8\beta+2)^2 + \frac{4}{3\sigma_{min}}}\right)^{-1}$ and contradicts (85). The above inequality has been obtained by rearranging (86) namely, dropping the squares and solving the resulting (quadratic) inequality with an unknown $1/\gamma$. Therefore, we conclude that $(x^{k+2}, \lambda^{k+2}, z^{k+2}) \in \mathcal{N}_s(\gamma, \beta)$. Earlier proved inequality (77) guarantees that iteration $k+1$ satisfies (56).

□

For sufficiently large n, we clearly have

$$\min\left\{1, \left((1+C_3)\omega^2\right)^{-1}, \frac{C_5}{n^5}\right\} = \frac{C_5}{n^5} < \frac{\bar{\bar{\delta}}}{n^2},$$

where $\bar{\bar{\delta}}$ comes from Lemma 10 and this guarantees that the hypotheses of this section hold. By (75) we have that both $\bar{\alpha}_k$ and $\bar{\alpha}$ are greater than or equal to $\frac{C_3^{-1}C_5}{n^5}$ hence

$$\mu_{k+1} \leq (1 - \alpha_{\text{dec}}\bar{\alpha}_k)\mu_k \leq \left(1 - \frac{\alpha_{\text{dec}} C_3^{-1} C_5}{n^5}\right)\mu_k,$$

for all $k = 0, 1, \ldots$, which yields $O(n^5)$ worst-case iteration complexity for the infeasible case of Algorithm 1.

Unlike in the feasible case, because of the close relation of the symmetric neighborhood and the $\mathcal{N}_{-\infty}(\gamma, \beta)$ neighborhood, we expect that a similar complexity result to that of Theorem 3 should hold for the algorithm which operates in $\mathcal{N}_{-\infty}(\gamma, \beta)$. However, to save space we do not derive it here.

5 Conclusions and final observations

This work provided theoretical tools to analyze the worst-case iteration complexity of quasi-Newton primal-dual interior point algorithms. A simplified algorithm was considered, which consisted of alternating Newton and quasi-Newton steps. The quasi-Newton approach was based on the Broyden "bad" low-rank update of the inverse of the unreduced system. Feasible and infeasible algorithms and well established neighborhoods of the central path have been considered.

The results showed that in all cases, the degree of the polynomial in the worst-case result has increased. This behavior has already been observed by [10], where the number of overall iterations increased, but the number of factorizations of the true unreduced system actually decreased.

Complexity results might also be obtained in the feasible case for convex quadratic programming problems and the $\mathcal{N}_2(\theta)$ neighborhood, following [8], for example. However, since Lemma 3 does not hold anymore and thus γ_1 is not eliminated from $\mu(\bar{\alpha})$, a further study needs to be carried out.

Another interesting and complicated question is the case $\ell > 1$, where we allow more than one quasi-Newton step after a Newton step. The authors in [10] observed a trade-off between the number of consecutive quasi-Newton iterations and the overall speed of convergence of the algorithm. Numerical results showed that $\ell = 5$ is a reasonable choice. We expect that similar worst-case complexity bounds should be obtained for the case $\ell > 1$, with possibly higher degrees of polynomials, but this still remains an open question.

The worst-case complexity results obtained when the $\mathcal{N}_s$ neighborhood was considered in both feasible and infeasible cases seem rather pessimistic. The high polynomial degrees originate from Lemma 9 for the feasible and from Lemma 13 for the infeasible cases. Finding new ways of reducing the degrees of such expressions would definitely result in better complexity results, since all the other terms are at most of order n.

References

[1] S. Bellavia, V. De Simone, D. di Serafino, and B. Morini, *Updating Constraint Preconditioners for KKT Systems in Quadratic Programming Via Low-Rank Corrections*, SIAM J. on Optim., 25 (2015), pp. 1787–1808.

[2] L. Bergamaschi, V. De Simone, D. di Serafino, and Á. Martínez, *BFGS-like updates of constraint preconditioners for sequences of KKT linear systems in quadratic programming*, Numer. Linear Algebr. with Appl., 25 (2018), p. e2144.

[3] M. Colombo and J. Gondzio, *Further development of multiple centrality correctors for interior point methods*, Comput. Optim. Appl., 41 (2008), pp. 277–305.

[4] M. D'Apuzzo, V. De Simone, and D. di Serafino, *On mutual impact of numerical linear algebra and large-scale optimization with focus on interior point methods*, Comput. Optim. Appl., pp. 283–310.

[5] J. E. Dennis Jr., A. M. Morshedi, and K. Turner, *A variable-metric variant of the Karmarkar algorithm for linear programming*, Math. Program., pp. 1–20.

[6] D. Ek and A. Forsgren, *A structured modified Newton approach for solving systems of nonlinear equations arising in interior-point methods for quadratic programming*, tech. rep.

[7] J. Gondzio, *HOPDM (version 2.12) — A fast LP solver based on a primal-dual interior point method*, Eur. J. Oper. Res., pp. 221–225.

[8] ——, *Interior point methods 25 years later*, Eur. J. Oper. Res., pp. 587–601.

[9] J. Gondzio and A. Grothey, *Exploiting structure in parallel implementation of interior point methods for optimization*, Comput. Manag. Sci., pp. 135–160.

[10] J. Gondzio and F. N. C. Sobral, *Quasi-Newton approaches to interior point methods for quadratic problems*, Comput. Optim. Appl., pp. 93–120.

[11] C. C. Gonzaga, *An Algorithm for Solving Linear Programming Problems in $O(n^3L)$ Operations*, in Progress in Mathematical Programming, N. Megiddo, ed., Springer New York, New York, NY, pp. 1–28.

[12] S. Gratton, S. Mercier, N. Tardieu, and X. Vasseur, *Limited memory preconditioners for symmetric indefinite problems with application to structural mechanics*, Numer. Linear Algebr. with Appl., pp. 865–887.

[13] S. Gratton, A. Sartenaer, and J. Tshimanga, *On A Class of Limited Memory Preconditioners For Large Scale Linear Systems With Multiple Right-Hand Sides*, SIAM J. on Optim., pp. 912–935.

[14] C. Greif, E. Moulding, and D. Orban, *Bounds on Eigenvalues of Matrices Arising from Interior-Point Methods*, SIAM J. on Optim., pp. 49–83.

[15] I. J. Lustig, R. E. Marsten, and D. F. Shanno, *Interior Point Methods for Linear Programming: Computational State of the Art*, ORSA J. on Comput., pp. 1–14.

[16] J. M. Martínez, *Practical quasi-Newton methods for solving nonlinear systems*, J. Comput. Appl. Math., pp. 97–121.

[17] R. D. C. Monteiro and I. Adler, *Interior path following primal-dual algorithms. part I: Linear programming*, Math. Program., pp. 27–41.

[18] B. Morini, V. Simoncini, and M. Tani, *A comparison of reduced and unreduced KKT systems arising from interior point methods*, Comput. Optim. Appl., pp. 1–27.

[19] Y. Nesterov and A. Nemirovskii, *Interior Point Polynomial Methods in Convex Programming*, Society for Industrial and Applied Mathematics, Philadelphia, 1994.

[20] S. H. Schmieta and F. Alizadeh, *Extension of primal-dual interior point algorithms to symmetric cones*, Math. Program., 96 (2003), pp. 409–438.

[21] S. J. Wright, *Primal-dual Interior Point Methods*, Society for Industrial and Applied Mathematics, Philadelphia, 1997.