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Chris Hurst

Derek Hurrell
University of Notre Dame Australia, derek.hurrell@nd.edu.au

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Chris Hurst
Curtin University
c.hurst@curtin.edu.au

Derek Hurrell
University of Notre Dame Australia
derek.hurrell@nd.edu.au

Multiplicative thinking is a 'big idea' of mathematics that underpins much of the mathematics learned beyond the early primary school years. This paper reports on a current study that utilises an interview tool and a written quiz to gather data about children's multiplicative thinking. The development of the tools and some of the research findings are described here. Findings suggest that middle and upper primary aged children often have a procedural level of understanding of aspects of multiplicative thinking and that various aspects of multiplicative thinking are partially known, and known in different ways by different children.

Background

Multiplicative thinking is considered a 'big idea' of mathematics that underpins mathematical understanding beyond middle primary years (Siemon, Bleckley & Neal, 2012; Hurst & Hurrell, 2014). It has been noted by Wright (2011) that children need to reconceptualise their thinking in order to make the conceptual leap from additive thinking. Both Watson (n.d.) and Siemon et al. (2012) have said that such a development is essential if children are to understand ratio, proportion, percentages, and concepts associated with algebraic thinking. Multiplicative thinking has been variously described and is well encapsulated in the definition provided by Siemon, Breed, Dole, Izard, & Virgona (2006) as being the ability to work flexibly and efficiently with an extended range of numbers, recognise and solve a range of problems involving multiplication and/or division including direct and indirect proportion, and communicate this effectively in various ways. Given its multi-faceted nature and critical position, it seems important to be able to comprehensively assess children's multiplicative thinking, and provide teachers with a tool for doing so.

In order to develop a useful instrument, it was necessary to identify key aspects of multiplicative thinking, or 'micro content' (Hurst & Hurrell, 2014), that could be assessed. Given the scope of this paper, what would normally be an extensive list has been paraphrased to contain the following:

- The multiplicative situation is understood in terms of group size and number of groups, the factor X factor = multiple relationship, and is represented by an array.
- Multiplicative arrays are used to visualise and represent multiplicative situations including commutivity and distributivity.
- Multiplicative situations can be represented as equal-groups problems, comparison problems, combinations (Cartesian) problems and area/array problems.
- The ten times multiplicative relationship between places extends to parts of wholes and is expressed as 'times as many' and 'how many times larger or smaller' a number is than another number.
- Numbers move a place each time they are multiplied or divided by 10.
- Basic number facts to 10 X 10 are recalled and can be extended by powers of ten.
- Properties such as distributivity, commutativity, and the inverse relationship between multiplication and division are known and understood

One-to-one interviews have been used for a number of years in mathematics education (Clarke, Clarke & Roche, 2011; Goldin, 1997) and they have the capacity to help teachers understand their students' learning, and develop tasks to support that learning (Sowder, 2007). They can be used to enhance the knowledge and skills of mathematics teachers by developing a deeper understanding and awareness of the way that children construct their mathematical understandings (Heng & Sudarshan, 2013). Indeed, one-to-one interviews are a way of joining research with educational practice (Goldin, 1997). They can illuminate student misconceptions which may be masked by what Heng and Sudarshan (2013) call the "whole class teaching experience" (p. 480). Clements and Ellerton (1995) observed that students may have a conceptual understanding of a topic which may become apparent through an interview, which would not necessarily be obvious through a written assessment task, a position supported through the work of Burns (2010). Burns encapsulated this well in her assertion that it is essential to ask children to explain how they arrived at an answer, even when that answer is correct.

Methodology

As part of a current study of primary school aged children's multiplicative thinking, semi-structured one-to-one interviews were conducted and audio recorded. Recordings were later transcribed to generate data and provide an opportunity for clarification and deeper reflection on what was said about the mathematics and 'around' the mathematics. During the initial phase of the research, interviews were conducted with sixteen (16) mixed ability female and male students in Year Six (aged either ten or eleven years) from two different classes and took between twenty five and thirty minutes. One issue with interviewing is the constraint of time needed to interview a class of children (Heng & Sudarshan, 2013) and while acknowledging this, Burns (2010) asserts that they are worthy of the investment of time. Nonetheless, it was felt that one important outcome of this research would be a usable tool for teachers and so, in addition to the interview proforma, a written quiz – the Multiplicative Thinking Quiz (MTQ) – was developed. This contained most of the interview questions and was able to be administered to a whole class in approximately thirty minutes, meaning that a large data set could be generated quickly. The MTQ was administered to a group of 22 Year Five students who were subsequently interviewed in order to establish that the MTQ provided the same data as did the interview. Since then, the format for both the interview and MTQ have been further refined and tested. A further 8 students have been interviewed and the MTQ has been administered to 411 students. For both instruments, students have ranged in age from Year Four to Year Six. Examples of specific questions from both instruments are embedded in the ensuing discussion of the results.

Results and Discussion

Discussion of results is embedded in this section at the point where results are presented. Two sets of results are included here. The first comprises individuals' responses to some of the interview questions and the second comprises large data sets generated from the administration of the MTQ across three year levels at different schools.

Data generated from one-to-one interviews

The first set of individual responses were made to the following interview question:

Table 1
Sample Question from Interview

Can you give an answer for this sum (6x17)?
Students were observed to see if they were able to calculate it mentally or if they needed to use an algorithm.
If the student did it mentally . . . "Please explain how you did it".
If the student used an algorithm . . . "Please explain what you did".
If the student could not provide an answer, a different combination (13x4) was offered.
If the student still could not produce an answer . . . "Can you use some of the materials (bundling sticks in sets of ten as well as singles) to help you show what is happening in the sum?"

Responses from two groups of students from different schools – Cohort A (Year Six students, n=16) and Cohort B (Year Five students, n=22) – are considered here and there are interesting contrasts between the two sets of results. In Cohort A, only one student offered a mental solution to the exercise and the other 15 opted to use a written algorithm, nine of whom did so correctly with standard place value partitioning being used. In Cohort B, 18 students offered a correct mental solution and none opted to do the exercise as a written problem. Four were unable to obtain a correct answer with mental computation. The seven students in Cohort A who did not correctly use a written method and the four students in Cohort B who did not obtain a correct answer were then asked to use the bundling sticks to show how to do the exercise. None of the 11 students was able to do so.

Typical responses from the students were to show a set of 17 bundling sticks with a multiplication sign and another set of six bundling sticks (Figure 1). None of the 11 students was able to depict the exercise as six sets of 17 bundling sticks (Figure 2) (Hurst & Hurrell, in press).
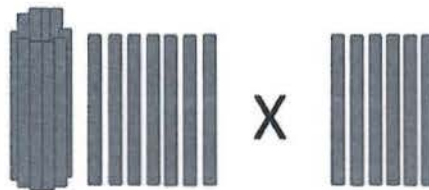

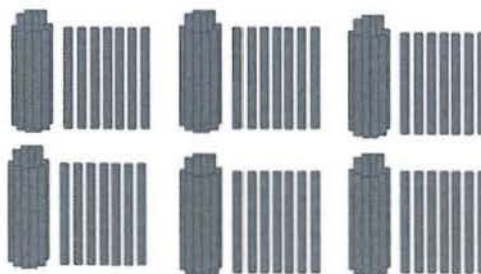*Figure 1.* 17 sticks alongside a group of six sticks

Figure 2. Six groups of 17 sticks

There are two observations worth making at this point. First, the different approaches taken by students in Cohort A and B might reflect that Cohort A had been shown

algorithmic procedures for calculating answers to multiplication exercises without necessarily having the underpinning understanding of how the algorithm relates to standard place value partitioning. This lack of understanding was highlighted by the inability to use bundling sticks Cohort B may well have been encouraged to use mental computations as a first resort. Second, if the underpinning understanding of place value partitioning is lacking, it is likely to hinder students' capacity to correctly calculate answers to multiplication exercises, using either mental or written methods. Of the seven students in Cohort A who did not arrive at a correct answer, most used a confused algorithm or some form of repeated addition, as shown by samples in Figure 3.
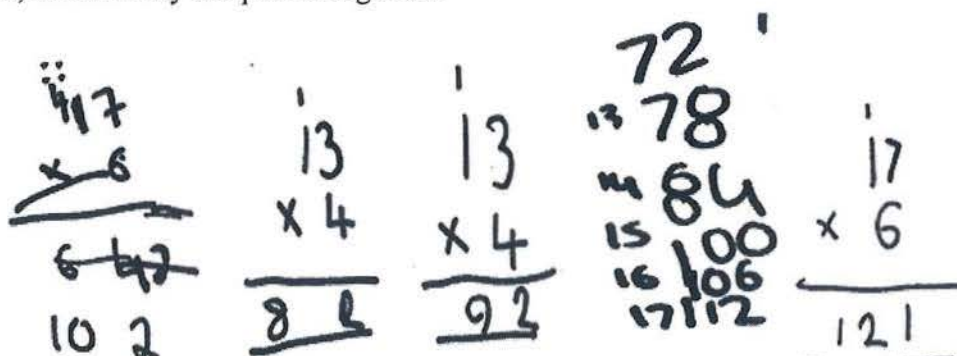

Figure 3. Cohort A students' attempts to use written methods

To further illustrate the variation in children's understanding, other data generated from Cohorts A and B are presented. In keeping with the notion of 'the multiplicative situation' (Hurst, 2015) – the multiplicative situation is understood in terms of number of groups and group size, factor X factor = multiple, and is represented by a multiplicative array – students were asked three questions (Table 2). They were also asked questions about the commutative property and the inverse relationship between multiplication and division.

Table 2
Sample Interview Questions

The multiplicative situation
7x6 – What does each number in the number sentence tell you?
4x3 – Use some of the tiles to show me this number fact.
4x3=12 – Which number/s is/are a factor/s and which number/s is/are a multiple/s?
My friend says that if you know the answer to 17x6, you must know the answer to 6x17. Is he correct? How do you know? (Commutative property)
My friend says that if you know that 17x6=102, then you must know the answer to 102÷17? Is he correct? How do you know? (Inverse relationship).

Responses to the questions are summarised below in Table 3. Responses to each question varied considerably between the two cohorts and also within each cohort. As well, no one students recorded a correct response to all five of the questions. In Cohort A, two students recorded four correct responses but they were not in response to the same questions. In Cohort B, six students provided correct responses to four questions but again they were not

all in response to the same questions. However, four students in Cohort B had the same four correct responses, the only incorrect response being to the question about group number and group size, which was generally not done well by that group. Also, in Cohort A, six students provided no correct responses and in Cohort B, one student failed to provide any correct responses.

Table 3
Comparison of Responses from Cohort A and Cohort B

Mathematical understanding demonstrated by responses to listed questions	Cohort A	Cohort B
Identifies numbers in multiplication fact as 'group size' and 'number of groups'.	56%	27%
Represents given multiplication fact as an array.	13%	50%
Defines 'factor' and 'multiple' and/or identifies factors and multiples in given number fact.	38%	73%
Explains commutative property in a conceptual way and/or demonstrates it using an array.	6%	41%
Explains inverse relationship in a conceptual way based on number of groups and group size	25%	32%

The most common explanations for the commutative property were that "They are just swapped around" or "You're switching the numbers". With regard to the inverse relationship, common responses were "They're the same family" and Multiplication and division are opposites, just like add and subtract". This suggests that many students may have learned about the properties in a procedural way, as is echoed in earlier comments about algorithms. However, some students were able to give a sound conceptual explanation such as the following:

  - "That's the multiple [102] and these are the factors [6, 17] and a multiple divided by a factor is another factor because factor times factor equals multiple" (Student Jimmy, explaining the inverse relationship).
  - "It's asking what is missing from my times table . . . something times six equals 102. Division means to share into equal groups and this is doing the same thing but backwards" (Student Kayley, explaining the inverse relationship).
  - "You've swapped them around. It's times tables. It doesn't matter which way – 17x6 or 6x17 – it's still 17 groups of six either way". [Student then referred to her two arrays for 2x6 and 6x2 and said it would give the same answer]. (Student Ellie, explaining the commutative property).

A number of students in Cohort A made what might initially be considered as 'slips of the tongue'. Student Izzy described 102÷6=17 as "102 goes into six, 17 times" and Student Abbie wrote "Six into 102 equals 17" as 6÷102= 17. Both students repeated the same type of mistake several times during the interview as did several other students in Cohort A. Perhaps this indicates that they are not familiar with the division algorithm being written in the way. These combined data seem to indicate that <u>some</u> students have developed <u>some</u> understanding of <u>some</u> key indicators of multiplicative thinking but their understanding is

either not robust and/or not connected to other key aspects of the micro content identified earlier.

Data generated from Multiplicative Thinking Written Quiz (MTQ)

The MTQ was administered with two cohorts of students from Years Four and Five, and one cohort from Years Four to Six. The total sample size was 411 students. As noted, earlier questions in the MTQ were the same as the core questions in the interview. One question in four parts sought to generate data about children's understanding of the notion about 'how many times bigger' is one number than another, as shown in Table 4.

Table 4
MTQ Question about 'Times Bigger'

How many times bigger is . . .
40 compared to 4
400 compared to 40
4000 compared to 400
400 compared to 4

Of the total cohort (n=411), only 17% provided four correct responses and 8% did not respond. There was also little variation across year levels with 17% of Year Four students, 15% of Year Five students, and 23% of Year Six students giving four correct answers. Similarly, there was little variation between different school cohorts. However, it is interesting to note the type of incorrect responses that were given and these are summarised in Table Five. In addition to the results shown in Table Five, 21% of students gave various incorrect responses.

Table 5
Summary of Student Responses to the 'Times Bigger' Question

Correct responses	Partially correct responses	Partially correct responses (3)		Additive responses
10 ✓	10 ✓	10 ✓	100	36
10 ✓	100	10 ✓	10 ✓	360
10 ✓	1000	10 ✓ or 10 ✓		3600
100 ✓	100 ✓	100	100 ✓	396
17% gave four correct responses	*13% gave these two correct responses*	*6% gave these three correct responses*		*35% gave this set of additive responses*

8% did not respond

21% gave a range of incorrect responses such as combinations of additive and one partially correct response

There are several important observations to be made concerning the data in Table Five. First, the most common response could be described as an additive response where students obtained their answer by subtracting the smaller number from the larger. It would appear that these students do not have an understanding of the notion of 'times bigger' or 'times as

many' when comparing two numbers. Again there was little variation across year levels with 40% of Year Fours, 39% of Year Fives, and 34% of Year Sixes giving an additive type response. It might be expected that more Year Six children would have developed an understanding of the 'times bigger' notion but that is not evident from this sample. Second, 13% of students provided the responses shown in the second column of Table Five. The two correct responses appear to be obtained from the first number in each comparison, that is, 40 (four tens) is 'ten times bigger' than four, and 400 (four hundreds) is 'a hundred times bigger' than four. The incorrect responses are likely to have been obtained in the same way, that is 400 (four hundreds) compared to 40 yielded a response of 'a hundred times' and 4000 (four thousands) compared to 400 yielded a response of 'a thousand times'. It would appear that students have focused on the size of the first number and are likely not to have an understanding of the 'times bigger' notion, even though they obtained two correct answers.

Another aspect of the MTQ which provides some basis for discussion relates to the inverse relationship between multiplication and division, the extension of number facts by powers of ten, and the identification of such number facts. The questions relevant to these ideas are contained in Table 6.

Table 6
MTQ Questions about Relationships and Number Facts

Please answer and explain why or why not.
My friend says that if you know that 24x6=144, you can work out the answer to 144÷6. Is she correct?
My friend says that if you know that 24x6=144, you can work out the answer to 240x6. Is she correct?
My friend says that if you know that 7x6=42, then you must know that 70x6=420. Write down lots of other things that you can know using 7x6=42 as a start.

From the whole cohort (n=411), 71% identified that the inverse relationship existed but only three students (less than 1%) gave an adequate conceptual explanation for it. Most children said something to the effect that 'multiplication is the opposite of division' or 'they are both related'. With regard to writing extended number facts, 50% identified that the statement was true yet only 9% could provide four or more extended multiplication facts and a further 19% could provide one, two or three extended multiplication facts. Only 2% could provide any extended division facts. No student could explain why extended number facts worked, beyond a procedural response based on 'adding a zero'.

Conclusions

The data presented here is only a small part of the total data generated from the interviews and the MTQ. However, it is possible to draw some conclusions about aspects of the multiplicative thinking of the students involved. First it is likely that students have been taught certain procedures before they have developed a conceptual understanding of the particular mathematics involved. This may apply to their use of algorithms, learning properties of multiplication including commutativity, the inverse relationship, and extension of number facts. Second, there is considerable variation in the level of conceptual understanding within particular student cohorts and between cohorts. Some students have particular aspects of knowledge not demonstrated by their peers, yet some of their peers demonstrate other aspects of knowledge. This might suggest that teaching of multiplicative

thinking concepts has not occurred in a connected way but has been ad hoc or piecemeal. Third, some key elements of multiplicative thinking such as the notion of 'times bigger' do not appear to be well understood by many students and they generally struggle to explain why certain things occur at any more than a procedural level (such as 'switching the numbers' for the commutative property and 'adding a zero' for the inverse relationship). Both the semi-structured interview tool and the written MTQ have been useful in identifying such lack of knowledge. The clear implication is that teaching might benefit from using the instruments as assessment tools which might lead to more connected teaching of multiplicative concepts and better outcomes for students.

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