Research Article

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The existence and uniqueness of solutions to a functional equation arising in psychological learning theory

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Abstract: The paradigm of choice practice represents the psychological theory of learning in the development of moral judgment. It is concerned with evaluating the implications of several choices and selecting one of them to implement. The goal of this work is to provide a generic functional equation to observe the behavior of animals in such circumstances. Our suggested functional equation can be employed to describe several well-known psychology and learning theories. The fixed point theorem proposed by Banach is utilized to show that the solution of a given functional problem exists and is unique. In addition, the stability of the given functional equation's solution is discussed in terms of the Hyers-Ulam and Hyers-Ulam-Rassias results. Furthermore, two examples are provided to highlight the relevance of the significant outcomes in the context of the literature.

Keywords: fixed points, functional equation, stability

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1 Preliminary discussion and introduction

The study of mathematical modeling of perceptual, intellectual, and cognitive processes is known as mathematical psychology. Alternatively, learning in animals and humans can be viewed as a sequence of decisions among several feedback possibilities. Preference sequences are typically surprising, indicating that reaction decisions are determined randomly even in basic recurrent research under well-controlled settings. As a result, it is instructive to include systematic changes in a succession of options to signify differences in response likelihood over trials. From this perspective, most of the learning study is devoted to elucidating the probability of the events that constitute a stochastic method over individual practices.

Recent mathematical learning research has proven that the most superficial learning experiment's behavior can be modeled using stochastic processes. So, it is a partially original thought (for further details, see [1,2]). Nevertheless, after 1950, two significant features became apparent, according to the research conducted primarily by Bush, Estes, and Mosteller. First, all recommended models have an inclusive learning process, which is crucial. Second, the statistical features of such models cannot be hidden from the analysis [3,4].

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In 1976, Istraţéscu [5] investigated the possible participation of predator animals eating two different types of food by the following equation:

$$H(x) = xH(\varpi_1 + (1 - \varpi_1)x) + (1 - x)H((1 - \varpi_2)x), \quad \forall x \in Y = [0, 1], \tag{1.1}$$

where $H : Y \to \mathbb{R}$ with $H(0) = 0$, $H(1) = 1$, and $0 < \varpi_1 \leq \varpi_2 < 1$. This behavior is theoretically defined by a Markov process with the state space Y and the probability of transition from state x to state $\varpi_1 + (1 - \varpi_1)x$ and from state x to state $(1 - \varpi_2)x$, which is provided by

$$\begin{cases} \mathbf{P}(x \to \varpi_1 + (1 - \varpi_1)x) = x, \\ \mathbf{P}(x \to (1 - \varpi_2)x) = 1 - x, \end{cases} \tag{1.2}$$

respectively. In the aforementioned functional equation (1.1), the solution H represents the ultimate probability of an occurrence when the predator is focused on a specific type of prey, considering that the starting probability for this class to be selected is equal to x.

Turab and Sintunavarat [6] addressed Bush and Mosteller's experimental studies [7] in 2019. They examined a paradise fish's behavior using their associated probabilities and presented the subsequent model

$$H(x) = xH(\varpi_1 x + 1 - \varpi_1) + (1 - x)H(\varpi_2 x), \quad \forall x \in Y, \tag{1.3}$$

where $H : Y \to \mathbb{R}$ such that $H(0) = 0$, $H(1) = 1$, and $0 < \varpi_1 \leq \varpi_2 < 1$.

In 2020, the authors employed the aforesaid approach and proposed the following functional equation (see [8]):

$$H(x) = xH(\varpi_1 x + (1 - \varpi_1)\eta_1) + (1 - x)H(\varpi_2 x + (1 - \varpi_2)\eta_2), \quad \forall x \in Y, \tag{1.4}$$

where $H : Y \to \mathbb{R}$, $0 < \varpi_1 \leq \varpi_2 < 1$, and $\eta_1, \eta_2 \in Y$. The outlined functional equation was used in this research to investigate a particular form of psychological barrier shown by dogs when confined in a tiny box.

Berinde and Khan, in [9], extended the concepts described above and discussed the existence of a unique solution to the following equation:

$$H(x) = xH(S_1(x)) + (1 - x)H(S_2(x)), \quad \forall x \in Y, \tag{1.5}$$

where $H : Y \to \mathbb{R}$ and $S_1, S_2 : Y \to Y$ are given contraction mappings. In addition, functional equations of this kind are utilized to explain the connection between predator animals and their two prey options. On the other hand, a few researchers discussed the functional equation's solution [10], stability [11], and convergence results [12].

For an arbitrary interval $[\xi_1, \xi_2]$, Turab and Sintunavarat [13], in 2020, proposed a functional equation that is an extension of the idea presented in [9]

$$H(x) = \left(\frac{x - \xi_1}{\xi_2 - \xi_1}\right)H(S_1(x)) + \left(1 - \frac{x - \xi_1}{\xi_2 - \xi_1}\right)H(S_2(x)), \quad \forall x \in [\xi_1, \xi_2], \tag{1.6}$$

where $H : [\xi_1, \xi_2] \to \mathbb{R}$ and $S_1, S_2 : [\xi_1, \xi_2] \to [\xi_1, \xi_2]$ are given contraction mappings.

Different conclusions have been drawn from various human and animal behavior investigations in probability-learning contexts (for further details, see [14–18]) using the transition operators given in Table 1.

Table 1: Reward-extinction and habit-formation operators

Reactions	Findings (left side)	Findings (right side)
Rewards-extinction model operators		
Reinforcement	$\varpi_1 x$	$\varpi_1 x + 1 - \varpi_1$
Non-reinforcement	$\varpi_2 x + 1 - \varpi_2$	$\varpi_2 x$
Habit formation model operators		
Reinforcement	$\varpi_1 x$	$\varpi_1 x + 1 - \varpi_1$
Non-reinforcement	$\varpi_2 x$	$\varpi_2 x + 1 - \varpi_2$

In 2022, by depending on the reward and selected side discussed by Bush and Wilson [7] (Table 1), Turab et al. [19] extended the aforementioned work by introducing the following functional equation:

$$H(x) = \zeta\kappa(x)H(S_1(x)) + (1-\zeta)\kappa(x)H(S_2(x)) + \zeta(1-\kappa(x))H(S_3(x)) + (1-\zeta)(1-\kappa(x))H(S_4(x)) \tag{1.7}$$

for all $x \in Y$, where $\kappa : Y \to Y$ stands for the proportional of the event probability occurring with the constant fraction of occurrences falling inside $0 \le \zeta \le 1$, $H : Y \to \mathbb{R}$ is an unknown, and $S_1, S_2, S_3, S_4 : Y \to Y$ are given mappings.

By following the work of Bush and Wilson [7], recently, Turab extended the above model (1.7) to the arbitrary interval $[\xi_1, \xi_2]$ by proposing the following functional equation (for the detail, see [20]):

$$\begin{aligned} H(x) = {} & \left(\frac{x-\xi_1}{\xi_2-\xi_1}\right)\left(\frac{\nu-\xi_1}{\xi_2-\xi_1}\right)H(S_1(x)) + \left(\frac{x-\xi_1}{\xi_2-\xi_1}\right)\left(1-\frac{\nu-\xi_1}{\xi_2-\xi_1}\right)H(S_2(x)) \\ & + \left(1-\frac{x-\xi_1}{\xi_2-\xi_1}\right)\left(\frac{\nu-\xi_1}{\xi_2-\xi_1}\right)H(S_3(x)) + \left(1-\frac{x-\xi_1}{\xi_2-\xi_1}\right)\left(1-\frac{\nu-\xi_1}{\xi_2-\xi_1}\right)H(S_4(x)) \end{aligned} \tag{1.8}$$

for all $x \in \tilde{Y} = [\xi_1, \xi_2]$, where $0 \le \nu \le 1$, $H : \tilde{Y} \to \mathbb{R}$ is an unknown, and $S_1, S_2, S_3, S_4 : \tilde{Y} \to \tilde{Y}$ are given mappings.

Various studies have examined how different animals respond to learning new probabilistic patterns, and their findings have changed considerably (for further details, see [21–24]). In these types of investigations, a typical issue that arises is

To what extent would the structure be affected if an animal refused to shift its position and remained in the middle?

A proper answer to this question requires a consideration of the innovative studies that Neimark has undertaken on human behavior in two-choice perspective scenarios (see [25]). During trials, the subjects were required to choose which of two lights would be illuminated, while others were given the option of doing nothing. Therefore, the "blank trials," as she introduced them, are the third kind of event.

In light of the notable study mentioned above, we offer a generic functional equation as follows:

$$H(x) = \upsilon_1(x)H(S_1(x)) + \upsilon_2(x)H(S_2(x)) + \upsilon_3(x)H(S_3(x)) + \upsilon_4(x)H(S_4(x)) + \upsilon_5(x)H(S_5(x)) \tag{1.9}$$

for all $x \in Y$, where $H : Y \to \mathbb{R}$, $\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5 : Y \to Y$ are corresponding probabilities of the events with $\upsilon_5(x) = (1 - \upsilon_1(x) - \upsilon_2(x) - \upsilon_3(x) - \upsilon_4(x))$, and $S_1, S_2, S_3, S_4, S_5 : Y \to Y$ are given mappings.

In this case, we shall employ the Banach fixed point theorem to demonstrate that the proposed equation (1.9) has a unique solution and that it exists. The stability of the recommended functional equation's solution shall be tested under the results proposed by Hyers-Ulam and Hyers-Ulam-Rassias. Two examples then show the significance of our research in this area.

Moreover, the philosophy of fixed point theory is engaged with the criteria that guarantee the existence of solutions to the fitted system. It includes methods that can be employed to address issues in several branches of mathematics. We recommend [26–33] and the references within it for the most current study done in this area.

For the progression to continue, the next outcome is required.

Theorem 1.1. [34] *Let (Y, d) is a complete metric space and a mapping $H : Y \to Y$ satisfy the following property with $\varrho < 1$, i.e., $d(H\mu, H\nu) \le \varrho d(\mu, \nu)$, $\forall \mu, \nu \in Y$. Then, H has a unique fixed point. Consequently, the Picard iteration $\{\mu_n\}$ in Y defined as $\mu_n = H\mu_{n-1}$, $\forall n \in \mathbb{N}$ and $\mu_0 \in Y$, converges to the unique fixed point of H.*

2 Existence of a unique solution

Throughout this work, we let $Y = [0, 1]$ and D is a category of all real-valued continuous functions $H : Y \to \mathbb{R}$ with $\sup_{\mu \ne \nu} \frac{|H(\mu) - H(\nu)|}{|\mu - \nu|} < \infty$ and $H(0) = 0$. There is no room for doubt in the fact that $(D, \|\cdot\|)$ is a Banach space (see [35]) with

$$\|H\| = \sup_{\mu\neq\nu} \frac{|H(\mu) - H(\nu)|}{|\mu - \nu|}, \quad \forall H \in D. \tag{2.1}$$

In this subsequent development, we shall use the following suppositions:

(λ_1): There exists $C_{\neq\varnothing} \subset \mathscr{T} \coloneqq \{H \in D | H(1) \leq 1\}$ such that $(C, \|\cdot\|)$ be a Banach space, where $\|\cdot\|$ is defined in (2.1).

(λ_2): The mappings $\upsilon_k : Y \to Y$ with $\upsilon_1(0) = 0 = \upsilon_2(0)$, for $k = 1, 2, 3, 4, 5$, satisfy the following properties:

$$d(\upsilon_k\mu, \upsilon_k\nu) \leq d(\mu, \nu), \quad \forall \mu, \nu \in Y \quad (\text{non-expansive}), \tag{2.2}$$

and

$$|\upsilon_k(x)| \leq \omega_k, \quad \forall x \in Y \quad \text{and} \quad \omega_k \geq 0 \quad (\text{boundedness}). \tag{2.3}$$

(λ_3): The mappings $S_k : Y \to Y$ with $S_3(0) = S_4(0) = S_5(0) = 0$, for $k = 1, 2, 3, 4, 5$, satisfy the following properties:

$$d(S_k\mu, S_k\nu) \leq \alpha_k d(\mu, \nu), \quad \forall \mu, \nu \in Y \quad \text{and} \quad \alpha_k < 1 \quad (\text{contraction}), \tag{2.4}$$

and

$$|S_k(x)| \leq \tau_k, \quad \forall x \in Y \quad \text{and} \quad \tau_k \geq 0 \quad (\text{boundedness}). \tag{2.5}$$

(λ_4): (Hyers-Ulam-Rassias stability [36]) For a function $\delta : C \to [0, \infty)$ and for $H \in C$ with $d(PH, H) \leq \delta(H)$, there exists a unique $H^* \in C$ such that $PH^* = H^*$ and $d(H, H^*) \leq \gamma\delta(H)$ with $\gamma > 0$.

(λ_5): (Hyers-Ulam stability [37]) For $\vartheta > 0$ and for $H \in C$ with $d(PH, H) \leq \vartheta$, there exists a unique $H^* \in C$ such that $PH^* = H^*$ and $d(H, H^*) \leq \gamma\vartheta$ with $\gamma > 0$.

Here is our first result.

Theorem 2.1. *Consider the functional equation* (1.9). *Suppose that* (λ_1)–(λ_3) *are satisfied with* $\Theta_1 < 1$, *where*

$$\Theta_1 \coloneqq [\alpha_1\omega_1 + \alpha_2\omega_2 + \alpha_3\omega_3 + \alpha_4\omega_4 + \alpha_5\omega_5 + \tau_1 + \tau_2 + \tau_3 + \tau_4 + \tau_5]. \tag{2.6}$$

Then, (1.9) *has a unique solution. Subsequently, the sequence* $\{H_n\}$ *in* C ($H_0 \in C$ *and* $\forall n \in \mathbb{N}$), *given below, converges to the unique solution of* (1.9) *with*

$$H_n(x) = \upsilon_1(x)H_{n-1}(S_1(x)) + \upsilon_2(x)H_{n-1}(S_2(x)) + \upsilon_3(x)H_{n-1}(S_3(x)) + \upsilon_4(x)H_{n-1}(S_4(x)) + \upsilon_5(x)H_{n-1}(S_5(x)). \tag{2.7}$$

Proof. Let $d : C \times C \to \mathbb{R}$ be an induced metric on C of $\|\cdot\|$. Consequently, the metric space (C, d) is complete. Here, we are concerned with the operator P, which is defined for all $H \in C$ as follows:

$$(PH)(x) = \upsilon_1(x)H(S_1(x)) + \upsilon_2(x)H(S_2(x)) + \upsilon_3(x)H(S_3(x)) + \upsilon_4(x)H(S_4(x)) + \upsilon_5(x)H(S_5(x)).$$

For each $H \in C$, we have $(PH)(0) = 0$. As $P : C \to C$ is continuous, $\|PH\| < \infty \ \forall H \in C$. In addition, the solution of equation (1.9) is unambiguously equal to the fixed point of the operator P.

For the linear mapping $P : C \to C$, we have

$$\|PH_1 - PH_2\| = \|P(H_1 - H_2)\| \quad \forall H_1, H_2 \in C.$$

Eventually, we construct the structure to investigate the term $\|PH_1 - PH_2\| \ \forall H_1, H_2 \in C$ as follows:

$$\Omega_{\rho,\wp} \coloneqq \frac{P(H_1 - H_2)(\rho) - P(H_1 - H_2)(\wp)}{\rho - \wp}, \quad \rho, \wp \in Y, \ \rho \neq \wp.$$

For each $\rho, \wp \in Y$ with $\rho \neq \wp$, we obtain

$$\begin{aligned}\Omega_{\rho,\wp} &= \frac{1}{\rho-\wp}[\upsilon_1(\rho)H(S_1(\rho)) + \upsilon_2(\rho)H(S_2(\rho)) + \upsilon_3(\rho)H(S_3(\rho)) + \upsilon_4(\rho)H(S_4(\rho))\\
&\quad + \upsilon_5(\rho)H(S_5(\rho)) - \upsilon_1(\wp)H(S_1(\wp)) - \upsilon_2(\wp)H(S_2(\wp)) - \upsilon_3(\wp)H(S_3(\wp))\\
&\quad - \upsilon_4(\wp)H(S_4(\wp)) - \upsilon_5(\wp)H(S_5(x))]\\
&= \frac{1}{\rho-\wp}[\upsilon_1(\rho)H(S_1(\rho)) - \upsilon_1(\rho)H(S_1(\wp)) + \upsilon_2(\rho)H(S_2(\rho)) - \upsilon_2(\rho)H(S_2(\wp))\\
&\quad + \upsilon_3(\rho)H(S_3(\rho)) - \upsilon_3(\rho)H(S_3(\wp)) + \upsilon_4(\rho)H(S_4(\rho)) - \upsilon_4(\rho)H(S_4(\wp))\\
&\quad + \upsilon_5(\rho)H(S_5(\rho)) - \upsilon_5(\rho)H(S_5(\wp)) + \upsilon_1(\rho)H(S_1(\wp)) - \upsilon_1(\wp)H(S_1(\wp))\\
&\quad + \upsilon_2(\rho)H(S_2(\wp)) - \upsilon_2(\wp)H(S_2(\wp)) + \upsilon_3(\rho)H(S_3(\wp)) - \upsilon_3(\wp)H(S_3(\wp))\\
&\quad + \upsilon_4(\rho)H(S_4(\wp)) - \upsilon_4(\wp)H(S_4(\wp)) + \upsilon_5(\rho)H(S_5(\wp)) - \upsilon_5(\wp)H(S_5(x))].\end{aligned}$$

Then, we have

$$\begin{aligned}|\Omega_{\rho,\wp}| &\le \left|\frac{1}{\rho-\wp}[\upsilon_1(\rho)H(S_1(\rho)) - \upsilon_1(\rho)H(S_1(\wp))]\right| + \left|\frac{1}{\rho-\wp}[\upsilon_2(\rho)H(S_2(\rho)) - \upsilon_2(\rho)H(S_2(\wp))]\right|\\
&\quad + \left|\frac{1}{\rho-\wp}[\upsilon_3(\rho)H(S_3(\rho)) - \upsilon_3(\rho)H(S_3(\wp))]\right| + \left|\frac{1}{\rho-\wp}[\upsilon_4(\rho)H(S_4(\rho)) - \upsilon_4(\rho)H(S_4(\wp))]\right|\\
&\quad + \left|\frac{1}{\rho-\wp}[\upsilon_5(\rho)H(S_5(\rho)) - \upsilon_5(\rho)H(S_5(\wp))]\right| + \left|\frac{1}{\rho-\wp}[\upsilon_1(\rho)H(S_1(\wp)) - \upsilon_1(\wp)H(S_1(\wp))]\right|\\
&\quad + \left|\frac{1}{\rho-\wp}[\upsilon_2(\rho)H(S_2(\wp)) - \upsilon_2(\wp)H(S_2(\wp))]\right| + \left|\frac{1}{\rho-\wp}[\upsilon_3(\rho)H(S_3(\wp)) - \upsilon_3(\wp)H(S_3(\wp))]\right|\\
&\quad + \left|\frac{1}{\rho-\wp}[\upsilon_4(\rho)H(S_4(\wp)) - \upsilon_4(\wp)H(S_4(\wp))]\right| + \left|\frac{1}{\rho-\wp}[\upsilon_5(\rho)H(S_5(\wp)) - \upsilon_5(\wp)H(S_5(x))]\right|.\end{aligned}$$

Here, our objective is to take advantage of the specification of the norm that is provided in equation (2.1). Therefore, we have

$$\begin{aligned}|\Omega_{\rho,\wp}| &\le \frac{|\upsilon_1(\rho)H(S_1(\rho)) - \upsilon_1(\rho)H(S_1(\wp))|}{|S_1(\rho) - S_1(\wp)|} \times \frac{|S_1(\rho) - S_1(\wp)|}{|\rho-\wp|} + \frac{|\upsilon_2(\rho)H(S_2(\rho)) - \upsilon_2(\rho)H(S_2(\wp))|}{|S_2(\rho) - S_2(\wp)|}\\
&\quad \times \frac{|S_2(\rho) - S_2(\wp)|}{|\rho-\wp|} + \frac{|\upsilon_3(\rho)H(S_3(\rho)) - \upsilon_3(\rho)H(S_3(\wp))|}{|S_3(\rho) - S_3(\wp)|} \times \frac{|S_3(\rho) - S_3(\wp)|}{|\rho-\wp|}\\
&\quad + \frac{|\upsilon_4(\rho)H(S_4(\rho)) - \upsilon_4(\rho)H(S_4(\wp))|}{|S_4(\rho) - S_4(\wp)|} \times \frac{|S_4(\rho) - S_4(\wp)|}{|\rho-\wp|} + \frac{|\upsilon_5(\rho)H(S_5(\rho)) - \upsilon_5(\rho)H(S_5(\wp))|}{|S_5(\rho) - S_5(\wp)|}\\
&\quad \times \frac{|S_5(\rho) - S_5(\wp)|}{|\rho-\wp|} + \frac{|\upsilon_1(\rho) - \upsilon_1(\wp)|}{|\rho-\wp|} \times \frac{|H(S_1(\wp)) - H(0)|}{|S_1(\wp) - 0|} \times |S_1(\wp)| + \frac{|\upsilon_2(\rho) - \upsilon_2(\wp)|}{|\rho-\wp|}\\
&\quad \times \frac{|H(S_2(\wp)) - H(0)|}{|S_2(\wp) - 0|} \times |S_2(\wp)| + \frac{|\upsilon_3(\rho) - \upsilon_3(\wp)|}{|\rho-\wp|} \times \frac{|H(S_3(\wp)) - H(0)|}{|S_3(\wp) - 0|} \times |S_3(\wp)| + \frac{|\upsilon_4(\rho) - \upsilon_4(\wp)|}{|\rho-\wp|}\\
&\quad \times \frac{|H(S_4(\wp)) - H(0)|}{|S_4(\wp) - 0|} \times |S_4(\wp)| + \frac{|\upsilon_5(\rho) - \upsilon_5(\wp)|}{|\rho-\wp|} \times \frac{|H(S_5(\wp)) - H(0)|}{|S_5(\wp) - 0|} \times |S_5(\wp)|.\end{aligned}$$

By using (λ_2) and (λ_3) with equation (2.1), we have

$$\begin{aligned}|\Omega_{\rho,\wp}| &\le \alpha_1\omega_1\|H_1 - H_2\| + \alpha_2\omega_2\|H_1 - H_2\| + \alpha_3\omega_3\|H_1 - H_2\| + \alpha_4\omega_4\|H_1 - H_2\| + \alpha_5\omega_5\|H_1 - H_2\|\\
&\quad + \tau_1\|H_1 - H_2\| + \tau_2\|H_1 - H_2\| + \tau_3\|H_1 - H_2\| + \tau_4\|H_1 - H_2\| + \tau_5\|H_1 - H_2\|\\
&= \Theta_1\|H_1 - H_2\|,\end{aligned}$$

where Θ_1 is given in (2.6). As a result of this, it is clear that

$$d(PH_1, PH_2) = \|PH_1 - PH_2\| \le \Theta_1\|H_1 - H_2\| = \Theta_1 d(H_1, H_2).$$

Since $0 < \Theta_1 < 1$, P is a contraction mapping. Because of the notable fixed point result proposed by Banach (Theorem 1.1), we deduce that the Picard iteration given in equation (2.7) converges to the unique solution of equation (1.9) in the interval Y. □

In contrast to the initial conditions (λ_2) and (λ_3), here, we present the following relaxed conditions:

$(\tilde{\lambda}_2)$: The mappings $\upsilon_k : Y \to Y$ with $\upsilon_1(0) = 0 = \upsilon_2(0)$, for $k = 1, 2, 3, 4, 5$, satisfy the following properties:

$$d(\upsilon_k\mu, \upsilon_k\nu) \le \beta_k d(\mu, \nu), \quad \forall \mu, \nu \in Y \quad \text{and} \quad \beta_k < 1 \quad (\text{contraction}), \tag{2.8}$$

and

$$|\upsilon_k(x)| \le \tilde{\omega}, \quad \forall x \in Y \quad \text{and} \quad \tilde{\omega} \ge 0 \quad (\text{boundedness}). \tag{2.9}$$

$(\tilde{\lambda}_3)$: The mappings $S_k : Y \to Y$ with $S_3(0) = S_4(0) = S_5(0) = 0$, for $k = 1, 2, 3, 4, 5$, satisfy the following properties:

$$d(S_k\mu, S_k\nu) \le \alpha_k d(\mu, \nu), \quad \forall \mu, \nu \in Y \quad \text{and} \quad \alpha_k < 1 \quad (\text{contraction}), \tag{2.10}$$

with $\alpha_1 \le \alpha_2 \le \alpha_3 \le \alpha_4 \le \alpha_5$ and

$$|S_k(x)| \le \tilde{\tau}, \quad \forall x \in Y \quad \text{and} \quad \tilde{\tau} \ge 0 \quad (\text{boundedness}). \tag{2.11}$$

Based on the above results, we can deduce the following conclusion.

Corollary 2.1. *Take into account the underlying functional equation* (1.9). *Suppose that* (λ_1), $(\tilde{\lambda}_2)$, *and* (λ_3) *are satisfied with* $\Theta_2 < 1$, *where*

$$\Theta_2 := [\tilde{\omega}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5) + \beta_1\tau_1 + \beta_2\tau_2 + \beta_3\tau_3 + \beta_4\tau_4 + \beta_5\tau_5]. \tag{2.12}$$

Then, equation (1.9) *has a unique solution. Moreover, the sequence* $\{H_n\}$ *in* C ($H_0 \in C$ *and* $\forall n \in \mathbb{N}$), *given below, converges to the unique solution of equation* (1.9) *with*

$$H_n(x) = \upsilon_1(x)H_{n-1}(S_1(x)) + \upsilon_2(x)H_{n-1}(S_2(x)) + \upsilon_3(x)H_{n-1}(S_3(x)) + \upsilon_4(x)H_{n-1}(S_4(x)) + \upsilon_5(x)H_{n-1}(S_5(x)). \tag{2.13}$$

Corollary 2.2. *Take into account the underlying functional equation* (1.9). *Suppose that* (λ_1), (λ_2), *and* $(\tilde{\lambda}_3)$ *are satisfied with* $\Theta_3 < 1$, *where*

$$\Theta_3 := [\tilde{\alpha}_5(\omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5) + \tilde{\tau}]. \tag{2.14}$$

Then, equation (1.9) *has a unique solution. Moreover, the sequence* $\{H_n\}$ *in* C ($H_0 \in C$ *and* $\forall n \in \mathbb{N}$), *given below, converges to the unique solution of equation* (1.9) *with*

$$H_n(x) = \upsilon_1(x)H_{n-1}(S_1(x)) + \upsilon_2(x)H_{n-1}(S_2(x)) + \upsilon_3(x)H_{n-1}(S_3(x)) + \upsilon_4(x)H_{n-1}(S_4(x)) + \upsilon_5(x)H_{n-1}(S_5(x)). \tag{2.15}$$

Next, we shall discuss a weaker criteria compared to $(\tilde{\lambda}_2)$, which can be utilized to establish the existence of a unique solution to equation (1.9).

$(\tilde{\tilde{\lambda}}_2)$: The mappings $\upsilon_k : Y \to Y$ with $\upsilon_1(0) = 0 = \upsilon_2(0)$, for $k = 1, 2, 3, 4, 5$, satisfy the following properties:

$$d(\upsilon_k\mu, \upsilon_k\nu) \le \beta_k d(\mu, \nu), \quad \forall \mu, \nu \in Y \quad \text{and} \quad \beta_k < 1 \quad (\text{contraction}), \tag{2.16}$$

with $\beta_1 \le \beta_2 \le \beta_3 \le \beta_4 \le \beta_5$, and

$$|\upsilon_k(x)| \le \tilde{\omega}, \quad \forall x \in Y \quad \text{and} \quad \tilde{\omega} \ge 0 \quad (\text{boundedness}). \tag{2.17}$$

The following corollary follows directly from the previous result.

Corollary 2.3. *Take into account the underlying functional equation* (1.9). *Suppose that* (λ_1), $(\tilde{\tilde{\lambda}}_2)$, *and* $(\tilde{\lambda}_3)$ *are satisfied with* $\Theta_4 < 1$, *where*

$$\Theta_4 := [\tilde{\alpha}_5\tilde{\omega} + \beta_5\tilde{\tau}]. \tag{2.18}$$

Then, (1.9) *has a unique solution. Subsequently, the sequence* $\{H_n\}$ in C $(H_0 \in C$ *and* $\forall n \in \mathbb{N})$, *given below, converges to the unique solution of equation* (1.9) *with*

$$H_n(x) = \upsilon_1(x)H_{n-1}(S_1(x)) + \upsilon_2(x)H_{n-1}(S_2(x)) + \upsilon_3(x)H_{n-1}(S_3(x)) + \upsilon_4(x)H_{n-1}(S_4(x)) + \upsilon_5(x)H_{n-1}(S_5(x)). \tag{2.19}$$

Remark 2.1. Since the Picard iteration converges at a linear rate, we cannot anticipate fast convergence based on the proposed iteration pattern. With the right accelerative strategy, we could get beyond this obstacle (for further research in this area, we refer [38– 40]).

Remark 2.2. The suggested equation (1.9) generalizes a wide variety of existing functional equations in the available literature, such as

(1) Let $\upsilon_1(x) = x$, $\upsilon_3(x) = 1 - x$, and $\upsilon_2(x) = \upsilon_4(x) = \upsilon_5(x) = 0$. As a result, it becomes abundantly evident that the suggested equation (1.9) is an extension of the functional equation (1.5) proposed by Berinde and Khan (see [9]) and Turab and Sintunavarat (see [10]). Likewise,
   - If we define $S_1(x) = \varpi_1 + (1 - \varpi_1)x$ and $S_3(x) = (1 - \varpi_2)x$, then the proposed equation (1.9) reduces to equation (1.1).

(2) For $\zeta \in Y$ and $\kappa : Y \to Y$ with $\kappa(0) = 0$, we define $\upsilon_1(x) = \zeta\kappa(x)$, $\upsilon_2(x) = (1 - \zeta)\kappa(x)$, $\upsilon_3(x) = \zeta(1 - \kappa(x))$, $\upsilon_4(x) = (1 - \zeta)(1 - \kappa(x))$, and $\upsilon_5(x) = 0$. Here, it can be clearly seen that the proposed equation (1.9) is a generalization of equation (1.7), which was presented by Turab et al. in [19].

3 Stability results

The following primary concern applies to almost all areas of mathematical computing: Does equivalence between a mathematical entity that approximatively satisfies a requirement and an entity that unambiguously fulfills that attribute always hold? When we shift our focus to functional equations, we may encounter the question of whether an equation's solution that varies marginally from an equation is approximate to the given equation's solution. Hence, the stability of the proposed equation (1.9) has to be discussed here (for further details, see [41–48]).

Theorem 3.1. *Assume that* Theorem 2.1 *holds. Then, the equation* $PH = H$ *satisfies* (λ_4), *where* $P : C \to C$ *is defined for all* $H \in C$ *and* $x \in Y$ *as follows*:

$$(PH)(x) = \upsilon_1(x)H(S_1(x)) + \upsilon_2(x)H(S_2(x)) + \upsilon_3(x)H(S_3(x)) + \upsilon_4(x)H(S_4(x)) + \upsilon_5(x)H(S_5(x)). \tag{3.1}$$

Proof. Let $H \in C$ with $d(PH, H) \leq \delta(H)$. Theorem 2.1 indicates that there is a unique $H^\star \in C$ with the property $PH^\star = H^\star$. Hence, we obtain

$$d(H, H^\star) \leq d(H, PH) + d(PH, H^\star) \leq \delta(H) + d(PH, PH^\star) \leq \delta(H) + \Theta_1 d(H, H^\star),$$

where Θ_1 is given in equation (2.6). Consequently,

$$d(H, H^\star) \leq \gamma\delta(H),$$

where $\gamma := \frac{1}{1-\Theta_1}$. □

From this investigation, we can infer the result regarding the stability of the Hyers-Ulam type.

Corollary 3.1. *Assume that the observations given in* Theorem 2.1 *hold. Then, the equation* $PH = H$ *satisfies* (λ_5), *where* $P : C \to C$ *is defined for all* $H \in C$ *and* $x \in Y$ *as follows*:

$$(PH)(x) = \upsilon_1(x)H(S_1(x)) + \upsilon_2(x)H(S_2(x)) + \upsilon_3(x)H(S_3(x)) + \upsilon_4(x)H(S_4(x)) + \upsilon_5(x)H(S_5(x)). \tag{3.2}$$

4 Some illustrative examples

Now, we highlight our results using the following examples.

Example 4.1. Take into account the following illustrative functional equation:

$$\begin{aligned} H(x) = {} & \left(\frac{\vartheta_1 x}{2}\right) H\left(\frac{\varpi_1}{9} + \frac{(1-\varpi_1)x}{9}\right) + \left(\frac{\vartheta_2 x}{2}\right) H\left(\frac{\varpi_2}{4} + \frac{(1-\varpi_2)x}{4}\right) + \left(\frac{(1-\vartheta_1)x}{2}\right) H\left(\frac{\varpi_3 x}{7}\right) \\ & + \left(\frac{(1-\vartheta_2)x}{2}\right) H\left(\frac{\varpi_4 x}{11}\right) + (1-x) H\left(\frac{\varpi_5 x}{17}\right), \quad \forall x \in Y, \end{aligned} \tag{4.1}$$

where $H : Y \to \mathbb{R}$. We define $\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5 : Y \to Y$ and $S_1, S_2, S_3, S_4, S_5 : Y \to Y$ by

$$\begin{cases} \upsilon_1(x) = \dfrac{\vartheta_1 x}{2}, \\ \upsilon_2(x) = \dfrac{\vartheta_2 x}{2}, \\ \upsilon_3(x) = \dfrac{(1-\vartheta_1)x}{2}, \\ \upsilon_4(x) = \dfrac{(1-\vartheta_2)x}{2}, \\ \upsilon_5(x) = 1 - x, \end{cases} \quad \text{and} \quad \begin{cases} S_1(x) = \dfrac{\varpi_1}{9} + \dfrac{(1-\varpi_1)x}{9}, \\ S_2(x) = \dfrac{\varpi_2}{4} + \dfrac{(1-\varpi_2)x}{4}, \\ S_3(x) = \dfrac{\varpi_3 x}{7}, \\ S_4(x) = \dfrac{\varpi_4 x}{11}, \\ S_5(x) = \dfrac{\varpi_5 x}{17}, \end{cases} \tag{4.2}$$

for all $x \in Y$, where $0 < \varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5 < 1$, and $\vartheta_1, \vartheta_2 \in Y$. It can be seen that the proposed equation (1.9) reduces to equation (4.1).

The functional equation (4.1) has enormous significance in the areas of psychology and theory of learning as it is utilized to examine the activity of a paradise fish (Figure 1) in a two-choice scenario (**A** or **B**).

In this regard, depending on the selected side and reward, S_1, S_2, S_3, S_4 and S_5 represent the five occurrences, while $\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4$ and υ_5 indicate their relative probabilities (Table 2).

Our goal is to employ Theorem 2.1 to prove that the proposed equation (4.1) has exactly one solution. It is clear that the mappings $\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4$, and υ_5, and S_1, S_2, S_3, S_4, and S_5 defined in (4.2) satisfy conditions (λ_2) and (λ_3), respectively, with coefficients

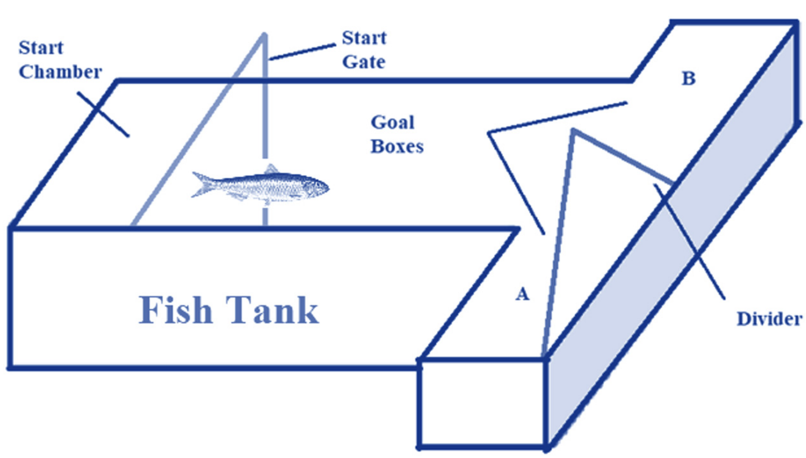

Figure 1: The activity of a paradise fish under two-option circumstances in a fish tank [6].

Table 2: Five possible events and their corresponding probabilities

Reactions (left or right side)	**Findings (reward or no reward)**	**Corresponding probabilities**
A (Left)	**Food side** (reward)	$\upsilon_1(x) \coloneqq \frac{\vartheta_1 x}{2}$
B (Right)	**Non-food side** (no reward)	$\upsilon_2(x) \coloneqq \frac{\vartheta_2 x}{2}$
A (Left)	**Non-food side** (no reward)	$\upsilon_3(x) \coloneqq \frac{(1\ \vartheta_1)x}{2}$
B (Right)	**Food side** (reward)	$\upsilon_4(x) \coloneqq \frac{(1\ \vartheta_2)x}{2}$
No side selection (**A** or **B**)	**Blank trial**	$\upsilon_5(x) \coloneqq 1 - x$

$$\begin{cases} |\upsilon_1(x)| \le \omega_1 \coloneqq \frac{\vartheta_1}{2}, \\ |\upsilon_2(x)| \le \omega_2 \coloneqq \frac{\vartheta_2}{2}, \\ |\upsilon_3(x)| \le \omega_3 \coloneqq \frac{1-\vartheta_1}{2}, \\ |\upsilon_4(x)| \le \omega_4 \coloneqq \frac{1-\vartheta_2}{2}, \\ |\upsilon_5(x)| \le \omega_5 \coloneqq 1, \end{cases} \quad \begin{cases} \alpha_1 \coloneqq \frac{1-\varpi_1}{9}, \\ \alpha_2 \coloneqq \frac{1-\varpi_2}{4}, \\ \alpha_3 \coloneqq \frac{\varpi_3}{7}, \\ \alpha_4 \coloneqq \frac{\varpi_4}{11}, \\ \alpha_5 \coloneqq \frac{\varpi_5}{17}, \end{cases} \quad \text{and} \quad \begin{cases} |S_1(x)| \le \tau_1 \coloneqq \frac{1}{9}, \\ |S_2(x)| \le \tau_2 \coloneqq \frac{1}{4}, \\ |S_3(x)| \le \tau_3 \coloneqq \frac{1}{7}, \\ |S_4(x)| \le \tau_4 \coloneqq \frac{1}{11}, \\ |S_5(x)| \le \tau_5 \coloneqq \frac{1}{17}, \end{cases} \quad \forall x, \vartheta_1, \vartheta_2 \in Y, \tag{4.3}$$

respectively. If $\Theta_1 \coloneqq \left[\frac{(1-\varpi_1)\vartheta_1}{18} + \frac{(1-\varpi_2)\vartheta_2}{8} + \frac{\varpi_3(1-\vartheta_1)}{14} + \frac{\varpi_4(1-\vartheta_2)}{22} + \frac{\varpi_5}{17} + \frac{30{,}805}{47{,}124}\right] < 1$, then the conditions of Theorem 2.1 are all met. This means that there is only one solution to equation (4.1).

In addition, suppose we take a preliminary estimate, say $H_0(x) = x, \ \forall x \in Y$, the subsequent iteration $(\forall n \in \mathbb{N})$ will converge to the solution of equation (4.1)

$$\begin{aligned} H_1(x) &= \left(\frac{\vartheta_1 x}{2}\right) H_0\left(\frac{\varpi_1}{9} + \frac{(1-\varpi_1)x}{9}\right) + \left(\frac{\vartheta_2 x}{2}\right) H_0\left(\frac{\varpi_2}{4} + \frac{(1-\varpi_2)x}{4}\right) + \left(\frac{(1-\vartheta_1)x}{2}\right) H_0\left(\frac{\varpi_3 x}{7}\right) \\ &\quad + \left(\frac{(1-\vartheta_2)x}{2}\right) H_0\left(\frac{\varpi_4 x}{11}\right) + (1-x) H_0\left(\frac{\varpi_5 x}{17}\right), \\ H_2(x) &= \left(\frac{\vartheta_1 x}{2}\right) H_1\left(\frac{\varpi_1}{9} + \frac{(1-\varpi_1)x}{9}\right) + \left(\frac{\vartheta_2 x}{2}\right) H_1\left(\frac{\varpi_2}{4} + \frac{(1-\varpi_2)x}{4}\right) + \left(\frac{(1-\vartheta_1)x}{2}\right) H_1\left(\frac{\varpi_3 x}{7}\right) \\ &\quad + \left(\frac{(1-\vartheta_2)x}{2}\right) H_1\left(\frac{\varpi_4 x}{11}\right) + (1-x) H_1\left(\frac{\varpi_5 x}{17}\right), \\ &\ \ \vdots \\ H_n(x) &= \left(\frac{\vartheta_1 x}{2}\right) H_{n-1}\left(\frac{\varpi_1}{9} + \frac{(1-\varpi_1)x}{9}\right) + \left(\frac{\vartheta_2 x}{2}\right) H_{n-1}\left(\frac{\varpi_2}{4} + \frac{(1-\varpi_2)x}{4}\right) + \left(\frac{(1-\vartheta_1)x}{2}\right) H_{n-1}\left(\frac{\varpi_3 x}{7}\right) \\ &\quad + \left(\frac{(1-\vartheta_2)x}{2}\right) H_{n-1}\left(\frac{\varpi_4 x}{11}\right) + (1-x) H_{n-1}\left(\frac{\varpi_5 x}{17}\right). \end{aligned}$$

Moreover, we have

$$\gamma \coloneqq \frac{1}{1-\Theta_1} = \frac{1}{1 - \left[\frac{(1-\varpi_1)\vartheta_1}{18} + \frac{(1-\varpi_2)\vartheta_2}{8} + \frac{\varpi_3(1-\vartheta_1)}{14} + \frac{\varpi_4(1-\vartheta_2)}{22} + \frac{\varpi_5}{17} + \frac{30{,}805}{47{,}124}\right]} > 0.$$

If a function $H \in Y$ meets the inequality

$$d(PH, H) \le \delta(H),$$

then, by employing Theorem 3.1, we deduce that there is a unique solution $H^* \in Y$ of the functional equation (4.1) with

$$PH^* = H^* \quad \text{and} \quad d(H, H^*) \le \gamma \delta(H).$$

Example 4.2. Consider the next illustrative functional equation in a two-choice case

$$H(x) = xH\left(\frac{5x+7}{35}\right) + (1-x)H\left(\frac{x}{21}\right) \quad \forall x \in Y, \tag{4.4}$$

where $H : Y \to \mathbb{R}$. Define $\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5 : Y \to Y$ and $S_1, S_2, S_3, S_4, S_5 : Y \to Y$ by

$$\begin{cases} \upsilon_1(x) = x, \\ \upsilon_3(x) = 1-x, \end{cases} \quad \text{and} \quad \begin{cases} S_1(x) = \dfrac{5x+7}{35}, \\ S_3(x) = \dfrac{x}{21}, \end{cases} \tag{4.5}$$

and $S_2(x) = S_4(x) = S_5(x) = 0$ for all $x \in Y$. The suggested equation (1.9) in this scenario simplifies to equation (4.2).

Our goal is to employ Theorem 2.1 to prove that the proposed equation (4.2) has exactly one solution. It can be seen that the mappings υ_1, υ_3 and S_1, S_3 defined in equation (4.2) satisfy conditions (λ_2) and (λ_3), respectively, with coefficients

$$\begin{cases} |\upsilon_1(x)| \le \omega_1 := 1, \\ |\upsilon_3(x)| \le \omega_3 := 1, \end{cases} \quad \begin{cases} \alpha_1 := \dfrac{1}{7}, \\ \alpha_3 := \dfrac{1}{21}, \end{cases} \quad \text{and} \quad \begin{cases} |S_1(x)| \le \tau_1 := \dfrac{12}{35}, \\ |S_3(x)| \le \tau_3 := \dfrac{1}{21}, \end{cases} \quad \forall x \in Y, \tag{4.6}$$

respectively. In addition, $\Theta_1 := \frac{61}{105} < 1$, so the conditions of Theorem 2.1 are all met. This means that there is only one solution to equation (4.2).

In addition, suppose we take a preliminary estimate, say $H_0(x) = x, \quad \forall x \in Y$, the subsequent iteration $(\forall\ n \in \mathbb{N})$ will converge to the solution of equation (4.2):

$$\begin{aligned} H_1(x) &= \frac{10x^2 + 26x}{105}, \\ H_2(x) &= \frac{400x^3 + 6{,}770x^2 + 15{,}078x}{231{,}525}, \\ &\vdots \\ H_n(x) &= xH_{n-1}\left(\frac{5x+7}{35}\right) + (1-x)H_{n-1}\left(\frac{x}{21}\right). \end{aligned}$$

Furthermore, we have

$$\gamma := \frac{1}{1-\Theta_1} = \frac{105}{44} > 0.$$

If a function $H \in Y$ meets the inequality

$$d(PH, H) \le \delta(H),$$

then by employing Theorem 3.1, we deduce that there is a unique solution $H^* \in Y$ of the functional equation (4.2) with the following property:

$$PH^* = H^* \quad \text{and} \quad d(H, H^*) \le \gamma\delta(H).$$

5 Conclusion

As a subfield of psychology, mathematical psychology is concerned with using mathematics to represent challenges in learning theory and psychology. From this vantage point, the majority of learning studies attempt to estimate the random chance that plays a role in a stochastic process by comparing results over several trials. Our study proposed a broad class of functional equations that can be applied to the analysis of a wide range of studies in the field of psychological learning theory. The existence of a unique solution to the given equation (1.9) has been examined using the Banach fixed point theorem. The main conclusions were illustrated using two examples. As a final result, we provided a brief stability analysis of a solution to the suggested functional equation.

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