The London School of Economics and Political Science

Essays in the Theory of Contracts and Organisations

Thomas Brzustowski

A thesis submitted to the Department of Economics for the degree of Doctor of Philosophy

February 21, 2023

Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

I warrant that this authorisation does not, to the best of my belief, infringe the rights of any third party.

I declare that my thesis consists of approximately 33,241 words.

Statement of co-authored work

I confirm that Chapter 2 was co-authored with Alkis Georgiadis-Harris and Balázs Szentes and I contributed 33% of this work.

I confirm that Chapter 3 was co-authored with Francesco Caselli and I contributed 50% of this work.

Acknowledgement

I am infinitely indebted to Balázs Szentes, who taught me so much, invariably supported me and provided priceless guidance on uncountably many occasions. I will be forever grateful that our paths crossed. I am very fortunate to have had the chance to collaborate with Francesco Caselli. His wisdom and work ethics has truly served as an inspiration to me. Alkis Georgiadis-Harris has been a wonderful fellow student, co-author and friend. My experience as a graduate student would not have been the same without him. I have greatly benefited from the time, support and precious advice of Daniel Gottlieb and Francesco Nava. I would also like to thank Ricardo Alonso, Andrew Ellis, Gilat Levy, Matt Levy, Ronny Razin and Christopher Sandmann, who always kept their door open to me. Many fellow students deserve to be thanked for their support and friendship, in particular Fabio Bertolotti, Carlo Cabrera, Daniel De Lima, Alex Desbuquois, Gaia Dossi, Maxi Guennewig, Xitong Hui, Evgeniya Kudinova, Peter Lambert, Will Matcham, Clément Minaudier, Veronica Salazar, Bilal Tabti, Di Song Tan and Cecilia Wood. Finally, I am grateful for my friends and family, and of course, Rachel.

Abstract

This thesis contains three essays in the theory of contracts and organisations. The first chapter examines the role of information in shaping the incentives of a decision maker who cares about passing a threshold. We present a model which can be applied to the case of a young employee in an organisation with an 'up-or-out' promotion system. We solve for the optimal design of the informational environment in which such a young employee operates, with the objective of encouraging hard work. The optimal information structure generates outcomes such that the promotion is allocated as if the young employee received full information. However, the young employee does not benefit from the information he receives and remains exactly indifferent between receiving advice or not. The second chapter (co-authored with Alkis Georgiadis-Harris and Balazs Szentes) analyses the sale of a durable good by a seller who cannot make intertemporal commitments to a buyer with private valuation for the good. Motivated by smart contracts used in digital markets, we allow the seller to offer general dynamic contracts. The main result is that the seller's expected payoff is bounded away from the lowest valuation, that is the Coase conjecture fails. The third chapter (co-authored with Francesco Caselli) develops a model of a dynamic economy in which production takes place in worker cooperatives. We formalise an equilibrium concept that applies to such an economy in an overlapping-generation environment. We illustrate its applicability under specific assumptions on preferences and technology. The cooperative economy follows a growth path qualitatively similar to the path followed by a capitalist economy, featuring gradual convergence to a steady state with constant output. However, the cooperative economy features a static inefficiency, in that, for a given aggregate capital stock, firm size is smaller than what a social planner would choose. On the other hand, the cooperative economy cannot be dynamically inefficient, and could accumulate capital at a rate that is higher or lower than the capitalist economy.

Contents

1 Encouraging a Go-Getter **7**

1.1 Introduction 7

1.2 Model 15

1.3 Preliminary Analysis 17

1.4 General Analysis 19

1.5 Optimal Information Structures 23

1.5.1 Convex Objective 23

1.5.2 Concave Objective 25

2 Smart Contracts and the Coase Conjecture **28**

2.1 Introduction 28

2.2 The Model and a Preview of the Results 34

2.3 Main Result 40

2.4 Discussion 51

3 Economic Growth in a Cooperative Economy **58**

3.1 Introduction 58

3.2 Physical Environment 64

3.3 Capitalist Economy 66

3.4 Cooperative Economy 67

3.4.1 Concept of a Cooperative 67

3.4.2 Continuation, Entry and Exit 69

3.4.3 General Equilibrium Concept for Cooperative Economies 69

3.4.4 Operational Equilibrium Concept for Cooperative Economies . . . 72

3.5 An Example with Closed Forms 74

3.5.1 Capitalist Economy 76
3.5.2 Cooperative Economy 76
3.5.3 Comparison 80
3.6 An Example with Numerical Computations 86
3.7 Comparative Statics 89
3.8 Conclusions 91

A Appendix to Chapter 1 **94**
A.1 Introductory Example Derivations 94
A.2 Proofs 95

B Appendix to Chapter 2 **107**
B.1 An Equilibrium when $\mathcal{C} = \mathcal{D}$ 107
B.1.1 Deployed Contracts 107
B.1.2 Auxiliary Games 117
B.1.3 Equilibrium Assessment 119
B.2 Continuous Types 121

C Appendix to Chapter 3 **132**
C.1 Appearances of "worker cooperatives" in digitized books 132
C.2 Proof that (3.4) holds under the Operational Equilibrium Concept 133
C.3 Algorithm to solve for the equilibrium 134
C.4 Derivation of capitalist equilibrium with log utility 134
C.5 Proof that cooperatives are smaller than capitalist firms 135
C.6 Capitalist dynamics with IES = 2 136
C.7 Fixed-cost Normalization 137
C.8 Sensitivity Analysis with τ recalculated 139
C.9 Endogenous Sharing Rule 139

List of Figures

1.1 Action Distribution Comparison 12

1.2 Posterior Belief Comparison 13

2.1 Comparison of the seller's profits when the discount factor varies. 52

3.1 Cooperative labour input as a function of initial capital stock – numerical solution to problem (3.4). 87

3.2 Cooperative capital investment as a function of initial capital stock – numerical solution to problem (3.4). 88

3.3 Steady-state output ratio $\frac{Y^*_{coop}}{Y^*_{cap}}$ and its static $\left(\left(\frac{Z_{coop}}{Z_{cap}}\right)^{\frac{1}{1-\alpha}}\right)$ and dynamic $\left(\left(\frac{s_{coop}}{s_{cap}}\right)^{\frac{\alpha}{1-\alpha}}\right)$ components, as functions of the model's parameters. 90

3.4 Amount of consumption to be given to agents in the cooperative economy to equalize their utility to agents in the capitalist economy, in steady state. 91

C.1 Appearances of "worker cooperatives" in digitized books 132

C.2 Steady-state output ratio $\frac{Y^*_{coop}}{Y^*_{cap}}$ and its static $\left(\left(\frac{Z_{coop}}{Z_{cap}}\right)^{\frac{1}{1-\alpha}}\right)$ and dynamic $\left(\left(\frac{s_{coop}}{s_{cap}}\right)^{\frac{\alpha}{1-\alpha}}\right)$ components, as functions of the model's parameters, with τ chosen to maximize steady state utility for each combination of parameters.139

List of Tables

3.1 Calibrated parameters 84
3.2 Numerical results 86

Chapter 1

Encouraging a Go-Getter

1.1 Introduction

Information disclosure is a crucial way to provide incentives. In accounting, management consulting, law or academia, where up-or-out systems are common, firms adopt strict rules of interim evaluations to encourage ambitious young workers to engage in higher effort. In this paper, we investigate the role of strategic information provision in shaping incentives for such go-getters who wish to be promoted.

We model a go-getter as a decision maker (he) who may receive an exogenously fixed reward if he takes an action above a threshold. Each action has a cost which can be identified with the action itself. The threshold is initially unknown to the decision maker. The information available to the decision maker is controlled by a designer (she), whose utility depends on the decision maker's action, in an increasing manner. The designer can commit to disclose arbitrary information about the true value of the threshold. Upon receiving that information, the decision maker updates his belief about the threshold and chooses an action.

Our main goal is to solve for the designer's optimal information disclosure policy. As is standard, her messages can be assumed to be action recommendations. The essence of the persuasion problem is to optimally attach information content to each recommendation. In particular, each action recommendation must be associated with a posterior belief such that, given this belief, the decision maker finds it optimal to follow the designer's recommendation. Moreover the induced distribution over those posterior beliefs must be consistent with Bayesian updating from the decision maker's prior. That is, the expected posterior belief is the prior.

We show that a solution to the designer's persuasion problem satisfies two general properties: *full-information rewarding* and *no-information indifference*. Let us explain each property. *Full-information rewarding* is defined in relation to the benchmark case in which the decision maker has full information about the realised threshold. We say that the reward is *within reach* if it would be optimal for the decision maker to match the threshold if he knew its true realisation. A disclosure policy is said to satisfy *full-information rewarding* if, with probability one, the threshold is strictly above the designer's recommendation only if it is not within reach. *No-information indifference* is defined in relation to the other extreme benchmark in which the decision maker receives no information beyond his prior belief. A disclosure policy is said to satisfy *no-information indifference* if the decision maker remains ex-ante indifferent between following the designer's recommendation or choosing an action without receiving information. Observe that information cannot hurt the decision maker, since he is free to ignore the designer's recommendation. *No-information indifference* means that he will not benefit either.

The consequences of those two properties for the outcomes induced by the designer's optimal information disclosure policy are the following. First, due to *full-information rewarding*, the decision maker always[1] takes an action (weakly) above the action he would have taken if he had known the true realisation of the threshold. To see this, observe that, with full information, the decision maker matches the threshold if it is within reach and takes the lowest action otherwise. Moreover, due to *no-information indifference*, the decision maker always takes an action above his no-information optimal action. The reason is that his no-information optimal action is always available to take ex-post. Since the decision maker is indifferent ex-ante between taking that action or following the designer's recommendation, the no-information optimal action cannot be strictly suboptimal ex-post. Therefore, if the designer's recommendation was ever below the decision maker's no-information optimal action, she could change her recommendation to be the no-information optimal action without needing to affect the decision maker's posterior beliefs.

Let us now describe the structure of the designer's optimal information disclosure

[1]If a property holds almost surely at the optimum, there exists a payoff equivalent information disclosure policy such that the property holds surely.

policy depending on her preferences. First observe that, if the designer's utility function is linear in the decision maker's action, any information disclosure policy satisfying *full-information rewarding* and *no-information indifference* is optimal. Indeed, *no-information indifference* implies that the decision maker's expected payoff is pinned down to his no-information expected payoff. Furthermore, *full-information rewarding* implies that the decision maker receives the reward with probability equal to the probability that the reward is within reach. Therefore, the expected cost incurred by the decision maker is constant across all information disclosure policies satisfying those two properties. However, if the designer's preferences are non-linear, *full-information rewarding* and *no-information indifference* are not sufficient to characterise optimal policies.

We use the following approach to solve the persuasion problem. At the optimum, each possible realisation of the threshold is assigned a shadow price, which captures how valuable that realisation is for incentive provision. We can then decompose the designer's problem as follows. First, for each action, among all posterior beliefs that would rationalise that action as optimal for the decision maker, the designer chooses a cheapest one, in the sense that it minimises the expected shadow price of the threshold. Intuitively, the decision maker's posterior beliefs place as little probability mass as possible on those realisations of the threshold that are the most valuable for incentive provision. In turn, the prior stock of probability mass on the most valuable realisations remains available to the designer to construct the posterior beliefs associated to alternative recommendations. Second, we use the value to this minimisation problem to define the *implementation cost* of each action. The designer's problem of choosing which actions to induce then reduces to a simple cost-benefit analysis, where the utility benefit of each action is traded-off against its implementation cost. In a final step, the probability allocated to each optimal action is set so that the associated optimal posterior beliefs are consistent with Bayesian updating from the decision maker's prior belief.

We present explicit solutions to the persuasion problem when the designer's objective is either convex or concave in the decision maker's action. In each case, the solution does not depend specifically on the designer's utility function beyond its convexity or concavity. The reason why the solution is the same for any increasing and concave (respectively convex) specification of the designer's utility function is that, among the distributions over the decision maker's actions which can be implemented by an information disclosure policy satisfying *full-information rewarding* and *no-information indifference*, there exists a maximal (respectively minimal) element, in terms of second-order stochastic dominance.

When the designer's utility function is convex, the actions she recommends are distributed between the decision maker's no-information optimal action and the costliest action that the decision maker may be willing to take, with an atom at the top of that interval. The decision maker's posterior beliefs are characterised by binding incentive constraints. In contrast, when the designer's utility function is concave, the lowest action she recommends, denoted a, is strictly above the decision maker's no-information optimal action. The designer's recommendations have full-support between a and the highest action the decision maker may be willing to take, with a unique atom at the bottom of that interval. The decision maker's posterior beliefs are characterised by a gap, located immediately below the designer's recommendation. In other words, when the decision maker is recommended to take an action x, she rules out the possibility that the threshold may be slightly below x.

In each case, a notable feature of the designer's optimal policy is that the decision maker's posterior beliefs keep him confused about which action he should take. Ex-post, there is a range of optimal actions, and in doubt the decision maker goes along with the designer's recommendation, which is the largest action in that range. In a sense, actions that are less favourable to the designer are pooled together with more favourable actions up until the decision maker is just indifferent. At this point, the designer can still recommend the most favourable action in an incentive compatible way.

The results are illustrated by the following example.

Example.— Normalise the value of the potential reward to 1. The decision maker chooses an action $x \geq 0$ at cost x and receives the reward if and only his action is (weakly) above the random threshold y. The designer has an increasing utility function of the decision maker's action. Assume that the decision maker and the designer share the prior belief that y is distributed on $[0, 1]$ with cumulative distribution function: $F(z) = \mathbb{P}(y \leq z) = \sqrt{z}$.

Our results imply that, irrespective of her utility function, the designer's optimal disclosure policy will satisfy *full-information rewarding* and *no-information indifference*.

If the decision maker had full information about the realisation of y, he would optimally choose action $x = y$ and receive the reward. Indeed, on the one hand, any higher action would also guarantee to receive the reward but would be costlier. On the other hand, since y only takes values below 1, the decision maker obtains a positive payoff from choosing $x = y$, while any action below would lead to a negative payoff. So *full-information rewarding* implies that it will be optimal for the designer to recommend actions so that the decision maker receives the reward with probability 1.

Now, consider the case in which the decision maker receives no information about the threshold beyond his prior. In this case, his expected payoff from taking any action $x \in [0, 1]$ writes as the prior probability that the threshold is below x minus the cost of action x, that is $\sqrt{x} - x$. As a result, there is a unique no-information optimal action $x^* = 1/4$, yielding expected payoff $\underline{u} = 1/4$. By *no-information indifference*, the designer's optimal disclosure policy will leave the decision maker with the same expected payoff $\underline{u}$.

Beyond those two properties, the designer's optimal information disclosure policy depends more specifically on her preferences. We present explicit descriptions of solutions when the designer's utility function is either convex or concave in the decision maker's action. A complete description of the solution in each case is in Appendix A.1. In either case, the solution does not depend on the specific functional form of the designer's utility beyond its convexity or concavity.[2]

Figure 1.1 plots the cumulative distribution function of the designer's action recommendations in each case. In the convex case, the distribution has full support between the no-information optimal action $x^* = 1/4$ and 1, with a unique atom at 1. In the concave case, the designer's lowest action recommendation is $a \approx 0.71$.[3] The distribution has an atom at a, followed by a continuous distribution until 1. Both distributions have the same mean. This fact is a consequence of *full-information indifference* and *no-information rewarding*. To see this, observe that the expected action taken by the decision maker writes as the ex-ante probability that he will get the reward minus his expected payoff, that is $1 - 1/4 = 3/4$.

Each action recommendation is associated with a posterior belief about the value

[2] A consequence is that, if the designer's utility is linear, then she is indifferent between the two information disclosure policies presented, and in fact many others.

[3] The exact value of a in this case is:

$$a = \frac{1}{4}\left(\frac{\sqrt{2} + 1/2}{\sqrt{2} - 1/2}\right)^{\sqrt{2}}.$$

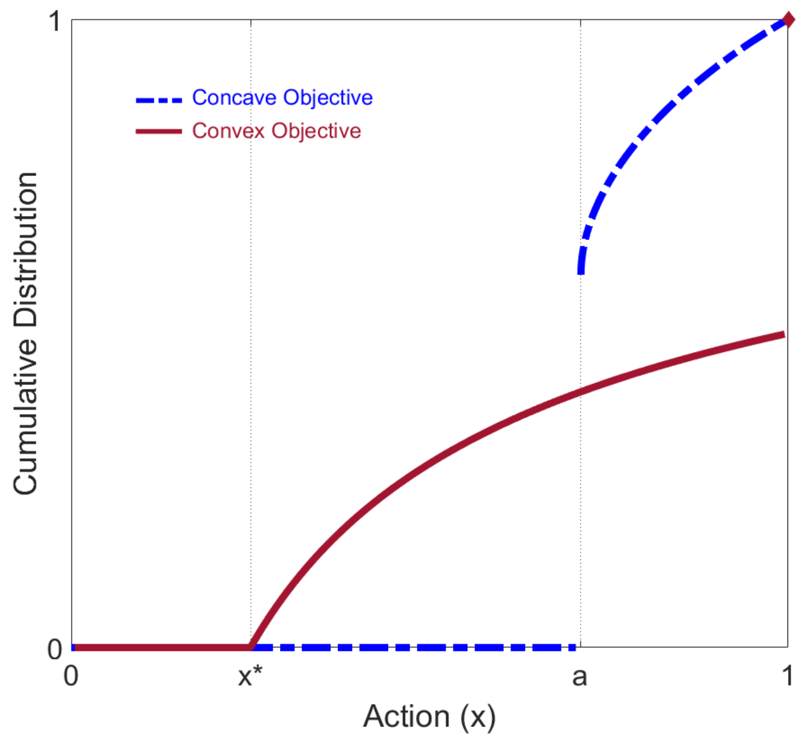

Figure 1.1: Action Distribution Comparison

Note: Cumulative distribution function of the decision maker's action induced by an optimal information structure when the designer's utility is concave (dashed blue) or convex (solid red) in the action, under the assumptions of the example.

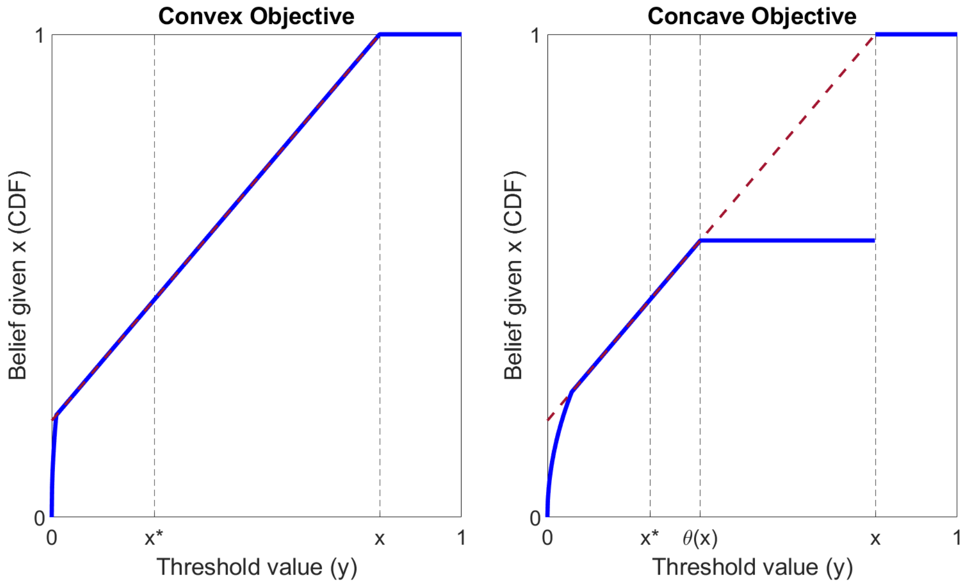

Figure 1.2: Posterior Belief Comparison

Note: Cumulative distribution function representing the decision maker's posterior belief upon receiving recommendation $x = 0.8$ when the designer's utility is convex (left) or concave (right) in the decision maker's action, under the assumptions of the example. In each case, the dashed red curve is the cumulative distribution function of the threshold that would keep the decision maker indifferent between any action in $[0, x]$.

of the threshold. Figure 1.2 represents the decision maker's posterior belief about the distribution of the threshold conditional on receiving recommendation $x = 0.8$, which belongs to the the support of the designer's action recommendations in both cases. A common feature is that the decision maker's posterior belief places probability 1 on the threshold being below the designer's recommendation. This is a consequence of *full-information rewarding*. A second common feature is that both posterior cumulative distribution functions remain weakly below the affine curve $z \mapsto 1 - x + z$. The reason is that the decision maker must find it better to follow the recommendation x, yielding expected payoff $1 - x$, rather than taking any alternative action z, yielding expected payoff $\mathbb{P}\big(y \leq z|x\big) - z$. It follows from *no-information indifference* that the decision maker is in fact indifferent between following the designer's recommendation and deviating to his no-information optimal action x^*, so the two posterior cumulative distribution functions coincide with the affine curve at that point. Actually, when the designer's utility function is convex, the decision maker's posterior belief keeps indifferent between any action in $[x^*, x]$. In contrast, when the designer's utility function is concave, the decision maker's posterior belief completely rules out that the threshold could take values in the interval $\big(\theta(x), x\big) \approx (0.37, 0.8)$, rendering deviations in that interval strictly sub-optimal.

The remaining of this introductory section is devoted to reviewing the related literature. Section 1.2 describes the model. Section 1.3 presents a preliminary analysis of the model, analysing the two benchmarks in which the decision maker either receives complete information about the threshold or no information at all, and showing that a solution satisfies *full-information rewarding*. Section 1.4 presents the general methodology that we use to solve the designer's persuasion problem and establishes *no-information indifference*. Section 1.5 describes, in turn, the solutions to the designer's problem when her utility function is either convex or concave.

Literature Review

We model information transmission following the tradition of information design and Bayesian persuasion as introduced[4] by Kamenica and Gentzkow (2011) and recently surveyed by Kamenica (2019) and Bergemann and Morris (2019). Operational tools to solve the persuasion problem when states of the world live in a continuum remain limited to special cases. Gentzkow and Kamenica (2016), Dworczak and Martini (2019), Arieli, Babichenko, Smorodinsky, and Yamashita (2023) and Kleiner, Moldovanu, and Strack (2021) develop approaches that can be applied when the designer's objective depends only on a single posterior moment of the decision maker's belief. More recently, Kolotilin, Corrao, and Wolitsky (2022) analyse the case in which the set of beliefs that rationalise an action as optimal for the decision maker can be characterised by a first-order condition. Those approaches are not useful in our setting due to the discontinuity in the decision maker's payoff introduced by the threshold.

Due to the aforementioned discontinuity, our environment is more closely related to that of Bergemann, Brooks, and Morris (2015).[5] In their model, properties equivalent to *full-information rewarding* and *no-information indifference* are sufficient for a consumer-surplus maximising market segmentation. As explained previously, those two properties are sufficient for a designer-optimal information disclosure policy when her utility function is linear. A contribution of our analysis is to go beyond the linear case.

[4]Early contributions to the topic include Aumann, Maschler, and Stearns (1995), Ostrovsky and Schwarz (2010) and Rayo and Segal (2010).

[5]It is possible to translate the set up of Bergemann et al. (2015) with our terminology as follows. The decision maker is a seller who chooses a price and receives a reward if and only if the price is below a random threshold — a buyer's willingness-to-pay. The seller incurs no direct cost for setting a price, but may have to reduce the size of the reward (the price itself) in order to increase the probability of getting it. Our analysis can be adapted to their environment to study optimal market segmentations from the perspective of a designer with non-linear preferences over the prices set in different segments. Details are available upon request.

Our main technical tool to establish optimality of a candidate information disclosure policy relies on assigning a shadow price to each possible realisation of the threshold. Duality methods have been used extensively to analyse other persuasion problems. Examples include Kolotilin (2018), Dworczak and Martini (2019), Dizdar and Kováč (2020), Perego, Galperti, and Levkun (2021), Kolotilin et al. (2022) and Dworczak and Kolotilin (2022).

More generally, the present paper is connected to a vast literature on the role of information in incentive provision. Sobel (1993), Lazear (2006), Jehiel (2014) and Ederer, Holden, and Meyer (2018) make the case that full transparency is often sub-optimal in moral hazard problems. Our results indeed imply that the exact realisation of the threshold is revealed to the decision maker with probability zero.

Our analysis complements that of Ely and Szydlowski (2020) who study optimal dynamic information disclosure to an agent with a payoff structure similar to ours. In their dynamic setting, the timing of information disclosure is a critical object of design.[6] In contrast, we focus on situations in which the designer has a one-off opportunity to disclose information and study the optimal content of her messages.

1.2 Model

A decision maker chooses a costly action $x \geq 0$ and receives a fixed reward of value normalised to 1 if his action is above a threshold $y \geq 0$. The decision maker's preferences are represented by the utility function:

$$u(x, y) = \mathbb{1}_{[0,x]}(y) - x,$$

where $\mathbb{1}_{[0,x]}$ is the indicator of the interval $[0, x]$. The threshold is initially unknown to the decision maker. We model y as the realisation of a positive random variable Y, with cumulative distribution function F. Observe that the decision maker would never rationally take an action whose cost is greater than the value of the reward. Therefore, we restrict the decision maker's choice set to the interval $[0, 1]$. The following assumption will be maintained throughout.

[6] See also Smolin (2021) and Liu (2021).

Assumption 1. *The threshold's cumulative distribution function F is strictly increasing and continuous on $[0, 1]$, with $F(0) = 0$ and $F(1) = 1$.*

Assumption 1 rules out gaps and atoms for simplicity. However, the arguments we present could be adapted to take into account those possibilities. Assumption 1 also requires that the highest possible realisation of the threshold coincides with the highest action that the decision maker may rationally be willing to take. Relaxing this assumption would not substantially alter the analysis presented in the main text.

The information available to the decision maker about the threshold is controlled by a designer. The designer's preferences depend only on the agent's action x and are represented by the increasing utility function $v : [0, 1] \to \mathbb{R}$.

In line with standard models of persuasion, we assume that the designer can fully commit at no cost to disclose arbitrary information about the threshold. That is, the designer publicly chooses a policy, which maps realisations $y \in \operatorname{supp} F = [0, 1]$ of the threshold Y to (possibly random) signals. The decision maker observes the policy as well as the signal realisation and chooses an action $x \in [0, 1]$. Both the designer and the decision maker share the prior F, update beliefs according to Bayes's rule and maximise their respective expected utility. An outcome is a joint distribution over the decision maker's actions and the realisations of the threshold induced by this game.

We formulate the designer's problem directly as a maximisation problem over outcomes. Denote $\mathcal{F}$ the set of cumulative distribution functions with support included in $[0, 1]$. The designer chooses an element $\langle (G_x)_x, H\rangle \in \mathcal{F}^{[0,1]} \times \mathcal{F}$, where for each $x \in [0, 1]$, G_x is the cumulative distribution function of the threshold conditional on the decision maker choosing action x; and H is the cumulative distribution function of the decision maker's actions. The decision maker's choices must be consistent with expected utility maximisation. That is, we must impose an incentive compatibility constraint. Note that, when the threshold is distributed according to G_x, the decision maker's expected utility from taking action $\hat{x}$ writes:

$$\int u(\hat{x}, y) dG_x(y) = \int \mathbb{1}_{[0,\hat{x}]}(y) dG_x(y) - \hat{x} = G_x(\hat{x}) - \hat{x}.$$

Therefore, we express the incentive compatibility constraint as:

$$\forall x, \hat{x} \in [0, 1], \quad G_x(\hat{x}) - \hat{x} \leq G_x(x) - x. \tag{IC}$$

In addition, the distribution of the threshold must be consistent with the prior F, leading to the Bayes plausibility constraint:

$$\forall y \in [0,1], \quad \int_0^1 G_x(y) dH(x) = F(y). \tag{BP}$$

To summarise, we analyse the following persuasion problem:

$$\max_{\langle (G_x)_x, H\rangle \in \mathcal{F}^{[0,1]} \times \mathcal{F}} \int_0^1 v(x) dH(x) \quad \text{s.t. (BP) and (IC).} \tag{1.1}$$

1.3 Preliminary Analysis

We discuss the two benchmarks in which the decision maker receives either no information or full information about the threshold.

No-Information Benchmark.— Consider first the case in which the designer provides no information to the decision maker. The decision maker's belief about the threshold remains fixed at the prior. His expected utility from taking action $x \in [0,1]$ writes:

$$F(x) - x.$$

By Assumption 1, $x \mapsto F(x) - x$ possesses a largest maximum $x^* \in [0,1]$. If $x^* = 1$, the designer's problem (1.1) has a trivial solution, since no information disclosure is necessary to induce the largest possible action. For the rest analysis we will maintain:

Assumption 2. $x^* < 1$.

Denote $\underline{u} = F(x^*) - x^*$. Note that, by Assumption 1, $\underline{u} \geq F(0) - 0 = 0$. Furthermore, information cannot hurt, that is $\underline{u}$ provides a lower bound on the decision maker's expected payoff from any choice of the designer.[7] Proposition 1 below will establish that, in fact, the decision maker does not benefit either from the designer's optimal choice. Let us define this property formally:

Definition 1 (No-Information Indifference)**.** *An outcome $\langle (G_x)_x, H\rangle$ satisfies no-information*

[7] To see this, observe that the decision maker's expected payoff writes: $\int_0^1 [G_x(x) - x] dH(x)$. Due to the incentive compatibility constraint: $\int_0^1 [G_x(x) - x] dH(x) \geq \int_0^1 [G_x(x^*) - x^*] dH(x)$. Due to the Bayes plausibility constraint, $\int_0^1 [G_x(x^*) - x^*] dH(x) = F(x^*) - x^* = \underline{u}$.

indifference if:

$$\int_0^1 [G_x(x) - x]dH(x) = \underline{u}.$$

Full-Information Benchmark.— Suppose now that the designer reveals perfectly the value y of the threshold to the decision maker. If $y \leq 1$, it is optimal for the decision maker to match exactly that value and choose action $x = y$. If instead, $y > 1$, the value of the reward is not worth the required cost and it is uniquely optimal for the decision maker to choose $x = 0$. In particular, with full information, the decision maker achieves the reward whenever it is within reach.

Definition 2 (Full-Information Rewarding)**.** *An outcome $\langle (G_x)_x, H\rangle$ satisfies full-information rewarding if:*

$$\forall x \in [0,1], \quad G_x(x) = G_x(1).$$

That is, an outcome satisfies full-information rewarding, if conditional on the decision maker choosing any action x, either the threshold is not within reach ($y > 1$) or the decision maker receives the reward. Moreover, under Assumption 1, the threshold is within reach with probability 1. Therefore, the decision maker is guaranteed to receive the reward if the designer's chosen outcome satisfies full-information rewarding.

Lemma 1 below implies that full-information rewarding must be a property of a solution to the designer's problem (1.1).[8] Before stating the result, let us introduce a piece of notation. For $x \in [0,1]$, denote $\mathcal{G}_x = \{G \in \mathcal{F} : G(x) = 1\}$. In addition, denote $\mathcal{G} = \bigtimes_{x\in[0,1]} \mathcal{G}_x$. Finally, let $\mathcal{O}$ the set of outcomes $\langle (G_x)_x, H\rangle \in \mathcal{G} \times \mathcal{F}$ satisfying (BP) and (IC). That is, $\mathcal{O}$ is the set of outcomes that are feasible for problem (1.1) and satisfy full-information rewarding.

Lemma 1. *For any outcome $\langle (G_x)_x, H\rangle$ feasible for problem (1.1), there exists and outcome $\langle (\hat{G}_x)_x, \hat{H}\rangle \in \mathcal{O}$ such that:*

$$\int_0^1 v(x)d\hat{H}(x) \geq \int_0^1 v(x)dH(x).$$

The proof is presented in Appendix A.2. Its logic is relatively straightforward. For any action taken by the decision maker, if the threshold is within reach but strictly above

[8] Due to indifferences over zero-probability events, there are also solutions to the designer's problem which do not satisfy full-information rewarding.

the decision maker's action, the designer could send a further message to the decision maker warning him about the true value of the threshold. Conditional on not receiving the warning message, the decision maker can only infer that the threshold is below the action he was planning to take, which makes him all the more confident in his choice. Instead, if he receives the warning message, the decision maker knows the true value of the threshold and optimally adjusts his action upwards, to the benefit of the designer.

In view of Lemma 1, we restrict the designer's choice set to $\mathcal{O}$. That is, we will consider the problem:

$$\max_{\langle (G_x)_x, H\rangle \in \mathcal{O}} \quad \int_0^1 v(x) dH(x). \tag{1.2}$$

1.4 General Analysis

This section presents three results. First, Theorem 1 provides sufficient conditions for the optimality of a candidate outcome. This will be our main tool to solve the designer's problem (1.2) depending on her preferences in the following section. Second, Proposition 1 establishes that the sufficient conditions of Theorem 1 not only imply optimality but also no-information indifference. As a consequence, the solutions presented in the following section will satisfy no-information indifference. Finally, Lemma 2 shows that outcomes satisfying no-information indifference need only be described above the no-information optimal action x^*, which allows to slightly simplify the analysis in the following section.

Theorem 1. *Let $\langle (G_x), H\rangle \in \mathcal{O}$. Suppose there exists a lower-semicontinuous function $\lambda : [0,1] \to \mathbb{R}$, such that:*

(i) For all $x \in [0,1]$ and $y \in [0,x]$, if $y \in \operatorname{supp} G_x$, then $\forall z \in [y,x]$, $\lambda(y) \leq \lambda(z)$.

(ii) For all $x \in [0,1]$ and $y \in [0,x]$, if $\forall z \in (y,x]$ $\lambda(y) < \lambda(z)$, then $G_x(y) - y = 1 - x$.

(iii) $\operatorname{supp} H \subseteq \arg\max_x \left\{ v(x) - (1-x) \min_{0 \leq z \leq x} \lambda(z) - \int_0^x \left[\min_{y \leq z \leq x} \lambda(z) \right] dy \right\}$.

Then $\langle (G_x), H\rangle$ is a solution to the designer's problem.

Let us explain the statement of the theorem. We consider a feasible outcome $\langle (G_x), H\rangle$ for the designer. For each $y \in [0,1]$, $\lambda(y)$ acts as a shadow price associated to the realisation y of the threshold. Condition (i) says that, for each x, the posterior distribution G_x places mass only at realisations y of the threshold that are the cheapest among all possible realisations in $[y, x]$. Furthermore, condition (ii) implies that, if the realisation y is strictly cheaper than any alternative in $(y, x]$, then the posterior distribution G_x places as much as possible below y, that is until the decision maker is just indifferent between taking action x and deviating to action y. Together, conditions (i) and (ii) are equivalent[9] to:

$$\forall x \in [0,1], \quad G_x \in \arg\min_{G \in \mathcal{G}_x} \int_0^x \lambda(y) dG(y) \qquad \text{subject to } \forall y \in [0,x], \quad G(y) - y \leq 1 - x. \tag{1.3}$$

That is, the posterior distribution of the threshold given action x is chosen to minimise the expected shadow price of the threshold. Moreover, the value of the minimisation problem in equation (1.3) defines an *implementation cost* $\Lambda(x)$ for each action $x \in [0,1]$. As we show in Appendix A.2, it follows from conditions (i) and (ii) that:

$$\Lambda(x) = \int_0^x \lambda(y) dG_x(y) = (1-x) \min_{0 \leq z \leq x} \lambda(z) + \int_0^x \big[\min_{y \leq z \leq x} \lambda(z)\big] dy. \tag{1.4}$$

Therefore, condition (iii) has a straightforward interpretation. It specifies that the support of the unconditional distribution over actions H should be found within the set of maximisers of $x \mapsto v(x) - \Lambda(x)$. That is, the designer implements actions optimally trading off their utility benefit to their implementation cost. Next, we present the proof of the theorem.

Proof. Suppose that $\langle (G_x), H \rangle$ and λ satisfy the assumptions of the theorem. In this proof, we take as given that equations (1.3) and (1.4) above are satisfied. Those are proven in Appendix A.2. We consider an arbitrary outcome $\langle (\tilde{G}_x), \tilde{H} \rangle \in \mathcal{O}$ and show that the designer must prefer $\langle (G_x), H \rangle$ over $\langle (\tilde{G}_x), \tilde{H} \rangle$. The designer's expected utility from $\langle (\tilde{G}_x), \tilde{H} \rangle$ writes:

$$\int_0^1 v(x) d\tilde{H}(x) = \int_0^1 [v(x) - \Lambda(x)] d\tilde{H}(x) + \int_0^1 \Lambda(x) d\tilde{H}(x). \tag{1.5}$$

Since $\Lambda(x)$ is the value of the minimisation problem in equation (1.3), it must be the case that:

$$\Lambda(x) \leq \int_0^x \lambda(y) d\tilde{G}_x(y).$$

[9] For a formal proof of this equivalence, see claim 1 in Appendix A.2.

Furthermore, since (BP) must be satisfied by all outcomes:

$$\int_0^1 \int_0^x \lambda(y) d\tilde{G}_x(y) d\tilde{H}(x) = \int_0^1 \lambda(y) dF(y) = \int_0^1 \int_0^x \lambda(y) dG_x(y) dH(x) = \int_0^1 \Lambda(x) dH(x),$$

where the last equality follows from equation (1.4). As a result, the second term in the designer's expected utility in equation (1.5) is bounded above:

$$\int_0^1 \Lambda(x) d\tilde{H}(x) \leq \int_0^1 \Lambda(x) dH(x).$$

Moreover, condition (iii) implies that the first term is also bounded above:

$$\int_0^1 [v(x) - \Lambda(x)] d\tilde{H}(x) \leq \int_0^1 [v(x) - \Lambda(x)] dH(x).$$

Summing the two inequalities, we conclude:

$$\int_0^1 v(x) d\tilde{H}(x) \leq \int_0^1 v(x) dH(x).$$

□

Conditions (i), (ii) and (iii) guarantee optimality of the outcome $\langle (G_x)_x, H \rangle$. Our next results shows that those conditions also guarantee no-information indifference.

Proposition 1. *Under the hypotheses of Theorem 1, $\langle (G_x)_x, H \rangle$ satisfies no-information indifference.*

The logic of the proof below is the following. We consider the cheapest realisation y^* of the threshold. The corresponding action must have the lowest implementation cost among all actions. The reason is that the posterior distribution placing probability 1 at y^* rationalises action y^* as optimal for the decision maker. Therefore, the designer would never find it optimal to implement actions below y^* — those would yield lower utility at a higher implementation cost. As a result, any action in the support of H is above y^*. Condition (ii) then implies that all posterior distributions G_x for $x \in \operatorname{supp} H$ make the decision maker exactly indifferent between taking action x and action y^*. Thus, from the ex-ante perspective, the decision maker must be indifferent between receiving the designer's information or simply taking action y^* without further information. Since information cannot hurt, it must be that y^* is a no-information optimal action.

Proof. Since it is lower-semicontinuous, λ has a minimum. Denote y^* the largest value at which λ attains its minimum. For all $y \leq y^*$, we have:

$$\min_{y \leq z \leq y^*} \lambda(z) = \lambda(y^*).$$

It follows that:

$$\Lambda(y^*) = (1 - y^*)\lambda(y^*) + \int_0^{y^*} \lambda(y^*)dy = \lambda(y^*).$$

Furthermore, for any $x \in [0, 1]$, we have:

$$\Lambda(x) = (1 - x) \min_{0 \leq z \leq x} \lambda(z) + \int_0^x \big[\min_{y \leq z \leq x} \lambda(z)\big] dy \geq \lambda(y^*) = \Lambda(y^*).$$

We use this fact to establish that any action in $\operatorname{supp} H$ must be weakly above y^*. Indeed, by condition (iii), for any $x \in \operatorname{supp} H$, it must be that:

$$v(x) - \Lambda(x) \geq v(y^*) - \Lambda(y^*),$$

then it follows that:

$$v(x) \geq v(y^*) + [\Lambda(x) - \Lambda(y^*)] \geq v(y^*).$$

Since v is increasing, we have indeed $x \geq y^*$.

As a consequence, for $x \in \operatorname{supp} H$ and by definition of y^*, it must be the case that $\forall z \in (y^*, x]$, $\lambda(y^*) < \lambda(z)$, implying, by condition (ii) that:

$$G_x(y^*) - y^* = 1 - x.$$

We use this fact to evaluate the decision maker's expected payoff:

$$\int_0^1 (1 - x)dH(x) = \int_0^1 [G_x(y^*) - y^*]dH(x) = F(y^*) - y^*.$$

Since $\int_0^1 (1 - x)dH(x) \geq \underline{u} = \max_x \big\{F(x) - x\big\} \geq F(y^*) - y^* = \int_0^1 (1 - x)dH(x)$, all the inequalities are equalities and we conclude that $\langle (G_x)_x, H\rangle$ satisfies no-information indifference. ☐

In light of Proposition 1, we will restrict the search of an optimal outcome to those satisfying no-information indifference. A consequence is that it will be sufficient to describe outcomes by their behaviour on $[x^*, 1]$. Let us first introduce a piece of notation and then state this result precisely.

Denote $\overline{\mathcal{O}}^{NII}$ the set of elements of the form $\langle(\Gamma_x)_{x\in[x^*,1]}, H\rangle$ where H is a cumulative distribution function with support included in $[x^*, 1]$ and, for each $x \in [x^*, 1]$, Γ_x is a non-decreasing and right-continuous function on $[x^*, x]$ with $\Gamma_x(x^*) = 1 - x + x^*$, $\Gamma_x(x) = 1$, for all $\hat{x} \in [x^*, x]$, $\Gamma_x(\hat{x}) \leq 1 - x + \hat{x}$, and:

$$\forall y \in [x^*, 1], \quad \int_{x^*}^{1} \Gamma_x(y) dH(x) = F(y).$$

We have the following result.

Lemma 2. *(A) If $\langle(G_x)_x, H\rangle \in \mathcal{O}$ satisfies no-information indifference, then (i) $\operatorname{supp} H \subseteq [x^*, 1]$, and (ii) there exists $(\tilde{G}_x)_x \in \mathcal{G}$ such that $\forall x \in [x^*, 1]$, $\tilde{G}_x(x^*) = 1 - x + x^*$, and $\langle(\tilde{G}_x)_x, H\rangle \in \mathcal{O}$.*
(B) If $\langle(\Gamma_x)_{x\in[x^*,1]}, H\rangle \in \overline{\mathcal{O}}^{NII}$, there exists $(G_x)_x \in \mathcal{G}$ such that $\langle(G_x)_x, H\rangle \in \mathcal{O}$ and:

$$\forall x \in [x^*, 1] \forall \hat{x} \in [x^*, x], \quad G_x(\hat{x}) = \Gamma_x(\hat{x}).$$

The proof is in Appendix A.2. In the following sections, we will describe solutions on $[x^*, 1]$. Then we can use the result of this lemma to construct complete solutions to the designer's problem. One remark is that, in part (B), the construction of (G_x) given $\langle(\Gamma_x)_{x\in[x^*,1]}, H\rangle$ is explicit.

1.5 Optimal Information Structures

The purpose of this section is to present explicit solutions to the designer's problem depending on her preferences. We start with the case in which the designer's utility function is convex. Then, we describe the case in which the designer's utility function is concave.

1.5.1 Convex Objective

Assume that the designer's utility function v is increasing and convex. We will construct a solution to her problem. Then, we prove its optimality by constructing shadow prices and using Theorem 1.

Intuitively, because her utility function is increasing, the designer wishes to induce the decision maker to take actions that are as high as possible. In order to choose a high action, the decision maker needs to be convinced that the threshold is likely to be high. In other words, the incentive compatibility constraint (IC) associated to the highest actions restricts the probability that the decision maker's posterior belief can place on lower realisations of the threshold. Therefore, due to the Bayes plausibility constraint, if the designer induces the highest actions with large probability, she may have to include lower actions in the support of her chosen outcome, in order to be able to include lower realisations of the threshold in the associated posterior distribution. It follows that the unconditional distribution over the decision maker's actions may place large probability on the highest actions only if the distribution is sufficiently spread. When the designer's utility is convex, she in fact benefits from this spread.

As a consequence, the construction of the candidate solution will focus on trying to induce the highest actions with as much probability as possible. The main restriction that the designer faces is the incentive compatibility constraint (IC). A first step in the construction may then be to assume that this constraint always holds with equality, that is to set: $G_x(y) = \min\left\{1, 1 - x + y\right\}$. This will allow to build a valid outcome only when the prior distribution of the threshold has a concave cumulative distribution function F. The reason is that the constraint (BP) will imply that F writes as a sum of concave functions. If F is not concave, we will have to adapt this argument by first concavifying F.

Let us now explain the construction formally. Denote $\overline{F} : [0, 1] \to [0, 1]$ the pointwise-smallest concave function that remains everywhere weakly above F. Note that the properties of F imposed by Assumption 1 are preserved by $\overline{F}$. By concavity, $\overline{F}$ has a well-defined right-derivative $\overline{F}'_+$ on $[0, 1)$. Furthermore, $\overline{F}'_+$ is non-increasing, positive and right-continuous on $[0, 1)$. We extend its definition by setting $\overline{F}'_+(1) = 0$. We define the unconditional cumulative distribution function over the decision maker's actions by:

$$H^{vex}(x) = \begin{cases} 0 & \text{if } x < x^*, \\ 1 - \overline{F}'_+(x) & \text{if } x \geq x^*. \end{cases} \tag{1.6}$$

Observe that equation (1.6) defines a valid cumulative distribution function with support in $[x^*, 1]$. To see this, in addition to the properties of $\overline{F}'_+$ described previously, note that F and $\overline{F}$ must coincide at x^* and that $\overline{F}'_+(x^*) \leq 1$.[10] Before describing the associated posterior distributions $(G_x)_x$, an important remark is that:[11]

$$\operatorname{supp} H^{vex} \subseteq \big\{x \in [x^*, 1] : F(x) = \overline{F}(x)\big\}.$$

With this remark in mind, we can define:

$$\forall x \in \operatorname{supp} H^{vex} \forall y \in [x^*, 1], \quad G_x^{vex}(y) = \min\left\{1, 1 - x + y - \frac{\overline{F}(y) - F(y)}{\overline{F}'_+(y)}\right\}. \tag{1.7}$$

Note that $\overline{F}'_+(y) = 0$ only if $y = 1$, at which point we must have $G_x^{vex}(y) = 1$, so the definition above is valid up to a slight abuse of notation. Recall also that, by Lemma 2, it is sufficient to describe G_x on $[x^*, 1]$. Since F and $\overline{F}$ coincide at x^* and at any $x \in \operatorname{supp} H^{vex}$, we have indeed $G_x^{vex}(x^*) = 1 - x + x^*$ and $G_x^{vex}(x) = 1$. In addition, the expression $y \mapsto y - \frac{\overline{F}(y) - F(y)}{\overline{F}'_+(y)}$ is increasing on $[x^*, 1]$.[12] For completeness, we should also describe G_x for $x \notin \operatorname{supp} H^{vex}$. In this case, set $G_x^{vex}(y) = \min\{1, 1 - x + y\}$.

We are ready to state the main result of this section. Irrespective of the specific utility function of the designer, as long as it is increasing and convex, the construction above provides a solution to her persuasion problem.

Proposition 2. *If v is convex, $\langle (G_x^{vex})_x, H^{vex}\rangle$ describe an optimal outcome for the designer.*

The proof is presented in Appendix A.2.

1.5.2 Concave Objective

Now assume that the designer's utility function v is increasing and concave. Again, we construct a solution to her problem and use Theorem 1 to establish its optimality.

In this case, the designer dislikes spread in the distribution of the decision maker's actions. Therefore, the designer will find it optimal to relax the incentive compatibility

[10] Indeed, $x \mapsto F(x^*) + x - x^*$ is a concave function remaining everywhere weakly above F, by definition of x^*. Therefore, for all x, $F(x) \leq \overline{F}(x) \leq F(x^*) + x - x^*$. Evaluating this chain of inequalities at x^* yields $F(x^*) = \overline{F}(x^*)$. Next, taking the limit as $x \to x^*$, $x > x^*$, yields $\overline{F}'_+(x^*) \leq 1$.

[11] If $F(x) < \overline{F}(x)$, define $a = \sup\{z \leq x : F(z) = \overline{F}(z)$ and $b = \inf\{z \geq x : F(z) = \overline{F}(z)$. Since F and $\overline{F}$ are both continuous, $a < x < b$. Furthermore, it is easy to see that $\overline{F}$ must be affine on $[a, b]$, so H^{vex} is constant on $[a, b)$.

[12] For y such that $F(y) \neq \overline{F}(y)$, define a and b as in the previous footnote. Then $\overline{F}$ is affine on $[a, b]$, that is of the form: $\overline{F}(y) = \alpha y + \beta$. Then:

$$y - \frac{\overline{F}(y) - F(y)}{\overline{F}'_+(y)} = -\frac{\beta}{\alpha} + \frac{F(y)}{\alpha}.$$

constraints locally. More precisely, the solution has the feature that, when the decision maker's action is x, the posterior distribution of the threshold places no mass on an interval $(\theta(x), x)$. As a result, realisations of the threshold in $(\theta(x), x)$ remain available to associate to posterior distributions conditional on alternative actions. This feature will allow the designer to induce the decision maker to take actions that are all relatively high, even though the highest among those will arise with small probability.

The main technical challenge of this section is to define $\theta(x)$. For clarity of the exposition, we make a further assumption on the prior. In this case, the construction will be relatively transparent.

Assumption 3. *The prior cumulative distribution function of the threshold F is twice continuously differentiable and strictly concave on* $[0, 1]$.

Under Assumption 3, we can define a function θ as a solution to the differential equation:

$$\theta'(x)F''\big(\theta(x)\big)\big(x - \theta(x)\big) = F'(x),$$

with initial condition:

$$\theta(1) = x^*.$$

We show in Appendix A.2 that there exists a constant $a \in (x^*, 1)$, such that a unique solution θ can be defined on $[a, 1]$, is decreasing on $[a, 1]$ and satisfies $\theta(a) = a$.

We can now use such a function θ to define a candidate solution $\langle (G_x^{cave})_x, H^{cave} \rangle$ to the designer's problem. Define:

$$H^{cave}(x) = \begin{cases} 0 & \text{if } x < a, \\ F'\big(\theta(x)\big) & \text{if } x \geq a. \end{cases} \tag{1.8}$$

H^{cave} has support $[a, 1]$, with an atom at a. Since both θ and F' are continuously decreasing, H^{cave} is continuously increasing on $[a, 1]$. Furthermore, $H^{cave}(1) = F'(x^*) = 1$, where the second equality is a consequence of the first-order condition of decision maker's no-information benchmark problem.[13] Now, for $x \geq a$ and $y \in [x^*, 1]$, define:

$$G_x^{cave}(y) = \begin{cases} 1 - x + y & \text{if } y \leq \theta(x), \\ 1 - x + \theta(x) & \text{if } \theta(x) \leq y < x, \\ 1 & \text{if } y \geq x. \end{cases} \tag{1.9}$$

Recall that by Lemma 2, it is sufficient to describe those posterior distributions above x^*. For all $x \geq a$, $\theta(x) \geq x^*$, so indeed $G_x^{cave}(x^*) = 1 - x + x^*$. The mapping θ assigns, to each action x above a, a "last tempting deviation" $\theta(x)$ below a. After $\theta(x)$, G_x^{cave} is flat, until reaching x. For completeness, we also define G_x^{cave} for $x < a$, in which case we set $G_x^{cave}(y) = \min\{1, 1 - x + y\}$.

Proposition 3. *If v is concave and F satisfies Assumption 3, then $\langle (G_x^{cave})_x, H^{cave} \rangle$ describes an optimal outcome for the designer.*

The proof is presented in Appendix A.2.

[13]Note that the first-order approach applies under Assumption 3.

Chapter 2

Smart Contracts and the Coase Conjecture

2.1 Introduction

The Coase Conjecture is a manifestation of the striking consequences of the lack of intertemporal commitment power: if the monopolist can post prices frequently, she clears the market quickly, at prices close to the lowest possible willingness-to-pay even when most consumers have high valuation for the good.[1] The goal of this paper is to examine the extent to which this conclusion is robust to considering more complex selling mechanisms than just price-posting. Motivated by *smart contracts* used in digital markets, we allow the seller to offer general dynamic contracts. Our main result is that if the monopolist has access to such contracts, the Coase Conjecture no longer holds.

In our model, there is a seller of a single good and a buyer. The buyer's valuation for the good is binary, high or low, and it is his private information.[2] We consider the case where the probability of high valuation is large enough for the static monopoly price to be the high valuation. Time is discrete and both parties discount the future at the same rate. In the initial period, the seller offers a contract from a space described below. If the buyer accepts the contract, it determines the probabilities of trade and the transfers

[1] This phenomenon was first described by Coase (1972), and later formalized by Stokey (1981), Fudenberg, Levine and Tirole (1985), and Gul, Sonnenschein and Wilson (1986).

[2] We focus on the "gap case" and assume that the seller's production cost, normalized to be zero, is smaller than the low valuation. In the "no-gap case", Ausubel and Deneckere (1989) show that the Coase Conjecture fails even with posted prices.

in subsequent periods until it is replaced. At the beginning of each period, the seller decides whether to proceed with the current contract or to void it and offer a new one.

We use the analogy of a mediator to describe a typical contract from our contract space. In each period, both the seller and the buyer may send messages to the mediator. In turn, the mediator sends private (and possibly public) signals to the contracting parties and implements an allocation. Perhaps the most notable feature of such a contract is that the mediator can possess information which the seller does not. When the seller abandons a contract, she loses that information. This feature is shown to be the driving force of our main result.

Some aspects of our contract space are reminiscent of the technologies developed in relation to the aforementioned smart contracts. First, smart contracts are automated in the sense that they execute trades without further consent from the contracting parties. Similarly, in our model, the allocation proposed by the mediator is implemented and cannot be renegotiated. Second, smart contracts in practice can be, and often are switched off just like the seller can abandon her current contract in our model. One of the reasons that contracts in digital markets are designed so that they can be switched off is to avoid the execution of unlawful transactions, for example, due to bankruptcy procedure against a contracting party.[3] Finally, we note that cryptographic encoding of a party's input can prevent the other contracting party to recover that input even if she has access to the contract's code. Such encoding also plays an important role in digital markets: smart contracts deployed on blockchain networks use cryptographically signed transactions.[4]

Since the seller's commitment power is limited, she may benefit from a small contract space. The reason is that removing contracts from the seller's action space makes the set of possible deviations shrink which, in turn, may enable the seller to stick with contracts which are advantageous from the ex ante perspective. This can be seen most vividly by considering the scenario when each contract available to the seller specifies trading at the high valuation. In this case, the seller could achieve the full-commitment profit because, even though she maybe tempted to lower the price if there is no trade, she cannot do so.

[3]Another context in which a contracting party retains her right to void smart contracts is where the issuer deploys these contracts in her private blockchains. Examples for such issuers include Walmart, Comcast, Spotify, DHL, JPMorgan and MetLife.

[4]While communications in many digital trading platforms are public, there are examples for protocols which allow for private communications. We discuss these examples and the implementation of such private communications in the concluding section.

So, in order to model the consequences of limited commitment in a meaningful way, the contract space should be rich enough. To this end, we assume that the seller has access to all *simple and direct* contracts defined as follows. A contract is called simple and direct if the contract elicits the buyer's valuation in the initial period of its deployment and does not communicate with the contracting parties ever after. Our main result holds as long as the seller's contract space is rich enough to include all such contracts.

We note that our model assumes a certain amount of commitment power of the monopolist. Namely, if the contract in place determines an allocation, that allocation will be implemented and the seller cannot take further actions. The same assumption is maintained in the standard Coasian model: if the buyer is willing to buy at the posted price, trade will take place and the seller cannot reneg on the price.[5]

Our main result is that the monopolist's payoff is bounded away from the low valuation irrespective of the discount factor. We prove this result by constructing a simple and direct but suboptimal mechanism which never reveals any information to the seller. In the initial period, the buyer reports his valuation and the high type trades at a price less than his valuation with probability less than one. The low-type buyer does not trade in the first period. Even though the seller receives no signal from the mechanism, she updates her prior about the buyer's type whenever trade does not occur. In every subsequent period, the probability of trade is constant and does not depend on the buyer's type, so the seller's posterior remains the same unless there is sale. Furthermore, the price is the buyer's reported valuation. This means that the low-value buyer's payoff is zero and the high-value buyer earns rent only in the initial period. Finally, from the second period onwards, the discounted present value of the seller's payoff is larger than the low valuation.

If the seller abandons this contract, she loses its information content. The trading probabilities in this mechanism are specified so that the optimal full-commitment mechanism in all but the initial period is clearing the market at the low valuation. Since the seller's expected payoff is larger than the low valuation, the constraint guaranteeing that she does not abandon the mechanism is satisfied in each future period. The mechanism we construct may not be optimal: In the initial period, the seller might prefer to choose a different mechanism. But that would only imply that her equilibrium payoff is even

[5]McAdams and Schwarz (2007) and Akbarpour and Li (2020) consider static mechanism design problems where the principal has even less committment power and she cannot credibly promise to follow the rules of her own mechanism.

larger than the one generated by the mechanism described above. Therefore, since the seller's expected profit is bounded away from the low valuation in our mechanism, her equilibrium payoff is also larger than the low valuation. That is, the Coase Conjecture fails.

From the methodological viewpoint, a contribution of our paper is the introduction of the aforementioned contracts into a dynamic principal-agent problem without intertemporal commitment. Following the tradition of Mechanism Design, we do not impose restrictions on the contract space and consider general dynamic contracts similar to those in the full-commitment benchmark. Of course, when the principal can commit to long-term contracts, the information revealed to her by the mechanism about the agent's prior communications is irrelevant. In contrast, when the principal lacks commitment power and re-optimizes in each period, such information is crucial in shaping the future relationship with the agent. Consequently, the information revealed by the contract should be part of the optimal contract design problem and hence, we allow for contracts that store more information than the principal has access to. We speculate that this approach may turn out to be useful in analyzing dynamic mechanism design problems with limited commitment in various environments. The application of this idea to the problem of a durable-good monopolist merely clarifies that the Coase Conjecture is not only due to the seller's lack of commitment power but also to her restricted contract space.

Literature Review

The literature on dynamic contracting in the absence of commitment probably started with the papers by Laffont and Tirole (1988 and 1990). The authors offer two related yet distinct approaches to model such environments. The first one is to consider one-period contracts. In each period, the principal offers a contract which, if accepted by the agent, determines the allocation in that period as a function of contractible variables.[6] The second approach is to allow dynamic contracts which can be voided and replaced if both parties wish to do so. That is, equilibrium contracts must be *renegotiation-proof*.[7]

[6] Examples for recent papers analysing dynamic screening problems with short-term contracts include Gerardi and Maestri (2020), Beccuti and Möller (2018), Acharya and Ortner (2017) and Tirole (2016).

[7] Among others, Battaglini (2007) and Maestri (2017) generalize the results of Laffont and Tirole (1990) in various ways. Strulovici (2017) provides a foundation for renegotiation-proof contracts in a bargaining environment. Hart and Tirole (1988) and Breig (2019) compare the two modelling approaches in a dynamic buyer-seller relationship.

The methodological contribution of our paper is to put forward another approach of modeling limited commitment which appears to be new. In order to explore the consequences of the absence of commitment in the context of a principal-agent relationship, it is desirable to consider a setting which differs from the full-commitment benchmark only in the assumption regarding the principal's commitment power. In the full-commitment benchmark, the principal has access to dynamic contracts and has full bargaining power. Therefore, our model combines the two approaches of Laffont and Tirole (1988 and 1990) in the following way. On the one hand, the set of mechanisms is not restricted to be one-period ones and the principal has access to infinite-horizon dynamic contracts. On the other hand, the principal can offer new contracts in each period and the agent's consent is not required to abandon the previous contract.

Doval and Skreta (2022) also consider mechanism design problems with limited commitment. They generalize the approach in Laffont and Tirole (1988) and consider one-period contracts. Their mechanisms do not only determine allocations but can also reveal public information. The authors develop a Revelation Principle and show that the information revealed by a mechanism can be assumed to be the principal's posterior about the agent's type.[8] In their companion paper, Doval and Skreta (2020) show that the Coase Conjecture still holds with such a contract space. Indeed, the authors demonstrate that in a Coasian environment, the seller optimally posts prices in each period.[9]

Our paper also contributes to the literature documenting failures of the Coase Conjecture in the 'gap case'. With multiple atomic buyers, Bagnoli, Salant and Swierzbinski (1989), von der Fehr and Kuhn (1995) and Montez (2013) show that the seller can maintain high posted prices until a trade occurs. Feinberg and Skrzypacz (2005) show that higher-order uncertainty can generate delay. Other papers demonstrate that the Coase Conjecture is not robust to the assumption that the seller's marginal cost of production is constant, see, for example, Kahn (1988), McAfee and Wiseman (2008), Karp (1993), and Ortner (2017).[10] Bulow (1982) argues that the monopolist benefits from renting the good

[8]Bester and Strausz (2001) also develop a Revelation Principle in finite-horizon environments and finite type spaces.

[9]Lomys and Yamashita (2022) introduce a mediator into the model of Doval and Skreta (2022) who controls the communication between the contracting parties. The mediator cannot be replaced by the principal and can possess information which the principal does not have. The authors demonstrate that such a mediator expands the set of implementable allocations of Doval and Skreta (2022). See also Fanning (2021a, 2021b) who explores how a mediator who can withhold information can improve equilibrium outcomes in a reputational bargaining problem.

[10]Also related is the literature on obsolescence or imperfect durability of the good, see Bulow (1986), Waldman (1993), and Fudenberg and Tirole (1998).

rather than selling it.[11]

Another approach to break the Coase Conjecture is to allow the seller to intratemporally screen, e.g. by producing a variety, see Wang (1998), Takeyama (2002), Hahn (2006), Inderst (2008), or Board and Pycia (2014). A notable contribution by Nava and Schiraldi (2019) demonstrates that all these results are consistent with the Coasian logic in the following sense. The seller's limit payoff is the maximal static monopoly profit subject to the market-clearing condition.

The Coase conjecture has been also proved to fail when market deterioration is prevented by the arrival of new buyers or stochastically changing values. Important contributions include Sobel (1991), Biehl (2001), and Fuchs and Skrzypacz (2010).

Our work is also related to the literature on smart contracts. The term 'smart contract' was first coined by Nick Szabo in the mid-90's, whose prototypical example of a vending machine highlights the ideas of automatic execution and immutability. Since then, with the advent of bitcoin and the popularization of blockchain technologies such as Ethereum, interest in smart contracts has heightened. For some recent papers on the blockchain, see Huberman, Leshno and Moallemi (2021) who provide an insightful analysis of the Bitcoin Payment System and Abadi and Brunnermeier (2018) who study the impossibility of any distributed ledger to satisfy certain desiderata.

Recent research on smart contracts has explored how these contracts can enlarge the space of implementable economic outcomes. Cong and He (2019) study the effects of smart contracts on industrial organization, while Tinn (2018) studies how financial contracting may be affected. Bakos and Halaburda (2019) delineate the effects of enhanced information generation of technologies dubbed the Internet-of-Things, and the automatic execution offered by smart contracts in a simple contracting game. Finally, Holden and Malani (2021) examine the use of smart contracts in the context of the hold-up problem. Two key properties of smart contracts underpin all of the above papers: (i) enhanced commitment power—for example, through lowering enforcement costs via automatic execution, or preventing renegotiation of terms altogether; and (ii) better information—for example, by reducing state-verification costs.

While we recognize that restoring some commitment power is possibly the main reason for the popularity of smart contracts, our paper intends to provide a different

[11] Hart and Tirole (1988) point out that the arguments of Bulow (1982) rely on buyer-anonymity and show and that renting may make the seller worse-off.

perspective. We take the view of Laffont and Tirole (1988) that the lack of intertemporal commitment is a form of contractual incompleteness. In other words, contracting parties may refrain from signing long-term, binding contracts due to potential unforeseen or non-contractible contingencies even if such contracts were feasible.[12] Although we do not model these contingencies, our assumption that the seller cannot commit not to switch off a deployed contract embodies the idea that she prefers a contract allowing for discretion in the future.[13] Our main result suggests that smart contracts may turn out to be useful even in such environments because they can store information securely.

2.2 The Model and a Preview of the Results

There is a seller of a durable, indivisible good and a buyer whose willingness-to-pay for the good is his private information. The buyer's valuation is either high, v_h, or low, v_l so that $v_h > v_l > 0$. The probability of high valuation, μ, is common knowledge. We assume that $\mu v_h > v_l$, so the static monopoly price is v_h. Time is discrete and indexed by $0, 1$,... In the initial period, the seller offers a contract from the set $\mathcal{C}$ described below. This contract then determines the allocation, i.e., the probability of trade and the transfer, in every period unless it is replaced. In each subsequent period, the seller decides whether to proceed with the previous period's contract or to deploy a new one. If the seller deploys a new contract, then it will determine the allocation in that period as well as in every future period until it is replaced. The game ends when the good is sold. We assume that both parties discount the future according to the common factor $\delta\, (\in (0,1))$. If the buyer's valuation is $v\, (\in \{v_l, v_h\})$, trade occurs in period T and the transfer is p_t at time t, then the payoffs of the buyer and seller are

$$\delta^T v - \sum_{t=0}^{\infty} \delta^t p_t \text{ and } \sum_{t=0}^{\infty} \delta^t p_t,$$

respectively. Moreover, both parties maximize their expected payoffs.

To complete the description of the model, we need to define the principal's contract

[12] Unexpected software security vulnerabilities, bugs, novel types of attack threats, the need for upgrades, and regulatory risk, are among the reasons one may willingly retain discretion over aspects of a smart contract, preventing its absolute immutability in practice.

[13] Such control can be exercised via 'admin keys' retained by the issuer. It is worth noting that 12 out of the 15 most popular Decentralized Finance protocols, governed by smart contracts, have such 'admin keys' (https://cointelegraph.com/news/how-many-defi-projects-still-have-god-mode-admin-keys-more-than-you-think).

space, $\mathcal{C}$. However, the formalism associated with general contracts may not be necessary to understand the main arguments leading to the failure of the Coase Conjecture. In order to spare the uninterested reader from technical details, we first describe a particular contract and explain informally how it can be used to bound the seller's largest equilibrium payoff away from v_l.

Preview of the Arguments.— We describe a dynamic contract by using the analogy of a long-lived mediator once again. In the initial period, the buyer can report a valuation from the set $\{v_l, v_h\}$ to the mediator privately. If the buyer reports v_h, the mediator executes trade immediately with probability $\alpha \in (0, 1)$ at price $p \in (v_l, v_h)$ and there is no trade if the buyer reports v_l. In any subsequent period, trade occurs with probability β with both types at a price equal to the buyer's report. Furthermore, to make the buyer's participation voluntary, in each period, he can tell the mediator that he rejects the contract. If so, neither the good nor any money changes hands in that period. In addition, rejecting the contract results in a one-period delay.[14] Moreover, the mediator does not communicate with the seller. Notably, the seller can learn about the buyer's valuation only from past allocations but receives no additional information from the mediator. Observe that such a contract is fully determined by the three parameters α, β and p.

Let us explain intuitively how the availability of such contracts may break the logic of the Coase Conjecture. The key feature of these contracts is that, after the initial period, they clear the market with a delay at a price equal to the buyer's valuation. When the seller is unable to commit not to clear the market, introducing delay is paramount in screening the buyer's willingness-to-pay. Indeed, if the delay is long enough (β is small), the high-type buyer finds it optimal to report his type truthfully and to trade immediately at price p with probability α instead of trading at price v_l in the future. In other words, setting β to be low makes the contract incentive compatible even when the price p is relatively high. The downside of the delay is that, just like in the standard price-posting model, the seller is tempted to deviate from her plan and clear the market faster. However, if she abandons the contract, she loses its information content and hence, she is unable to trade at the buyer's true valuation. So, being able to trade at high prices enables the seller to resist the temptation to abandon the contract and to clear the market at v_l.

[14] That is, the mediator's plan at the beginning of a period after a rejection is the same as at the beginning of the previous period.

The proof of our main theorem consists of two steps. First, based on the intuitions of the previous paragraph, we demonstrate that, for each δ, the triple (α, p, β) can be chosen so that the corresponding contract satisfies the following four properties: (i) the seller's expected payoff generated by this contract is larger than a constant, say $\underline{\pi}$, and $\underline{\pi} > v_l$, (ii) the contract is incentive compatible, (iii) after the initial period, conditional on no trade, the seller's posterior regarding the high type is so low that the static monopoly price becomes v_l, and finally (iv) the seller's continuation payoff is larger than v_l in each period. Let us denote a contract having these properties by d_δ.

The second step of the proof is to show that if, for a given δ, there is an equilibrium in which the seller's payoff is less than $\underline{\pi}$, then there also exists an equilibrium in which her payoff is larger than $\underline{\pi}$. Observe that the failure of the Coase Conjecture follows from this result because $\underline{\pi} > v_l$. Let us now explain the arguments related to the second step. To this end, consider an equilibrium in which the seller's payoff is less than $\underline{\pi}$. By modifying this equilibrium, we construct a new one in which the contract d_δ is deployed forever, so the seller's payoff is larger than $\underline{\pi}$. On the new equilibrium path, the seller always deploys d_δ, the buyer always accepts it and reports his valuation truthfully. Of course, one must also specify what happens off the equilibrium path. The rough idea is to define the equilibrium strategy at a certain information set to be the same as in the original equilibrium at a similar information set. Importantly, if the seller deviates in the initial period and offers a contract different from d_δ, the continuation play will be identical to that in the original equilibrium.

It remains to argue that properties (i)-(iv) imply that neither the seller nor the buyer can profitably deviate. The seller's payoff from offering a contract different from d_δ in the initial period is the same as from offering that contract in the original equilibrium. Hence, her deviation payoff is weakly less than the original equilibrium payoff which, in turn, is less than $\underline{\pi}$. Since, by property (i), the seller's new equilibrium payoff is $\underline{\pi}$, such a deviation is not profitable. In addition, by property (ii), the buyer optimally reports his valuation truthfully at the beginning of the game. In subsequent periods, when d_δ was already deployed a number of times, the seller's continuation payoff exceeds the full-commitment profit by properties (iii) and (iv). Of course, if the seller abandons d_δ and loses its information content, her continuation payoff cannot exceed the full-commitment profit. So even in later periods, the seller has no incentive to deviate from offering d_δ. In fact, since the seller expects the buyer to always accept d_δ, she optimally offers this contract even after many periods of rejection. In turn, knowing this, the buyer best-responds by accepting the seller's offer.

In the remainder of this section, we describe the contract space formally and define the equilibrium concept.

The Contract Space $\mathcal{C}$.— We describe a typical contract, c, from the seller's contract space $\mathcal{C}$. The contract specifies both the communication and the implemented allocation in each period when the contract is deployed. Formally, $c = \left(M_T^b, M_T^s, S_T^b, S_T^s, \mathbf{x}_T, \mathbf{p}_T, \rho_T\right)_{T=0}^{\infty}$, where M_T^b and M_T^s are the messages available to the buyer and the seller in a given period if the contract was already deployed T consecutive periods immediately preceding that period.[15] The sets S_T^b and S_T^s are the signals the buyer and the seller may receive privately. The functions $\mathbf{x}_T : \left(M_\gamma^b, M_\gamma^s\right)_{\gamma=0}^{T} \times \left(S_\gamma^b, S_\gamma^s\right)_{\gamma=0}^{T-1} \rightarrow [0,1]$ and $\mathbf{p}_T : \left(M_\gamma^b, M_\gamma^s\right)_{\gamma=0}^{T} \times \left(S_\gamma^b, S_\gamma^s\right)_{\gamma=0}^{T-1} \rightarrow \mathbb{R}$ specify the probability of trade and the transfer conditional on sale[16] as a function of histories of messages and signals. Finally, the function $\rho_T = \left(\rho_T^b, \rho_T^s\right) : \left(M_\gamma^b, M_\gamma^s\right)_{\gamma=0}^{T} \times \left(S_\gamma^b, S_\gamma^s\right)_{\gamma=0}^{T-1} \rightarrow \Delta\left(S_T^b, S_T^s\right)$ specifies the distributions of the signals revealed to the buyer and the seller as a function of the history of messages and signals. To model the buyer's participation decision, we assume that, for each T, the buyer's message space, M_T^b, includes a special message, r, which triggers no trade.[17] Sending this message is interpreted as *rejecting the contract*. If the buyer rejects the contract, $m_T^b = r$, then $\mathbf{x}_T = \mathbf{p}_T = 0$.[18] We say that the contract c is *actively deployed* in a given period, if the seller deploys c and the buyer does not reject it in that period. The seller's contract space $\mathcal{C}$ is a set of contracts described above.

Note first that the signals revealed to the contracting parties are assumed to be private. However, when the signals are perfectly correlated, they are effectively public. In fact, contracts are defined to be general enough to also allow the mixture of private and public communication; signals may have both private and public components. Second, despite the seller having no private information to start with, it is important to allow a contract to condition on the seller's messages. The reason is that the seller learns over

[15]For an example, suppose that c is deployed at $t = 0, 2, 3$ but not at $t = 1$. Then, $T = 0$ at $t = 0, 2$ and $T = 1$ at $t = 3$.

[16]For notational simplicity, we assume that transfers are deterministic and paid only if there is trade. Allowing random transfers has no impact on our results.

[17]We further discuss our modeling choice of the buyer's interim participation in the Discussion Section.

[18]One may also find it natural to assume that the seller is informed about the buyer's rejection of the contract. Our main result holds irrespective of such an assumption.

time and when she decides to deploy a contract, she may benefit from inputting her posterior and making the implemented allocations dependent on it.

Simple and Direct Contracts.— We are not making any additional assumption on the contract space except that it contains all those mechanisms which ask the buyer to report her valuation in the initial period of deployment but involve no additional meaningful communication. We call such contracts *simple and direct* and define them formally below. Again, let us describe a typical simple and direct contract, d. First, if d is deployed repeatedly then sending the message r only triggers a one-period delay. So, the easiest way to describe d is to index the message and signal spaces as well as the allocations defining these contracts by the number of those consecutive periods of deployment in which the contract d was not rejected. More precisely, at each history, let τ denote the number of previous periods in which d was actively deployed since a different contract was deployed.[19] Then, with a slight abuse of notation, the contract d is defined by the collection $(\mathbf{x}_\tau, \mathbf{p}_\tau)_{\tau=0}^{\infty}$, where $\mathbf{x}_\tau : \{v_l, v_h\} \to [0,1]$ and $\mathbf{p}_\tau : \{v_l, v_h\} \to \mathbb{R}$. In the initial period of deployment, and in every other period in which $\tau = 0$, the buyer is asked to report his valuation, so $M_0^b = \{v_l, v_h, r\}$. If the buyer reports $v \in \{v_l, v_h\}$ then trade occurs with probability $\mathbf{x}_0(v)$ at price $\mathbf{p}_0(v)$. If the buyer sends the message r and the seller deploys d in the next period, the buyer's message space is again $\{v_l, v_h, r\}$ and the allocation is determined by $(\mathbf{x}_0, \mathbf{p}_0)$. After the buyer does not reject d and reports a valuation, he can only accept or reject the contract, that is, $M_\tau^b = \{a, r\}$ for all $\tau > 0$. The seller is only informed whether or not the buyer rejected the contract, that is, $S_\tau^s = \{a, r\}$ for all τ and $\rho_\tau^s(m_\tau^b) = r$ if, and only if, $m_\tau^b = r$. The seller does not communicate to the contract and the buyer does not receive any information, so the seller's message spaces and the buyer's signal spaces are singletons. The set of such simple and direct contracts is denoted by $\mathcal{D}$ and we assume that $\mathcal{D} \subset \mathcal{C}$.

We point out that the set $\mathcal{D}$ is different from the set of contracts one may wish to call *direct* in our environment. In general, a contract should be defined to be direct if its message spaces in each period of its deployment are rich enough to allow the seller and the buyer to report their private information. Since such a contract may send signals to both parties and information may also evolve in those periods when the contract is not deployed, a direct contract must allow the reporting of hierarchies of beliefs. For

[19]For example, if d was deployed at $t = 0, 1, 2, 3$ and was rejected only at $t = 1$, then $\tau = 2$ at $t = 3$.

example, the seller's type includes his posterior about the buyer's valuation, her belief about the buyer's belief about her posterior, etc.

Equilibrium Concept and Existence.— We focus on Weak Perfect Bayesian Equilibria. That is, an equilibrium is defined as an assessment: a pair of system of beliefs and a (possibly mixed) strategy profile. The belief system specifies for each information set of the game a probability distribution over the set of nodes in that set, which is then interpreted as the belief of the contracting party who moves at that information set. An assessment is a Weak Perfect Bayesian Equilibrium if (i) the strategy profile is sequentially rational at each information set and (ii) beliefs are derived by Bayes' rule at those information sets which are reached with positive probability.

The concept of Weak Perfect Bayesian Equilibrium places little restrictions on the players' out-of-equilibrium beliefs. Since we provide a lower bound on the seller's equilibrium payoffs, one may suspect that this result is supported by constructing beliefs off the equilibrium path which may appear arbitrary. For example, if the seller believes that the buyer's willingness-to-pay is surely v_h whenever he rejects a contract, she would rationally offer a contract which specifies trade only at price v_h in subsequent periods. In fact, the seller may maintain this belief even after the buyer rejects contracts arbitrarily many times. This, in turn, may deter the buyer from rejecting an otherwise unattractive contract in the first place if it generates non-negative payoffs. We emphasize that our analysis does not rely on such arguments and we impose further restrictions on the seller's off-equilibrium beliefs. First, we require the assessment to satisfy a version of the "*no-signaling-what-you-don't-know*" condition. In particular, the seller's posterior regarding the buyer's type cannot change arbitrarily after her own deviation. Specifically, at the seller's information sets followed by such deviations, her new posterior must be computed by Bayes' Rule using the buyer's equilibrium strategy. Second, in the spirit of the concept of Sequential Equilibrium,[20] special care is taken to construct the seller's beliefs so that they are limit points of beliefs derived by Bayes' rule along a sequence of totally mixed strategy profiles converging to the equilibrium strategy profile.

It is not hard to show that equilibria exist in a discretized version of our model, i.e., the set of contracts, the message and signal spaces are all finite.[21] We also prove existence

[20]Myerson and Reny (2020) discuss the difficulty to extend the definition of Sequential Equilibrium to games with infinite sets of signals and actions, and propose the new concept of Perfect Conditional ε-Equilibrium.

[21]See Fudenberg and Levine (1983).

for the case when the seller only has access to simple and direct contracts, that is, $\mathcal{C} = \mathcal{D}$, see the Online Appendix. In the rest of this paper, we assume that an equilibrium exists irrespective of the discount factor.

2.3 Main Result

In order to state our main theorem, let $\pi(\mathcal{C}, \delta)$ denote the supremum of the seller's payoff across all equilibria if the contract space is $\mathcal{C}$ and the discount factor is δ.

Theorem 2. *There exists a $\underline{\pi} > v_l$, such that for all $\delta \in (0, 1)$,*

$$\pi(\mathcal{C}, \delta) \geq \underline{\pi}.$$

We remark that this theorem implies the failure of the Coase Conjecture: no matter how close the discount factor is to one, the largest equilibrium payoff of the seller is bounded from below by a constant, $\underline{\pi}$, which is larger than the low valuation, v_l.

The key to the arguments leading to the statement of Theorem 2 is to analyze a particular set of simple and direct contracts, coined as *abiding contracts*. The identifying feature of these contracts is that if they are actively deployed forever then (i) the buyer's continuation payoff is weakly positive in each period irrespective of his type and (ii) the seller's expected continuation payoff is larger than her full-commitment profit in all but the initial periods. The proof of the theorem consists of two steps. We first show that the seller's largest equilibrium payoff cannot be smaller than her payoff generated by any of the abiding contracts. The second step is to construct an abiding contract generating a payoff to the seller which is larger than v_l and does not depend on the discount factor.

Next, we define incentive compatible and abiding contracts formally.

Incentive Compatible Simple and Direct Contracts.— Whether the buyer has incentive to report his willingness-to-pay truthfully after accepting a simple and direct contract depends on what contracts he expects to be deployed in the future. Moreover, it also depends on the discount factor, δ. In what follows, we define incentive compatibility conditional on the same contract being deployed forever. Before presenting the formal definition, recall that the allocation determined by a simple and direct contract, $(\mathbf{x}_\tau, \mathbf{p}_\tau)_{\tau=0}^{\infty}$, depends only on the initial report of the buyer. Observe that if a simple and direct contract is actively deployed in each period, the buyer's report, v, determines the unconditional probability of trade, $X_\tau(v)$, and the expected transfer, $P_\tau(v)$, in each period by

$$X_\tau(v) = \mathbf{x}_\tau(v)\,\Pi_{t=0}^{\tau-1}(1-\mathbf{x}_t(v)) \text{ and } P_\tau(v) = \mathbf{p}_\tau(v)\,\mathbf{x}_\tau(v)\,\Pi_{t=0}^{\tau-1}(1-\mathbf{x}_t(v)).$$

Vice versa, each simple and direct contract $d \in \mathcal{D}$ can be described by $(X_\tau, P_\tau)_{\tau=0}^{\infty}$, where $X_\tau : \{v_l, v_h\} \to [0,1]$ denotes the probability that trade occurs in period τ conditional d being actively deployed in each period and $P_\tau : \{v_l, v_h\} \to \mathbb{R}$ is the expected transfer in that period.

Note that if a simple and direct contract is deployed forever, the buyer may maximize his payoff by misreporting his type in the initial period of deployment and optimizing with respect to his rejection-acceptance strategy in the future. Let $U(v, \widehat{v}, d, \delta)$ denote the buyer's value if the contract d is deployed forever, the discount factor is δ, the buyer's valuation is v and he reported $\widehat{v}$ in the initial period. Recall that if the buyer rejects a simple and direct contract, he only induces a one-period delay. Therefore, he only rejects the contract if his continuation payoff is negative, in which case, he would reject it forever. Consequently,

$$U(v, \widehat{v}, d, \delta) = \sup_{T \geq 0} \sum_{t=0}^{T} \delta^t \left[X_t(\widehat{v})\, v - P_t(\widehat{v})\right],$$

where T denotes the time period after which the buyer rejects the contract forever. We are now ready to define incentive compatibility.

Definition 3. *The contract $d = (X_\tau, P_\tau)_{\tau=0}^{\infty} \in \mathcal{D}$ is δ-incentive compatible if for $v \in \{v_l, v_h\}$*

$$v \in \arg\max_{\widehat{v} \in \{v_l, v_h\}} U(v, \widehat{v}, d, \delta).$$

Abiding Contracts.— As mentioned before, we intend to call a contract abiding if, conditional on the contract being actively deployed forever, the buyer's continuation payoff is non-negative and the seller's continuation payoff exceeds her full-commitment profit in each period. More precisely, we require that an abiding contract specifies trading probabilities with each type of the buyer so that, after the initial period, the static monopoly price becomes the low valuation. That is, conditional on not trading, the seller becomes so pessimistic regarding the buyer's willingness-to-pay that she would optimally clear the market at price v_l. Before providing the formal definition, let us introduce an additional piece of notation. If an incentive compatible, simple and direct contract is actively deployed in each period then $\mu_t(d)$ denotes the posterior probability that the buyer's willingness-to-pay is v_h in period t.

Definition 4. *The contract $d = (X_\tau, P_\tau)_{\tau=0}^{\infty} \in \mathcal{D}$ is δ-abiding if it is δ-incentive compatible and, in addition,*

(i) $\sum_{t=T}^{\infty} \delta^{t-T} [X_t(v) v - P_t(v)] \geq 0$ for all $v \in \{v_l, v_h\}, T \geq 0$,

(ii) $\mu_t(d) \leq v_l/v_h$ for all $t \geq 1$, and

(iii) $\mu_T(d) \sum_{t=T}^{\infty} \delta^{t-T} P_t(v_h) + (1 - \mu_T(d)) \sum_{t=T}^{\infty} \delta^{t-T} P_t(v_l) \geq v_l$ for all $T \geq 1$.

Condition (i) implies that if a δ-abiding contract is deployed forever, accepting the contract in each period is an optimal strategy of the buyer if his discount factor is δ. Conditions (ii) and (iii) require that the static monopoly price is v_l and the seller's continuation value is larger than v_l in all but the initial period if d is actively deployed forever.

Let $v(d, \delta)$ denote the seller's payoff if the incentive compatible, simple and direct contract $d = (X_\tau, P_\tau)_{\tau=0}^{\infty} \in \mathcal{D}$ is actively deployed forever, that is,

$$v(d, \delta) = \mu \sum_{t=0}^{\infty} \delta^t P_t(v_h) + (1 - \mu) \sum_{t=0}^{\infty} \delta^t P_t(v_l).$$

We are ready to state that the seller's value generated by any abiding contract is a lower bound on her largest equilibrium payoff.

Lemma 3. *Suppose that $d \in \mathcal{D}$ is a δ-abiding contract. Then $\pi(\mathcal{C}, \delta) \geq v(d, \delta)$.*

Let us explain the main arguments leading to this result. If the statement was false, the seller's payoff in each equilibrium would be strictly less than $v(d, \delta)$. Therefore, to prove the lemma, it is enough to argue that each such equilibrium can be modified so that, in the new equilibrium, the contract d is actively deployed forever. On the modified equilibrium path, the seller always deploys d and the buyer always accepts it. Off the equilibrium path the new equilibrium assessment is constructed based on the original equilibrium. In particular, the seller's payoff from offering a contract different from d in the initial period is the same as from offering that contract in the original equilibrium. Since the seller's payoff from offering any contract in the initial period is smaller than

$v(d, \delta)$ in the original equilibrium, such deviations are not profitable. In subsequent periods, when d was already deployed a number of times, the seller's continuation payoff from offering it exceeds the full-commitment profit because d is abiding. Of course, if the seller abandons d and loses its information content, her continuation payoff cannot exceed the full-commitment profit. So even in later periods, the seller has no incentive to deviate from offering d. If the buyer ever rejects the contract, the seller's posterior belief remains the same.[22] Given this belief and that the buyer is expected to accept d, the seller rationally offers this contract even after many periods of rejection. In turn, knowing this, the buyer best-responds by accepting the seller's offer because d is abiding so it provides him with a non-negative continuation payoff irrespective of his willingness-to-pay.

Proof. We prove this lemma by contradiction. Suppose that the seller's payoff in each equilibrium is strictly smaller than $v(d, \delta)$. In what follows, we fix such an equilibrium and, by modifying it, we construct a new equilibrium so that the contract d is actively deployed forever and, consequently, the seller's payoff is $v(d, \delta)$, yielding a contradiction.

Let us first define the new equilibrium assessment at those information sets which are reached by paths along which no contract was offered but d. The seller always offers d and the buyer never rejects it. Moreover, in the initial period, the buyer reports his type truthfully. So, the equilibrium path, and hence payoffs, are determined by the repeated active deployment of d. If the seller moves at such an information set, her belief is defined to be $\mu_\tau(d)$ if d was actively deployed τ times before reaching that information set, irrespective of the number of times the buyer rejected the contract. In other words, when the buyer rejects d along a path where no other contract was offered, the seller does not update her belief.

Next, we define the assessment at each information set which is reached by a path along which a contract $c \neq d$ is offered. Observe that if d is actively deployed τ times before the seller deviates for the first time, her posterior is $\mu_\tau(d)$. Next, we show that even in the original equilibrium assessment there are information sets at which the seller's posterior is exactly $\mu_\tau(d)$. We accomplish this by demonstrating the existence of a simple and direct contract $c(d, \tau) = (X_\tau, P_\tau)_{\tau=0}^{\infty}$, with the following properties. In each

[22] This belief is the limit of beliefs derived by Bayes' rule along a sequence of mixed strategies of the buyer over rejecting and accepting the contract, along which the probability of rejection goes to zero and does not depend on the buyer's valuation.

equilibrium,

(i) the buyer accepts $c(d,\tau)$ in the initial period,

(ii) the buyer truthfully reports his type in the initial period if $c(d,\tau)$ is deployed and

(iii) the seller's posterior belief is $\mu_\tau(d)$ after the initial period if there is no trade.

To this end, let $P_0(v_l) = -v_h$, $X_0(v_l) = 1/2$, $P_0(v_h) = -v_h + q(v_l + \varepsilon)$ and $X_0(v_h) = 1/2 + q$, so that

$$\mu_\tau(d) = \frac{\mu\left(\frac{1}{2} - q\right)}{\mu\left(\frac{1}{2} - q\right) + (1-\mu)\frac{1}{2}},$$

and $\varepsilon\,(>0)$ is small enough so that $v_h - (v_l + \varepsilon) > \delta(v_h - v_l)$. Moreover, let $P_\tau(v) = X_\tau(v) = 0$ for all $\tau > 0$ and $v \in \{v_l, v_h\}$. One interpretation of this contract is that each type trades with probability half at price $-2v_h$ in the initial period. If the buyer reports v_h, he trades with an additional probability of q at a price just above v_l. After the initial period, the contract prescribes autarky. Note that accepting this contract generates an instantaneous payoff of at least v_h to the buyer. The sequential rationality of the seller implies that the expected continuation payoff of the buyer cannot exceed v_h, so the buyer accepts this contract in every equilibrium, yielding (i). To see part (ii), first recall that reporting v_h triggers trade with an additional probability of q at $v_l + \varepsilon$. Observe that the sequential rationality of the seller implies that the object is never sold at a price lower than v_l so the high-value buyer is better off trading at $v_l + \varepsilon$ with an additional probability of q whereas the low-value buyer is not. Hence, the buyer reports his value truthfully. To obtain part (iii), observe that, by *no-signaling-what-you-don't-know*, the seller's posterior must be computed by Bayes' rule. In addition, q is defined such that this posterior is $\mu_\tau(d)$.

We are now ready to specify the new assessment at those information sets which are reached by paths along which a contract $c \neq d$ is offered. To this end, consider an information set at which c is offered and along the paths reaching this set the contract d was actively deployed τ times and no other contract was ever offered. Therefore, the seller's posterior belief when offering c is $\mu_\tau(d)$. Of course, the continuation game starting at this information set is isomorphic to the continuation game in which the seller offers $c(d,\tau)$ in the initial period, the buyer accepts it and reports his type truthfully, trade does not occur and the seller offers c in the next period. Indeed, the seller's posterior is also $\mu_\tau(d)$ by the definition of $c(d,\tau)$. Therefore, we define the equilibrium assessment in the continuation game starting from offering c to be the same as the original equilibrium assessment in the continuation game in which the seller offers $c(d,\tau)$ and c in the first and second periods, respectively.

It remains to prove that the new assessment defined above is indeed an equilibrium assessment. We first argue that players are sequentially rational at each information set. In the initial period, the seller's payoff from offering $c(\neq d)$ is at most as large as her payoff in the original equilibrium. Since, $v(d,\delta)$ is larger than that, the seller rationally offers d. At those information sets which are reached by paths along which only d was offered, the seller's continuation payoff is larger than her full-commitment payoff given her posterior. So, even at those information sets, the seller rationally offers d. At any other information set, the seller's strategy is sequentially rational because it is defined by the sequentially rational original equilibrium assessment in the corresponding isomorphic continuation game. The buyer's strategy is also sequentially rational at those information sets which are reached by those paths along which no contract other than d was offered. The reason is that d provides the buyer with a non-negative payoff and rejecting d would only delay those payoffs given that the seller offers it again after any number of rejections. Since d is incentive compatible, the buyer rationally reports his type truthfully in the first period. At any other information set, the buyer's strategy is sequentially rational because it is defined by the original equilibrium assessment in the corresponding isomorphic continuation game. Also note that the seller's belief is defined by Bayes' rule at each information set which is reached with positive probability. Indeed, the seller's belief after the contract d was actively deployed τ times is $\mu_\tau(d)$. $\square$

Having established Lemma 3, in order to prove Theorem 2, we need to demonstrate the existence of a δ-abiding contract for each δ which generates a payoff to the seller which is larger than a bound which is bigger than v_l. The next lemma states that such contracts exist.

Lemma 4. *For all $\delta \in (0,1)$, there exists a δ-abiding contract $d_\delta \in \mathcal{D}$ so that $v(d_\delta,\delta) \geq \underline{\pi} > v_l$.*

In what follows, we construct a contract for each δ satisfying this lemma's statement. This contract will have the following properties. In its initial period of deployment, if the buyer reports v_h, the contract specifies a positive probability of trade, α, at price $p \in [v_l, v_h]$ and the buyer does not trade if he reports v_l. In any subsequent period, trade occurs with probability β with both types at a price equal to the buyer's report. In other words, this simple and direct contract depends on three parameters $(\alpha, \beta, p) \in$

$[0,1]^2 \times [v_l, v_h]$ and can be formally defined as follows. In the initial period, $x_0(v_h) = \alpha$, $x_0(v_l) = 0$ and $p_0(v_h) = p_0(v_l) = p$.[23] For each $\tau > 0$, $x_\tau(v) = \beta$ and $p_\tau(v) = v$ for $v \in \{v_l, v_h\}$. In what follows, we express the constraints guaranteeing that the contract is not only incentive compatible but also abiding in terms of these parameters.

Let us first discuss incentive compatibility. First, note that the buyer with type v_l weakly prefers to report his willingness-to-pay irrespective of the parameter values. The reason is that if he does so, he always trades at price v_l and hence, his expected payoff is zero. On the other hand, if he reports v_h, the price is always weakly larger than v_l and hence, his expected payoff is non-positive. Consider now the buyer whose valuation is v_h. If he reports his true valuation, his payoff is $\alpha(v_h - p)$ because he trades with probability α at price p in the initial period and, any time in the future, the price is v_h. If he reports v_l, he does not trade in the initial period and, conditional on not trading before, he trades with probability β at price v_l in every period in the future. Therefore, if the high-type buyer misreports his type, the expected discounted present value of his payoff is

$$\delta \sum_{t=0}^{\infty} \delta^t (1-\beta)^t \beta (v_h - v_l) = \frac{\beta\delta}{1-\delta+\beta\delta}(v_h - v_l).$$

So, the incentive constraint of the buyer with type v_h is satisfied if

$$\alpha(v_h - p) \geq \frac{\beta\delta}{1-\delta+\beta\delta}(v_h - v_l). \tag{2.1}$$

Next, we investigate the set of those parameters for which the contract is abiding. First, note that whenever the buyer trades, the price is weakly smaller than his willingness to pay. Therefore, if such a contract is actively deployed forever, the continuation value of each type is weakly larger than zero, so the contract satisfies part (i) of Definition 4 for any parameters. Let us now describe the constraint corresponding to part (ii) of Definition 4. That is, we describe conditions under which the seller's posterior in each future period is such that the full-commitment monopoly price is v_l and her continuation payoff exceeds v_l if the contract is deployed forever. To this end, observe that, conditional on no trade, the seller's posterior remains the same after the initial period because the probability of trade in future periods, β, does not depend on the buyer's report. This posterior depends only on the probability of trade in the initial period, α,

[23]Since $x_0(v_l) = 0$, $p_0(v_l)$ can be defined arbitrarily.

and we denote it by $\widetilde{\mu}(\alpha)$. It can be computed by Bayes' Rule,

$$\widetilde{\mu}(\alpha) = \frac{(1-\alpha)\mu}{1-\mu+(1-\alpha)\mu}. \tag{2.2}$$

So, condition (ii) of Definition 4 holds and the static monopoly price is v_l if, and only if,

$$v_l \geq \widetilde{\mu}(\alpha)\, v_h. \tag{2.3}$$

We now compute the seller's continuation payoff in each period after the first deployment of the contract. Since neither the posterior distribution of types nor the probability of trade depend on time, this continuation payoff is also independent of time and can be expressed as

$$\sum_{t=T}^{\infty} \delta^{t-T}(1-\beta)^{t-T}\beta[\widetilde{\mu}(\alpha)\, v_h + (1-\widetilde{\mu}(\alpha))v_l] = \frac{\beta}{1-\delta+\beta\delta}[\widetilde{\mu}(\alpha)\, v_h + (1-\widetilde{\mu}(\alpha))v_l].$$

So, part (iii) of Definition 4 holds if this payoff is larger than v_l. In fact, we will construct parameter values so that that the seller's continuation value also exceeds his payoff from deploying this contract for one more period and selling the good at v_l immediately if the contract does not recommend trade, that is,

$$\frac{\beta}{1-\delta+\beta\delta}[\widetilde{\mu}(\alpha)\, v_h + (1-\widetilde{\mu}(\alpha))v_l] \geq \beta[\widetilde{\mu}(\alpha)\, v_h + (1-\widetilde{\mu}(\alpha))v_l] + (1-\beta)\, v_l. \tag{2.4}$$

In order to prove Lemma 4, for each large enough δ, it is enough to show the existence of a triple, $(\alpha^*, \beta^*, p^*) \in [0,1]^2 \times [v_l, v_h]$, such that the constraints (2.1), (2.3) and (2.4) are satisfied. Furthermore, we need to demonstrate that the seller's payoff is bounded away from v_l uniformly.

Proof of Lemma 4. First, we construct a δ-abiding contract for small discount factors. For each δ, consider the contract which specifies trade with the high-type buyer in the initial period at a price $v_h - \delta(v_h - v_l)$ and specifies trade with the low-type buyer in the next period at price v_l. This contract corresponds to the parameter triple where $\alpha = \beta = 1$ and $p = v_h - \delta(v_h - v_l)$. Note that this price makes the high-type buyer indifferent between buying the good immediately and trading at v_l a period later and hence, the constraint (2.1) is satisfied. Furthermore, since $\alpha = \beta = 1$ and $\widetilde{\mu}(1) = 0$, this contract is obviously

δ-abiding and satisfies the constraint (2.4). Let π_δ denote the seller's value generated by this contract and note that

$$\pi_\delta = \mu\left[v_h - \delta\left(v_h - v_l\right)\right] + \left(1-\mu\right)\delta v_l = \mu v_h\left(1-\delta\right) + \delta v_l > v_l\left(1-\delta\right) + \delta v_l = v_l,$$

where the inequality follows from $\mu > v_l/v_h$. Moreover, observe that π_δ decreases in δ and converges to v_l as δ goes to one. Therefore, it is enough to prove the lemma's statement for large δ's. That is, we show that there exists a $\overline{\delta} \in (0,1)$, such that for all $\delta \geq \overline{\delta}$, there exists a δ-abiding contract $d_\delta \in \mathcal{D}$ so that $v\left(d_\delta, \delta\right) \geq \widehat{\pi} > v_l$. Then, setting $\underline{\pi}$ to be $\underline{\pi} = \min\left\{\pi_{\overline{\delta}}, \widehat{\pi}\right\}$, the lemma follows.

Let us explain how we construct the aforementioned triple of parameters for large δ. First, for each α we define $\widetilde{\beta}\left(\alpha\right)$ so that the abiding constraint (2.4) evaluated at $\beta = \widetilde{\beta}\left(\alpha\right)$ binds and ignore the constraint that $\widetilde{\beta}\left(\alpha\right)$ must be a probability. Second, we define $\widetilde{p}\left(\alpha\right)$ so that the incentive constraint (2.1) evaluated at $\left(\beta, p\right) = \left(\widetilde{\beta}\left(\alpha\right), \widetilde{p}\left(\alpha\right)\right)$ binds and ignore the constraint that $\widetilde{p}\left(\alpha\right) \in \left[v_l, v_h\right]$. Then, we consider the functional form of the seller's payoff at $\left(\alpha, \widetilde{\beta}\left(\alpha\right), \widetilde{p}\left(\alpha\right)\right)$, maximize it with respect to α subject to the constraint (2.3) and define α^* to be the maximizer. Finally, we show that, if δ is large enough, the parameters $\widetilde{\beta}\left(\alpha^*\right), \widetilde{p}\left(\alpha^*\right)$ are feasible, that is, $\left(\widetilde{\beta}\left(\alpha^*\right), \widetilde{p}\left(\alpha^*\right)\right) \in [0,1] \times \left[v_l, v_h\right]$. Moreover, the seller's payoff generated by the contract corresponding to $\left(\alpha^*, \widetilde{\beta}\left(\alpha^*\right), \widetilde{p}\left(\alpha^*\right)\right)$ is strictly larger than v_l and does not depend on δ.

For each $\alpha \in [0,1]$, let $\widetilde{\beta}\left(\alpha\right)$ be defined so that the constraint (2.4) binds, that is,

$$\widetilde{\beta}\left(\alpha\right) = \beta = \frac{1-\delta}{\delta} \cdot \frac{v_l}{\widetilde{\mu}\left(\alpha\right)\left(v_h - v_l\right)}. \tag{2.5}$$

In addition, let us define $\widetilde{p}\left(\alpha\right)$ for each $\alpha \in [0,1]$ so that the high-type buyer's incentive constraint, (2.1) binds, that is

$$\alpha(v_h - \widetilde{p}\left(\alpha\right)) = \frac{\widetilde{\beta}\left(\alpha\right)\delta}{1-\delta+\widetilde{\beta}\left(\alpha\right)\delta}(v_h - v_l). \tag{2.6}$$

We now turn our attention to the seller's payoff generated by the contract corresponding to the triple $\left(\alpha, \widetilde{\beta}\left(\alpha\right), \widetilde{p}\left(\alpha\right)\right)$. We first compute the seller's continuation payoff in each period $t > 0$. As mentioned above, this continuation payoff does not depend on t. Since the abiding constraint (2.4) binds at $\beta = \widetilde{\beta}\left(\alpha\right)$, this payoff can be computed by plugging $\widetilde{\beta}(\alpha)$ into the right-hand side of this constraint,

$$\begin{aligned} & \frac{1-\delta}{\delta}\cdot\frac{v_l}{\widetilde{\mu}(\alpha)(v_h-v_l)}[\widetilde{\mu}(\alpha)v_h+(1-\widetilde{\mu}(\alpha))v_l]+\left(1-\frac{1-\delta}{\delta}\cdot\frac{v_l}{\widetilde{\mu}(\alpha)(v_h-v_l)}\right)v_l \\ = & \frac{1-\delta}{\delta}\cdot\frac{v_lv_h}{(v_h-v_l)}-\frac{1-\delta}{\delta}\cdot\frac{v_l^2}{(v_h-v_l)}+v_l=\frac{v_l}{\delta}. \end{aligned}$$

We are now ready to compute the seller's payoff generated by the contract defined by $\left(\alpha,\widetilde{\beta}(\alpha),\widetilde{p}(\alpha)\right)$. Let $\nu(\alpha)$ denote this payoff. Observe that, in the initial period, the seller receives $\widetilde{p}(\alpha)$ with probability $\mu\alpha$ and, in the next period, her continuation payoff is v_l/δ. Therefore,

$$\nu(\alpha)=\mu\alpha\widetilde{p}(\alpha)+\delta(1-\mu\alpha)\frac{v_l}{\delta}=\mu\alpha\widetilde{p}(\alpha)+(1-\mu\alpha)v_l. \tag{2.7}$$

Substituting $\widetilde{p}(\alpha)$ from equation (2.6) and using equation (2.2) yield

$$\nu(\alpha)=v_l+(v_h-v_l)\left(1-\frac{1-\mu}{1-\widetilde{\mu}(\alpha)}-\frac{\mu v_l}{\widetilde{\mu}(\alpha)v_h+(1-\widetilde{\mu}(\alpha))v_l}\right). \tag{2.8}$$

Finally, we define α^* to maximize $\nu(\alpha)$, subject to the constraint (2.3). That is, α^* solves

$$\max\left\{\nu(\alpha):\alpha\in[0,1],\widetilde{\mu}(\alpha)\leq v_l/v_h\right\}. \tag{2.9}$$

We now show that α^* is uniquely determined. To this end, note that ν depends on α only through $\widetilde{\mu}(\alpha)$. Also note that, by (2.2), the function $\widetilde{\mu}$ is continuous, strictly decreasing in α and, in addition, $\widetilde{\mu}(0)=\mu$ and $\widetilde{\mu}(1)=0$. Let $\Pi(\hat{\mu})$ denote $\nu\left(\widetilde{\mu}^{-1}(\hat{\mu})\right)$. In what follows, we characterize the unique solution, $\hat{\mu}^*$, of the following maximization problem

$$\max_{\hat{\mu}\in[0,v_l/v_h]}\Pi(\hat{\mu}).$$

Then, it follows that $\alpha^*=\widetilde{\mu}^{-1}(\hat{\mu}^*)$ is the unique solution of the problem (2.9). Note that

$$\Pi'(\hat{\mu})=-(v_h-v_l)\left(\frac{1-\mu}{(1-\hat{\mu})^2}-\frac{\mu v_l(v_h-v_l)}{(\hat{\mu}v_h+(1-\hat{\mu})v_l)^2}\right),$$

so $\Pi'(\hat{\mu}) \geq 0$ if, and only if, $\hat{\mu} \leq \left[\sqrt{\frac{\mu}{1-\mu}} - \sqrt{\frac{v_l}{v_h - v_l}}\right] / \left[\sqrt{\frac{\mu}{1-\mu}} + \sqrt{\frac{v_h - v_l}{v_l}}\right]$. Therefore,

$$\hat{\mu}^* = \min\left\{\frac{\sqrt{\frac{\mu}{1-\mu}} - \sqrt{\frac{v_l}{v_h - v_l}}}{\sqrt{\frac{\mu}{1-\mu}} + \sqrt{\frac{v_h - v_l}{v_l}}}, \frac{v_l}{v_h}\right\} \tag{2.10}$$

and note that $\hat{\mu}^* \in (0, v_l/v_h]$ because $\mu \in (v_l/v_h, 1)$.

Let us now return to examine whether $\left(\widetilde{\beta}(\alpha^*), \widetilde{p}(\alpha^*)\right) \in [0,1] \times [v_l, v_h]$ if $\alpha^* = \widetilde{\mu}^{-1}(\hat{\mu}^*)$. Observe that, by equation (2.5), $\widetilde{\beta}(\alpha^*) \in [0,1]$ if, and only if,

$$\hat{\mu}^* (= \widetilde{\mu}(\alpha^*)) \geq \frac{1-\delta}{\delta} \cdot \frac{v_l}{v_h - v_l}.$$

Since the right-hand side is decreasing in δ and converges to zero as δ goes to one, there exists $\overline{\delta}$ such that $\widetilde{\beta}(\alpha^*) \in [0,1]$ whenever $\delta \in (\overline{\delta}, 1)$. Let us turn our attention to the first period's transfer, $\widetilde{p}(\alpha^*)$. By the definition of the function $\widetilde{p}$, it follows that $\widetilde{p}(\alpha^*) \leq v_h$ for all $\alpha \in [0,1]$. Furthermore, equation (2.7) implies that the seller's payoff, $\nu(\alpha)$, can be expressed as a convex combination of $\widetilde{p}(\alpha^*)$ and v_l. Therefore, in order to establish that $\widetilde{p}(\alpha^*) \in [v_l, v_h]$ we only need to show that $\nu(\alpha^*) > v_l$, which we do next.

Before proceeding, we note that the construction of the parameters depends on the prior distribution of types, $\mu \in (v_l/v_h, 1)$. We now make this dependency explicit and express the seller's payoff induced by the contract constructed above as a function of μ. To this end, let us write the seller's posterior defined by (2.10) as a function of μ, $\hat{\mu}^*(\mu)$. Now, observe that, by equation (2.8), the seller's payoff can be written as

$$V(\mu) = v_l + (v_h - v_l)\left(1 - \frac{1-\mu}{1-\hat{\mu}^*(\mu)} - \frac{\mu v_l}{\hat{\mu}^*(\mu)\, v_h + (1-\hat{\mu}^*(\mu))v_l}\right). \tag{2.11}$$

In order to prove that $\nu(\alpha^*) > v_l$, it is enough to show that V is strictly increasing on $(v_l/v_h, 1)$ and
$\lim_{\mu \to v_l/v_h} V(\mu) = v_l$. To this end, note that V is continuous on $(v_l/v_h, 1)$. From equation (2.10), it follows that there is a cutoff value of μ, $\bar{\mu} \in (v_l/v_h, 1)$,[24] such that $\hat{\mu}^*(\mu) = v_l/v_h$ whenever $\mu \in (\bar{\mu}, 1)$. On this domain, $V(\mu) = \mu v_h^2/(2v_h - v_l)$, which is indeed strictly increasing. Since $\hat{\mu}^*$ was chosen to maximize the seller's payoff, on the domain

[24]It can be shown that

$$\bar{\mu} = \frac{v_l(2v_h - v_l)^2}{v_l(2v_h - v_l)^2 + (v_h - v_l)^3}.$$

$(v_l/v_h, \bar{\mu})$, the Envelope Theorem implies that

$$\begin{aligned} V'(\mu) &= (v_h - v_l)\left[\frac{1}{1-\hat{\mu}^*(\mu)} - \frac{v_l}{\hat{\mu}^*(\mu)v_h + (1-\hat{\mu}^*(\mu))\, v_l}\right] \\ &= (v_h - v_l)\left[1 - 2\frac{v_l}{v_h} + \sqrt{\frac{v_l}{v_h}\left(1-\frac{v_l}{v_h}\right)}\left(\sqrt{\frac{\mu}{1-\mu}} - \sqrt{\frac{1-\mu}{\mu}}\right)\right]. \end{aligned}$$

It is clear from inspecting the expression in the second line that V' is strictly increasing on $(v_l/v_h, \bar{\mu})$. Furthermore, since $\lim_{\mu\to v_l/v_h} \hat{\mu}^*(\mu) = 0$ by (2.10), the first line of the previous equality chain implies that $\lim_{\mu\to v_l/v_h} V'(\mu) = 0$. Therefore, V' is strictly positive on $(v_l/v_h, \bar{\mu})$. Recall that V' is also strictly positive on $(\bar{\mu}, 1)$ and continuous on $(v_l/v_h, 1)$. Then, by noting that $\lim_{\mu\to v_l/v_h} V(\mu) = v_l$, we conclude that $V > v_l$ on $(v_l/v_h, 1)$.

To summarize, we have constructed a triple of parameters,

$$(\alpha^*, \beta^*, p^*) = \left(\alpha^*, \widetilde{\beta}\,(\alpha^*)\,, \widetilde{p}\,(\alpha^*)\right).$$

We have demonstrated the existence of $\overline{\delta}$ such that $\left(\widetilde{\beta}\,(\alpha^*)\,, \widetilde{p}\,(\alpha^*)\right) \in [0,1]\times[v_l, v_h]$, so these parameters indeed define a contract. By equations (2.5) and (2.7), this contract is incentive compatible and abiding. Finally, we have proved that the seller's value from deploying this contract forever is strictly larger than v_l. To conclude the lemma's statement, all is left to do is to argue that, provided that $\delta > \overline{\delta}$, the seller's value does not depend on δ. This, however, is evident from equations (2.11) and (2.10). □

We are ready to argue that the statement of Theorem 2 follows from Lemmas 3 and 4.

Proof of Theorem 2. Recall that Lemma 4 guarantees the existence of a δ-abiding contract $d_\delta \in \mathcal{D}$ for each $\delta \in (0,1)$ such that seller's value generated by d_δ is bounded away from v_l, that is, $v\,(d_\delta, \delta) \geq \underline{\pi} > v_l$. Then Lemma 3 implies that, for all $\delta \in (0,1)$, the seller's largest equilibrium payoff exceeds $\underline{\pi}$, that is, $\pi\,(\mathcal{C}, \delta) \geq \underline{\pi}$. □

2.4 Discussion

Optimal Contracts.— Theorem 2 above states that the seller's largest equilibrium payoff is bounded away from v_l but it provides no further information about this payoff. In fact, we do not know what the seller's optimal contract is, nor her largest equilibrium

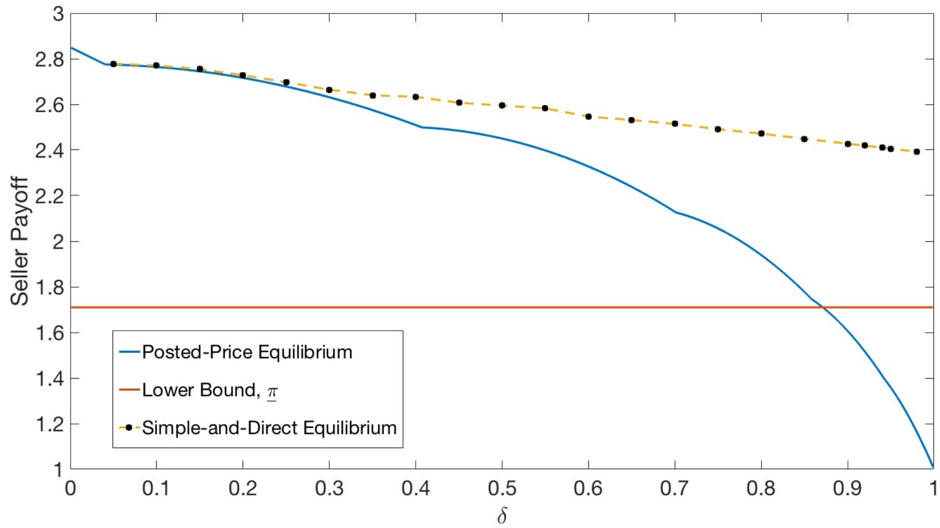

Figure 2.1: Comparison of the seller's profits when the discount factor varies.

profit. However, when we prove equilibrium existence for the case of $\mathcal{C} = \mathcal{D}$, we construct an equilibrium contract which induces a payoff to the seller which is significantly larger than the bound provided by Lemma 4 (see the Online Appendix). For each δ, this equilibrium contract specifies trade before a certain date, $T(\delta)$, with probability one. In the initial period, only the buyer with valuation v_h trades with a positive probability at a price in (v_l, v_h). Ever after, the price is always the reported valuation, just like in the case of the contract described in the proof of Lemma 4. In early time periods, only the high-type buyer trades and the seller is becoming more and more pessimistic. The low-type buyer only trades in the last and in the penultimate periods. Of course, as δ goes to one, $T(\delta)$ converges to infinity. Figure 2.1 plots the seller's payoff generated by this contract as a function of δ for the example where $v_l = 1$, $v_h = 3$ and $\mu = .95$.

What happens if the seller's contract space is larger than $\mathcal{D}$? It is not hard to show that in general principal-agent models, the principal typically benefits from having access to contracts which reveal information to both contracting parties. It is also possible to construct examples where the principal is worse off if her contract space includes such contracts. Unfortunately, we were unable to establish whether the seller benefits or is hurt by enlarging the set $\mathcal{D}$.

Other Contracting Games.— As mentioned in the Introduction, a common approach to model the lack of intertemporal commitment is to restrict the contract space to be the set of one-period contracts. Doval and Skreta (2020) pursue this approach in the context of a durable-good monopolist. In their setup, a contract of a given period determines the probability of trade and transfer in that period and reveals a public signal which can be assumed to be the seller's posterior. The authors show that the largest equilibrium payoff the seller can get, can be generated by a sequence of posted prices. Moreover, the seller's equilibrium profit converges to v_l as the discount factor goes to one, that is, the Coase Conjecture holds. Our main theorem highlights that the Coase Conjecture in Doval and Skreta (2020) is not only the consequence of the seller's limited commitment power, but also of the restricted contract space.

Figure 2.1 also plots the seller's largest equilibrium profit in the model of Doval and Skreta (2020) as a function of the discount factor. Note that when the discount factor is small, this profit level is larger than the one generated by the stationary contract described in Lemma 4. However, the profit induced by the aforementioned equilibrium contract is larger than the maximum profit in Doval and Skreta (2020) irrespective of δ.

Lomys and Yamashita (2022) consider a contracting game similar to ours. Let us paraphrase the description of their model using the terminology of our paper. In the initial period, their seller has access to the same set of contracts as our seller does. However, once the initial contract is abandoned, the seller in Lomys and Yamashita (2022) loses access to dynamic contracts and can offer only one-period contracts. A natural question to ask is: How does the seller's largest equilibrium payoffs compare in the two models? Observe that the seller's willingness to redeploy a contract depends on the payoff she expects in the continuation game following the abandonment of that contract. Intuitively, if the seller's smallest equilibrium payoff in such a continuation game is low, the set of those contracts she can credibly promise to deploy forever is large and, in turn, her highest equilibrium payoff is also large. In general, we don't know what the worst payoff of the seller is in either of these models except if the seller's posterior is so low that the static monopoly price is v_l. In that case, clearing the market at v_l is the unique equilibrium in both contracting games. Lomys and Yamashita (2022) show that if the fraction of high-type buyers is low enough, the optimal contract is the solution of a relaxed problem where the seller's payoff from abandoning the contract is assumed to be

v_l. Moreover, this contract specifies trade with the high-type buyer with a large enough probability in the initial period, so that the static monopoly price becomes v_l ever after. Consequently, the seller's payoff from abandoning this contract is indeed v_l. Therefore, in this case, the seller's largest equilibrium payoff coincides in the two models.

Implementation.— As explained in the Introduction, the contracts considered in our model have features resembling those of smart contracts used in digital markets. For our main result to hold, it is essential that the buyer communicates his willingness-to-pay to the contract privately. While communications on blockchain-based software platforms, such as Ethereum, are typically public, there are numerous examples for smart contracts involving private communication.[25] Implementing such communication is not hard using the cryptographic technology already employed in those markets. One way to do this is to encrypt the buyer's messages and let the buyer retain the decryption key. Then, in each period, the buyer can input the decryption key and an allocation is determined. Inputting an incorrect key would simply be treated as rejecting the contract.[26]

Continuous Types.— We now explain that the arguments in our paper can be extended to the case where the buyer's type is continuously distributed on an interval $[\underline{v}, \overline{v}] \subset \mathbb{R}_{++}$.[27] The idea is to construct a contract similar to the abiding contract of Lemma 4 which treats the buyer as if his type was binary. More specifically, we consider the same set of contracts described by parameter triples, (α, p, β)'s, except that v_l and v_h are also determined by design. In particular, v_l is set to be $\underline{v}$ and $v_h \in (\underline{v}, \overline{v})$. Effectively, the contract asks the buyer to report whether his valuation is below or above the threshold v_h. That is, sending the message v_l (v_h) in the initial period is interpreted such that the buyer's valuation is below (above) v_h. If the buyer reports v_h the contract specifies trade with probability α at price $p \in (\underline{v}, v_h)$ immediately. In any subsequent period, the buyer trades with probability β at a price equal to the initial report. Of course, the parameters can be chosen so that the threshold type, v_h, is indifferent between sending either of the messages. Moreover, it is not hard to show that, if α is large enough and v_h is sufficiently close to $\underline{v}$, the static monopoly price becomes $\underline{v}$ conditional on not trading in the initial period. In this sense, such a contract is abiding and a result analogous to Lemma 3

[25] Examples for digital protocols where such private communication is implemented in practice include Tornado.Cash and Aztec.Network.

[26] For discussions of implementing private communications on public blockchains, see Kerber, Kiayias, and Kohlweiss (2021), Steffen et al. (2019) or Bünz et al. (2020).

[27] For formal proofs, see the Online Appendix.

implies that the seller's value generated by it is a lower bound of her largest equilibrium payoff. Finally, observe that if such a contract is deployed forever, the seller's payoff is the same as if the buyer's valuation was binary: v_l with the probability that his true valuation is below v_h and v_h with the remaining probability. Therefore, a computation similar to the one in the proof of Lemma 4 implies that the contract can be defined so that the seller's payoff is $\underline{\pi}\,(> \underline{v})$ irrespective of the discount factor.

Side-Contracts.— If the seller decides not to proceed with the previous period's contract, the information content of the contract is lost. This feature enables the seller to redeploy contracts which implement allocations which are dominated from the viewpoints of both contracting parties. For example, in the context of the simple and direct contract of Lemma 4, d_δ, the buyer trades with probability $\beta^*\,(< 1)$ at the price of the reported valuation in all but the initial periods. Of course, both the seller and the buyer would be (weakly) better off if this probability was larger. However, if the seller wants to replace the contract with another one with larger trading probabilities, she would need to pay information rent to the buyer again which makes such a deviation non-profitable. A possible way to circumvent such ex-post inefficiencies associated to a contract would be to consider the possibility of writing side-contracts. That is, the seller continues to redeploy the previous period's contract but can offer a side-contract which conditions on the outcome of the redeployed contract. We next discuss the robustness of our main result to the introduction of such side-contracts.

Our arguments do not depend on the exact details of a model of side-contracts. However, one must specify what happens if the original contract and a side-contract recommend to implement different allocations in a certain period. In what follows, we assume that contracts can be written so that they have *priority* over contracts offered later. That is, if the seller offers such a priority contract and later a side-contract, the side-contract can affect the terms of trade only if the original contract is abandoned.

To argue that the Coase Conjecture fails even in a model with side-contracts, it is enough to establish that the value generated by d_δ, $v\,(d_\delta, \delta)$, is still a lower bound on the seller's largest equilibrium payoff. In our model without side-contracts, Lemma 3 implies that, if the seller's payoff is less than $v\,(d_\delta, \delta)$ in an equilibrium, there is another equilibrium in which the contract d_δ is offered and accepted in each period, so the seller's payoff is $v\,(d_\delta, \delta)$. We extend the equilibrium construction in the proof of Lemma 3 by specifying an assessment at information sets following the offer of a side-contract. To this end, consider the continuation game following a period of no trade. Recall, we constructed equilibrium strategies so that the seller deploys d_δ even after multiple periods of rejection and the buyer always accepts it. There is, however, another equilibrium in this continuation game. Observe that, if the buyer expects the seller to always clear the market at price v_l, he rejects any contract that generates a payoff less than that from buying the object at v_l a period later. Since, conditional on not trading in the initial period, the static monopoly price is v_l, the seller best-responds by trading with both types at v_l immediately. Let us now assume that this continuation play ensues whenever a side-contract is offered and that d_δ is offered with priority. Then, in each period, the seller's temptation to offer a side-contract is maximal right after the buyer accepted d_δ but before it determines the allocation of that period. Since d_δ has priority, the optimal side-contract of that moment would implement trade at price v_l if the outcome of the original contract is no trade.[28] Observe now that the abiding constraint (2.4) implies that, conditional on not trading in any given period, the seller's continuation value generated by d_δ exceeds her payoff from clearing the market at v_l. Therefore, the seller cannot gain from offering a side-contract.

Buyer-participation.— In our model, the buyer's participation is voluntary even at interim stages because he can send a message in each period that triggers autarky in that period. Notably, sending this message does not force the seller to abandon the rejected contract and consequently, its information content is not necessarily lost. In a sense, our modelling choice provides the seller with maximal bargaining power while still respecting the buyer's interim participation constraints. This is consistent with our primary objective: to model dynamic principal-agent relationships without intertemporal commitment. In particular, our contracting game is meant to capture the principal's identifying feature of having full bargaining power.

Let us also point out that our assumption regarding the buyer's voluntary participation may appear to be particularly plausible in the context of dynamic option contracts. Indeed, by relabelling the buyer's action of sending the reject message to *not sending a message*, our contracts can be reinterpreted as general dynamic options. In each period,

[28]If d_δ had no priority, it can be used as a signal generating device. For example, a side-contract offered in the initial period could ask the buyer to report his type. If the buyer reports v_l and d_δ recommends trade at p^* the side-contract could punish the buyer for inconsistent reports by implementing trade at an arbitrary high price.

the buyer must send a message to exercise the option. The terms of trade in that period, i.e., probability and transfer, may depend on the entire history of messages. If, in a certain period, the buyer does not send a message, the option is not triggered and trade does not take place. In this case, the seller may decide to abandon the current option contract and offer a new one. However, assuming that the buyer forces her to do so just by not exercising the option might be controversial.

Finally, we explain that whether our main result remains valid in a model where the buyer's rejection decision forces the seller to abandon her contract may depend on the equilibrium concept of the model. To this end, consider the abiding contract, d_δ, described in the proof of Lemma 4. In order to show that the seller's largest equilibrium payoff is bounded away from v_l, we need to modify the proof of Lemma 3 and construct an equilibrium in which d_δ is offered in each period. Recall that, conditional on not trading in the initial period of deployment of d_δ, the seller becomes so pessimistic that the static monopoly price becomes v_l. The high-type buyer may be tempted to exploit this fact by rejecting d_δ and hope that the seller will clear the marker at v_l in the next period. However, since the low-type buyer has no strict incentive to reject, the seller may make an inference from observing the rejection about the buyer's willingness-to-pay. Therefore, one can specify the assessment so that, after the buyer rejects d_δ, the seller becomes fully convinced that the buyer is of high type and sets price v_h ever after. Moreover, the high-type buyer responds by accepting trade at v_h while the low-type buyer rejects it. In turn, the high-type buyer no longer benefits from rejecting d_δ. This modified assessment is a Weak Perfect Bayesian Equilibrium. Furthermore, it is not hard to show that the seller's beliefs are limit points of beliefs derived by Bayes' rule along a sequence of totally mixed strategies of the buyer converging to his equilibrium strategy profile. However, the convergence is not uniform across the seller's information sets, so the construction above may not survive stronger equilibrium refinements.[29]

[29] Myerson and Reny (2020) suggest that, in the context of infinite games, the refinement concept corresponding to sequential equilibrium should require uniform convergence. Otherwise, the fully mixed strategies in the sequence may not be close to the limit strategy.

Chapter 3

Economic Growth in a Cooperative Economy

3.1 Introduction

For the first time in many decades the capitalist organization of production is under discussion in several Western societies. In the United States, avowed socialists are among the most popular politicians in the country - and one of them has been a leading candidate to be nominated by a major party in the last two Presidential elections. Meanwhile, historically unprecedented percentages of opinion-poll respondents express positive views of socialism. Perhaps more significantly for future developments, socialism is viewed more favorably than capitalism among the youngest cohorts.[1] In the United Kingdom, leaders with a Marxist background, and with a recent history of advocating worker ownership of the means of production, have recently led the major opposition party, and might have succeeded in winning power had they not chosen an unpopular stance on Brexit. Disaffection with capitalism is also affecting political dynamics in several other countries.

A similar, vigorous debate is taking place among academics and public intellectuals. New books about the failures of capitalism appear on a monthly basis, and columns on the same topic are featured daily on the major newspapers. Major research programs, involving management scientists, sociologists, political scientists, and economists repu-

[1] E.g. Pew Research centre, 2019

diate Friedmanite shareholder value and attempt to redefine the role and purpose of corporations.[2] Campaigns to redistribute power from shareholders to workers attract support from thousands of academics in social science disciplines.[3]

Macroeconomists have yet to make significant contributions to this important debate, and yet the institutional changes under discussion cry out for rigorous analysis of their general equilibrium and dynamic implications. What do they imply for aggregate productivity? How do they affect economic growth? This paper attempts to take a first step towards filling this gap.[4]

The alternative to shareholder capitalism that we study in this paper is the worker cooperative. This is a natural starting point for several reasons. First, cooperatives are frequently cited as possible remedies to the perceived crisis of capitalism, making an assessment of their growth implications directly relevant for the ongoing debate. Recent influential books which contain expressions of support for producer cooperatives as part of the needed revamp of the economic system include Block (2018), Cass (2018), and Collier (2018), which all appeared within a few months of each other. The earlier blockbuster on the consequences of inequality by Wilkinson and Pickett (2010) devotes its *entire* "normative" section to producer cooperatives, to the exclusion of all other remedies for the problems the book highlights. As we show in Appendix C.1, the phrase "worker cooperatives" has appeared more and more frequently among newly published (digitized) books since the mid-2000s. Positive media coverage of producer cooperatives seems also to have become more frequent, with stories centering on their ability to support the income and employment of their members during recent crisis periods; or on owner-managers transferring ownership to the workers as their individual contribution towards the transition to a post-capitalist model [for examples in prominent media, see Financial Times (2019) and New York Times (2021)].

Second, worker cooperatives have existed for nearly 200 years, and continue to exist virtually everywhere in the world.[5] This provides a real-world basis to build the model on, and some confidence that the alternative to capitalism being studied has a chance

[2] E.g. the British Academy's *Future of the Corporation* programme, lead by Colin Mayer.

[3] E.g. the *Democratizing Work* campaign of the Summer of 2020.

[4] Microeconomists have been quicker to the mark, and have produced important normative insights in a partial equilibrium context (e.g. Magill, Quinzii, and Rochet (2015) and Hart and Zingales (2017)). But these contributions cannot substitute for positive assessments of the dynamic and general equilibrium consequences of alternative arrangements.

[5] The worker cooperative movement has its origins in the industrial revolution. Then as now it emerged as a response to the perceived shortcomings of subordinate-labour capitalism.

to survive impact with reality. Third, as we discuss shortly, there is a pre-existing (if largely forgotten) tradition of economic modelling of worker cooperatives which we can relate our work to. Fourth, and perhaps most importantly, worker cooperatives can be thought of as a limiting case of many of the more nuanced ideas advanced by would-be reformers, which typically include less complete reallocations of control and ownership rights away from shareholders (or, essentially equivalently, reallocations of the weights of different stakeholders in corporation decision making). We submit that studying this limiting case is a useful first step towards a framework suitable for the study of more "interior" forms of organization.[6]

We study a production economy inhabited by two-period lived overlapping generations, where only the young work, while both old and young consume. The capitalist version of this economy, characterized by individual property of capital and profit-maximizing firms, is entirely standard and its dynamic properties are well known. Consistent with real-world arrangements, we conceptualize cooperatives as *labor-managed* entities which allow *no individual ownership* of their assets. In our model this implies that cooperatives, and not any individuals, own their own capital stock, and that young workers come together to produce and collectively choose investment plans. Given that the firm is managed by young workers, its objective is to maximize the present value of their (common) lifetime utility. As in the capitalist economy, these cooperatives supply their output on a perfectly competitive product market.

Real-world cooperatives differ in the claims former workers have in the distribution of income, with traditional cooperatives tending to severe all links upon a worker's retirement or withdrawal from the membership, and other, often more successful cooperatives where former workers continue to receive payments. Coops in the celebrated Mondragon system, for example, which employs nearly 100,000 people in the Basque region of Spain, belong to the latter category.[7] Our modelling choices mimic this model: old workers continue to participate in the distribution of income of the cooperative to which they were attached when young. As was noted in the early economics literature on labour-managed firms, in traditional cooperatives members' horizon when voting over investment is limited to their expected remaining time with the coop, and this tends

[6]Early statements of the view that worker cooperatives are limiting cases of models of codetermination and/or collective bargaining include Law (1977), Aoki (1980), Svejnar (1982), and Miyazaki (1984).

[7]Classic economic analyses of the (still thriving) Mondragon experience include Bradley and Gelb (1983) and Whyte and Whyte (2014).

to depress cooperative investment. By lengthening the planning horizon of young workers our institutional setup encourages greater investment by the coop, and potentially explains the apparent greater success of those coops which continue to confer distribution rights to former workers. Importantly, in an appendix we endogenize this arrangement and show that it can emerge as a feature of the equilibrium in the dynamic inter-generational game among subsequent cohorts of workers in the coop.

One of our main goals is to identify an appropriate equilibrium concept for a dynamic cooperative economy. This is challenging because it is not a priori obvious how young workers will sort themselves into the cooperatives that exist when they join the labor market, and also under what conditions they will decide to form new cooperatives rather than joining an existing one. Furthermore, any worker allocation mechanism has repercussions for investment, as a cooperative's current workers incentive to invest depends on the expected employment of the cooperative in the future. We solve these challenges by developing a "minimum rationality" constraint on the admissible allocations. Part of this criterion is that workers in one cooperative cannot improve their lifetime utility by attracting a *willing* worker from another cooperative. After establishing the general equilibrium notion, we also provide an equilibrium-selection criterion which minimizes informational requirements. We explain later how our equilibrium concept borrows from and extends existing ideas in cooperative game theory, as well as how it relates to the literature on matching.

After developing the framework and the equilibrium concepts for a cooperative economy, we study a couple of examples. In these examples, we characterize the growth path of the cooperative economy and compare it to the growth path of the same economy when production takes place in the "standard" capitalist firms which feature in neoclassical growth theory. Our analysis is based on choices of technology and preferences for which we are able to develop qualitative, or at least quantitative results.

In our examples, the cooperative economy converges to a steady state level of income per efficiency unit of labor - just as the capitalist economy is well known to do. In general, steady state income, consumption, and welfare can be higher or lower in the cooperative economy, depending on parameter values. Still, there are some systematic differences. We uncover a form of *static inefficiency* in the cooperative economy: for a given aggregate capital stock, worker cooperatives are inefficiently small (or, equivalently, there are too many firms in the cooperative economy). On the other hand, in our one fully-solved example the capitalist economy features potential over-accumulation of capital, while the cooperative economy is always dynamically efficient. We provide an exact decomposition of steady-state income differences between the cooperative and the capitalist economy into a *static efficiency* component and a *capital accumulation* component. The static efficiency component always favours the capitalist economy but, if the cooperative economy saves considerably more than the capitalist one, this can more than compensate for its lesser static efficiency, resulting in higher steady-state output and welfare.

We calibrate our model's preference and technology parameters by matching the capitalist version of the model to relevant US data moments. In our baseline calibration the steady state output of the cooperative economy is 73% of what it is in the capitalist economy, resulting in a 28% welfare loss. All of this output gap is due to the static inefficiency of cooperatives: the aggregate saving rate is in fact slightly higher in the cooperative economy. Needless to say these results are illustrative and more in the nature of a "proof of concept" for the modelling framework. As we discuss in the Conclusions, their robustness will have to be assessed against a number of modelling extensions.

The Golden Era of the theoretical economic analysis of worker cooperatives was the period between the late 1950s and the late 1970s, when some of the stars of the profession took an interest in the topic. Ward (1958), Domar (1966), and Sen (1966) set up static, partial equilibrium models focused on the determination of cooperative labor input (on the extensive and/or intensive margin). Vanek (1970), Drèze (1976, 1989), Ichiishi (1977), Greenberg (1979), Drèze and Greenberg (1980), and Laffont and M. Moreaux (1983) provided general equilibrium analyses, and established conditions for the existence and Pareto optimality of equilibria in economies constituted by worker cooperatives.[8] However, their analyses were still static. Furubotn and Peyovich (1973) and Furubotn (1976) argued that this gave them a blind spot for the anti-investment bias arising from the limited planning horizons of traditional cooperative members, who lose property rights in the cooperative's assets when they leave the firm.[9]

[8] Or "labor-managed firms," as earlier writers prefer to call them, or the less politically-loaded "partnerships" which features most frequently in post-1990 writings. We use "worker cooperative," "producer cooperative," and "labor-managed firm" interchangeably.

[9] Atkinson (1973) and Sapir (1980) also attempted to inject dynamic considerations in the Ward (1958) model, but were not able to produce significant insights. The contributions cited here are only the landmarks of what became a huge literature full of extensions and generalizations of the results in the key papers. The *Journal of Comparative Economics*, in particular, was largely devoted to the study of labor-managed organizations well

Conceptually, our paper can be understood as a step towards marrying Vanek and Dreze's general equilibrium analysis with Furubotn and Peyovich's dynamic (but partial equilibrium) one - while at the same time proposing a solution to the Furubotn and Peyovich critique (in the form of giving former workers a claim on current distributions).[10] However the modelling framework is completely different and much more in line with recognizable modern macroeconomic practice.

Subsequent theoretical developments have returned to concerns, originally voiced by Alchian and Demsetz (1972), with cooperative members' incentives to provide effort (e.g. Holmström, 1982, Kremer, 1997). Solutions to this problem have been identified in peer monitoring (e.g. Mirrlees, 1976, Putterman, 1982),[11] and the repeated nature of the interaction among coop members, giving rise to the extension of the Folk Theorem to so-called "partnership games" (Radner, 1986, Radner, Myerson, and Maskin, 1986, Fudenberg, Levine, and Maskin, 1994, and a conspicuous following). In order to keep the focus on the macroeconomic implications, in this paper we abstract from the intensive margin of effort. We do however note that, as pointed out by Bonin, Jones, and Putterman (1993), and confirmed in many successive surveys, shirking by workers or managers is virtually never reported as a concern in studies of real-world producer cooperatives.[12]

In the last two or three decades the focus of the research effort on worker cooperatives (and more generally of forms of worker participation in profit and/or management) has shifted from the development of theoretical models to the mobilization of empirical evidence. Excellent recent surveys of this large literature, which collectively covers a large variety of countries and industries, can be found in Pencavel (2013), and Jones (2018). Generally speaking, the evidence suggests that worker cooperatives tend to be

into the 1980s. A very comprehensive review of this literature (up to the mid-1980s) is in Bonin and Putterman (1987).

[10]The implicit assumption being that, if our society turns to the cooperative mode of production, it will do so based on the best practice available.

[11]In particular, cooperative workers have much greater incentives to monitor each other's effort than subordinate employees on a fixed salary.

[12]Our deterministic environment also means that we abstract from differences in risk diversification between capitalist and cooperative economies. The theoretical literature has generally pointed to countervailing risk-diversification mechanisms operating in the two economies. Capitalist firms do a superior job with the diversification of capital income (e.g. Meade, 1972), but cooperatives are more likely to insulate workers from labor income volatility, particularly as arising from unemployment risk (Steinherr and Thisse, 1979, Miyazaki and Neary, 1983, Bonin, 1984) and, thanks to their more equalitarian pay structure, provide better insurance against idiosyncratic productivity shocks (e.g. Lang and Gordon, 1995, Kremer, 1997). Hansmann (1996) reviews empirical evidence showing that cooperatives have more stable employment and that they are often found in highly capital-intensive and high-volatility industries, and concludes that, on balance, differences in risk diversification are probably not first order in comparing the two types of institutions. See also Drèze (1989) for equivalence results between stochastic capitalist and cooperative economies.

(somewhat) more productive than conventional firms, to afford their workers greater income stability and job satisfaction, and to display comparable exit and investment rates. It must be acknowledged, however, that only rarely are these empirical results immune from concerns regarding selectivity.[13]

The paper is organized as follows. Section 3.2 describes the physical environment, including technology, demographics, and preferences. Section 3.3 describes the institutional setup with which we represent the "capitalist" system, and the maximization problems and equilibrium conditions that derive from it. These are familiar to all economic students. Section 3.4 sets out institutions, maximization problems, and equilibrium conditions for a cooperative economy. This is the main conceptual and methodological contribution of the paper. Section 3.5 solves the model, both under capitalist and under cooperative institutions, for the case in which individuals derive log utility from consumption and production is Cobb-Douglas. For this example we are able to develop closed from solutions and make a number of general statements about the comparative growth paths of the two economies. Section 3.6 presents a calibration of the model with slightly more realistic preferences and derive the main quantitative results. Section 3.7 evaluates the dependence of our numerical results on variations in the parameters and performs comparative statics exercises. Section 3.8 discusses some of the many directions in which we hope to take this project in future work, both to probe the robustness of our preliminary results and to investigate additional issues, such as inequality.

3.2 Physical Environment

As noted in the Introduction, a critical economic feature of producer cooperatives is the finite planning horizon of self-managing workers. These workers know that benefits accruing to the coop after they have left may escape them, potentially leading to severe under-investment (and failure to implement other choices with back-loaded returns). These considerations need to be taken into account when choosing the appropriate modelling of demographic. The simplest option is a two-period overlapping-generations

[13] We should mention a healthy parallel literature on other types of cooperatives. For example, Rey and Tirole (2007) study cooperative investment by groups of firms, and Hart and Moore (1996, 1998) study consumer cooperatives. We should also cite an important 1980s research program on profit sharing (e.g. Weitzman, 1984, 1985, Meade, 1986), which had a particular focus on its potential role in dealing with stagflation.

framework, in which agents only work when young (for a profit-maximizing firm in the capitalist economy; as members of a cooperative in the cooperative economy), but consume when both young and old. This means that young cooperative workers make decisions which will affect cooperative outcomes after they have stepped down from their membership, as is the case in real-world cooperatives. The fact that all the workers making decisions within a cooperative are identical allows to identify an unambiguous objective for the coop, namely the maximisation of the utility of all its current workers.[14]

Formally, we endow agents with utility function

$$U(c^Y, c^O),$$

where c^Y (c^O) is consumption of the economy's final good when young (old). All agents in a generation are identical, and each young agent supplies one unit of labour inelastically. For simplicity, we assume that the population is constant and denote L the mass of each generation.

There exists a technology that uses capital and labour as inputs to produce the final good according to the production function

$$F(k, l).$$

The capital used for production fully depreciates across periods, while investment of the final good generates new capital on a one-for-one basis.

[14]In a model with a more general dynamic structure different workers in the same cooperative would have different horizons and different employment histories. Since workers accumulate claims on their previous cooperatives' revenues, these would create heterogeneity in preferences within the decision makers of a cooperative regarding investment decisions. For example workers closer to retirement may have different preferences vis-à-vis investment to workers further away from retirement. The problem of heterogeneous horizons might be solved within an infinite-horizon framework by using a dynastic model à la Barro or a perpetual-youth model à la Blanchard, but still additional "tricks" would be required to make former employment histories (namely the fact that workers will have accumulated claims of potentially differing expected value against previous employers) irrelevant for their preferences regarding the firm's current investment decisions. While clever devices along these lines could certainly be introduced, we do not think that pursuing them would move the model in the direction of greater realism. Having said all this, it has to be acknowledged that conflict of interest among workers has been stressed as a key weakness of cooperatives by Hansmann (1996), though we don't know of compelling empirical evidence in support of this view. Other than the omission of an analysis of such potential conflicts, we struggle to conceive of insights about growth in a cooperative economy which would be fundamentally different in a more complex demographic framework, or an infinite-horizon one, from those we identify in our simpler OLG model.

3.3 Capitalist Economy

Our capitalist benchmark is a standard competitive equilibrium where profit-maximizing firms can enter and exit freely; young workers supply labour, consume and save in the form of capital; old workers rent out the capital they saved and use the proceeds to finance consumption; and all agents are price takers.

The prices of labour and capital at time t are denoted w_t and r_t, and they are in terms of the final good, which acts as numéraire. Conditional on entry, individual firms maximize profits taking current prices as given:

$$\pi(r_t, w_t) = \max_{k,l}\{F(k,l) - r_t k - w_t l\},$$

with factor demands denoted: $k(r_t, w_t)$ and $l(r_t, w_t)$. We assume that these factor demands are single-valued so that all active firms behave symmetrically, and we can omit firm subscripts.

Capital is owned by individuals. We assume that each period-0 old agent is endowed with some initial capital stock κ_0.[15] In each subsequent period, old workers can sell their savings in the form of capital stock κ_t at the market price r_t. At the same time, young workers become old capitalists by saving some of their labor income. In particular, the young solve the following program:

$$\begin{aligned} &\max_{c^Y, c^O, \kappa_{t+1}} U(c^Y, c^O) \\ &\text{s.t. } c^Y + \kappa_{t+1} = w_t \\ &\qquad c^O = r_{t+1}\kappa_{t+1}. \end{aligned}$$

The solution to this problem defines the optimal capital investment as a function of the prices w_t and r_{t+1}:

$$\kappa_{t+1} = \mathfrak{K}(w_t, r_{t+1}).$$

In equilibrium, markets for capital, labour and the final good clear, and free entry and exit drive firms' profits to zero. Denoting N_t the equilibrium measure of operating firms,

[15]Heterogeneous capital endowments among the initial old could trivially be allowed for, but all heterogeneity would immediately disappear with the first young generation.

the competitive equilibrium in each period is characterised by the following system:

$$\pi_t(r_t, w_t) = 0$$
$$N_t l(r_t, w_t) = L$$
$$N_t k(r_t, w_t) = L\kappa_t.$$

A solution to this system defines the equilibrium prices and the number of firms as functions of the state variable κ_t: $r_t = r(\kappa_t)$, $w_t = w(\kappa_t)$, $N_t = N(\kappa_t)$. It follows that the dynamics in this economy are characterised by the following capital accumulation equation:

$$\kappa_{t+1} = \mathfrak{K}\big(w(\kappa_t), r(\kappa_{t+1})\big).$$

3.4 Cooperative Economy

This section works its way to the construction of a general-equilibrium concept for a dynamic cooperative economy. We begin by formalizing the concept of cooperative, and identifying the decisions which cooperative members make. Then we take up the more complex task of analysing how these decisions interact at the aggregate level and, in particular, we discuss the allocation of labour in the absence of a wage rate.

3.4.1 Concept of a Cooperative

Our conceptualization of cooperatives stresses two features which seem to most clearly distinguish this mode of organization from standard, externally-owned corporations: collective decision making by workers (labour management) and the non-tradability of productive assets. Self-management implies that decisions concerning the cooperative's size and investment are made collectively by the current workers of the cooperative. In our simplified context, where all workers are identical, this means that the objective function of the cooperative is the maximization of the present value of the lifetime utility of its current workers. Non-tradability means that capital is directly owned by the cooperative.

Any period t begins with a set of *incumbent* cooperatives, indexed by i. An incumbent cooperative i is characterized by an inherited capital stock k_{it} and a set of former workers l_{it-1}. Each incumbent cooperative is allocated a set of workers l_{it} via a mechanism which we describe later (Section 3.4.3). These workers produce output $y_{it} = F(k_{it}, l_{it})$. A share τ of this output is immediately distributed to the former workers. Next, the current-period workers decide how much of the cash flow (net of payments to the old) should be invested to put in place capital to be used in the next period, k_{it+1}. All non-retained earnings are distributed equally among current-period workers. These assumptions result in the following consumption levels for a representative young worker of incumbent cooperative i in period t:

$$\begin{aligned} c_{it}^{Y} &= \frac{(1-\tau)y_{it} - k_{it+1}}{l_{it}}, \\ c_{it+1}^{O} &= \frac{\tau y_{it+1}}{l_{it}}. \end{aligned}$$

The sharing rule τ provides young workers with an incentive to agree on the retention of earnings for the purposes of investment. It should be clear from the equations above that if $\tau = 0$ young workers will wish to set k_{it+1} to 0 as well. This is the Furubotn-Peyovich critique of traditional cooperatives as it manifests itself in our model. In our view this critique largely explains why many traditional cooperatives tend to remain small over their life cycle, while those with post-retirement attachment (such as those in the Mondragon system) tend to flourish.

In the main text we treat τ as a constitutional principle of the cooperative, which entitles former workers to keep sharing in the coop's distributions. However in Appendix C.9 we generalize our model to nest an intergenerational game in which, in each period, the decisions whether to honor the payment τ is taken optimally by the young. The construct is in the spirit of a literature on endogenous pay-as-you-go social-security systems, conceived as time consistent equilibria in infinite-horizon intergenerational games (e.g. Kandori (1992), Cooley and Soares (1999)). In those models, as in ours, these transfers from the young to the old are supported by trigger strategies: young workers not making the transfer forego access to the transfer themselves when old. The difference from that literature is that our arrangement is essentially a *within firm* pay-as-you go system - rather than a society-wide one. We show that *all* of the analysis presented in the text of the paper is robust to the generalized version with endogenous τ. In particular, the cooperative problem is subject to an additional constraint which ensures adherence by the young to the payment τ. We state conditions under which this constraint is not binding and verify that these hold in the quantitative analysis.

We focus on symmetric equilibria in which cooperatives adopt a perfectly egalitarian pay structure (within current workers and within former workers). We conjecture that standard arguments used in the context of capitalist economies could still be deployed to rule out equilibria with inequality within generations. In particular, workers receiving below-average pay in one cooperative could offer to undercut workers receiving above-average pay at another cooperative.

3.4.2 Continuation, Entry and Exit

The previous subsection describes the consumption of workers allocated to a *continuing* incumbent cooperative, which in our framework is an incumbent cooperative which is allocated some positive young membership l_{it}.

Our model also allows for entry and exit of cooperatives. Entering cooperatives have no capital stock, so they produce with labour only. They also have no former workers. Hence, young-worker consumption in an entering cooperative is $c_{0t}^Y = \frac{F(0,l_{0t})-k_{0t+1}}{l_{0t}}$ - where we use the subscript 0 for workers belonging to, or inputs and outputs of, entering cooperatives.

An exiting cooperative at time t is an incumbent which is assigned no workers by the worker-allocation mechanism. Such a cooperative produces zero output and its capital stock is left idle. Because of full depreciation this cooperative does not continue to period $t+1$. Note that the consumption of old workers attached to exiting cooperatives is 0.

3.4.3 General Equilibrium Concept for Cooperative Economies

We now discuss, jointly, how labour is allocated to cooperatives and how cooperatives make their investment decisions. Informally, we have in mind a decentralized mechanism in which workers are able to move freely into cooperatives, as long as these are willing to accept them. Therefore, workers sort into the cooperatives which generate highest utility levels until the market clears. This process takes into account the possibility that groups of workers might create a new cooperative without any initial capital. On the other hand, any remaining cooperative without any worker willing to join exits. Once workers have been allocated to cooperatives and production has taken place, workers collectively decide on the amount of earnings that should be retained to put in place as capital for the next period. In making this decision workers take into account the implications of the worker-allocation mechanism for the number of young workers joining the cooperative in that period.

Formally, in each period, the economy is characterised by a set of incumbent cooperatives I_t, and by a distribution of initial capital stocks: $\{k_{it}\}_{i \in I_t}$. For convenience, we assume the set of cooperatives is located on a continuum, and that in each period the set of incumbents I_t is a subset of the real line with finite Lebesgue measure. Denote $\bar{I} = \mathbb{R}$, with the interpretation that $\bar{I} \setminus I_t$ is the set of potential entrants. An arbitrary allocation of workers is a measurable function $l : \bar{I} \to \mathbb{R}_+$ with support of finite measure, such that:

$$\int_{\bar{I}} l_i di = L.$$

Note that entry and exit are captured by the fact that the support of l is not restricted to coincide with I_t. Note also that we impose full employment, assuming that any group of unemployed workers would optimally create a new cooperative.[16] The set of all such allocations is denoted $\mathfrak{L}$. Our relevant equilibrium object is a worker allocation mechanism, that is a mapping:

$$(I_t, \{k_{i,t}\}_{i \in I_t}) \mapsto \mathbb{L}(I_t, \{k_{i,t}\}_{i \in I_t}) \in \mathfrak{L}.$$

Though the state of the economy $(I_t, \{k_{i,t}\}_{i \in I_t})$ is a complex object, we emphasise that we restrict attention to stationary worker allocation mechanisms.

Given such a mechanism and any current allocation of workers, the continuation of the economy is characterised by optimal investment decisions taking as given the behaviour of other cooperatives. Denote $\mathcal{U}_i(l)$ the continuation utility of the young workers assigned to cooperative $i \in \bar{I}$ by allocation l, with the convention that $\mathcal{U}_i(l) = -\infty$ if i is not allocated workers by l. That is, for an incumbent cooperative:

$$\mathcal{U}_i(l) = \max_{k_{i,t+1}} U\left(\frac{(1-\tau)F(k_{i,t}, l_i) - k_{i,t+1}}{l_i}, \frac{\tau F(k_{i,t+1}, \mathbb{L}_i(I_{t+1}, \{k_{j,t+1}\}_{j \in I_{t+1}}))}{l_i}\right),$$

[16] We do not mean to suggest that a cooperative economy would be less (or more) prone to some frictional unemployment than a private-ownership economy. Our omission of search frictions is purely to focus on long-run analysis, as is standard in growth theory. We regard the addition of search and matching frictions to a model of a cooperative-based economy as an interesting area of research to learn more about the business-cycle properties of these economies. Such an endeavor would be particularly fruitful since, as mentioned earlier, empirical evidence suggests that employment is less cyclical in worker cooperatives.

while for an entering cooperative:

$$\mathcal{U}_i(l) = \max_{k_{i,t+1}} U\left(\frac{F(0,l_i) - k_{i,t+1}}{l_i}, \frac{\tau F(k_{i,t+1}, \mathbb{L}_i(I_{t+1}, \{k_{j,t+1}\}_{j\in I_{t+1}}))}{l_i}\right).$$

The last two expressions define the utility level workers can rationally expect by joining the various cooperatives in the economy. At the same time, they provide information to workers who have joined a particular cooperative about the consequences of allowing further workers to join in. Hence, we can use these objects to define an equilibrium as one in which there exist no reallocation in which the transfer of a worker to a different cooperative makes *both* this worker *and* the original members of this cooperative better off.

Formally, for any state $(I_t, \{k_{\iota,t}\}_{\iota\in I_t})$, any allocation $l \in \mathfrak{L}$, and any two cooperatives $i, j \in \bar{I}$, if $l_i < \mathbb{L}_i(I_t, \{k_{\iota,t}\}_{\iota\in I_t})$ and $l_j > \mathbb{L}_j(I_t, \{k_{\iota,t}\}_{\iota\in I_t})$, then:

$$\mathcal{U}_i\Big(\mathbb{L}(I_t, \{k_{\iota,t}\}_{\iota\in I_t})\Big) \geq \mathcal{U}_i(l), \tag{3.1}$$

and

$$\text{either} \quad \mathcal{U}_i\Big(\mathbb{L}(I_t, \{k_{\iota,t}\}_{\iota\in I_t})\Big) \geq \mathcal{U}_j(l), \tag{3.2}$$

$$\text{or} \quad \mathcal{U}_j\Big(\mathbb{L}(I_t, \{k_{\iota,t}\}_{\iota\in I_t})\Big) > \mathcal{U}_j(l). \tag{3.3}$$

In words, we are considering a feasible reallocation of workers from cooperative i to cooperative j. Condition (3.1) says that in an equilibrium this reallocation must not be beneficial to the remaining workers of cooperative i (or these workers would wish to reduce the membership). Furthermore, either the reallocation does not make the reallocated workers better off [condition (3.2)], or it makes the workers of the receiving cooperative worse off [condition (3.3)]. Note that the subscripts i and j can equally apply to continuing, entering, and exiting cooperatives.

Our equilibrium concept for a dynamic cooperative economy has elements in common with equilibrium concepts in cooperative game theory as well as in models of matching. Cooperative game theorists study coalition formation and typically seek stable coalition structures which, like in our model, are robust to defection from subsets of agents.[17] However in our model the "coalition formation game" is re-played in every

[17] Indeed the tools of cooperative game theory have been deployed for the study of (static) cooperative economies. See, e.g., Ichiishi (1977), Greenberg (1979), Drèze and Greenberg (1980), and Farrell and Scotchmer

period by a new set of agents and, more importantly, the entire distribution of investment decisions taken by the coalitions that exist at time t operate as state variables for the time $t+1$ game, and in turn this game's outcome is payoff relevant for agents making decisions at time t. In this sense, the coalition-formation aspect of the model is much more complex than in typical cooperative games, and the definition of equilibrium had to be generalized accordingly. This is compensated to a considerable extent by the fact that we work with a homogeneous-agent model.

In the previous paragraph the first reference to a "coalition formation game" was hedged by quotation marks, because this terminology is arguably slightly misleading. In typical cooperative games coalitions are formed in a sort of vacuum, and the output of the coalition depends exclusively on its size and composition. In our model, however, workers form coops around and inside existing lumps of capital. They don't so much form coalitions but they attach themselves to an existing coop – represented by the stock of capital inherited from the past (and its former workers). In this sense, our equilibrium concept is as much about forming coalition as it is about matching workers to incumbent coops – hence the link with the matching literature. Compared to the matching literature, however, we offer a somewhat axiomatic definition of equilibrium (based on stability from deviations) rather than the more standard description of a search environment.[18]

3.4.4 Operational Equilibrium Concept for Cooperative Economies

The general concept of equilibrium in the previous section is inspired by minimal requirements of rationality and efficiency. Needless to say, these general principles are hardly sufficient as a basis for a study of economic growth in a cooperative economy. What is needed is a more operational refinement allowing us to focus on a subset of equilibria which are tractable for the modeller, and do not impose unrealistic information requirements on the agents in the model. In particular, the generic decentralization of the equilibrium definition in the previous section requires knowledge by each agent of the strategies of all agents in all future generations. This is in sharp contrast to the equilibrium in the capitalist economy where agents need only know current wages and interest rates.

(1988).

[18] See, however, Sasaki and Toda (1996), and a small following literature on matching with externalities. Our mechanism to assign workers to coops is similar to the concept of *Optimistic Stability* in the working paper version of their article.

The particular restriction we impose on our equilibria is as follows: the worker allocation mechanism assigns to each incumbent cooperative a number of workers which depends only on that cooperative's capital stock k_{it}. Formally, we only consider equilibria in which, for $t > 0$, there is a mapping $\mathcal{L}(k_{jt})$ such that $\mathbb{L}_j\big(I_t, \{k_{it}\}_{i \in I_t}\big) = \mathcal{L}(k_{jt})$ for $j \in I_t$. Note that the above is a statement about the allocation of workers only to incumbent cooperatives on path. We do not impose restrictions on the allocation of workers to entering cooperatives.

It can easily be seen that if $\mathcal{L}(k_{jt})$ is an allocation mechanism in an equilibrium as defined in the previous section, then each cooperative has an investment policy rule which also depends only on that cooperative's capital stock, $\mathcal{K}(k_{jt})$. Furthermore, in Appendix C.2 we establish the following hugely useful property of $\mathcal{L}(k_{jt})$ and $\mathcal{K}(k_{jt})$:

$$\big(\mathcal{L}(k_{jt}), \mathcal{K}(k_{jt})\big) \in \arg\max_{l,k} U\left(\frac{(1-\tau)F(k_{jt}, l) - k}{l}, \frac{\tau F\big(k, \mathcal{L}(k)\big)}{l}\right). \tag{3.4}$$

In words, focusing only on equilibria in which an incumbent's allocation of workers depends only on that incumbent's initial capital stock is equivalent to focusing on equilibria in which each incumbent cooperative chooses current employment and investment so as to maximize the utility of current young workers, taking as given the fact that all future generations will follow the same strategy. Importantly, this maximization is unconstrained.

It is important to stress some implications and limitations of our operational equilibrium concept. Equation (3.4) implies that, for $t > 0$, incumbent cooperatives are never constrained in the number of members they can attract, i.e. we are implicitly ruling out growth paths along which cooperatives would like to attract more members, but are prevented from doing so because all workers are already "taken up" by other coops. Nevertheless, in Sections 3.5 and 3.6 we show by example that under standard growth-theoretic assumptions about preferences and technology equilibria fulfilling our operational concept emerge naturally.

Importantly, in our operational equilibrium concept the independence of the labour allocation from the full distribution of capital stocks to incumbents only applies for $t > 0$. Hence, we allow for the possibility that, at time 0, there are "too many coops for too few workers," in the sense that the unconstrained optimal membership of at least some coops exceeds the number of members the coop can attract. In the examples we work out later, we will see that this can result in a burst of exit at time 0. The possibility of exit at time 0 due to insufficient access to workers is generally useful because it makes the existence of equilibria independent of the initial distribution and size of the capital stock, and thus makes it potentially possible to study "MIT-type" shocks, i.e. unanticipated permanent changes in endowments or technology.

Another important feature of our operational equilibrium concept is that it is fully consistent with entry and imposes no restriction on the allocation of workers to cooperatives - other than the restriction imposed by the aggregate labour supply. In particular, it must be the case that:

$$\forall t, \quad \int_{I_t} \mathcal{L}(k_{it}) di \leq L.$$

Therefore, in any period, once incumbents have been allocated workers, new entering cooperatives are created and allocated workers. This allocation of workers to new cooperatives follows the restrictions imposed in section 3.4.3, and in particular takes into account the economy's resource constraint in terms of labour supply. Importantly though, cooperatives may be constrained only upon entry, but expect to be allocated workers as incumbents in future periods according to the mapping $\mathcal{L}$.

In an equilibrium as defined in this section all behaviour is pinned down by an initial distribution of capital stocks and the mappings $\mathcal{L}$ and $\mathcal{K}$. The capital accumulation dynamics within a cooperative are pinned down by the equilibrium mapping:

$$k_{jt+1} = \mathcal{K}(k_{jt}).$$

As cooperatives in the economy may differ only in their capital stock, we can then easily study aggregate dynamics as resulting from the sum of individual independent decisions using the same mapping $\mathcal{K}$. The precise algorithm we follow to solve for the equilibrium is detailed in Appendix C.3.

3.5 An Example with Closed Forms

In this section, we use specific functional forms for preferences and the production technology which allow us to characterise analytically employment as well as the capital accumulation dynamics both in the capitalist and cooperative economies. We use these results to compare the two economies in terms of output, efficiency, and welfare.

The production function for production units with positive inputs ($k, l > 0$) takes the form

$$F(k,l) = Ak^{\alpha}(l - \underline{l})^{\beta}, \tag{3.5}$$

where $A > 0$, $\underline{l} \in (0, L)$, $\alpha > 0$ and $\beta > 0$ are constant parameters.

Relative to the familiar neoclassical growth model, this production function features the slightly unusual property that there is a fixed cost, in the form of a minimum of $\underline{l}$ units of labor which are required independently of the scale of operation. This assumption is a direct legacy of the older static literature on cooperatives, which showed that in the absence of a fixed cost of production there is no equilibrium with positive cooperative size.[19] The intuition will be apparent below. Needless to say the assumption that production involves fixed costs is entirely realistic.

Since fixed costs of production introduce a form of increasing returns to scale, in order for the model to have an equilibrium under the capitalist form of organization we need decreasing returns to scale in the variable inputs, i.e.

$$\alpha + \beta < 1.$$

The assumption of decreasing returns to variable inputs is also realistic, and it is usually motivated by span-of-control considerations.

Since the cooperative model features potential entry, we will also need an assumption for production in production units with $k = 0$. However, it will turn out that we do not need a specific functional form. Hence, for now we simply assume that $F(0, l) > 0$. We will add some mild restrictions to this below.

As for preferences, in order to derive closed form results we assume for now that agents obtain log-utility from consumption, with a discount factor $\delta \in (0, 1]$:

$$U(c^Y, c^O) = \log c^Y + \delta \log c^O.$$

[19] More accurately the existence of cooperatives requires that at low levels of membership the marginal product of labour exceeds average income. In models of capitalist economies the omission of fixed costs of production is without loss of generality due to the replication argument. This is not the case in modelling cooperatives.

3.5.1 Capitalist Economy

Using these functional forms, we can solve for the capitalist equilibrium as outlined in section 3.3. The procedure to find the equilibrium is entirely standard and hence we relegate the details to Appendix C.4. Here we only discuss the main aspects of the equilibrium.

The only slightly unfamiliar feature of the capitalist equilibrium is that, because of the fixed production cost, it features an optimal firm size:

$$l_{cap} = \frac{1-\alpha}{1-\alpha-\beta}\underline{l}, \tag{3.6}$$

where the subscript cap will be helpful later to distinguish firm size in a capitalist equilibrium from firm size in the cooperative economy. The optimal firm size would generally depend on state variables, such as the aggregate capital stock. This will be the case in the example in the next section. However, under the particular combination of functional forms in this section, the optimal firm size is constant over time both under capitalist and under cooperative arrangements. It is this constancy that allows us to solve the model in closed form, and hence it is a valuable simplification.

Despite this slightly unfamiliar feature the dynamics of the economy are qualitatively the ones we have come to expect from standard growth models. In particular, individual capital holdings evolve according to

$$\kappa_{t+1} = \frac{\delta}{1+\delta} A(1-\alpha)^{\alpha}\beta^{\beta}\Big(\frac{1-\alpha-\beta}{\underline{l}}\Big)^{1-\alpha-\beta}\kappa_t^{\alpha}, \tag{3.7}$$

where, recall, κ_t is the savings decision by a member of the period t young. It follows from this functional form that κ_t converges to a steady-state value.

3.5.2 Cooperative Economy

We study the cooperative economy following the approach presented in section 3.4.4. We follow a "conjecture and verify" strategy. The conjecture is that in the equilibrium, if one exists, cooperative firm size is constant, or $\mathcal{L}(k) = l_{coop}$. If this is so, then the cooperative solves the problem:

$$\max_{l_t, k_{t+1}} \log\left(\frac{(1-\tau)Ak_t^{\alpha}(l_t-\underline{l})^{\beta}-k_{t+1}}{l_t}\right)+\delta\log\left(\frac{\tau Ak_{t+1}^{\alpha}(l_{coop}-\underline{l})^{\beta}}{l_t}\right). \tag{3.8}$$

The necessary and sufficient first-order conditions for this problem are:

$$-\frac{1}{(1-\tau)Ak_t^{\alpha}(l_t-\underline{l})^{\beta}-k_{t+1}}+\frac{\alpha\delta}{k_{t+1}}=0, \tag{3.9}$$

$$\frac{\beta(1-\tau)Ak_t^{\alpha}(l_t-\underline{l})^{\beta-1}}{(1-\tau)Ak_t^{\alpha}(l_t-\underline{l})^{\beta}-k_{t+1}}-\frac{1+\delta}{l_t}=0. \tag{3.10}$$

Equation (3.9) describes the optimal reinvestment of earnings. The first term is the marginal utility loss from diminished current consumption from an extra unit of investment, while the second term is the marginal utility gain from the extra output that investment will deliver next period. First order condition (3.10) determines the optimal current employment level l_t. Here the trade-off is that an extra worker has a positive marginal impact on current output (first term) but also a negative marginal impact on the share of other workers both in the current period and in the next period, both of which effects are captured in the second term.

This system is easy to solve and yields:

$$\begin{aligned} l_t &= \frac{1+\delta}{1+\delta-\beta(1+\alpha\delta)}\underline{l} \equiv l_{coop}, \\ k_{t+1} &= \frac{\alpha\delta}{1+\alpha\delta}(1-\tau)Ak_t^{\alpha}(l_t-\underline{l})^{\beta}. \end{aligned} \tag{3.11}$$

The first of these two equations shows that, when expecting a constant labour input in the next period, cooperatives choose a constant labour input in the current period. This both verifies our conjecture and defines the equilibrium cooperative size, l_{coop}. The second equation characterizes the investment policy of cooperatives. This policy inherits the conventional proportionality to current income associated with log utility. Plugging in the form of $l_t = l_{coop}$, we obtain the capital accumulation equation for a single cooperative:

$$k_{t+1}=\frac{\alpha\delta}{1+\alpha\delta}(1-\tau)A\left(\frac{\beta(1+\alpha\delta)}{1+\delta-\beta(1+\alpha\delta)}\underline{l}\right)^{\beta}k_t^{\alpha}, \tag{3.12}$$

which has the same qualitative features as those derived for the capital accumulation process of individuals in the capitalist economy. We define k^*_{coop} the steady state cooperative capital implied by (3.12). For later reference we also define $\mathcal{U}(k_{it})$ as the maximized value of (3.8). It is trivial (but important) to see that $\mathcal{U}(k_{it})$ is an increasing function: workers prefer joining incumbents with larger capital stocks.

To move now to a full characterization of the dynamics of the economy, as well as to complete the argument that the equilibrium sketched thus far exists, we must now consider the possibility of entry. It is easy to see that, in the equilibrium we are constructing, the allocation of labour to an entrant and the entrant's investment policy must maximize the objective

$$\log\left(\frac{F(0,l)-k}{l}\right)+\delta\log\left(\frac{\tau Ak^{\alpha}\left(l_{coop}-\underline{l}\right)^{\beta}}{l}\right). \tag{3.13}$$

Note that this problem is time invariant, so both entry size and the utility afforded to a young worker who helps forming a new cooperative are also time invariant. To facilitate the discussion of dynamics we label $\mathcal{L}_e$ the size of an entrant, $\mathcal{K}_e$ its investment policy, and $\mathcal{U}_e$ the utility experienced by a worker joining an entrant.

At any time t, it may conceivably be the case that $\mathcal{U}_e > \mathcal{U}(k_{it})$ for some incumbents i with sufficiently low capital stock. In this case, these incumbents will not be able to attract any workers and will have to exit. We define as $I_t^+ \subseteq I_t$ the set of incumbents at time t such that $\mathcal{U}_e \leq \mathcal{U}(k_{it})$. We can think of I_t^+ as the set of *viable* incumbents. As we will soon see, the key assumption we need to make to insure the existence of a cooperative equilibrium fulfilling put operational criteria is that $\mathcal{U}_e \leq \mathcal{U}(k^*_{coop})$. In other words, an incumbent endowed with the steady state level of capital is viable. We refer to this as Assumption 1.[20]

Define

$$N_{coop} \equiv \frac{L}{l_{coop}}$$

as the measure of incumbent cooperatives consistent with full employment when each cooperative operates at its optimal size l_{coop}. The dynamics of the economy, as well as the further assumptions (if any) required to establish the existence of the equilibrium, are slightly different in the case in which N_{coop} is smaller or larger than the initial endowment of viable cooperatives, $|I_0^+|$, where we use $|x|$ for the measure of set x.

Case 1: $N_{coop} \leq |I_0^+|$

In this case the N_{coop} incumbents with the largest capital stocks will scoop up all the

[20]This is an assumption that $F(0,l)$ is not too productive. If $F(0,l) = BG(l)$ one can always choose B low enough that Assumption 1 is verified.

workers in the economy at time 0, and each of them will employ l_{coop} workers. The $|I_0| - N_{coop}$ coops with the *smallest* capital stock (including some viable ones) will exit. No entry will occur as all continuing incumbents afford workers more utility. Moving to period 1, there are no non-viable incumbents. Those incumbents which had capital stock less than k^*_{coop} have experienced capital growth, so they are a fortiori viable in period 1. Even those incumbents which started in period 0 with capital in excess of k^*_{coop} are still viable in light of Assumption 1. Furthermore, since the existing viable incumbents are exactly N_{coop}, there are no workers left out and forced to create a new cooperative. Hence, there is neither entry nor exit, and the same is true in all subsequent periods. Hence, each coop's capital stock evolves according to (3.12), and eventually, *the entire measure* N_{coop} *of cooperatives converge to the identical steady state level* k^*_{coop}. Note that no further assumptions on $F(0,l)$ were required.

Case 2: $N_{coop} > |I_0^+|$

In this case the economy is not initially endowed with a measure of viable incumbents sufficient to absorb the entire young-worker population. Hence, while each viable incumbent will be assigned l_{coop} workers, there will have to be entry to employ the remaining $L - |I_0^+| l_{coop}$ workers. To fully describe the dynamics and establish existence of the equilibrium we then need further restrictions on $F(0,l)$. The first restriction (Assumption 2) is that the size of entrants is no less than the size of incumbents, or $\mathcal{L}_e \geq l_{coop}$.[21] The second restriction (Assumption 3) is that entrants become viable incumbents in the period after entry, or $\mathcal{U}(\mathcal{K}_e) \geq \mathcal{U}_e$.[22]

With these assumptions, consider first the special case in which $\mathcal{L}_e = l_{coop}$. In this case there will be exactly a measure $N_{coop} - |I_0^+|$ of entrants at time 0. From there, just as in Case 1, there is no further entry or exit, and each coop once again converges to the capital stock k^*_{coop}. If instead $\mathcal{L}_e > l_{coop}$, the size of period-0 entrants will drop to l_{coop} in period 1, necessitating a further round of entry in that period to insure full-employment. This pattern of residual entry and subsequent shrinkage will continue until the measure of entrants shrinks to 0. From then on no further entry or exit occurs and once again we

[21] It may seem counter-intuitive to have entrants which are larger than incumbents, but our intuitions are based on observations of capitalist economies. There is no empirical basis to form a prior on whether entering cooperatives would be larger or smaller than incumbent ones.

[22] A sufficient condition for Assumption 2 is that $F(0,l) = B(l - \underline{l}_e)^\gamma$, $\gamma \in (0, (1+\alpha)/(1+\alpha\delta))$, and $\underline{l}_e \geq [1 + \delta - \gamma(1+\alpha\delta)] / [1 + \delta - \beta(1+\alpha\delta)]\, \underline{l}$. This can be verified by substituting these assumptions into (3.13) and solving the maximization problem. If $\gamma = \beta$ and $\underline{l}_e = \underline{l}$ then $\mathcal{L}_e = l_{coop}$. As regards Assumption 3, if $F(0,l) = BG(l)$ one can always find a B small enough that the assumption is verified. Notice that since $\mathcal{L}_e$ does not depend on B there is no possible tension between Assumptions 1, 2, and 3.

converge to a steady state with N_{coop} identical cooperatives, all with capital k^*_{coop}.[23,24]

3.5.3 Comparison

In this section, we compare economic performance in the two models along two dimensions: (i) static organization of production and efficiency, and (ii) capital accumulation. We also show how differences in steady state output can be exactly decomposed into two terms reflecting differences in these dimensions. We also include a quantitative comparison as a prelude to the subsequent quantitative section, which uses more realistic preferences.

Firm Size and Static Efficiency

Consider the choices of a planner whose intention is to make the economy *statically efficient*, i.e. to maximize aggregate output *for a given aggregate stock of capital K*. Because of the concavity of the production function, the planner will distribute the capital and labor endowments equally across whatever number of production units she chooses to have, so her problem is equivalent to identifying the optimal firm size.[25] In this subsection we identify this statically-efficient firm size, and compare it to firm sizes in the capitalist and cooperative economies. Of course the welfare significance of static efficiency is limited, because overall efficiency also depends on the amount of capital in the economy, which in turn depends on dynamic considerations (which we take up in the next sub-section). Still, in our quantitative exercises differences in static efficiency turn out to play an important role.

Using our functional assumption, the statically-efficient firm size is the solution to

$$\max_{l} \quad A\frac{(l-\underline{l})^{\beta}}{l^{1-\alpha}}K^{\alpha}L^{1-\alpha}.$$

[23] To understand why Assumption 2 is needed consider the consequences of entrants having scale smaller than l_{coop}. These entrants would have to *grow* in size to satisfy the conjectured equilibrium property that all incumbents are allocated l_{coop} workers. But this is clearly incompatible with the labor resource constraint, because there are not enough workers in the economy to allow all entrants to grow to size l_{coop}. Similarly, a violation of Assumption 3 would imply re-exit at time 1 of cooperatives which entered at time 0, but this violates the equilibrium requirement that all incumbents have membership l_{coop} for $t > 0$.

[24] In the text we have implicitly assumed that incumbent firms do not have access to technology $F(0, l)$. This is clearly immaterial for Case 1. In Case 2 one could wonder whether non-viable incumbents might be able to avoid exit by switching to the labor-only technology. The answer is no, as any young worker joining an incumbent "inherits" the incumbent's stock of former workers and is thus subject to the sharing rule. She is thus always better off striking out with a new venture.

[25] Aggregate output is $NF(K/N, L/N)$, where N is the number of production units the planner chooses to have. Using $l = L/N$ this rewrites as $L/lF(lK/L, l)$, and static efficiency is achieved by maximizing this with respect to l.

As a result, the aggregate variables K and L do not affect the maximisation problem, and we can define

$$Z(l) = A\frac{(l - \underline{l})^{\beta}}{l^{1-\alpha}}$$

as a measure of static efficiency associated with any arbitrary firm size. Indeed, the social planner's objective is simply to maximise $Z(l)$ with respect to l. The larger $Z(l)$, the more statically efficient the economy. The socially optimal firm size trades off the following considerations: smaller firms allows the economy to spread variable inputs across more units, thereby reducing the impact of diminishing returns to variable inputs. On the other hand, the larger the measure of firms, the larger the amount of labor "wasted" because of the fixed cost $\underline{l}$.

The firm size l_{eff} (for "efficient") which maximizes $Z(l)$ is easily derived from the first order condition, and the verdict on static effciency is as follows:

$$l_{eff} = l_{cap} > l_{coop},$$

(where the last inequality is proved in Appendix C.5). Hence, the capitalist economy is statically efficient, but the cooperative economy features firm sizes which are inefficiently small.

There are two reasons why cooperatives are inefficiently small. First, unlike the social planner, cooperatives take their *current* capital stock as given. When they consider adding extra workers they only perceive the impact on the average product of labor. Instead, the social planner also takes into account that an extra worker increases the marginal product of capital, and that he can therefore counter the decline in the marginal product of labor by reallocating some extra capital to the production unit. The same happens in the capitalist economy, because extra workers induce the firm to rent extra capital.

The second reason why cooperatives are inefficiently small (in a static sense) is the existence of the sharing rule. An extra worker today is an extra claimant to the payments that will accrue to old workers tomorrow. This is why the firm size in the cooperative economy is decreasing in the weight agents give to old-age consumption, δ.[26,27]

Capital Accumulation and Dynamic Efficiency

Aggregating the capitalist law of motion (3.7) over individuals and the cooperative law of motion (3.12) over cooperatives, and making the appropriate substitutions, we easily see that both economies have laws of motion for the aggregate capital stock K_t of the form

$$K_{t+1} = sF(K_t, L),$$

with the corresponding aggregate savings rates

$$s_{cap} = \frac{\delta}{1+\delta}(1-\alpha), \tag{3.14}$$

and

$$s_{coop} = \frac{\alpha\delta}{1+\alpha\delta}(1-\tau). \tag{3.15}$$

It is worth discussing the qualitative similarities and differences between these two saving rates.

In both economies, a higher preference for the future increases the saving rate – which is hardly surprising. However, a higher elasticity of output to capital reduces the saving rate in the capitalist economy, while it increases it in the cooperative economy. In the capitalist economy all savings are financed out of labor income, so a larger capital share reduces resources available for saving. In the cooperative economy, the share of income received by young workers is $1-\tau$, independent of α. However, workers in the cooperative economy internalize the concavity of the production function. The less steeply the marginal product of capital declines with the capital stock (i.e. the higher is α) the more they wish to invest.[28]

[26] We can confirm these intuitions by considering the case $\alpha = \delta = 0$, i.e. when labor is the only input and agents discount old age completely. In this case, we can readily check that $l_{coop} = l_{cap} = l_{eff} = \underline{l}/(1-\beta)$. Firm size in all scenarios depends exclusively on how rapidly diminishing returns to labor set in (the more so, the smaller the firm size). If $\delta = 0$ but $\alpha > 0$, firm size in the cooperative economy is still $\underline{l}/(1-\beta)$, which maximizes firm output per worker keeping firm capital constant, but the efficient and capitalist firm size is the larger expression we have derived above, and is increasing in α. Finally, if $\alpha = 0$ but $\delta > 0$ the efficient size is $\underline{l}/(1-\beta)$, but the cooperative size drops to $\underline{l}(1+\delta)/(1+\delta-\beta)$.

[27] An additional known reason why cooperative size may be inefficiently small is worker heterogeneity in the presence of strictly egalitarian pay rules (Farrell and Scotchmer, 1988; Levin and Tadelis, 2005).

[28] The following stylized version of the problems faced by workers in the two economies further clarifies this point. In the capitalist economy workers essentially maximize $\log(w-k) + \delta\log(rk)$, which as is well known means that r, and hence α, is irrelevant to the chosen level of k, since income and substitution effect cancel each other out. Instead, if workers maximize $\log(w-k) + \delta\log(Ak^{\alpha})$ the solution will directly depend on α and indeed it is clear that the term $\alpha\delta$ will be critical. Outside of the log case (e.g. in the next section), α affects

It is well established that capitalist economies can exhibit dynamic inefficiency, in the sense that a reduction in saving can improve the consumption and hence the welfare of all generations. This of course applies to the capitalist version of the economy studied here. But can the cooperative economy also be dynamically inefficient?

The standard analysis of dynamic efficiency begins by establishing a *golden rule* level of the capital stock (or, equivalently, of the saving rate) which maximizes total consumption (the sum of the consumption of the young and of the old) subject to enough output being reinvested to keep the total capital stock constant. In our context this problem can be stated as

$$\max_{K,N} c^Y + c^O$$

subject to

$$NF\left(\frac{K}{N}, \frac{L}{N}\right) = L(c^Y + c^O) + K.$$

Now it is clear that, for any K, the optimal N in the problem just stated must be the output-maximizing one which we identified in the previous subsection. Using this and maximizing with respect to K we find the familiar Cobb-Douglas golden rule $K = \alpha Y$. It follows from comparison with (3.15) that *the cooperative economy can never be dynamically inefficient* as $s_{coop} < \alpha$. (Comparison with (3.14) confirms that the capitalist economy can be.)

The intuition is closely linked to our discussion of saving in the two economies in the earlier part of this subsection. As is (now) well understood, in the capitalist economy the potential dynamic inefficiency is due to a pecuniary externality: the young do not internalize the fact that by increasing saving they lower the return to capital for everyone.[29] In contrast, as we have seen, young cooperative members fully take into account the consequences of their accumulation decision on the marginal product of capital, and this prevents them from over-accumulating.[30]

individual saving decisions in the capitalist economy as well, indirectly through r.

[29] Acemoglu (2009, pp. 338-339) discusses the evolution of thinking about the sources of dynamic inefficiency in OLG economies.

[30] It is well known that introducing a pay-as-you-go social security system in a capitalist OLG economy can reduce excess savings and, depending on the quantitative strength of this effect, lessen the risk of dynamic inefficiency. Since our cooperatives operate an internal pay-as-you go system, it may be tempting to interpret our finding that they are dynamically efficient as arising from the same mechanism. But this would be inaccurate: in the capitalist economy the reduction in savings occurs simply because the existence of the system reduces the young workers' perceived need (and income available) to save, and thus depends on the size of the social-security tax. In the cooperative economy, as discussed, dynamic efficiency arises from the internalization of the effect of investment on the rate of return, and is independent of the size of the transfer τ.

Table 3.1: Calibrated parameters

Concept	Parameter	Target	Data	Value for $\sigma = 1$	Value for $\sigma = 2$
Capital share	α	rK/Y	0.33	0.33	0.33
Variable labor elasticity	β	$\underline{l}/l_{cap}$	0.18	0.55	0.55
Discount rate	δ	K/Y	3/25	0.22	0.13
Sharing Rule	τ	Max U		0.12	0.15

Note: σ is the elasticity of intertemporal substitution in preferences over consumption paths (see Equation (3.17)), so $\sigma = 1$ is the log-utility case.

Steady State Output

If an economy with our Cobb-Douglas technology features a steady state in which all firms are identical and operate with constant inputs k^* and l^*, then steady-state aggregate output per worker can be written as

$$\frac{Y^*}{L} = (s^*)^{\frac{\alpha}{1-\alpha}} (Z^*)^{\frac{1}{1-\alpha}} , \tag{3.16}$$

where Z is the measure of *static efficiency* we derived in Section 3.5.3, and $s^* = K^*/Y^*$ is the saving rate in steady state.[31]

The interpretation is straightforward after the discussions in the last two subsections. An economy's steady-state output per worker is driven by two factors: how efficiently it produces in a static sense, and how much it saves.

Under our log-utility assumption both the capitalist and the cooperative economy have constant saving rates and constant firm sizes l, the latter implying constant Zs. We have seen that we cannot sign the difference between s_{coop} and s_{cap}, and that $Z_{coop} \leq Z_{cap}$. Despite this disadvantage, because cooperative economies could potentially save at a higher rate, which economy has a higher output per worker is a quantitative matter.

Quantification

In this subsection we calibrate the log-utility economy for a first set of quantitative insights on the comparison between capitalist and cooperative economies. In the next section we quantify an example with more realistic preferences (but no closed-form solutions).

We do not observe a cooperative-based economy but we do observe economies which

[31] To see this write aggregate output as $Y^* = \frac{L}{l} F(k^*, l^*)$. The capital input of each individual firm is given by: $k^* = \frac{l^*}{L} K^* = \frac{l^*}{L} s^* Y^*$. Plugging this into Y^* and using the functional form for F we get the decomposition in the text.

are broadly organized according to capitalist principles. Hence, we calibrate the parameters of the model so that the capitalist economy in steady-state matches corresponding moments of the US economy in recent decades. Table 3.1 summarises the values chosen for the parameters of the model in the column titled "Value for $\sigma = 1$" (σ being defined later as the intertemporal elasticity of substitution in consumption, and hence $\sigma = 1$ being the log case). The parameter α maps as usual into the share of capital in national income. Given α, a choice of β in (3.6) uniquely determines the share of fixed labour in total firm employment, $\underline{l}/l_{cap}$. We match this to the share of non-production workers in the economy, from the Bureau of Labor Statistics. Finally, given α a choice of δ uniquely determines the saving rate in the capitalist economy, and in turn this saving rate equals the capital-output ratio in steady state, for which we use the standard value of 3 (with an adjustment for a putative 25-year duration of a model period.) The parameter τ is unique to the cooperative economy and thus cannot be calibrated on any kind of data. Hence, we select the value of τ that maximises steady-state lifetime utility of the representative consumer in the cooperative economy.[32] Simple calculations show that this value is:

$$\tau = \frac{\delta}{1+\delta}(1-\alpha).$$

(This happens to also be the saving rate in the capitalist economy - but we do not have a compelling intuition for this coincidence.)[33]

The implications of this calibration are presented in Table 3.2. First, cooperatives are only around a third as large as capitalist firms, or $l_{coop}/l_{cap} = 0.35$. This large size difference implies a significant disparity in static efficiency: $(Z_{coop}/Z_{cap})^{1/(1-\alpha)} = 0.78$. Second, the cooperative economy also saves half as much as the capitalist one, as we have $s_{cap} = 0.12$ and $s_{coop} = 0.06$. (Note that the capitalist economy is dynamically efficient). Hence, the contribution of saving to the output gap is $(s_{coop}/s_{cap})^{\frac{\alpha}{1-\alpha}} = 0.71$. When combined, the static inefficiency and the lower saving rate of the cooperative economy imply that steady state output per worker is 55% of steady state output per worker in the capitalist economy.

We can also evaluate the welfare consequences of a transition to a cooperative econ-

[32]In the version of the model with endogenous τ presented in the Appendix, there is a continuum of values of τ which can be sustained in equilibrium. The implicit assumption in our calibration is thus that the first generation of young who set the value of τ do so with steady-state welfare in mind.

[33]The fixed cost $\underline{l}$ cancels out in all the ratios of capitalist-to-cooperative outcomes we wish to present, so it does not need to be calibrated here. Similarly, there is no need to choose values for the size of the population L and the productivity factor A.

Table 3.2: Numerical results

	Value for $\sigma = 1$	Value for $\sigma = 2$
$\left(\frac{Z_{coop}}{Z_{cap}}\right)^{\frac{1}{1-\alpha}}$	0.78	0.69
$\left(\frac{s_{coop}}{s_{cap}}\right)^{\frac{\alpha}{1-\alpha}}$	0.71	1.05
$\left(\frac{Y_{coop}}{Y_{cap}}\right)$	0.55	0.73

Note: σ is the elasticity of intertemporal substitution in preferences over consumption paths (see Equation (3.17); $\sigma = 1$ is the log-utility case).

omy. In particular, we can compute the total amount of consumption which a worker in the cooperative economy would need to be given to obtain the same utility as in the capitalist economy. This calculation is made on the assumption that the worker is free to allocate this transfer as she wishes over her lifetime. In steady state, this welfare loss as a percentage of GDP in the coop economy is 51% (in the current log case).[34]

3.6 An Example with Numerical Computations

The assumption of log preferences in the previous section was extremely useful in deriving a closed-form characterisation of the equilibrium, and analytical formulas to compare steady states in the cooperative and capitalist economies. However, most macroeconomic applications use

$$U(c^Y, c^O) = \frac{(c^Y)^{1-\sigma}}{1-\sigma} + \delta \frac{(c^O)^{1-\sigma}}{1-\sigma}, \tag{3.17}$$

with an elasticity of intertemporal substitution σ closer to 2. It turns out that, if σ is exactly 2, we can still produce analytical solutions for the capitalist steady state, which is extremely useful for calibration purposes. Hence, this is the case we study in this section.

The competitive equilibrium in the capitalist economy is unaffected by the assumption on preferences, which only affects the saving rule. As a result, the derivation of equilibrium prices and number of firms in the previous section is still valid. Importantly,

[34]We can also compute the dynamic path of output and welfare following the introduction of the cooperative organisation of production. Suppose that we begin with a capitalist economy initially in steady-state. At some initial period, all capital is seized and distributed equally to $N = L/l_{coop}$ cooperatives. Old agents in the initial period receive a share τ of output. From then on, the economy evolves according to our model of cooperative economy. Under our baseline calibration, relative output of the cooperative economy steadily and gradually declines towards its steady state level, while the compensation required by the current generation to be as well off as in the steady state of the capitalist economy steadily rises towards the steady state.

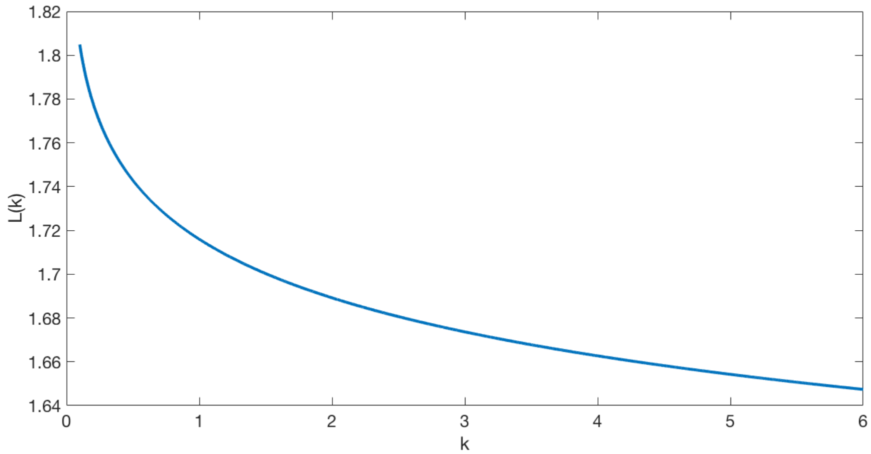

Figure 3.1: Cooperative labour input as a function of initial capital stock – numerical solution to problem (3.4).

this implies that firm size is still constant and takes the same value derived above, l_{cap}. Among other things this means that α and β do not need to be re-calibrated.

In Appendix C.6 we study the consumption-saving decision of young workers in the capitalist economy. We show that the capitalist economy converges to a steady state, and, for the case $\sigma = 2$, the aggregate steady state saving rate is

$$s^*_{cap} = \frac{4(1-\alpha)}{\left(\left(\frac{\alpha}{\delta(1-\alpha)}\right)^{1/2} + \left(4 + \frac{\alpha}{\delta(1-\alpha)}\right)^{1/2}\right)^2}.$$

As in the previous example, $s^* = K^*/Y^*$, and as we already have a calibration for α, this equation can be used to re-calibrate δ, as reported in the last column of Table 3.1.

While we have closed form characterizations of the (steady state of the) capitalist economy, for the cooperative economy we must proceed numerically. We begin as before with the problem of an incumbent. In particular, we use policy function iteration to find a (numerical) fixed point for the mapping $\mathcal{L}(k)$ which solves problem (3.4). This is done using the already calibrated α, β, and δ, as well as normalized values for $\underline{l}$, A and L. (Appendix C.7 shows that comparison of steady state values among the two economies is independent of $\underline{l}$. That A and L can be normalized is obvious.) We also re-calibrate τ, to maximise, as before, the steady-state lifetime utility of the representative agent under the new preferences (and the new value of δ).

The policies $\mathcal{L}(k)$ and $\mathcal{K}(k)$ implied by our calibration are plotted in Figures 3.1 and

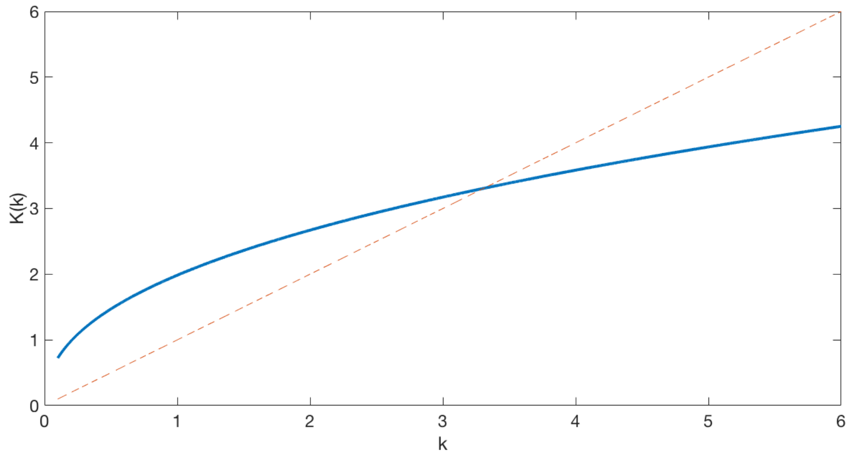

Figure 3.2: Cooperative capital investment as a function of initial capital stock – numerical solution to problem (3.4).

3.2. $\mathcal{L}(k)$ is decreasing in k, while $\mathcal{K}(k)$ is increasing and concave. This last property implies that there exists a steady state for the cooperative economy in which all coops have the same capital stock k^*, the same membership $\mathcal{L}(k^*)$, and their measure is $L/\mathcal{L}(k^*)$.[35]

Given the existence of a steady state where identical incumbents maximize the unconstrained -cooperative problem, and given the allocation criterion $\mathcal{L}(k)$ and the investment function $\mathcal{K}(k)$, sufficient conditions for convergence to this steady state can be identified using a reasoning similar to the one we used in Section 3.5.2. In particular, if (i) all initial incumbents start with a capital stock $k_{i0} \leq k^*$; (ii) in any period, entrants' worker allocation l_e and optimal investment k_e satisfy (a) $l_e \geq \mathcal{L}(k_e)$ and (b) $k_e \leq k^*$; and (iii) one-period-old coops are viable; then every coop's capital stock grows over time towards the steady-state level, while every coop's labour input decreases over time towards the steady-state, generating entry but no exit.

Recall now that that decomposition (3.16) is valid for any economy featuring a steady state with identical firm sizes, and is thus still valid – with the same interpretation – in the current example. The terms of the decomposition are reported in the last column of Table 3.2. With the alternative choice of preferences the cooperative economy features an even stronger bias towards small firms, meaning that its static inefficiency cause an even greater disadvantage relative to the capitalist economy: the term in Z drops to 0.69. On the other hand, the higher elasticity of intertemporal substitution boosts the

[35]The existence of this steady state is established only numerically via the numerical properties of the policy function. Its uniqueness is not established in any formal sense. All we can say is that our policy function iteration converges to the same fixed point from a wide variety of initial guesses we have attempted.

relative savings rate of the cooperative economy, which is now 10 per cent higher than the capitalist one (resulting in a 5 per cent higher term in s). As a consequence of this latter feature, the relative output of the cooperative economy rises to 0.73. Welfare is correspondingly much less impacted than in the log case: the welfare loss from moving to a coop economy is now 28% of the coop economy's GDP.

3.7 Comparative Statics

In this section we explore the dependence of relative output and welfare to changes in some of the parameters of the model. These exercises can be be interpreted as robustness checks on the benchmark numerical results of the previous section or, perhaps more usefully, as numerical comparative-static results for our model of cooperatives.

When varying the parameters of technology or preferences, a choice needs to be made about whether to hold τ constant at its benchmark level, or allow it to vary so that, for each configuration of parameters, the inter-generational transfer is always the one that maximizes steady state welfare. Because both strategies are defensible, we do both.

In Figure 3.3 we present a series of plots showing the numerical dependence of the output ratio $\frac{Y^*_{coop}}{Y^*_{cap}}$, as well as its two components, on the parameters α, β, γ, *without* changing the value of τ. We also of course look at the impact of different values of τ holding the other parameters constant. In Appendix C.8 we show the analogous sensitivity graphs for α, β, and γ when we re-calculate the value of τ as the other parameters vary. The qualitative patterns are virtually identical and even quantitatively the sensitivities are quite similar to those in Figure 3.3. Hence, the commentary which follows applies almost equally well to comparative statics with and without re-optimization.

The top-left plots reveal that relative steady state cooperative output is first decreasing and then increasing in the elasticity of output to capital α. Looking at the two subcomponents reveals why: static efficiency steadily declines with α, while the saving rate increases with it - for the reasons we discussed in Section 3.5.3. Clearly the former effect dominates at low level of the output elasticity of capital, while the latter dominates for larger values.

The top-right panel shows relative cooperative output to be monotonically decreasing in the elasticity to variable labour, β. Inspection of equations (3.6) and (3.11) shows

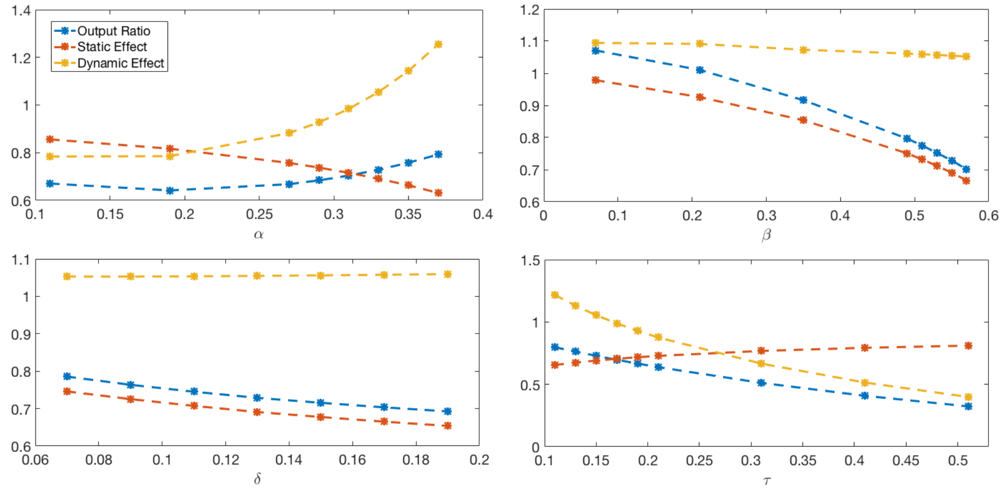

Figure 3.3: Steady-state output ratio $\frac{Y^*_{coop}}{Y^*_{cap}}$ and its static $\left(\left(\frac{Z_{coop}}{Z_{cap}}\right)^{\frac{1}{1-\alpha}}\right)$ and dynamic $\left(\left(\frac{s_{coop}}{s_{cap}}\right)^{\frac{\alpha}{1-\alpha}}\right)$ components, as functions of the model's parameters.

that, as β increases, firm size increases in both economies, but proportionally more so in the capitalist (and statically efficient) economy. This leads to an exacerbation of the static inefficiency of cooperative economies. Quantitatively this is clearly the main driver of the decline of the output ratio with β, though the graph shows that the relative saving rate is also slightly decreasing in this parameter. Another interesting feature of this panel is that it confirms that there actually exist *combinations of parameter values such that steady state output in the cooperative economy is higher than in the capitalist economy*. In this particular case, this happens when the static inefficiency is minimized (through a very low value of β) so that the entire difference in incomes is due to the higher saving rate of the cooperative economy.

The static inefficiency also dominates the dependence of relative output on the discount factor δ. As seen in Section 3.5.3, the more importance workers give to the future, the more they wish to limit current employment. This negative effect is quantitatively much stronger than the positive effect of δ on relative saving, which goes in the opposite direction.

Finally, a larger share of output devoted to former workers, τ, directly reduces the cooperative economy's saving rate, leading again to a reduction in relative cooperative-economy output. This is despite the effect that an increase in τ improves somewhat the cooperative economy's static allocative efficiency.

The corresponding sensitivity plots for the welfare loss from moving to a cooperative

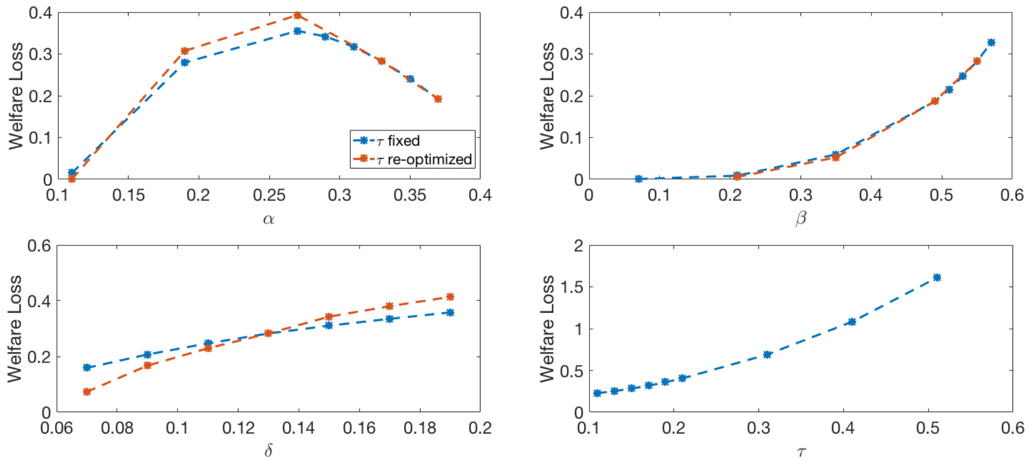

Figure 3.4: Amount of consumption to be given to agents in the cooperative economy to equalize their utility to agents in the capitalist economy, in steady state.

economy are presented, both with and without re-calculation of τ, in Figure 3.4. Qualitatively the welfare losses tend to be mirror-images of the output-ratio graphs in Figure 3.3: the lower the relative output of the cooperative economy, the larger the welfare loss to adopting this growth model. Quantitatively however the welfare losses are quite sensitive to parameter values. For example, agents are virtually indifferent between the two economies if α is very small, even though there is a significant difference in GDP. This is due to the very poor consumption-smoothing properties of capitalism when the capital share is very small. The strong dependence of the welfare loss on τ is also likely related to consumption smoothing (this time in the coop economy).[36]

3.8 Conclusions

In light of the current crisis in the perceived legitimacy of corporation-based capitalism it is important to investigate the macroeconomic consequences of alternative institutional arrangements for the production of goods and services. This paper has taken a first step towards developing a theoretical and quantitative framework towards this goal, with a particular focus on worker cooperatives as the engine of economic activity. We have also provided quantitative examples of comparisons of macroeconomic outcomes under corporation-based capitalism and under labor-management.

[36]For the avoidance of confusion, there is no contradiction between the fact that the optimal τ in the cooperative economy (0.15) is not the τ which minimizes the welfare loss to moving to a cooperative economy. The objective functions are conceptually completely different.

Much work needs to be done for a proper qualitative and quantitative comparison of capitalist economies and cooperative-based economies. In the rest of these Conclusions we outline the agenda for future research.

Our cooperative economy differs from the capitalist economy in the following main respects: (i) there is a non-wage mechanism which assigns workers to firms in a manner that is collectively rational and yet decentralized; (ii) former workers retain rights to the distribution of the cooperative's income; (iii) investment decisions are made by worker collectives to maximize the lifetime utility of current workers; (iv) the capital used in production by each cooperative is the result of past cooperative investments from retained earnings.

We don't think there is much scope to investigate alternatives to (i) if the productive units in our economy must continue to be recognizable as worker cooperatives. Indeed we think of the conceptualization of the worker-allocation mechanism in a cooperative economy as one of the key contributions of the paper. Similarly, dropping (ii) while leaving (iii) in place would trivially lead to an economy with zero investment.

A more feasible alternative might seem to be to drop (iii) and return the investment decision to the individuals. In particular, we could have young workers save in the form of capital, and cooperatives renting capital from old individuals. It is apparent, however, that such an alternative would be isomorphic to the capitalist model – at least in our OLG framework. This is because the rental rate on capital would be the marginal product of capital, so young workers would be the residual claimants of the same share of income as in the capitalist version.

This leaves us with (iv), and it is here that a truly important and fruitful alternative could potentially lie. In particular, it would be useful to investigate the consequences of opening up a market on which cooperatives could rent capital from each other. We have noted earlier that one reason for the inefficiently small size of cooperatives is that they take their capital stock as given. The existence of a rental market for capital might therefore lead to different decisions. Unfortunately, extending our framework to feature an inter-cooperative rental market for capital is challenging, as the worker-allocation rule for each cooperative would have to depend on the indefinite future history of rental rates. Hence, we leave this task for future work.

From a quantitative perspective, future work should also assess the implications of canonical variants of the OLG framework, such as economies with alternative stores of value, like money, or with social security. Always within the current setup, it would also be interesting to identify strategies to study the possible coexistence of capitalist and cooperative firms - or whether one type of firm would necessarily drive the other from the market.

However, the true, long-term payoff of this research agenda will only come from much richer qualitative and quantitative descriptions of the economy. A more complex demographic structure is only a minor aspect of this quest. Introducing realistic distortions to the capitalist economy (monopoly power, monopsony power, short-termism in decision-making, etc.) would put the comparison of efficiency and production on a more even playing field. Considerations of externalities (e.g. pollution) would similarly be informative on the relative welfare properties of the two systems. Most important of all, introducing realistic sources of heterogeneity (in skills, in initial wealth, in access to schooling and high-return assets) would allow to compare corporation-based capitalism and cooperative-based alternatives not only on their implications for aggregate productivity but also on their implications for income and wealth inequality. Since it is aversion to the consequences of extreme inequality which has fostered much of the current push back against capitalism, it is essential that efficiency losses associated with a cooperative-based system (if any) be evaluated against the likely benefits in terms of lower inequality. We hope that our paper will prove to be a first step on this (long) road.

Appendix A

Appendix to Chapter 1

A.1 Introductory Example Derivations

This appendix section provides details of the derivations of the results presented in the introductory example. The set up fits the description of the model in Section 1.2 with the functional assumption $F(y) = \sqrt{y}$. We apply the constructions of Sections 1.5.1 and 1.5.2 to describe solutions to the designer's problem when her utility function is convex and concave respectively.

Consider first the case in which the designer's utility function is convex. Since the prior cumulative distribution function F is concave, it coincides with $\overline{F}$. Recall also that the no-information optimal action is $x^* = 1/4$ in this case. The optimal unconditional distribution over the decision maker's actions is given by equation (1.6), yielding in this case:

$$H^{vex}(x) = \begin{cases} 0 & \text{if } x < 1/4, \\ 1 - \frac{1}{2\sqrt{x}} & \text{if } 1/4 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Next, we construct the posterior distributions $(G_x^{vex})_x$ for $x \geq 1/4$. Using equation (1.7) and the construction in Lemma 2, we obtain:

$$G_x^{vex}(y) = \min\left\{2\sqrt{y}, 1 - x + y, 1\right\}.$$

Next, we describe the construction of $\langle (G_x^{cave})_x, H^{cave} \rangle$, solving the designer's problem when her utility function is concave. Observe that F satisfies Assumption 3. The mapping θ is solution to:

$$-\theta'(x)\frac{x-\theta(x)}{2\theta(x)\sqrt{\theta(x)}} = \frac{1}{\sqrt{x}}, \quad \theta(1) = \frac{1}{4}.$$

It is easy to verify that θ can be implicitly described by:

$$\frac{x}{\theta(x)} = \left(\sqrt{2} - 1 + 2\sqrt{2}\frac{\frac{3-\sqrt{2}}{3+\sqrt{2}}}{\big(4\theta(x)\big)^{\sqrt{2}/2} - \frac{3-\sqrt{2}}{3+\sqrt{2}}}\right)^2.$$

Furthermore, we define the lowest action a by $\theta(a) = a$. We can compute:

$$a = \frac{1}{4}\left(\frac{\sqrt{2}+1/2}{\sqrt{2}-1/2}\right)^{\sqrt{2}}.$$

We can then define:

$$H^{cave}(x) = \begin{cases} 0 & \text{if } x < a, \\ \frac{1}{2\sqrt{\theta(x)}} & \text{if } x \geq a. \end{cases}$$

In particular, H^{cave} possesses an atom of size ≈ 0.59 at a. Let us now describe the posterior cumulative distribution functions $(G_x^{cave})_x$ for $x \geq a$. Using the construction of Lemma 2, define γ on $[0, 1/4]$ by:

$$\sqrt{y} - y = \frac{1}{4} + \sqrt{1+y-\gamma(y)} - \frac{1+y-\gamma(y)}{2\sqrt{\theta\big(1+y-\gamma(y)\big)}} - \frac{\sqrt{\theta\big(1+y-\gamma(y)\big)}}{2}.$$

Then:

$$G_x^{cave}(y) = \begin{cases} \min\big\{1-x+y, \gamma(y)\big\} & \text{if } y \leq 1/4, \\ 1-x+y & \text{if } 1/4 \leq y \leq \theta(x), \\ 1-x+\theta(x) & \text{if } \theta(x) \leq y < x, \\ 1 & \text{if } y \geq x. \end{cases}$$

A.2 Proofs

Proof of Lemma 1.— Let $\langle (G_x)_x, H\rangle$ an outcome satisfying (BP) and (IC) and (X, Y) the random vector formed by the decision maker's action X and the threshold Y. We construct the outcome $\langle(\hat{G}_x)_x, \hat{H}\rangle$ associated to $(\hat{X}, Y)$ where $\hat{X} = X + \mathbb{1}(X \leq Y \leq 1)(Y - X)$. Then, we show that it is an element of $\mathcal{O}$ and is weakly better for the seller.

For $x \in [0, 1]$, let:

$$\hat{H}(x) = \int_0^x [1 - G_z(1) + G_z(x)]dH(z).$$

The alternative unconditional distribution over actions is such that the decision maker's action is below x whenever the initial outcome specified an action below x and the threshold was not in $(x, 1]$. Next we define the associated conditional distributions of the threshold given each action. Observe that $d\hat{H}(x) \geq [1 - G_x(1) + G_x(x)]dH(x) \geq xdH(x)$, where the second inequality follows from (IC). With the convention $0/0 = 0$, define:

$$\hat{G}_x(y) = \begin{cases} G_x(y)dH(x)/[d\hat{H}(x)] & \text{if } y < x, \\ 1 - [1 - G_x(1)]dH(x)/[d\hat{H}(x)] & \text{if } x \leq y \leq 1, \\ 1 - [1 - G_x(y)]dH(x)/[d\hat{H}(x)] & \text{if } y > 1. \end{cases}$$

Note that $\hat{H}$ first-order stochastically dominate H. So, if $\langle(\hat{G}_x)_x, \hat{H}\rangle$ is feasible, it would be preferred to $\langle(G_x)_x, H\rangle$ by the designer. The following arguments establish that it is indeed feasible by verifying that $\langle(\hat{G}_x)_x, \hat{H}\rangle$ satisfies the constraints (BP) and (IC).

Let us first show that (BP) is satisfied. We check two cases. If $y \leq 1$, then:

$$\begin{aligned}
\int_0^1 \hat{G}_x(y)d\hat{H}(x) &= \int_0^y \left(1 - \big[1 - G_x(1)\big]\frac{dH(x)}{d\hat{H}(x)}\right)d\hat{H}(x) + \int_y^1 G_x(y)\frac{dH(x)}{d\hat{H}(x)}d\hat{H}(x) \\
&= \hat{H}(y) - \int_0^y \big[1 - G_x(1)\big]dH(x) + \int_y^1 G_x(y)dH(x) \\
&= \int_0^y \big[1 - G_x(1) + G_x(y)\big]dH(x) - \int_0^y \big[1 - G_x(1)\big]dH(x) + \int_y^1 G_x(y)dH(x) \\
&= \int_0^1 G_x(y)dH(x) \\
&= F(y),
\end{aligned}$$

where the first equality uses the definition of $(\hat{G}_x)_x$, the second simplifies the expression, the third uses the definition of $\hat{H}$, the fourth collects terms and the fifth concludes using the fact that $\langle(G_x)_x, H\rangle$ satisfies (BP). Similar steps are used when $y > 1$. Computations go as follows:

$$\begin{aligned}\int_0^1 \hat{G}_x(y)d\hat{H}(x) &= \int_0^1 \left(1-\big[1-G_x(y)\big]\frac{dH(x)}{d\hat{H}(x)}\right)d\hat{H}(x)\\ &= \hat{H}(1) - \int_0^1 \big[1-G_x(y)\big]dH(x)\\ &= \int_0^1 dH(x) - \int_0^1 \big[1-G_x(y)\big]dH(x)\\ &= \int_0^1 G_x(y)dH(x)\\ &= F(y).\end{aligned}$$

Next, we show that (IC) is satisfied. Due to the fact that, for each $x \in [0,1]$, $\hat{G}_x$ places no mass between x and 1, upward deviations cannot be tempting for the decision maker. Therefore, it suffices to check that downward deviations are not tempting either. That is, we need to show that for $\tilde{x} < x$, we have:

$$\hat{G}_x(\tilde{x}) - \tilde{x} \leq \hat{G}_x(x) - x.$$

Using the definition of $\hat{G}_x$, the above inequality rewrites:

$$\frac{dH(x)}{d\hat{H}(x)}G_x(\tilde{x}) - \tilde{x} \leq 1 - \big[1-G_x(1)\big]\frac{dH(x)}{d\hat{H}(x)} - x,$$

or equivalently:

$$\frac{dH(x)}{d\hat{H}(x)}\big[1-G_x(1)+G_x(\tilde{x})\big] \leq 1 - x + \tilde{x}.$$

The inequality is trivially satisfied if $dH(x)/[d\hat{H}(x)] = 0$. Otherwise, we use the fact that $d\hat{H}(x) \geq [1-G_x(1)+G_x(x)]dH(x)$ to obtain:

$$\frac{dH(x)}{d\hat{H}(x)}\big[1-G_x(1)+G_x(\tilde{x})\big] \leq \frac{1-G_x(1)+G_x(\tilde{x})}{1-G_x(1)+G_x(x)}.$$

Since $\langle (G_x)_x, H\rangle$ satisfies (IC), it must also be the case that:

$$\frac{1-G_x(1)+G_x(\tilde{x})}{1-G_x(1)+G_x(x)} \leq \frac{1-G_x(1)+G_x(x)-x+\tilde{x}}{1-G_x(1)+G_x(x)} = 1 - \frac{x-\tilde{x}}{1-G_x(1)+G_x(x)}.$$

Finally, since $x > \tilde{x}$ and $1 - G_x(1) + G_x(x)$, it must be that:

$$1 - \frac{x - \tilde{x}}{1 - G_x(1) + G_x(x)} \leq 1 - x + \tilde{x}.$$

The chain of inequalities allows to conclude that (IC) is indeed satisfied by $\langle (\hat{G}_x)_x, \hat{H} \rangle$.

Complement to the Proof of Theorem 1.— We made the following claim after the statement of Theorem 1.

Claim 1. *For $x \in [0,1]$,*

$$G^* \in \arg\min_{G \in \mathcal{G}_x} \int_0^x \lambda(y) dG(y)$$
$$\textit{subject to } \forall y \in [0,x], \quad G(y) - y \leq 1 - x,$$

if and only if:

(i) For all $y \in [0,x]$, if $y \in \operatorname{supp} G^*$, then $\forall z \in [y,x]$, $\lambda(y) \leq \lambda(z)$.

(ii) For all $y \in [0,x]$, if $\forall z \in (y,x]$ $\lambda(y) < \lambda(z)$, then $G^*(y) - y = 1 - x$.

Furthermore, the value to the minimisation problem writes:

$$\Lambda(x) = (1-x) \min_{0 \leq z \leq x} \lambda(z) + \int_0^x \big[\min_{y \leq z \leq x} \lambda(z)\big] dy.$$

Proof. Suppose that the two conditions (i) and (ii) are satisfied. For $G \in \mathcal{G}_x$ satisfying the constraints, since $\lambda(y) \geq \min_{y \leq z \leq x} \lambda(z)$, it must be that:

$$\int_0^x \lambda(y) dG(y) \geq \int_0^x \big[\min_{y \leq z \leq x} \lambda(z)\big] dG(y).$$

In the case of G^*, the inequality must be an equality by condition (i). Performing integration by parts:

$$\int_0^x \big[\min_{y \leq z \leq x} \lambda(z)\big] dG(y) = G(x)\lambda(x) - \int_0^x G(y) d\big[\min_{y \leq z \leq x} \lambda(z)\big].$$

Furthermore, $G(y) \leq G(x) - x + y$, so:

$$\int_0^x \big[\min_{y \leq z \leq x} \lambda(z)\big] dG(y) \geq G(x)\lambda(x) - \int_0^x [G(x) - x + y] d\big[\min_{y \leq z \leq x} \lambda(z)\big].$$

In the case of G^*, this is again an equality by condition (ii). Performing integration by parts again:

$$G(x)\lambda(x) - \int_0^x [G(x) - x + y] d\big[\min_{y \le z \le x} \lambda(z)\big] = [G(x) - x] \min_{0 \le z \le x} \lambda(z) + \int_0^x \big[\min_{y \le z \le x} \lambda(z)\big] dy.$$

Since $G(x) = G_x(x)$, this final expression on the right-hand side does not depend on the specific choice of G. Therefore, for an arbitrary G, we have obtained:

$$\int_0^x \lambda(y) dG(y) \ge [G_x(x) - x] \min_{0 \le z \le x} \lambda(z) + \int_0^x \big[\min_{y \le z \le x} \lambda(z)\big] dy = \int_0^x \lambda(y) dG^*(y).$$

Reciprocally, we show that conditions (i) and (ii) are necessary. Let us start with condition (i). Suppose there exists y such that $\lambda(y) > \min_{y \le z \le x} \lambda(z)$. If $y \in \operatorname{supp} G^*$, then G^* places positive mass on the neighbourhoods of y. By lower-semicontinuity, $\lambda(\hat{y}) > \min_{\hat{y} \le z \le x} \lambda(z)$ if $\hat{y}$ is in a sufficiently small neighbourhood of y. As a result, the inequality:

$$\int_0^x \lambda(u) dG^*(u) > \int_0^x \big[\min_{u \le z \le x} \lambda(z)\big] dG^*(u)$$

must be strict.

Next, suppose that there exists y such that $\lambda(y) < \lambda(z)$ for all $z \in (y, x]$. If $G^*(y) < G^*(x) - x + y$, let:

$$\overline{y} = \min\big\{z \in [0, x] : G^*(z) \ge G^*(x) - x + y\big\}.$$

Note that $\overline{y}$ exists since $G^*(x) \ge G^*(x) - x + y$ when $y \le x$. By assumption, $y < \overline{y}$. So we can move a mass $G^*(x) - x + y - G^*(y)$ contained in $(y, \overline{y}]$ to y. That is, create a $\tilde{G}$ which behaves like G^* below y, binds the constraint at y, is flat on $[y, \overline{y})$, and behaves like G^* above $\overline{y}$. Then, it is easy to see that $\tilde{G}$ is a strict improvement upon G^*. □

Proof of Lemma 2.— For part (A), note that no-information indifference implies that:

$$\int_0^1 (1 - x) dH(x) = \int_0^1 [G_x(x^*) - x^*] dH(x).$$

Since $G_x(x^*) \le 1$, this equality can only hold if $x \ge x^*$ on the support of H. Furthermore, the set of elements x such that $G_x(x^*) - x^* < 1 - x$ must have zero mass under H. Therefore, replacing those by $\tilde{G}_x(y) = 1 - x + y$ on $[0, x]$ does not affect the feasibility of the outcome.

For part (B), we set, for each x, $G_x = \Gamma_x$ on $[x^*, x]$. Then, we need to complete G_x below x^*. Let:

$$\Psi : C \mapsto \int \min\{1 - x, C\} dH(x).$$

Note that Ψ is non-decreasing and concave on $\mathbb{R}$. Furthermore, for C sufficiently large, Ψ becomes stationary at $\int[1 - x]dH(x) = F(x^*) - x^*$. Therefore, for any $z \in (-\infty, F(x^*) - x^*]$, $\Psi^{-1}(z) = \min\{C : \Psi(C) \geq z\}$ is well-defined and continuous. Denote $\overline{F}$ the concave envelope of F on $[0, x^*]$. Note that $\overline{F}'(x^*) \geq 1$. For $y \in [0, x^*]$, we define:

$$\gamma(y) = y - \frac{\overline{F}(y) - F(y)}{\overline{F}'(y)} + \Psi^{-1}\left(F(y) - y + \frac{\overline{F}(y) - F(y)}{\overline{F}'(y)}\right).$$

Note that $y \mapsto F(y) - y + \frac{\overline{F}(y)-F(y)}{\overline{F}'(y)}$ is non-decreasing in y and reaches $F(x^*) - x^*$ at $y = x^*$. So γ is well-defined. In addition, $y \mapsto y - \frac{\overline{F}(y)-F(y)}{\overline{F}'(y)}$ is also non-decreasing in y, so γ is non-decreasing. Moreover, $\gamma(0) = 0 + \Psi^{-1}(0) = 0$, where the second equality follows from $1 - x \geq 0$ for all x. We use γ to define, for all $x \in [x^*, 1]$ and $y \in [0, x^*]$:

$$G_x(y) = \min\left\{1 - x + y - \frac{\overline{F}(y) - F(y)}{\overline{F}'(y)}, \gamma(y)\right\}.$$

Note that (IC) is satisfied. Furthermore, $G_x(x^*) = 1 - x + x^*$ so the definition is consistent with the construction of G_x above x^*. To see this, note that $\Psi^{-1}\big(F(x^*) - x^*\big) \geq G_x(x) - x$ for all x. Moreover, G_x is indeed non-decreasing and continuous on $[0, x^*]$. It suffices to check (BP):

$$\begin{aligned}
\int G_x(y)dH(x) &= y - \frac{\overline{F}(y) - F(y)}{\overline{F}'(y)} + \int \min\left\{1 - x, \gamma(y) - y + \frac{\overline{F}(y) - F(y)}{\overline{F}'(y)}\right\} dH(x) \\
&= y - \frac{\overline{F}(y) - F(y)}{\overline{F}'(y)} + \Psi \circ \Psi^{-1}\left(F(y) - y + \frac{\overline{F}(y) - F(y)}{\overline{F}'(y)}\right) \\
&= F(y).
\end{aligned}$$

Proof of Proposition 2.— We first verify that the constraints (IC) and (BP) are satisfied on $[x^*, 1]$ (which is sufficient by Lemma 2). Then, we construct shadow prices $\lambda : [0, 1] \to \mathbb{R}$ and verify the conditions of Theorem 1 to conclude.

It is obvious from the definition of G_x^{vex} that (IC) holds, since F is always below $\overline{F}$.

Now, for $y \in [x^*, 1]$, we have:

$$\int_{x^*}^{1} G_x^{vex}(y)dH^{vex}(x) = \int_{x^*}^{1} \min\left\{1, 1 - x + y - \frac{\overline{F}(y) - F(y)}{\overline{F}'_+(y)}\right\} dH^{vex}(x)$$
$$= H^{vex}(y) + \int_{y}^{1} \left(1 - x + y - \frac{\overline{F}(y) - F(y)}{\overline{F}'_+(y)}\right) dH^{vex}(x).$$

Using integration by parts, we can then compute:

$$\int_{x^*}^{1} G_x^{vex}(y)dH^{vex}(x) = H^{vex}(y) + y - \frac{\overline{F}(y) - F(y)}{\overline{F}'_+(y)} - \left(1 - \frac{\overline{F}(y) - F(y)}{\overline{F}'_+(y)}\right) H^{vex}(y) + \int_{y}^{1} H^{vex}(x)dx.$$

Using equation (1.6), we obtain:

$$\int_{x^*}^{1} G_x^{vex}(y)dH^{vex}(x) = y - \frac{\overline{F}(y) - F(y)}{\overline{F}'_+(y)} \overline{F}'_+(y) + \int_{y}^{1} [1 - \overline{F}'_+(x)]dx = y - \overline{F}(y) + F(y) - y + \overline{F}(y) = F(y).$$

It follows that (BP) is also satisfied.

Next, we construct $\lambda : [0, 1] \to \mathbb{R}$. For $y \leq x^*$, we set $\lambda(y) = 0$. For $y > x^*$, consider three cases. First, if $F(y) < \overline{F}(y)$, define $a = \sup\{z \leq y : F(y) = \overline{F}(y)\}$ and $b = \sup\{z \geq y : F(y) = \overline{F}(y)\}$. Since F and $\overline{F}$ are both continuous, $a < y < b$. Therefore, we can define:

$$\lambda(y) = \frac{v(b) - v(a)}{b - a}.$$

Second, if $F(y) = \overline{F}(y)$ and there exists $\varepsilon > 0$ such that for all $z \in (y - \varepsilon, y)$, $F(z) < \overline{F}(z)$, set:

$$\lambda(y) = \lim_{z \to_- y} \lambda(z),$$

where for $z \in (y - \varepsilon, y)$, $\lambda(z)$ was defined by the previous case and constant, so the definition is unambiguous. Finally, in all other cases, set $\lambda(y) = v'_-(y)$, which is the left-derivative of v at y, which is well-defined since v is convex.

Observe that the above construction guarantees that λ is lower-semicontinuous. Furthermore, since v is convex, λ is weakly increasing on $[0, 1]$. As a result, condition (i) of Theorem 1 is automatically satisfied. Next, we show that conditions (ii) and (iii) also hold.

For $x \in [0, 1]$ and $y \in [0, x]$, if $\forall z \in (y, x]$ $\lambda(y) < \lambda(z)$, then either $y = x$ or $F(y) = \overline{F}(y)$. In either case, we have $G_x^{vex}(y) = 1 - x + y$, so condition (ii) is satisfied.

Finally, since λ is non-decreasing, we have:

$$\Lambda(x) = (1-x)\min_{0\leq z\leq x}\lambda(z) + \int_0^x [\min_{y\leq z\leq x}\lambda(z)]dy$$
$$= \int_0^x \lambda(y)dy$$

Observe that Λ is flat on $[0,x^*]$. Since v is increasing, the maximisers of $v-\Lambda$ must be weakly above x^*. On an interval $[a,b]$ such that $\forall y\in(a,b)$, $\overline{F}(y) > F(y)$, and with equality at the boundaries, Λ increases at constant rate $\frac{v(b)-v(a)}{b-a}$. Therefore, since v is convex, we have:

$$\forall x\in[a,b],\quad v(x)-\Lambda(x)\leq v(a)-\Lambda(a) = v(b)-\Lambda(b).$$

Finally, a point where $\overline{F}(x) = F(x)$ is either a right boundary of an interval of the form above or satisfies $\Lambda'(x) = \lambda(x) = v'_-(x)$. Therefore, it must be the case that:

$$\arg\max_x\{v(x)-\Lambda(x)\} = \{x\geq x^* : F(x) = \overline{F}(x)\}.$$

Therefore, condition (iii) is satisfied.

Complement to the Construction of Section 1.5.2.— Let us show that a and θ are well defined. Consider first the Cauchy problem:

$$Z'(u) = F''(u)\Big(F^{-1}\big(Z(u)\big)-u\Big),\quad Z(x^*) = 1,$$

where F^{-1} is the inverse of F on $[0,1]$. Under Assumption 3, the Cauchy–Lipschitz theorem applies, guaranteeing the existence of solutions on an interval of the form $[x^*,\overline{u}]$, where $Z(\overline{u}) = F(\overline{u})$. Furthermore, such a solution is strictly decreasing on $[x^*,\overline{u}]$. Therefore, define $a = F(\overline{u})$ and:

$$\forall x\in[a,1],\quad \theta(x) = Z^{-1}\circ F(x).$$

It is easy to verify that θ satisfies the properties required for the construction of Section 1.5.2.

Proof of Proposition 3.— We first verify that the constraints (IC) and (BP) are satisfied. Then, we construct shadow prices λ and use Theorem 1 to conclude.

It is clear from the definition in equation (1.9) that the constraint (IC) is satisfied. Let us now verify that (BP) holds on $[x^*, 1]$. There are two cases. If $y \geq a$, we have:

$$\int_0^1 G_x^{cave}(y)dH^{cave}(x) = 1 - \int_y^1 [x - \theta(x)]dH^{cave}(x),$$

where we have used the definition in equation (1.9). Since $a = \theta(a)$ and H^{cave} has density $\theta'(x)F''\big(\theta(x)\big)$ on $(a, 1)$, we can rewrite this expression:

$$\int_0^1 G_x^{cave}(y)dH^{cave}(x) = 1 - \int_y^1 [x - \theta(x)]\theta'(x)F''\big(\theta(x)\big)dx.$$

We recognise under the integral the left-hand-side of the differential equation defining θ. Therefore:

$$\int_0^1 G_x^{cave}(y)dH^{cave}(x) = 1 - \int_y^1 F'(x)dx = F(y).$$

Now suppose $y < a$. Since θ is monotonic and continuous, its inverse is well-defined. Using equation (1.9), we can write:

$$\int_0^1 G_x^{cave}(y)dH^{cave}(x) = \int_0^1 \big[1 - x + \theta(x)\big]dH^{cave}(x) + \int_0^{\theta^{-1}(y)} \big[y - \theta(x)\big]dH^{cave}(x).$$

Observe that the first term in the sum corresponds to $\int_0^1 \big[1 - x + \theta(x)\big]dH^{cave}(x) = \int_0^1 G_x^{cave}(a)dH^{cave}(x) = F(a)$ by the treatment of the case $y \geq a$ above. The second term can be computed using integration by parts. We obtain:

$$\int_0^1 G_x^{cave}(y)dH^{cave}(x) = F(a) + \int_0^{\theta^{-1}(y)} H^{cave}(x)\theta'(x)dx = F(a) + \int_a^{\theta^{-1}(y)} F'\big(\theta(x)\big)\theta'(x)dx.$$

The integral can now be easily computed and we find:

$$\int_0^1 G_x^{cave}(y)dH^{cave}(x) = F(a) + F(y) - F(a) = F(y).$$

We conclude that (BP) holds for all $y \geq x^*$.

Let us now explain how to construct shadow prices $\lambda : [0, 1] \to \mathbb{R}$ such that conditions (i), (ii) and (iii) of Theorem 1 are satisfied. First, we set $\lambda(x) = 0$ if $x \leq x^*$. On $(x^*, 1]$, we will construct λ to be increasing until a and then decreasing, such that it satisfies:

$$\forall x \geq a, \quad \lambda(x) = \lambda\big(\theta(x)\big).$$

This feature guarantees that conditions (i) and (ii) of the Theorem are satisfied. Indeed, condition (i) requires that $y \in \operatorname{supp} G_x^{cave}$ only if $\lambda(y) = \min_{y \leq z \leq x} \lambda(z)$, which will be the case exactly for $y \leq \theta(x)$ or $y = x$, consistently with the definition of G_x^{cave}. Moreover, condition (ii) requires that $G_x^{cave}(y) = 1 - x + y$ when $\lambda(y) < \lambda(z)$ for all $z \in (y, x]$, that is for $y \in \big[x^*, \theta(x)\big)$ or $y = x$, again consistently with the definition of G_x^{cave} in equation (1.9). Next, we discuss how to construct λ satisfying condition (iii).

If we can construct λ as specified in the previous paragraph, the implementation cost of any action x will write:

$$\Lambda(x) = (1-x) \min_{0 \leq z \leq x} \lambda(z) + \int_0^x \big[\min_{y \leq z \leq x} \lambda(z)\big] dy = \begin{cases} 0 & \text{if } x \leq x^*, \\ \int_{x^*}^x \lambda(y) dy & \text{if } x^* \leq x \leq a, \\ \int_{x^*}^{\theta(x)} \lambda(y) dy + [x - \theta(x)]\lambda(x) & \text{if } x \geq a. \end{cases}$$

In particular, for $x \in (a, 1)$, we have:

$$\Lambda(x) = \Lambda\big(\theta(x)\big) + \big[x - \theta(x)\big]\Lambda'\big(\theta(x)\big).$$

In order to satisfy condition (iii), $x \mapsto v(x) - \Lambda(x)$ must be maximised on $[a, 1]$. Define α to be the constant value of $v(x) - \Lambda(x)$ on $[a, 1]$. The previous equation rewrites:

$$\forall x \in (a, 1), \quad v(x) - \alpha = \Lambda\big(\theta(x)\big) + \big[x - \theta(x)\big]\Lambda'\big(\theta(x)\big).$$

This motivates considering the differential equation:

$$\forall t \in (x^*, a), \quad v\big(\theta^{-1}(t)\big) = q(t) + \big[\theta^{-1}(t) - t\big]q'(t). \tag{A.1}$$

For a solution q on (x^*, a), we can set $\Lambda(t) = q(t) - \alpha$. It will follow that $\lambda(t) = q'(t)$ on (x^*, a). Since we wish to construct λ increasing on $[x^*, a]$ with $\lambda(x^*) = 0$, the solution q must be increasing and convex. Furthermore, $q'(t)$ must remain bounded as t approaches a, so we can define $\lambda(a) = \lim_{t \to a} q'(t)$. Then for $x \in (a, 1)$, it suffices to set $\lambda(x) = \lambda\big(\theta(x)\big)$, and $\lambda(1) = \lim_{t \to x^*} q'(t)$. Note that since q' remains bounded, it will have to be the case that $\lim_{t\to a} q(t) = v\big(\theta^{-1}(a)\big) = v(a)$. Finally, in order to guarantee that condition (iii) holds, we must have:

$$\forall t \in (x^*, a), \quad v(t) - \Lambda(t) \leq \alpha,$$

or equivalently:

$$\forall t \in (x^*, a), \quad v(t) \leq q(t).$$

Since q will be convex with $\lim_{t\to a} q(t) = v(a)$, a sufficient condition for this inequality to hold is:

$$\forall t \in (x^*, a), \quad v(t) \leq v(a) + (t-a) \lim_{s\to a} q'(s),$$

and since v is concave, it will be sufficient to have $\lim_{s\to a} q'(s) \leq v'_-(a)$, the left-derivative of v at a.

To summarise, it suffices to show that the differential equation (A.1) possesses an increasing and convex solution q with $\lim_{s\to a} q'(s) \leq v'_-(a)$. This is what we do next by exhibiting a solution. Let us first introduce a notation. For $t \in (x^*, a)$, define:

$$\zeta(t) = \int_{x^*}^{t} \frac{du}{\theta^{-1}(u) - u}.$$

Observe that $t \mapsto \big[\theta^{-1}(t) - t\big]e^{\zeta(t)}$ is positive and decreasing on (x^*, a).[1] Therefore its limit as $t \to a$ is well defined, and we denote it $\ell \geq 0$. Then we can define:

$$q(t) = v\big(\theta^{-1}(t)\big) - e^{-\zeta(t)}\left(\ell v'_+(a) - \int_t^a \frac{F''(u)}{F'\big(\theta^{-1}(u)\big)}\big(\theta^{-1}(u) - u\big)e^{\zeta(u)} v'_+\big(\theta^{-1}(u)\big)\,du\right). \tag{A.2}$$

In the integral, $\frac{F''(u)}{F'\big(\theta^{-1}(u)\big)}$, $\big(\theta^{-1}(u) - u\big)e^{\zeta(u)}$ and $v'_+\big(\theta^{-1}(u)\big)$ are all bounded, so the integral is well-defined. Furthermore, q is everywhere right-differentiable. Noting that:

$$\big(\theta^{-1}\big)'(u) = \frac{F''(u)}{F'\big(\theta^{-1}(u)\big)}\big(\theta^{-1}(u) - u\big),$$

by the differentiable equation defining θ, simple algebra establishes that:

$$q'_+(t) = \frac{v\big(\theta^{-1}(t)\big) - q(t)}{\theta^{-1}(t) - t},$$

[1] To see this, differentiate to obtain: $\frac{d}{dt}[(\theta^{-1}(t) - t)e^{\zeta(t)}] = (\theta^{-1})'(t)e^{\zeta(t)} < 0$.

for all $t \in (x^*, a)$. It follows that q solves equation (A.1). Furthermore, from equation (A.2), we have:

$$v\big(\theta^{-1}(t)\big) - q(t) = e^{-\zeta(t)}\ell v'_+(a) + e^{-\zeta(t)} \int_t^a \frac{-F''(u)}{F'\big(\theta^{-1}(u)\big)} \big(\theta^{-1}(u) - u\big) e^{\zeta(u)} v'_+\big(\theta^{-1}(u)\big) du,$$

which is a sum of positive terms. As a result, $q'_+ \geq 0$, so q is increasing. Using this expression, we write:

$$q'_+(t) = \frac{\ell v'_+(a) + \int_t^a \frac{-F''(u)}{F'[\theta^{-1}(u)]} \big(\theta^{-1}(u) - u\big) e^{\zeta(u)} v'_+\big(\theta^{-1}(u)\big) du}{\big(\theta^{-1}(t) - t\big) e^{\zeta(t)}}.$$

As explained previously, the denominator in this ratio is a positive and decreasing term in t.

Appendix B

Appendix to Chapter 2

B.1 An Equilibrium when $\mathcal{C} = \mathcal{D}$

This appendix is concerned with constructing an equilibrium when the seller's contract space is restricted to include only *simple and direct* contracts. In section B.1.1, we use a fixed-point argument to define a set of contracts $D^*(\mu)$ parameterised by $\mu \in [0,1]$, such that the seller deploys a contract in $D^*(\mu)$ when her belief about the buyer having high valuation is μ. In section B.1.2, we describe and analyse a class of auxiliary games whose sequential equilibria map to off-path parts of our equilibrium assessment. Finally in section B.1.3, we use the results from sections B.1.1 and B.1.2 to build a complete assessment and prove that it constitutes an equilibrium.

B.1.1 Deployed Contracts

Let $v_h > v_l > 0$ and $\delta \in (0,1)$. For $\mu \in [0,1]$, let $\bar{J}(\mu) = \max\{v_l, \mu v_h\}$ and

$$\underline{J}(\mu) = \begin{cases} v_l & \text{if } \mu < 1, \\ v_h & \text{if } \mu = 1. \end{cases}$$

A function $J : [0,1] \to \mathbb{R}$ is piece-wise linear if $[0,1]$ can be partitioned into countably many intervals such that J is affine on each of these intervals. We denote by $\mathcal{J}$ the set of non-decreasing, piece-wise linear and convex functions $J : [0,1] \to [v_l, v_h]$ such that, for all $\mu \in [0,1]$, $\bar{J}(\mu) \geq J(\mu) \geq \underline{J}(\mu)$.

Given $J \in \mathcal{J}$, we construct a function $\mathcal{A}J \in \mathcal{J}$. We set $\mathcal{A}J(0) = v_l$ and $\mathcal{A}J(1) = v_h$. For $\mu \in (0,1)$, $\mathcal{A}J(\mu)$ is the value to the maximisation problem described below.

Fix $\mu \in (0,1)$. A trading time s is a random time whose distribution $\langle s \rangle$ depends on the buyer's valuation $v \in \{v_l, v_h\}$. We identify $\langle s \rangle = (q^h, q^l) \in \mathcal{Q} \times \mathcal{Q}$, where $\mathcal{Q} = \big\{q \in [0,1]^{\mathbb{N}} : \sum_{t \geq 0} q_t \leq 1\big\}$. For $t \geq 0$ and $i \in \{h, l\}$, q_t^i is interpreted as the probability that trade occurs in period t if the buyer's valuation in v_i. Given $\langle s \rangle \in \mathcal{Q} \times \mathcal{Q}$ and $t \geq 0$, if $\mathbb{P}(s \geq t) = 1 - \sum_{k=0}^{t-1} \big(\mu q_k^h + (1-\mu) q_k^l\big) > 0$, we define:

$$\mu_t = \mathbb{P}(v = v_h | s \geq t) = \frac{\mu\Big(1 - \sum_{k=0}^{t-1} q_k^h\Big)}{\mu\Big(1 - \sum_{k=0}^{t-1} q_k^h\Big) + (1-\mu)\Big(1 - \sum_{k=0}^{t-1} q_k^l\Big)}.$$

Define, for $\langle s \rangle \in \mathcal{Q} \times \mathcal{Q}$, the objective function:

$$\Omega\big(\langle s \rangle \big| \mu\big) = \sum_{k=0}^{\infty} \delta^k \Big(q_k^h \mu v_h + q_k^l (v_l - \mu v_h)\Big),$$

and for $t \geq 1$, the constraint mapping:

$$G_t\big(\langle s \rangle \big| \mu, J\big) = \begin{cases} \mathbb{P}\big(s \geq t\big) J(\mu_t) - \sum_{k=t}^{\infty} \delta^{k-t} \Big(q_k^h \mu v_h + q_k^l (1-\mu) v_l\Big) & \text{if } \mathbb{P}\big(s \geq t\big) > 0, \\ 0 & \text{if } \mathbb{P}\big(s \geq t\big) = 0. \end{cases}$$

Note that $\Omega(\cdot|\mu)$ is linear and, for each $t \geq 1$, $G_t(\cdot|\mu, J)$ is convex on the convex set $\mathcal{Q} \times \mathcal{Q}$. Denoting $G\big(\langle s \rangle \big| \mu, J\big) = \Big(G_t\big(\langle s \rangle \big| \mu, J\big)\Big)_{t \geq 1}$, the maximisation problem is given by:

$$\begin{aligned} \mathcal{A}J(\mu) = \max_{\langle s \rangle \in \mathcal{Q} \times \mathcal{Q}} & \ \Omega\big(\langle s \rangle \big| \mu\big) \\ \text{s.t. } & G\big(\langle s \rangle \big| \mu, J\big) \leq 0. \end{aligned} \tag{B.1}$$

If $\mu \leq \frac{v_l}{v_h}$, an obvious solution is given by $s = 0$, that is $q_0^h = q_0^l = 1$. Thus, in this case, $\mathcal{A}J(\mu) = v_l$. Therefore, we focus on the case $\mu > \frac{v_l}{v_h}$. We first prove preliminary results which help describe a candidate solution $\langle s^* \rangle(\mu, J)$ to (B.1). Then, we prove that $\langle s^* \rangle(\mu, J)$ indeed solves the maximisation problem, and we verify that $\mathcal{A}J$ indeed belongs to $\mathcal{J}$.

Lemma 5. *For $\tilde{\mu} \in (0,1)$, there exists $\tilde{\mu}' \in [0, \mu)$ such that:*

$$\delta J(\tilde{\mu}') = v_h - \frac{1 - \tilde{\mu}'}{1 - \tilde{\mu}}\big(v_h - J(\tilde{\mu})\big),$$

if and only if $\tilde{\mu} \geq \frac{(1-\delta)v_l}{v_h - \delta v_l} \equiv \bar{\mu} \in (0, \frac{v_l}{v_h})$. *In this case,* $\tilde{\mu}'$ *is unique.*

Proof. Uniqueness is guaranteed by the fact that J is convex, and $\delta J(\tilde{\mu}) < J(\tilde{\mu})$. If $\delta J(0) \geq v_h - \frac{v_h - J(\tilde{\mu})}{1-\tilde{\mu}}$, then a solution exists by the intermediate value theorem. Otherwise, no solution exists, by convexity. Therefore, a solution exists if and only if:

$$J(\tilde{\mu}) \leq \tilde{\mu} v_h + (1-\tilde{\mu})\delta v_l.$$

This inequality is satisfied if $\tilde{\mu} \geq \frac{v_l}{v_h}$ since $J(\tilde{\mu}) \leq \max\{\tilde{\mu} v_h, v_l\}$. For $\tilde{\mu} < \frac{v_l}{v_h}$, $J(\tilde{\mu}) = v_l$ and the inequality is equivalent to $\tilde{\mu} \geq \bar{\mu}$. □

In view of lemma 5, define on $[0, 1)$ the function $\tilde{\mu}'$ such that $\tilde{\mu}'(\tilde{\mu}) = 0$ if $\tilde{\mu} \leq \bar{\mu}$, and:

$$\delta J(\tilde{\mu}') = v_h - \frac{1-\tilde{\mu}'}{1-\tilde{\mu}}\big(v_h - J(\tilde{\mu})\big),$$

otherwise. Now, given $\mu_1 \in (0, 1)$, we construct the non-increasing sequence $(\mu_k)_{k\geq 1}$ such that, for $k \geq 1$, $\mu_{k+1} = \tilde{\mu}'(\mu_k)$.

Lemma 6. *There exists* $T \geq 1$ *such that* $\mu_T < \bar{\mu}$.

Proof. Otherwise, $(\mu_k)_{k\geq 1}$ has a limit $\mu_\infty \in [\bar{\mu}, 1)$. Since J is continuous on $[0, 1)$, the limit must satisfy:

$$\delta J(\mu_\infty) = J(\mu_\infty),$$

which is impossible since $J \geq v_l > 0$ and $\delta < 1$. □

We define $T = \min\{t \geq 1 : \mu_t < \bar{\mu}\}$. Now, let:

$$\rho^J(\mu_1) = \frac{v_l}{v_h - v_l} \prod_{t=1}^{T} \left(1 + \frac{1-\delta}{\delta} \frac{v_h}{v_h - J(\mu_t) - (1-\mu_t)J'_+(\mu_t)}\right) \in (0, \infty],$$

where J'_+ is the right-derivative of J. By convexity and piece-wise linearity of J, ρ^J is a non-decreasing and right-continuous step function, which may take infinite values if $v_h = J(\mu_1) + (1-\mu_1)J'_+(\mu_1)$.

Lemma 7. *For any* $\mu_1 \in (0, 1)$, $\rho^J(\mu_1) > \frac{\mu_1}{1-\mu_1}$.

Proof. If $T = 1$, then:

$$\frac{\mu_1}{1-\mu_1} < \frac{\bar{\mu}}{1-\bar{\mu}} = \frac{(1-\delta)v_l}{v_h - v_l} = \frac{\rho^J(\mu_1)}{\frac{1}{1-\delta} + \frac{1}{\delta}\frac{v_h}{v_h - v_l}} < \rho^J(\mu_1).$$

If $T > 1$, note that for $t < T$, since $\mu_{t+1} < \mu_t$ and J is convex:

$$J'_+(\mu_t) \geq \frac{J(\mu_t) - J(\mu_{t+1})}{\mu_t - \mu_{t+1}}.$$

As a result:

$$1+\frac{1-\delta}{\delta}\frac{v_h}{v_h - J(\mu_t) - (1-\mu_t)J'_+(\mu_t)} \geq \frac{(\mu_t - \mu_{t+1})v_h - \delta(1-\mu_{t+1})J(\mu_t) + \delta(1-\mu_t)J(\mu_{t+1})}{\delta\big[(\mu_t - \mu_{t+1})v_h - (1-\mu_{t+1})J(\mu_t) + (1-\mu_t)J(\mu_{t+1})\big]}.$$

Substitute in the numerator $\delta J(\mu_{t+1}) = v_h - \frac{1-\mu_{t+1}}{1-\mu_t}\big(v_h - J(\mu_t)\big)$, and in the denominator $J(\mu_t) = v_h - \frac{1-\mu_t}{1-\mu_{t+1}}\big(v_h - \delta J(\mu_{t+1})\big)$, to obtain:

$$1 + \frac{1-\delta}{\delta}\frac{v_h}{v_h - J(\mu_t) - (1-\mu_t)J'_+(\mu_t)} \geq \frac{(1-\mu_{t+1})J(\mu_t)}{\delta(1-\mu_t)J(\mu_{t+1})}.$$

As a result:

$$\rho^J(\mu_1) \geq \frac{1-\mu_T}{\delta^T}\frac{v_h - \delta v_l}{(v_h - v_l)^2}\frac{J(\mu_1)}{1-\mu_1}.$$

Since $J(\mu_1) = v_h - \frac{1-\mu_1}{1-\mu_2}\big(v_h - \delta J(\mu_2)\big)$, we have:

$$\frac{1-\mu_T}{\delta^T}\frac{v_h - \delta v_l}{(v_h - v_l)^2}\frac{J(\mu_1)}{1-\mu_1} - \frac{\mu_1}{1-\mu_1}$$

$$= \frac{1-\mu_T}{\delta^{T-1}}\frac{v_h - \delta v_l}{(v_h - v_l)^2}\frac{J(\mu_2)}{1-\mu_2} - \frac{\mu_2}{1-\mu_2} + \left(\frac{1-\mu_T}{\delta^T}\frac{v_h - \delta v_l}{(v_h - v_l)^2}v_h - 1\right)\left(\frac{1}{1-\mu_1} - \frac{1}{1-\mu_2}\right).$$

Now, since $\mu_T < \bar{\mu} = \frac{(1-\delta)v_l}{v_h - \delta v_l}$:

$$\frac{1-\mu_T}{\delta^T}\frac{v_h - \delta v_l}{(v_h - v_l)^2}v_h - 1 > \frac{1}{\delta^T}\frac{v_h}{v_h - v_l} - 1 > 0,$$

and since $\mu_1 > \mu_2$, we obtain:

$$\frac{1-\mu_T}{\delta^T}\frac{v_h - \delta v_l}{(v_h - v_l)^2}\frac{J(\mu_1)}{1-\mu_1} - \frac{\mu_1}{1-\mu_1} > \frac{1-\mu_T}{\delta^{T-1}}\frac{v_h - \delta v_l}{(v_h - v_l)^2}\frac{J(\mu_2)}{1-\mu_2} - \frac{\mu_2}{1-\mu_2}.$$

The same argument applies by induction to establish:

$$\frac{1-\mu_T}{\delta^T}\frac{v_h - \delta v_l}{(v_h - v_l)^2}\frac{J(\mu_1)}{1-\mu_1} - \frac{\mu_1}{1-\mu_1} > \frac{1}{\delta}\frac{v_h - \delta v_l}{(v_h - v_l)^2}v_l - \frac{\mu_T}{1-\mu_T}$$

$$> \frac{v_l}{v_h - v_l}\frac{(1-\delta)v_h + \delta^2(v_h - v_l)}{\delta(v_h - v_l)} > 0.$$

To summarise:

$$\rho^J(\mu_1) > \frac{1-\mu_T}{\delta^T}\frac{v_h-\delta v_l}{(v_h-v_l)^2}\frac{J(\mu_1)}{1-\mu_1} > \frac{\mu_1}{1-\mu_1},$$

which proves the claim. □

Now, given the prior $\mu \in (\frac{v_l}{v_h}, 1)$, let:

$$\mu_1^* = \min\Big\{\mu_1 \in [0,1) : \rho^J(\mu_1) > \frac{\mu}{1-\mu}\Big\}.$$

As above, we iterate on $\tilde{\mu}'$ to construct the path $(\mu_1^*, \mu_2^*, ..., \mu_{T^*}^*)$, where $\mu_{T^*}^* \in [0, \bar{\mu})$. Defining $\mu_0^* = \mu$ and $\mu_{T^*+1}^* = 0$, the candidate solution $\langle s^* \rangle(\mu, J)$ is characterised by:

$$\forall t \in \{0, ..., T^*\}, \quad q_t^{*h} = \frac{1-\mu}{\mu}\Big(\frac{1}{1-\mu_t^*} - \frac{1}{1-\mu_{t+1}^*}\Big),$$

$$q_{T^*}^{*l} = 1 - q_{T^*+1}^{*l} = \frac{(1-\delta)v_l - \mu_{T^*}^*(v_h - \delta v_l)}{(1-\delta)(1-\mu_{T^*}^*)v_l}.$$

Proposition 4. *$\langle s^* \rangle(\mu, J)$ solves problem (B.1).*

Proof. To simplify notations, we omit the dependence on μ and J of $\langle s^* \rangle$, Ω and G. We first define a sequence of non-negative Lagrange multipliers $(\lambda_t)_{t\geq 1}$ as follows.

If:

$$\frac{v_l}{v_h-v_l}\prod_{t=1}^{T^*}\left(1+\frac{1-\delta}{\delta}\frac{v_h}{v_h - J(\mu_t^*) - (1-\mu_t^*)J'_-(\mu_t^*)}\right) \leq \frac{\mu}{1-\mu}$$

$$< \frac{v_l}{v_h-v_l}\prod_{t=1}^{T^*}\left(1+\frac{1-\delta}{\delta}\frac{v_h}{v_h - J(\mu_t^*) - (1-\mu_t^*)J'_+(\mu_t^*)}\right),$$

where $J'_-(0) = 0$, then there exists $\big(J'_*(\mu_t^*)\big)_{1\leq t\leq T^*} \in \prod_{1\leq t\leq T^*}\big[J'_-(\mu_t^*), J'_+(\mu_t^*)\big]$ such that:

$$\frac{\mu}{1-\mu} = \frac{v_l}{v_h-v_l}\prod_{t=1}^{T^*}\left(1+\frac{1-\delta}{\delta}\frac{v_h}{v_h - J(\mu_t^*) - (1-\mu_t^*)J'_*(\mu_t^*)}\right).$$

Otherwise, it must be that $\mu_{T^*}^* = 0$, and:

$$\frac{v_l}{v_h-v_l}\prod_{t=1}^{T^*-1}\left(1+\frac{1-\delta}{\delta}\frac{v_h}{v_h - J(\mu_t^*) - (1-\mu_t^*)J'_-(\mu_t^*)}\right) \leq \frac{\mu}{1-\mu}$$

$$< \frac{v_l}{v_h-v_l}\prod_{t=1}^{T^*}\left(1+\frac{1-\delta}{\delta}\frac{v_h}{v_h - J(\mu_t^*) - (1-\mu_t^*)J'_-(\mu_t^*)}\right).$$

Then, we define $J'_*(\mu_t^*) = J'_-(\mu_t^*)$ for all $t \in \{1, ..., T^*\}$.

In both cases, for $t \in \{1, ..., T^*\}$, let:

$$\lambda_t = \frac{1-\delta}{\delta} \frac{v_h}{v_h - J(\mu_t^*) - (1-\mu_t^*)J_*'(\mu_t^*)} \prod_{k=1}^{t-1} \left(1 + \frac{1-\delta}{\delta} \frac{v_h}{v_h - J(\mu_k^*) - (1-\mu_k^*)J_*'(\mu_k^*)}\right),$$

and for $t > T^*$, let $\lambda_t = 0$. We also introduce the notation, for $t \geq 0$:

$$\Lambda_t = \sum_{k=1}^{t} \lambda_k = -1 + \prod_{k=1}^{\min\{T^*,t\}} \left(1 + \frac{1-\delta}{\delta} \frac{v_h}{v_h - J(\mu_k^*) - (1-\mu_k^*)J_*'(\mu_k^*)}\right).$$

With these definitions, we establish below that $\langle s^* \rangle$ maximises on $\mathcal{Q} \times \mathcal{Q}$ the Lagrangian:

$$\Omega\big(\langle s \rangle\big) - \sum_{t=1}^{\infty} \delta^t \lambda_t G_t\big(\langle s \rangle\big).$$

It follows that $\langle s^* \rangle$ solves problem (B.1), since for any $\langle s \rangle$ feasible, $G\big(\langle s \rangle\big) \leq 0$, so:

$$\Omega\big(\langle s \rangle\big) \leq \Omega\big(\langle s \rangle\big) - \sum_{t=1}^{\infty} \delta^t \lambda_t G_t\big(\langle s \rangle\big) \leq \Omega\big(\langle s^* \rangle\big) - \sum_{t=1}^{\infty} \delta^t \lambda_t G_t\big(\langle s^* \rangle\big) = \Omega\big(\langle s^* \rangle\big).$$

For $\langle s \rangle \in \mathcal{Q} \times \mathcal{Q}$, let:

$$\begin{aligned} f_{\langle s \rangle} &: [0,1] \to \mathbb{R} \\ &\alpha \mapsto \Omega\big(\alpha \langle s \rangle + (1-\alpha)\langle s^* \rangle\big) - \sum_{t=1}^{\infty} \delta^t \lambda_t G_t\big(\alpha \langle s \rangle + (1-\alpha)\langle s^* \rangle\big). \end{aligned}$$

The desired result is implied if, for any $\langle s \rangle \in \mathcal{Q} \times \mathcal{Q}$, $f_{\langle s \rangle}$ is maximised at $\alpha = 0$. Since $f_{\langle s \rangle}$ is concave, it is sufficient to show that $f_{\langle s \rangle}$ is differentiable at $\alpha = 0$, with $f'_{\langle s \rangle}(0) \leq 0$.

Fix $\langle s \rangle \in \mathcal{Q} \times \mathcal{Q}$ and denote $\langle s_\alpha \rangle = \alpha \langle s \rangle + (1-\alpha)\langle s^* \rangle$. Ω is linear and $\lambda_t = 0$ when $t > T^*$, thus it is sufficient to show differentiability of the term $\alpha \mapsto G_t\big(\langle s_\alpha \rangle\big)$, when $t \in \{1, ..., T^*\}$. In this case, if $\mathbb{P}(s \geq t) > 0$, then:

$$\begin{aligned} G_t\big(\langle s_\alpha \rangle\big) =& \mathbb{P}(s_\alpha \geq t) J\left(\alpha \frac{\mathbb{P}(s \geq t)}{\mathbb{P}(s_\alpha \geq t)} \mu_t + (1-\alpha) \frac{\mathbb{P}(s^* \geq t)}{\mathbb{P}(s_\alpha \geq t)} \mu_t^*\right) - \alpha\Big(\mathbb{P}(s \geq t) J(\mu_t) - G_t\big(\langle s \rangle\big)\Big) \\ &- (1-\alpha)\Big(\mathbb{P}(s^* \geq t) J(\mu_t^*) - G_t\big(\langle s^* \rangle\big)\Big), \end{aligned}$$

where $\mathbb{P}(s_\alpha \geq t) = \alpha \mathbb{P}(s \geq t) + (1-\alpha)\mathbb{P}(s^* \geq t)$. This expression has a right- and left-derivative at any α, which coincide if $\alpha \frac{\mathbb{P}(s \geq t)}{\mathbb{P}(s_\alpha \geq t)} \mu_t + (1-\alpha) \frac{\mathbb{P}(s^* \geq t)}{\mathbb{P}(s_\alpha \geq t)} \mu_t^*$ is not at a kink of J

and write:

$$\begin{aligned}\frac{\partial G_t\big(\langle s_\alpha\rangle\big)}{\partial\alpha} =&\frac{\mathbb{P}(s\geq t)\mathbb{P}(s^*\geq t)}{\mathbb{P}(s_\alpha\geq t)}(\mu_t-\mu_t^*)J'\left(\alpha\frac{\mathbb{P}(s\geq t)}{\mathbb{P}(s_\alpha\geq t)}\mu_t+(1-\alpha)\frac{\mathbb{P}(s^*\geq t)}{\mathbb{P}(s_\alpha\geq t)}\mu_t^*\right)\\&+\Big(\mathbb{P}(s\geq t)-\mathbb{P}(s^*\geq t)\Big)J\left(\alpha\frac{\mathbb{P}(s\geq t)}{\mathbb{P}(s_\alpha\geq t)}\mu_t+(1-\alpha)\frac{\mathbb{P}(s^*\geq t)}{\mathbb{P}(s_\alpha\geq t)}\mu_t^*\right)\\&-\left(\Big(\mathbb{P}(s\geq t)J(\mu_t)-G_t\big(\langle s\rangle\big)\Big)-\Big(\mathbb{P}(s^*\geq t)J(\mu_t^*)-G_t\big(\langle s^*\rangle\big)\Big)\right),\end{aligned}$$

Now, using the fact that $G_t\big(\langle s\rangle\big)=\mathbb{P}\big(s\geq t\big)J(\mu_t)-\sum_{k=t}^{\infty}\delta^{k-t}\Big(q_k^h\mu v_h+q_k^l(1-\mu)v_l\Big)$ and $G_t\big(\langle s^*\rangle\big)=0$, we obtain:

$$\left.\frac{\partial G_t\big(\langle s_\alpha\rangle\big)}{\partial\alpha}\right|_{\alpha=0}=\mathbb{P}(s\geq t)\Big(J(\mu_t^*)+(\mu_t-\mu_t^*)J'_{\to\mu_t}(\mu_t^*)\Big)-\sum_{k=t}^{\infty}\delta^{k-t}\Big(q_k^h\mu v_h+q_k^l(1-\mu)v_l\Big), \tag{B.2}$$

or equivalently:

$$\begin{aligned}\left.\frac{\partial G_t\big(\langle s_\alpha\rangle\big)}{\partial\alpha}\right|_{\alpha=0}=&\mu\Big(1-\sum_{k=0}^{t-1}q_k^h\Big)\Big(J(\mu_t^*)\\&+(1-\mu_t^*)J'_{\to\mu_t}(\mu_t^*)\Big)+(1-\mu)\Big(1-\sum_{k=0}^{t-1}q_k^l\Big)\Big(J(\mu_t^*)-\mu_t^*J'_{\to\mu_t}(\mu_t^*)\Big)\\&-\sum_{k=t}^{\infty}\delta^{k-t}\Big(q_k^h\mu v_h+q_k^l(1-\mu)v_l\Big),\end{aligned}$$

where:

$$J'_{\to\mu_t}(\mu_t^*)=\begin{cases}J'_-(\mu_t^*) & \text{if } \mu_t<\mu_t^*,\\ J'_+(\mu_t^*) & \text{if } \mu_t>\mu_t^*,\\ J'_*(\mu_t^*) & \text{if } \mu_t=\mu_t^*.\end{cases}$$

This expression is valid if $\mathbb{P}(s\geq t)=0$, in which case $G_t\big(\langle s_\alpha\rangle\big)=(1-\alpha)G_t\big(\langle s^*\rangle\big)=0$, if we extend the definition $J'_{\to\mu_t}(\mu_t^*)=J'_*(\mu_t^*)$ when $\mathbb{P}(s\geq t)=0$. Thus, the derivative at $\alpha=0$ of $f_{\langle s\rangle}$ writes:

$$\begin{aligned}f'_{\langle s\rangle}(0)=&-\Omega\big(\langle s^*\rangle\big)+\sum_{k=0}^{\infty}\delta^k\Big(q_k^h\mu v_h+q_k^l(v_l-\mu v_h)\Big)\\&-\sum_{t=1}^{T^*}\delta^t\lambda_t\left(\mu\Big(1-\sum_{k=0}^{t-1}q_k^h\Big)\Big(J(\mu_t^*)+(1-\mu_t^*)J'_{\to\mu_t}(\mu_t^*)\Big)\right.\\&\left.+(1-\mu)\Big(1-\sum_{k=0}^{t-1}q_k^l\Big)\Big(J(\mu_t^*)-\mu_t^*J'_{\to\mu_t}(\mu_t^*)\Big)-\sum_{k=t}^{\infty}\delta^{k-t}\Big(q_k^h\mu v_h+q_k^l(1-\mu)v_l\Big)\right).\end{aligned}$$

Rearranging, we get:

$$
\begin{aligned}
f'_{\langle s\rangle}(0) = &-\Omega\big(\langle s^*\rangle\big) + \sum_{k=0}^{\infty}\delta^k\Big[q_k^h\mu v_h(1+\Lambda_k) + q_k^l\big(v_l - \mu v_h + (1-\mu)v_l\Lambda_k\big)\Big] \\
&+ \sum_{k=0}^{T^*-1}\Bigg(\mu q_k^h\sum_{t=k+1}^{T^*}\delta^t\lambda_t\Big(J(\mu_t^*) \\
&\quad + (1-\mu_t^*)J'_{\to\mu_t}(\mu_t^*)\Big) + (1-\mu)q_k^l\sum_{t=k+1}^{T^*}\delta^t\lambda_t\Big(J(\mu_t^*) - \mu_t^*J'_{\to\mu_t}(\mu_t^*)\Big)\Bigg) \\
&- \sum_{t=1}^{T^*}\delta^t\lambda_t\Big(J(\mu_t^*) + (\mu-\mu_t^*)J'_{\to\mu_t}(\mu_t^*)\Big).
\end{aligned}
\tag{B.3}
$$

Using equation (B.2), note that: $f'_{\langle s\rangle}(0) \leq H\big(\langle s\rangle\big)$, where:

$$
\begin{aligned}
H\big(\langle s\rangle\big) = &-\Omega\big(\langle s^*\rangle\big) + \sum_{k=0}^{\infty}\delta^k\Big[q_k^h\mu v_h(1+\Lambda_k) + q_k^l\big(v_l - \mu v_h + (1-\mu)v_l\Lambda_k\big)\Big] \\
&+ \sum_{k=0}^{T^*-1}\Bigg(\mu q_k^h\sum_{t=k+1}^{T^*}\delta^t\lambda_t\Big(J(\mu_t^*) \\
&\quad + (1-\mu_t^*)J'_*(\mu_t^*)\Big) + (1-\mu)q_k^l\sum_{t=k+1}^{T^*}\delta^t\lambda_t\Big(J(\mu_t^*) - \mu_t^*J'_*(\mu_t^*)\Big)\Bigg) \\
&- \sum_{t=1}^{T^*}\delta^t\lambda_t\Big(J(\mu_t^*) + (\mu-\mu_t^*)J'_*(\mu_t^*)\Big).
\end{aligned}
\tag{B.4}
$$

H is linear on $\mathcal{Q}\times\mathcal{Q}$. Denote, for all $t \geq 0$, γ_t^h and γ_t^l the terms multiplying q_t^h and q_t^l respectively. If $t \geq T^*$, $\gamma_t^h = \mu v_h\delta^t(1+\Lambda_{T^*})$ is positive and decreasing in t. If $t \in \{1, ..., T^*\}$:

$$
\begin{aligned}
\gamma_t^h &= \mu v_h\delta^t(1+\Lambda_t) + \mu\sum_{k=t+1}^{T^*}\delta^k\lambda_k\Big(J(\mu_k^*) + (1-\mu_k^*)J'_*(\mu_k^*)\Big) \\
&= \gamma_{t-1}^h - (1-\delta)\mu v_h\delta^{t-1}(1+\Lambda_{t-1}) + \mu\delta^t\Big(v_h - J(\mu_t^*) - (1-\mu_t^*)J'_*(\mu_t^*)\Big)\lambda_t \\
&= \gamma_{t-1}^h,
\end{aligned}
$$

where we have used the fact that:

$$
\lambda_t = 1+\Lambda_t - (1+\Lambda_{t-1}) = \frac{1-\delta}{\delta}\frac{v_h}{v_h - J(\mu_t^*) - (1-\mu_t^*)J'(\mu_t^*)}(1+\Lambda_{t-1}).
$$

Now, for $t \in \{1, ..., T^*\}$:

$$\begin{aligned}\gamma_t^l &= \delta^t\big(v_l - \mu v_h + (1-\mu)v_l\Lambda_t\big) + (1-\mu)\sum_{k=t+1}^{T^*}\delta^k\lambda_k\Big(J(\mu_k^*) - \mu_k^* J_*'(\mu_k^*)\Big)\\ &= \gamma_{t-1}^l + (1-\mu)\delta^t\lambda_t\Big(v_l - J(\mu_t^*) + \mu_t^* J_*'(\mu_t^*)\Big)\\ &\qquad + (1-\delta)\delta^{t-1}\Big(\mu(v_h - v_l) - (1-\mu)v_l(1+\Lambda_{t-1})\Big).\end{aligned}$$

By convexity, $v_l = J(0) \geq J(\mu_t^*) - \mu_t^* J_*'(\mu_t^*)$. In addition:

$$1 + \Lambda_{t-1} \leq 1 + \Lambda_{T^*-1} \leq \frac{v_h - v_l}{v_l}\frac{\mu}{1-\mu}.$$

It follows that $\gamma_t^l \geq \gamma_{t-1}^l$. If $t \geq T^*$, $\gamma_t^l = \delta^t\big(v_l - \mu v_h + (1-\mu)v_l\Lambda_{T^*}\big)$. When $\mu_{T^*}^* > 0$, $1 + \Lambda_{T^*} = \frac{v_h - v_l}{v_l}\frac{\mu}{1-\mu}$, so $\gamma_t^l = 0$ for all $t \geq T^*$. In any case, $1 + \Lambda_{T^*} \geq \frac{v_h - v_l}{v_l}\frac{\mu}{1-\mu}$, and γ_t^l is non-negative and non-increasing in t for $t \geq T^*$.

It follows that H is maximised at $\langle s^*\rangle$, that is, for any $\langle s\rangle \in \mathcal{Q} \times \mathcal{Q}$:

$$f_{\langle s\rangle}'(0) \leq H\big(\langle s\rangle\big) \leq H\big(\langle s^*\rangle\big) = f_{\langle s^*\rangle}'(0) = 0,$$

which proves the desired result. □

Remark 1. *For each μ and J, we have described a solution $\langle s^*\rangle(\mu, J)$. If there exists $\mu_1 \in [0,1)$ such that $\rho^J(\mu_1) = \frac{\mu}{1-\mu}$, we can describe an alternative solution $\langle \hat{s}^*\rangle(\mu, J)$ by setting:*

$$\hat{\mu}_1^* = \min\Big\{\mu_1 \in [0,1) : \rho^J(\mu_1) = \frac{\mu}{1-\mu}\Big\},$$

and the continuing path of beliefs as before. All the arguments in the proof of proposition (4) directly apply. Since ρ^J is a step function, $\langle \hat{s}^*\rangle(\mu, J)$ is only defined for a discrete set of priors μ.

Remark 2. *For each μ and J, there exists a direct and simple contract that implements $\langle s^*\rangle(\mu, J)$. The probabilities of trade for each type of the buyer are given by q^{*h} and q^{*l}. For every $t \geq 1$, the trading price if the buyer's report was v_i is $p_t^i = v_i$. In the initial period of deployment, the low-valuation buyer trades at price $p_0^l = v_l$, while the price for the high-valuation buyer is such that:*

$$q_0^{*h}(v_h - p_0^h) = \delta^{T^*}\big(q_{T^*}^{*l} + \delta(1 - q_{T^*}^{*l})\big)(v_h - v_l).$$

The same applies to $\langle \hat{s}^*\rangle(\mu, J)$. We denote $D(\mu, J)$ the set of those contracts (which contains *either one or two elements).*

Proposition 5. *$\mathcal{A}J$ belongs to $\mathcal{J}$.*

Proof. For $\mu \leq \frac{v_l}{v_h}$, $\mathcal{A}J(\mu) = v_l$. For $\mu \in \left(\frac{v_l}{v_h}, \frac{\rho^J(0)}{1+\rho^J(0)}\right]$, $\mathcal{A}J(\mu) = (1-\delta)\mu v_h + \delta v_l$. Note that $\mathcal{A}J$ is thus continuous at $\frac{v_l}{v_h}$. Similarly, if μ_1^* is a point of discontinuity of ρ^J, and $(\mu_1^*, ..., \mu_{T^*}^*)$ the corresponding path of beliefs obtained by iteration on $\tilde{\mu}'$, then for every $\mu \in \left[\frac{\rho_-^J(\mu_1^*)}{1+\rho_-^J(\mu_1^*)}, \frac{\rho^J(\mu_1^*)}{1+\rho^J(\mu_1^*)}\right]$, where $\rho_-^J(\mu_1^*)$ denotes the left limit of ρ^J at μ_1^*, we have:

$$\begin{aligned}\mathcal{A}J(\mu) =&\left((1-\delta)\sum_{t=1}^{T^*}\frac{\delta^{t-1}}{1-\mu_t^*} + \delta^{T^*}\frac{\mu_{T^*}^*}{1-\mu_{T^*}^*}\frac{v_h-v_l}{v_l}\right)\mu v_h \\ &+ \left(1-(1-\delta)\sum_{t=1}^{T^*}\frac{\delta^{t-1}}{1-\mu_t^*} - \frac{\delta^{T^*}}{1-\mu_{T^*}^*}\frac{v_h-v_l}{v_h}\right)v_h.\end{aligned}$$

Thus $\mathcal{A}J$ is piece-wise linear. In addition, by continuity at the boundaries, $\mathcal{A}J$ is non-decreasing. The term $\left((1-\delta)\sum_{t=1}^{T^*}\frac{\delta^{t-1}}{1-\mu_t^*} + \delta^{T^*}\frac{\mu_{T^*}^*}{1-\mu_{T^*}^*}\frac{v_h-v_l}{v_l}\right)$ is increasing in μ_1^*, therefore $\mathcal{A}J$ is convex. It is clear that $\mathcal{A}J \geq v_l$. Now, for $\mu > \frac{v_l}{v_h}$ and $\langle s\rangle \in \mathcal{Q}\times\mathcal{Q}$:

$$\mu v_h - \Omega\big(\langle s\rangle\big|\mu\big) = \left(1-\sum_{k=0}^{\infty}\delta^k q_k^h\right)\mu v_h + \sum_{k=0}^{\infty}\delta^k q_k^l(\mu v_h - v_l) \geq 0,$$

so $\mathcal{A}J \leq \bar{J}$. ∎

Lemma 8. *For all $J, \hat{J} \in \mathcal{J}$, if for all $\mu \in [0,1]$, $J(\mu) \geq \hat{J}(\mu)$, then for all $\mu \in [0,1]$, $\mathcal{A}J(\mu) \leq \mathcal{A}\hat{J}(\mu)$.*

Proof. Under the assumption of the lemma, for any $\langle s\rangle \in \mathcal{Q}\times\mathcal{Q}$ and $\mu \in [0,1]$, $G\big(\langle s\rangle\big|\mu, J\big) \geq G\big(\langle s\rangle\big|\mu, \hat{J}\big)$. Thus $\langle s^*\rangle(\mu, J)$ is feasible in problem (B.1) for $\hat{J}$, from which the conclusion follows. ∎

Proposition 6. *The operator $\mathcal{A} : \mathcal{J} \to \mathcal{J}$ has a unique fixed point J^*.*

Proof. By lemma 8, and since $\bar{J}$ is an upper bound on any element of $\mathcal{J}$, any fixed point J^* of $\mathcal{A}$ must satisfy:

$$\mathcal{A}\bar{J} \leq J^* \leq \bar{J},$$

and by immediate induction:

$$\forall n \geq 0, \quad \mathcal{A}^{2n+1}\bar{J} \leq J^* \leq \mathcal{A}^{2n}\bar{J}.$$

Therefore, existence and uniqueness of a fixed point are guaranteed if $(\mathcal{A}^n\bar{J})_{n\geq 0}$ converges in $\mathcal{J}$. Note that for all $n,m \geq 0$, $\mathcal{A}^n\bar{J}$ and $\mathcal{A}^m\bar{J}$ coincide on $[0,\hat{\mu}_0)$, where $\hat{\mu}_0 = \frac{v_l}{v_h}$. Therefore, $\rho^{\mathcal{A}^n\bar{J}}$ and $\rho^{\mathcal{A}^m\bar{J}}$ also coincide on $[0,\hat{\mu}_0)$, thus $\mathcal{A}^{n+1}\bar{J}$ and $\mathcal{A}^{m+1}\bar{J}$ coincide on $[0,\hat{\mu}_1)$, where $\hat{\mu}_1 = \frac{\rho_-^{\bar{J}}(\hat{\mu}_0)}{1+\rho_-^{\bar{J}}(\hat{\mu}_0)} > \hat{\mu}_0$. By immediate induction, for every $k \geq 0$ and $n,m \geq k$, $\mathcal{A}^n\bar{J}$ and $\mathcal{A}^m\bar{J}$ coincide on $[0,\hat{\mu}_k)$, where the sequence $(\hat{\mu}_k)_{k\geq 0}$ is constructed such that $\hat{\mu}_{k+1} = \frac{\rho_-^{\mathcal{A}^k\bar{J}}(\hat{\mu}_k)}{1+\rho_-^{\mathcal{A}^k\bar{J}}(\hat{\mu}_k)}$. $(\hat{\mu}_k)_{k\geq 0}$ is an increasing sequence. By lemma 7, its limit must be $\hat{\mu}_\infty = 1$. It follows that for every $\mu \in [0,1]$, $\mathcal{A}^n\bar{J}(\mu)$ remains constant for n sufficiently large (recall that $\mathcal{A}^n\bar{J}(1) = v_h$ for all $n \geq 0$). Denote $J^*(\mu)$ this constant.

By construction, $J^* \in \mathcal{J}$ and $\mathcal{A}J^* = J^*$. □

For $\mu \in [0,1]$, we denote $D^*(\mu) = D(\mu, J^*)$. If a contract $d \in D^*(\mu)$ is deployed actively and truthfully in every period, the seller's payoff is $J^*(\mu)$ and the low-valuation buyer's payoff is always $U_l(\mu) = 0$. The high-valuation buyer's payoff depends on the contract, and we denote $U_h(\mu)$ the convex-hull of those payoffs. Finally, denote $d^*(\mu) \in D^*(\mu)$ the contract such that $\max U_h(\mu)$ is achieved.

B.1.2 Auxiliary Games

Given a contract $d = (x_\tau, p_\tau)_{\tau=0}^{\infty} \in \mathcal{D}$ and belief $\mu \in [0,1]$, let $\Gamma(d,\mu)$ be the discontinuous dynamic psychological game defined as follows. The two players are the seller and the buyer. As in the main text, the buyer has two types v_l or v_h, and μ is the seller's common knowledge belief that the buyer has a high valuation. The game is played in discrete time. In the initial period, the contract d is deployed with $\tau = 0$. That is, the buyer chooses among $\{h, l, r\}$. If r is chosen, it is observed by the seller and the game proceeds to the next period. If instead $i \in \{h, l\}$ is selected, trade occurs with probability $x_0(v_i)$ at price $p_0(v_i)$, in which case the game ends. If trade does not occur, the game proceeds to the next period and the index in d is updated to $\tau = 1$. In any following period, the seller chooses among $\{C, S\}$. If S is selected, the game ends. If C is selected, the contract d is deployed again. That is, the buyer makes a report in $\{h, l, r\}$ if $\tau = 0$ or in $\{a, r\}$ if $\tau \geq 1$. As above, when r is selected, it is observed by the seller and the game proceeds to the next period. Otherwise, trade may occur or not as specified by the contract d.

The game ends either when trade occurs or when the seller chooses S. If trade occurs in period $t \geq 0$ at price p, the seller's payoff is $\delta^t p$ and the buyer's payoff is $\delta^t(v - p)$, where $v \in \{v_h, v_l\}$ is his valuation. If the game ends when the seller chooses S in period t, the payoffs are partially determined and depend on the seller's belief about the buyer's valuation at that information set $\hat{\mu}$. In particular, we use the approach of Simon and Zame (1990) and specify payoffs if the seller chooses S in period t with belief $\hat{\mu}$ as $\left(\delta^t J^*(\hat{\mu}), \delta^t U_l(\hat{\mu}), \delta^t U_h(\hat{\mu})\right)$ for the seller, low-valuation buyer and high-valuation buyer respectively. Since U_h is a correspondence, which element of $U_h(\hat{\mu})$ actually determines the high-valuation buyer's payoff is part of the solution concept.

As a result, we define a *payoff selection* u to be a mapping from terminal nodes following S to the real numbers. An *augmented assessment* is a triple (σ, α, u), where σ is a strategy profile, α is the seller's belief system and u is a payoff selection for the high-valuation buyer. u is said to be *consistent* with α if, at every terminal history $\mathfrak{h}$ following S, if the seller's belief that the buyer has a high valuation is $\mu_{\mathfrak{h}}$, then $u(\mathfrak{h}) \in U_h(\mu_{\mathfrak{h}})$. A *sequential equilibrium* of $\Gamma(d, \mu)$ is an augmented assessment (σ, α, u) such that (i) α is consistent with σ and the prior μ in the usual sense, (ii) u is consistent with α and (iii) σ is sequentially rational[1] given α and u.

Proposition 7. *Suppose that d specifies bounded transfers $(p_\tau)_{\tau \geq 0}$. Then $\Gamma(d, \mu)$ has a sequential equilibrium.*

Proof. We first consider the truncated version of the game $\Gamma_T(d, \mu)$ such that the game coincides with $\Gamma(d, \mu)$ until period T is reached, and for any period $t \geq T$, the seller's action space is restricted to $\{C\}$ and the buyer's action space is restricted to $\{r\}$.

For any $\varepsilon > 0$, there exists a continuous function $U_h^\varepsilon : [0,1] \to [0, v_h - v_l]$ such that any point $\left(\tilde{\mu}, U_h^\varepsilon(\tilde{\mu})\right)$ in the graph of U_h^ε is at a distance less than ε to the graph of U_h. Let $\Gamma_T^\varepsilon(\mu, d)$ be the psychological game corresponding to $\Gamma_T(\mu, d)$ in which the high-valuation buyer's payoff after the seller chooses S with belief $\hat{\mu}$ is $U_h^\varepsilon(\hat{\mu})$. By Theorem 9 of Battigalli and Dufwenberg (2009), $\Gamma_T^\varepsilon(d, \mu)$ has a sequential equilibrium $(\sigma^\varepsilon, \alpha^\varepsilon)$ (where the first component refers to the strategy profile and the second to the belief system).

Given $(\sigma^\varepsilon, \alpha^\varepsilon)$, let $\vec{\mu}^\varepsilon$ the vector listing the seller's beliefs that the buyer has a high valuation at all her information sets in $\Gamma_T^\varepsilon(d, \mu)$, and $U_h^\varepsilon(\vec{\mu}^\varepsilon)$ the vector of high-valuation payoffs whenever S is chosen. Since $\left(\sigma^\varepsilon, \alpha^\varepsilon, U_h^\varepsilon(\vec{\mu}^\varepsilon)\right)_\varepsilon$ lives in a compact set, it possesses an accumulation point $(\sigma_T, \alpha_T, u_T)$ as $\varepsilon \to 0$.

We claim that $(\sigma_T, \alpha_T, u_T)$ is a sequential equilibrium of $\Gamma_T(d, \mu)$. Indeed, α must be consistent for σ since α^ε is consistent for σ^ε for any $\varepsilon > 0$. Moreover, at every terminal

[1] The notion of sequential rationality extends naturally to psychological games. The reader is referred to Battigalli and Dufwenberg (2009) for details.

node following S, the seller's belief $\hat{\mu}^\varepsilon$ converges to $\hat{\mu}$, while the limit of $\left(\hat{\mu}^\varepsilon, U_h^\varepsilon(\hat{\mu}^\varepsilon)\right)$ must belong to the graph of U_h. Thus u_T must be a payoff selection for the high-valuation buyer. Finally, since the terminal payoffs of any player converge together with the assessment, for any strategy s_i of any player i, the evaluation of player i's expected payoff at any of her information sets under $(s_i, \sigma_{-i}^\varepsilon)$ also converges to that under (s_i, σ_{-i}). Since no profitable deviation exists under σ^ε, the same is true in the limit under σ.

σ_T specifies a full profile in $\Gamma(d, \mu)$. We can also extend α_T and u_T to construct a full assessment and payoff selection in $\Gamma(d, \mu)$. For the belief system α, take the limit of the belief system induced by a fully mixed buyer strategy after period T such that both types make a mistake with the same small probability at every decision node. Given the completed infinite vector $\vec{\mu}$, whenever the seller chooses S after period T with belief $\hat{\mu}$, select $U_h^*(\hat{\mu}) = \max U_h(\mu)$. Denote $\left(\bar{\sigma}_T, \bar{\alpha}_T, \bar{u}_T\right)$ the completed assessment and payoff selection in the infinite-horizon game $\Gamma(d, \mu)$.

Since transfers are bounded, discounting guarantees that $\left(\bar{\sigma}_T, \bar{\alpha}_T, \bar{u}_T\right)$ is a $\bar{p}\delta^T$ - equilibrium in $\Gamma(d, \mu)$. In addition, the set of augmented assessments is compact in the topology of Fudenberg and Levine (1983) (note that, given a belief system, a payoff selection is isomorphic to choosing a mixture between $\max U_h(\hat{\mu})$ and $\min U_h(\hat{\mu})$ at every information set of the seller, where $\hat{\mu}$ is her belief at that information set). Thus $\left(\bar{\sigma}_T, \bar{\alpha}_T, \bar{u}_T\right)$ has a converging subsequence, which must be a sequential equilibrium in $\Gamma(d, \mu)$. $\square$

B.1.3 Equilibrium Assessment

We construct an equilibrium in the version of the game presented in the main text in which the seller's contract space is restricted to include only simple and direct contracts, with the additional assumption that all transfers are bounded. Specifically, we assume that any transfer must be in the set $[\underline{p}, \bar{p}]$, where $\underline{p} < 0$ and $\bar{p} > v_h$.

The seller's prior is μ. Throughout we maintain that the seller's belief does not update following her own deviations. At the beginning of the game, the seller deploys the contract $d^*(\mu)$. Consider a history $\mathfrak{h}$ at which the seller's belief is $\mu_{\mathfrak{h}}$ and the seller offers a new contract d. If $d \in D^*(\mu_{\mathfrak{h}})$, the buyer accepts the contract and reports his valuation truthfully. In every following period, as long as trade does not occur, the seller's belief updates according to Bayes' rule, she deploys again d and the buyer accepts. If the buyer ever rejects d, the seller's belief does not update and she deploys d again. If $d \notin D^*(\mu_{\mathfrak{h}})$, the continuing assessment until the game ends or d is replaced is constructed as follows.

First note that, by the axiom of choice and proposition 7, we can select a sequential equilibrium of the auxiliary game $\Gamma(d, \mu_{\mathfrak{h}})$ for any $(d, \mu_{\mathfrak{h}})$. Let (σ, α, u) the chosen equilibrium. Note that, as long as d is deployed, all the decision nodes of the buyer are identical in $\Gamma(d, \mu_{\mathfrak{h}})$ and in the original game. Thus, the buyer's strategy at those nodes can be taken directly from the profile σ. Now, we translate the seller's strategy in σ to a strategy in the original game. We interpret the action C as deploying the contract d again. Note that, as long as d has been deployed, the seller's belief at every information set can be taken directly from α. Finally, when the seller chooses S in the auxiliary game, with belief $\hat{\mu}$, u specifies a payoff selection for the high-valuation buyer $\hat{u} \in U_h(\hat{\mu})$. We specify that the contract d is replaced by a contract $\hat{d} \in D^*(\hat{\mu})$, which results from the unique mixture across the contracts in $D^*(\hat{\mu})$ that delivers expected payoff $\hat{u}$ to the high-valuation buyer (recall that $D^*(\hat{\mu})$ contains at most two contracts), if $\hat{d}$ is to be deployed actively and truthfully forever after. Once d is replaced, the continuing assessment is described in the previous paragraph.

Next, we prove that the above assessment is indeed an equilibrium. By construction, the assessment satisfies updating consistency in the sense of Perea (2002). Therefore the one-shot deviation principle applies. It is clear that the buyer has no incentive to deviate. Next, we establish that the seller has no profitable one-shot deviation at any information set.

It is sufficient to show that, at any history $\mathfrak{h}$ with belief $\mu_{\mathfrak{h}}$, the seller's payoff from deploying a new contract cannot exceed $J^*(\mu_{\mathfrak{h}})$ given the buyer's strategy. Then, by construction, it is clear that the seller's behaviour is sequentially rational.

Suppose that the seller's belief is $\mu_{\mathfrak{h}}$ and a new contract is offered. This is a one-shot deviation, thus the continuation path is that specified in the above assessment. In particular, at every subsequent history $\hat{\mathfrak{h}}$, if the seller's belief is $\mu_{\hat{\mathfrak{h}}}$, her continuation payoff is at least $J^*(\mu_{\hat{\mathfrak{h}}})$. Let s the random time at which trade occurs induced by the continuation path. Denoting by V, θ_h and θ_l the continuation payoffs for the seller, the high-valuation buyer and the low-valuation buyer respectively, we can express the total continuation surplus as:

$$V + \mu_{\mathfrak{h}}\theta_h + (1 - \mu_{\mathfrak{h}})\theta_l = \mathbb{E}_0[\delta^s|h]\mu_{\mathfrak{h}}v_h + \mathbb{E}_0[\delta^s|l](1 - \mu_{\mathfrak{h}})v_l.$$

Since a new contract is deployed, the high-valuation buyer could mimic the low-valuation buyer's strategy, thus:

$$\theta_h \geq \theta_l + \mathbb{E}_0[\delta^s|l](v_h - v_l).$$

In addition, the buyer can guarantee a non-negative payoff by rejecting the contract in every period, thus:

$$\theta_l \geq 0.$$

Together, these inequalities imply that the seller's continuation payoff satisfies:

$$V \leq \mathbb{E}_0[\delta^s|h]\mu_{\mathfrak{h}} v_h + \mathbb{E}_0[\delta^s|l](v_l - \mu_{\mathfrak{h}} v_h).$$

Finally, since J^* is convex, in any period, the total continuation surplus must exceed $J^*(\mu_t)$, where μ_t is the average belief of the seller across all histories reaching period t. That is:

$$\mathbb{E}_t\big[\delta^s|h\big]\mu_t v_h + \mathbb{E}_t\big[\delta^s|l\big](1-\mu_t)v_l \geq J^*(\mu_t).$$

In other words, it must be the case that $V \leq \mathcal{A}J^*(\mu_{\mathfrak{h}}) = J^*(\mu_{\mathfrak{h}})$. Thus the seller has no profitable one-shot deviation, which concludes the proof.

B.2 Continuous Types

In this appendix we will show how to modify the analysis in the main text to demonstrate that our result is robust in the case of continuous types. In particular, we first construct an abiding contract which achieves a payoff strictly bounded above the market clearing profit. This contract resembles the contract of Lemma 2 in the main text by treating the buyer as if his type was binary. In particular, the contract specifies a cut-off type $\hat{v} \in (\underline{v}, \overline{v})$ and treats all types $v < \hat{v}$ as the 'low type,' and all types $v \geq \hat{v}$ as the 'high type.' The allocation probabilities and prices across 'high' and 'low' types are structured similarly to the abiding contract in Lemma 2: only 'high' types trade in the initial period at a discount. Conditional on no trade in the initial period, all types trade at the same random time. 'High' types trade at the 'high' valuation, $\hat{v}$, while 'low' types trade at the 'low' valuation, $\underline{v}$.

The only significant difference is the abidance constraint. After no trade in the initial period the seller faces a residual market and we need to guarantee that the optimal monopoly price is the lowest valuation, $\underline{v}$. This is accomplished by choosing the cut-off type, $\hat{v}$, and the probability of trade in the initial period appropriately, so that the residual market is sufficiently deteriorated. Moreover, the payoff from continuing with the contract must exceed $\underline{v}$.

Proposition 8 shows that for each δ such an abiding contract can be found which delivers payoff greater than some $\underline{\pi} > \underline{v}$. Finally, in Proposition 9 we show that this $\underline{\pi}$ is a lower bound to the seller's best equilibrium profit. The argument proceeds similarly to the proof of Lemma 1 in the main text.

The Contract

Let F be the cdf of valuations on $[\underline{v}, \overline{v}]$, with $\underline{v} > 0$. We assume that $f(v) > 0$, for all $v \in [\underline{v}, \overline{v}]$ and that the revenue function $R(p) := p \cdot (1 - F(p))$ satisfies $R'(\underline{v}) > 0$. Note that this implies that market clearing is not the optimal monopoly price, $\max_p R(p) > \underline{v}$.

We consider the following stationary cut-off mechanism.

- The buyer submits a report $v \in [\underline{v}, \overline{v}]$.
- If $v \geq \hat{v}$, there is trade with probability α in the first period, at price p.
- In any future period trade occurs with probability β at price $\hat{v}$.
- If $v < \hat{v}$, there is no trade in the initial period.
- In any future period trade occurs with probability β at price $\underline{v}$.

We introduce the notation:

$$\psi = \frac{\beta\delta}{1 - \delta + \beta\delta} \in [0, \delta]$$

for the expected present value of the discount factor at the time of trade.

Incentive Compatibility

First, we note that this contract is IC whenever:

$$\alpha \cdot (v - p) + (1 - \alpha) \cdot \max\Big\{\psi \cdot (v - \hat{v}), 0\Big\} \geq \psi \cdot (v - \underline{v}) \quad \iff \quad v \geq \hat{v}$$

where the $\max\{\cdot\}$ controls for the buyer's participation decision in future periods.

Then, the IC is written as:

$$\alpha \cdot (v - p) + (1 - \alpha) \cdot \psi \cdot (v - \hat{v}) \geq \psi \cdot (v - \underline{v}) \quad \text{for all } v \geq \hat{v}$$

and

$$\psi \cdot (v - \underline{v}) \geq \alpha \cdot (v - p) \quad \text{for all } v \leq \hat{v}$$

The following lemma gives sufficient conditions for incentive compatibility.

Lemma 9. *The contract* $(\hat{v}, \alpha, \psi, p) \in (\underline{v}, \overline{v}) \times [0, 1] \times [0, \delta] \times \mathbb{R}^+$ *is IC if:*

$$\alpha \cdot (\hat{v} - p) = \psi \cdot (\hat{v} - \underline{v}) \tag{IC+}$$

and

$$\alpha \geq \psi \tag{IC-}$$

Proof. Let $\Delta_+(v) = \alpha \cdot (v - p) + (1 - \alpha) \cdot \psi \cdot (v - \hat{v}) - \psi \cdot (v - \underline{v})$. The IC for $v \geq \hat{v}$ is satisfied whenever $\Delta_+(v) \geq 0$ for $v \geq \hat{v}$. First, note that (IC+) implies that $\Delta_+(\hat{v}) = 0$. Hence, it is sufficient to verify that $\Delta_+(v)$ is increasing in v. We have:

$$\Delta'_+(v) = \alpha + (1 - \alpha)\psi - \psi = \alpha \cdot (1 - \psi) \geq 0$$

Therefore, the IC for $v \geq \hat{v}$ is satisfied.

Let $\Delta_-(v) = \psi \cdot (v - \underline{v}) - \alpha \cdot (v - p)$, so that the IC for $v \leq \hat{v}$ is satisfied if $\Delta_-(v) \geq 0$ for $v \leq \hat{v}$. Given (IC+) we have that the price p must be given by:

$$\alpha \cdot p = (\alpha - \psi) \cdot \hat{v} + \psi \cdot \underline{v}$$

and thus:

$$\Delta_-(v) = (\psi - \alpha) \cdot v - \psi\underline{v} + \alpha p = (\alpha - \psi) \cdot (\hat{v} - v) \geq 0$$

for $v \leq \hat{v}$, since $\alpha - \psi \geq 0$ by (IC-).

□

Abidance

Abidance comprises of two features: (i) The monopoly price on the residual market is $\underline{v}$ (market clearing); and (ii) the seller's continuation payoff within the mechanism is greater than $\underline{v}$.

First, we compute the seller's continuation payoff. This is given by:

$$\psi \cdot \Big[\mathbb{P}(v < \hat{v} | \neg \text{ trade}) \cdot \underline{v} + \mathbb{P}(v \geq \hat{v} | \neg \text{ trade}) \cdot \hat{v}\Big]$$

Given the trading probabilities specified above, we have:

$$\mathbb{P}(v \geq \hat{v} | \neg \text{ trade}) = \frac{(1-\alpha) \cdot (1 - F(\hat{v}))}{F(\hat{v}) + (1-\alpha) \cdot (1 - F(\hat{v}))}$$

The abidance constraint is hence given by:[2]

$$\psi \cdot (\delta - \psi) \left[\underline{v} + \frac{(1-\alpha) \cdot (1 - F(\hat{v}))}{F(\hat{v}) + (1-\alpha) \cdot (1 - F(\hat{v}))} \cdot (\hat{v} - \underline{v})\right] \geq (\delta - \psi) \cdot \underline{v} \tag{AC1}$$

Residual Market

We now compute the residual market conditional on no trade in the initial period. In particular, we are interested in $\mathbb{P}(v \geq p | \neg \text{ trade})$, which is given by:

$$\mathbb{P}(v \geq p | \neg \text{ trade}) = \frac{\int_p^{\overline{v}} \Big[1 - \alpha \cdot \mathbb{1}_{v \geq \hat{v}}\Big] dF(v)}{\int_{\underline{v}}^{\overline{v}} \Big[1 - \alpha \cdot \mathbb{1}_{v \geq \hat{v}}\Big] dF(v)}$$

Hence, we get the residual market:

$$D(p \,| \alpha, \hat{v}) = \mathbb{P}(v \geq p | \neg \text{ trade}) = \begin{cases} \frac{(1-\alpha) \cdot (1 - F(p))}{F(\hat{v}) + (1-\alpha) \cdot (1 - F(\hat{v}))}, & \text{if } p \geq \hat{v} \\ \\ \frac{(1-\alpha) \cdot (1 - F(\hat{v})) + F(\hat{v}) - F(p)}{F(\hat{v}) + (1-\alpha) \cdot (1 - F(\hat{v}))} & \text{if } p < \hat{v} \end{cases}$$

[2]Note that this is equivalent to the constraint in Lemma 2 of the main text. That is, continuing in the mechanism delivers more than the payoff from running the mechanism for a period more, and clearing immediately conditional on no trade.

Ex-ante payoffs

We now compute the seller's ex-ante payoff from a contract $(\hat{v}, \alpha, \psi, p) \in (\underline{v}, \overline{v}) \times [0, 1] \times [0, \delta] \times \mathbb{R}^+$:

$$\Pi(\hat{v}, \alpha, \psi, p) = (1 - F(\hat{v})) \cdot (\alpha \cdot p + (1 - \alpha)\psi\hat{v}) + F(\hat{v}) \cdot \psi\underline{v}$$

We can use (IC+) to eliminate the price, and re-write the problem as:

$$\Pi(\hat{v}, \alpha, \psi) = (1 - \psi)\alpha \cdot R(\hat{v}) + \psi\underline{v}$$

with the following constraints:

$$\alpha \geq \psi \tag{IC-}$$

$$\psi \cdot (\delta - \psi) \left[F(\hat{v}) \cdot \underline{v} + (1 - \alpha) \cdot R(\hat{v})\right] \geq \Big[1 - \alpha \cdot \big(1 - F(\hat{v})\big)\Big](\delta - \psi) \cdot \underline{v} \tag{AC1}$$

$$\sup_{p} \; p \cdot D(p \,|\alpha, \hat{v}) = \underline{v} \tag{AC2a}$$

$$D(p \,|\alpha, \hat{v}) = \begin{cases} \frac{(1-\alpha)\cdot(1-F(p))}{F(\hat{v})+(1-\alpha)\cdot(1-F(\hat{v}))}, & \text{if } p \geq \hat{v} \\ \\ \frac{(1-\alpha)\cdot(1-F(\hat{v}))+F(\hat{v})-F(p)}{F(\hat{v})+(1-\alpha)\cdot(1-F(\hat{v}))} & \text{if } p < \hat{v} \end{cases} \tag{AC2b}$$

Parameters

We are looking for parameters $(\hat{v}, \alpha, \psi) \in (\underline{v}, \overline{v}) \times [0, 1] \times [0, \delta]$ satisfying the following conditions:

$$\alpha \cdot R(\hat{v}) > \underline{v}$$

$$\psi < 1$$

$$\alpha \geq \psi$$

$$\psi \cdot [F(\hat{v}) \cdot \underline{v} + (1-\alpha) \cdot R(\hat{v})] \geq \Big[1 - \alpha \cdot \big(1 - F(\hat{v})\big)\Big]\underline{v}$$

$$\sup_p \; p \cdot D(p \,|\alpha, \hat{v}) = \underline{v}$$

The first two conditions guarantee the failure of the Coase Conjecture. The third ensures incentive compatibility for $v \leq \hat{v}$. The fourth is the abidance constraint for the seller. Note that the way this is written implicitly assumes that we can take $\psi < \delta$. This will turn out to be the case for sufficiently high δ—which is the relevant case. Finally, the last condition guarantees that market clearing is the monopoly price on the residual market.

Intuitively for all sufficiently low $\hat{v}$, there is a high enough α which deteriorates the residual market enough for market clearing to be the monopoly price.

We use this idea to specify for each $\hat{v}$, the following:

$$1 - \psi_{\hat{v}} = (\hat{v} - \underline{v})^2$$

$$1 - \alpha_{\hat{v}} = (\hat{v} - \underline{v})^2$$

with the intention of converging as $\hat{v} \downarrow \underline{v}$.

With this specification as $\hat{v}$ converges to $\underline{v}$, then $\hat{v} - \underline{v}$ converges to 0, slower than α and ψ are converging to 1. Note that $\alpha_{\hat{v}} = \psi_{\hat{v}}$ so (IC-) is taken care of;[3] while for $\hat{v} > \underline{v}$, $\psi_{\hat{v}} < 1$.

We now consider the remaining conditions.

Lemma 10. *There exists $\varepsilon_1 > 0$ such that for all $\hat{v} > \underline{v}$ with $\hat{v} - \underline{v} \leq \varepsilon_1$,*

$$\alpha_{\hat{v}} \cdot R(\hat{v}) > \underline{v}$$

Proof. Suppose that the inequality fails for all $\hat{v} > \underline{v}$. That is:

[3] In particular, this means that $p = \underline{v}$.

$$
\begin{aligned}
\alpha_{\hat{v}} \cdot R(\hat{v}) &\leq \underline{v} \\
\alpha_{\hat{v}} \cdot \Big[R(\hat{v}) - R(\underline{v})\Big] &\leq (1-\alpha_{\hat{v}})\underline{v} \\
\alpha_{\hat{v}} \cdot \Big[R(\hat{v}) - R(\underline{v})\Big] &\leq (\hat{v}-\underline{v})^2\underline{v} \\
\alpha_{\hat{v}} \cdot \frac{R(\hat{v}) - R(\underline{v})}{(\hat{v}-\underline{v})} &\leq (\hat{v}-\underline{v})\underline{v}
\end{aligned}
$$

where $R(\underline{v}) = \underline{v}$. Taking limits as $\hat{v} \downarrow \underline{v}$ we arrive at:

$$
R'(\underline{v}) \leq 0
$$

which contradicts the hypothesis that $R'(\underline{v}) > 0$.

□

Lemma 11. *There exists $\varepsilon_2 > 0$ such that for all $\hat{v} > \underline{v}$ such that $\hat{v} - \underline{v} \leq \varepsilon_2$,*

$$
\psi_{\hat{v}} \cdot [F(\hat{v}) \cdot \underline{v} + (1-\alpha_{\hat{v}}) \cdot R(\hat{v})] \geq \Big[1 - \alpha_{\hat{v}} \cdot \big(1 - F(\hat{v})\big)\Big]\underline{v}
$$

Proof. Re-writing the inequality we have:

$$
1 - \psi_{\hat{v}} \leq (1-\alpha_{\hat{v}}) \frac{(\hat{v}-\underline{v}) \cdot (1 - F(\hat{v})}{[F(\hat{v}) \cdot \underline{v} + (1-\alpha_{\hat{v}}) \cdot R(\hat{v})]}
$$

Suppose the inequality fails for all $\hat{v} > \underline{v}$. We then have:

$$
1 - \psi_{\hat{v}} > (1-\alpha_{\hat{v}}) \frac{(\hat{v}-\underline{v}) \cdot (1 - F(\hat{v})}{[F(\hat{v}) \cdot \underline{v} + (1-\alpha_{\hat{v}}) \cdot R(\hat{v})]} \Rightarrow 1 > \frac{(\hat{v}-\underline{v}) \cdot (1 - F(\hat{v}))}{[F(\hat{v}) \cdot \underline{v} + (\hat{v}-\underline{v})^2 \cdot R(\hat{v})]}
$$

which can be re-arranged to:

$$
\begin{aligned}
&\left[\frac{F(\hat{v})}{(\hat{v}-\underline{v})} \cdot \underline{v} + (\hat{v}-\underline{v}) \cdot R(\hat{v})\right] > (1 - F(\hat{v})) \\
\Rightarrow &\left[f(\underline{v}) \cdot \underline{v} + \frac{o((\hat{v}-\underline{v}))}{(\hat{v}-\underline{v})} + (\hat{v}-\underline{v}) \cdot R(\hat{v})\right] > (1 - F(\hat{v}))
\end{aligned}
$$

Taking limits as $\hat{v} \downarrow \underline{v}$ we arrive at:

$$f(\underline{v}) \cdot \underline{v} \geq 1$$

This is a contradiction to $R'(\underline{v}) > 0$ since:

$$R'(x) = 1 - F(x) - f(x) \cdot x \Rightarrow R'(\underline{v}) = 1 - f(\underline{v}) \cdot \underline{v} > 0$$

□

Lemma 12. *There exists $\varepsilon_3 > 0$ such that for all $\hat{v} > \underline{v}$ such that $\hat{v} - \underline{v} \leq \varepsilon_3$,*

$$\sup_p \; p \cdot D(p \,|\alpha_{\hat{v}}, \hat{v}) = \underline{v}$$

Proof. First, we have that for any $p \geq \hat{v}$,

$$R(p|\hat{v}) := p \cdot D(p|\alpha_{\hat{v}}, \hat{v}) \leq \overline{v} \cdot \frac{(\hat{v} - \underline{v})^2 \cdot (1 - F(\hat{v}))}{F(\hat{v}) + (\hat{v} - \underline{v})^2 \cdot (1 - F(\hat{v}))}$$

since the RHS is the profit from trading with all $v \geq \hat{v}$ at a price $p = \overline{v}$. Dividing both the numerator and the denominator by $(\hat{v} - \underline{v}) > 0$, we have:

$$\overline{v} \cdot \frac{(\hat{v} - \underline{v}) \cdot (1 - F(\hat{v}))}{F(\hat{v})/(\hat{v} - \underline{v}) + (\hat{v} - \underline{v}) \cdot (1 - F(\hat{v}))} \to 0$$

since $F(\hat{v})/(\hat{v} - \underline{v})$ converges to $f(\underline{v}) > 0$. Consequently, profits from prices $p \geq \hat{v}$ become arbitrarily small as $\hat{v} \downarrow \underline{v}$, so for sufficiently small $(\hat{v} - \underline{v}) > 0$ the optimal prices must be below $\hat{v}$.

We now consider prices $p < \hat{v}$. First note that:

$$R'(\underline{v}|\hat{v}) \cdot (\hat{v} - \underline{v}) = \frac{\hat{v} - \underline{v}}{F(\hat{v}) + (\hat{v} - \underline{v})^2 \cdot (1 - F(\hat{v}))} \cdot \Big[(\hat{v} - \underline{v})^2 \cdot (1 - F(\hat{v}))$$
$$+ F(\hat{v}) - F(\underline{v}) - \underline{v} \cdot f(\underline{v})\Big]$$
$$= \frac{1}{\frac{F(\hat{v})}{\hat{v} - \underline{v}} + (\hat{v} - \underline{v}) \cdot (1 - F(\hat{v}))} \cdot \Big[(\hat{v} - \underline{v})^2 \cdot (1 - F(\hat{v}))$$
$$+ F(\hat{v}) - F(\underline{v}) - \underline{v} \cdot f(\underline{v})\Big]$$
$$\to \frac{1}{f(\underline{v})} \cdot \Big[- \underline{v} \cdot f(\underline{v})\Big] = -\underline{v} \quad \text{as } \hat{v} \downarrow \underline{v}$$

Moreover, for $\underline{v} < p < \hat{v}$, we have:

$$R(p|\hat{v}) - R(\underline{v}|\hat{v}) = R'(\underline{v}|\hat{v}) \cdot (p - \underline{v}) + o(p - \underline{v})$$
$$< R'(\underline{v}|\hat{v}) \cdot (\hat{v} - \underline{v}) + o(p - \underline{v})$$

Since the RHS converges to a strictly negative limit as $\hat{v} \downarrow \underline{v}$, we can find $\varepsilon_3 > 0$, such that for all $\hat{v} > \underline{v}$ with $\hat{v} - \underline{v} \leq \varepsilon_3$,

$$R(p|\hat{v}) - R(\underline{v}|\hat{v}) < 0, \quad \text{for all } \underline{v} < p < \hat{v}$$

The claim follows since $R(\underline{v}|\hat{v}) = \underline{v}$.

□

We now put lemmata 6-8 together to prove the failure of the Coase Conjecture:

Lemma 13. *There exists $\bar{\delta} < 1$ such that for all $\delta \geq \bar{\delta}$, we can pick a single abiding contract $(\hat{v}^*, \alpha^*, \psi^*, p^*) \in (\underline{v}, \overline{v}) \times [0,1] \times [0,\delta] \times \mathbb{R}^+$ with $\Pi(\hat{v}^*, \alpha^*, \psi^*, p^*) > \underline{v}$.*

Proof. We pick $\hat{v}^*$ such that $0 < (\hat{v}^* - \underline{v}) < \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ and set $\alpha^* = 1 - (\hat{v}^* - \underline{v})^2 = \psi^*$, and $p^* = \underline{v}$ given by (IC+). Lemma 3 and Lemma 4 imply that the contract is abiding. Lemma 2 delivers $\Pi(\hat{v}^*, \alpha^*, \psi^*, p^*) > \underline{v}$.

Finally, setting $\bar{\delta} = 1 - (\hat{v}^* - \underline{v})^2$ the feasibility constraint $\psi^* \leq \delta$, is satisfied for all $\delta \geq \bar{\delta}$.

□

Finally, we prove the analogue of Lemma 2 in the main text:

Proposition 8. *For each $\delta \in (0,1)$, there exists a δ-abiding contract $d_\delta \in \mathcal{D}$ such that $v(d_\delta, \delta) > \underline{\pi} > \underline{v}$.*

Proof. If $\delta \geq \bar{\delta}$ where $\bar{\delta}$ the cut-off in Lemma 13 we use the contract $(\hat{v}^*, \alpha^*, \psi^*, p^*)$. If $\delta < \bar{\delta}$, we consider the contract $(\hat{v}, \alpha, \psi, p) = (\hat{v}^*, 1, \delta, \tilde{p})$, with $\tilde{p} = \hat{v} - \delta \cdot (\hat{v} - \underline{v})$. This is abiding and delivers payoff:

$$(1-\delta)R(\hat{v}) + \delta\underline{v} > \underline{v}$$

The payoff from the contract $(\hat{v}^*, 1, \delta, \tilde{p})$ is strictly decreasing in δ so the payoff it generates for $\delta < \bar{\delta}$ is at least as large as $\Pi(\hat{v}^*, 1, \bar{\delta}, \tilde{p})$.

We set $\underline{\pi} = \min\{\Pi(\hat{v}^*, 1, \bar{\delta}, \tilde{p}), \Pi(\hat{v}^*, \alpha^*, \psi^*, p^*)\} > \underline{v}$. Consequently, for each δ we can construct an abiding contract d_δ such that $v(d_\delta, \delta) \geq \underline{\pi} > \underline{v}$.

□

Lower bound

We now prove that the best equilibrium payoff to the seller is bounded below by $\underline{\pi}$. To prove this we need to strengthen the equilibrium concept. In particular, an assessment is an equilibrium if it is a Weak PBE, and in addition, after *any* initial contract offer the continuation assessment forms a Weak PBE.[4] We will assume that for any prior distribution of types $G \in \Delta([\underline{v}, \overline{v}])$ such an equilibrium exists.

[4]Note that this is equivalent to a subgame-perfect Weak PBE assessment in a game-tree where Nature draws the type *after* every contract offer—at which point a well-specified subgame starts.

Proposition 9. *We have* $\pi(\mathcal{C}, \delta) \geq \underline{\pi}$.

Proof. Suppose in anticipation of a contradiction that $\pi(\mathcal{C}, \delta) < \underline{\pi}$. The argument follows exactly the same steps of Lemma 1 in the main text, apart from what happens along a history where only d_δ has been offered.

Note that d_δ is abiding so no matter what payoffs the seller gets from deviating to $c \neq d_\delta$, as long as they are consistent with sequential rationality, they cannot exceed market clearing which is the commitment payoff. So the best deviation payoff consistent with equilibrium in the continuation game is $\underline{v}$. Consequently, it is enough to show that *some* equilibrium assessment can be defined after any offer of contract $c \neq d_\delta$ when the seller's belief (residual demand) is given by $D(p \,|\alpha^*, \hat{v}^*)$.

The continuation game where $c \neq d_\delta$ is offered after τ periods of deployment of d_δ, is clearly isomorphic to the one arising from an initial offer of c, in a game where the prior is $D(p \,|\alpha^*, \hat{v}^*)$. By assumption, an assessment exists in such a game which induces a Weak PBE after the contract offer $c \neq d_\delta$. We can therefore specify the corresponding assessment to complete the proof.

□

Appendix C

Appendix to Chapter 3

C.1 Appearances of "worker cooperatives" in digitized books

Figure C.1 shows an increasing trend in the occurrence of the phrase "worker cooperatives" among newly published digitized books since the mid-2000s. For reference, we also added "stakeholder capitalism". We do not interpret the figure as showing that worker cooperatives are necessarily a more prominent alternative than stakeholder capitalism, as other terms are probably used to refer to the concept. However, cooperatives are certainly as prominent nowadays as they were in the heydays of work on the subject, i.e. the 1980s.

[1] The figure was generated using the Google Ngram Viewer (https://books.google.com/ngrams). Details on the corpus are presented by Michel et al (2011).

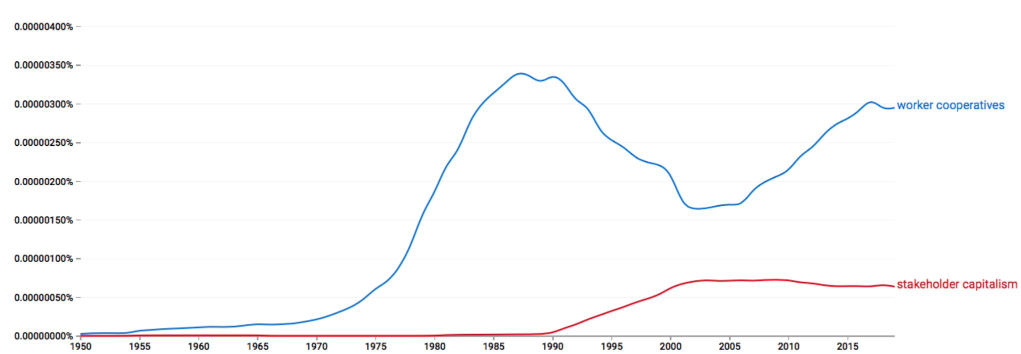

Figure C.1: Appearances of "worker cooperatives" in digitized books

Note: Frequency of the (case-insensitive) bigrams "worker cooperatives" and "stakeholder capitalism" among all bigrams contained in the sample of English-language books digitized by Google, by date of publication.[1]

C.2 Proof that (3.4) holds under the Operational Equilibrium Concept

For $k > 0$ and $l > 0$, denote:

$$\mathcal{U}(k,l) = \max_{k'} U\left(\frac{(1-\tau)F(k,l) - k'}{l}, \frac{\tau F\big(k', \mathcal{L}(k')\big)}{l}\right).$$

The result we wish to prove is that

$$\mathcal{L}(k) \in \arg\max_{l} \mathcal{U}(k,l).$$

We will prove this by contradiction.

Consider an incumbent $\bar{i} \in I_t$, and suppose that $\mathcal{L}(k_{\bar{i}t})$ does *not* coincide with the argmax. Condition (3.1) implies that if $l \leq \mathcal{L}(k_{it})$ then $\mathcal{U}(k_{it}, l) \leq \mathcal{U}\big(k_{it}, \mathcal{L}(k_{it})\big)$, so the argmax must be strictly greater than $\mathcal{L}(k_{\bar{i}t})$.

Now apply conditions (3.2) and (3.3) to $j = \bar{i}$ and $i \in I_{t+1}$ an arbitrary cooperative. If there is a feasible reallocation in which $\bar{i}$ is allocated $l > \mathcal{L}(k_{\bar{i}t})$ workers and i fewer workers than in the original allocation $\mathbb{L}_i$, then either:

$$\mathcal{U}(k_{\bar{i}t}, l) < \mathcal{U}(k_{\bar{i}t}, \mathcal{L}(k_{\bar{i}t})),$$

or:

$$\mathcal{U}(k_{it}, \mathbb{L}_i) \geq \mathcal{U}(k_{\bar{i}t}, l).$$

Since $\mathcal{L}(k_{\bar{i}t})$ is strictly less than the argmax, there must exist $l > \mathcal{L}(k_{\bar{i}t})$ such that the first condition is violated. It follows that the second condition must hold for any other cooperative $i \in I_{t+1}$. In particular, since we take l such that $\mathcal{U}(k_{\bar{i}t}, \mathcal{L}(k_{\bar{i}t})) < \mathcal{U}(k_{\bar{i}t}, l)$, it follows that:

$$\forall i \neq \bar{i} \in I_{t+1}, \quad \mathcal{U}(k_{it}, \mathbb{L}_i) > \mathcal{U}(k_{\bar{i}t}, \mathcal{L}(k_{\bar{i}t})).$$

So any incumbent that is not allocated its optimal labour input must be the cooperative that provides the lowest utility level to its workers among all cooperatives in the economy. But being such a cooperative depends not only on that cooperative's own capital stock, but on the capital stock of all other cooperatives. This then contradicts the premise that the allocation of workers to incumbent cooperatives depends exclusively on each cooperative's own initial capital.

C.3 Algorithm to solve for the equilibrium

In practice, we solve for equilibria as follows. The first step is to obtain the equilibrium choices of labour input and capital investment of a cooperative:

$$\big(\mathcal{L}(k), \mathcal{K}(k)\big) \in \arg\max_{l,k'} U\bigg(\frac{(1-\tau)F(k,l) - k'}{l}, \frac{\tau F\big(k', \mathcal{L}(k')\big)}{l}\bigg),$$

as well as the optimal capital investment level of an entering cooperative with an arbitrary labour input l:

$$\mathbf{K}(l) \in \arg\max_{k'} U\bigg(\frac{(1-\tau)F(0,l) - k'}{l}, \frac{\tau F\big(k', \mathcal{L}(k')\big)}{l}\bigg).$$

The value of this problem is denoted $\mathcal{U}_0(l)$. Then, given any initial distribution of capital $\{k_{i0}\}_{i \in I_0}$, one can construct the growth path of the economy and check for feasibility.

Specifically, in each period, all incumbents $i \in I_t$ are allocated $\mathcal{L}(k_{it})$ workers. If $\int_{I_t} \mathcal{L}(k_{it})di > L$, feasibility is violated so the initial distribution cannot lead to an equilibrium satisfying the requirement. If $\int_{I_t} \mathcal{L}(k_{it})di = L$, then $I_{t+1} = I_t$ and $k_{i,t+1} = \mathcal{K}(k_{it})$. If $\int_{I_t} \mathcal{L}(k_{it})di < L$, define:

$$l^* \in \arg\max_{l} \mathcal{U}_0(l).$$

Then a set of entrants E_t of measure $|E_t| = \frac{L - \int_{I_t} \mathcal{L}(k_{it})di}{l^*}$ is created. That is, new cooperatives are created with $l_i = l^*$ workers. Finally, $I_{t+1} = I_t \cup E_t$ where E_t is the set of newly created cooperatives, and for $i \in I_t$, $k_{i,t+1} = \mathcal{K}(k_{it})$, while for $i \in E_t$, $k_{i,t+1} = \mathbf{K}(l^*)$.

C.4 Derivation of capitalist equilibrium with log utility

Conditional factor demands from individual firms take the form:

$$\begin{aligned} k(r_t, w_t) =& \bigg[A\Big(\frac{\alpha}{r_t}\Big)^{1-\beta}\Big(\frac{\beta}{w_t}\Big)^{\beta}\bigg]^{\frac{1}{1-\alpha-\beta}}, \\ l(r_t, w_t) =& \underline{l} + \bigg[A\Big(\frac{\alpha}{r_t}\Big)^{\alpha}\Big(\frac{\beta}{w_t}\Big)^{1-\alpha}\bigg]^{\frac{1}{1-\alpha-\beta}}, \end{aligned}$$

while profits write:

$$\pi(r_t, w_t) = (1-\alpha-\beta)\Big[A\Big(\frac{\alpha}{r_t}\Big)^{\alpha}\Big(\frac{\beta}{w_t}\Big)^{\beta}\Big]^{\frac{1}{1-\alpha-\beta}} - w_t\underline{l}.$$

As a result, we can solve the system of equilibrium conditions to derive:

$$\begin{aligned}
r(\kappa_t) &= A\frac{\alpha}{(1-\alpha)^{1-\alpha}}\beta^{\beta}\Big(\frac{1-\alpha-\beta}{\underline{l}}\Big)^{1-\alpha-\beta}\kappa_t^{\alpha-1},\\
w(\kappa_t) &= A(1-\alpha)^{\alpha}\beta^{\beta}\Big(\frac{1-\alpha-\beta}{\underline{l}}\Big)^{1-\alpha-\beta}\kappa_t^{\alpha},\\
N(\kappa_t) &\equiv N = \frac{1-\alpha-\beta}{1-\alpha}\Big(\frac{L}{\underline{l}}\Big).
\end{aligned}$$

Note that the number of firms is constant over time and hence independent of the size of the capital stock, or equivalently, capitalist firms have the constant size given in equation (3.6).

The solution to the Young's consumption-saving problem leads to the well known log-utility saving rule

$$\kappa_{t+1} = \frac{\delta}{1+\delta}w_t.$$

Substituting from the equations above this delivers the capital accumulation equation (3.7).

C.5 Proof that cooperatives are smaller than capitalist firms

Capitalist firms have $\frac{1-\alpha}{1-\alpha-\beta}\underline{l}$ employees, while cooperatives have $\frac{1+\delta}{1+\delta-\beta(1+\delta\alpha)}\underline{l}$ workers. Since $\alpha \in (0,1)$, it must be the case that:

$$1+\delta \geq (1-\alpha)(1+\delta\alpha).$$

It follows that $\frac{1}{1-\alpha} \geq \frac{1+\delta\alpha}{1+\delta}$, which implies that:

$$1-\frac{\beta}{1-\alpha} \leq 1-\frac{\beta(1+\delta\alpha)}{1+\delta}.$$

Therefore:

$$\frac{1-\alpha}{1-\alpha-\beta}\underline{l} \geq \frac{1+\delta}{1+\delta-\beta(1+\delta\alpha)}\underline{l}.$$

C.6 Capitalist dynamics with IES = 2

Solving the consumption-saving problem of young agents yields the following saving rule:

$$\kappa_{t+1} = \frac{\delta^{\frac{1}{\sigma}} r_{t+1}^{\frac{1-\sigma}{\sigma}}}{1+\delta^{\frac{1}{\sigma}} r_{t+1}^{\frac{1-\sigma}{\sigma}}} w_t.$$

As a result, capital accumulation dynamics are characterised by the following equation:

$$\left[1+\delta^{-\frac{1}{\sigma}}\left(A\frac{\alpha}{(1-\alpha)^{1-\alpha}}\beta^{\beta}\left(\frac{1-\alpha-\beta}{\underline{l}}\right)^{1-\alpha-\beta}\kappa_{t+1}^{\alpha-1}\right)^{-\frac{1-\sigma}{\sigma}}\right]\kappa_{t+1} = A(1-\alpha)^{\alpha}\beta^{\beta}\left(\frac{1-\alpha-\beta}{\underline{l}}\right)^{1-\alpha-\beta}\kappa_t^{\alpha}. \tag{C.1}$$

If $\sigma > 1$ and $\alpha \in (0,1)$, equation (C.1) defines κ_{t+1} as an increasing and concave function of κ_t, with a first-order derivative which is infinite at 0 and vanishes at infinity.

To simplify notations, denote:

$$a = \delta^{-\frac{1}{\sigma}}\left(A\frac{\alpha}{(1-\alpha)^{1-\alpha}}\beta^{\beta}\left(\frac{1-\alpha-\beta}{\underline{l}}\right)^{1-\alpha-\beta}\right)^{-\frac{1-\sigma}{\sigma}},$$

$$b = A(1-\alpha)^{\alpha}\beta^{\beta}\left(\frac{1-\alpha-\beta}{\underline{l}}\right)^{1-\alpha-\beta},$$

$$\theta = \alpha + \frac{1-\alpha}{\sigma}.$$

Then equation (C.1) rewrites:

$$\kappa_{t+1} + a\kappa_{t+1}^{\theta} = b\kappa_t^{\alpha},$$

or equivalently:

$$\kappa_{t+1} = f^{-1}(\kappa_t),$$

where $f(x) = \left(\frac{1}{b}\right)^{\frac{1}{\alpha}}\left(x + ax^{\theta}\right)^{\frac{1}{\alpha}}$ is strictly increasing and strictly convex on $(0,\infty)$, and satisfies:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} f'(x) = 0,$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} f'(x) = \infty,$$

if we restrict attention to the case $\sigma > 1$, so that $\theta \in (\frac{1}{\sigma}, 1)$.

Indeed, these properties are easily derived from differentiating twice, which yields:

$$f'(x) = \Big(\frac{1}{b}\Big)^{\frac{1}{\alpha}} \big(1 + a\theta x^{\theta-1}\big)\big(x + ax^{\theta}\big)^{\frac{1-\alpha}{\alpha}},$$

and:

$$f''(x) = \Big(\frac{1}{b}\Big)^{\frac{1}{\alpha}} \big(x + ax^{\theta}\big)^{\frac{1-\alpha}{\alpha}-1}\Big[\frac{1-\alpha}{\alpha} + \theta\big(\theta - 3 + \frac{2}{\alpha}\big)ax^{\theta-1} + \theta\big(\frac{\theta}{\alpha} - 1\big)\big(ax^{\theta-1}\big)^2\Big],$$

where $\frac{\theta}{\alpha} - 1 = \frac{1-\alpha}{\alpha\sigma} > 0$, and $\theta - 3 + \frac{2}{\alpha} = (1-\alpha)\big(\frac{2-\alpha}{\alpha} + \frac{1}{\sigma}\big) > 0$.

It follows that capital accumulation follows standard dynamics with a unique strictly positive attractive steady-state.

In the special case where $\sigma = 2$, the steady-state capital stock per old worker takes a simple algebraic form:

$$\kappa^* = \left(\frac{\left(4A(1-\alpha)^{\alpha}\beta^{\beta}\left(\frac{1-\alpha-\beta}{\underline{l}}\right)^{1-\alpha-\beta}\right)^{1/2}}{\left(\frac{\alpha}{\delta(1-\alpha)}\right)^{1/2} + \left(4 + \frac{\alpha}{\delta(1-\alpha)}\right)^{1/2}}\right)^{\frac{2}{1-\alpha}}.$$

It follows that the discount factor δ can still be identified from the targeting of the capital-output ratio:

$$\frac{K}{Y} = \frac{4(1-\alpha)}{\left(\left(\frac{\alpha}{\delta(1-\alpha)}\right)^{1/2} + \left(4 + \frac{\alpha}{\delta(1-\alpha)}\right)^{1/2}\right)^2}.$$

C.7 Fixed-cost Normalization

We use the following functional form for the production technology:

$$F_{\underline{l}}(k, l) = Ak^{\alpha}(l - \underline{l})^{\beta}.$$

In this appendix, we show that the specific value of the parameter $\underline{l}$ does not affect the quantitative comparison between the two economies.

For any given $\underline{l}$, we can implement the following change of variables. Any quantity of labour l can be renormalized as:

$$\tilde{l} = \frac{l}{\underline{l}},$$

while any quantity of capital k can be renormalized as:

$$\tilde{k} = \underline{l}^{-\frac{\beta}{1-\alpha}} k.$$

It follows that the production output is also renormalized as:

$$F_{\underline{l}}(k, l) = \underline{l}^{\frac{\beta}{1-\alpha}} F_1(\tilde{k}, \tilde{l}).$$

In our model of capitalist economy, the problem of the firm is completely unchanged as long as the wage is suitably renormalized to:

$$\tilde{w} = \underline{l}^{1-\frac{\beta}{1-\alpha}} w.$$

The consumer's problem is unchanged either (note that each consumer now supplies $1/\underline{l}$ units of labour). The renormalization implies that consumption level c is to be renormalized as: $\tilde{c} = \underline{l}^{-\frac{\beta}{1-\alpha}} c$. Given that preferences are of the form:

$$U(c^Y, c^O) = \frac{(c^Y)^{1-\sigma}}{1-\sigma} + \delta \frac{(c^O)^{1-\sigma}}{1-\sigma},$$

the renormalization amounts to multiplying the utility function by a positive constant, thus does not affect choices. Note also that in the log case, the renormalization simply corresponds to adding a constant to the utility function, so the argument is also valid.

Similarly, in the cooperative model, consumption levels per consumer write:

$$c^Y = \underline{l}^{\frac{\beta}{1-\alpha}} \frac{(1-\tau)F_1(\tilde{k}, \tilde{l}) - \tilde{k}'}{\tilde{l}},$$

$$c^O = \underline{l}^{\frac{\beta}{1-\alpha}} \frac{\tau F_1(\tilde{k}', \tilde{l}')}{\tilde{l}}.$$

Therefore, choices are unaffected by the same renormalization. Since the normalization affects the levels of relevant quantities in the same way in the two models, no quantitative comparison is affected by the level of $\underline{l}$.

C.8 Sensitivity Analysis with τ recalculated

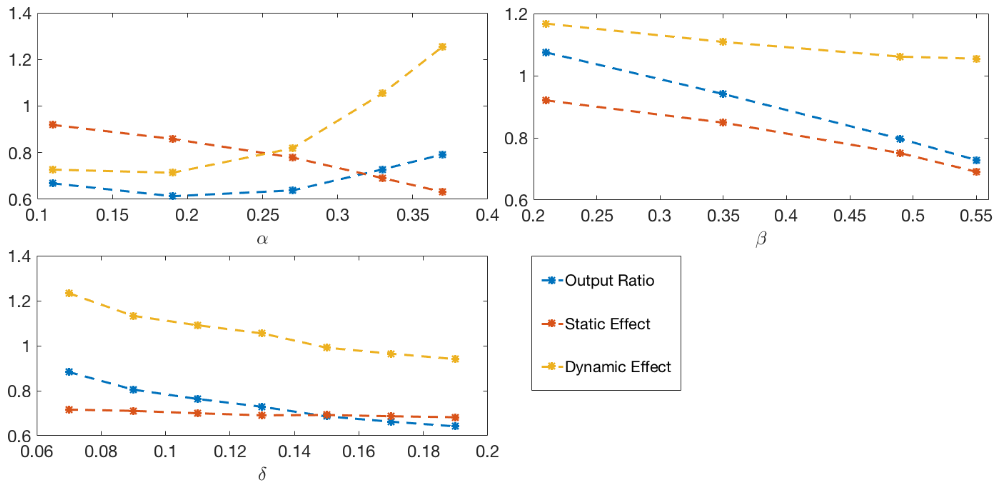

Figure C.2: Steady-state output ratio $\frac{Y^*_{coop}}{Y^*_{cap}}$ and its static $\left(\left(\frac{Z_{coop}}{Z_{cap}}\right)^{\frac{1}{1-\alpha}}\right)$ and dynamic $\left(\left(\frac{s_{coop}}{s_{cap}}\right)^{\frac{\alpha}{1-\alpha}}\right)$ components, as functions of the model's parameters, with τ chosen to maximize steady state utility for each combination of parameters.

C.9 Endogenous Sharing Rule

In this appendix, we present an extended version of our model of cooperatives in which we relax the assumption that old workers automatically receive a share of current revenues. Instead, in each period, current workers decide by a vote whether to implement a sharing-rule. We describe those sharing-rules that can be sustained as an equilibrium of the dynamic game played by the different generations of workers within a cooperative. In equilibrium, each generation expects to receive payments when old only if they agree to pay their old workers when young. Therefore agreement to a sharing-rule arises endogenously. Our approach follows closely that of Cooley and Soares (1999), who introduce endogenous pay-as-you-go social security systems in a general-equilibrium overlapping-generations model.

Specifically, we assume that, upon creation of a cooperative i, the initial workers choose a linear sharing rule $\tau_i \in [0, 1]$. Following generations of workers are not committed to abide by the policy designed by their predecessors, but may vote only for or against its implementation. That is, in each following period, if cooperative i is assigned workers, those workers choose, once production has taken place, whether to distribute a share τ_i of revenues to the old workers or to keep all revenues. Then, the maintained share is split between investment and payment to the young workers as in the main text.

Our general equilibrium concept can be adapted to include decisions regarding the sharing-rule. We call a sharing-rule *sustainable* if its implementation in every period can be supported by trigger strategies such that every generation of workers in cooperative i chooses to implement the sharing rule τ_i as long as every previous generation has done so. If workers of cooperative i in period t deviate from τ_i, they expect to be punished by the following generation of workers and not to receive any payment as old. As a result, they have no incentive to invest at all. It follows that the sharing-rule τ_i is sustainable if, on path:

$$U\left(\frac{(1-\tau_i)F(k_{i,t},l_{i,t})-k_{i,t+1}}{l_{i,t}},\frac{\tau_i F(k_{i,t+1},l_{i,t+1})}{l_{i,t}}\right)\geq U\left(\frac{F(k_{i,t},l_{i,t})}{l_{i,t}},0\right). \qquad (*)$$

An important consequence is that no sharing rule is sustainable in a cooperative that exits in equilibrium, and conversely, a cooperative with non-sustainable sharing rule immediately exits after its first period of existence[2].

Now, the definition of an equilibrium follows naturally from that of section 4.3. An equilibrium is characterised by a worker allocation mechanism, investment decisions, and sharing-rules. Investment decisions are required to be optimal subject to condition $(*)$, taking as given the worker allocation mechanism. In turn, the worker allocation mechanism takes as input the set of incumbent cooperatives, their current capital stock and the sharing rule in place in each of them, and operates according to the same requirements as in the main text. In particular, when we consider reallocations, we take into account the potential creation of a new cooperative i with any arbitrary sharing rule τ_i. In this sense, sharing rules are chosen optimally upon the creation of a cooperative.

As in section 4.4, we may impose restrictions in order to define an operational equilibrium concept. First, we require that an incumbent cooperative's allocation of workers depends only on its capital stock and its sharing rule. That is, given the sharing rule τ_i, there is a mapping $\mathcal{L}_{\tau_i}(k_{i,t})$ such that, on path, incumbent cooperative i is allocated $\mathcal{L}_{\tau_i}(k_{i,t})$ in any state in which its capital stock is $k_{i,t}$ and its sharing rule τ_i. Second, we impose that, given the mapping $\mathcal{L}_{\tau_i}(\cdot)$, the resulting optimal investment path implies that τ_i is a sustainable sharing rule. That is, condition $(*)$ is satisfied in every period,

[2] This follows from noting that the last generation of workers before a cooperative exits would have no incentive to distribute revenues to old workers. In anticipation, the previous generation would have no incentive to implement a sharing rule either, and so on by backward induction.

when every generation of workers invests as if τ_i was to be automatically implemented in every future period. This second restriction is in line with the limited rationality requirements that motivate our operational equilibrium concept in the main text, as workers do not need to anticipate the behaviour of every future generation when they invest, only that of the next generation when they vote on the implementation of the sharing rule. Importantly though, it limits the set of sharing rules that can be selected by the first generation of workers in a cooperative, since those must choose among those sharing rules that are indeed sustainable given the optimal investment path. If they do not choose such a sharing rule, they must understand it and the cooperative immediately exits. Then, as mentioned above, exit does not have to be ruled out for an operational equilibrium concept, but it may occur only to newly created cooperatives.

It is easy to establish under these restrictions that equation (4) is still valid for a sustainable sharing rule τ. If we further impose symmetry across cooperatives regarding their initial choice of sharing rule, the rest of our analysis applies without modification to this model with endogenous sharing rules. In particular, in the examples of sections 5 and 6, any sharing rule $\tau \in (0, 1)$ is sustainable. This result follows from the fact that the workers' utility goes to $-\infty$ if consumption goes to 0 in any period. Therefore condition $(*)$ has to hold.

References

Abadi, J., & Brunnermeier, M. (2018). *Blockchain Economics.* National Bureau of Economic Research.

Acemoglu, D. (2009). *Introduction to Modern Economic Growth*. Princeton: Princeton University Press.

Acharya, A., & Ortner, J. (2017). Progressive Learning. *Econometrica*, *85*(6), 1965–1990.

Akbarpour, M., & Li, S. (2020). Credible Auctions: A Trilemma. *Econometrica*, *88*(2), pp. 425–467.

Alchian, A. A., & Demsetz, H. (1972). Production, Information Costs, and Economic Organization. *The American Economic Review*, *62*(5), 777–795.

Aoki, M. (1980). A Model of the Firm as a Stockholder-Employee Cooperative Game. *The American Economic Review*, *70*(4), 600–610.

Arieli, I., Babichenko, Y., Smorodinsky, R., & Yamashita, T. (2023). Optimal Persuasion via Bi-pooling. *Theoretical Economics*, *18*(1), 15–36.

Atkinson, A. (1973). Worker Management and the Modern Industrial Enterprise. *The Quarterly Journal of Economics*, *87*(3), 375-392.

Aumann, R. J., Maschler, M., & Stearns, R. E. (1995). *Repeated Games with Incomplete Information* . Cambridge, Mass: MIT Press.

Ausubel, L. M., & Deneckere, R. J. (1989). Reputation in Bargaining and Durable Goods Monopoly. *Econometrica*, *57*(3), 511–531.

Bagnoli, M., Salant, S. W., & Swierzbinski, J. E. (1989). Durable-Goods Monopoly with Discrete Demand. *Journal of Political Economy*, *97*(6), 1459–1478.

Bakos, Y., & Halaburda, H. (2019). Smart Contracts, IoT Sensors and Efficiency: Automated Execution vs. Better Information. *NYU Stern School of Business, available at https://ssrn. com/abstract*, *3394546*.

Battaglini, M. (2007). Optimality and Renegotiation in Dynamic Contracting. *Games and Economic Behavior*, *60*(2), 213–246.

Battigalli, P., & Dufwenberg, M. (2009). Dynamic Psychological Games. *Journal of Economic Theory*, *144*(1), 1–35.

Beccuti, J., & Möller, M. (2018). Dynamic Adverse Selection with a Patient Seller. *Journal of Economic Theory*, *173*, 95–117.

Bergemann, D., Brooks, B., & Morris, S. (2015). The Limits of Price Discrimination. *American Economic Review*, *105*(3), 921-57.

Bergemann, D., & Morris, S. (2019). Information Design: A Unified Perspective. *Journal of Economic Literature*, *57*(1), 44-95.

Bester, H., & Strausz, R. (2001). Contracting with Imperfect Commitment and the Revelation Principle: The Single Agent Case. *Econometrica*, *69*(4), 1077–1098.

Biehl, A. R. (2001). Durable-Goods Monopoly with Stochastic Values. *The RAND Journal of Economics*, *32*(3), 565–577.

Block, F. L. (2018). *Capitalism: The Future of an Illusion* (1st ed.). University of California Press.

Board, S., & Pycia, M. (2014). Outside Options and the Failure of the Coase Conjecture. *The American Economic Review*, *104*(2), 656–671.

Bonin, J. P. (1984). Membership and Employment in an Egalitarian Cooperative. *Economica*, *51*(203), 295–305.

Bonin, J. P., Jones, D. C., & Putterman, L. (1993). Theoretical and empirical studies of producer cooperatives: Will ever the twain meet? *Journal of Economic Literature*, *31*(3), 1290–1320.

Bonin, J. P., & Putterman, L. (1987). *Economics of Cooperation and the Labor-managed Economy*. Taylor & Francis.

Bradley, K., & Gelb, A. (1983). *Cooperation at Work: The Mondragon Experience*. Heinemann Educational.

Breig, Z. (2019). Endogenous and Exogenous Commitment. *Economics Letters*, *183*, 108577.

Bulow, J. I. (1982). Durable-Goods Monopolists. *Journal of Political Economy*, *90*(2), 314–332.

Bulow, J. I. (1986). An Economic Theory of Planned Obsolescence. *The Quarterly Journal* *of Economics*, *101*(4), 729–750.

Bünz, B., Agrawal, S., Zamani, M., & Boneh, D. (2020). Zether: Towards privacy in a smart contract world. In *Financial Cryptography and Data Security: 24th International Conference, FC 2020, Kota Kinabalu, Malaysia, February 10–14, 2020 Revised Selected Papers* (pp. 423–443).

Cass, O. (2018). *The Once and Future Worker: A Vision for the Renewal of Work in America*. Encounter Books.

Coase, R. H. (1972). Durability and Monopoly. *The Journal of Law & Economics*, *15*(1), 143–149.

Collier, P. (2018). *The Future of Capitalism: Facing the New Anxieties*. Harper New York.

Cong, L. W., & He, Z. (2019). Blockchain Disruption and Smart Contracts. *The Review of Financial Studies*, *32*(5), pp. 1754–1797.

Cooley, T. F., & Soares, J. (1999). A Positive Theory of Social Security Based on Reputation. *Journal of Political Economy*, *107*(1), 135–160.

Dizdar, D., & Kováč, E. (2020). A Simple Proof of Strong Duality in the Linear Persuasion Problem. *Games and Economic Behavior*, *122*, 407-412.

Domar, E. D. (1966). The Soviet Collective Farm as a Producer Cooperative. *The American Economic Review*, *56*(4), 734–757.

Doval, L., & Skreta, V. (2020). *Optimal Mechanism for the Sale of a Durable Good.* (arXiv preprint arXiv:1904.07456)

Doval, L., & Skreta, V. (2022). Mechanism Design with Limited Commitment. *Econometrica*, *90*(4), 1463–1500.

Drèze, J. H. (1976). Some Theory of Labor Management and Participation. *Econometrica: Journal of the Econometric Society*, 1125–1139.

Drèze, J. H. (1989). *Labour Management, Contracts and Capital Markets.* Basil Blackwell, Oxford.

Drèze, J. H., & Greenberg, J. (1980). Hedonic Coalitions: Optimality and Stability. *Econometrica: Journal of the Econometric Society*, 987–1003.

Dworczak, P., & Kolotilin, A. (2022). *The Persuasion Duality.*

Dworczak, P., & Martini, G. (2019). The Simple Economics of Optimal Persuasion. *Journal of Political Economy*, *127*(5), 1993-2048.

Ederer, F., Holden, R., & Meyer, M. (2018). Gaming and Strategic Opacity in Incentive

Provision. *The RAND Journal of Economics*, *49*(4), 819-854.

Ely, J. C., & Szydlowski, M. (2020). Moving the Goalposts. *Journal of Political Economy*, *128*(2), 468-506.

Fanning, J. (2021a). Mediation in Reputational Bargaining. *American Economic Review*, *111*(8), 2444–72.

Fanning, J. (2021b). Optimal Dynamic Mediation. *Unpublished Manuscript*.

Farrell, J., & Scotchmer, S. (1988). Partnerships. *The Quarterly Journal of Economics*, *103*(2), 279–297.

Feinberg, Y., & Skrzypacz, A. (2005). Uncertainty about Uncertainty and Delay in Bargaining. *Econometrica*, *73*(1), 69–91.

Fuchs, W., & Skrzypacz, A. (2010). Bargaining with Arrival of New Traders. *The American Economic Review*, *100*(3), 802–836.

Fudenberg, D., & Levine, D. (1983). Subgame-Perfect Equilibria of Finite-and Infinite-Horizon Games. *Journal of Economic Theory*, *31*(2), 251–268.

Fudenberg, D., Levine, D., & Maskin, E. (1994). The Folk Theorem with Imperfect Public Information. *Econometrica: Journal of the Econometric Society*, 997–1039.

Fudenberg, D., Levine, D., & Tirole, J. (1985). Infinite Horizon Models of Bargaining with One-Sided Uncertainty. In A. Roth (Ed.), *Game Theoretic Models of Bargaining* (pp. 73–98). Cambridge University Press.

Fudenberg, D., & Tirole, J. (1998). Upgrades, Tradeins, and Buybacks. *The RAND Journal of Economics*, *29*(2), 235–258.

Furubotn, E. G. (1976). The Long-run Analysis of the Labor-managed Firm: An Alternative Interpretation. *The American Economic Review*, *66*(1), 104–123.

Furubotn, E. G., & Pejovich, S. (1973). Property Rights, Economic Decentralization, and the Evolution of the Yugoslav Firm, 1965-1972. *The Journal of Law and Economics*, *16*(2), 275–302.

Gentzkow, M., & Kamenica, E. (2016). A Rothschild-Stiglitz Approach to Bayesian Persuasion. *American Economic Review*, *106*(5), 597-601.

Gerardi, D., & Maestri, L. (2020). Dynamic Contracting with Limited Commitment and the Ratchet Effect. *Theoretical Economics*, *15*(2), 583–623.

Greenberg, J. (1979). Existence and Optimality of Equilibrium in Labour-managed Economies. *The Review of Economic Studies*, *46*(3), 419–433.

Gul, F., Sonnenschein, H., & Wilson, R. (1986). Foundations of Dynamic Monopoly and the Coase Conjecture. *Journal of Economic Theory*, *39*(1), 155–190.

Hahn, J.-H. (2006). Damaged Durable Goods. *The RAND Journal of Economics*, *37*(1), 121–133.

Hansmann, H. (1996). *The Ownership of Enterprise*. Harvard University Press.

Hart, O., & Moore, J. (1996). The Governance of Exchanges: Members' Cooperatives versus Outside Ownership. *Oxford Review of Economic Policy*, *12*(4), 53–69.

Hart, O., & Zingales, L. (2017). Companies Should Maximize Shareholder Welfare Not Market Value. *Journal of Law, Finance, and Accounting*, *2*(2), 247–275.

Hart, O. D., & Moore, J. (1998). *Cooperatives vs. Outside Ownership.* National Bureau of Economic Research Cambridge, Mass., USA.

Hart, O. D., & Tirole, J. (1988). Contract Renegotiation and Coasian Dynamics. *The Review of Economic Studies*, *55*(4), 509–540.

Holden, R., & Malani, A. (2021). *Can Blockchain Solve the Hold-up Problem in Contracts?* Cambridge University Press.

Holmström, B. (1982). Moral Hazard in Teams. *The Bell Journal of Economics*, 324–340.

Huberman, G., Leshno, J. D., & Moallemi, C. (2021). Monopoly Without a Monopolist: An Economic Analysis of the Bitcoin Payment System. *The Review of Economic Studies*, *88*(6), 3011–3040.

Ichiishi, T. (1977). Coalition Structure in a Labor-managed Market Economy. *Econometrica: Journal of the Econometric Society*, 341–360.

Inderst, R. (2008). Durable Goods with Quality Differentiation. *Economics Letters*, *100*(2), 173–177.

Jehiel, P. (2014). On Transparency in Organizations. *The Review of Economic Studies*, *82*(2), 736-761.

Jones, D. C. (2018). The Economics of Participation and Employee Ownership (PEO): an Assessment. *Journal of Participation and Employee Ownership*, *1*(1), 4–37.

Kahn, C. (1988). The Durable Goods Monopolist and Consistency with Increasing Costs. *Econometrica*, *56*(3), 754–754.

Kamenica, E. (2019). Bayesian Persuasion and Information Design. *Annual Review of Economics*, *11*(1), 249-272.

Kamenica, E., & Gentzkow, M. (2011). Bayesian Persuasion. *American Economic Review*, *101*(6), 2590-2615.

Kandori, M. (1992). Repeated Games Played by Overlapping Generations of Players. *The Review of Economic Studies*, *59*(1), 81–92.

Karp, L. S. (1993). Monopoly Extraction of a Durable Non-Renewable Resource: Failure of the Coase Conjecture. *Economica*, *60*(237), 1–11.

Katrin, T. (2018). *'Smart' Contracts and External Financing.* (https://ssrn.com/abstract=3072854)

Kerber, T., Kiayias, A., & Kohlweiss, M. (2021). Kachina — Foundations of Private Smart Contracts. In *2021 IEEE 34th Computer Security Foundations Symposium (CSF)* (pp. 1–16).

Kleiner, A., Moldovanu, B., & Strack, P. (2021). Extreme Points and Majorization: Economic Applications. *Econometrica*, *89*(4), 1557-1593.

Kolotilin, A. (2018). Optimal Information Disclosure: a Linear Programming Approach. *Theoretical Economics*, *13*(2), 607-636.

Kolotilin, A., Corrao, R., & Wolitsky, A. (2022). *Persuasion as Matching.*

Kremer, M. (1997). *Why Are Worker Cooperatives So Rare?* National Bureau of Economic Research Cambridge, Mass., USA.

Laffont, J.-J., & Moreaux, M. (1983). The Nonexistence of a Free Entry Cournot Equilibrium in Labor-managed Economies. *Econometrica: Journal of the Econometric Society*, 455–462.

Laffont, J.-J., & Tirole, J. (1988). The Dynamics of Incentive Contracts. *Econometrica*, *56*(5), 1153–1175.

Laffont, J.-J., & Tirole, J. (1990). Adverse Selection and Renegotiation in Procurement. *The Review of Economic Studies*, *57*(4), 597–625.

Lang, K., & Gordon, P.-J. (1995). Partnerships as Insurance Devices: Theory and Evidence. *The RAND Journal of Economics*, 614–629.

Law, P. J. (1977). The Illyrian Firm and Fellner's Union-Management Model. *Journal of Economic Studies*, *4*(1), 29–37.

Lazear, E. P. (2006). Speeding, Terrorism, and Teaching to the Test. *The Quarterly Journal of Economics*, *121*(3), 1029-1061.

Levin, J., & Tadelis, S. (2005). Profit Sharing and the Role of Professional Partnerships. *The Quarterly Journal of Economics*, *120*(1), 131–171.

Liu, C. (2021). *Motivating Effort with Information about Future Rewards.* arXiv. doi: 10.48550/ARXIV.2110.05643

Lomys, N., & Yamashita, T. (2022). *A Mediator Approach to Mechanism Design with Limited Commitment.* (Available at SSRN 4116543)

Maestri, L. (2017). Dynamic Contracting Under Adverse Selection and Renegotiation. *Journal of Economic Theory*, *171*, 136–173.

Magill, M., Quinzii, M., & Rochet, J.-C. (2015). A Theory of the Stakeholder Corporation. *Econometrica*, *83*(5), 1685–1725.

McAdams, D., & Schwarz, M. (2007). Credible Sales Mechanisms and Intermediaries. *The American Economic Review*, *97*(1), 260–276.

McAfee, R. P., & Wiseman, T. (2008). Capacity Choice Counters the Coase Conjecture. *The Review of Economic Studies*, *75*(1), 317–332.

Meade, J. E. (1972). The Theory of Labour-managed Firms and of Profit Sharing. *The Economic Journal*, *82*(325), 402–428.

Meade, J. E. (1986). *Alternative Systems of Business Organization and of Workers' Renumeration*. Allen & Unwich.

Michel, J.-B., Shen, Y. K., Aiden, A. P., Veres, A., Gray, M. K., Team, G. B., … Aiden, E. L. (2011). Quantitative Analysis of Culture Using Millions of Digitized Books. *Science*, *331*(6014), 176–182.

Mirrlees, J. A. (1976). The Optimal Structure of Incentives and Authority within an Organization. *The Bell Journal of Economics*, 105–131.

Miyazaki, H. (1984). Internal Bargaining, Labor Contracts, and a Marshallian Theory of the Firm. *The American Economic Review*, *74*(3), 381–393.

Miyazaki, H., & Neary, H. M. (1983). The Illyrian Firm Revisited. *The Bell Journal of Economics*, 259–270.

Montez, J. (2013). Inefficient Sales Delays by a Durable-Good Monopoly Facing a Finite Number of Buyers. *The RAND Journal of Economics*, *44*(3), 425–437.

Myerson, R. B., & Reny, P. J. (2020). Perfect Conditional ε-Equilibria of Multi-Stage Games with Infinite Sets of Signals and Actions. *Econometrica*, *88*(2), pp. 495–531.

Nava, F., & Schiraldi, P. (2019, may). Differentiated Durable Goods Monopoly: A Robust Coase Conjecture. *American Economic Review*, *109*(5), 1930–1968.

Ortner, J. (2017). Durable Goods Monopoly with Stochastic Costs. *Theoretical Economics*, *12*(2), 817–861.

Ostrovsky, M., & Schwarz, M. (2010). Information Disclosure and Unraveling in Matching Markets. *American Economic Journal: Microeconomics*, *2*(2), 34-63.

Pencavel, J. H. (2013). The Economics of Worker Cooperatives. In *The Economics of Worker Cooperatives.* Edward Elgar Publishing Limited.

Perea, A. (2002). A Note on the One-deviation Property in Extensive Form Games. *Games and Economic Behavior*, *40*(2), 322–338.

Perego, J., Galperti, S., & Levkun, A. (2021, September). *The Value of Data Records.* doi: dx.doi.org/10.2139/ssrn.3866625

Putterman, L. (1982). Some Behavioral Perspectives on the Dominance of Hierarchical over Democratic Forms of Enterprise. *Journal of Economic Behavior & Organization*, *3*(2-3), 139–160.

Radner, R. (1986). Repeated Partnership Games with Imperfect Monitoring and No Discounting. *The Review of Economic Studies*, *53*(1), 43–57.

Radner, R., Myerson, R., & Maskin, E. (1986). An Example of a Repeated Partnership Game with Discounting and with Uniformly Inefficient Equilibria. *The Review of Economic Studies*, *53*(1), 59–69.

Rayo, L., & Segal, I. (2010). Optimal Information Disclosure. *Journal of Political Economy*, *118*(5), 949-987.

Rey, P., & Tirole, J. (2007). Financing and Access in Cooperatives. *International Journal of Industrial Organization*, *25*(5), 1061–1088.

Sapir, A. (1980). A Growth Model for a Tenured-labor-managed Firm. *The Quarterly Journal of Economics*, *95*(3), 387–402.

Sasaki, H., & Toda, M. (1996). Two-sided Matching Problems with Externalities. *Journal of Economic Theory*, *70*(1), 93–108.

Sen, A. K. (1966). Labour Allocation in a Cooperative Enterprise. *The Review of Economic Studies*, *33*(4), 361–371.

Simon, L. K., & Zame, W. R. (1990). Discontinuous Games and Endogenous Sharing Rules. *Econometrica: Journal of the Econometric Society*, 861–872.

Smolin, A. (2021). Dynamic Evaluation Design. *American Economic Journal: Microeconomics*, *13*(4), 300-331.

Sobel, J. (1991). Durable Goods Monopoly with Entry of New Consumers. *Econometrica*, *59*(5), 1455–1485.

Sobel, J. (1993). Information Control in the Principal-Agent Problem. *International Economic Review*, *34*(2), 259–269.

Steffen, S., Bichsel, B., Gersbach, M., Melchior, N., Tsankov, P., & Vechev, M. (2019). zkay: Specifying and Enforcing Data Privacy in Smart Contracts. In *Proceedings of the 2019 ACM SIGSAC conference on computer and communications security* (pp. 1759–1776).

Steinherr, A., & Thisse, J.-F. (1979). Are Labor-managers Really Perverse? *Economics Letters*, *2*(2), 137–142.

Stokey, N. L. (1981). Rational Expectations and Durable Goods Pricing. *The Bell Journal of Economics*, *12*(1), 112–128.

Strulovici, B. (2017). Contract Renegotiation and the Coase Conjecture: A Strategic Foundation for Renegotiation-Proof Contracts. *Econometrica*, *85*(2), 585–616.

Svejnar, J. (1982). On the Theory of a Participatory Firm. *Journal of Economic Theory*, *27*(2), 313–330.

Takeyama, L. N. (2002). Strategic Vertical Differentiation and Durable Goods Monopoly. *The Journal of Industrial Economics*, *50*(1), 43–56.

Tirole, J. (2016). From the Bottom of the Barrel to the Cream of the Crop: Sequential Screening with Positive Selection. *Econometrica*, *84*(4), 1291–1343.

Vanek, J. (1970). *The General Theory of Labor-Managed Market Economies.* Ithaca, NY: Cornell University Press.

von der Fehr, N.-H., & Kuhn, K.-U. (1995). Coase versus Pacman: Who Eats Whom in the Durable-Goods Monopoly? *Journal of Political Economy*, *103*(4), 785–812.

Waldman, M. (1993). A New Perspective on Planned Obsolescence. *The Quarterly Journal of Economics*, *108*(1), 273–283.

Wang, G. H. (1998). Bargaining over a Menu of Wage Contracts. *The Review of Economic Studies*, *65*(2), 295–305.

Ward, B. (1958). The Firm in Illyria: Market Syndicalism. *The American Economic Review*, *48*(4), 566–589.

Weitzman, M. L. (1984). *The Share Economy: Conquering Stagflation*. Harvard University Press.

Weitzman, M. L. (1985). The Simple Macroeconomics of Profit Sharing. *The American*

Economic Review, *75*(5), 937–953.

Whyte, W. F., & Whyte, K. K. (2014). *Making Mondragon: The Growth and Dynamics of the Worker Cooperative Complex*. Cornell university press.

Wilkinson, R., & Pickett, K. (2010). *The Spirit Level: Why Equality is Better for Everyone*. Penguin Books Limited.