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Evaluation of a Special Hankel Determinant of Binomial Coefficients

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This paper makes use of the recently introduced technique of γ-operators to evaluate the Hankel determinant with binomial coefficient entries $a_k = (3k)!/(2k)!k!$. We actually evaluate the determinant of a class of polynomials $a_k(x)$ having this binomial coefficient as constant term. The evaluation in the polynomial case is as an almost product, i.e. as a sum of a small number of products. The γ-operator technique to find the explicit form of the almost product relies on differential-convolution equations and establishes a second order differential equation for the determinant.

In addition to $x = 0$, product form evaluations for $x = \frac{3}{5}, \frac{3}{4}, \frac{3}{2}, 3$ are also presented. At $x = 1$, we obtain another almost product evaluation for the Hankel determinant with $a_k = (3k+1)!/(2k+1)!k!$.

Keywords: Hankel determinants, binomial coefficients, almost product form evaluations, differential equations, γ-operators.

1 Introduction

Certain classes of Hankel determinants with combinatorially interesting entries $a_k = a_{i+j}$ have product representations with surprising evaluations. An example that was proved in (1) is

$$\det\left[\binom{3(i+j)+2}{i+j}\right]_{0\leq i,j\leq n} = \prod_{i=1}^{n} \frac{(6i+4)!(2i+1)!}{2(4i+2)!(4i+3)!} .$$

A number of evaluations of this type appear in Gessel and Xin (4), and a comprehensive list can be found in Krattenthaler ((6), Theorem 31). For product form evaluations, LU decomposition, continued fractions and Dodgson condensation are the standard tools. There is an extensive literature on this topic, and a compilation of the state of affairs of the theory of determinants up to 2005 is in Krattenthaler (5; 6).

It appears that the evaluation of Hankel determinants with one of the simplest looking binomial entries among the lot, namely the one corresponding to

$$a_k = \binom{3k}{k}$$

does not appear in these compilations. In this paper we prove that

$$\det\left[\binom{3(i+j)}{i+j}\right]_{0\le i,j\le n} = \prod_{i=1}^{n} \frac{3(3i+1)(6i)!(2i)!}{(4i)!(4i+1)!} \ . \tag{1}$$

However, this evaluation is only one of the many results that follows from our method. We actually evaluate the determinant of the Hankel matrix with polynomial entries

$$a_k(x) = \sum_{m=0}^{k} \binom{3k-m}{k-m} x^m \tag{2}$$

as an *almost product* (2; 3), in this case as a sum of $n+1$ simple products. Put

$$H_0(n,x) = \det[a_{i+j}(x)]_{0\le i,j\le n} \ . \tag{3}$$

For small parameters, $a_k(x)$ and $H_0(n,x)$ are as follows:

$$\begin{aligned} a_0(x) &= 1 \\ a_1(x) &= 3+x \\ a_2(x) &= 15+5x+x^2 \\ a_3(x) &= 84+28x+7x^2+x^3 \\ a_4(x) &= 495+165x+45x^2+9x^3+x^4 \end{aligned}$$

and

$$\begin{aligned} H_0(0,x) &= 1 \\ H_0(1,x) &= 6-x \\ H_0(2,x) &= 99-24x-x^2 \\ H_0(3,x) &= 4590-1242x-252x^2+62x^3 \\ H_0(4,x) &= 601749-161082x-82080x^2+29640x^3-2090x^4 \ . \end{aligned}$$

The polynomials in (2) are of the form

$$a_k^{(\beta,\alpha)}(x) = \sum_{m=0}^{k} \binom{\beta k+\alpha-m}{k-m} x^m \ .$$

Following (2; 3), we refer to this as the (β,α)-*case* of the Hankel determinant evaluation problem.

In addition to the specialization at $x=0$, the method to prove the $(3,0)$-case provides product evaluations similar to (1) for $x = \frac{3}{5}, \frac{3}{4}, \frac{3}{2}, 3$. These are given in Corollaries 1 and 2. The evaluation of the $(3,0)$-case uses the γ-operator technique that we introduced in (3), and the tables therein. The γ-operators bypass the trace calculations of (2) that were used to evaluate the in-between $(3,1)$-case and consequently the evaluation of the Hankel determinant of binomial coefficients

$$a_k = \binom{3k+1}{k} \ .$$

This Hankel determinant was evaluated as an almost product in (2). Taking $x = 1$ in (2) and using Theorem 1 below, we obtain yet another formula for this evaluation.

The technique presented in (2) and further developed in (3) to find the explicit form of the almost product for $H_0(n, x)$ relies on establishing a second order ODE satisfied by $H_0(n, x)$, constructing the polynomial solution of this ODE by the method of Frobenius, and evaluating it at $x = 0$. Sometimes the constant of integration can be obtained more easily if the Frobenius solution is sought at some point other than $x = 0$. In the proof of Theorem 1, for example, we use $x = \frac{3}{2}$.

We give the elements of the application of γ-operators by working through the proof of the following theorem. The evaluations at special points $x = 0, \frac{3}{5}, \frac{3}{4}, \frac{3}{2}, 3$ are obtained as byproducts along the way to obtaining the ODE for $H_0(n, x)$.

Theorem 1 *Suppose a_k and the $H_0(n, x)$ are as defined in (2) and (3). Then*

$$H_0(n,x) = \prod_{i=1}^{n} \frac{9(2i)!(6i-2)!}{2(4i)!(4i-2)!} \sum_{k=0}^{n} \frac{n(n-1)\cdots(n-k+1)p_k(n)}{9^k k!\,(4n+1)(4n)\cdots(4n-k+3)} (2x-3)^k \tag{4}$$

where p_k are integral polynomials satisfying the recurrence relation

$$p_k(x) = -2(2x+k)p_{k-1}(x) - 15(k-1)(4x+4-k)p_{k-2}(x)$$

for $k > 2$ with $p_0(x) = 1$, $p_1(x) = -1$, $p_2(x) = 4x - 11$.

The outline of this paper is as follows: In section 2, we define the determinant H_λ for partitions λ obtained from a given Hankel matrix and the γ-operators. For the proofs of the combinatorial properties of the γ-operators and their compiled tables of values we refer the reader to (3). This is followed in section 3 by the three identities that are typical of our methods, and the derivation of the equations satisfied by the various H_λ that arise in the calculations. We obtain a system of first order ODE which results in a second order ODE for $H_0(n, x)$ in section 4. Evaluation at special points are discussed in section 5, and the general solution of the differential equation is derived in section 6, followed by remarks. The proofs of the three identities used can be found in the Appendix.

2 Preliminaries

A *partition* λ of an integer m is a weakly decreasing sequence of nonnegative integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m)$ with $m = \lambda_1 + \lambda_2 + \cdots + \lambda_m$. Each of the integers $\lambda_i > 0$ is a *part* of λ. For example $\lambda = (3, 2, 2)$ is a partition of $m = 7$ into three parts.

We use the notation $\lambda = m^{\alpha_m} \cdots 2^{\alpha_2} 1^{\alpha_1}$ for integer partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0)$, indicating that λ has α_i parts of size i. Thus for example, $\lambda = 3^2 2 1^3$ denotes the partition $3 + 3 + 2 + 1 + 1 + 1$ of 11. We use the special notation 0 to denote the partition of zero. Each partition $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n+1})$ defines a determinant of a matrix obtained from the $(n + 1) \times (n + 1)$ Hankel matrix $A_n = [a_{i+j}]_{0 \leq i,j \leq n}$ in the symbols a_k, by shifting the column indices of the entries up according to λ as follows:

$$H_\lambda = \det[a_{i+j+\lambda_{n+1-j}}]_{0 \leq i,j \leq n}\ .$$

For example when $n = 3$,

$$H_0 = \det \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \end{bmatrix}, \; H_2 = \det \begin{bmatrix} a_0 & a_1 & a_2 & a_5 \\ a_1 & a_2 & a_3 & a_6 \\ a_2 & a_3 & a_4 & a_7 \\ a_3 & a_4 & a_5 & a_8 \end{bmatrix}, \; H_{31^2} = \det \begin{bmatrix} a_0 & a_2 & a_3 & a_6 \\ a_1 & a_3 & a_4 & a_7 \\ a_2 & a_4 & a_5 & a_8 \\ a_3 & a_5 & a_6 & a_{10} \end{bmatrix}$$

We remark that these are obtained in a way similar to the expansion of Schur functions in terms of the homogeneous symmetric functions by the Jacobi-Trudi identity (7). When it is clear from the context, we use H_λ for the $(n+1) \times (n+1)$ Hankel determinant $H_\lambda(n, x)$.

The γ-operator is a multilinear operator defined on m-tuples of matrices:

Definition 1 *Given $(n+1)\times(n+1)$ matrices A and $X_1, X_2, \ldots, X_m$ with $m \geq 1$, define $\gamma_A(\,) = \det(A)$ and*

$$\gamma_A(X_1, \ldots, X_m) = \partial_{t_1} \partial_{t_2} \cdots \partial_{t_m} \det(A + t_1 X_1 + t_2 X_2 + \cdots + t_m X_m)|_{t_1 = \cdots = t_m = 0}$$

where $t_1, t_2, \ldots, t_m$ are variables that do not appear in A or $X_1, X_2, \ldots, X_m$.

One of our motivations for using the γ-operators is that they differentiate nicely; the derivative of a γ is a sum of γ's.

Proposition 1 *For $m \leq n$,*

$$\frac{d}{dx}\gamma_A(X_1, \ldots, X_m) = \gamma_A(\frac{d}{dx}A, X_1, \ldots, X_m) + \sum_{j=1}^{m} \gamma_A(X_1, \ldots, X_{j-1}, \frac{d}{dx}X_j, X_{j+1}, \ldots, X_m)\ .$$

The reader is referred to (3) for the proofs of various properties of γ-operators. It is worth mentioning that the values of the γ-operators need not be calculated from scratch for different Hankel determinant evaluations. Tables of of values of γ-operators (as well as a computationally feasible combinatorial interpretation of $\gamma_A(X_1, \ldots, X_m)$ for small m) are given in (3) (Sections 3, 4 and Appendix III).

Let $a_k(x)$ be as in (2) and define the convolution polynomials

$$c_n = \sum_{k=0}^{n} a_k a_{n-k}$$

with $c_{-1} = 0$.

3 Identities and expansions

The bulk of the work for the proof of Theorem 1 is contained in obtaining the ODE for $H_0(n, x)$, and this part of the argument itself relies on three essential identities, which are characteristic of our method.

Lemma 1 *(First Identity (FI))*[(i)]

$$\begin{aligned} 3(x-3)x(4x-3)\frac{d}{dx}a_n \quad = \quad & (4(2x-3)n + 2(2x-5))\, a_{n+1} \\ & - \big(27(2x-3)n + 3(4x^2 - 3x - 9)\big)\, a_n \\ & + 4(x-1)c_{n+1} - 27(x-1)c_n\ . \end{aligned} \tag{5}$$

(i) In (2) p. 47, where this identity appears, there is a typo and the constant 4 in front of the c_{n+1} term is missing.

Lemma 2 *(Second Identity (SI))*

$$\begin{aligned}&\big(4(2x-3)^2(5x-3)n+2(2x-3)(5x-3)(6x-11)\big)\,a_{n+2}\\&-\big(81(8x^3-24x^2+27x-9)n+18(37x^3-123x^2+153x-54)\big)\,a_{n+1}\\&+(729x^3n+486x^3)a_n+4(x-1)(2x-3)(5x-3)c_{n+2}\\&-3(40x^4-30x^3-207x^2+270x-81)c_{n+1}+162x^2(5x^2-15x+9)c_n=0\,.\end{aligned}\tag{6}$$

Lemma 3 *(Third Identity)*

$$\sum_{j=0}^{n+2} w_{n,j}(x)a_{i+j}(x)=0 \tag{7}$$

for $i=0,1,\dots,n$ where $w_{n,j}(x)$ are certain polynomials weights with

$$\begin{aligned}w_{n,n+2} &= 8(3+4n)(5+4n)(5x-3)\,,\\ w_{n,n+1} &= 4(3+4n)(150+219n+81n^2-250x-365nx-135n^2x-50x^2-40nx^2)\,,\\ w_{n,n} &= 3(-942n-2733n^2-2484n^3-729n^4+1570nx+4555n^2x+4140n^3x\\ &\qquad +1215n^4x+720x^2+2196nx^2+2188n^2x^2+720n^3x^2)\,.\end{aligned}\tag{8}$$

We will use the third identity in a determinantal form as given in (12). The proofs of these three lemmas are given in the Appendix. The first two are straightforward generating function calculations, whereas the proof of the third identity uses an alternate form of the generating function of the a_k and requires a new technique. Similar to the third identity proofs of $(3,1)$ and $(2,1)$-cases given in (2), we prove that weights $w_{n,j}(x)$ of Lemma 3 exist without explicitly constructing them except for the three in (8) that we need.

Theorem 2 *Suppose the polynomials $a_k(x)$ and the $(n+1)\times(n+1)$ Hankel determinant $H_0=H_0(n,x)$ are as defined in (2) and (3). Then*

$$\begin{aligned}&(x-3)(2x-3)(5x-3)\frac{d^2}{dx^2}H_0-2\left(10nx^2-10x^2-27nx+36x-9n-45\right)\frac{d}{dx}H_0\\&\qquad+n(10xn-3n-10x+21)H_0=0\,.\end{aligned}\tag{9}$$

Proof: The proof is made up of a number of different sections. First we derive two equations that relate the determinants H_2, H_{1^2}, H_1, H_0. These are used to express H_2 and H_{1^2} in terms of H_1, H_0. Then we find expressions for the derivatives of H_0 and H_1 in terms of H_0 and H_1.

3.1 Equation from $\gamma_A([SI(i+j)])$

Apply $\gamma_A(*)$ to the $(n+1)\times(n+1)$ matrix whose (i,j)-th entry is obtained from the second identity (6) evaluated at $i+j$ and expand using linearity. If we denote the matrix so obtained from the second identity by $[SI(i+j)]$, then the computation is the expansion of $\gamma_A([SI(i+j)])=0$. Making use of the entries in the $\gamma_A(*)$ computations from Table 2 of (3), we get

$$\begin{aligned}0 &= 4(2x-3)^2(5x-3)(2nH_2-2(n-1)H_{1^2})\\ &\quad +2(2x-3)(5x-3)(6x-11)(H_2-H_{1^2})-81(8x^3-24x^2+27x-9)(2nH_1)\end{aligned}$$

$$
\begin{aligned}
&-18(37x^3-123x^2+153x-54)H_1+729x^3n(n+1)H_0+486x^3(n+1)H_0\\
&+4(x-1)(2x-3)(5x-3)(2H_2-2H_{1^2}+2(x+3)H_1+(2n-1)(x^2+5x+15)H_0)\\
&-3(40x^4-30x^3-207x^2+270x-81)(2H_1+2n(x+3)H_0)\\
&+162x^2(5x^2-15x+9)(2n+1)H_0\ .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&2(5+4n)(2x-3)^2(5x-3)H_2-2(1+4n)(2x-3)^2(5x-3)H_{1^2}\\
&-2(-621-729n+1863x+2187nx-1476x^2-1944nx^2+247x^3+648nx^3+80x^4)H_1\\
&+(540+378n-1620x-1134nx+2754x^2+2430nx^2-2044x^3-1663nx^3+729n^2x^3\\
&\qquad+734x^4+1232nx^4-40x^5-160nx^5)H_0=0\ .
\end{aligned}
\tag{10}
$$

3.2 Equation from the third identity

Define the column vector

$$v_j=[a_j,a_{j+1},\ldots,a_{j+n}]^T\ .$$

The third identity (7) says that the vectors $v_0, v_1, \ldots, v_{n+2}$ are linearly dependent with the weights $w_{n,j}$ i.e.

$$\sum_{j=0}^{n+2} w_{n,j}v_j=0\ . \tag{11}$$

Now consider the determinant of the $(n+1)\times(n+1)$ matrix whose first n columns are the columns of A, and whose last column is the zero vector. Writing the zero vector in the form (11) and expanding the determinant by linearity, we find

$$w_{n,n+2}H_2+w_{n,n+1}H_1+w_{n,n}H_0=0\ . \tag{12}$$

Substituting the weights from (8), this gives the equation

$$
\begin{aligned}
&8(3+4n)(5+4n)(5x-3)H_2\\
&+4(3+4n)(150+219n+81n^2-250x-365nx-135n^2x-50x^2-40nx^2)H_1\\
&+3(-942n-2733n^2-2484n^3-729n^4+1570nx+4555n^2x+4140n^3x\\
&\qquad+1215n^4x+720x^2+2196nx^2+2188n^2x^2+720n^3x^2)H_0=0\ .
\end{aligned}
\tag{13}
$$

This is the second equation we need. Equations (10) and (13) form a linear system which can be solved to express the determinants H_2, H_{1^2} in terms of the determinants H_0, H_1. We obtain

$$
\begin{aligned}
&8(3+4n)(5+4n)(5x-3)H_2=\\
&-3\big(-942n-2733n^2-2484n^3-729n^4+1570nx+4555n^2x+4140n^3x+1215n^4x+720x^2\\
&+2196nx^2+2188n^2x^2+720n^3x^2\big)H_0\\
&+4(3+4n)\big(-150-219n-81n^2+250x+365nx+135n^2x+50x^2+40nx^2\big)H_1\ ,
\end{aligned}
\tag{14}
$$

$$8(1+4n)(3+4n)(2x-3)^2 H_{1^2} = \tag{15}$$
$$\big(-2160-12870n-26613n^2-22356n^3-6561n^4+2880x+17160nx+35484n^2x$$
$$+29808n^3x+8748n^4x+264x^2+1348nx^2+280n^2x^2-3456n^3x^2-2916n^4x^2-24x^3$$
$$-272nx^3-1616n^2x^3-1728n^3x^3-96x^4-512nx^4-512n^2x^4\big)H_0+4(3+4n)\big(36$$
$$+171n+243n^2-48x-228nx-324n^2x-14x^2-44nx^2+108n^2x^2+8x^3+32nx^3\big)H_1 \ .$$

4 The derivatives of H_0 and H_1

We now proceed with the calculation of the derivatives of H_0 and H_1.

4.1 *The derivative of* H_0

From Definition 1, $H_0 = \gamma_A(\ )$. Therefore by Proposition 1

$$\frac{d}{dx}H_0 = \gamma_A([\frac{d}{dx}a_{i+j}]) \ .$$

Using $FI(i+j)$,

$$\begin{aligned} 3(x-3)x(4x-3)\frac{d}{dx}H_0 \quad = \quad & 4(2x-3)\gamma_A([(i+j)a_{i+j+1}]) \\ & +2(2x-5)\gamma_A([a_{i+j+1}]) \\ & -27(2x-3)\gamma_A([(i+j)a_{i+j}]) \\ & -3(4x^2-3x-9)\gamma_A([a_{i+j}]) \\ & +4(x-1)\gamma_A([c_{i+j+1}]) \\ & -27(x-1)\gamma_A([c_{i+j}]) \ . \end{aligned}$$

The values for $\gamma_A(*)$ from Table 2 of (3) give

$$\begin{aligned} 3(x-3)x(4x-3)\frac{d}{dx}H_0 \quad = \quad & 4(2x-3)2nH_1 \\ & +2(2x-5)H_1 \\ & -27(2x-3)n(n+1)H_0 \\ & -3(4x^2-3x-9)(n+1)H_0 \\ & +4(x-1)(2H_1+2n(x+3)H_0) \\ & -27(x-1)(2n+1)H_0 \ . \end{aligned}$$

Therefore

$$3(x-3)x(4x-3)\frac{d}{dx}H_0 = \tag{16}$$
$$2(4n+3)(2x-3)H_1+(54+138n+81n^2-18x-83nx-54n^2x-12x^2-4nx^2)H_0 \ .$$

4.2 The derivative of H_1

To differentiate H_1 we use the expression $H_1 = \gamma_A([a_{i+j+1}])$ from Table 2 of (3). From Proposition 1 we have

$$\frac{d}{dx}H_1 = \gamma_A([a_{i+j+1}],[\frac{d}{dx}a_{i+j}]) + \gamma_A([\frac{d}{dx}a_{i+j+1}]) \ .$$

Therefore, to compute $\frac{d}{dx}H_1$

$$\gamma_A([a_{i+j+1}],[FI(i+j)]) \ \text{ and } \ \gamma_A([FI(i+j+1)])$$

are needed. Using the entries in Table 3 of (3) for the $\gamma_A([a_{i+j+1}],*)$ computations, we get for the first one of these

$$\begin{aligned} 3(x-3)x(4x-3)\gamma_A([a_{i+j+1}],[FI(i+j)]) \ &= \ 4(2x-3)(2(2n-1)H_{1^2}) \\ &\quad +2(2x-5)2H_{1^2} \\ &\quad -27(2x-3)n(n-1)H_1 \\ &\quad -3(4x^2-3x-9)nH_1 \\ &\quad +4(x-1)(4H_{1^2}+2(n-1)(x+3)H_1 \\ &\qquad -2(n-1)(x^2+5x+15)H_0) \\ &\quad -27(x-1)((2n-1)H_1-(2n-1)(x+3)H_0) \ , \end{aligned}$$

and the second one by using Table 2 of (3) as

$$\begin{aligned} 3(x-3)x(4x-3)\gamma_A([FI(i+j+1)]) \ &= \ 4(2x-3)(2nH_2-2(n-1)H_{1^2}) \\ &\quad +2(6x-11)(H_2-H_{1^2}) \\ &\quad -27(2x-3)(2nH_1) \\ &\quad -3(4x^2+15x-36)H_1 \\ &\quad +4(x-1)(2H_2-2H_{1^2}+2(x+3)H_1 \\ &\qquad +(2n-1)(x^2+5x+15)H_0) \\ &\quad -27(x-1)(2H_1+2n(x+3)H_0). \end{aligned}$$

Adding, we get

$$\begin{aligned} 3(x-3)x(4x-3)\frac{d}{dx}H_1 \ &= \ 2(4n+5)(2x-3)H_2 \\ &\quad +2(4n+1)(2x-3)H_{1^2} \\ &\quad +(135+138n+81n^2-72x-83nx-54n^2x-12x^2-4nx^2)H_1 \\ &\quad +(x-1)(4x^2-7x-21)H_0 \ . \end{aligned} \tag{17}$$

Note that we obtained the expressions for the derivative of H_0 in (16) and the derivative of H_1 in (17) by a direct application of γ-operators. The derivations do not require the expansions (14) and (15) which used the third identity (7) for their derivation.

We can use (14) and (15) to express $\frac{d}{dx}H_1$ as a linear combination of H_0, H_1:

$$\begin{aligned}
&6(4n+3)(x-3)x(2x-3)(4x-3)(5x-3)\frac{d}{dx}H_1 = \\
&2(4n+3)\big(280nx^4+120x^4+540n^2x^3-82nx^3-162x^3-1944n^2x^2-2817nx^2\\
&-594x^2+2187n^2x+3807nx+891x-729n^2-1242n-243\big)H_1\\
&+\big(-1280n^2x^5-960nx^5-8640n^3x^4-16400n^2x^4-14640nx^4-5400x^4-14580n^4x^3\\
&-17928n^3x^3+15178n^2x^3+34902nx^3+14418x^3+52488n^4x^2+159408n^3x^2\\
&+157140n^2x^2+48384nx^2-486x^2-59049n^4x-201204n^3x-230445n^2x-100602nx\\
&-13122x+19683n^4+67068n^3+76815n^2+33534n+4374\big)H_0\ .
\end{aligned} \tag{18}$$

Therefore we have a first order linear system of equations of the form

$$\begin{aligned}
Q\frac{d}{dx}H_0 &= Q_0H_0+Q_1H_1\\
U\frac{d}{dx}H_1 &= U_0H_0+U_1H_1
\end{aligned} \tag{19}$$

where the coefficient polynomials are as given in (16) and (18). First differentiate the first equation in (19) and substitute the expansion of $\frac{d}{dx}H_0$ and $\frac{d}{dx}H_1$ in terms of H_0 and H_1. After that, H_1 can be eliminated from the resulting equation for $\frac{d^2}{dx^2}H_0$ and the equation for $\frac{d}{dx}H_0$ that we already have. This proves Theorem 2. □

5 Product evaluations at special points

At this point we have enough information to evaluate $H_0(n,x)$ at special points. The evaluations do not use the ODE (9) for $H_0(n,x)$. We recall the following general result on Hankel determinants from (2) ((2), Section 3, Proposition 1):

Proposition 2

$$H_0(n-1,x)H_0(n+1,x) = H_0(n,x)H_2(n,x)+H_0(n,x)H_{1^2}(n,x)-H_1(n,x)^2\ . \tag{20}$$

Note that for our problem we know both $H_2(n,x)$ and $H_{1^2}(n,x)$ as a linear combination of $H_1(n,x)$ and $H_0(n,x)$. This means that for any $x=x_0$ for which we can evaluate $H_2(n,x_0)$, $H_{1^2}(n,x_0)$ and $H_1(n,x_0)$ in terms of $H_0(n,x_0)$, we obtain a recursion of the form

$$H_0(n-1,x_0)H_0(n+1,x_0) = f(n,x_0)H_0(n,x_0)^2\ , \tag{21}$$

where $f(n,x_0)$ is a rational function of n. Since this is a recursion in $H_0(n,x_0)/H_0(n-1,x_0)$ with $H_0(1,x_0)/H_0(0,x_0)=6-x_0$, it can be solved to evaluate $H_0(n,x_0)$ in product form. Note that in particular, we can easily evaluate $H_1(n,x_0)/H_0(n,x_0)$ for any x_0 for which the right hand side of (18) (or (16)) vanishes. We first show that there are product form evaluations of $H_0(n,x)$ at the points $x=0,\frac{3}{4},3$.

Corollary 1 *Suppose $a_k(x)$ is as defined in (2). Then*

$$\det[a_{i+j}(0)]_{0\le i,j\le n} = \prod_{i=1}^{n} \frac{3(3i+1)(2i)!(6i)!}{(4i)!(4i+1)!}, \tag{22}$$

$$\det[a_{i+j}(\tfrac{3}{4})]_{0\le i,j\le n} = \prod_{i=1}^{n} \frac{3(2i)!(6i+1)!}{2(4i)!(4i+1)!}, \tag{23}$$

$$\det[a_{i+j}(3)]_{0\le i,j\le n} = \frac{(3n+2)!}{2}\prod_{i=1}^{n} \frac{3(2i)!(6i-2)!}{(4i)!(4i+1)!}. \tag{24}$$

Proof: Evaluating the expansions (14) and (15) and the derivative (18) at the points $x = 0, \frac{3}{4}, 3$, we obtain the factor $f(n, x_0)$ in the recursion (21) explicitly as:

$$\begin{aligned} f(n,0) &= \frac{9(3n+2)(3n+4)(6n+1)(6n+5)}{4(4n+1)(4n+3)^2(4n+5)}, \\ f(n,\tfrac{3}{4}) &= \frac{9(3n+1)(3n+2)(6n+5)(6n+7)}{4(4n+1)(4n+3)^2(4n+5)}, \\ f(n,3) &= \frac{9(3n+4)(3n+5)(6n-1)(6n+1)}{4(4n+1)(4n+3)^2(4n+5)}. \end{aligned}$$

As an example of the steps involved in these derivations, we consider the case of the point $x = 0$. Specialize the identities (10), (16) and (17) at $x = 0$ and solve for H_2, H_{1^2}, H_1 to obtain

$$\begin{aligned} H_1 &= \frac{27n^2+46n+18}{2(4n+3)} H_0 \\ H_2 &= \frac{729n^4+3942n^3+7655n^2+6286n+1800}{8(4n+3)(4n+5)} H_0 \\ H_{1^2} &= \frac{3\left(243n^4+342n^3-7n^2-126n-32\right)}{8(4n+1)(4n+3)} H_0. \end{aligned} \tag{25}$$

Substituting these expressions in (20),

$$H_0(n-1,0)H_0(n+1,0) = \frac{9(3n+2)(3n+4)(6n+1)(6n+5)}{4(4n+1)(4n+3)^2(4n+5)} H_0(n,0)^2.$$

This recurrence gives

$$H_0(n,0) = \prod_{m=0}^{n-1} 6 \prod_{i=1}^{m} \frac{9(3i+2)(3i+4)(6i+1)(6i+5)}{4(4i+1)(4i+3)^2(4i+5)},$$

which can in turn be rewritten as (23). The proofs of the other two evaluations are similar. Specializing (10), (16) and (17) at $x = \frac{3}{4}$ we obtain

$$H_1 = \frac{54n^2+98n+45}{4(4n+3)} H_0$$

$$
\begin{aligned}
H_2 &= \frac{1458n^4 + 8208n^3 + 16756n^2 + 14654n + 4635}{16(4n+3)(4n+5)} H_0 \qquad (26)\\
H_{1^2} &= \frac{3\left(243n^4 + 396n^3 + 104n^2 - 69n - 14\right)}{8(4n+1)(4n+3)} H_0 \ ,
\end{aligned}
$$

and at $x = 3$, we obtain

$$
\begin{aligned}
H_1 &= \frac{27n^2 + 49n + 36}{2(4n+3)} H_0\\
H_2 &= \frac{729n^4 + 4104n^3 + 9107n^2 + 10108n + 4680}{8(4n+3)(4n+5)} H_0 \qquad (27)\\
H_{1^2} &= \frac{3\left(243n^4 + 396n^3 + 347n^2 - 114n - 32\right)}{8(4n+1)(4n+3)} H_0 \ .
\end{aligned}
$$

□

6 The differential equation solution

We now indicate briefly the solution to the ODE (9) for $H_0(n,x)$. Let b_k be the coefficient of the Frobenius solution at the regular singular point $x = \frac{3}{2}$. The exponents are computed to be $r = 0, 4n+3$. The polynomial solution is for $r = 0$, and we obtain the recursion for the coefficients

$$
b_k = \frac{4(k-n-1)}{27k(k-4n-3)}\big(-3(k+2n)b_{k-1} + 5(k-n-2)b_{k-2}\big) \qquad (28)
$$

with

$$
b_0 = H_0(n, \tfrac{3}{2}) \ , \quad b_1 = -\frac{2}{9} n \, b_0
$$

where the expression for b_1 follows from specializing identity (16) at $x = \frac{3}{2}$. We can show by induction using (28) that for $k > 0$

$$
b_k = \left(\frac{2}{9}\right)^k \frac{n(n-1)\cdots(n-k+1)}{k! \ (4n+1)(4n)\cdots(4n-k+3)} p_k(n) \ b_0
$$

where $p_k = p_k(x)$ is an integral polynomial of degree $k-1$ satisfying the recurrence relation in Theorem 1. We omit the proof of this step. Using the product form of $b_0 = H_0(n, \frac{3}{2})$ from (30) we obtain (4). This completes the proof of Theorem 1.

7 Remarks and additional results

We have made use of the γ-operators of (3) to evaluate the Hankel determinant of the polynomials in (2) and obtained a number of product form evaluations at special points as corollaries of the method.

We remark that the polynomials $p_{4k}, p_{4k-1}, p_{4k-2}, p_{4k-3}$ for $k \geq 1$ that appear in the almost product evaluation in (4) are divisible in $\mathbb{Z}[x]$ by

$$
\prod_{i=1}^{k-1} (2x - 2i + 1) \ .
$$

It should also be possible to write (4) in alternate forms.

The method also allows for the product form evaluations of the Hankel determinant at $x = \frac{3}{5}, \frac{3}{2}$, and we give a sketch of the proof for these two. In addition to (14) and (15) we evaluate the identity we obtain from the expansion of $\gamma_A([SI(i+j+1)])$ at $x = \frac{3}{5}$. These give $H_2(n, \frac{3}{5})$, $H_{1^2}(n, \frac{3}{5})$ and $H_1(n, \frac{3}{5})$ in terms of $H_0(n, \frac{3}{5})$, and we find

$$f(n, \tfrac{3}{5}) = \frac{9(3n+4)(3n+5)(6n+5)(6n+7)}{4(4n+3)(4n+5)^2(4n+7)}\ .$$

Similarly, at $x = \frac{3}{2}$ we evaluate (14) and (15) together with the identity we obtain from the expansion of $\gamma_A([a_{i+j+1}], [SI(i+j)])$. This gives

$$f(n, \tfrac{3}{2}) = \frac{9(3n+1)(3n+2)(6n-1)(6n+1)}{4(4n-1)(4n+1)^2(4n+3)}\ .$$

Therefore

Corollary 2 *Suppose $a_k(x)$ is as defined in (2). Then*

$$\det[a_{i+j}(\tfrac{3}{5})]_{0\le i,j\le n} = \prod_{i=1}^{n} \frac{9(2i)!(6i+4)!}{20(4i+1)!(4i+3)!}\ , \tag{29}$$

$$\det[a_{i+j}(\tfrac{3}{2})]_{0\le i,j\le n} = \prod_{i=1}^{n} \frac{9(2i)!(6i-2)!}{2(4i)!(4i-2)!}\ . \tag{30}$$

The evaluation of $\gamma_A([SI(i+j+1)])$ and $\gamma_A([a_{i+j+1}], [SI(i+j)])$ using the tables of γ-operators in (3) can be found in the Appendix.

Since $a_k(1) = \binom{3k+1}{k}$, we also get the following evaluation

Corollary 3

$$\det\left[\binom{3(i+j)+1}{i+j}\right]_{0\le i,j\le n} = \prod_{i=1}^{n} \frac{9(2i)!(6i-2)!}{2(4i)!(4i-2)!} \sum_{k=0}^{n} \frac{n(n-1)\cdots(n-k+1)p_k(n)(-1)^k}{9^k k!\ (4n+1)(4n)\cdots(4n-k+3)}$$

where $p_k(x)$ are the integral polynomials defined in Theorem 1.

The $(3,1)$-case was already evaluated as an almost product in two different ways in (2) (see (2), (5) and (6)). The above expression is a third evaluation of this determinant.

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8 Appendix

The generating function of the $a_k(x)$ defined in (2) is given by

$$f(x,y) = \sum_{k\geq 0} a_k(x)y^k = \frac{t}{(3-2t)\,(1-t^2xy)} \tag{31}$$

where

$$t = \sum_{k\geq 0} \frac{(3k)!}{(2k+1)!k!} y^k = 1 + y + 3y^2 + 12y^3 + \cdots \tag{32}$$

satisfies

$$yt^3 = t - 1 \; . \tag{33}$$

A general case of this generating function can be found in (8).

8.1 First and Second identities

The proofs of Lemma 1, Lemma 2 are based on generating function manipulations.

Using $\frac{d}{dy}t = t^3/(1-3yt^2)$ in the computation of $\frac{d}{dy}f$ and using the resulting expressions for $\frac{d}{dx}f$ and $f' = \frac{d}{dy}f$, we make the substitutions

$$\begin{aligned}
\frac{d}{dx}a_n &\rightarrow \frac{d}{dx}f \\
a_n &\rightarrow f \\
na_n &\rightarrow yf' \\
a_{n+1} &\rightarrow (f-1)/y \\
na_{n+1} &\rightarrow y((f-1)/y)'
\end{aligned}$$

$$\begin{aligned}
a_{n+2} &\rightarrow (f-1-(3+x)y)/y^2\\
na_{n+2} &\rightarrow y((f-1-(3+x)y)/y^2)'\\
c_n &\rightarrow f^2\\
c_{n+1} &\rightarrow (f^2-1)/y\\
c_{n+2} &\rightarrow (f^2-1-2(3+x)y)/y^2
\end{aligned}$$

The generating function of the left hand side minus the right hand side of (5) factors as

$$\frac{3\left(yt^3-t+1\right)}{(2t-3)^2y\left(3t^2y-1\right)\left(t^2xy-1\right)^2}\Big(-8x^2y^2t^5+24x^2y^2t^4+18x^2y^2t^3-54xy^2t^3 +8x^2yt^3+4xyt^3-4x^2yt^2-18xyt^2-4xt-12x^2yt+9xyt+27yt+6\Big).$$

The generating function of the left hand side of (6) factors as

$$\frac{3\left(yt^3-t+1\right)}{(2t-3)^2y^2\left(3t^2y-1\right)\left(t^2xy-1\right)^2}\Big(-324x^4y^3t^5+324x^3y^3t^5+80x^4y^2t^5-168x^3y^2t^5 +72x^2y^2t^5+972x^4y^3t^4-972x^3y^3t^4-240x^4y^2t^4+504x^3y^2t^4-216x^2y^2t^4 -243x^4y^3t^3+729x^3y^3t^3+360x^4y^2t^3-270x^3y^2t^3-162x^2y^2t^3-80x^4yt^3 +128x^3yt^3+12x^2yt^3-36xyt^3-270x^4y^2t^2+162x^3y^2t^2-729x^2y^2t^2+729xy^2t^2 +40x^4yt^2+96x^3yt^2-342x^2yt^2+162xyt^2+40x^3t-84x^2t-486x^3y^2t+36xt +18x^3yt-351x^2yt+648xyt-243yt-60x^2+126x+243x^2y-243xy-54\Big).$$

Since yt^3-t+1 is a factor in each numerator, they are both zero by (33) and Lemma 1 and Lemma 2 hold as stated.

8.2 Third identity

For the proof of the third identity we need the following form of the generating function:

Lemma 4 *The generating function in (31) has the alternate expression*

$$f(x,y)=\frac{t(3-2x)-3x}{(x-3)(4x-3)+t\left((9y-4)x^2+10x-6\right)} \tag{34}$$

where $yt^3=t-1$.

Proof:

$$\frac{t}{(3-2t)\left(1-t^2xy\right)}-\frac{t(3-2x)-3x}{(x-3)(4x-3)+\left((9y-4)x^2+10x-6\right)t} = \frac{x(4xt-6t+9)\left(yt^3-t+1\right)}{(2t-3)\left(t^2xy-1\right)\left(-4tx^2+9tyx^2+4x^2+10tx-15x-6t+9\right)}$$

and therefore the right hand side vanishes by (33). □

We prove the existence of weights $w_{n,0}, w_{n,1}, \ldots, w_{n,n+2}$ satisfying the third identity (7) where $w_{n,n+2}, w_{n,n+1}, w_{n,n}$ are as in (8).

There is a nontrivial polynomial $Q_0 = Q_0(y)$ of degree $n+1$ such that

$$tQ_0 = Q_1 + y^{2n+3}\Psi_0 \tag{35}$$

where $Q_1 = Q_1(y)$ is a polynomial of degree $n+1$, and $\Psi_0 = \Psi_0(y)$ is a power series in y; i.e. the coefficients of y^k in tQ_0 vanish for $n+2 \le k \le 2n+2$. Such a nontrivial Q_0 exists because there are $n+2$ coefficients to determine in Q_0 and only $n+1$ linear homogeneous equations these coefficients need to satisfy. In the next step, put

$$Q_2 = \big((9y-4)x^2 + 10x - 6\big)\, Q_1 + (x-3)(4x-3)Q_0\ . \tag{36}$$

Then $Q_2 = Q_2(x,y)$ is a polynomial in x and y of y-degree $n+2$. All three polynomials Q_0, Q_1, Q_2 are nontrivial. We claim that the coefficients of Q_2 are the weights we want. In other words, the coefficients of the terms y^{n+2} through y^{2n+2} in fQ_2 vanish. Writing (34) in the form

$$f(x,y)\Big(\big((9y-4)x^2 + 10x - 6\big)\, t + (x-3)(4x-3)\Big) = t(3-2x) - 3x$$

and multiplying through by Q_0, we get

$$f(x,y)\Big(\big((9y-4)x^2 + 10x - 6\big)\,(Q_1 + y^{2n+3}\Psi_0) + (x-3)(4x-3)Q_0\Big) = (3-2x)(Q_1 + y^{2n+3}\Psi_0) - 3xQ_0$$

Therefore

$$fQ_2 = Q_1 + y^{2n+3}\Psi_1 \tag{37}$$

where $\Psi_1 = \Psi(y)$ is a power series in y. This last statement (37) is equivalent to

$$\sum_{j=0}^{n+2} \mathcal{C}_{n+2-j}(Q_2)a_{i+j} \ = 0 \tag{38}$$

for $i = 0, 1, \ldots, n$, where by $\mathcal{C}_k(\Psi)$ we denote the coefficient of the term y^k in a power series Ψ. Thus (7) holds with

$$w_{n,j} = \mathcal{C}_{n+2-j}(Q_2)$$

for $j = 0, 1, \ldots, n+2$. Therefore

$$\mathcal{C}_0(Q_2)H_2 + \mathcal{C}_1(Q_2)H_1 + \mathcal{C}_2(Q_2)H_0 = 0\ . \tag{39}$$

This identity is not trivial, for otherwise we would have a nontrivial linear relationship among the $n+1$ columns $v_0, v_1, \ldots, v_n$ of H_n, but H_n does not identically vanish. Rewrite (39) in terms of $\mathcal{C}_0(Q_0), \mathcal{C}_1(Q_0), \mathcal{C}_2(Q_0)$ which are pure constants, independent on x and y. We will express the coefficients in (39) in terms of these. Using the expansion in (32) and comparing coefficients in (35) and (36), we obtain

$$\begin{aligned}
\mathcal{C}_0(Q_1) &= \mathcal{C}_0(Q_0) \\
\mathcal{C}_1(Q_1) &= \mathcal{C}_1(Q_0) + \mathcal{C}_0(Q_0) \\
\mathcal{C}_2(Q_1) &= \mathcal{C}_2(Q_0) + \mathcal{C}_1(Q_0) + 3\mathcal{C}_0(Q_0) \\
\mathcal{C}_0(Q_2) &= (-4x^2 + 10x - 6)\mathcal{C}_0(Q_1) + (x-3)(4x-3)\mathcal{C}_0(Q_0) \\
\mathcal{C}_1(Q_2) &= (-4x^2 + 10x - 6)\mathcal{C}_1(Q_1) + 9x^2\mathcal{C}_0(Q_1) + (x-3)(4x-3)\mathcal{C}_1(Q_0) \\
\mathcal{C}_2(Q_2) &= (-4x^2 + 10x - 6)\mathcal{C}_2(Q_1) + 9x^2\mathcal{C}_1(Q_1) + (x-3)(4x-3)\mathcal{C}_2(Q_0)\ .
\end{aligned}$$

Therefore

$$
\begin{aligned}
\mathcal{C}_0(Q_2) &= (3-5x)\mathcal{C}_0(Q_0) \\
\mathcal{C}_1(Q_2) &= (3-5x)\mathcal{C}_1(Q_0) + (5x^2+10x-6)\mathcal{C}_0(Q_0) \\
\mathcal{C}_2(Q_2) &= (3-5x)\mathcal{C}_2(Q_0) + (5x^2+10x-6)\mathcal{C}_1(Q_0) - 3(x^2-10x+6)\mathcal{C}_0(Q_0)\,.
\end{aligned} \tag{40}
$$

Substituting back in (39) we have

$$
\begin{aligned}
&(3-5x)H_0\,\mathcal{C}_2(Q_0) - \big((5x^2+10x-6)H_0 + (3-5x)H_1\big)\,\mathcal{C}_1(Q_0) \\
&+\big(3(x^2-10x+6)H_0 - (5x^2+10x-6)H_1 - (3-5x)H_2\big)\,\mathcal{C}_0(Q_0) = 0\,.
\end{aligned} \tag{41}
$$

We need two linearly independent relations between $\mathcal{C}_2(Q_0), \mathcal{C}_1(Q_0), \mathcal{C}_0(Q_0)$. These are obtained by evaluating at any two of the special points $\{0, \frac{3}{4}, 3\}$. Using the expressions (25) for H_1, H_2, H_{1^2} in terms of n and H_0 in (41) we obtain

$$
\begin{aligned}
&8(3+4n)(5+4n)\mathcal{C}_2(Q_0) - 12(5+4n)(2+10n+9n^2)\mathcal{C}_1(Q_0) \\
&\qquad -3(120+778n+1445n^2+1026n^3+243n^4)\mathcal{C}_0(Q_0) = 0\,.
\end{aligned} \tag{42}
$$

This is the first equation we need. For the next point we use $x = 3$. Using the expressions from (27) for H_1, H_2, H_{1^2} in terms of n and H_0 in (41) we obtain

$$
\begin{aligned}
&8(3+4n)(5+4n)\mathcal{C}_2(Q_0) - 6(5+4n)(1+2n+18n^2)\mathcal{C}_1(Q_0) \\
&\qquad -3(30+67n+338n^2+540n^3+243n^4)\mathcal{C}_0(Q_0) = 0\,.
\end{aligned} \tag{43}
$$

Solving (42) and (43) for $\mathcal{C}_2(Q_0)$ and $\mathcal{C}_1(Q_0)$ in terms of the parameter $\mathcal{C}_0(Q_0)$, we get

$$
\begin{aligned}
\mathcal{C}_2(Q_0) &= -\frac{3n\left(243n^3+540n^2+343n+50\right)}{8(4n+3)(4n+5)}\mathcal{C}_0(Q_0) \\
\mathcal{C}_1(Q_0) &= -\frac{3(n+1)(9n+10)}{2(4n+5)}\mathcal{C}_0(Q_0)
\end{aligned}
$$

Substituting back into (40) we obtain

$$
\begin{aligned}
\mathcal{C}_0(Q_2) &= (3-5x)\mathcal{C}_0(Q_0) \\
\mathcal{C}_1(Q_2) &= \big(135xn^2 - 81n^2 + 40x^2n + 365xn - 219n + 50x^2 + 250x - 150\big)\,\mathcal{C}_0(Q_0)/2(4n+5) \\
\mathcal{C}_2(Q_2) &= 3\big(1215xn^4 - 729n^4 + 720x^2n^3 + 4140xn^3 - 2484n^3 + 2188x^2n^2 + 4555xn^2 \\
&\qquad -2733n^2 + 2196x^2n + 1570xn - 942n + 720x^2\big)\mathcal{C}_0(Q_0)/8(4n+3)(4n+5)
\end{aligned}
$$

Taking

$$\mathcal{C}_0(Q_0) = -8(3+4n)(5+4n)$$

these are exactly the weights $w_{n,n+2}, w_{n,n+1}, w_{n,n}$ as claimed in (8).

8.3 The expansions $\gamma_A([SI(i+j+1)])$ and $\gamma_A([a_{i+j+1}],[SI(i+j)])$

The expansion of $\gamma_A([SI(i+j+1)])$ using Table 2 of (3) is

$$
\begin{aligned}
&2(7+4n)(2x-3)^2(5x-3)H_3\\
&-2(3+4n)(2x-3)^2(5x-3)H_{21}+2(4n-1)(2x-3)^2(5x-3)H_{1^3}\\
&+(1971+1458n-5913x-4374nx+4896x^2+3888nx^2-1142x^3-1296nx^3-160x^4)H_2\\
&+(-513-1458n+1539x+4374nx-1008x^2-3888nx^2-154x^3+1296nx^3+160x^4)H_{1^2}\\
&+(378-1134x+2430x^2-1663x^3+1458nx^3+1232x^4-160x^5)H_1\\
&+(2403+1242n-7209x-3726nx+9108x^2+5148nx^2-5029x^3-2990nx^3\\
&\qquad-15x^4-714nx^4+198x^5+912nx^5+40x^6-160nx^6)H_0=0\,,
\end{aligned}
$$

and the expansion of $\gamma_A([a_{i+j+1}],[SI(i+j)])$ using Table 3 of (3) is

$$
\begin{aligned}
&2(5+4n)(2x-3)^2(5x-3)H_{21}-4(4n-1)(2x-3)^2(5x-3)H_{1^3}\\
&+(1026+2916n-3078x-8748nx+2016x^2+7776nx^2+308x^3-2592nx^3-320x^4)H_{1^2}\\
&+(162+378n-486x-1134nx+324x^2+2430nx^2+348x^3-3121nx^3+729n^2x^3-498x^4\\
&\qquad+1232nx^4+120x^5-160nx^5)H_1\\
&+(-1782-1242n+5346x+3726nx-6534x^2-5148nx^2+3534x^3+2990nx^3-342x^4\\
&\qquad+714nx^4+258x^5-912nx^5-120x^6+160nx^6)H_0=0\,.
\end{aligned}
$$