

# Record statistics in integer compositions

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A composition  $\sigma = a_1 a_2 \dots a_m$  of  $n$  is an ordered collection of positive integers whose sum is  $n$ . An element  $a_i$  in  $\sigma$  is a strong (weak) record if  $a_i > a_j$  ( $a_i \geq a_j$ ) for all  $j = 1, 2, \dots, i - 1$ . Furthermore, the position of this record is  $i$ . We derive generating functions for the total number of strong (weak) records in all compositions of  $n$ , as well as for the sum of the positions of the records in all compositions of  $n$ , where the parts  $a_i$  belong to a fixed subset  $A$  of the natural numbers. In particular when  $A = \mathbb{N}$ , we find the asymptotic mean values for the number, and for the sum of positions, of records in compositions of  $n$ .

**Keywords:** Composition, Record, Left-to-right maxima, Generating function, Mellin transforms, Asymptotic estimates

## Introduction

Let  $\pi = a_1 a_2 \dots a_n$  be any permutation of length  $n$ , an element  $a_i$  in  $\pi$  is a record if  $a_i > a_j$  for all  $j = 1, 2, \dots, i - 1$ . Furthermore, the position of this record is  $i$ . The number of records was first studied by Rényi [13], compare also [7]. A survey of results on this topic can be found in [2]. In the literature records are also referred to as a *left-to-right maxima* or *outstanding elements*. In particular the study of records has applications to observations of extreme weather problems, test of randomness, determination of minimal failure, and stresses of electronic components. The recent paper by Kortchemski [8] defines a new statistic *srec*, where  $srec(\pi)$  is the sum over the positions of all records in  $\pi$ . For instance, the permutation  $\pi = 451632$  has 3 records 4, 5, 6 and  $srec(\pi) = 1 + 2 + 4 = 7$ .

A word over an alphabet  $A$ , a set of positive integers, is defined as any ordered sequence of possibly repeated elements of  $A$ . Recently, Prodinger [12] studied the statistic *srec* for words over the alphabet  $\mathbb{N} = \{1, 2, 3, \dots\}$ , equipped with geometric probabilities  $p, pq, pq^2, \dots$ , with  $p + q = 1$ . In the case of words there two versions: A *strong record* in a word  $a = a_1 \dots a_n$  is an element  $a_i$  such that  $a_i > a_j$  for all  $j = 1, 2, \dots, i - 1$  (that is, must be strictly larger than elements to the left) and *weak record* is an element  $a_i \geq a_j$  for all  $j = 1, 2, \dots, i - 1$  (must be only larger or equal to elements to the left). Furthermore, the position  $i$  is called the position of the strong record (weak record). We denote the sum of

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the positions of all strong (respectively, weak) records in a word  $a$  by  $ssrec$  (respectively,  $wsrec$ ). In [12], Prodinger found the expected value of the sum of the positions of strong records, in random geometrically distributed words of length  $n$ . Previously, Prodinger [10] also studied the number of strong and weak records, in samples of geometrically distributed random variables. He also studied further properties of such records in papers such as [11] and references therein.

A composition  $\sigma = \sigma_1\sigma_2 \dots \sigma_m$  of  $n$  is an ordered collection of positive integers whose sum is  $n$ . Thus a composition  $\sigma$  of  $n$  with parts in  $A$  is a restricted word over the alphabet  $A$ . We denote the set of all compositions of  $n$  with  $m$  parts in  $A$  by  $C_A(n, m)$ . It is well known that the number of compositions of  $n \geq 1$  with  $m$  parts in  $\mathbb{N}$  is given by  $\binom{n-1}{m-1}$  and that the total number of compositions of  $n$  is  $2^{n-1}$ .

In this paper we find generating functions for these parameters, number of strong records, number of weak records, sum of positions of strong records, and sum positions of weak records in a random composition of  $n$  with parts in  $A = [d] := \{1, 2, \dots, d\}$  or  $A = \mathbb{N}$ . We also study the mean values of these parameters as  $n \rightarrow \infty$  in the case  $A = \mathbb{N}$  by means of rational function asymptotics and Mellin transforms. Details of some of the lengthier proofs will be left to the full version of the paper. We remark that in [5], an asymptotic correspondence is established between compositions of  $n$  and samples of geometric variable of parameter  $p = 1/2$  and length  $n/2$ . By exploiting this correspondence, and using the already established results of Prodinger for samples of geometric random variables, alternative derivations of our asymptotic results can be obtained.

## 1 Number strong records and weak records

Let  $NSR_A(z, y, q)$  and  $NWR_A(z, y, q)$  be the generating function for the number of compositions of  $n$  with  $m$  parts in  $A$  according to the number of strong and weak records, respectively, that is,

$$NSR_A(z, y, q) = \sum_{n, m \geq 0} \sum_{\sigma \in C_A(n, m)} z^n y^m q^{nsr(\sigma)},$$

$$NWR_A(z, y, q) = \sum_{n, m \geq 0} \sum_{\sigma \in C_A(n, m)} z^n y^m q^{nwr(\sigma)},$$

where  $nsr(\sigma)$  and  $nwr(\sigma)$  is the number of strong and weak records in composition  $\sigma$ , respectively. In this section we find an explicit formulas for those generating functions.

**Theorem 1.1** *The generating function  $NSR_{[d]}(z, y, q)$  is given by*

$$NSR_{[d]}(z, y, q) = \prod_{j=1}^d \left( 1 + \frac{z^j y q}{1 - y \sum_{i=1}^j z^i} \right).$$

**Proof:** We denote the number of occurrences of the part  $d$  in the composition  $\sigma \in C_{[d]}(n, m)$  by  $\ell(\sigma)$ . Now let us write equation for the generating function  $NSR_{[d]}(z, y, q)$ . The contribution of the case  $\ell(\sigma) = 0$  is given by  $NSR_{[d-1]}(z, y, q)$ . Assume  $\ell(\sigma) > 0$ , then  $\sigma$  can be decomposed as  $\sigma' d \sigma''$ , where  $\sigma'$  is a composition with parts in  $[d-1]$  and  $\sigma''$  is a composition with parts in  $[d]$ . Thus, the contribution of the case  $\ell(\sigma) > 0$  equals  $z^d y q NSR_{[d-1]}(z, y, q) NSR_{[d]}(z, y, 1)$ . Therefore,

$$NSR_{[d]}(z, y, q) = NSR_{[d-1]}(z, y, q) + z^d y q NSR_{[d-1]}(z, y, q) NSR_{[d]}(z, y, 1).$$

For  $q = 1$  and by induction we have that

$$NSR_{[d]}(z, y, 1) = \frac{1}{1 - y \sum_{j=1}^d z^j}.$$

Hence,

$$NSR_{[d]}(z, y, q) = \prod_{j=1}^d \left( 1 + \frac{z^j y q}{1 - y \sum_{i=1}^j z^i} \right),$$

as claimed.  $\square$  Theorem 1.1 with  $q = 1$  gives that the generating function for the number of compositions of  $n$  with  $m$  parts in  $[d]$  is given by  $NSR_{[d]}(z, y, 1) = \frac{1}{1 - y \sum_{i=1}^d z^i}$ , see for example [4].

Also from Theorem 1.1 we get that

$$\begin{aligned} \frac{\partial}{\partial q} NSR_{[d]}(z, y, 1) &= \prod_{j=1}^d \left( 1 + \frac{z^j y}{1 - y \sum_{i=1}^j z^i} \right) \left( \sum_{j=1}^d \frac{z^j y}{1 - y \sum_{i=1}^{j-1} z^i} \right) \\ &= \frac{1}{1 - y \sum_{i=1}^d z^i} \left( \sum_{j=1}^d \frac{z^j y}{1 - y \sum_{i=1}^{j-1} z^i} \right). \end{aligned}$$

Hence, the generating function for the number strong records in all compositions of  $n$  with parts in  $\mathbb{N}$  is given by

$$f(z) := \frac{1}{1 - \sum_{i \geq 1} z^i} \sum_{j \geq 1} \frac{z^j}{1 - \sum_{i=1}^{j-1} z^i} = \frac{1 - z}{1 - 2z} \sum_{j \geq 1} \frac{z^j}{1 - \sum_{i=1}^{j-1} z^i}.$$

**Theorem 1.2** *The average number  $E_n^s$  of strong left-to-right maxima in the context of compositions of  $n$  has the asymptotic expansion*

$$E_n^s = \frac{1}{2} \left[ \log_2 n - \frac{1}{2} + \frac{\gamma}{L} - \delta(\log_2 n) \right] + o(1).$$

Here and in the rest of the paper,  $L = \log 2$ ;  $\gamma$  is Euler's constant and  $\delta(x)$  is a periodic function of period 1 and mean 0 and small amplitude, which is given by the Fourier series

$$\delta(x) = \frac{1}{L} \sum_{k \neq 0} \Gamma(-\chi_k) e^{2k\pi i x}.$$

The complex numbers  $\chi_k$  are given by  $\chi_k = 2k\pi i/L$ .

**Proof:** Firstly by summing the finite geometric series and using partial fraction decomposition,

$$f(z) = \frac{z - z^2}{1 - 2z} + (1 - z)^2 \sum_{k \geq 2} \left[ \frac{1}{1 - 2z} - \frac{1}{1 - 2z + z^k} \right].$$

Hence the average number  $E_n^s$  of strong left-to-right maxima in compositions of  $n$  satisfies

$$E_n^s = \frac{1}{2^{n-1}} [z^n] f(z) = \frac{1}{2} + \frac{1}{2^{n-1}} [z^n] (1-z)^2 \sum_{k=2}^n \left[ \frac{1}{1-2z} - \frac{1}{1-2z+z^k} \right].$$

Let  $\rho_k$  be the smallest positive root of the denominator polynomial  $1 - 2z + z^k$  that lies between  $1/2$  and  $1$ . An application of the principle of the argument or Rouché's Theorem shows such a root to exist with all other roots of modulus greater than  $3/4$ . By dominant pole analysis,

$$q_{n,k} := [z^n] \frac{(1-z)^2}{1-2z+z^k} = c_k \rho_k^{-n} + O\left(\left(\frac{4}{3}\right)^n\right) \quad \text{with} \quad c_k = \frac{(1-\rho_k)^2}{\rho_k(2-k\rho_k^{k-1})},$$

for large  $n$  but fixed  $k$ . The denominator polynomial  $1 - 2z + z^k$  behaves like a perturbation of  $1 - 2z$  near  $z = 1/2$ . By "bootstrapping" we find that

$$\rho_k = \frac{1}{2} + 2^{-k-1} + O(k2^{-2k})$$

and hence  $c_k = \frac{1}{4} + O(k2^{-k})$ . The use of this approximation can be justified for a wide range of values of  $k$  and  $n$  (see for example [3] or [6]).

Let us now restrict our attention to those  $k$  for which  $n^{-3} \leq 2^{-k} \leq \frac{\log n}{n}$ . For such  $k$  we can show that

$$q_{n,k} = 2^{n-2} \left( \exp\left(-\frac{n}{2^k}\right) + O\left(\frac{\log^3 n}{n}\right) \right). \tag{1}$$

Turning next to smaller values of  $k \geq 2$ , that is,  $k$  such that  $2^{-k} > \frac{\log n}{n}$ , we find that now the coefficients  $q_{n,k}$  are relatively small, since for such  $k$ ,  $q_{n,k} = O\left(\frac{2^n}{n}\right)$  as  $n \rightarrow \infty$ . Finally we must consider larger values of  $k \leq n$  that is,  $k$  for which  $n^{-3} > 2^{-k}$ , or equivalently,  $k \geq 3 \log_2 n$ . In this range we find that

$$q_{n,k} = 2^{n-2} \left( \exp\left(-\frac{n}{2^k}\right) + O\left(\frac{1}{n^2}\right) \right). \tag{2}$$

Then combining the estimates for  $q_{n,k}$  over the range  $2 \leq k \leq n$  above,

$$\begin{aligned} E_n^s - \frac{1}{2} &= \frac{1}{2} \sum_{k=2}^n \left( 1 - \frac{q_{n,k}}{2^{n-2}} \right) \\ &\sim \frac{1}{2} \sum_{k \geq 0} \left( 1 - \exp\left(-\frac{n}{2^k}\right) \right) - 1, \end{aligned}$$

as the additional tail sum  $\sum_{k > n} (1 - \exp(-\frac{n}{2^k}))$  is exponentially small. It remains to estimate

$$h(n) := \sum_{k \geq 0} \left( 1 - \exp\left(-\frac{n}{2^k}\right) \right),$$

as  $n \rightarrow \infty$ . For this we use Mellin transforms and find (see [1, Appendix B.7, equation (48)])

$$h(n) = \log_2 n + \frac{1}{2} + \frac{\gamma}{L} - \delta(\log_2 n) + O(1/n). \tag{3}$$

The asymptotic estimate for  $E_n^s$  follows. □

**Remarks** Asymptotically we find that the expected number of strict left-to-right maxima is half the expected size of the *largest part* in a random composition of  $n$  (see [9]). Also, as mentioned in the introduction, the asymptotic correspondence established in [5] would allow one to use the results of Prodinger [10] in the case  $p = 1/2$ , to give an alternative proof of Theorem 1.2.

A similar approach to that of Theorem 1.3 leads to

**Theorem 1.3** *The generating function  $NWR_{[d]}(z, y, q)$  is given by*

$$NWR_{[d]}(z, y, q) = \prod_{j=1}^d \frac{1}{1 - \frac{z^j y q}{1 - y \sum_{i=1}^{j-1} z^i}}.$$

The generating function for the total number of weak records in compositions over  $\mathbb{N}$  is then

$$\begin{aligned} g(z) &:= \left. \frac{\partial NWR_{[\mathbb{N}]}(z, 1, q)}{\partial q} \right|_{q=1} = \frac{1-z}{1-2z} \sum_{k \geq 1} \frac{z^k}{1 - \sum_{i=1}^k z^i} \\ &= \frac{(1-z)^2}{z} \sum_{k \geq 2} \left[ \frac{1}{1-2z} - \frac{1}{1-2z+z^k} \right]. \end{aligned}$$

**Theorem 1.4** *The average number  $E_n^w$  of weak left-to-right maxima in the context of compositions of  $n$  has the asymptotic expansion*

$$E_n^w = \log_2 n - \frac{3}{2} + \frac{\gamma}{L} - \delta(\log_2 n) + o(1).$$

**Proof:** The average number  $E_n^w$  of weak left-to-right maxima in compositions of  $n$  satisfies

$$E_n^w = \frac{1}{2^{n-1}} [z^n] g(z) = \frac{1}{2^{n-1}} [z^{n+1}] (1-z)^2 \sum_{k=2}^n \left[ \frac{1}{1-2z} - \frac{1}{1-2z+z^k} \right].$$

Then using the  $q_{n,k}$  notation in the proof of Theorem 1.2

$$E_n^w = \sum_{k=2}^{n+1} \left( 1 - \frac{q_{n+1,k}}{2^{n-1}} \right) = 2E_{n+1}^s - 1.$$

The asymptotic estimate then follows from that of Theorem 1.2. □

## 2 The statistics $ssrec$ and $wsrec$ on the set of compositions

Let  $NSR_A(z, y, q)$  and  $NWR_A(z, y, q)$  be the generating function for the number of compositions of  $n$  with  $m$  parts in  $A$  according to the statistic  $ssrec$  and  $wsrec$ , respectively, that is,

$$\begin{aligned} PSR_A(z, y, q) &= \sum_{n,m \geq 0} \sum_{\sigma \in C_A(n,m)} z^n y^m q^{ssrec(\sigma)}, \\ PWR_A(z, y, q) &= \sum_{n,m \geq 0} \sum_{\sigma \in C_A(n,m)} z^n y^m q^{wsrec(\sigma)}. \end{aligned}$$

**Theorem 2.1** The generating function  $PSR_{[d]}(z, y, q)$  is given by

$$1 + \sum_{k=1}^d q^k \left( \sum_{d \geq j_1 > j_2 > \dots > j_k \geq 1} \prod_{i=1}^k \frac{z^{j_i} y q^{i-1}}{1 - y q^{i-1} \sum_{\ell=1}^{j_i} z^\ell} \right).$$

**Proof:** We denote the number of occurrences of the part  $d$  in the composition  $\sigma \in C_{[d]}(n, m)$  by  $\ell(\sigma)$ . Decomposing according to  $\ell(\sigma) = 0$  and  $\ell(\sigma) > 0$  leads to

$$PSR_{[d]}(z, y, q) = PSR_{[d-1]}(z, y, q) + z^d y q PSR_{[d-1]}(z, qy, q) PSR_{[d]}(z, y, 1). \quad (4)$$

For  $q = 1$ ,  $PSR_{[d]}(z, y, 1) = \frac{1}{1 - y \sum_{j=1}^d z^j}$ . Hence,

$$\begin{aligned} PSR_{[d]}(z, y, q) &= PSR_{[d-1]}(z, y, q) + \frac{z^d y q}{1 - y \sum_{i=1}^d z^i} PSR_{[d-1]}(z, qy, q) \\ &= PSR_{[d-2]}(z, y, q) + \sum_{j=d-1}^d \frac{z^j y q}{1 - y \sum_{i=1}^j z^i} PSR_{[j-1]}(z, qy, q) \\ &\vdots \\ &= 1 + \sum_{j=1}^d \frac{z^j y q}{1 - y \sum_{i=1}^j z^i} PSR_{[j-1]}(z, qy, q). \end{aligned}$$

Iterating the above recurrence relation  $d$  times we get the desired result.  $\square$

From this we derive

**Corollary 2.2** The generating function  $v_d(z) = \frac{\partial}{\partial q} PSR_{[d]}(z, 1, q) |_{q=1}$  is given by

$$\frac{z}{1 - \sum_{j=1}^d z^j} \sum_{j=0}^{d-1} \frac{z^j}{\left(1 - \sum_{i=1}^j z^i\right)^2}.$$

The above corollary gives that the generating function for the number of compositions of  $n$  according to the total of the statistic  $ssrec$  is given by

$$v(z) := \frac{z(1-z)}{1-2z} \sum_{j \geq 0} \frac{z^j}{\left(1 - \sum_{i=1}^j z^i\right)^2}.$$

The rather lengthy proof of the asymptotic behaviour of the coefficients of  $v(z)$  will be left for the journal version of the paper. We obtain

**Theorem 2.3** The average sum of the positions of the strong records  $e_n^s$  in compositions of  $n$  has the asymptotic expansion

$$e_n^s = \frac{n}{4 \log 2} (1 + \delta_2 (\log_2 n)) + o(n),$$

where  $\delta_2(x)$  is a periodic function of period 1, mean zero and small amplitude, which is given by the Fourier series

$$\delta_2(x) = \sum_{k \neq 0} \chi_k \Gamma(-1 - \chi_k) e^{2k\pi i x}.$$

With reference again to [5], Theorem 2.3 is seen to correspond to the  $p = 1/2$  case of the results of Prodinger [12].

The corresponding results for  $PWR_{[d]}(z, y, q)$  are as follows.

**Theorem 2.4** *The generating function  $PWR_{[d]}(z, y, q)$  satisfies the following recurrence relation*

$$PWR_{[d]}(z, y, q) = PWR_{[d-1]}(z, y, q) + \frac{z^d y q}{1 - y \sum_{i=1}^{d-1} z^i} PWR_{[d]}(z, qy, q).$$

**Corollary 2.5** *The generating function  $w_d(z) = \frac{\partial}{\partial q} PWR_{[d]}(z, 1, q) |_{q=1}$  is given by*

$$w_d(z) = \frac{1}{1 - \sum_{j=1}^d z^j} \sum_{j \geq 1} \frac{z^j}{\left(1 - \sum_{i=1}^j z^i\right)^2}.$$

The above corollary gives that the generating function for the number of compositions of  $n$  according to the total of the statistic *swrec* is given by

$$w(z) := \frac{1 - z}{1 - 2z} \sum_{j \geq 1} \frac{z^j}{\left(1 - \sum_{i=1}^j z^i\right)^2}.$$

**Theorem 2.6** *The average sum of the positions of the weak records  $e_n^w$  in compositions of  $n$  has the asymptotic expansion*

$$e_n^w = \frac{n}{2 \log 2} (1 + \delta_2(\log_2 n)) + o(n),$$

where  $\delta_2(x)$  is the same periodic function that occurred in Theorem 2.3.

**Proof:** The generating functions  $v(z)$  and  $w(z)$  are related as follows,

$$v(z) = \frac{z(1 - z)}{1 - 2z} + zw(z).$$

From this we see that

$$[z^{n+1}]v(z) = 2^{n-1} + [z^n]w(z).$$

So that  $e_n^w = 2e_{n+1}^s - 1$ . The result then follows from Theorem 2.3. □

Now, our aim is to present a combinatorial explanation for the fact that the number (sum) of the positions of weak records in all compositions of  $n$  plus  $2^{n-1}$  equals the number (sum) of the positions of strong records in all compositions of  $n + 1$ , for  $n \geq 1$ . In order to do that we need the following notations.

Let  $sw_{n,r}$  (respectively,  $ss_{n,r}$ ) be the sum of  $r$ -th power of the positions of weak (respectively, strong) records in all the compositions of  $n$ , namely,

$$\begin{aligned} sw_{n,r} &= \sum_{\sigma \in C_n} \sum_{\sigma_i \text{ is a weak record of } \sigma} i^r, \\ ss_{n,r} &= \sum_{\sigma \in C_n} \sum_{\sigma_i \text{ is a strong record of } \sigma} i^r, \\ sw'_{n,r} &= \sum_{\sigma \in C_n(A)} \sum_{\sigma_i \text{ is a weak record of } \sigma, i > 1} i^r, \\ ss'_{n,r} &= \sum_{\sigma \in C_n(A)} \sum_{\sigma_i \text{ is a strong record of } \sigma, i > 1} i^r, \end{aligned}$$

where  $C_n = \cup_{m=1}^n C_{n,m}$  is the set of all compositions of  $n$ . From the definitions, each first letter is a weak (strong) record. Therefore,

$$sw_{n,r} = |C_n| + sw'_{n,r} \text{ and } ss_{n,r} = |C_n| + ss'_{n,r}, \quad (5)$$

where  $|C_n| = 2^{n-1}$  is the number of compositions of  $n$ .

**Theorem 2.7** For all  $n \geq 1$ ,

$$ss_{n+1,r} = sw_{n,r} + 2^{n-1}.$$

**Proof:** It is not hard to see that  $\sigma_1 \cdots \sigma_m$  is a composition of  $n$  and  $\sigma_i, i > 1$ , is a weak record if and only if  $\sigma_1 \cdots \sigma_{i-1}(\sigma_i + 1)\sigma_{i+1} \cdots \sigma_m$  is a composition of  $n$  and  $\sigma_i + 1, i > 1$ , is a strong record. Therefore, the multiset of all positions  $i, i > 1$ , of the weak records in all compositions of  $n$  is the same multiset as all positions  $i, i > 1$ , of the strong records in all compositions on  $n + 1$ . In other words,  $ss'_{n+1,r} = sw'_{n,r}$  for all  $n$  and  $r$ .

Hence, by (5) we have

$$ss_{n+1,r} = 2^n + ss'_{n+1,r} = 2^n + sw'_{n,r} = 2^{n-1} + 2^{n-1} + sw'_{n,r} = 2^{n-1} + sw_{n,r},$$

as requested.  $\square$

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