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# Counting occurrences of some subword patterns

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We find generating functions for the number of strings (words) containing a specified number of occurrences of certain types of order-isomorphic classes of substrings called subword patterns. In particular, we find generating functions for the number of strings containing a specified number of occurrences of a given 3-letter subword pattern.

**Keywords:** Generalized patterns, subword patterns

## 1 Introduction

Counting the number of words which contain a set of given strings as substrings a certain number of times is a classical problem in combinatorics. This problem can, for example, be attacked using the transfer matrix method (see [20, Section 4.7]). In particular, it is a well-known fact that the generating function of such words is always rational. For example, in [20, Example 4.7.5] it is shown that the generating function for the number of words in $[3]^n$ where neither 11 nor 23 appear as two consecutive digits is given by

$$\frac{3+x-x^2}{1-2x-x^2+x^3}.$$

In this paper, we present, in several cases, a complete solution for the problem of the enumeration of words containing a *subword pattern* (see below for the precise definition) of length $l$ exactly $r$ times. For example, we find the number of words in $[3]^n$ containing the subword pattern 111 exactly $r$ times, that is, the number of words which contain 111, 222, and 333 as substrings a total of $r$ times.

Régnier and Szpankowski [18] used a combinatorial approach to study the frequency of occurrences of strings (which they also called a “pattern”) from a given set in a random word, when overlapping copies of the “patterns” are counted separately (see [18, Theorem 2.1]). We note that the term “pattern” in [18] is used to denote an exact string rather than its type with respect to order isomorphism. For example, the “pattern” 112 in [18] is the actual string 112, whereas in our setting an occurrence of the subword pattern 112 is any substring $aab$ of the ambient string with $a < b$. Although, in principle, it is possible to deduce our results from the result by Régnier and Szpankowski, our direct derivations are much simpler.

Goulden and Jackson [12] also consider sequences with distinguished substrings and use the term "pattern of a sequence". However, their "pattern" is more locally defined than in this paper in that only order relations between adjacent elements of a string are considered, rather than order relations between any pair of elements of a string, as is done in this paper. For example, the pattern "rise, non-rise" (or $(<,\geq)$, or $\pi_1\pi_2$) as defined in [12] includes the subword patterns $121, 122, 132, 231$ as defined in this paper. However, we show that each of the subword patterns $121, 122, 132$ is avoided by a different number of words (of a given length on a given alphabet) than the other two patterns.

In what follows, we use analytical and combinatorial means to find a complete answer for several cases of counting strings with a specified number of occurrences of certain patterns.

## 1.1 Classical patterns in permutations

Let $\pi \in S_n$ and $\tau \in S_m$ be two permutations. An *occurrence* of $\tau$ in $\pi$ is a subsequence $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ such that $(\pi(i_1), \ldots, \pi(i_m))$ is order-isomorphic to $\tau$. In this context, $\tau$ is usually called a *pattern*. We denote the number of occurrences of $\tau$ in $\pi$ by $\pi(\tau)$.

Recently, much attention has been paid to the problem of counting the number of permutations of length $n$ containing a given number $r \geq 0$ of occurrences of a certain pattern $\tau$. Most of the authors consider only the case $r = 0$, thus studying permutations *avoiding* a given pattern. Only a few papers consider the case $r > 0$, usually restricting themselves to the patterns of length 3. In fact, simple algebraic considerations show that there are only two essentially different cases for $\tau \in S_3$, namely, $\tau = 123$ and $\tau = 132$. Noonan [16] has proved that the number of permutations in $S_n$ containing 123 exactly once is given by $\frac{3}{n}\binom{2n}{n-3}$. A general approach to the problem was suggested by Noonan and Zeilberger [17]; they gave another proof of Noonan's result, and conjectured that the number of permutations in $S_n$ containing 123 exactly twice is given by $\frac{59n^2+117n+100}{2n(2n-1)(n+5)}\binom{2n}{n-4}$ and the number of permutations in $S_n$ containing 132 exactly once is given by $\binom{2n-3}{n-3}$. The first conjecture was proved by Fulmek [11] and the second conjecture was proved by Bóna in [3]. A general conjecture of Noonan and Zeilberger states that the number of permutations in $S_n$ containing $\tau$ exactly $r$ times is $P$-recursive in $n$ for any $r$ and $\tau$. It was proved by Bóna [2] for $\tau = 132$. However, as stated in [2], a challenging question is to describe the number of permutations in $S_n$ containing $\tau \in S_3$ exactly $r$ times, explicitly for any given $r$. Later, Mansour and Vainshtein [15] suggested a new approach to this problem in the case $\tau = 132$, which allows one to get an explicit expression for the number of permutations in $S_n$ containing 132 exactly $r$ times for any given $r$.

## 1.2 Generalized patterns in permutations

In [1], Babson and Steingrímsson introduced generalized permutation patterns that allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. For example, an occurrence of a generalized pattern 12-3 in a permutation $\pi = a_1a_2\cdots a_n$ is a subword $a_ia_{i+1}a_j$ of $\pi$ such that $a_i < a_{i+1} < a_j$.

**Notation 1.1** Unfortunately, there is a bit of confusion in denoting classical and generalized patterns. Before generalized patterns were introduced, the hyphens were unnecessary, hence classical patterns (those with all possible hyphens) are often written with no hyphens when generalized patterns are not considered, and with all hyphens when they are. For example, a classical pattern 123 is now denoted by 1-2-3 when considered as a generalized pattern (using the notation of [1, 7]). Unless otherwise stated, all patterns under consideration from now on are generalized patterns.

In [10] Elizalde and Noy presented the following theorem regarding the distribution of the number of occurrences of any generalized pattern of length 3 without hyphens.

**Theorem 1.2** (Elizalde and Noy [10]) *Let* $h(x) = \sqrt{(x-1)(x+3)}$. *Then*

$$\sum_{n\geq 0}\sum_{\pi\in S_n} x^{\pi(123)}\frac{t^n}{n!} = \frac{2h(x)e^{\frac{1}{2}(h(x)-x+1)t}}{h(x)+x+1+(h(x)-x-1)e^{h(x)t}},$$

$$\sum_{n\geq 0}\sum_{\pi\in S_n} x^{\pi(213)}\frac{t^n}{n!} = \frac{1}{1-\int_0^t e^{(x-1)z^2/2}dz},$$

*where* $\pi(123)$ *(respectively,* $\pi(213)$*) is the number of occurrences of the* generalized *pattern* 123 *(respectively,* 213*) without hyphens in* $\pi$*.*

On the other hand, Claesson [7] gave a complete answer for the number of permutations avoiding a generalized pattern of the form $xy$-$z$ where $xyz \in S_3$. Later, Claesson and Mansour [8] presented an algorithm to count the number of permutations containing a generalized pattern of the form $xy$-$z$ exactly $r$ times for any given $r \geq 0$, where $xyz \in S_3$.

**Theorem 1.3** (Claesson and Mansour [8]) *The ordinary generating function for the number of permutations of length n avoiding the generalized pattern* 12-3 *(or* 23-1*) is*

$$\sum_{k\geq 0}\frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

*The ordinary generating function for the number of permutations of length n avoiding the generalized pattern* 2-13 *is*

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

*The ordinary generating function for the number of permutations of length n containing exactly one occurrence of the generalized pattern* 12-3 *is*

$$\sum_{n\geq 1}\frac{x}{1-nx}\sum_{k\geq 0}\frac{kx^{k+n}}{(1-x)(1-2x)\cdots(1-(k+n)x)}.$$

*The ordinary generating function for the number of permutations of length n containing exactly one occurrence of the generalized pattern* 23-1 *is*

$$\sum_{n\geq 1}\frac{x}{1-(n-1)x}\sum_{k\geq 0}\frac{kx^{k+n}}{(1-x)(1-2x)\cdots(1-(k+n)x)}.$$

*The ordinary generating function for the number of permutations of length n containing exactly one occurrence of the generalized pattern* 2-13 *is*

$$\frac{x^3C(x)^7}{1-tC(x)^2}.$$

### 1.3 Generalized patterns in words

A *generalized pattern* $\tau$ is a (possibly hyphenated) string in $[\ell]^m$ which contains all letters from $[\ell] = \{1,\dots,\ell\}$. We say that the string $\sigma \in [k]^n$ *contains* a generalized pattern $\tau$ exactly $r$ times (denoted by $r = \sigma(\tau)$) if $\sigma$ contains $r$ different subsequences isomorphic to $\tau$ in which the entries corresponding to consecutive entries of $\tau$ not separated by a hyphen must be adjacent. We call the generalized patterns without hyphens *subword patterns*. If $r = 0$, we say that $\sigma$ *avoids* $\tau$ and write $\sigma \in [k]^n(\tau)$. Thus, $[k]^n(\tau)$ denotes the set of strings in $[k]^n$ (i.e., $n$-long $k$-ary strings) which avoid $\tau$. For example, a string $\pi = a_1a_2\dots a_n$ avoids the generalized pattern 12-1 if $\pi$ has no subsequence $a_ia_{i+1}a_j$ with $j > i+1$ and $a_i = a_j < a_{i+1}$.

**Example 1.4** Davenport-Schinzel sequences [9] can be defined in terms of subword pattern avoidance as follows. For any $d \geq 1$, let $T_d$ be the set of all the subword patterns $\pi = a_1a_2\cdots a_{d+1} \in [d+1]^{d+1}$ such that either $a_{2j} < a_{2j+1} > a_{2j+2}$ for all $j$, or $a_{2j-1} < a_{2j} > a_{2j+1}$ for all $j$. For example, $T_2 = \{121, 132, 231, 212, 213, 312\}$. An $k$-ary $n$-long sequence avoiding the subword pattern 11 (i.e., with no equal consecutive letters) and avoiding all the subword patterns in $T_d$ (there are no alternating subwords of length greater than $d+1$) is called a Davenport-Schinzel sequence if $n$ is maximal.

Let $f_{\tau;r}(n,k)$ be the number of words $\sigma \in [k]^n$ such that $\sigma(\tau) = r$. Denote the corresponding bivariate generating function by $F_\tau(x,y;k)$, in other words,

$$F_\tau(x,y;k) = \sum_{n\geq 0}\sum_{r\geq 0} f_{\tau;r}(n,k)x^ny^r.$$

Burstein [4] gave a complete answer for the numbers $f_{\tau;0}(n,k)$ where $\tau$ is a 3-letter classical pattern. Later, Burstein and Mansour [5, 6] presented a complete answer for the number $f_{\tau;0}(n,k)$ where $\tau$ is a generalized pattern of length 3 (a word of length 3).

In this paper, we present a complete answer for several cases of $f_{\tau;r}(n,k)$ where $\tau$ is a subword pattern of length $l$ (which is the analogue of the results by Elizalde and Noy in [10]). In particular, we find a complete answer for the case $l = 3$.

## 2 Counting a subword pattern of length $l$

In this section we find $F_\tau(x,y;k)$ for several cases of $\tau$. Burstein and Mansour [6] found $F_\tau(x,y;k)$ for the subword pattern $\tau = 11\dots1 \in [1]^l$ and proved the following theorem.

**Theorem 2.1** (Burstein and Mansour [6, Th. 2.1]) *Let $\tau = 11\dots1 \in [1]^l$ be a subword pattern. Then*

$$F_\tau(x,y;k) = \frac{1+(1-y)x\sum_{j=0}^{l-2}(kx)^j - (1-y)(k-1)\sum_{d=2}^{l-1}x^d\sum_{j=0}^{l-1-d}(kx)^j}{1-(k-1+y)x-(k-1)(1-y)(1-x^{l-2})\frac{x^2}{1-x}}.$$

### 2.1 The subword pattern $\tau = 11\dots12$

Let $\tau = 11\dots12 \in [2]^l$ be a subword pattern. Define $d_{\tau;r}(n,k)$ to be the number of words $\beta \in [k]^n$ such that $(\beta, k+1)$ contains $\tau$ exactly $r$ times, and denote the corresponding generating function by

$$D_\tau(x,y;k) = \sum_{n,r\geq 0} d_{\tau;r}(n,k)x^ny^r.$$

Let us find a recurrence for $F_\tau$.

Let $\sigma \in [k]^n$ ($k \geq 2$), with $\sigma(\tau) = r$, contain exactly $d$ copies of the letter $k$. If $d = 0$, then $\sigma \in [k-1]^n$ and $\sigma(\tau) = r$. If $d \geq 1$, then $\sigma = \sigma_1 k \sigma_2$, where $\sigma_1 \in [k-1]^{n_1}$, $\sigma_2 \in [k]^{n_2}$, $n_1 + n_2 + 1 = n$ and $(\sigma_1, k)(\tau) + \sigma_2(\tau) = r$. Taking generating functions, we see that the above translates into

$$F_\tau(x,y;k) = F_\tau(x,y;k-1) + xD_\tau(x,y;k-1)F_\tau(x,y;k),$$

or, equivalently,

$$F_\tau(x,y;k) = \frac{F_\tau(x,y;k-1)}{1 - xD_\tau(x,y;k-1)}.$$

Let us now find the recurrence for $D_\tau$. Let $\sigma \in [k]^n$ be such that $(\sigma, k+1)$ contains $\tau$ exactly $r$ times, and has exactly $d$ letters $k$. Then $\sigma = \sigma_1 k \sigma_2 k \ldots k \sigma_d k \sigma_{d+1}$ for some $\sigma_i \in [k-1]^{n_i}$, $1 \leq i \leq d+1$, where $n_1 + \cdots + n_{d+1} = n - d$ and

$$(\sigma_1, k)(\tau) + \cdots + (\sigma_d, k)(\tau) + (\sigma_{d+1}, k+1)(\tau) + \delta(\sigma \text{ ends on } l-1\ k\text{'s}) = r.$$

Taking generating functions, we obtain

$$\begin{aligned} D_\tau(x,y;k) = \sum_{d=0}^{l-2} x^d D_\tau^{d+1}(x,y;k-1) \\ + \sum_{d=l-1}^{\infty} x^d \Big( D_\tau^{d+1}(x,y;k-1) - D_\tau^{(d+1)-(l-1)}(x,y;k-1) \\ + yD_\tau^{(d+1)-(l-1)}(x,y;k-1) \Big), \end{aligned}$$

which, after summing over $d$, yields

$$D_\tau(x,y;k-1) = \frac{(1 - x^{l-1}(1-y))D_\tau(x,y;k-1)}{1 - xD_\tau(x,y;k-1)}.$$

These two recurrences, together with $D_\tau(x,y;0) = F_\tau(x,y;0) = 1$ and induction on $k$, yield the following theorem.

**Theorem 2.2** *Let $\tau = 11\ldots12 \in [2]^l$ be a subword pattern such that $l \geq 1$; then*

$$F_\tau(x,y;k) = \frac{1-y}{1 - x^{2-l} - y + x^{2-l}(1 - x^{l-1}(1-y))^k}.$$

**Example 2.3** (see Burstein and Mansour [6, Th. 3.10]) Letting $l = 3$ and $y = 0$ in Theorem 2.2, we get that the generating function for the number of words in $[k]^n$ avoiding the subword pattern 112 is given by

$$\frac{1}{1 - \frac{1}{x} + \frac{1}{x}(1-x^2)^k}.$$

In the special case of $l = 3$, we get from Theorem 2.2 the following result.

**Corollary 2.4** *The generating function for the number of words in $[2]^n$ containing the subword pattern* 112 *exactly r times is given by*

$$\frac{x^{3r}}{(1-x)^{r+1}(1-x-x^2)^{r+1}}.$$

**Proof:** Let $\tau = 112$ be a subword pattern. It is easy to see that a word $\sigma \in [2]^n$ with $\sigma(\tau) = r$ must have the form $\sigma = \sigma_1\tau\sigma_2\tau\dots\tau\sigma_{r+1}$, for some $\sigma_1,\dots,\sigma_{r+1} \in [2]^n(\tau)$. Now, from Example 2.3 with $k = 2$, we have that $F_\tau(x;2) = \frac{1}{(1-x)(1-x-x^2)}$, hence the result follows. □

## 2.2 The subword pattern $\tau = 211\dots112$

Let $\tau = 211\dots112 \in [2]^l$ be a subword pattern. We define $d_\tau(n,r;k)$ to be the number of words $\beta \in [k]^n$ such that $(k+1,\beta,k+1)$ contains $\tau$ exactly $r$ times, and denote the corresponding generating function by $D_\tau(x,y;k)$. Let $\sigma \in [k]^n$ such that $\sigma(\tau) = r$, and such that $\sigma$ contains $d$ occurrences of the letter $k$. For $d = 0$, the generating function for the number of such words $\sigma$ is given by $F_\tau(x,y;k-1)$, and for $d \geq 1$, by $x^d F_\tau^2(x,y;k-1)D_\tau^{d-1}(x,y;k-1)$ (since in that case $\sigma = \sigma_0 k\sigma_1 k\dots k\sigma_{d-1}k\sigma_d$, where all $\sigma_i \in [k-1]^{n_i}$, $\sum n_i = n-d$, and $\sigma(\tau) = \sigma_0(\tau)+(k,\sigma_1,k)(\tau)+\cdots+(k,\sigma_{d-1},k)(\tau)+\sigma_d(\tau)$). Hence, if we sum over all $d \geq 0$, we get

$$F_\tau(x,y;k) = F_\tau(x,y;k-1) + \frac{xF_\tau^2(x,y;k-1)}{1-xD_\tau(x,y;k-1)}.$$

On the other hand, the word $(k+1,\beta,k+1)$, with $\beta$ as above, contains an occurrence of $\tau$ involving the two letters $k+1$ if and only if $\beta$ is a constant string of length $l-2$, otherwise, $(k+1,\beta,k+1)(\tau) = \beta(\tau)$. Taking generating functions, we obtain

$$D_\tau(x,y;k) = kx^{l-2}y + F_\tau(x,y;k) - kx^{l-2}.$$

Therefore, using the initial conditions $F_\tau(x,y;0) = D_\tau(x,y;0) = 1$ and induction on $k$, we get the following theorem.

**Theorem 2.5** *Let $\tau = 211\dots112 \in [2]^l$ be a subword pattern and $l \geq 2$, then*

$$F_\tau(x,y;k) = \frac{1}{1-x-x\sum_{j=0}^{k}\frac{1}{1+jx^{l-1}(1-y)}}.$$

**Example 2.6** (Burstein and Mansour [6, Th. 3.12]) Letting $l = 3$ and $y = 0$ in Theorem 2.5, we get that the generating function for the number of words in $[k]^n$ avoiding the subword pattern 212 is given by

$$\frac{1}{1-x-x\sum_{j=0}^{k-1}\frac{1}{1+jx^2}}.$$

## 2.3 The subword pattern $\tau = m\tau' m$

**Theorem 2.7** *Let $\tau = m\tau' m \in [m]^l$ be a subword pattern, where $\tau'$ does not contain m. Then for $k \geq m$,*

$$F_\tau(x,y;k) = \frac{1}{1-(m-1)x-x\sum_{j=m-1}^{k-1}\frac{1}{1+\binom{j}{m-1}x^{l-1}(1-y)}}.$$

**Proof:** Let $\sigma \in [k]^n$. The generating function for the number of words $\sigma$ which do not contain $m$ and contain $\tau$ exactly $r$ times is given by $F_\tau(x,y;k-1)$. Now assume that the leftmost $m$ in $\sigma$ is at position $i$. Then $\sigma = \sigma_1 m \sigma_2$ and $\sigma(\tau) = \sigma_1(\tau) + (m,\sigma_2)(\tau)$, so the generating function for the number of such words $\sigma$ is given by $xF_\tau(x,y;k-1)D_\tau(x,y;k)$, where $D_\tau(x,y;k)$ is the generating function for the number of words $\sigma \in [k]^n$ such that $(m,\sigma)$ contains $\tau$ exactly $r$ times. Therefore,

$$F_\tau(x,y;k) = F_\tau(x,y;k-1) + xF_\tau(x,y;k-1)D_\tau(x,y;k).$$

On the other hand, let $\sigma' = (m,\sigma) \in [k]^{n+1}$. If $\sigma$ does not contain $m$, then the generating function for the number of such $\sigma$ is given by $F_\tau(x,y;k-1)$. Otherwise, let $i$ be the position of the leftmost letter $m$ and let $\sigma|_i$ be the left prefix of $\sigma$ of length $i$, then the generating function for these words is given by

$$x\left(F_\tau(x,y;k-1) - x^{l-2}\binom{k-1}{m-1}\right)D_\tau(x,y;k)$$

if $(m,\sigma|_i)$ is not order-isomorphic to $\tau$, or by

$$xyx^{l-2}\binom{k-1}{m-1}D_\tau(x,y;k)$$

if $(m,\sigma|_i)$ is order-isomorphic to $\tau$. Therefore,

$$\begin{aligned} D_\tau(x,y;k) = F_\tau(x,y;k-1) + x\left(F_\tau(x,y;k-1) - x^{l-2}\binom{k-1}{m-1}\right)D_\tau(x,y;k) \\ + yx^{l-1}\binom{k-1}{m-1}D_\tau(x,y;k). \end{aligned}$$

Hence, from the above two equations, we obtain

$$F_\tau(x,y;k) = \frac{\left(1 + x^{l-1}\binom{k-1}{m-1}(1-y)\right)F_\tau(x,y;k-1)}{1 + x^{l-1}\binom{k-1}{m-1}(1-y) - xF_\tau(x,y;k-1)},$$

so, by induction on $k$ with the initial condition $F_\tau(x,y;m-1) = \frac{1}{1-(m-1)x}$, we get the desired result. $\square$

**Example 2.8** Applying Theorem 2.7 to the subword patterns 2112 and 3123, we get

$$\begin{aligned} F_{2112}(x,y;k) &= \frac{1}{1 - x - x\sum_{j=0}^{k-1}\frac{1}{1+jx^3(1-y)}}, \\ F_{3123}(x,y;k) &= \frac{1}{1 - 2x - x\sum_{j=2}^{k-1}\frac{1}{1+j(j-1)x^3(1-y)/2}}. \end{aligned}$$

**Definition 2.9** *We say that the patterns* $\beta$ *and* $\gamma$ *are* strongly Wilf-equivalent, *or are in the same* strong Wilf class, *if the number of words in* $[k]^n$ *containing* $\beta$ *exactly* $r$ *times is the same as the number of words in* $[k]^n$ *containing* $\gamma$ *exactly* $r$ *times, for any* $r \geq 0$.

By Theorem 2.7 and the symmetry operations "reversal" and "complement," we immediately get the following corollary.

**Corollary 2.10** *The subword patterns* 1121 *and* 1221 *are in the same strong Wilf class.*

## 2.4 The subword pattern $\tau = m\tau'(m+1)$

Let $\tau = m\tau'(m+1)$ be a subword pattern, where $\tau'$ does not contain $m$ or $m+1$. Note that $\tau$ is in the same symmetry class as $r(c(\tau)) = 1r(c(\tau'))2$. This case is treated in a similarl manner as the case of $\tau = m\tau'm$. As a result, we obtain the theorem below.

**Theorem 2.11** *Let $\tau = m\tau'(m+1) \in [m+1]^l$ be a subword pattern, where $\tau'$ does not contain $m$ or $m+1$. Then for $k \geq m$,*

$$F_\tau(x,y;k) = \frac{1}{1-(m-1)x - x\sum_{i=m-2}^{k-2}\prod_{j=m-2}^{i}\left(1-\binom{j}{m-1}x^{l-1}(1-y)\right)}.$$

# 3 Subword patterns of length 3

The symmetry class representatives of 3-letter subword patterns are 111, 112, 212, 123, 213. In the current subsection, we find explicit formulas for $F_\tau(x,y;k)$ for each of these representatives $\tau$. Theorem 2.1 yields the answer for the first class.

**Theorem 3.1** *Let $\tau = 111$ be a subword pattern. Then, for all $k \geq 0$, we have*

$$F_\tau(x,y;k) = \frac{1+x(1+x)(1-y)}{1-(k-1+y)x-(k-1)(1-y)x^2}.$$

Theorems 2.2 and 2.5 contain already the answers for the second and the third classes, respectively. Let us summarize the corresponding results in the theorem below.

**Theorem 3.2** *Let 112 and 212 be subword patterns. For $k \geq 0$,*

$$F_{112}(x,y;k) = \frac{1-y}{1-\frac{1}{x}-y+\frac{1}{x}(1-x^2(1-y))^k},$$
$$F_{212}(x,y;k) = \frac{1}{1-x-x\sum_{j=0}^{k}\frac{1}{1-jx^2(1-y)}}.$$

Now let us find the generating function for the fourth class, $F_\tau(x,y;k)$ where $\tau = 123$ is a subword pattern. Let $D_\tau(x,y;k)$ be the generating function for the number of words $\sigma \in [k]^n$ such that $(\sigma, k+1)$ contains the subword pattern 123 exactly $r$ times. Suppose a word $\sigma \in [k]^n(\tau)$ has exactly $d$ letters $k$. Then $\sigma = \sigma_0 k \sigma_1 k \dots k \sigma_d$, where all $\sigma_i \in [k-1]^n$, and any occurrence of $\tau$ in $\sigma$ must be either in $(\sigma_i, k)$ for some $i = 0,1,\dots,d-1$, or in $\sigma_d$. Therefore, the generating function for the number of such words $\sigma$ is $(xD_\tau(x,y;k-1))^d F_\tau(x,y;k-1)$, so

$$F_\tau(x,y;k) = \sum_{d\geq 0}(xD_\tau(x,y;k-1))^d F_\tau(x,y;k-1).$$

On the other hand, suppose $\sigma \in [k]^n$ is counted by $D_\tau(x,y;k)$. Then $(\sigma, k+1) = \sigma_0 k \sigma_1 k \dots k \sigma_d k+1$ (where $\sigma_i \in [k-1]^n$ for all $i$) contains the pattern $\tau$ exactly $r$ times. If $d = 0$, then $\sigma \in [k-1]^n$. If $d \geq 1$, there are several possibilities. If $\sigma_d \neq \emptyset$ or $\sigma_d = \sigma_{d-1} = \emptyset$, then all occurrences of the pattern $\tau$ in $(\sigma, k+1)$ are in $(\sigma_i, k)$ for some $i = 0,1,\dots,d-1$, or in $(\sigma_d, k+1)$. If $\sigma_d = \emptyset$ and $\sigma_{d-1} \neq \emptyset$, then there is one extra occurrence of $\tau$ since $(\sigma, k+1)$ ends by $(a, k, k+1)$ for some $a < k$. Taking generating functions, we obtain

$$\begin{aligned} D_\tau(x,y;k) =& D_\tau(x,y;k-1) \\ &+ \sum_{d\geq 1} x^d D_\tau^d(x,y;k-1)(D_\tau(x,y;k-1)-1) \\ &+ \sum_{d\geq 1} x^d D_\tau^{d-1}(x,y;k-1)) \\ &+ \sum_{d\geq 1} x^d y D_\tau^{d-1}(x,y;k-1)(D_\tau(x,y;k-1)-1). \end{aligned}$$

Hence,

$$\begin{aligned} F_\tau(x,y;k) &= \frac{F_\tau(x,y;k-1)}{1-xD_\tau(x,y;k-1)}, \\ D_\tau(x,y;k) &= \frac{(1-x+xy)D_\tau(x,y;k-1)+x(1-y)}{1-xD_\tau(x,y;k-1)}. \end{aligned}$$

Together with $F_\tau(x,y;0) = D_\tau(x,y;0) = 1$, $F_\tau(x,y;1) = D_\tau(x,y;1) = 1/(1-x)$ and induction on $k$, this yields the following result.

**Theorem 3.3** *Let $\tau = 123$ be a subword pattern. For all $k \geq 2$, we have*

$$F_\tau(x,y;k) = \frac{1}{1-kx-\sum_{j=3}^{k}(-x)^j\binom{k}{j}(1-y)^{\lfloor j/2\rfloor}U_{j-3}(y)},$$

*where $U_0(y) = U_1(y) = 1$, $U_{2n}(y) = (1-y)U_{2n-1}(y) - U_{2n-2}(y)$, and $U_{2n+1}(y) = U_{2n}(y) - U_{2n-1}(y)$. Furthermore, the generating function for $U_n(y)$ is given by*

$$\sum_{n\geq 0} U_n(y)z^n = \frac{1+z+z^2}{1+(1+y)z^2+z^4}.$$

Finally, Theorem 2.11 for $l = 3$ and $m = 2$ provides already the answer for the last class. The corresponding result is summarized below.

**Theorem 3.4** *Let $\tau = 213$ be a subword pattern. Then for all $k \geq 2$, we have*

$$F_\tau(x,y;k) = \frac{1}{1-x-x\sum_{i=0}^{k-2}\prod_{j=0}^{i}(1-jx^2(1-y))}.$$

# 4 Further results

We say that a subword pattern $\tau \in [m]^l$ is *primitive* if any two distinct occurrences of $\tau$ may overlap by at most one letter. For example, the subword patterns 112, 121, 122, 132, 211, 212, 213, 221, 231, and 312 are all the primitive patterns of length three.

**Theorem 4.1** *Let $\tau, \tau' \in [m]^l$ be two primitive subword patterns such that there exists a permutation $\Phi \in S_l$ with $\Phi(1) = 1$, $\Phi(l) = l$ and $\tau' = \Phi \circ \tau$. In other words, $\tau$ and $\tau'$ have the same supply of each letter, the same first letter and the same last letter. Then $\tau$ and $\tau'$ are in the same strong Wilf class.*

**Proof:** Let $\sigma \in [k]^n$ contain $\tau$ exactly $r$ times. Since $\tau$ is a primitive subword pattern, we can define a function $f$ which changes any occurrence of $\tau$ in $\sigma$ to an occurrence of $\tau'$. It is easy to see from the definition of primitive patterns that $f$ is a bijection, hence the theorem follows. □

An immediate corollary is the following.

**Corollary 4.2** *The subword patterns* 1232 *and* 1322 *are in the same strong Wilf class.*

**Theorem 4.3** *All primitive subword patterns $\tau \in [m]^l$ such that $\tau(1) = a$ and $\tau(l) = b$, where $a < b$, are in the same strong Wilf class.*

**Proof:** Similarly as in the proof of Theorem 2.7, we get

$$F_\tau(x,y;k) = F_\tau(x,y;k-1) + xF_\tau(x,y;k-1)D_\tau(x,y;k;a),$$

where $D_\tau(x,y;k;h)$ is the generating function for the number of words $\sigma \in [k]^n$ such that $(h,\sigma)$ contains $\tau$ exactly $r$ times.

Now let us consider the case $h = a + p(b-a)$. Let $\sigma = (\sigma', h+b-a, \sigma'')$, where $\sigma'$ is a word on the letters in $[k]$ which does not contain $h+b-a$. Using the fact that $\tau$ is a primitive subword pattern, we get

$$\begin{aligned}
&D_\tau(x,y;k;h) = D_\tau(x,y;k-1;h) \\
&+ x^{l-1}y\binom{h-1}{a-1}\binom{k-(h+b-a)}{m-b}D_\tau(x,y;k;h+b-a) \\
&+ x\left[F_\tau(x,y;k-1) - x^{l-2}\binom{h-1}{a-1}\binom{k-(h+b-a)}{m-b}\right]D_\tau(x,y;k;h+b-a).
\end{aligned}$$

Hence, by induction on $p$ and $k$, using $F_\tau(x,y;m-1) = 1/(1-(m-1)x)$ and $D_\tau(x,y;k;h) = 0$ for $h > k$, we get the desired result. □

Using the proof of the above theorem, we get the following generalization.

**Corollary 4.4** *Let $\tau, \tau' \in [m]^l$ be two primitive subword patterns such that $\tau(1) = \tau'(1) = a$ and $\tau(l) = \tau'(l) = b$, where $a < b$. Then the subword patterns*

$$\underbrace{aa\dots a}_{p}\tau\underbrace{b\dots bb}_{p} \quad \text{and} \quad \underbrace{aa\dots a}_{p}\tau'\underbrace{b\dots bb}_{p}$$

*are in the same strong Wilf class.*

Theorem 4.3 implies as well the following corollary.

**Corollary 4.5**

1. *The subword patterns* 1132, 1232, 1322, *and* 1332 *are in the same strong Wilf class.*
2. *The subword patterns* 1432 *and* 1342 *are in the same strong Wilf class.*

**Theorem 4.6** *Let $12\tau1 \in [m]^{l+3}$ be a primitive subword pattern, and let $\tau'$ be the same pattern $\tau$ with* 1 *replaced by* 2*. Then the subword patterns $12\tau13$ and $12\tau'23$ are in the same strong Wilf class.*

**Proof:** If $\sigma \in [k]^n$ contains $12\tau13$ exactly $r$ times, then we define $\sigma'$ as follows. If $(\sigma_i, \dots, \sigma_{i+l+3})$ is an occurrence of $12\tau13$, then we define $\sigma'_{i+j} = \sigma_i + 1$ for all $j$ such that $\sigma_{i+j} = \sigma_i$. The function $f$ defined by $f(\sigma) = \sigma'$ is a bijection since $12\tau1$ is a primitive subword pattern. □

**Corollary 4.7** *The subword patterns* 1213 *and* 1223 *are in the same strong Wilf class.*

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# References

[1] E. Babson, E. Steingrímsson, Generalized permutation patterns and a classification of the Mahonian statistics, *Séminaire Lotharingien de Combinatoire*, B44b:18pp, 2000.

[2] M. Bona, The number of permutations with exactly $r$ 132-subsequences is $P$-recursive in the size!, *Adv. Appl. Math.* **18** (1997), 510–522.

[3] M. Bona, Permutations with one or two 132-subsequences, *Discrete Math.* **181** (1998), 267–274.

[4] A. Burstein, Enumeration of words with forbidden patterns, Ph.D. thesis, University of Pennsylvania, 1998.

[5] A. Burstein, T. Mansour, Words restricted by patterns with at most 2 distinct letters, *Electronic J. of Combinatorics* **9(2)** (2002), #R3.

[6] A. Burstein, T. Mansour, Words restricted by 3-letter generalized multipermutation patterns, preprint. Available at `www.arXiv.org/ps/math.CO/0112281`.

[7] A. Claesson, Generalized pattern avoidance, *European J. of Combinatorics* **22:7** (2001), 961–971.

[8] A. Claesson and T. Mansour, Permutations avoiding a pair of generalized patterns of the form *x*-*yz* or *xy*-*z*, preprint. Available at `www.arXiv.org/ps/math.CO/0107044`.

[9] H. Davenport and A. Schinzel, A combinatorial problem connected with differential equations, *Amer. J. Math.* **87** (1965), 684–690.

[10] S. Elizalde and M. Noy, Enumeration of subwords in permutations, Local Proceedings of *Formal Power Series and Algebraic Combinatorics (FPSAC'01), 13th International Conference*, ed. by H. Barcelo and V. Welker, Tempe, Arizona (USA), May 20-26, 2001, 179–189.

[11] M. Fulmek, Enumeration of permutations containing a prescribed number of occurrences of a length-3-pattern, Adv. Appl. Math. (to appear) Available at `www.arXiv.org/ps/math.CO/0112092`.

[12] I.P. Goulden and D.M. Jackson, Combinatorial enumeration, John Wiley and Sons, New York, NY, 1983.

[13] D.E. Knuth, The Art of Computer Programming, vols. 1, 3, Addison-Wesley, NY, 1968, 1973.

[14] A.M. Odlyzko, Asymptotic enumeration methods, in *Handbook of Combinatorics*, ed. by R.L. Graham, M. Grötschel and L. Lovász, MIT Press, 1995, vol. 2, pp. 1063–1230.

[15] T. Mansour and A. Vainshtein, Counting occurrences of 132 in a permutation, *Adv. Appl. Math.* **28:2** (2002), 185–195.

[16] J. Noonan, The number of permutations containing exactly one increasing subsequence of length three, *Discrete Math.* **152** (1996), 307–313.

[17] J. Noonan and D. Zeilberger, The enumeration of permutations with a prescribed number of "forbidden" patterns, *Adv. Appl. Math.* **17** (1996), 381–407.

[18] M. Régnier and W. Szpankowski, On the approximate pattern occurrences in a text, in "Compression and Complexity of Sequences 1997", *IEEE Computer Society*, 1998, pp. 253–264.

[19] R. Simion and F.W. Schmidt, Restricted permutations, *Europ. J. of Combinatorics* **6** (1985), 383–406.

[20] R. Stanley, Enumerative Combinatrics, vol. 1, Cambridge University Press, 1997.