A FIRST-ORDER LOGIC CHARACTERIZATION OF SAFETY AND CO-SAFETY LANGUAGES

ALESSANDRO CIMATTI ●[a], LUCA GEATTI ●[b], NICOLA GIGANTE ●[c], ANGELO MONTANARI ●[b], AND STEFANO TONETTA ●[a]

[a] Fondazione Bruno Kessler, Via Sommarive, 18, Povo, Trento, 38123, Italy
e-mail address: {cimatti,tonettas}@fbk.eu

[b] University of Udine, Via delle Scienze 206, Udine, 33100, Italy
e-mail address: {luca.geatti,angelo.montanari}@uniud.it

[c] Free University of Bozen-Bolzano, Piazza Università, 1, Bolzano, 39100, Italy
e-mail address: gigante@inf.unibz.it

ABSTRACT. *Linear Temporal Logic* (LTL) is one of the most popular temporal logics and comes into play in a variety of branches of computer science. Among the various reasons of its widespread use there are its strong foundational properties: LTL is equivalent to counter-free ω-automata, to star-free ω-regular expressions, and (by Kamp's theorem) to the *First-Order Theory of Linear Orders* (FO-TLO). Safety and co-safety languages, where a finite prefix suffices to establish whether a word does not belong or belongs to the language, respectively, play a crucial role in lowering the complexity of problems like model checking and reactive synthesis for LTL. Safety-LTL (resp., coSafety-LTL) is a fragment of LTL where only the *tomorrow*, the *weak tomorrow* and the *until* temporal modalities (resp., the *tomorrow*, the *weak tomorrow* and the *release* temporal modalities) are allowed, that recognises safety (resp., co-safety) languages only.

The main contribution of this paper is the introduction of a fragment of FO-TLO, called Safety-FO, and of its dual coSafety-FO, which are *expressively complete* with respect to the LTL-definable safety and co-safety languages. We prove that they exactly characterize Safety-LTL and coSafety-LTL, respectively, a result that joins Kamp's theorem, and provides a clearer view of the characterization of (fragments of) LTL in terms of first-order languages. In addition, it gives a direct, compact, and self-contained proof that any safety language definable in LTL is definable in Safety-LTL as well. As a by-product, we obtain some interesting results on the expressive power of the *weak tomorrow* operator of Safety-LTL, interpreted over finite and infinite words. Moreover, we prove that, when interpreted over finite words, Safety-LTL (resp. coSafety-LTL) devoid of the *tomorrow* (resp., *weak tomorrow*) operator captures the safety (resp., co-safety) fragment of LTL over finite words.

We then investigate some formal properties of Safety-FO and coSafety-FO: (i) we study their succinctness with respect to their modal counterparts, namely, Safety-LTL and coSafety-LTL; (ii) we illustrate an important practical application of them in the context of reactive synthesis; (iii) we compare them with expressively equivalent first-order fragments.

Last but not least, we provide different characterizations of the (co-)safety fragment of LTL in terms of temporal logics, automata, and regular expressions.

1. Introduction

Linear Temporal Logic (LTL) is the de-facto standard logic for system specifications [Pnu77]. It is a modal logic that is usually interpreted over infinite state sequences, but the finite-words semantics has recently gained attention as well [DV13, DV15]. The widespread use of LTL is due to its simple syntax and semantics, and to its strong foundational properties. Among them, we would like to mention the seminal work by Kamp [Kam68] and Gabbay *et al.* [GPSS80] on its expressive completeness, that is, LTL-definable languages are exactly those definable in the first-order fragment of the monadic second-order theory of linear orders [Büc90] (FO-TLO for short).

In formal verification, an important class of specifications is that of *safety languages*. They are languages of infinite words where a finite prefix suffices to establish whether a word does not belong to the language. As an example, the set of all and only those infinite sequences where some particular bad event never happens can be regarded as a safety language. In the dual *co-safety* languages (sometimes called *guarantee* languages), a finite prefix is sufficient to tell whether a word *belongs* to the language, *e.g.*, when some desired event is mandated to eventually happen. Safety and co-safety languages are important for verification, model-checking, monitoring, and automated synthesis, because they capture a variety of real-world requirements while being much simpler to deal with algorithmically [KV01, BAS02, ZTL+17].

Safety-LTL is the fragment of LTL where only the *tomorrow*, the *weak tomorrow* and the *until* temporal modalities are allowed. Similarly, its dual coSafety-LTL is obtained by only allowing the *tomorrow*, the *weak tomorrow* and the *release* modalities. It has been proved by Chang *et al.* [CMP92] that Safety-LTL and coSafety-LTL define exactly the safety and co-safety languages that are definable in LTL, respectively.

The paper consists of four parts.

In the first part, we provide a novel characterization of LTL-definable safety languages, and of their duals, in terms of a fragment of FO-TLO, called Safety-FO, and of its dual coSafety-FO. We argue that they have a very natural syntax, and we prove that they are *expressively complete* with respect to LTL-definable safety and co-safety languages. We first prove the correspondence between coSafety-FO and coSafety-LTL, which extends naturally to their duals and can be viewed as a version of Kamp's theorem [Kam68] specialized for safety and co-safety properties. Such a result provides a clearer picture of the correspondence between (fragments of) temporal and first-order logics. Then, we exploit it to prove the correspondence between co-safety languages definable in LTL and coSafety-FO, thus establishing also the equivalence between the former and coSafety-LTL. This gives a new proof of the fact that Safety-LTL captures exactly the set of LTL-definable safety languages [CMP92], which can be viewed as another contribution of the paper.

The interest of the latter proof is twofold: on the one hand, the original proof by Chang *et al.* [CMP92] is only sketched and it relies on two non-trivial translations scattered across different sources [Zuc86, SPH84]; on the other hand, such an equivalence result seems not to be very much known, as some authors presented the problem as open as lately as 2021 [ZTL+17, DGDST+21]. Thus, a compact and self-contained proof of the result seems to be a useful contribution for the community. It is worth to note that both proofs build on the fact that safety/co-safety languages can be captured by formulas of the form $G\alpha/F\alpha$ with α pure-past, but, after that, the two proofs significantly diverge. At the end of this

part, as a by-product, we give some results that assess the expressive power of the *weak tomorrow* operator of Safety-LTL when interpreted over finite *vs.* infinite words.

The second part is devoted to the safety and co-safety fragments of LTL interpreted over finite words. We show that the logic obtained from Safety-LTL (resp. coSafety-LTL) by forbidding the *tomorrow* (resp., *weak-tomorrow*) operator captures the set of safety (resp., co-safety) properties of LTL over finite words. This provides a clearer view of which fragments of LTL over finite words characterize the safety and co-safety fragments.

In the third part, we study some formal properties of coSafety-FO and Safety-FO. We begin by studying the succinctness of coSafety-FO with respect to coSafety-LTL. We first show that there is a linear-size equivalence-preserving translation from coSafety-LTL to coSafety-FO. Then, we show that the proposed translation from coSafety-LTL to coSafety-FO that we exploit to prove the expressive equivalence between the two formalisms is nonelementary. Next, we illustrate an interesting practical application of coSafety-FO to reactive synthesis from temporal specifications. Finally, we compare coSafety-FO with another fragment of FO-TLO that has been proved to be expressively equivalent to the co-safety fragment of LTL [Tho88]. Naturally, all the above results can be dualized for the case of Safety-FO.

In the fourth and last part, we summarize the other characterizations of the (co)safety fragment of LTL that have been proposed in the literature so far, that is, those in terms of (i) temporal logics, (ii) automata, and (iii) regular expressions.

The paper is organized as follows. Section 2 provides some background knowledge. Section 3 introduces Safety-FO and coSafety-FO and proves their correspondence with Safety-LTL and coSafety-LTL, respectively. Then, Section 4 proves their correspondence with the set of safety and co-safety languages definable in LTL, thus providing a compact and self-contained proof of the equivalence between Safety-LTL and LTL-definable safety languages. Some properties of the *weak next* operator are outlined as well. Section 5 proves the expressive completeness of the fragment of LTL over finite words devoid of the *tomorrow* (resp., *weak-tomorrow*) operator and the safety (resp., co-safety) fragment of LTL over finite words. Section 6 compares coSafety-FO with related fragments and describes a practical application of coSafety-FO to reactive synthesis. Section 7 summarizes the state of the art about different characterizations of the (co)safety fragment of LTL. Finally, Section 8 provides an assessment of the work done and discusses future work.

The paper is a revised and largely extended version of [CGG$^+$22]. In particular, the whole second, third, and fourth parts of the paper were not present in [CGG$^+$22].

2. Preliminaries

Let A be a finite alphabet. We denote by A^* and A^ω the set of all finite and infinite words over A, respectively. Moreover, we let $A^+ = A^* \setminus \{\varepsilon\}$, where ε is the empty word. Given a word $\sigma \in A^*$, we denote by $|\sigma|$ the length of σ. For an infinite word $\sigma \in A^\omega$, $|\sigma| = \omega$. Given a (finite or infinite) word σ, we denote by $\sigma_i \in A$, for $0 \le i < |\sigma|$, the letter at the i-th position of the word. For $0 \le i \le j < |\sigma|$, we denote by $\sigma_{[i,j]}$ the subword that starts at the i-th position (letter) of the word and ends at the j-th one, extrema included. By $\sigma_{[i,\infty]}$ we denote the suffix of σ starting at the i-th position. Given a word $\sigma \in A^*$ and $\sigma' \in A^* \cup A^\omega$, we denote the *concatenation* of the two words as $\sigma \cdot \sigma'$, or simply $\sigma\sigma'$. A *language* $\mathcal{L}$, with $\mathcal{L} \subseteq A^*$ or $\mathcal{L} \subseteq A^\omega$, is a set of words. Given two languages $\mathcal{L}$ and $\mathcal{L}'$ with $\mathcal{L} \subseteq A^*$ and either $\mathcal{L}' \subseteq A^*$ or $\mathcal{L}' \subseteq A^\omega$, we define $\mathcal{L} \cdot \mathcal{L}'$ as the set $\{\sigma \cdot \sigma' \mid \sigma \in \mathcal{L} \text{ and } \sigma' \in \mathcal{L}'\}$. Given a finite

word $\sigma = \sigma_0 \ldots \sigma_k$, let $\sigma^r = \sigma_k \ldots \sigma_0$ be the reverse of σ, and given a language of finite words $\mathcal{L}$, let $\mathcal{L}^r = \{\sigma^r \mid \sigma \in \mathcal{L}\}$. We are now ready to define *safety* and *co-safety* languages.

Definition 2.1 (Safety language [KV01, Tho88]). Let $\mathcal{L} \subseteq A^\omega$. We say that $\mathcal{L}$ is a *safety language* if and only if for all $\sigma \in A^\omega$, it holds that if $\sigma \notin \mathcal{L}$, then there exists $i \in \mathbb{N}$ such that, for all $\sigma' \in A^\omega$, $\sigma_{[0,i]} \cdot \sigma' \notin \mathcal{L}$. The class of safety languages is denoted by SAFETY.

Definition 2.2 (Co-safety language [KV01, Tho88]). Let $\mathcal{L} \subseteq A^\omega$. We say that $\mathcal{L}$ is a *co-safety language* if and only if for all $\sigma \in A^\omega$, it holds that if $\sigma \in \mathcal{L}$, then there exists $i \in \mathbb{N}$ such that, for all $\sigma' \in A^\omega$, $\sigma_{[0,i]} \cdot \sigma' \in \mathcal{L}$. The class of co-safety languages is denoted by coSAFETY.

Linear Temporal Logic with Past (LTL+P) is a temporal logic interpreted over infinite or finite words. Given a set of proposition letters Σ, the set of LTL+P formulas ϕ is generated by the following grammar:

$$\phi ::= p \mid \neg\phi \mid \phi \vee \phi \mid \phi \wedge \phi \qquad \text{Boolean connectives}$$
$$\mid \mathsf{X}\phi \mid \widetilde{\mathsf{X}}\phi \mid \phi\,\mathcal{U}\,\phi \mid \phi\,\mathcal{R}\,\phi \qquad \text{future modalities}$$
$$\mid \mathsf{Y}\phi \mid \widetilde{\mathsf{Y}}\phi \mid \phi\,\mathcal{S}\,\phi \mid \phi\,\mathcal{T}\,\phi \qquad \text{past modalities}$$

where $p \in \Sigma$ and ϕ is an LTL+P formula. We say that an LTL+P formula is a *pure future* formula if it does not make use of past modalities, and that it is *pure past* if it does not make use of future modalities. Let us denote by LTL the set of pure future formulas, and by LTL$_\mathsf{P}$ the set of pure past formulas.

LTL+P is interpreted over *state sequences*, which are finite or infinite words over 2^Σ. Given a state sequence $\sigma \in (2^\Sigma)^+$ or $\sigma \in (2^\Sigma)^\omega$, the *satisfaction* of a formula ϕ by σ at a time point $i \geq 0$, denoted by $\sigma, i \models \phi$, is defined as follows:

1. $\sigma, i \models p$ iff $p \in \sigma_i$;
2. $\sigma, i \models \neg\phi$ iff $\sigma, i \not\models \phi$;
3. $\sigma, i \models \phi_1 \vee \phi_2$ iff $\sigma, i \models \phi_1$ or $\sigma, i \models \phi_2$;
4. $\sigma, i \models \phi_1 \wedge \phi_2$ iff $\sigma, i \models \phi_1$ and $\sigma, i \models \phi_2$;
5. $\sigma, i \models \mathsf{X}\phi$ iff $i + 1 < |\sigma|$ and $\sigma, i + 1 \models \phi$;
6. $\sigma, i \models \widetilde{\mathsf{X}}\phi$ iff   either $i + 1 = |\sigma|$ or $\sigma, i + 1 \models \phi$;
7. $\sigma, i \models \mathsf{Y}\phi$ iff $i > 0$ and $\sigma, i - 1 \models \phi$;
8. $\sigma, i \models \widetilde{\mathsf{Y}}\phi$ iff   either $i = 0$ or $\sigma, i - 1 \models \phi$;
9. $\sigma, i \models \phi_1\,\mathcal{U}\,\phi_2$ iff   there exists $i \leq j < |\sigma|$ such that $\sigma, j \models \phi_2$,
                                  and $\sigma, k \models \phi_1$ for all k, with $i \leq k < j$;
10. $\sigma, i \models \phi_1\,\mathcal{S}\,\phi_2$ iff   there exists $j \leq i$ such that $\sigma, j \models \phi_2$,
                                  and $\sigma, k \models \phi_1$ for all k, with $j < k \leq i$;
11. $\sigma, i \models \phi_1\,\mathcal{R}\,\phi_2$ iff   either $\sigma, j \models \phi_2$ for all $i \leq j < |\sigma|$, or there exists
                                  $k \geq i$ such that $\sigma, k \models \phi_1$ and
                                  $\sigma, j \models \phi_2$ for all $i \leq j \leq k$;
12. $\sigma, i \models \phi_1\,\mathcal{T}\,\phi_2$ iff   either $\sigma, j \models \phi_2$ for all $0 \leq j \leq i$, or there exists
                                  $k \leq i$ such that $\sigma, k \models \phi_1$ and
                                  $\sigma, j \models \phi_2$ for all $i \geq j \geq k$

Some connectives/operators of the language can be defined in terms of a small number of basic ones. In particular, $\phi_1 \wedge \phi_2$ (conjunction) can be defined in terms of disjunction as $\neg(\neg\phi_1 \vee \neg\phi_2)$, $\phi_1 \mathcal{R} \phi_2$ (the *release* operator) in terms of the *until* one as $\neg(\neg\phi_1 \mathcal{U} \neg\phi_2)$, and $\phi_1 \mathcal{T} \phi_2$ (the *triggered* operator) in terms of the *since* one as $\neg(\neg\phi_1 \mathcal{S} \neg\phi_2)$. Nevertheless,

we consider all these connectives and operators as primitive ones in order to be able to put any formula in *negated normal form* (NNF), that is, a form where negation is only applied to proposition letters. Note that the syntax includes both a *tomorrow* ($\mathsf{X}\phi$) and a *weak tomorrow* ($\widetilde{\mathsf{X}}\phi$) operator, and, similarly, a *yesterday* ($\mathsf{Y}\phi$) and a *weak yesterday* ($\widetilde{\mathsf{Y}}\phi$) operator. Finally, standard shortcut operators are available such as the *eventually* ($\mathsf{F}\phi \equiv \top\,\mathcal{U}\,\phi$) and *always* ($\mathsf{G}\phi \equiv \neg\mathsf{F}\neg\phi$) future modalities, and the *once* ($\mathsf{O}\phi \equiv \top\,\mathcal{S}\,\phi$) and *historically* ($\mathsf{H}\phi \equiv \neg\mathsf{O}\neg\phi$) past modalities.

We say that a state sequence σ satisfies ϕ, written $\sigma \models \phi$, if $\sigma, 0 \models \phi$. If ϕ belongs to $\mathsf{LTL_P}$, that is, the pure past fragment of $\mathsf{LTL+P}$, then we interpret ϕ only on *finite* state sequences and we say that $\sigma \in (2^\Sigma)^+$ is a model of ϕ if and only if $\sigma, |\sigma| - 1 \models \phi$, *i.e.*, each ϕ in $\mathsf{LTL_P}$ is interpreted at the *last* state of a finite state sequence.

Notice that, when interpreted over an infinite word, the semantics of the *tomorrow* and *weak tomorrow* operators is the same. The *language* of ϕ, denoted by $\mathcal{L}(\phi)$, is the set of words $\sigma \in (2^\Sigma)^\omega$ such that $\sigma \models \phi$. The *language of finite words* of ϕ, denoted by $\mathcal{L}^{<\omega}(\phi)$, is the set of finite words $\sigma \in (2^\Sigma)^+$ such that $\sigma \models \phi$. Given a logic L, we denote by $[\![\mathsf{L}]\!]$ the set of languages $\mathcal{L}$ such that there is a formula $\phi \in \mathsf{L}$ such that $\mathcal{L} = \mathcal{L}(\phi)$, and by $[\![\mathsf{L}]\!]^{<\omega}$ the set of languages of finite words $\mathcal{L}$ such that there is a formula $\phi \in \mathsf{L}$ such that $\mathcal{L} = \mathcal{L}^{<\omega}(\phi)$ ($[\![\mathsf{LTL}]\!]^{<\omega}$ is usually referred to by LTLf in the literature [DV13]). It is known that LTLf and pure past LTL ($\mathsf{LTL_P}$) have the same expressive power [LPZ85, Tho88].

Proposition 2.3. $[\![\mathsf{LTL}]\!]^{<\omega} = [\![\mathsf{LTL_P}]\!]^{<\omega}$.

We now define the two fragments of LTL that are the subject of this paper.

Definition 2.4 (Safety-LTL and coSafety-LTL [Sis94])**.** The logic $\mathsf{Safety\text{-}LTL}$ (resp., the logic $\mathsf{coSafety\text{-}LTL}$) is the fragment of LTL where, for formulas in *negated normal form*, only the *tomorrow*, *weak tomorrow*, and *release* (resp., *until*) temporal modalities are allowed.

Note that both $\mathsf{Safety\text{-}LTL}$ and $\mathsf{coSafety\text{-}LTL}$ contain only *future* temporal operators. We also define the logic $\mathsf{coSafety\text{-}LTL}(-\widetilde{\mathsf{X}})$ as the logic $\mathsf{coSafety\text{-}LTL}$ devoid of the *weak tomorrow* ($\widetilde{\mathsf{X}}$) operator (this logic will play a central role in our proofs). Similarly, we define $\mathsf{Safety\text{-}LTL}(-\mathsf{X})$ as the logic $\mathsf{Safety\text{-}LTL}$ devoid of the *tomorrow* (X) operator.

In the next section, we introduce two fragments of the *First-Order Theory of Linear Orders* [Buc63, Büc90], namely $\mathsf{FO\text{-}TLO}$ (or simply FO for short). Given an alphabet Σ, FO is a first-order language with equality over the signature $\langle <, \{P\}_{p\in\Sigma}\rangle$, and is interpreted over structures $\mathcal{M} = \langle D^\mathcal{M}, <^\mathcal{M}, \{P^\mathcal{M}\}_{p\in\Sigma}\rangle$, where $D^\mathcal{M}$ is either the set $\mathbb{N}$ of natural numbers or a prefix $\{0, \ldots, n\}$ thereof, and $<^\mathcal{M}$ is the usual ordering relation over natural numbers. A *sentence* of FO is a formula of FO with no free variables. Given an FO formula $\phi(x_0, \ldots, x_m)$, with $m + 1$ free variables, the satisfaction of ϕ by a first-order structure $\mathcal{M}$ when $x_0 = n_0, \ldots, x_m = n_m$, denoted by $\mathcal{M}, n_0, \ldots, n_m \models \phi(x_0, \ldots, x_m)$, is defined according the standard first-order semantics. State sequences over Σ map naturally into such structures. Given a word $\sigma \in (2^\Sigma)^*$ or $\sigma \in (2^\Sigma)^\omega$, we denote by $(\sigma)^s$ the corresponding first-order structure. Given a formula $\phi(x)$ with exactly one free variable x, the *language* of ϕ, denoted by $\mathcal{L}(\phi)$, is the set of words $\sigma \in (2^\Sigma)^\omega$ such that $(\sigma)^s, 0 \models \phi$. Similarly, the *language of finite words* of ϕ, denoted by $\mathcal{L}^{<\omega}(\phi)$, is the set of finite words $\sigma \in (2^\Sigma)^+$ such that $(\sigma)^s \models \phi$. We denote by $[\![\mathsf{FO}]\!]$ and $[\![\mathsf{FO}]\!]^{<\omega}$ the set of languages of respectively infinite and finite words definable by a FO formula.

$$\llbracket\mathsf{LTL}\rrbracket \cap \mathsf{SAFETY} \quad = \quad \llbracket\mathsf{Safety\text{-}LTL}\rrbracket$$
$$\llbracket\mathsf{LTL}\rrbracket \cap \mathsf{coSAFETY} \quad = \quad \llbracket\mathsf{coSafety\text{-}LTL}\rrbracket$$

Chang *et al.* [CMP92]

Lemmas 4.1 and 4.2 — Lemma 3.5 — Kamp — Lemma 4.2

$$\llbracket\mathsf{coSafety\text{-}LTL}\rrbracket^{<\omega} \cdot (2^{\Sigma})^{\omega} = \llbracket\mathsf{coSafety\text{-}LTL}(-\widetilde{\mathsf{X}})\rrbracket^{<\omega} \cdot (2^{\Sigma})^{\omega}$$

$$\llbracket\mathsf{LTL}\rrbracket^{<\omega}$$

$$\llbracket\mathsf{coSafety\text{-}LTL}\rrbracket^{<\omega} \supsetneq \llbracket\mathsf{coSafety\text{-}LTL}(-\widetilde{\mathsf{X}})\rrbracket^{<\omega}$$

Theorem 4.6

Corollary 3.10

Lemma 4.3 — Corollary 3.11 — Corollary 3.10

$$\llbracket\mathsf{coSafety\text{-}FO}\rrbracket$$

$$\llbracket\mathsf{FO}\rrbracket^{<\omega} \quad \supsetneq \quad \llbracket\mathsf{coSafety\text{-}FO}\rrbracket^{<\omega}$$
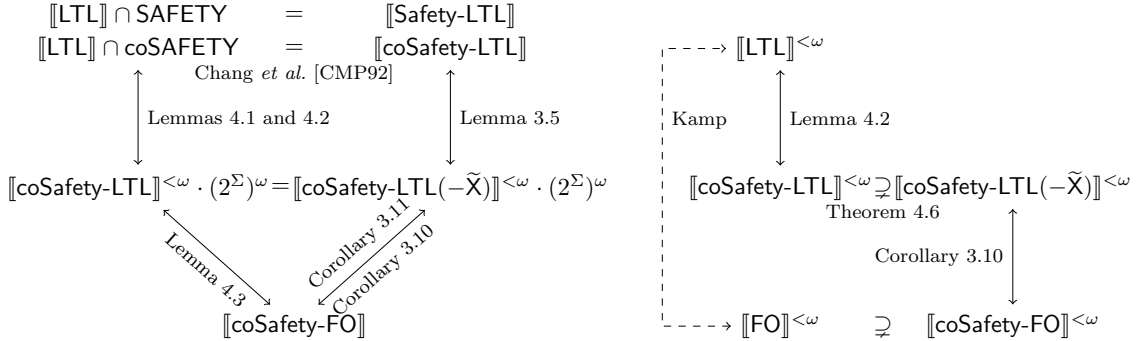
Figure 1. Summary of the results about languages over infinite words on the left and over finite words on the right. Solid arrows are own results; dashed arrows are known from literature.

Given a class of languages of finite words $\llbracket\mathsf{L}\rrbracket^{<\omega}$, we denote by $\llbracket\mathsf{L}\rrbracket^{<\omega} \cdot (2^{\Sigma})^{\omega}$ the set of languages $\{\mathcal{L}\cdot(2^{\Sigma})^{\omega} \mid \mathcal{L} \in \llbracket\mathsf{L}\rrbracket^{<\omega}\}$. From now on, given a formula $\phi \in L$, we denote by $|\phi|$ the number of symbols in ϕ.

We conclude the section by recalling some fundamental known results.

Proposition 2.5 (Kamp [Kam68] and Gabbay [GPSS80]).
$\llbracket\mathsf{LTL}\rrbracket = \llbracket\mathsf{FO}\rrbracket$ *and* $\llbracket\mathsf{LTL}\rrbracket^{<\omega} = \llbracket\mathsf{FO}\rrbracket^{<\omega}$.

Finally, we state a normal form for LTL-definable safety/co-safety languages.

Proposition 2.6 (Chang *et al.* [CMP92], Thomas [Tho88]). *A language* $\mathcal{L} \in \llbracket\mathsf{LTL}\rrbracket$ *is* safety *(resp.,* co-safety*) if and only if it is the language of a formula of the form* $\mathsf{G}\alpha$ *(resp.,* $\mathsf{F}\alpha$*), where* $\alpha \in \mathsf{LTL_P}$.

3. Safety-FO and coSafety-FO

In this section, we state and prove the main results of the paper: we define two simple fragments of FO and we show that they precisely capture Safety-LTL and coSafety-LTL, respectively. A summary of the achieved results is given in Fig. 1.

Definition 3.1 (Safety-FO). The logic Safety-FO is generated by the following grammar:

$$atomic ::= x < y \mid x = y \mid x \neq y \mid P(x) \mid \neg P(x)$$
$$\phi ::= atomic \mid \phi \vee \phi \mid \phi \wedge \phi \mid \exists y(x < y < z \wedge \phi) \mid \forall y(x < y \to \phi)$$

where x, y, and z are first-order variables, P is a unary predicate, and ϕ_1 and ϕ_2 are Safety-FO formulas.

Definition 3.2 (coSafety-FO). The logic coSafety-FO is generated by the following grammar:

$$atomic ::= x < y \mid x = y \mid x \neq y \mid P(x) \mid \neg P(x)$$
$$\phi ::= atomic \mid \phi \vee \phi \mid \phi \wedge \phi \mid \exists y(x < y \wedge \phi) \mid \forall y(x < y < z \to \phi)$$

where x, y, and z are first-order variables, P is a unary predicate, and ϕ_1 and ϕ_2 are coSafety-FO formulas.

We need to make a few observations on the syntax of the two fragments. First of all, note how any formula of Safety-FO is the negation of a formula of coSafety-FO and *vice versa*, and how any formula in this fragments has at least one free variable. Then, note that the two fragments are defined in *negated normal form*, *i.e.*, negation only appears on atomic formulas. The particular kind of existential and universal quantifications allowed are the culprit of these fragments. In particular, Safety-FO restricts any existentially quantified variable to be bounded between two free variables. The same applies to universal quantification in coSafety-FO. Moreover Safety-FO and coSafety-FO formulas are *future formulas*, *i.e.*, the quantifiers can only range over values *greater* than some free variables. These two features are essential to precisely capture Safety-LTL and coSafety-LTL. Finally, note that the comparisons in the guards of the quantifiers are strict, but non-strict comparisons can be used as well. In particular, $\exists y(x \leq y \wedge \phi)$ can be rewritten as $\phi[y/x] \vee \exists y(x < y \wedge \phi)$, where $\phi[y/x]$ is the formula obtained by renaming all the free occurrences of y in ϕ with x. Similarly, $\forall z(x \leq z \leq y \rightarrow \phi)$ can be rewritten as $\phi[z/x] \wedge \phi[z/y] \wedge \forall z(x < z < y \rightarrow \phi)$.

To prove the relationship between Safety-LTL, coSafety-LTL, and these fragments, we focus now on coSafety-FO. By duality, all the results transfer to Safety-FO. We focus on coSafety-FO because the unbounded quantification is existential, and it is easier to reason about the existence of prefixes than on all the prefixes at once. We start by observing that, since the *weak tomorrow* operator, over infinite words, coincides with the *tomorrow* operator, the following holds.

Observation 3.3. $[\![\text{coSafety-LTL}]\!] = [\![\text{coSafety-LTL}(-\widetilde{X})]\!]$

When reasoning over finite words, the *weak tomorrow* operator plays a crucial role, since it can be used to recognize when we are at the last position of a word. In fact, the formula $\sigma, i \models \widetilde{X}\bot$ is true if and only if $i = |\sigma| - 1$, for any $\sigma \in (2^\Sigma)^*$.

Now, let us note that, thanks to the absence of the *weak tomorrow* operator, the coSafety-LTL$(-\widetilde{X})$ logic is such that the concatenation of any (finite or infinite) suffix to a finite model of a coSafety-LTL$(-\widetilde{X})$ formula results in a correct model of a formula. Lemmas 3.4 and 3.5 prove this result for finite and infinite suffixes, respectively.

Lemma 3.4. $[\![\text{coSafety-LTL}(-\widetilde{X})]\!]^{<\omega} = [\![\text{coSafety-LTL}(-\widetilde{X})]\!]^{<\omega} \cdot (2^\Sigma)^*$

Proof. We have to prove that, for each formula $\phi \in \text{coSafety-LTL}(-\widetilde{X})$, it holds that:
$$\mathcal{L}^{<\omega}(\phi) = \mathcal{L}^{<\omega}(\phi) \cdot (2^\Sigma)^*$$
We proceed by induction on the structure of ϕ. For the base case, consider $\phi = p \in \Sigma$. The case for $\phi = \neg p$ is similar. Let $\sigma \in (2^\Sigma)^*$. It holds that $\sigma \in \mathcal{L}^{<\omega}(p)$ iff $\sigma_0 \in \mathcal{L}(p)$ iff $\sigma_0 \cdot \sigma' \in \mathcal{L}(p)$ (for any $\sigma' \in (2^\Sigma)^*$) iff $\sigma \in \mathcal{L}^{<\omega}(p) \cdot (2^\Sigma)^*$.

For the inductive step:
(1) Let $\phi = \phi_1 \wedge \phi_2$. It holds that $\sigma \in \mathcal{L}^{<\omega}(\phi_1 \wedge \phi_2)$ iff $\sigma \in \mathcal{L}^{<\omega}(\phi_1)$ and $\sigma \in \mathcal{L}^{<\omega}(\phi_2)$. By inductive hypothesis, this is equivalent to $\sigma \in \mathcal{L}^{<\omega}(\phi_1) \cdot (2^\Sigma)^*$ and $\sigma \in \mathcal{L}^{<\omega}(\phi_2) \cdot (2^\Sigma)^*$. This holds iff $\sigma \in \mathcal{L}^{<\omega}(\phi_1 \wedge \phi_2) \cdot (2^\Sigma)^*$.
(2) Let $\phi = \phi_1 \vee \phi_2$. It holds that $\sigma \in \mathcal{L}^{<\omega}(\phi_1 \vee \phi_2)$ iff $\sigma \in \mathcal{L}^{<\omega}(\phi_1)$ or $\sigma \in \mathcal{L}^{<\omega}(\phi_2)$. By inductive hypothesis, this is equivalent to $\sigma \in \mathcal{L}^{<\omega}(\phi_1) \cdot (2^\Sigma)^*$ or $\sigma \in \mathcal{L}^{<\omega}(\phi_2) \cdot (2^\Sigma)^*$. This holds iff $\sigma \in \mathcal{L}^{<\omega}(\phi_1 \vee \phi_2) \cdot (2^\Sigma)^*$.
(3) Let $\phi = X\phi_1$. It holds that $\sigma \in \mathcal{L}^{<\omega}(X\phi_1)$ iff $\sigma_{[1,|\sigma|-1]} \in \mathcal{L}^{<\omega}(\phi_1)$. By inductive hypothesis, this is equivalent to $\sigma_{[1,|\sigma|-1]} \in \mathcal{L}^{<\omega}(\phi_1) \cdot (2^\Sigma)^*$. This holds iff $\sigma \in \mathcal{L}^{<\omega}(X\phi_1) \cdot (2^\Sigma)^*$.

(4) Let $\phi = \phi_1 \mathcal{U} \phi_2$. It holds that $\sigma_{[i,|\sigma|-1]} \in \mathcal{L}^{<\omega}(\phi_1)$ and $\sigma_{[j,|\sigma|-1]} \in \mathcal{L}^{<\omega}(\phi_2)$, for some $0 \le i < |\sigma|$ and for all $0 \le j < i$. By inductive hypothesis, $\sigma_{[i,|\sigma|-1]} \in \mathcal{L}^{<\omega}(\phi_1) \cdot (2^\Sigma)^*$ and $\sigma_{[j,|\sigma|-1]} \in \mathcal{L}^{<\omega}(\phi_2) \cdot (2^\Sigma)^*$ (for some $0 \le i < |\sigma|$ and for all $0 \le j < i$). This is equivalent to $\sigma \in \mathcal{L}^{<\omega}(\phi_1 \mathcal{U} \phi_2) \cdot (2^\Sigma)^*$. $\square$

The following lemma generalizes Lemma 3.4 to the case of infinite words.

Lemma 3.5. $[\![\mathsf{coSafety\text{-}LTL}(-\widetilde{\mathsf{X}})]\!] = [\![\mathsf{coSafety\text{-}LTL}(-\widetilde{\mathsf{X}})]\!]^{<\omega} \cdot (2^\Sigma)^\omega$

Proof. We have to prove that, for each formula $\phi \in \mathsf{coSafety\text{-}LTL}(-\widetilde{\mathsf{X}})$, it holds that:

$$\mathcal{L}(\phi) = \mathcal{L}^{<\omega}(\phi) \cdot (2^\Sigma)^\omega$$

We proceed by induction on the structure of ϕ. For the base case, consider $\phi = p \in \Sigma$. The case for $\phi = \neg p$ is similar. Let $\sigma \in (2^\Sigma)^\omega$. It holds that $\sigma \in \mathcal{L}(p)$ iff $\sigma_0 \in \mathcal{L}(p)$ iff $\sigma_0 \cdot \sigma' \in \mathcal{L}(p)$ (for any $\sigma' \in (2^\Sigma)^\omega$) iff $\sigma \in \mathcal{L}^{<\omega}(p) \cdot (2^\Sigma)^\omega$.

For the inductive step:

(1) Let $\phi = \phi_1 \wedge \phi_2$. It holds that $\sigma \in \mathcal{L}(\phi)$ iff $\sigma \in \mathcal{L}(\phi_1)$ and $\sigma \in \mathcal{L}(\phi_2)$. By the inductive hypothesis, this is equivalent to $\sigma \in \mathcal{L}^{<\omega}(\phi_1) \cdot (2^\Sigma)^\omega$ and $\sigma \in \mathcal{L}^{<\omega}(\phi_2) \cdot (2^\Sigma)^\omega$. This means that there exist two indices $i, j \in \mathbb{N}$ such that $\sigma_{[0,i]} \in \mathcal{L}^{<\omega}(\phi_1)$ and $\sigma_{[0,j]} \in \mathcal{L}^{<\omega}(\phi_2)$. Let m be the greatest between i and j. By Lemma 3.4 it holds that $\sigma_{[0,m]} \in \mathcal{L}^{<\omega}(\phi_1)$ and $\sigma_{[0,m]} \in \mathcal{L}^{<\omega}(\phi_2)$, i.e., $\sigma_{[0,m]} \in \mathcal{L}^{<\omega}(\phi_1 \wedge \phi_2)$. Therefore $\sigma \in \mathcal{L}^{<\omega}(\phi_1 \wedge \phi_2) \cdot (2^\Sigma)^\omega$.

(2) Let $\phi = \phi_1 \vee \phi_2$. It holds that $\sigma \in \mathcal{L}(\phi)$ iff $\sigma \in \mathcal{L}(\phi_1)$ or $\sigma \in \mathcal{L}(\phi_2)$. Without loss of generality, we consider the case that $\sigma \in \mathcal{L}(\phi_1)$ (the other case is specular). By the inductive hypothesis, this is equivalent to $\sigma \in \mathcal{L}^{<\omega}(\phi_1) \cdot (2^\Sigma)^\omega$. Therefore, $\sigma = \sigma' \cdot \sigma''$ where $\sigma' \in \mathcal{L}^{<\omega}(\phi_1)$ and $\sigma'' \in (2^\Sigma)^\omega$. Since $\mathcal{L}^{<\omega}(\phi_1) \subseteq \mathcal{L}^{<\omega}(\phi_1 \vee \phi_2)$, this is equivalent to $\sigma \in \mathcal{L}^{<\omega}(\phi_1 \vee \phi_2) \cdot (2^\Sigma)^\omega$.

(3) Let $\phi = \mathsf{X}\phi_1$. It holds that $\sigma \in \mathcal{L}(\mathsf{X}\phi_1)$ iff $\sigma_{[1,\infty)} \in \mathcal{L}(\phi_1)$. By inductive hypothesis, this is equivalent to $\sigma_{[1,\infty)} \in \mathcal{L}^{<\omega}(\phi_1) \cdot (2^\Sigma)^\omega$. This holds iff $\sigma \in \mathcal{L}^{<\omega}(\mathsf{X}\phi_1) \cdot (2^\Sigma)^\omega$.

(4) Let $\phi = \phi_1 \mathcal{U} \phi_2$. By the semantics of the *until* operator, it holds that $\sigma \in \mathcal{L}(\phi)$ iff there exists an index $i \in \mathbb{N}$ such that $\sigma_{[i,\infty)} \in \mathcal{L}(\phi_2)$ and $\sigma_{[j,\infty)} \in \mathcal{L}(\phi_1)$ for all $0 \le j < i$. By the inductive hypothesis, this is equivalent to $\sigma_{[i,\infty)} \in \mathcal{L}^{<\omega}(\phi_2) \cdot (2^\Sigma)^\omega$ and $\sigma_{[j,\infty)} \in \mathcal{L}^{<\omega}(\phi_1) \cdot (2^\Sigma)^\omega$ for all $0 \le j < i$. This means that there exists an index $i \in \mathbb{N}$ and $i + 1$ indices $k_0, \dots, k_i \in \mathbb{N}$ such that $\sigma_{[i,k_i]} \in \mathcal{L}^{<\omega}(\phi_2)$ and $\sigma_{[j,k_j]} \in \mathcal{L}^{<\omega}(\phi_1)$ for all $0 \le j < i$. Let m be the greatest between $k_0, \dots, k_i$. By Lemma 3.4, it holds that there exists an index $i \in \mathbb{N}$ such that $\sigma_{[i,m]} \models \phi_2$ and $\sigma_{[j,m]} \models \phi_1$ for all $0 \le j < i$. Therefore, this is equivalent to $\sigma \in \mathcal{L}^{<\omega}(\phi_1 \mathcal{U} \phi_2) \cdot (2^\Sigma)^\omega$. $\square$

Note that an equality similar to Lemmas 3.4 and 3.5 does not hold if we allow the *weak tomorrow* operator, because if $\sigma \in \mathcal{L}^{<\omega}(\widetilde{\mathsf{X}}\phi)$, it might still be that $|\sigma| = 1$ and ϕ is unsatisfiable, hence there is no way to extend σ to an infinite word while still satisfying the formula.

In [GMM14], De Giacomo *et al.* define the notion of *insensitive to infiniteness* as a way to compare the finite and the infinite word semantics of fragments of LTL. They define a formula ϕ (over an alphabet Σ) to be *insensitive to infiniteness* if, and only if, for any finite word $\sigma \in \Sigma^+$, it holds that $\sigma \models \phi$ iff $\sigma \cdot \{\mathsf{e}^\omega\} \models \phi$, where e is a fresh proposition letter ($\mathsf{e} \notin \Sigma$). By Lemma 3.5, it follows that every formula of $\mathsf{coSafety\text{-}LTL}(-\widetilde{\mathsf{X}})$ is insensitive to infiniteness.

Then, we can focus on $\mathsf{coSafety\text{-}LTL}(-\widetilde{\mathsf{X}})$ and $\mathsf{coSafety\text{-}FO}$ on finite words. If we can prove that $[\![\mathsf{coSafety\text{-}LTL}(-\widetilde{\mathsf{X}})]\!]^{<\omega} = [\![\mathsf{coSafety\text{-}FO}]\!]^{<\omega}$, we are done. At first, we show how to encode $\mathsf{coSafety\text{-}LTL}(-\widetilde{\mathsf{X}})$ formulas into $\mathsf{coSafety\text{-}FO}$ with exactly one free variable.

Lemma 3.6. $[\![\mathsf{coSafety\text{-}LTL}(-\widetilde{\mathsf{X}})]\!]^{<\omega} \subseteq [\![\mathsf{coSafety\text{-}FO}]\!]^{<\omega}$

Proof. Let $\mathcal{L} \in [\![\mathsf{coSafety\text{-}LTL}(-\widetilde{\mathsf{X}})]\!]^{<\omega}$, and let $\phi \in \mathsf{coSafety\text{-}LTL}(-\widetilde{\mathsf{X}})$ such that $\mathcal{L} = \mathcal{L}^{<\omega}(\phi)$. By following the semantics of the operators in ϕ, we can obtain an equivalent $\mathsf{coSafety\text{-}FO}$ formula ϕ_{FO}. We inductively define the formula $FO(\phi, x)$, where x is a variable, as follows:

- $FO(p, x) = P(x)$, for each $p \in \Sigma$
- $FO(\neg p, x) = \neg P(x)$, for each $p \in \Sigma$
- $FO(\phi_1 \wedge \phi_2, x) = FO(\phi_1, x) \wedge FO(\phi_2, x)$
- $FO(\phi_1 \vee \phi_2, x) = FO(\phi_1, x) \vee FO(\phi_2, x)$
- $FO(\mathsf{X}\phi_1, x) = \exists y(x < y \wedge y = x + 1 \wedge FO(\phi_1, y))$
  where $y = x + 1$ can be expressed as $\forall z(x < z < y \to \bot)$.
- $FO(\phi_1 \, \mathcal{U} \, \phi_2, x) = \exists y(x \leq y \wedge FO(\phi_2, y) \wedge \forall z(x \leq z < y \to FO(\phi_1, z)))$

For each $\phi \in \mathsf{coSafety\text{-}LTL}(-\widetilde{\mathsf{X}})$, the formula $FO(\phi, x)$ has exactly one free variable x. It is easy to see that for all finite state sequences $\sigma \in (2^\Sigma)^*$, it holds that $\sigma \models \phi$ if and only if $(\sigma)^s, 0 \models FO(\phi, x)$, and $FO(\phi, x) \in \mathsf{coSafety\text{-}FO}$. Therefore, $\mathcal{L} \in [\![\mathsf{coSafety\text{-}FO}]\!]^{<\omega}$.   □

It is time to show the opposite direction, *i.e.*, that any $\mathsf{coSafety\text{-}FO}$ formula can be translated into a $\mathsf{coSafety\text{-}LTL}(-\widetilde{\mathsf{X}})$ formula which is equivalent over finite words. To prove this fact we adapt a proof of Kamp's theorem by Rabinovich [Rab14]. Kamp's theorem is one of the fundamental results about temporal logics, which states that LTL corresponds to FO in terms of expressiveness. Here, we prove a similar result in the context of co-safety languages. The proof goes by introducing a *normal form* for FO formulas, and showing that (i) any $\mathsf{coSafety\text{-}FO}$ formula can be translated into such normal form and (ii) any formula in normal form can be straightforwardly translated into a $\mathsf{coSafety\text{-}LTL}(-\widetilde{\mathsf{X}})$ formula. We start by introducing such a normal form.

Definition 3.7 ($\exists\forall$-formulas). An $\exists\forall$-formula $\phi(z_0, \ldots, z_m)$ with m free variables is a formula of this form:

$$\phi(z_0, \ldots, z_m) \coloneqq \exists x_0 \ldots \exists x_n \big($$

$$x_0 < x_1 < \cdots < x_n \qquad\qquad \text{ordering constraints}$$

$$\wedge \; z_0 = x_0 \wedge \bigwedge_{k=1}^{m} (z_k = x_{i_k}) \qquad\qquad \text{binding constraints}$$

$$\wedge \; \bigwedge_{j=0}^{n} \alpha_j(x_j) \qquad\qquad \text{punctual constraints}$$

$$\wedge \; \bigwedge_{j=1}^{n} \forall y(x_{j-1} < y < x_j \to \beta_j(y))) \qquad\qquad \text{interval constraints}$$

where $i_k \in \{0, \ldots, n\}$ for each $0 \leq k \leq m$, and α_j and β_j, for each $1 \leq j \leq n$, are quantifier-free formulas with exactly one free variable.

Some explanations are due. Each $\exists\forall$-formula states a number of requirements for its free variables and for its quantified variables. Through the binding constraints, the free variables are identified with a subset of the quantified variables in order to uniformly state

the punctual and interval constraints, and the ordering constraints which sort all the variable in a total order. Note that there is no relationship between n and m: there might be more quantified variables than free variables, or less. Note as well that the binding constraint $z_0 = x_0$ is always present, *i.e.*, at least one free variable has to be the minimal element of the ordering. This ensures that $\exists\forall$-formulas constrain only positions of the word that are greater than the value of x_0.

We say that a formula of coSafety-FO is in *normal form* if and only if it is a disjunction of $\exists\forall$-formulas. To see how formulas in normal form make sense, let us immediately show how to translate them into coSafety-LTL$(-\widetilde{X})$ formulas.

Lemma 3.8. *For any formula $\phi(z) \in$ coSafety-FO in normal form, with a single free variable, there exists a formula $\psi \in$ coSafety-LTL$(-\widetilde{X})$ such that $\mathcal{L}^{<\omega}(\phi(z)) = \mathcal{L}^{<\omega}(\psi)$.*

Proof. We show how any $\exists\forall$-formula is equivalent to a coSafety-LTL$(-\widetilde{X})$-formula, over finite words. Since each formula in normal form is a disjunction of $\exists\forall$-formulas, and since coSafety-LTL$(-\widetilde{X})$ is closed under disjunction, this implies the proposition. Let $\phi(z)$ be a $\exists\forall$-formula with a single free variable. Having only one free variable, $\phi(z)$ is of the form:

$$\exists x_0 \dots \exists x_n \big(x_0 < \dots < x_n \wedge z = x_0$$
$$\wedge \bigwedge_{j=0}^{n} \alpha_j(x_j) \wedge \bigwedge_{j=1}^{n} \forall y (x_{j-1} < y < x_j \to \beta_j(y)) \big)$$

Now, let A_i be the temporal formulas corresponding to α_i and B_i be the ones corresponding to β_i. Recall that α_i and β_i are quantifier free with only one free variable, hence this correspondence is trivial. Since z is the first time point of the ordering mandated by the formula, we only need future temporal operators to encode ϕ into a coSafety-LTL$(-\widetilde{X})$ formula ψ defined as follows:

$$\psi = A_0 \wedge \mathsf{X}(B_0 \,\mathcal{U}\, (A_1 \wedge \mathsf{X}(B_1 \,\mathcal{U}\, A_2 \wedge \dots \mathsf{X}(B_{n-1} \,\mathcal{U}\, A_n) \dots)))$$

It can be seen that $\sigma, k \models \psi$ if and only if $(\sigma)^s, k \models \phi(z)$, for each $\sigma \in (2^\Sigma)^+$ and each $k \geq 0$. Thus, $\mathcal{L}^{<\omega}(\phi(z)) = \mathcal{L}^{<\omega}(\psi)$. $\square$

Two differences between our $\exists\forall$-formulas and those used by Rabinovich [Rab14] are crucial: first, we do not have unbounded universal requirements, but all interval constraints use bounded quantifications, hence we do not need the *always* operator to encode them; second, our $\exists\forall$-formulas are *future* formulas, hence we only need future operators to encode them.

We now show that any coSafety-FO formula can be translated into normal form, that is, into a *disjunction* of $\exists\forall$-formulas.

Lemma 3.9. *Any coSafety-FO formula is equivalent to a disjunction of $\exists\forall$-formulas.*

Proof. Let ϕ be a coSafety-FO formula. We proceed by structural induction on ϕ. For the base case, for each atomic formula $\phi(z_0, z_1)$ we provide an equivalent $\exists\forall$-formula $\psi(z_0, z_1)$:

(1) if $\phi = (z_0 < z_1)$ then $\psi := \exists x_0 \exists x_1 (z_0 = x_0 \wedge z_1 = x_1 \wedge x_0 < x_1)$;
(2) if $\phi = (z_0 = z_1)$, then $\psi := \exists x_0 (z_0 = x_0 \wedge z_1 = x_0)$.
(3) if $\phi = (z_0 \neq z_1)$, we can note that $\phi \equiv z_0 < z_1 \vee z_1 < z_0$ and then apply Item 1;
(4) if $\phi = P(z_0)$ then we define $\psi := \exists x_0 (z_0 = x_0 \wedge P(x_0))$. Similarly if $\phi = \neg P(z_0)$.

For the inductive step:

(1) The case of a disjunction is trivial.

(2) If $\phi(z_0, \ldots, z_k)$ is a conjunction, by the inductive hypothesis each conjunct is equivalent to a disjunction of $\exists\forall$-formulas. By distributing the conjunction over the disjunction we can reduce ourselves to the case of a conjunction $\psi_1(z_0, \ldots, z_k) \wedge \psi_2(z_0, \ldots, z_k)$ of two $\exists\forall$-formulas [1]. In this case we have that:

$$\psi_1 \equiv \exists x_0 \ldots \exists x_n \big(x_0 < \cdots < x_n \wedge z_0 = x_0 \wedge \ldots \big)$$
$$\psi_2 \equiv \exists x_{n+1} \ldots \exists x_m (x_{n+1} < \cdots < x_m \wedge z_0 = x_{n+1} \wedge \ldots)$$

Since the set of quantified variables in ψ_1 is disjoint from the set of quantified variables in ψ_2, we can distribute the existential quantifiers over the conjunction $\psi_1 \wedge \psi_2$, obtaining:

$$\psi_1 \wedge \psi_2 \equiv \exists x_0 \ldots \exists x_n \exists x_{n+1} \ldots \exists x_m$$
$$\big(x_0 < \cdots < x_n \wedge x_{n+1} < \cdots < x_m \wedge z_0 = x_0 \wedge z_0 = x_{n+1} \wedge \ldots \big)$$

Note that we can identify x_0 and x_{n+1}, obtaining:

$$\psi_1 \wedge \psi_2 \equiv \exists x_0 \ldots \exists x_n \exists x_{n+2}, \ldots \exists x_m$$
$$\big(x_0 < \cdots < x_n \wedge x_0 < x_{n+2} < \cdots < x_m \wedge$$
$$z_0 = x_0 \wedge \bigwedge_{i=1}^{k} (z_i = x_{j_i}) \wedge \bigwedge_{\substack{i=0, i \neq n+1}}^{m} \alpha_i(x_i) \wedge$$
$$\bigwedge_{\substack{i=1, i \neq n+1 \\ i \neq n+2}}^{m} \forall y(x_{i-1} < y < x_i \rightarrow \beta_i(y)) \wedge \forall y(x_0 < y < x_{n+2} \rightarrow \beta_{n+2})\big)$$

where $j_i \in \{0, \ldots, k\}$, for each $0 \leq i \leq k$. Now, to turn this formula into a disjunction of $\exists\forall$-formulas, we consider all the possible interleavings of the variables that respect the two imposed orderings and explode the formula into a disjunction that consider each such interleaving. Let $X = \{x_0, \ldots, x_n, x_{n+2}, \ldots, x_m\}$ and let Π be the set of all the permutations of X compatible with the orderings $x_0 < \cdots < x_n$ and $x_0 < x_{n+1} < \cdots < x_m$. For any $\pi \in \Pi$, $\pi(0) = x_0$. Now, $\psi_1 \wedge \psi_2$ becomes the disjunction of a set of $\exists\forall$-formulas ψ_π, for each $\pi \in \Pi$, defined as:

$$\psi_\pi \equiv \exists x_{\pi(0)} \ldots \exists x_{\pi(m)}$$
$$\big(x_{\pi(0)} < \cdots < x_{\pi(m)} \wedge$$
$$z_0 = x_0 \wedge \bigwedge_{i=1}^{k} (z_i = x_{\pi(j_i)}) \wedge \bigwedge_{i=0}^{m} \alpha_i(x_i) \wedge$$
$$\bigwedge_{i=0}^{m} \forall y(x_{\pi(i-1)} < y < x_{\pi(i)} \rightarrow \beta_i^*(y)))$$

where β_i^* suitably combines the formulas β according to the interleaving of the orderings of the original variables, and is defined as follows:

$$\beta_i^* = \begin{cases} \beta_{\pi(i)} & \text{if both } \pi(i), \pi(i-1) \leq n \text{ or both } \pi(i), \pi(i-1) > n \\ \beta_{\pi(i)} \wedge \beta_{\pi(i-1)} & \text{if } \pi(i) \leq n \text{ and } \pi(i-1) > n \text{ or } \textit{vice versa} \end{cases}$$

---

[1]Note that, without loss of generality, we can assume that ψ_1 and ψ_2 have the same free variables $z_1, \ldots, z_k$. In the case one of the two is not using a variable (say z_i), then its binding constraint will not bind any variable to z_i.

Then we have that $\psi_1 \wedge \psi_2 \equiv \bigvee_{\pi \in \Pi}(\psi_\pi)$, which is a disjunction of $\exists\forall$-formulas.

(3) Let $\phi(z_0, \ldots, z_m) = \exists z_{m+1} . (z_i < z_{m+1} \wedge \phi_1(z_0, \ldots, z_m, z_{m+1}))$, for some $0 \leq i \leq m$. By the inductive hypothesis, this is equivalent to the formula $\exists z_{m+1}(z_i < z_{m+1} \wedge \bigvee_{k=0}^{j} \psi_k(z_0, \ldots, z_m, z_{m+1}))$, where $\psi_k(z_0, \ldots, z_m, z_{m+1})$ is a $\exists\forall$-formula, for each $0 \leq k \leq j$, that is:

$$\exists z_{m+1} . (z_i < z_{m+1} \wedge \bigvee_{k=0}^{j} (\exists x_0 \ldots \exists x_{n_k} \psi'_k(z_0, \ldots, z_{m+1}, x_0, \ldots, x_{n_k})))$$

By distributing the conjunction over the disjunction, we obtain:

$$\exists z_{m+1} . (\bigvee_{k=0}^{j} ((z_i < z_{m+1}) \wedge \exists x_0 \ldots \exists x_{n_k} \psi'_k(z_0, \ldots, z_{m+1}, x_0, \ldots, x_{n_k})))$$

and by distributing the existential quantifier over the disjunction, we have:

$$\bigvee_{k=0}^{j} (\exists z_{m+1}((z_i < z_{m+1}) \wedge \exists x_0 \ldots \exists x_{n_k} \psi'_k(z_0, \ldots, z_{m+1}, x_0, \ldots, x_{n_k})))$$

Since the subformula $z_i < z_{m+1}$ does not contain the variables $x_0, \ldots, x_n$, we can push it inside the existential quantification, obtaining:

$$\bigvee_{k=0}^{j} (\exists z_{m+1} . \exists x_0 \ldots \exists x_{n_k} . ((z_i < z_{m+1}) \wedge \psi'_k(z_0, \ldots, z_{m+1}, x_0, \ldots, x_{n_k})))$$

Now we divide in cases:

(a) suppose that the formula $\psi'_k(z_0, \ldots, z_{m+1}, x_0, \ldots, x_{n_k})$ contains the following conjuncts: $z_i = x_{l_i}$ and $z_{m+1} = x_{l_{m+1}}$, with $l_i = l_{m+1}$. It holds that these formulas are in contradiction with the formula $z_i < z_{m+1}$, that is:

$$(z_i < z_{m+1}) \wedge (z_i = x_{l_i}) \wedge (z_{m+1} = x_{l_{m+1}}) \equiv \bot$$

Therefore, the disjunct $(z_i < z_{m+1}) \wedge \psi'_k(z_0, \ldots, z_{m+1}, x_0, \ldots, x_{n_k})$ is equivalent to $\bot$, and thus can be safely removed from the disjunction.

(b) suppose that the formula $\psi'_k(z_0, \ldots, z_{m+1}, x_0, \ldots, x_{n_k})$ contains the following conjuncts: $z_i = x_{l_i}$, $z_{m+1} = x_{l_{m+1}}$ (with $l_i \neq l_{m+1}$), and $x_{l_{m+1}} < \cdots < x_{l_i}$. As in the previous case, it holds that:

$$(z_i < z_{m+1}) \wedge (z_i = x_{l_i}) \wedge (z_{m+1} = x_{l_{m+1}}) \wedge (x_{l_{m+1}} < \cdots < x_{l_i}) \equiv \bot$$

Thus, also in this case, this disjunct can be safely removed from the disjunction.

(c) otherwise, it holds that the formula $\psi'_k(z_0, \ldots, z_{m+1}, x_0, \ldots, x_{n_k})$ contains the following conjuncts: $z_i = x_{l_i}$, $z_{m+1} = x_{l_{m+1}}$ (with $l_i \neq l_{m+1}$), and $x_{l_i} < \cdots < x_{l_{m+1}}$. Therefore, the subformula $z_i < z_{m+1}$ is redundant, and can be safely removed from $\psi'_k(z_0, \ldots, z_{m+1}, x_0, \ldots, x_{n_k})$. The resulting formula is a $\exists\forall$-formula.

After the previous transformation, we obtain:

$$\bigvee_{k=0}^{j'} (\exists z_{m+1} . \exists x_0 \ldots \exists x_{n_k} . \psi''_k(z_0, \ldots, z_{m+1}, x_0, \ldots, x_{n_k}))$$

Finally, since each formula $\psi_k''(z_0, \ldots, z_{m+1}, x_0, \ldots, x_{n_k})$ contains the conjunct $z_{m+1} = x_{l_{m+1}}$, we can safely remove the quantifier $\exists z_{m+1}$. We obtain the formula:

$$\bigvee_{k=0}^{j'} (\exists x_0 \ldots \exists x_{n_k} \cdot \psi_k''(z_0, \ldots, z_m, x_0, \ldots, x_{n_k}))$$

which is a disjunction of $\exists\forall$-formulas.

(4) Let $\phi(z_0, \ldots, z_m) = \forall z_{m+1}(z_i < z_{m+1} < z_j \rightarrow \phi_1(z_0, \ldots, z_m, z_{m+1}))$, for some $0 \leq i, j \leq m$. By the induction hypothesis we know that ϕ_1 is equivalent to a disjunction $\bigvee_k \psi_k$ where ψ_k are $\exists\forall$-formulas, i.e., each ψ_k is of the form:

$$\psi_k \equiv \exists x_0, \ldots, x_n \big(x_0 < \ldots < x_n \wedge z_0 = x_0 \wedge \bigwedge_{l=1}^{m+1} (z_l = x_{u_l}) \wedge$$

$$\bigwedge_{l=0}^{n} \alpha_l(x_l) \wedge \bigwedge_{l=1}^{n} \forall y(x_{l-1} < y < x_l \rightarrow \beta_l(y)))$$

Without loss of generality, we can suppose that z_i, z_{m+1} and z_j are binded to some variables x_{u_i}, $x_{u_{m+1}}$ and x_{u_j} that are ordered consecutively, i.e., $x_{u_i} < x_{u_{m+1}} < x_{u_j}$ with no other variable in between. That is because otherwise the ordering constraints and the binding constraints would be in conflict with the guard $z_i < z_{m+1} < z_j$ of the universal quantification, and the disjunct ψ_k could be removed from the disjunction. As a matter of fact, take for example a disjunct of $\bigvee_k \psi_k$ with ordering constraints inducing the order $z_i < z_h < z_{m+1}$, for some h. The existence of such a z_h is not guaranteed *for each* value of z_{m+1} between z_i and z_j because when $z_{m+1} = z_i + 1$ there is no value between z_i and $z_i + 1$ (we are on discrete time models), and thus such a disjunct can be safely removed from $\bigvee_k \psi_k$. That said, we can now isolate all the parts of ψ_k that talk about z_{m+1}, bringing them out of the existential quantification, obtaining $\psi_k \equiv \theta_k \wedge \eta_k$, where:

$$\theta_k \equiv z_i < z_{m+1} < z_j$$
$$\wedge \, \alpha(z_{m+1}) \wedge \forall y(z_i < y < z_{m+1} \rightarrow \beta(y)) \wedge \forall y(z_{m+1} < y < z_i \rightarrow \beta'(y))$$

$$\eta_k \equiv \exists x_0, \ldots, x_n \big(x_0 < \ldots < x_n \wedge z_0 = x_0 \wedge \bigwedge_{l=1}^{m} (z_l = x_{u_l}) \wedge$$

$$\bigwedge_{\substack{l=0 \\ l \neq u_{m+1}}}^{n} \alpha_l(x_l) \wedge \bigwedge_{\substack{l=1 \\ l-1 \neq u_i \\ l \neq u_j}}^{n} \forall y(x_{l-1} < y < x_l \rightarrow \beta_l(y)))$$

Now, we have $\phi \equiv \forall z_{m+1}(z_i < z_{m+1} < z_j \rightarrow \bigvee_k(\theta_k \wedge \eta_k))$. We can distribute the head of the implication over the disjunction:

$$\phi \equiv \forall z_{m+1}(\bigvee_k (z_i < z_{m+1} < z_j \rightarrow (\theta_k \wedge \eta_k)))$$

and then over the conjunction, obtaining:

$$\phi \equiv \forall z_{m+1}\big(\bigvee_k ((z_i < z_{m+1} < z_j \rightarrow \theta_k) \wedge (z_i < z_{m+1} < z_j \rightarrow \eta_k)))$$

In order to simplify the exposition, we now show how to proceed in the case of two disjuncts, which is easily generalizable. So suppose we have:

$$\phi \equiv \forall z_{m+1} \left(\begin{array}{l} (z_i < z_{m+1} < z_j \rightarrow \theta_1) \wedge (z_i < z_{m+1} < z_j \rightarrow \eta_1) \\ \vee \\ (z_i < z_{m+1} < z_j \rightarrow \theta_2) \wedge (z_i < z_{m+1} < z_j \rightarrow \eta_2) \end{array} \right)$$

We can a) distribute the disjunction over the conjunction (*i.e.*, convert in conjunctive normal form in the case of multiple disjuncts):

$$\phi \equiv \forall z_{m+1} \left(\begin{array}{l} ((z_i < z_{m+1} < z_j \rightarrow \theta_1) \vee (z_i < z_{m+1} < z_j \rightarrow \theta_2)) \\ \wedge ((z_i < z_{m+1} < z_j \rightarrow \theta_1) \vee (z_i < z_{m+1} < z_j \rightarrow \eta_2)) \\ \wedge ((z_i < z_{m+1} < z_j \rightarrow \eta_1) \vee (z_i < z_{m+1} < z_j \rightarrow \theta_2)) \\ \wedge ((z_i < z_{m+1} < z_j \rightarrow \eta_1) \vee (z_i < z_{m+1} < z_j \rightarrow \eta_2)) \end{array} \right)$$

b) factor out the head of the implications:

$$\phi \equiv \forall z_{m+1} \left(\begin{array}{l} (z_i < z_{m+1} < z_j \rightarrow \theta_1 \vee \theta_2) \\ \wedge (z_i < z_{m+1} < z_j \rightarrow \theta_1 \vee \eta_2) \\ \wedge (z_i < z_{m+1} < z_j \rightarrow \eta_1 \vee \theta_2) \\ \wedge (z_i < z_{m+1} < z_j \rightarrow \eta_1 \vee \eta_2) \end{array} \right)$$

and c) distribute the universal quantification over the conjunction, obtaining:

$$\phi \equiv \left(\begin{array}{l} \forall z_{m+1}(z_i < z_{m+1} < z_j \rightarrow \theta_1 \vee \theta_2) \\ \wedge \forall z_{m+1}(z_i < z_{m+1} < z_j \rightarrow \theta_1 \vee \eta_2) \\ \wedge \forall z_{m+1}(z_i < z_{m+1} < z_j \rightarrow \eta_1 \vee \theta_2) \\ \wedge \forall z_{m+1}(z_i < z_{m+1} < z_j \rightarrow \eta_1 \vee \eta_2) \end{array} \right)$$

Now, note that η_1 and η_2 do not contain z_{m+1} as a free variable, because we factored out all the parts mentioning z_{m+1} into θ_1 and θ_2 before. Therefore we can push them out from the universal quantifications, obtaining:

$$\phi \equiv \left(\begin{array}{l} \forall z_{m+1}(z_i < z_{m+1} < z_j \rightarrow \theta_1 \vee \theta_2) \\ \wedge \forall z_{m+1}(z_i < z_{m+1} < z_j \rightarrow \theta_1) \vee \eta_2 \\ \wedge \forall z_{m+1}(z_i < z_{m+1} < z_j \rightarrow \theta_2) \vee \eta_1 \\ \wedge \neg \exists z_{m+1}(z_i < z_{m+1} < z_j) \vee \eta_1 \vee \eta_2 \end{array} \right)$$

Now, note that $\neg \exists z_{m+1}(z_i < z_{m+1} < z_j)$ is equivalent to $z_i = z_j \vee z_j = z_i + 1$, which is the disjunction of two formulas that can be turned into $\exists \forall$-formulas. Since both η_1 and η_2 are already $\exists \forall$-formulas and since we already know how to deal with conjunctions and disjunctions of $\exists \forall$-formulas, it remains to show that the universal quantifications in the formula above can be turned into $\exists \forall$-formulas. Take $\forall z_{m+1}(z_i < z_{m+1} < z_j \rightarrow \theta_1)$, *i.e.*:

$$\forall z_{m+1} \left(z_i < z_{m+1} < z_j \rightarrow \begin{array}{l} z_i < z_{m+1} < z_j \\ \wedge \alpha(z_{m+1}) \\ \wedge \forall y(z_i < y < z_{m+1} \rightarrow \beta(y)) \\ \wedge \forall y(z_{m+1} < y < z_j \rightarrow \beta'(y)) \end{array} \right)$$

Note that the first conjunct of the consequent can be removed, since it is redundant. Now, this formula is requesting $\beta(y)$ for all y between z_i and z_{m+1}, but with z_{m+1} that

ranges between z_i and $z_j - 1$, hence effectively requesting $\beta(y)$ to hold between z_i and z_j. Similarly for $\beta'(y)$, which has to hold for all y between $z_i + 1$ and z_j.

Hence, it is equivalent to:

$$
\begin{aligned}
& z_i = z_j \\
& \vee\, z_j = z_i + 1 \\
& \vee\, \exists x_{i+1}(z_i < x_{i+1} \wedge x_{i+1} = z_i + 1 \wedge z_j = x_{i+1} + 1 \wedge \alpha(x_{i+1})) \\
& \vee\, \exists x_i \exists x_{i+1} \exists x_{j-1} \exists x_j
\left(
\begin{array}{l}
x_i < x_{i+1} < x_{j-1} < x_j \\
\wedge\, z_i = x_i \wedge z_j = x_j \\
\wedge\, \alpha(x_{i+1}) \wedge \alpha(x_{j-1}) \\
\wedge\, \forall y(x_i < y < x_{i+1} \to \bot) \\
\wedge\, \forall y(x_{j-1} < y < x_j \to \bot) \\
\wedge\, \forall y(x_i < y < x_{j-1} \to \alpha(y) \wedge \beta(y)) \\
\wedge\, \forall y(x_{i+1} < y < x_j \to \alpha(y) \wedge \beta'(y))
\end{array}
\right)
\end{aligned}
$$

which is a disjunction of a $\exists\forall$-formula and others that can be turned into disjunctions of $\exists\forall$-formulas. The reasoning is at all similar for $\forall z_{m+1}(z_i < z_{m+1} < z_j \to \theta_1 \vee \theta_2)$. $\qquad\square$

Any coSafety-FO formula can be translated into a disjunction of $\exists\forall$-formulas by Lemma 3.9, and then to a coSafety-LTL$(-\widetilde{X})$ formula by Lemma 3.8. Together with Lemma 3.6, we obtain the following.

Corollary 3.10. $[\![\mathsf{coSafety\text{-}FO}]\!]^{<\omega} = [\![\mathsf{coSafety\text{-}LTL}(-\widetilde{X})]\!]^{<\omega}$

Moreover, Corollary 3.10 and Lemmas 3.4 and 3.5, imply the following corollary.

Corollary 3.11. *It holds that:*
- $[\![\mathsf{coSafety\text{-}FO}]\!]^{<\omega} = [\![\mathsf{coSafety\text{-}FO}]\!]^{<\omega} \cdot (2^\Sigma)^*$
- $[\![\mathsf{coSafety\text{-}FO}]\!] = [\![\mathsf{coSafety\text{-}FO}]\!]^{<\omega} \cdot (2^\Sigma)^\omega$

We are now ready to state the main result of this section.

Theorem 3.12. $[\![\mathsf{coSafety\text{-}LTL}]\!] = [\![\mathsf{coSafety\text{-}FO}]\!]$

Proof. We know that $[\![\mathsf{coSafety\text{-}LTL}]\!] = [\![\mathsf{coSafety\text{-}LTL}(-\widetilde{X})]\!]^{<\omega} \cdot (2^\Sigma)^\omega$ by Observation 3.3 and Lemma 3.5. Since $[\![\mathsf{coSafety\text{-}LTL}(-\widetilde{X})]\!]^{<\omega} = [\![\mathsf{coSafety\text{-}FO}]\!]^{<\omega}$ by Corollary 3.10, we have that $[\![\mathsf{coSafety\text{-}LTL}(-\widetilde{X})]\!]^{<\omega} \cdot (2^\Sigma)^\omega = [\![\mathsf{coSafety\text{-}FO}]\!]^{<\omega} \cdot (2^\Sigma)^\omega$. Then, by Corollary 3.11 we have that $[\![\mathsf{coSafety\text{-}FO}]\!]^{<\omega} \cdot (2^\Sigma)^\omega = [\![\mathsf{coSafety\text{-}FO}]\!]$, hence $[\![\mathsf{coSafety\text{-}LTL}]\!] = [\![\mathsf{coSafety\text{-}FO}]\!]$. $\qquad\square$

Corollary 3.13. $[\![\mathsf{Safety\text{-}LTL}]\!] = [\![\mathsf{Safety\text{-}FO}]\!]$

4. Safety-FO captures LTL-definable safety languages

In this section, we prove that coSafety-FO captures LTL-definable co-safety languages. By duality, we have that Safety-FO captures LTL-definable safety languages, and by the equivalence shown in the previous section, this provides a novel proof of the fact that Safety-LTL captures LTL-definable safety languages. We start by characterizing co-safety languages in terms of LTL over finite words.

Lemma 4.1. $[\![\mathsf{LTL}]\!] \cap \mathsf{coSAFETY} = [\![\mathsf{LTL}]\!]^{<\omega} \cdot (2^\Sigma)^\omega$

Proof. ($\subseteq$) By Proposition 2.6 we know that each language $\mathcal{L} \in [\![\mathsf{LTL}]\!] \cap \mathsf{coSAFETY}$ is definable by a formula of the form $\mathsf{F}\alpha$ where $\alpha \in \mathsf{LTL_P}$. Hence for each $\sigma \in \mathcal{L}$ there exists an n such that $\sigma, n \models \alpha$, hence $\sigma_{[0,n]}, n \models \alpha$. Note that $\sigma_{[n+1,\infty]}$ is unconstrained. By replacing all the *since/yesterday/weak yesterday* operators in α with *until/tomorrow/weak tomorrow* operators, we obtain an LTL formula α^r such that $(\sigma_{[0,n]})^r, 0 \models \alpha^r$ (where σ^r is the reverse of σ). Since LTL captures star-free languages [LPZ85] and star-free languages are closed by reversal, there is also an LTL formula β such that $\sigma_{[0,n]}, 0 \models \beta$. Hence $\mathcal{L} = \mathcal{L}^{<\omega}(\beta) \cdot (2^\Sigma)^\omega$, and we proved that $[\![\mathsf{LTL}]\!] \cap \mathsf{coSAFETY} \subseteq [\![\mathsf{LTL}]\!]^{<\omega} \cdot (2^\Sigma)^\omega$.

($\supseteq$) Given $\mathcal{L} \in [\![\mathsf{LTL}]\!]^{<\omega} \cdot (2^\Sigma)^\omega$, we know $\mathcal{L} = \mathcal{L}^{<\omega}(\beta) \cdot (2^\Sigma)^\omega$ for some LTL formula β. Hence, for each $\sigma \in \mathcal{L}$ there is an n such that $\sigma_{[0,n]}, 0 \models \beta$. Since LTL captures star-free languages and star-free languages are closed by reversal, there is an LTL formula α^r such that $(\sigma_{[0,n]})^r, 0 \models \alpha^r$. Now, by replacing all the *until/tomorrow/weak tomorrow* operators in α^r with *since/yesterday/weak yesterday* operators, we obtain an $\mathsf{LTL_P}$ formula α such that $\sigma_{[0,n]}, n \models \alpha$. Hence, σ is such that there is an n such that $\sigma, n \models \alpha$, *i.e.*, $\sigma \models \mathsf{F}\alpha$. Therefore, by Proposition 2.6, $\mathcal{L} \in [\![\mathsf{LTL}]\!] \cap \mathsf{coSAFETY}$, and this in turn implies that $[\![\mathsf{LTL}]\!]^{<\omega} \cdot (2^\Sigma)^\omega \subseteq [\![\mathsf{LTL}]\!] \cap \mathsf{coSAFETY}$. $\square$

Now, we show that, over finite words, the *release* and the *globally* modalities can be defined only in terms of the *weak tomorrow*, the *until* and the *eventually* modalities. Similarly, we also show that, over finite trace, the *until* and the *eventually* operators can be defined only in terms of the *tomorrow*, the *release* and the *globally* modalities.

Lemma 4.2. $[\![\mathsf{LTL}]\!]^{<\omega} = [\![\mathsf{Safety\text{-}LTL}]\!]^{<\omega} = [\![\mathsf{coSafety\text{-}LTL}]\!]^{<\omega}$

Proof. Since $\mathsf{Safety\text{-}LTL}$ and $\mathsf{coSafety\text{-}LTL}$ are fragments of LTL, we only need to show one direction, *i.e.*, that $[\![\mathsf{LTL}]\!]^{<\omega} \subseteq [\![\mathsf{Safety\text{-}LTL}]\!]^{<\omega}$ and $[\![\mathsf{LTL}]\!]^{<\omega} \subseteq [\![\mathsf{coSafety\text{-}LTL}]\!]^{<\omega}$. At first, we show the case of $\mathsf{coSafety\text{-}LTL}$. For each LTL formula ϕ, we can build a $\mathsf{coSafety\text{-}LTL}$ formula whose language over finite words is exactly $\mathcal{L}^{<\omega}(\phi)$. The *globally* operator can be replaced by means of an *until* operator whose existential part always refers to the last position of the word. In turn, this can be done with the formula $\widetilde{\mathsf{X}}\bot$, which is true only at the final position:

$$\mathsf{G}\phi \equiv \phi\,\mathcal{U}\,(\phi \wedge \widetilde{\mathsf{X}}\bot)$$

Similarly, the *release* operator can be expressed by means of a *globally* operator in disjunction with an *until* operator:

$$\phi_1\,\mathcal{R}\,\phi_2 \equiv \mathsf{G}\phi_2 \vee (\phi_2\,\mathcal{U}\,(\phi_1 \wedge \phi_2)) \equiv \big(\phi_2\,\mathcal{U}\,(\phi_2 \wedge \widetilde{\mathsf{X}}\bot)\big) \vee \big(\phi_2\,\mathcal{U}\,(\phi_1 \wedge \phi_2)\big)$$

Hence, $[\![\mathsf{LTL}]\!]^{<\omega} = [\![\mathsf{coSafety\text{-}LTL}]\!]^{<\omega}$. Now, if we exploit the duality between the *eventually/until* and the *globally/release* operators, we obtain:

$$\mathsf{F}\phi \equiv \phi\,\mathcal{R}\,(\phi \vee \mathsf{X}\top)$$
$$\phi_1\,\mathcal{U}\,\phi_2 \equiv \phi_2\,\mathcal{R}\,(\phi_2 \vee \mathsf{X}\top) \wedge \phi_2\,\mathcal{R}\,(\phi_1 \vee \phi_2)$$

Hence, since we showed that any *eventually* operator and any *until* operator can be defined only in terms of the *tomorrow*, the *globally*, and the *release* operators, we have that $[\![\mathsf{LTL}]\!]^{<\omega} = [\![\mathsf{Safety\text{-}LTL}]\!]^{<\omega}$. $\square$

Then, we relate $\mathsf{coSafety\text{-}LTL}$ on finite words and $\mathsf{coSafety\text{-}FO}$.

Lemma 4.3. $[\![\mathsf{coSafety\text{-}LTL}]\!]^{<\omega} \cdot (2^\Sigma)^\omega = [\![\mathsf{coSafety\text{-}FO}]\!]$

Proof. ($\subseteq$) We have that $[\![\mathsf{coSafety\text{-}LTL}]\!]^{<\omega} = [\![\mathsf{LTL}]\!]^{<\omega}$ by Lemma 4.2, and this implies that $[\![\mathsf{coSafety\text{-}LTL}]\!]^{<\omega} \cdot (2^\Sigma)^\omega = [\![\mathsf{LTL}]\!]^{<\omega} \cdot (2^\Sigma)^\omega$, and $[\![\mathsf{coSafety\text{-}LTL}]\!]^{<\omega} \cdot (2^\Sigma)^\omega = [\![\mathsf{FO}]\!]^{<\omega} \cdot (2^\Sigma)^\omega$ by Proposition 2.5. Now, let $\phi \in \mathsf{FO}$, and suppose *w.l.o.g.* that ϕ is in *negated normal form*. We define the formula $\phi'(x, y)$, where x and y are two fresh variables that do not occur in ϕ, as the formula obtained from ϕ by a) replacing each subformula of ϕ of type $\exists z \phi_1$ with $\exists z(x \leq z \wedge \phi_1)$, and b) by replacing each subformula of ϕ of type $\forall z \phi_1$ with $\forall z(x \leq z < y \rightarrow \phi_1)$. Now, consider the formula $\psi = \exists y(x \leq y \wedge \phi'(x, y))$. Note that ψ is a $\mathsf{coSafety\text{-}FO}$ formula. When interpreted over *infinite* words, the models of ψ are exactly those containing a prefix that belongs to $\mathcal{L}^{<\omega}(\phi)$, with the remaining suffix unconstrained, that is $\mathcal{L}(\psi) = \mathcal{L}^{<\omega}(\phi) \cdot (2^\Sigma)^\omega$, hence $[\![\mathsf{FO}]\!]^{<\omega} \cdot (2^\Sigma)^\omega \subseteq [\![\mathsf{coSafety\text{-}FO}]\!]$, and this implies that $[\![\mathsf{coSafety\text{-}LTL}]\!]^{<\omega} \cdot (2^\Sigma)^\omega \subseteq [\![\mathsf{coSafety\text{-}FO}]\!]$.

($\supseteq$) We know by Corollary 3.11 that $[\![\mathsf{coSafety\text{-}FO}]\!] = [\![\mathsf{coSafety\text{-}FO}]\!]^{<\omega} \cdot (2^\Sigma)^\omega$. Since $\mathsf{coSafety\text{-}FO}$ formulas are also FO formulas, we have $[\![\mathsf{coSafety\text{-}FO}]\!] \subseteq [\![\mathsf{FO}]\!]^{<\omega} \cdot (2^\Sigma)^\omega$. By Proposition 2.5 and Lemma 4.2, we obtain that $[\![\mathsf{coSafety\text{-}FO}]\!] \subseteq [\![\mathsf{coSafety\text{-}LTL}]\!]^{<\omega} \cdot (2^\Sigma)^\omega$. □

We are ready now to state the main result.

Theorem 4.4. $[\![\mathsf{LTL}]\!] \cap \mathsf{coSAFETY} = [\![\mathsf{coSafety\text{-}FO}]\!]$

Proof. We know that $[\![\mathsf{LTL}]\!] \cap \mathsf{coSAFETY} = [\![\mathsf{LTL}]\!]^{<\omega} \cdot (2^\Sigma)^\omega$ by Lemma 4.1. Then, by Lemma 4.2 we know that $[\![\mathsf{LTL}]\!]^{<\omega} = [\![\mathsf{coSafety\text{-}LTL}]\!]^{<\omega}$, and this in turn implies that $[\![\mathsf{LTL}]\!]^{<\omega} \cdot (2^\Sigma)^\omega = [\![\mathsf{coSafety\text{-}LTL}]\!]^{<\omega} \cdot (2^\Sigma)^\omega$. Since $[\![\mathsf{coSafety\text{-}LTL}]\!]^{<\omega} \cdot (2^\Sigma)^\omega = [\![\mathsf{coSafety\text{-}FO}]\!]$ by Lemma 4.3, we conclude that $[\![\mathsf{LTL}]\!] \cap \mathsf{coSAFETY} = [\![\mathsf{coSafety\text{-}FO}]\!]$. □

This result together with Theorem 3.12 allow us to conclude the following.

Theorem 4.5. $[\![\mathsf{Safety\text{-}LTL}]\!] = [\![\mathsf{LTL}]\!] \cap \mathsf{SAFETY}$

Note that by Observation 3.3 and Lemma 3.5 on one hand, and by Lemmas 4.1 and 4.2 on the other, the question of whether $[\![\mathsf{Safety\text{-}LTL}]\!] = [\![\mathsf{LTL}]\!] \cap \mathsf{SAFETY}$ can be reduced to whether $[\![\mathsf{coSafety\text{-}LTL}]\!]^{<\omega} \cdot (2^\Sigma)^\omega = [\![\mathsf{coSafety\text{-}LTL}(-\widetilde{\mathsf{X}})]\!]^{<\omega} \cdot (2^\Sigma)^\omega$. If $\mathsf{coSafety\text{-}LTL}$ and $\mathsf{coSafety\text{-}LTL}(-\widetilde{\mathsf{X}})$ were equivalent over finite words, this would already prove Theorem 4.5. However, the next theorem states that this is not the case.

Theorem 4.6. $[\![\mathsf{coSafety\text{-}LTL}]\!]^{<\omega} \neq [\![\mathsf{coSafety\text{-}LTL}(-\widetilde{\mathsf{X}})]\!]^{<\omega}$

Proof. Note that in $\mathsf{coSafety\text{-}LTL}(-\widetilde{\mathsf{X}})$ we cannot hook the final position of the word without the *weak tomorrow* operator. For these reasons, given a $\mathsf{coSafety\text{-}LTL}(-\widetilde{\mathsf{X}})$ formula ϕ, with a simple structural induction we can prove that for each $\sigma \in (2^\Sigma)^+$ such that $\sigma \models \phi$, it holds that $\sigma\sigma' \models \phi$ for any $\sigma' \in (2^\Sigma)^+$, *i.e.*, all the extensions of σ satisfy ϕ as well. This implies that $\mathcal{L}^{<\omega}(\phi)$ is either empty (*i.e.*, if ϕ is unsatisfiable) or infinite. Instead, by using the *weak tomorrow* operator to hook the last position of the word, we can describe a finite non-empty language, for example as in the formula $\phi = a \wedge \mathsf{X}(a \wedge \widetilde{\mathsf{X}}\bot)$. The language of ϕ is $\mathcal{L}(\phi) = \{\mathsf{aa}\}$, including exactly one word, hence $\mathcal{L}(\phi)$ cannot be described without the *weak tomorrow* operator. □

Note that Theorem 4.6 does *not* contradict Theorem 4.5, that is, it does not imply that $[\![\mathsf{coSafety\text{-}LTL}]\!]^{<\omega} \cdot (2^\Sigma)^\omega \neq [\![\mathsf{coSafety\text{-}LTL}(-\widetilde{\mathsf{X}})]\!]^{<\omega} \cdot (2^\Sigma)^\omega$. For example, consider again the formula $a \wedge \mathsf{X}(a \wedge \widetilde{\mathsf{X}}\bot)$. It cannot be expressed without the *weak tomorrow* operator, yet it holds that: $\mathcal{L}^{<\omega}(a \wedge \mathsf{X}(a \wedge \widetilde{\mathsf{X}}\bot)) \cdot (2^\Sigma)^\omega = \mathcal{L}^{<\omega}(a \wedge \mathsf{X}a) \cdot (2^\Sigma)^\omega$.

5. The (co)safety fragment of LTL over finite words

So far, we focused primarily on safety and co-safety languages of infinite words. Naturally, safety and co-safety languages of *finite words* deserve attention as well. In this section, we define the notion of (co-)safety languages of finite words and we prove that coSafety-LTL($-\widetilde{X}$) (resp., Safety-LTL($-X$)), *i.e.*, the logic obtained from coSafety-LTL (resp., Safety-LTL) by forbidding the $\widetilde{X}$ (resp., the X) operator, captures the set of co-safety (resp., safety) languages of LTL interpreted over finite words.

We start with the definitions of safety and co-safety languages of finite words, which (unsurprisingly) are the natural restriction of Definitions 2.1 and 2.2 to finite words.

Definition 5.1. Let $\mathcal{L} \subseteq A^*$ be a language of finite words. We say that $\mathcal{L}$ is a *safety language* if and only if for all the words $\sigma \in A^*$ it holds that, if $\sigma \notin \mathcal{L}$, then there exists an $i < |\sigma|$ such that, for all $\sigma' \in A^*$, $\sigma_{[0,i]} \cdot \sigma' \notin \mathcal{L}$. The class of safety languages of finite words is denoted as $\mathsf{SAFETY}^{<\omega}$.

Definition 5.2. Let $\mathcal{L} \subseteq A^*$ be a language of finite words. We say that $\mathcal{L}$ is a *co-safety language* if and only if for all the words $\sigma \in A^*$ it holds that, if $\sigma \in \mathcal{L}$, then there exists an $i < |\sigma|$ such that, for all $\sigma' \in A^*$, $\sigma_{[0,i]} \cdot \sigma' \in \mathcal{L}$. The class of co-safety languages of finite words is denoted as $\mathsf{coSAFETY}^{<\omega}$.

The remaining part of the section is devoted to the proof of the following theorem, which gives two characterizations of the safety and co-safety fragments of LTL over finite words, one in terms of temporal logics and one in terms of first-order logics.

Theorem 5.3. *It holds that:*
- $[\![\mathsf{LTL}]\!]^{<\omega} \cap \mathsf{coSAFETY}^{<\omega} = [\![\mathsf{coSafety\text{-}LTL}(-\widetilde{X})]\!]^{<\omega} = [\![\mathsf{coSafety\text{-}FO}]\!]^{<\omega}$
- $[\![\mathsf{LTL}]\!]^{<\omega} \cap \mathsf{SAFETY}^{<\omega} = [\![\mathsf{Safety\text{-}LTL}(-X)]\!]^{<\omega} = [\![\mathsf{Safety\text{-}FO}]\!]^{<\omega}$

We first prove the following auxiliary lemma.

Lemma 5.4. *For any formula $\phi(x) \in \mathsf{FO}$ with one free variable, there exists a formula $\phi'(x) \in \mathsf{coSafety\text{-}FO}$ such that $\mathcal{L}^{<\omega}(\phi'(x)) = \mathcal{L}^{<\omega}(\phi(x)) \cdot (2^\Sigma)^*$.*

Proof. Let $\phi(x)$ be a formula in FO in negation normal form with one free variable. We define $\phi'(x)$ as the formula $\exists y \, . \, (x \le y \wedge \psi(x,y))$, where $\psi(x,y)$ is the formula with free variables x and y (where y is a fresh variable that does not appear in $\phi(x)$) obtained from $\phi(x)$ by replacing each subformula of type $\exists z \, . \, \phi_1$ with $\exists z \, . \, (x \le z < y \wedge \phi_1)$ and each subformula of type $\forall z \, . \, \phi_1$ with $\forall z \, . \, (x \le z < y \to \phi_1)$. It is simple to see that $\phi'(x) \in \mathsf{coSafety\text{-}FO}$ and $\mathcal{L}^{<\omega}(\phi'(x)) = \mathcal{L}^{<\omega}(\phi(x)) \cdot (2^\Sigma)^*$. $\qquad\square$

We now prove that coSafety-FO (interpreted over finite words) captures $[\![\mathsf{FO}]\!]^{<\omega} \cdot (2^\Sigma)^*$, as stated by the following Lemma.

Lemma 5.5. $[\![\mathsf{coSafety\text{-}FO}]\!]^{<\omega} = [\![\mathsf{FO}]\!]^{<\omega} \cdot (2^\Sigma)^*$.

Proof. We first prove the inclusion $[\![\mathsf{coSafety\text{-}FO}]\!]^{<\omega} \subseteq [\![\mathsf{FO}]\!]^{<\omega} \cdot (2^\Sigma)^*$. By Corollary 3.11, it holds that $[\![\mathsf{coSafety\text{-}FO}]\!]^{<\omega} = [\![\mathsf{coSafety\text{-}FO}]\!]^{<\omega} \cdot (2^\Sigma)^*$. Since coSafety-FO is a syntactic fragment of FO, it also holds that $[\![\mathsf{coSafety\text{-}FO}]\!]^{<\omega} \subseteq [\![\mathsf{FO}]\!]^{<\omega}$. It follows that $[\![\mathsf{coSafety\text{-}FO}]\!]^{<\omega} \subseteq [\![\mathsf{FO}]\!]^{<\omega} \cdot (2^\Sigma)^*$.

We now prove the inclusion $[\![\mathsf{FO}]\!]^{<\omega} \cdot (2^\Sigma)^* \subseteq [\![\mathsf{coSafety\text{-}FO}]\!]^{<\omega}$. Let ϕ be a formula of FO. By Lemma 5.4, there exists a formula $\phi'(x)$ such that $\mathcal{L}^{<\omega}(\phi'(x)) = \mathcal{L}^{<\omega}(\phi) \cdot (2^\Sigma)^*$.

Since $\mathcal{L}^{<\omega}(\phi) \cdot (2^\Sigma)^* \in \llbracket\mathsf{FO}\rrbracket^{<\omega} \cdot (2^\Sigma)^*$ and $\mathcal{L}^{<\omega}(\phi'(x)) \in \llbracket\mathsf{coSafety\text{-}FO}\rrbracket^{<\omega}$, this proves that $\llbracket\mathsf{FO}\rrbracket^{<\omega} \cdot (2^\Sigma)^* \subseteq \llbracket\mathsf{coSafety\text{-}FO}\rrbracket^{<\omega}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

We can now prove that coSafety-FO and coSafety-LTL($-\widetilde{\mathsf{X}}$) capture the co-safety fragment of LTL interpreted over finite words, *i.e.*, $\llbracket\mathsf{LTL}\rrbracket^{<\omega} \cap \mathsf{coSAFETY}$. By dualization, it also holds that Safety-FO and Safety-LTL($-\mathsf{X}$) are characterizations of the safety fragment of LTL over finite words in terms of temporal logics and first-order logics, respectively.

Theorem 5.3. *It holds that:*
- $\llbracket\mathsf{LTL}\rrbracket^{<\omega} \cap \mathsf{coSAFETY}^{<\omega} = \llbracket\mathsf{coSafety\text{-}LTL}(-\widetilde{\mathsf{X}})\rrbracket^{<\omega} = \llbracket\mathsf{coSafety\text{-}FO}\rrbracket^{<\omega}$
- $\llbracket\mathsf{LTL}\rrbracket^{<\omega} \cap \mathsf{SAFETY}^{<\omega} = \llbracket\mathsf{Safety\text{-}LTL}(-\mathsf{X})\rrbracket^{<\omega} = \llbracket\mathsf{Safety\text{-}FO}\rrbracket^{<\omega}$

Proof. We first prove the case for the co-safety fragment. The following equivalences are true:

$$\llbracket\mathsf{LTL}\rrbracket^{<\omega} \cap \mathsf{coSAFETY}^{<\omega}$$
$$= \quad \llbracket\mathsf{FO}\rrbracket^{<\omega} \cdot (2^\Sigma)^* \qquad\qquad\qquad\qquad \text{by Propositions 2.3, 2.5 and 2.6}$$
$$= \quad \llbracket\mathsf{coSafety\text{-}FO}\rrbracket^{<\omega} \qquad\qquad\qquad\qquad\qquad\qquad \text{by Lemma 5.5}$$
$$= \quad \llbracket\mathsf{coSafety\text{-}LTL}(-\widetilde{\mathsf{X}})\rrbracket^{<\omega} \qquad\qquad\qquad\qquad \text{by Corollary 3.10}$$

Exploiting the duality between safety and co-safety fragments, one can directly obtain the proof for the safety case. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

6. Comparison with related fragments

In this section, we compare coSafety-FO with two related fragments, that is coSafety-LTL and EB-FO, another first-order logic characterization of LTL-definable co-safety properties. We also point out a practical application of the translation of coSafety-LTL formulas into coSafety-FO. As before, all the results can be dualized to the safety case.

6.1. **Succinctness of** coSafety-FO **with respect to** coSafety-LTL. We show that there exists an equivalence-preserving translation from coSafety-LTL into coSafety-FO that involves only a linear blowup.

Proposition 6.1. *For all $\phi \in$ coSafety-LTL, there exists $\phi' \in$ coSafety-FO such that:* (i) $\mathcal{L}(\phi) = \mathcal{L}(\phi')$*; and* (ii) $|\phi'| \in \mathcal{O}(|\phi|)$*.*

Proof. The transformation of coSafety-LTL into coSafety-FO is the same as the transformation of coSafety-LTL($-\widetilde{\mathsf{X}}$) into coSafety-FO maintaing the equivalence over finite words (see Lemma 3.6). For sake of clarity, we report here the transformation. We inductively define the formula $FO(\phi, x)$, where x is a variable, as follows:
- $FO(p, x) = P(x)$, for each $p \in \Sigma$
- $FO(\neg p, x) = \neg P(x)$, for each $p \in \Sigma$
- $FO(\phi_1 \wedge \phi_2, x) = FO(\phi_1, x) \wedge FO(\phi_2, x)$
- $FO(\phi_1 \vee \phi_2, x) = FO(\phi_1, x) \vee FO(\phi_2, x)$
- $FO(\mathsf{X}\phi_1, x) = \exists y(x < y \wedge y = x + 1 \wedge FO(\phi_1, y))$
  where $y = x + 1$ can be expressed as $\forall z(x < z < y \rightarrow \bot)$.
- $FO(\phi_1 \, \mathcal{U} \, \phi_2, x) = \exists y(x \leq y \wedge FO(\phi_2, y) \wedge \forall z(x \leq z < y \rightarrow FO(\phi_1, z)))$

For each $\phi \in \mathsf{coSafety\text{-}LTL}(-\widetilde{\mathsf{X}})$, the formula $FO(\phi, x)$ has exactly one free variable x. By the semantics of the operators in $\mathsf{coSafety\text{-}LTL}$, it is immediate to see that for all infinite state sequences $\sigma \in (2^\Sigma)^\omega$, it holds that $\sigma \models \phi$ if and only if $(\sigma)^s, 0 \models FO(\phi, x)$, and $FO(\phi, x) \in \mathsf{coSafety\text{-}FO}$. Therefore, $\mathcal{L}(FO(\phi, x)) \in [\![\mathsf{coSafety\text{-}FO}]\!]$.

Now, we study the size of $FO(\phi, x)$ in terms of the size of ϕ. From now on, let $n = |\phi|$. If ϕ is an atomic formula, then $FO(\phi, x)$ is of constant size. If instead $\phi \equiv \mathsf{X}\phi_1$, then $|FO(\phi, x)| = \mathcal{O}(1) + |\phi_1|$. Otherwise, if $\phi \equiv \phi_1 \vee \phi_2$ or $\phi \equiv \phi_1 \wedge \phi_2$ or $\phi \equiv \phi_1 \mathcal{U} \phi_2$, then without loss of generality we can suppose that $|\phi_1| = |\phi_2| = \lceil \frac{|\phi|-1}{2} \rceil$ and thus $|FO(\phi, x)| = \mathcal{O}(1) + 2 \cdot |FO(\phi_1, x)|$.

Therefore, the size of $FO(\phi, x)$ is described by the following recurrence equation:

$$S(n) = \begin{cases} \mathcal{O}(1) & \text{if } n = 1 \\ \max\{\mathcal{O}(1) + 2 \cdot S(\frac{n}{2}), \mathcal{O}(1) + S(n-1)\} & \text{otherwise} \end{cases}$$

We have that:

$$S(n) \le (2^i \cdot S(\frac{n}{2^i}) + i \cdot \mathcal{O}(1)) + (S(n-1-j) + j \cdot \mathcal{O}(1))$$

For $i = \log_2(n)$ and for $j = n - 2$, we obtain:

$$S(n) \le 2^{\log_2(n)} \cdot S(\frac{n}{2^{\log_2(n)}}) + \log_2(n) \cdot \mathcal{O}(1) + S(n-1-n+2) + (n-2) \cdot \mathcal{O}(1)$$
$$\le n \cdot S(1) + \mathcal{O}(\log_2(n)) + S(1) + \mathcal{O}(n)$$
$$\le n \cdot \mathcal{O}(1) + \mathcal{O}(\log_2(n)) + \mathcal{O}(1) + \mathcal{O}(n)$$
$$\in \mathcal{O}(n)$$

Therefore $|FO(\phi, x)| \in \mathcal{O}(|\phi|)$. ∎

Of course, also in this case, the result can be dualized, having that for all $\phi \in \mathsf{Safety\text{-}LTL}$, there exists $\phi' \in \mathsf{Safety\text{-}FO}$ such that: (i) $\mathcal{L}(\phi) = \mathcal{L}(\phi')$; and (ii) $|\phi'| \in \mathcal{O}(|\phi|)$.

The other direction of Proposition 6.1 is less obvious. The translation of any $\mathsf{coSafety\text{-}FO}$ formula into an equivalent one in $\mathsf{coSafety\text{-}LTL}$ described in this paper (Section 3) follows two main steps: (i) the transformation of $\mathsf{coSafety\text{-}FO}$ into normal form (Lemma 3.9); (ii) the transformation of the normal form to $\mathsf{coSafety\text{-}LTL}$ (Lemma 3.8). While the second step requires only a linear size increase, the first step, in the general case, can produce a formula of nonelementary size with respect to the size of the initial formula. This is mainly due to how the case of *conjunctions* is managed by the proof of Lemma 3.9: the resulting formula, in this case, contains a subformula for each interleaving π in the set of all possible interleavings Π; since this set is exponentially larger than the size of the starting formula, the formula resulting from the case of conjunctions causes an exponential blow-up in the worst case. As a consequence, the equivalence-preserving translation from $\mathsf{coSafety\text{-}FO}$ to $\mathsf{coSafety\text{-}LTL}$ shown in this paper is nonelementary in the size of the final formula. Of course, this gives an upper bound to the succinctness of $\mathsf{coSafety\text{-}FO}$ with respect to $\mathsf{coSafety\text{-}LTL}$: a still open question is about the lower bound, in particular whether there exists a translation from any $\mathsf{coSafety\text{-}FO}$ formula to an equivalent $\mathsf{coSafety\text{-}LTL}$ one of polynomial size.

6.2. **A practical feedback of $\mathsf{coSafety\text{-}FO}$.** Interestingly, the succinctness of $\mathsf{coSafety\text{-}FO}$ with respect to $\mathsf{coSafety\text{-}LTL}$ described in Section 6.1 has a practical feedback in the context of realizability and reactive synthesis.

Given a formula in LTL over a set of *controllable* and *uncontrollable* variables, realizability is the problem of establishing whether, given any sequence Unc of uncontrollable variables,

there exists a strategy s choosing the value of the controllable variables in such a way to guarantee that any sequence generated by s responding to Unc is a model of the initial formula. Reactive Synthesis is the problem of computing such a strategy (if any).

In [ZTL$^+$17], Zhu *et al.* consider the realizability from Safety-LTL specifications. The first steps of their algorithm consist in negating the starting formula (thus obtaining a formula in coSafety-LTL, after the transformation into negation normal form), and the consequent translation into FO. This last step is used in order to exploit the tool MONA [HJJ$^+$95], an efficient tool for the construction and manipulation of automata. Interestingly, the formula resulting from this step is a formula of coSafety-FO of linear size with respect to the starting one, although Zhu *et al.* never explicitly identified it as such.

6.3. An alternative first-order logic characterization of (co)safety LTL properties.
We start by giving a brief account of a different first-order logic characterization of safety and co-safety LTL properties, proposed by Thomas in [Tho88].

Given a formula $\phi(x)$ in the language of FO with one free variable (recall Section 2), we say that $\phi(x)$ is *bounded* if and only if all quantifiers in $\phi(x)$ are either of the form $\exists y(y \leq x \wedge \dots)$ or $\forall y(y \leq x \rightarrow \dots)$. The two fragments of FO proposed by Thomas [Tho88] for capturing the safety and co-safety fragment of LTL are defined as follows.[2]

Definition 6.2. The *Existential Bounded* fragment of FO (EB-FO, for short) is the set of FO sentences of type $\exists x . \phi(x)$, such that $\phi(x)$ is a bounded formula. The *Universal Bounded* fragment of FO (UB-FO, for short) is the set of FO sentences of type $\forall x . \phi(x)$, such that $\phi(x)$ is a bounded formula.

Note that, on the contrary of coSafety-FO and Safety-FO, formulas of EB-FO and UB-FO do not contain any free variable. For this reason, the definition of *language* for EB-FO and UB-FO formulas differs from the case for coSafety-FO and Safety-FO. We define the language of a formula ϕ in EB-FO or UB-FO, denoted as $\mathcal{L}(\phi)$, as the set of words $\sigma \in (2^\Sigma)^\omega$ such that $(\sigma)^s \models \phi$.

The EB-FO and UB-FO fragments are heavily based on FO and the Fα and Gα normal forms (Proposition 2.6); in particular, we recall that:

- the set of LTL-definable co-safety (resp. safety) properties is captured by the set of formulas of type Fα (resp. Gα), where $\alpha \in$ LTL$_P$;
- by Propositions 2.3 and 2.5, we have that $[\![$LTL$_P]\!] = [\![$FO$]\!]$.

Take for example the EB-FO fragment. The structure of its formulas naturally resembles the Fα normal form: the power of FO is used for representing all and only the formulas α in LTL$_P$, while the initial existential quantifier $\exists x . (\dots)$ together with the bound $\dots \leq x$ on all the other quantifiers is used for modeling the *eventually* (F) operator. A similar rationale holds for UB-FO. It follows that the EB-FO (resp. UB-FO) fragment is expressively complete with respect to the co-safety (resp. safety) fragment of LTL, that is [Tho88, Proposition 2.1]:

- $[\![$EB-FO$]\!] = [\![$LTL$]\!] \cap$ coSAFETY
- $[\![$UB-FO$]\!] = [\![$LTL$]\!] \cap$ SAFETY

---

[2]Thomas did not give a name to these fragments. We chose to call them the *Existential* and the *Universal Bounded* fragment of FO.

6.4. **Comparison between** coSafety-FO **and** EB-FO. Since both coSafety-FO and EB-FO capture the co-safety fragment of LTL, it follows that EB-FO and the fragment of coSafety-FO with exactly one free variable have the same expressive power. Clearly, the same holds for the safety fragment, having that UB-FO and the fragment of Safety-FO with exactly one free variable are expressively equivalent.

  We now show that, in addition of being expressively equivalent, there is a linear-size translation between the fragment of coSafety-FO with only one free variable and EB-FO, and *vice versa*.

Proposition 6.3. *For any formula* $\phi(x) \in$ coSafety*-FO, there exists a formula* $\phi' \in$ EB*-FO such that:* (i) $\mathcal{L}(\phi(x)) = \mathcal{L}(\phi')$; *and* (ii) $|\phi'| \in \mathcal{O}(|\phi(x)|)$.

Proof. Let $\phi(x) \in$ coSafety-FO. Since the language of the formula $\phi(x)$ is defined as the set of state sequences that are models of $\phi(x)$ when x is interpreted as 0 (recall Section 2), it suffices to define a formula in EB-FO that extends $\phi(x)$ by forcing x to be 0. Formally, we define the formula ϕ' as follows:

$$\exists y \,.\, (\exists x \,.\, (x \le y \wedge \forall z \,.\, (z \le y \rightarrow (z < x \rightarrow \bot)) \wedge \phi_b(x)))$$

where $\phi_b(x)$ is obtained from $\phi(x)$ by replacing any quantifier of the form $\exists k(\dots)$ (resp. $\forall k(\dots)$) with $\exists k(k \le y \wedge \dots)$ (resp. $\forall k(k \le y \rightarrow \dots)$). It is easy to see that:

  (i) ϕ' is a formula of EB-FO;
  (ii) $\mathcal{L}(\phi(x)) = \mathcal{L}(\phi')$; and
  (iii) $|\phi'| \in \mathcal{O}(|\phi(x)|)$.   □

  The converse direction holds as well.

Proposition 6.4. *For any formula* $\phi \in$ EB*-FO, there exists a formula* $\phi'(x) \in$ coSafety*-FO such that:* (i) $\mathcal{L}(\phi'(x)) = \mathcal{L}(\phi)$; *and* (ii) $|\phi'(x)| \in \mathcal{O}(|\phi|)$.
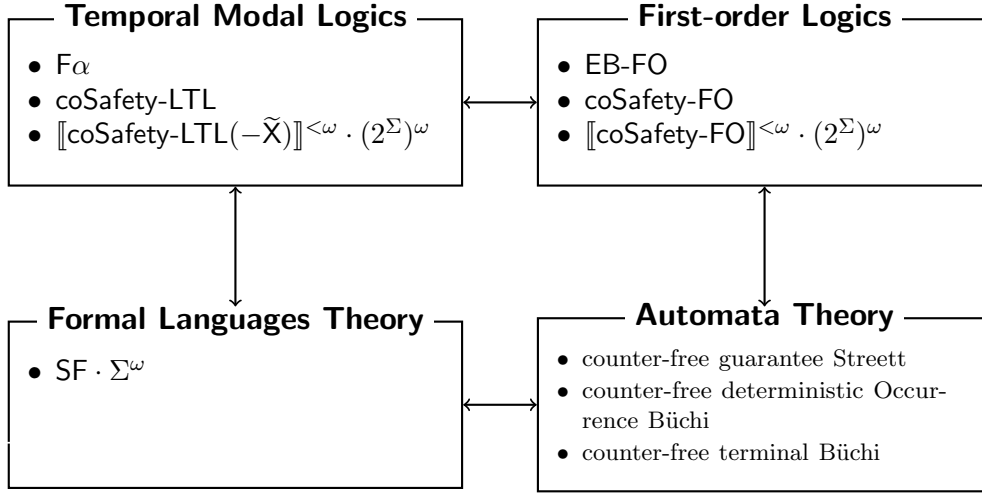
Proof. We first prove that, for any formula $\phi \in$ EB-FO, there exists a formula $\phi'(x) \in$ coSafety-FO such that: (i) $\mathcal{L}(\phi'(x)) = \mathcal{L}^{<\omega}(\phi) \cdot (2^\Sigma)^\omega$; and (ii) $|\phi'(x)| \in \mathcal{O}(|\phi|)$. We define $\phi'(x)$ as the following formula:
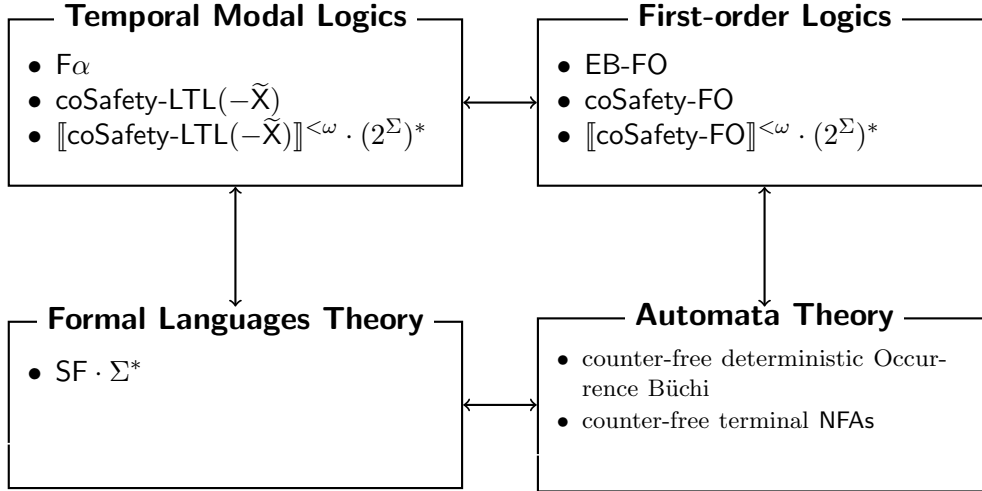
$$\exists y \,.\, (x \le y \wedge \psi(x, y))$$

where $\psi(x, y)$ is the formula obtained from ϕ by replacing each subformula of type $\exists z \,.\, \phi_1$ with $\exists z \,.\, (x \le z \wedge \phi_1)$ and each subformula of type $\forall z \,.\, \phi_1$ with $\forall z \,.\, (x \le z < y \rightarrow \phi_1)$. It is simple to see that $\phi'(x) \in$ coSafety-FO, $\mathcal{L}(\phi'(x)) = \mathcal{L}^{<\omega}(\phi) \cdot (2^\Sigma)^\omega$ and $|\phi'(x)| \in \mathcal{O}(|\phi|)$.

  Now, with a simple induction, one can prove that any formula $\phi \in$ EB-FO is such that $\mathcal{L}(\phi) = \mathcal{L}^{<\omega}(\phi) \cdot (2^\Sigma)^\omega$. Therefore, we have that $\mathcal{L}(\phi'(x)) = \mathcal{L}(\phi)$, which concludes the proof.   □

  The expressively equivalence between the fragment of coSafety-FO with only one free variable, EB-FO and the co-safety fragment of LTL, together with the linear-size transformation of coSafety-FO into EB-FO (Proposition 6.3), allow for the following consideration: in order to capture the whole co-safety fragment of LTL, it is not necessary to have the full power of FO, on which, as noted above, EB-FO is strongly based; on the contrary, it suffices to use the syntax of coSafety-FO, *i.e.*, with existential quantifiers of type $\exists y(x < y \wedge \dots)$ and with universal quantifiers of type $\forall y(x < y < z \rightarrow \dots)$.

Temporal Modal Logics

- $F\alpha$
- coSafety-LTL
- $[\![\text{coSafety-LTL}(-\widetilde{X})]\!]^{<\omega} \cdot (2^\Sigma)^\omega$

First-order Logics

- EB-FO
- coSafety-FO
- $[\![\text{coSafety-FO}]\!]^{<\omega} \cdot (2^\Sigma)^\omega$

Formal Languages Theory

- $\text{SF} \cdot \Sigma^\omega$

Automata Theory

- counter-free guarantee Streett
- counter-free deterministic Occurrence Büchi
- counter-free terminal Büchi

(A) The co-safety fragment of LTL on infinite words semantics.

Temporal Modal Logics

- $F\alpha$
- coSafety-LTL$(-\widetilde{X})$
- $[\![\text{coSafety-LTL}(-\widetilde{X})]\!]^{<\omega} \cdot (2^\Sigma)^*$

First-order Logics

- EB-FO
- coSafety-FO
- $[\![\text{coSafety-FO}]\!]^{<\omega} \cdot (2^\Sigma)^*$

Formal Languages Theory

- $\text{SF} \cdot \Sigma^*$

Automata Theory

- counter-free deterministic Occurrence Büchi
- counter-free terminal NFAs

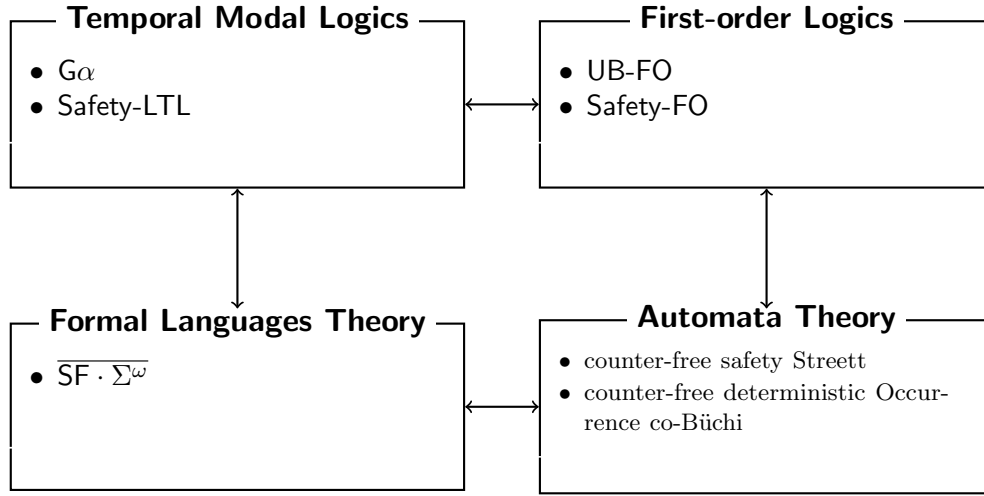(B) The co-safety fragment of LTL on finite words semantics.

FIGURE 2. Different characterizations for the co-safety fragment of LTL over (a) infinite words and (b) finite words.

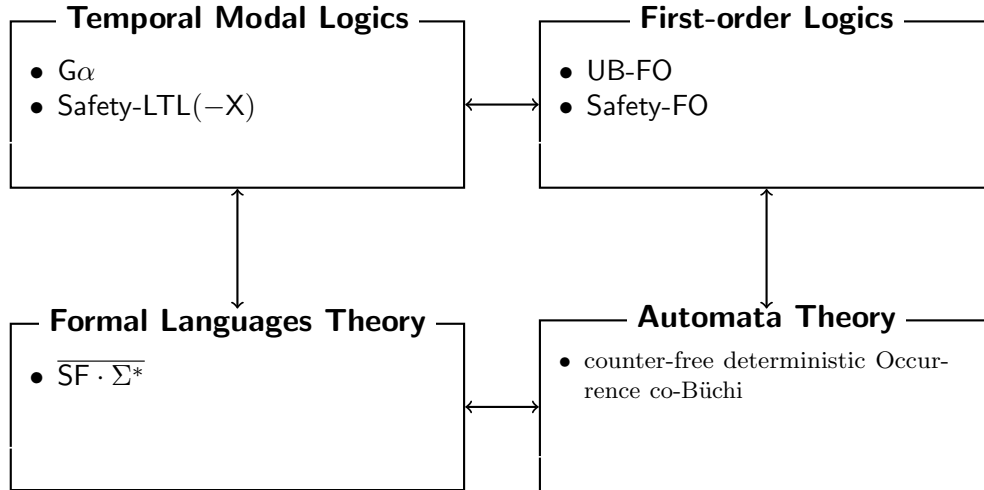## 7. OTHER CHARACTERIZATIONS OF THE (CO-)SAFETY FRAGMENT OF LTL

In this section, we give an overview of the other characterizations that are present in the literature of the safety and co-safety fragments of LTL, both on infinite and finite words.

We start by recalling that there are four main characterizations of the set of LTL-definable ω-languages:

- in terms of temporal modal logics, $[\![\text{LTL}]\!]$ is of course definable by LTL and LTL+P [Pnu77];
- in terms of first-order logics, $[\![\text{LTL}]\!]$ is captured by FO-TLO [Kam68];
- in terms of regular expressions, $[\![\text{LTL}]\!]$ is characterized by *star-free* ω-regular expressions [Tho79];

(A) The safety fragment of LTL on infinite words semantics.

(B) The safety fragment of LTL on finite words semantics.

Figure 3. Different characterizations for the safety fragment of LTL over (a) infinite words and (b) finite words.

- in terms of automata, $\llbracket\textsf{LTL}\rrbracket$ is captured by *counter-free* Büchi automata [MP71].

Over finite words, the characterizations of LTL are the same, except that instead of star-free ω-regular expressions and counter-free Büchi automata, we consider star-free regular expressions and counter-free nondeterministic finite automata.

In Figures 2 and 3, we summarize the characterizations of the co-safety and safety fragments of LTL, both over infinite and finite words, in terms of: (i) temporal logics; (ii) first-order logics; (iii) regular expressions; (iv) automata.

7.1. **Temporal and first-order logics.** We first recall the characterizations in terms of temporal and first-order logics. In terms of temporal logics, the co-safety fragment of LTL is captured:

- over infinite words, by $\mathsf{F}\alpha$, coSafety-LTL, and $[\![\text{coSafety-LTL}(-\widetilde{\mathsf{X}})]\!]^{<\omega} \cdot (2^\Sigma)^\omega$ (*i.e.*, the finite-words interpretation of coSafety-LTL($-\widetilde{\mathsf{X}}$) when concatenated to any possible infinite word);
- over finite words, by $\mathsf{F}\alpha$, coSafety-LTL($-\widetilde{\mathsf{X}}$) and $[\![\text{coSafety-LTL}(-\widetilde{\mathsf{X}})]\!]^{<\omega} \cdot (2^\Sigma)^*$ (*i.e.*, the finite-words interpretation of coSafety-LTL($-\widetilde{\mathsf{X}}$) when concatenated to any possible finite word).

Dually, the safety fragment of LTL is captured by $\mathsf{G}\alpha$ and Safety-LTL, for the case of infinite words interpretation, and by $\mathsf{G}\alpha$ and Safety-LTL($-\mathsf{X}$), for the case of finite words interpretation.

As for first-order logics, $[\![\text{LTL}]\!] \cap \text{coSAFETY}$ (*i.e.*, the co-safety fragment of LTL over infinite words) is captured by EB-FO, coSafety-FO and $[\![\text{coSafety-FO}]\!]^{<\omega} \cdot (2^\Sigma)^\omega$, while $[\![\text{LTL}]\!]^{<\omega} \cap \text{coSAFETY}^{<\omega}$ (*i.e.*, the co-safety fragment of LTL over finite words) is captured by EB-FO, coSafety-FO and $[\![\text{coSafety-FO}]\!]^{<\omega} \cdot (2^\Sigma)^*$. The characterizations for the safety fragment of LTL over finite and infinite words is dual.

7.2. **Regular and ω-regular expressions.** Consider now the characterization in terms of (ω-)regular expressions. We recall that a regular expression is an expression built starting from the symbols in a finite alphabet Σ using the operations of union ($\cup$), complementation ($\overline{S}$), concatenation ($\cdot$) and the Kleene'star (*). ω-regular expressions extend regular expressions by admitting also the operation S^ω, which is the ω-closure of the set S. *Star-free* (ω-)regular expressions are (ω-)regular expressions devoid of the Kleene'star. We denote with SF the set of star-free regular expressions. It is known that $[\![\text{LTL}]\!]^{<\omega}$ (resp. $[\![\text{LTL}]\!]$) is captured by *star-free* regular (resp. ω-regular) expressions [MP71].

We start with the co-safety fragment of LTL over infinite words. Recall that, by Proposition 2.6, $[\![\text{LTL}]\!] \cap \text{coSAFETY}$ is captured by $\mathsf{F}\alpha$, where $\alpha \in \text{LTL}_\mathsf{P}$. Moreover, by Proposition 2.3, LTL_P (*i.e.*, pure past LTL+P) is expressively equivalent to LTL over finite words, *i.e.*, $[\![\text{LTL}]\!]^{<\omega} = [\![\text{LTL}_\mathsf{P}]\!]^{<\omega}$. Now, since $[\![\text{LTL}]\!]^{<\omega}$ is captured by star-free regular expressions, by the semantics of the *eventually* (F) operator, we have that $[\![\text{LTL}]\!] \cap \text{coSAFETY}$ is captured by $\mathsf{SF} \cdot (\Sigma)^\omega$. For finite words, by the same kind of reasoning, it follows that $[\![\text{LTL}]\!]^{<\omega} \cap \text{coSAFETY}^{<\omega}$ is captured by $\mathsf{SF} \cdot (\Sigma)^*$. By duality, $[\![\text{LTL}]\!] \cap \text{SAFETY}$ and $[\![\text{LTL}]\!]^{<\omega} \cap \text{SAFETY}^{<\omega}$ are captured by $\overline{\mathsf{SF} \cdot (\Sigma)^\omega}$ and $\overline{\mathsf{SF} \cdot (\Sigma)^*}$, respectively.

7.3. **Automata.** In this part, we give an overview of some automata-based characterizations proposed in the literature for the safety and co-safety fragments of LTL. We first recall some basic notions of automata theory.

Definition 7.1 (Semi-automata). A *nondeterministic semi-automaton* $\mathcal{A}_s$ is a tuple (Σ, Q, q_0, δ) such that: (i) Σ is a finite alphabet; (ii) Q is a set of states; (iii) $q_0 \in Q$ is the initial state; (iv) $\delta : Q \times \Sigma \to 2^Q$ is the transition function.

Given a finite alphabet Σ and $\delta : Q \times \Sigma \to 2^Q$, we can extend δ to $\delta^* : Q \times \Sigma^* \to 2^Q$ in the natural way. Given a semi-automaton $\mathcal{A}_s = (\Sigma, Q, q_0, \delta)$, we say that the word $\sigma \in \Sigma^*$ defines a *nontrivial cycle* in $\mathcal{A}$ if and only if there exists a state $q \in Q$ such that $q \notin \delta^*(q, \sigma)$ and $q \in \delta^*(q, \sigma^i)$ for some $i > 1$ [MP71, ST96].

A semi-automaton $\mathcal{A}_s = (\Sigma, Q, q_0, \delta)$ is said to be:

- *deterministic* if and only if $\delta(q, s)$ is a singleton set, for each $q \in Q$ and each $s \in \Sigma$.
- *counter-free* if and only if it does *not* contain any nontrivial cycle.

Given a semi-automaton $\mathcal{A}_s = (\Sigma, Q, q_0, \delta)$ and a (finite or infinite) word $\sigma = \langle \sigma_0, \sigma_1, \ldots \rangle \in \Sigma^* \cup \Sigma^\omega$, a run π over σ is a (finite or infinite) sequence of states $\langle q_0, q_1, \ldots \rangle \in Q^* \cup Q^\omega$ such that $q_{i+1} \in \delta(q_i, \sigma_i)$, for any $i \geq 0$. We denote with $\inf(\pi)$ the set of states that occur infinitely often in π, and with $\mathrm{occ}(\pi)$ the set of states that occur at least once in π.

An automaton $\mathcal{A}$ is a tuple $(\Sigma, Q, q_0, \delta, \alpha)$ such that (Σ, Q, q_0, δ) is a semi-automaton and α is an *accepting condition*. Starting from semi-automata, we can obtain many types of *automata* by defining different accepting conditions.

- In nondeterministic finite automata (NFAs, for short), α is a subset of Q and is called the set of *final state*. A run π is accepting iff there exists an $i \geq 0$ such that $\pi_i \in \alpha$.
- In deterministic Streett automata (DSAs, for short), $\alpha = \{(G_1, R_1), \ldots, (G_n, R_n)\}$ where $G_i, R_i \subseteq Q$, for each $1 \leq i \leq n$ and some $n \in \mathbb{N}$. A run π is accepting iff, for all $1 \leq i \leq n$, either $\inf(\pi) \cap G_i = \varnothing$ or $\inf(\pi) \cap R_i \neq \varnothing$.
- A Büchi automaton is a Streett automaton in which $\alpha = \{(Q, R_1)\}$. In this case, R_1 is called the set of *final states* of the automaton.
- A co-Büchi automaton is a Streett automaton in which $\alpha = \{(G_1, \varnothing)\}$. In this case, G_1 is called the set of *rejecting states* of the automaton.
- An Occurrence Streett automaton is a Streett automaton with accepting condition $\alpha = \{(G_1, R_1), \ldots, (G_n, R_n)\}$ in which a run π is accepting iff either $\mathrm{occ}(\pi) \cap G_i = \varnothing$ or $\mathrm{occ}(\pi) \cap R_i \neq \varnothing$, for all $1 \leq i \leq n$.
- The definitions of Occurrence Büchi and Occurrence co-Büchi follow from the definition of Occurrence Streett automaton.

For all types of automata, an automaton $\mathcal{A}$ accepts a word σ if and only if there exists an accepting run induced by σ in $\mathcal{A}$. The language recognized by $\mathcal{A}$ is the set of words that are accepted by $\mathcal{A}$. It is known that each ω-language definable in LTL is recognized by a counter-free Büchi automaton, and *vice versa* [MP71]. Similary, a language of finite words is definable in LTL iff it is recognized by a counter-free NFA [MP71].

In [MP90], Manna and Pnueli characterize the set of all co-safety regular properties in terms of *guarantee (deterministic) Streett automata*. A guarantee Streett automaton $\mathcal{A} = (\Sigma, Q, q_0, \delta, \alpha)$ is a Streett automaton such that:

- $\alpha = \{(G_1, R_1)\}$;
- $\mathrm{Good} = (Q \setminus G_1) \cup R_1$ and $\mathrm{Bad} = Q \setminus \mathrm{Good}$;
- $\forall q \in \mathrm{Good} . \forall q' \in \mathrm{Bad} . \forall \sigma \in \Sigma . (q' \notin \delta(q, \sigma))$.

Intuitively, any accepting run of a guarantee Streett automaton can visit the states in G_1 only a finite number of times, after which it is forced to visit only states in the Good region. Crucially, once a run enters the Good region, each extension of it will result into an accepting run, since it will never visit the states in G_1. For this reason, guarantee Street automata capture REG $\cap$ coSAFETY. In order to characterize $[\![\mathrm{LTL}]\!] \cap$ coSAFETY, by exploiting the equivalence between counter-free automata and LTL, Manna and Pnueli [MP90] prove that *counter-free guarantee Streett automata* capture the co-safety fragment of LTL over infinite words. By a simple dualization, they define *safety Streett automata* as Streett automata in which there is no transition from the Bad to the Good region, *i.e.*:

- $\alpha = \{(G_1, R_1)\}$;
- $\mathrm{Good} = (Q \setminus G_1) \cup R_1$ and $\mathrm{Bad} = Q \setminus \mathrm{Good}$;

- $\forall q \in \text{Bad} \,.\, \forall q' \in \text{Good} \,.\, \forall \sigma \in \Sigma \,.\, (q' \notin \delta(q, \sigma))$.

It holds that *counter-free safety Streett automata* capture $[\![\mathsf{LTL}]\!] \cap \mathsf{SAFETY}$.

In [CP03], Cerná and Pelánek prove that *deterministic Occurrence Büchi automata* are equivalent to guarantee Streett automata, thus proving also that the formers characterize the co-safety fragment of regular languages. The intuition behind this characterization is simple. A run π of a deterministic Occurrence Büchi automaton is accepting if and only if it reaches a final state (say at position i). Now, by definition of *Occurrence Büchi automaton*, every run that agrees with π from 0 to i and then goes on arbitrarly is accepting as well. It is not difficult to see that, in order to capture $[\![\mathsf{LTL}]\!] \cap \mathsf{coSAFETY}$, it suffices to add the counter-free condition to deterministic Occurrence Büchi automata. By dualization, Cerná and Pelánek [CP03] obtain that counter-free deterministic Occurrence co-Büchi automata capture $[\![\mathsf{LTL}]\!] \cap \mathsf{SAFETY}$. It is simple to see that this characterization of both the co-safety and the safety fragment of LTL in terms of counter-free deterministic Occurrence Büchi and co-Büchi automata holds for finite words as well.

Last but not least, the co-safety fragment of LTL can be captured by *counter-free terminal automata* [BRS99, CP03]. Terminal automata are nondeterministic automata such that each final state $q \in \alpha$ is such that $\delta(q, \sigma) \subseteq \alpha$ (for any $\sigma \in \Sigma$), *i.e.*, any run, once reached a final state, cannot reach a state which is not final. It holds that [CP03]: (i) $[\![\mathsf{LTL}]\!] \cap \mathsf{coSAFETY}$ is captured by counter-free terminal Büchi automata; (ii) $[\![\mathsf{LTL}]\!]^{<\omega} \cap \mathsf{coSAFETY}^{<\omega}$ is captured by counter-free terminal NFAs.

8. Conclusions

In this paper, we gave a first-order characterization of safety and co-safety languages, by means of two fragments of first-order logic, $\mathsf{Safety\text{-}FO}$ and $\mathsf{coSafety\text{-}FO}$. These fragments of $\mathsf{FO\text{-}TLO}$ provide a very natural syntax and are *expressively complete* with regards to LTL-definable safety and co-safety languages.

The core theorem establishes a correspondence between $\mathsf{Safety\text{-}FO}$ (resp., $\mathsf{coSafety\text{-}FO}$) and $\mathsf{Safety\text{-}LTL}$ (resp., $\mathsf{coSafety\text{-}LTL}$), and thus it can be viewed as a special version of Kamp's theorem for safety (resp., co-safety) properties. Thanks to these new fragments, we were able to provide a novel, compact, and self-contained proof of the fact that $\mathsf{Safety\text{-}LTL}$ captures LTL-definable safety languages. Such a result was previously proved by Chang *et al.* [CMP92], but in terms of the properties of a non-trivial transformation from star-free languages to LTL by Zuck [Zuc86]. As a by-product, we provided a number of results that relate the considered languages when interpreted over finite and infinite words. In particular, we highlighted the expressive power of the *weak tomorrow* temporal modality, showing it to be essential in $\mathsf{coSafety\text{-}LTL}$ over finite words. Last but not least, we show that $\mathsf{coSafety\text{-}LTL}(-\widetilde{\mathsf{X}})$ and $\mathsf{Safety\text{-}LTL}(-\mathsf{X})$ capture the set of co-safety and safety languages of finite words definable in LTL, respectively.

The equivalence-preserving translation from $\mathsf{coSafety\text{-}FO}$ to $\mathsf{coSafety\text{-}LTL}$ shown in this paper can, in the worst case, produce formulas of nonelementary size. An interesting future direction is to investigate whether more efficient (even polynomial) translations are possible.

As we have seen, different fragments of LTL can capture the (co-)safety fragment. It is interesting to study the succinctness of these fragments, in particular of $\mathsf{coSafety\text{-}LTL}$ and $\mathsf{F}\alpha$, and to ask whether one can be exponentially more succinct than the other, or whether they are incomparable as far as succinctness is considered. Last but not least, a natural

related question is whether the previous results generalize to the case of finite words as well, *i.e.*, for the logics coSafety-LTL($-\widetilde{\mathsf{X}}$) and $\mathsf{F}\alpha$.

Acknowledgements

References

[BAS02]   Armin Biere, Cyrille Artho, and Viktor Schuppan. Liveness checking as safety checking. *Electronic Notes in Theoretical Computer Science*, 66(2):160–177, 2002.

[BRS99]   Roderick Bloem, Kavita Ravi, and Fabio Somenzi. Efficient decision procedures for model checking of linear time logic properties. In *International Conference on Computer Aided Verification*, pages 222–235. Springer, 1999.

[Buc63]   J Richard Buchi. Weak second-order arithmetic and finite automata. *Journal of Symbolic Logic*, 28(1), 1963.

[Büc90]   J Richard Büchi. On a decision method in restricted second order arithmetic. In *The collected works of J. Richard Büchi*, pages 425–435. Springer, 1990.

[CGG⁺22]   Alessandro Cimatti, Luca Geatti, Nicola Gigante, Angelo Montanari, and Stefano Tonetta. A first-order logic characterisation of safety and co-safety languages. In Patricia Bouyer and Lutz Schröder, editors, *Foundations of Software Science and Computation Structures - 25th International Conference, FOSSACS 2022, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2022, Munich, Germany, April 2-7, 2022, Proceedings*, volume 13242 of *Lecture Notes in Computer Science*, pages 244–263. Springer, 2022. $\overline{T}$doi:10.1007/978-3-030-99253-8_13.

[CMP92]   Edward Y. Chang, Zohar Manna, and Amir Pnueli. Characterization of temporal property classes. In Werner Kuich, editor, *Proceedings of the 19th International Colloquium on Automata, Languages and Programming*, volume 623 of *Lecture Notes in Computer Science*, pages 474–486. Springer, 1992. $\overline{T}$doi:10.1007/3-540-55719-9_97.

[CP03]   Ivana Cerná and Radek Pelánek. Relating hierarchy of temporal properties to model checking. In Branislav Rovan and Peter Vojtás, editors, *Proceedings of the 28th International Symposium on Mathematical Foundations of Computer Science 2003*, volume 2747 of *Lecture Notes in Computer Science*, pages 318–327. Springer, 2003. $\overline{T}$doi:10.1007/978-3-540-45138-9_26.

[DGDST⁺21]   Giuseppe De Giacomo, Antonio Di Stasio, Lucas M Tabajara, Moshe Y Vardi, and Shufang Zhu. Finite-trace and generalized-reactivity specifications in temporal synthesis. In *IJCAI*, pages 1852–1858, 2021.

[DV13]   Giuseppe De Giacomo and Moshe Y. Vardi. Linear temporal logic and linear dynamic logic on finite traces. In Francesca Rossi, editor, *Proceedings of the 23rd International Joint Conference on Artificial Intelligence*, pages 854–860. IJCAI/AAAI, 2013.

[DV15]   Giuseppe De Giacomo and Moshe Y. Vardi. Synthesis for LTL and LDL on finite traces. In Qiang Yang and Michael J. Wooldridge, editors, *Proceedings of the Twenty-Fourth International Joint Conference on Artificial Intelligence*, pages 1558–1564. AAAI Press, 2015.

[GMM14]   Giuseppe De Giacomo, Riccardo De Masellis, and Marco Montali. Reasoning on LTL on finite traces: Insensitivity to infiniteness. In Carla E. Brodley and Peter Stone, editors, *Proceedings of the Twenty-Eighth AAAI Conference on Artificial Intelligence*, pages 1027–1033. AAAI Press, 2014.

[GPSS80]   Dov Gabbay, Amir Pnueli, Saharon Shelah, and Jonathan Stavi. On the temporal analysis of fairness. In *Proceedings of the 7th ACM SIGPLAN-SIGACT symposium on Principles of programming languages*, pages 163–173, 1980.

[HJJ$^+$95]   Jesper G Henriksen, Jakob Jensen, Michael Jørgensen, Nils Klarlund, Robert Paige, Theis Rauhe, and Anders Sandholm. Mona: Monadic second-order logic in practice. In *International Workshop on Tools and Algorithms for the Construction and Analysis of Systems*, pages 89–110. Springer, 1995.

[Kam68]   Johan Anthony Wilem Kamp. *Tense logic and the theory of linear order*. University of California, Los Angeles, 1968.

[KV01]   Orna Kupferman and Moshe Y Vardi. Model checking of safety properties. *Formal Methods in System Design*, 19(3):291–314, 2001.

[LPZ85]   Orna Lichtenstein, Amir Pnueli, and Lenore Zuck. The glory of the past. In *Workshop on Logic of Programs*, pages 196–218. Springer, 1985.

[MP71]   Robert McNaughton and Seymour A Papert. *Counter-Free Automata (MIT research monograph no. 65)*. The MIT Press, 1971.

[MP90]   Zohar Manna and Amir Pnueli. A hierarchy of temporal properties (invited paper, 1989). In *Proceedings of the 9th annual ACM symposium on Principles of distributed computing*, pages 377–410, 1990.

[Pnu77]   Amir Pnueli. The temporal logic of programs. In *18th Annual Symposium on Foundations of Computer Science (sfcs 1977)*, pages 46–57. IEEE, 1977.

[Rab14]   Alexander Rabinovich. A Proof of Kamp's theorem. *Logical Methods in Computer Science*, Volume 10, Issue 1, February 2014. $\overline{T}$doi:10.2168/LMCS-10(1:14)2014.

[Sis94]   A Prasad Sistla. Safety, liveness and fairness in temporal logic. *Formal Aspects of Computing*, 6(5):495–511, 1994.

[SPH84]   Rivi Sherman, Amir Pnueli, and David Harel. Is the interesting part of process logic uninteresting? A translation from PL to PDL. *SIAM J. Comput.*, 13(4):825–839, 1984. $\overline{T}$doi:10.1137/0213051.

[ST96]   Ina Schiering and Wolfgang Thomas. Counter-free automata, first-order logic, and star-free expressions extended by prefix oracles. *Developments in Language Theory, II (Magdeburg, 1995), Worl Sci. Publishing, River Edge, NJ*, pages 166–175, 1996.

[Tho79]   Wolfgang Thomas. Star-free regular sets of ω-sequences. *Information and Control*, 42(2):148–156, 1979.

[Tho88]   Wolfgang Thomas. Safety-and liveness-properties in propositional temporal logic: characterizations and decidability. *Banach Center Publications*, 1(21):403–417, 1988.

[ZTL$^+$17] Shufang Zhu, Lucas M. Tabajara, Jianwen Li, Geguang Pu, and Moshe Y. Vardi. A Symbolic Approach to Safety LTL Synthesis. In Ofer Strichman and Rachel Tzoref-Brill, editors, *Proceedings of the 13th International Haifa Verification Conference*, volume 10629 of *Lecture Notes in Computer Science*, pages 147–162. Springer, 2017. $\overline{T}$doi:10.1007/978-3-319-70389-3_10.

[Zuc86]   Lenore Zuck. Past temporal logic. *Weizmann Institute of Science*, 67, 1986.