AXISYMMETRIC FREE VIBRATION OF THIN PROLATE SPHEROIDAL SHELLS

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ABSTRACT

In this paper a detailed study of the theory of free axisymmetric vibration of thin isotropic prolate spheroidal shells is presented. The analysis is performed according to Rayleigh – Ritz method. This method as well as an approximate modeling technique were attempted to estimate the natural frequencies for the shell. This technique is based on considering the prolate spheroidal as a continuous system constructed from two spherical shell elements matched at the continuous boundaries. Through out the obtained results it is found that this method predicted fairly well the natural frequencies of a prolate spheroidal shell for all values of eccentricities.

الخلاصة

يتناول هذا البحث الدراسة النظرية للاهتزازات الحرة للقشريات نحيفة الجدران البيضوية الشكل المتطاولة المتناظرة المحور المتشابهة الخواص في جميع الاتجاهات ، وقد أجري التحليل النظري بطريقة رايلي-رتز .إن هذه الطريقه بالرغم من كونها تقريبية ، ولكن بالإمكان الاعتماد عليها لحساب الذبذبة الطبيعية لهذه القشريات .

من خلال النتائج وجد إن تلك الطريقة أعطت نتائج جيدة للترددات الطبيعية للقشرة البيضوية المتناظرة المحور لكل قيم أللامركزية.

KEYWORDS

Spheroidal Shells, Thin Prolate , Free Vibration, Axisymmetric

INTRODUCTION

Prolate spheroid shells can be obtained by rotating an ellipse around its major axis, see **Fig.1**. It is worthy to indicate the industrial applications and importance of shell structures. This interest was appreciated today in aerospace, sea vehicles industry and the structure of rocket can be considered as a prolate shells. In such structures the resonance problem may occur, therefore the study of free vibration become very important to prevent the resonance appearance.

The study of free vibration of prolate shells take a considerable attempt in the published literature. several investigators, using a variety of mathematical techniques, have obtained approximate solutions for the natural frequencies of axisymmetric vibrations of thin prolate spheroidal shells.

(De Maggio and Silibiger 1961) obtained a solution for the torsional vibrations of thin prolate spheroidal shell in terms of spheroidal angle functions. **(Kanins 1963)** was concerned with the vibration analysis of spheroidal shells, closed at one pole and open at the other, by means of the linear classical bending theory of shells. Frequency equations are derived in terms of Legender function with complex indices, and axisymmetric vibration of the natural frequencies and mode shapes are deduced for all opining angles ranging from a shallow to closed spherical shell. It was found that for all opening angles the frequency spectrum consist of two coupled infinite sets of modes that can be labeled as bending (or flexural) and membrane modes. It was also found that membrane modes are practically independent of thickness, whereas the bending modes vary with the thickness. The same author concerned with a theoretical investigation of the free vibration of arbitrary shells of revolution by means of the classical bending theory of shells.

A method is developed that is applicable to rotationally symmetric shells with meridional variations (including discontinuities) in Young’s modulus, Poisson’s ratio, radii of curvature, and thickness. The natural frequencies and the corresponding mode shapes of axisymmetric free vibration of rotationally symmetric shell can be obtained without any limitation on the length of the meridian of the shell. The results of free vibration of spherical and conical shells obtained earlier by means of the bending theory. In addition, parapoloidal shells and sphere-one shell combination are considered, which have been previously analyzed by means of the inextentional theory of shells, and natural frequencies and mode shapes predicted by the bending theory are given.

(Numergut and Brand 1965) determined the lower axisymmetric modes of prolate shell with five values of eccentricity. **(DiMaggio and Rand 1966)** using membrane shell theory in which the effects of bending resistance are ignored. Their work was distinguished by applying their solution to constant thickness membrane shell by means of integrating numerically the equations of motion. It was found that the frequencies associated with higher modes are strongly dependent on the eccentricity ratio.

(Zhu 1995) based upon general thin shell theory and basic equations of fluid-mechanics; the Rayleigh-Ritz's method for coupled fluid-structure free vibrations is developed for arbitrary fully or partially filled in viscid, irrigational and compressible or incompressible fluid, by means of the generalized orthogonality relations of wet modes and the associated Rayleigh quotients.

(Wasmi 1997) used the finite element and modal analysis techniques to investigate the static and dynamic behavior of oblate spheroidal dishes, prolate and the relevant structures. Different types of elements were considered in one dimension, two dimensions and three dimensions.

For framed structures, Euler Bernouilli theory, Tiomshenko theory, integrated Tiomshenko and improved Tiomshenko theories were applied. While for plates and shells, Kerchief's, Zienkiewicz and Mindlin theories were applied. The capability of these trenchancies was investigated in this work to predict the natural frequencies and mode shapes, as well as the static analysis of framed structures and spheroidal dishes. It was found that the natural frequencies of oblate and prolate shells have two types of behavior against increasing the shell thickness and eccentricity, which are the membrane and bending modes. The membrane modes natural frequencies tend to increase with increasing the eccentricity of oblate, while the bending mode natural frequencies decrease with increasing the value of eccentricity.

(Aleksandr Korjanik et al. 2001) investigated the free damped vibrations of sandwich shells of revolution. As special cases the vibration analysis under consideration of damping of cylindrical, conical and spherical sandwich shells is performed. A specific sandwich shell finite element with 54 degrees of freedom is employed. Starting from the energy method the damping model is developed. Numerical examples for the free vibration analysis with damping based on the proposed finite element approach are discussed. Results for sandwich shells show a satisfactory agreement with various references solutions.

(Antoine et, al 2002) investigated the linear and nonlonear vibrations of shallow spherical shells with free edge experimentally and numerically. Combination resonances due to quadratic nonlinearities are studied, for a harmonic forcing of the shell. Identification of the excited modes is achieved through symmetric comparisons between spatial results obtained from a finite element modelling, and spectral information derived from experiments.

This investigation deals with the free vibration characteristics of thin elastic prolate spheroidal shell. The shell is assumed be of isotropic material. The analysis depends on the Rayleigh _ Ritz method.

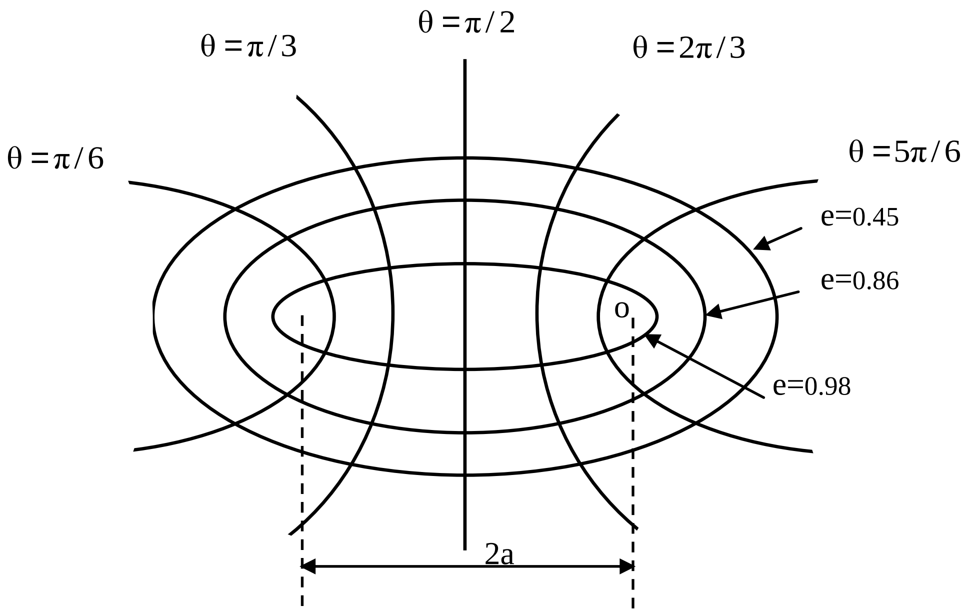


Fig. (1): Prolate spheroidal co-ordinates

MATHEMATICAL ANALYSIS

Through out the review of literature, it is found that even though the governing equations for shells of revolution are well spelt out, nevertheless, the governing equations for prolate spheroidal shells are not available, therefore the approximate energy procedure will be followed.

For a shell undergoing deformation in which the normal to the middle surface of undeformed shell remains straight and of a constant length under deformation, the shell displacements can be expressed as, (**Burroughs 1978)**:

$$w_{\Phi}(\Phi', t) = W(\Phi')e^{iwt}$$

$$u_{\Phi}(\Phi', t) = U_{\Phi}(\Phi')e^{iwt} \quad (1)$$

where, ω denotes the circular frequency. The stress resultants and couples are related to the displacement of the reference surface by the same expressions derived in appendix A with the eccentricity set equal to zero. **(Kalnins 1963)** show that the actual Φ-dependent coefficients of the variables can be written as:

$$W=\sum_{i=1}^{3}\left[A_i P_{ni}(x)+B_i Q_{ni}(x)\right] \quad (2)$$

$$U_\Phi=\sum_{i=1}^{3}-(1+\nu)C_i\left[A_i P'_{ni}(x)+B_i Q'_{ni}(x)\right] \quad (3)$$

$$U_\theta=\sum_{i=1}^{3}-(1+\nu)C_i\left[A_i P'_{ni}(x)+B_i Q'_{ni}(x)\right] \quad (4)$$

$$N_\Phi=\frac{E\,h}{(1-\nu)R_\Phi}\sum_{i=1}^{3}\{(1+C_i\beta_i)\left[A_i P_{ni}(x)+B_i Q_{ni}(x)\right]$$

$$+(1-\upsilon)C_i\cot\Phi\left[A_i P'_{ni}(x)+B_i Q'_{ni}(x)\right]\} \quad (5)$$

$$N_\theta=\frac{E.h}{(1-\nu)R_\Phi}\sum_{i=1}^{3}\{(1+\nu C_i\beta_i).\left[A_i P_{ni}(x)+B_i Q_{ni}(x)\right]$$

$$-(1-\upsilon)C_i\cot\Phi\left[A_i P_{ni}'(x)+B_i Q_{ni}'(x)\right] \quad (6)$$

$$M_\Phi=\frac{D_b}{R_\Phi^2}\sum_{i=1}^{3}[1+(1+\nu)C_i]\{\beta_i\left[A_i P_{ni}(x)+B_i Q_{ni}(x)\right]$$

$$+(1-\nu)C_i\cot\Phi\left[A_i P'_{ni}(x)+B_i Q'_{ni}(x)\right] \quad (7)$$

$$M_\theta=\frac{D_b}{R_\Phi^2}\sum_{i=1}^{3}[1+(1+\upsilon)C_i]\{\nu\beta_i$$

$$-(1-\nu)\cot\Phi\left[A_i P'_{ni}(x)+B_i Q'_{ni}(x)\right]\} \quad (8)$$

$$Q_\Phi=\frac{D_b}{R_\Phi^2}\sum_{i=1}^{3}[1+(1+\nu)C_i](\nu+\beta_i-1)\left[A_i P'_{ni}(x)+B_i Q'_{ni}(x)\right] \quad (9)$$

Where,

$$C_i=\frac{1+(\beta_i-2)/[(1+\nu)(1+\xi)]}{1-\nu-\beta_i+\xi(1-\nu^2)\Omega^2/(1+\xi^2)} \quad (10)$$

$$\xi=\frac{12R_\Phi^2}{h^2} \quad (11)$$

$$n_i=-\frac{1}{2}+\sqrt{1/4+\beta_i} \quad (12)$$

$$x=\cos\Phi \quad (13)$$

The value of β_i's are the three roots of the cubic equation:

$$\beta^3 - \left[4+(1-\nu\ )\Omega^2\right]\beta^2 + \left[4+((1-\nu^2)\Omega^2+(1+\xi)(1-\nu^2)\right]$$
$$(1-\Omega^2)\beta + (1-\nu)(1-\nu^2)\left[\Omega^2 - \frac{2}{1-\nu}\right]\left[1+(1+\nu^2)\left[\Omega^2 - \frac{1}{1+\nu}\right]\right] = 0 \quad (14)$$

Where,

$$\Omega^2 = \frac{\rho\omega^2 R_\Phi^2}{E} \quad (15)$$

And:

$$D_b = \frac{Eh}{12(1-\nu^2)} \quad (16)$$

In the above equations *Pn(x), Qn(x)* are the Legendre functions of the first and second kinds, respectively, $P_n'(x), Q_n'(x)$ are the derivatives with respect to (Φ) for the Legendre functions of the first and the second kinds, respectively. A_i*'s* & B_i*'s* are arbitrary constants.

The above solutions can be applied to study the free vibration of an elastic spherical shell bounded in general by any two concentric openings.

As the shell is taken to be closed at the apex (Φ=0), and since the Legendre function for the second kind is singular at this point, then the arbitrary constants (B_i*'s*) are set equal to zero. For this reason all terms involving $Q_n(x)$ are omitted.

RAYLEIGH-RITZ METHOD

Rayleigh-Ritz method, which is an extension of the Raleigh's method, helps to determine the natural frequencies and mode shapes with general boundary condition in approximate form.

The Rayleigh-Ritz procedure is essentially statement on the ratio of potential energy to the kinetic energy. At the natural frequency (ω), and assuming separation of variables, the shell displacements may be written as give by **Eq. (1)**. Substituting these in the strain energy expression gives:

$$P = \int_{-h/2}^{h/2}\int_0^{2\pi}\int_0^{2\pi}\frac{1}{2}\left[\sigma_\Phi \in_\Phi' + \sigma_\theta \in_\theta'\right]R_\Phi R_\theta \sin\Phi' d\Phi' d\theta dz \quad (17)$$

Where,

$$\sigma_{\Phi'} = \frac{E}{(1-\nu^2)}\left[\in'_{\Phi'} + \upsilon \in'_{\theta}\right] \quad , \ \sigma_{\theta} = \frac{E}{(1-\nu^2)}\left[\in'_{\theta} + \nu \in'_{\Phi'}\right] \tag{18}$$

and

$$\in'_{\Phi} = \in^{o}_{\Phi} + zk_{\Phi} \quad , \in'_{\theta} = \in'_{\theta} + zk_{\theta} \tag{19}$$

An expression for the maximum strain energy ($P_{\max}$) may be obtained upon taking e^{iwt} to be unity and applying the appropriate expressions for σ_{Φ} , σ_{θ} , $\in'_{\Phi}$ and $\in'_{\theta}$ to given by:

$$\begin{aligned} P_{\max} = \; & \frac{E\,h}{2(1-\nu^2)} \int_0^{2\pi}\int_0^{2\pi} \left\{ \frac{h^2}{12}\left[\frac{1}{R_{\Phi}^2}\left[\frac{\partial}{\partial\Phi'}\left[\frac{U_{\Phi}}{R_{\Phi}} - \frac{\partial W}{R_{\Phi}\,\partial\Phi'}\right]\right]\right.\right. \\ & + \frac{\cos^2\Phi'}{R_{\Phi}^2\ R_{\theta}^2\ \sin^2\Phi'}\left[U_{\Phi} - \frac{\partial W}{\partial\Phi'}\right]^2 + 2\,\nu\frac{\cos\Phi'}{R_{\theta}\ R_{\Phi}^2\ \sin\Phi'}\left[U_{\Phi} - \frac{\partial W}{\partial\Phi'}\right]. \\ & \left.\frac{\partial}{\partial\Phi'}\left[\frac{U_{\Phi}}{R_{\Phi}} - \frac{\partial W}{R_{\Phi}\,\partial\Phi'}\right]\right] + \frac{1}{R_{\Phi}^2}\left[\frac{\partial U_{\Phi}}{\partial\Phi'} + W\right]^2 \\ & + \frac{1}{(R_{\theta}\ \sin\Phi')^2}\left(U_{\Phi}\ \cos\Phi' + W\ \sin\Phi'\right)^2 \\ & \left. + \frac{2\,\nu}{R_{\theta}\ R_{\Phi}\sin\Phi'}\left[\frac{\partial U_{\Phi}}{\partial\Phi'} + W\right]\left(U_{\Phi}\cos\Phi' + W\ \sin\Phi'\right)\right\} \\ & R_{\Phi}\ R_{\theta}\ \sin\Phi'\,d\Phi'\ d\theta \end{aligned} \tag{20}$$

The kinetic energy is:

$$K = \int_{h/2}^{h/2}\int_0^{2\pi}\int_0^{2\pi} \frac{1}{2}\rho\left[\left[\frac{\partial u_{\theta}}{\partial t}\right]^2 + \left[\frac{\partial w}{\partial t}\right]^2\right] R_{\Phi} R_{\theta} \sin\Phi' d\Phi' \, d\theta \tag{21}$$

After integration with respect to (z) and substituting for the appropriate expression, the maximum kinetic will take the form:

$$K_{\max} = \frac{\omega^2 \rho h}{2}\int_0^{2\pi}\int_0^{2\pi} (u_{\Phi}^2 + w^2) R_{\Phi} R_{\theta} \sin\Phi' d\Phi' \, d\theta \tag{22}$$

Equating the maximum kinetic energy to the maximum potential energy, the natural frequency can be written as:

$$\omega_r^2 = \frac{P_{\max}}{K_{\max}} = \frac{N}{D} \tag{23}$$

Where, N and D represent the equations in numerator and denominator, respectively.
Following the procedure of Rayleigh-Ritz's method, the radial (or transverse) and tangential displacements can be written in power series form as:

$$w(\Phi') = \sum_{i=1}^{3} a_i w_i(\Phi') \qquad u_\Phi(\Phi') = \sum_{i=1}^{3} b_i u_{\Phi i}(\Phi') \tag{24}$$

Where, the a_i's and b_i's are coefficients to be determined.

The functions $W_i(\Phi')$, $U(\Phi')$ satisfy all the geometry boundary conditions of the system. **Eq.(23)** is an exact expression for the frequency according to Rayleigh quotient. In order to use the procedure of Rayleigh-Ritz's method, **Eq. (24)** is substituted into **Eq.(20)** and **(22)**, then the results are used in **Eq. (23).**

Now substituting **Eq. (24)** into **Eqs. (20)** and **(22)**, and after some mathematical manipulations, the following equation will results:

$$\omega_r^2 = \frac{\alpha}{\Psi} \tag{25}$$

Where,

$$\alpha = \sum_{i=1}^{n}\sum_{i=1}^{n} c_i c_j \frac{Eh\pi}{(1-v^2)} \int_{o}^{2\pi} \Bigg\{ \frac{h^2}{12R_\Phi^4}\Big[U_{\Phi i}''U_{\Phi i}'' - 2U_{\Phi i}'' + W_i''W_j''''\Big]\sin\Phi'$$
$$+\frac{vh^2}{6R_\theta R_\Phi^3}\Big[U_{\Phi i}U_{\Phi i}'' - U_{\Phi i}W'' - U_{\Phi i}W_i' + W'W_i''\Big]\cos\Phi'$$
$$+\frac{h^2}{12R_\theta^2 R_\Phi^2}\Big[U_{\Phi i}U_{\Phi j}' - 2U_{\Phi i}W_i' + W_i'W_j'\Big]\frac{\cos^2\Phi'}{\sin\Phi'}$$
$$+\frac{1}{R_\Phi^2}\Big[U_{\Phi i}'U_{\Phi j}' + 2U_{\Phi i}W_i + W_i'W_j'\Big]\sin\Phi'$$
$$+\frac{1}{R_\theta^2}\left[U_{\Phi i}U_{\Phi j}\frac{\cos^2\Phi'}{\sin\Phi'} + 2U_{\Phi i}W_i\cos\Phi' + W_iW_j\sin\Phi'\right]$$
$$+\frac{2v}{R_\Phi R_\theta}\Big[U_{\Phi i}U_{\Phi i}'\cos\Phi' + U_{\Phi i}'W_i\sin\Phi' - U_{\Phi i}W_i\cos\Phi' + W_iW_j\sin\Phi'\Big]\Bigg\}$$
$$R_\Phi R_\theta\, d\Phi' \tag{26}$$

$$\Psi = \sum_{i=1}^{n}\sum_{i=1}^{n} c_i c_j \int_{0}^{2\pi}\Big[U_iU_j + W_iW_j\Big]R_\Phi R_\theta \sin\Phi' d\Phi' \tag{27}$$

An n-term finite sum leads to the estimation of the first frequencies. **Eqs. (26)** and **(27)** gives the physical properties of the shell from the stiffness and mass distribution point of view.

The stiffness and mass of the shell are given by the following two equations respectively:

$$k_{ij} = \frac{E\,h\,\pi}{(1-\nu^2)} \int_0^{2\pi} \Big\{ \frac{h^2}{12\,R_\Phi^4} \left[U'_{\Phi i} U'_{\Phi i} - 2U'_{\Phi i}\, W_i'' + W_i''\, W_j'' \right] \sin\Phi'$$

$$+ \frac{\nu\; h^2}{6\,R_\theta\, R_\Phi^3} \left[U_{\Phi i}\, U'_{\Phi i} - U_{\Phi i}\, W_i'' - U'_{\Phi i}\; W_i' + W_i'\, W_i'' \right] \cos\Phi'$$

$$+ \frac{h^2}{12\,R_\Phi^2\, R_\theta^2} \left[U'_{\Phi i}\, U'_{\Phi j} - 2U'_{\Phi i} W_i' + W_i' W_j' \right] \frac{\cos^2\Phi'}{\sin\Phi'}$$

$$+ \frac{1}{R_\Phi^2} \left[U'_{\Phi i} U'_{\Phi j} + 2U'_{\Phi i}\, W_j + W_i \;\; W_j \right] \sin\Phi'$$

$$+ \frac{1}{R_\theta^2} \left[\; U_{\Phi i}\, U_{\Phi j} \, \frac{\cos^2\Phi'}{\sin\Phi'} + 2U_{\Phi i} \;\, W_i \cos\Phi' \; + W_i\, W_j \, \sin\Phi' \; \right]$$

$$+ \frac{2\nu}{R_\Phi\, R_\theta} \left[U_{\Phi i}\, U'_{\Phi i} \cos\Phi' + U'_{\Phi i} \;\, W_i \sin\Phi' \; + U_{\Phi i}\, W_i \cos\Phi' \; + W_i\, W_i \sin\Phi' \; \right] \Big\}.$$

$$.\; R_\Phi\, R_\theta\, d\Phi'$$

and

(28)

$$m_{ij} = \int_0^{2\pi} \rho h \pi \left[U_i U_j + W_i W_j \right] R_\Phi R_\theta \sin\Phi' d\Phi' \tag{29}$$

Then

$$\omega_r^2 = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j k_{ij}}{\sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j m_{ij}} \tag{30}$$

The exact frequency is always smaller than the approximate value. In order to minimize the approximate value, which given by **Eq.(30)**, it should be differentiated with respect to c_i and equating the resulting expression to zero, that is:

$$\frac{\partial}{\partial c_i} = \frac{D \partial N / \partial c_i - N \partial D / \partial c_i}{D^2} = 0 \qquad ,i=1,2,3,\ldots\ldots n \tag{31}$$

The only way in which this equation can equal is zero if the numerator equals zero, since D is never equal to zero. The numerator can be written in a more useful form as:

$$\frac{\partial N}{\partial c_i} - \frac{N}{D} \frac{\partial D}{\partial c_i} = 0 \;, \qquad i = 1,2,3,\ldots,n \tag{32}$$

$=N/D$

It is as given by equation (23) $\omega_r^2 = \frac{N}{D}$, and n is the number of terms in the approximate solution. The infinite degrees of freedom system has been replaced by an n degree of freedom system. Therefore, **Eq. (31)** can be written in a matrix form as:

$$\left[\{K\} - \omega^2 \{M\}\right]\{c\} = \{0\} \quad (33)$$

The stiffness and mass are determined at the edge ($\Phi = \Phi_0$) of the spherical shell using (28 and 29) respectively. The values equated in above equations are then substituted in the following determinant:

$$\begin{vmatrix} k_{11} - \Omega^2 m_{11} & k_{12} - \Omega^2 m_{12} & k_{13} - \Omega^2 m_{13} \\ k_{21} - \Omega^2 m_{21} & k_{22} - \Omega^2 m_{22} & k_{23} - \Omega^2 m_{23} \\ k_{31} - \Omega^2 m_{31} & k_{32} - \Omega^2 m_{32} & k_{33} - \Omega^2 m_{33} \end{vmatrix} = 0 \quad (34)$$

The value of Ω^2 which make the determinant equal zero represent the natural frequency of the shell.

RESULTS AND DISCUSSION

In order to confirm the accuracy of the theoretical results, these results are compared with the available literature due complexity of obtaining a closed form solution for the free vibration characteristics of a prolate spheroid shell. From **Table (1)** it can be noted that the variation of the natural frequencies of bending modes increase with thickness and with the mode number. This phenomenon can be elaborated due to the fact that the strain energy increased with increasing the ratio of thickness for larger eccentricities ratio.

The non-dimensional frequency coefficients for the first three flexural modes which computed from the present work with (h/a=0.05) are presented in **Table (2)** along with the results of **(Burroghs and Magrab 1978)**. From this table it is seen that there is reasonable agreement between these results, which provide the accuracy of the formulation and results.

Fig.2 shows the non-dimensional natural frequency ($\lambda = \sqrt{\rho / E}\omega a$) of the first three modes of vibration as a function of the eccentricity ratio obtained by the Rayleigh- Ritz's method using the non-shallow shell theory. It is clearly shown that the tendency of the natural frequencies towards higher values as the eccentricity ratio increases. This behavior could be explained by the mode shapes of a closed spherical shell would resemble those of a prolate spheroid up to certain eccentricity. As the eccentricity increases, the bending stress increased and the potential energy increased. Another reason is that the geometry of the prolate shape is stiffer than the spherical shape.

Fig.3 gives the first few natural frequencies as a function of the thickness ratio for a prolate spheroid with (e=0) obtained by RRM. **Fig.4** show the first few natural frequencies as a function of thickness ratio with (e=0.7). All these figures are obtained for $(\nu = 0.3)$ and they depend on the bending as well as the membrane modes using the non-shallow shell theory. It can be noted that the variation of the natural frequency of the bending modes increases with thickness and with the mode number. This phenomenon can be elaborated due to the fact that the strain energy increased with increasing the thickness ratio. Also, for larger eccentricity ratio, the variations are more pronounced than for smaller eccentricities.

Fig.5 shows the effect of eccentricity ratio on the first membrane mode. It is seen that the natural frequency increased with increasing the eccentricity ratio. The eccentricity ratio affects the natural frequency hardly at the lower range, while this effect decreased when the eccentricity ratio beyond 0.8.

The mode shapes of the first three modes of the prolate spheroid shell are shown in **Fig.8**, in which both the transverse and tangential displacements are illustrated. This figure shows that the modes of the transverse displacement occurs at a position in which the tangential; displacement has maxima and vice versa.

CONCLUTIONS

The main conclutions from the present work can be summarized as;

1- Natural frequencies are seen to have two types of behaviour against increasing the thickness to major radius ratio. One type, which is associated with the bending modes, tends to increase with thickness, while the other type, which is associated with membrane mode, remains unaffected by the thickness variation.
2- Both bending and membrane modes natural frequencies tend to increase with increasing eccentricity ratio.
3- The natural frequency tends to increase with increasing the ratio of thickness of the shell.

APPENDIX

The principal curvatures of the surface as a function of the angle of inclination (Φ) in the following form.

$$R_{\Phi} = \frac{a(1-e^2)}{(1-e^2\cos^2\Phi')^{3/2}}$$

$$R_{\theta} = \frac{a}{(1-e^2\cos^2\Phi')}$$

Where (Φ') is the angle in the space between the vertical axis and the normal vector, it is given by

$$\cos\Phi' = \frac{\sin\beta}{\sqrt{1-e^2\cos^2\beta}},$$

(e) is the eccentricity ratio of the spheroidal shell , which is given by;

$$e = \frac{1}{\cosh\alpha} = \frac{1}{c}$$

The strains, expressed in terms of displacement can be written as:

$$\varepsilon_{\Phi}^{\circ} = \frac{1}{R_{\Phi}}\left[\frac{\partial u_{\Phi}}{\partial\Phi'} + w\right]$$

$$\varepsilon_{\theta}^{\circ} = \frac{1}{R_{\theta}\sin\Phi'}\left[u_{\Phi}\cos\Phi' + w\sin\Phi'\right] = \frac{1}{R_{\theta}}\left[u_{\Phi}\cot\Phi' + w\right]$$

$$k_{\Phi} = \frac{1}{R_{\Phi}}\frac{\partial}{\partial\Phi'}\left[\frac{1}{R_{\Phi}}\left(u_{\Phi} - \frac{\partial w}{\partial\Phi'}\right)\right]$$

$$k_{\theta} = \frac{1}{R_{\Phi}\sin\Phi'}\left[\frac{\cos\Phi'}{R_{\Phi}}\left(u_{\Phi} - \frac{\partial w}{\partial\Phi'}\right)\right]$$

If E, ν are as in nomenclature then, the forces and moments per unit length will be

$$N_{\Phi} = \frac{Eh}{1-\nu^2}\left[\varepsilon_{\Phi}^{o} + \varepsilon_{\theta}^{o}\right]$$

$$N_{\theta} = \frac{Eh}{1-\nu^2}\left[\varepsilon_{\theta}^{o} + \nu\varepsilon_{\Phi}^{o}\right]$$

$$M_{\Phi} = \frac{Eh^2}{12(1-\nu^2)}\left[K \quad + \nu K \quad\right]$$

$$M_{\theta} = \frac{Eh^2}{12(1-\nu^2)}\left[K_{\theta} + \nu K_{\Phi}\right]$$

Substituting the relevant expressions get:-

$$N_\Phi = \frac{E\,h}{1-\upsilon^2}\left[\frac{1}{R_\Phi}\left(\frac{\partial u_\Phi}{\partial \Phi'} + w\right) + \frac{\upsilon}{R_\Phi}\left(u_\Phi \cot\Phi' + w\right)\right]$$

$$N_\theta = \frac{E\,h}{1-\upsilon^2}\left[\frac{1}{R_\theta}\left(u_\Phi \cot\Phi' + w\right) + \frac{\upsilon}{R_\Phi}\left(\frac{\partial u_\Phi}{\partial \Phi'} + w\right)\right]$$

$$M_\Phi = \frac{E\,h^3}{12(1-\upsilon^2)}\left[\frac{1}{R_\Phi}\frac{\partial}{\partial \Phi'}\frac{1}{R_\theta}\left(u_\Phi - \frac{\partial w}{\partial \Phi'}\right) + \frac{\upsilon \cos\Phi'}{R_\Phi\,R_\theta \sin\Phi'}\left(u_\Phi - \frac{\partial w}{\partial \Phi'}\right)\right]$$

$$M_\theta = \frac{E\,h^3}{12(1-\upsilon^2)}\left[\frac{\cos\Phi'}{R_\Phi\,R_\theta \sin\Phi'}\left(u_\Phi - \frac{\partial w}{\partial \Phi'}\right) + \frac{\upsilon}{R_\Phi}\frac{\partial}{\partial \Phi'}\left(\frac{1}{R_\theta}\left(u_\Phi - \frac{\partial w}{\partial \Phi'}\right)\right)\right]$$

Table (1): Dimensionless natural frequency coefficients for the axisymmetric free vibration of a prolate spheroidal shell.

Mode Number	E=0.3	e=0.7		
	h/a=0.01	h/a=0.05	h/a=0.01	h/a=0.05
1	0.0	0.0	0.0	0.0
2	0.16	0.16	0.725	0.725
3	0.18	0.19	0.87	0.89
4	0.2	0.23	0.91	0.93

Table (2): Comparison of other estimates of Ω for the flexural modes of a thin prolate spheroidal shell with e=0.7

Mode Number	Present Work h/a=0.05	Reference [9]
2	0.725	0.73
3	0.89	0.90
4	0.93	0.95

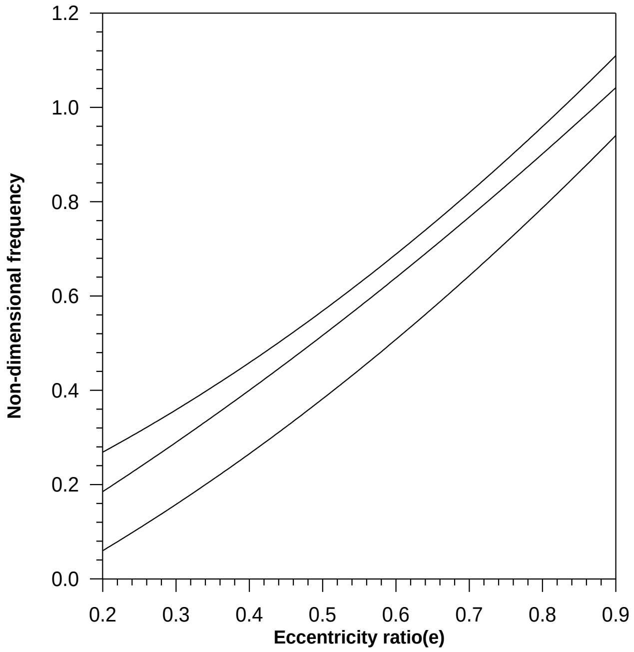


Fig.(2): Effect of eccentricity on the first three bending modes obtained by RRM

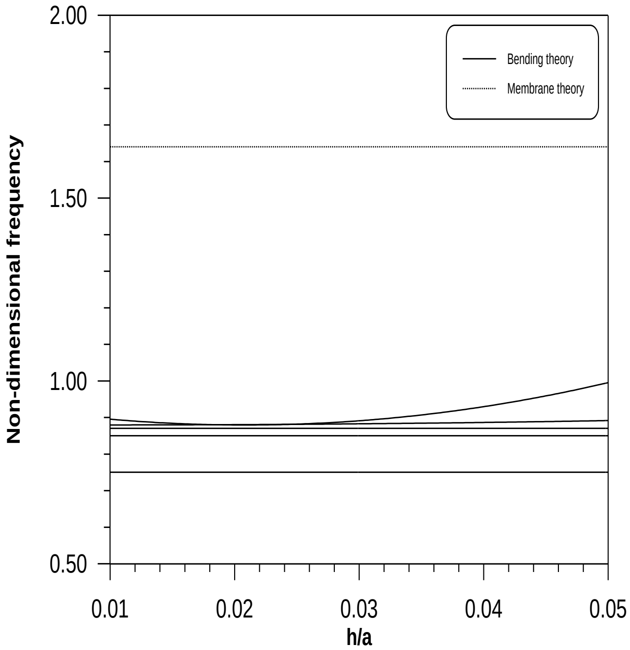


Fig. (3): Effect of the eccentricity ratio on the natural frequency of a full sphere (e=0) obtained by RRM

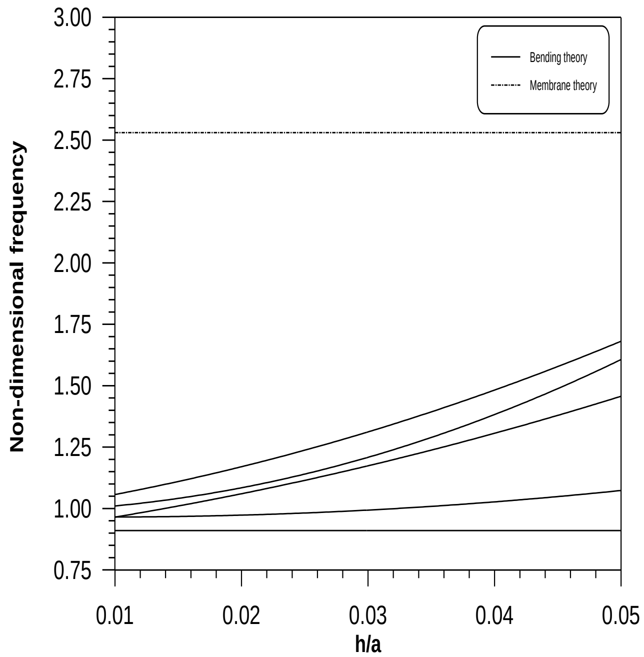


Fig. (4): Effect of the thickness ratio on the natural frequency of a prolate spheroidal shell (e=0.7) obtained by RRM

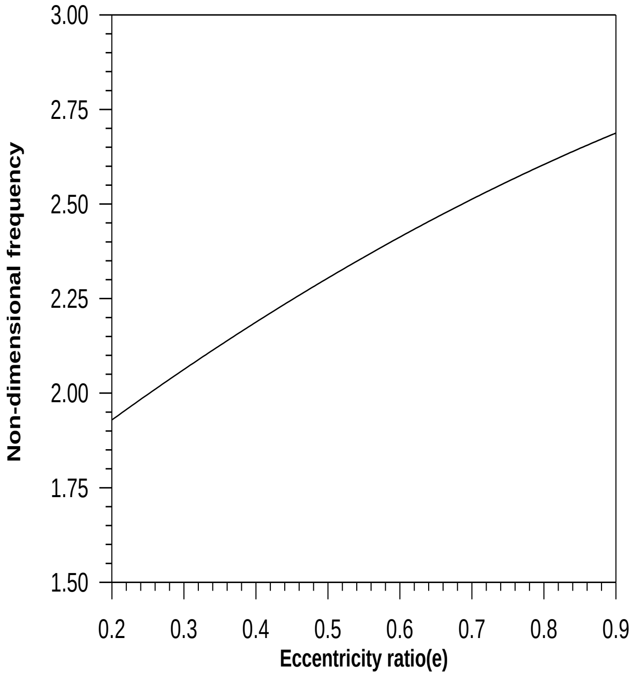
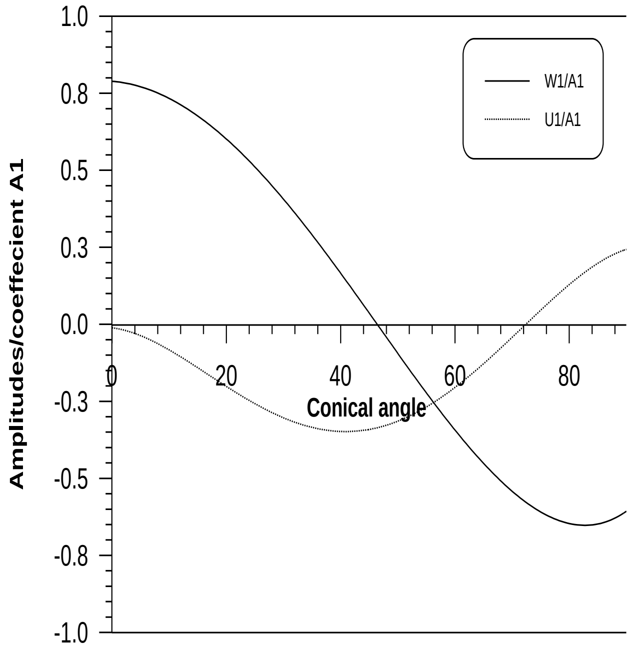
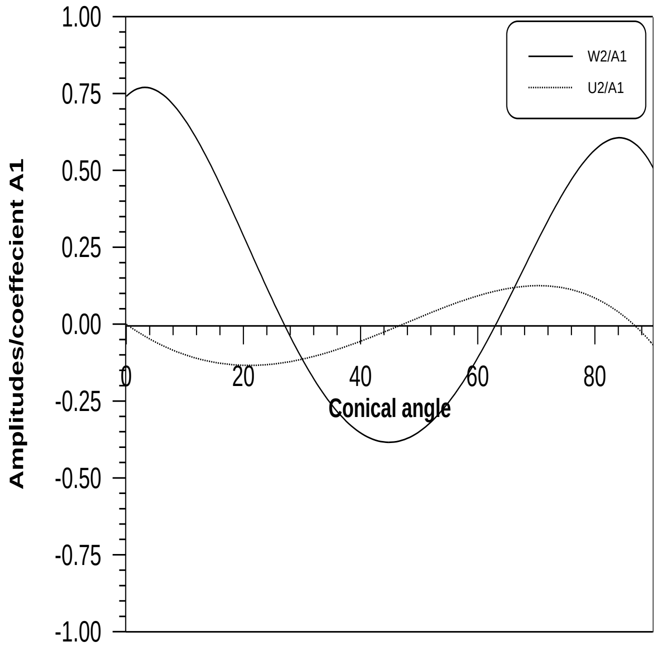


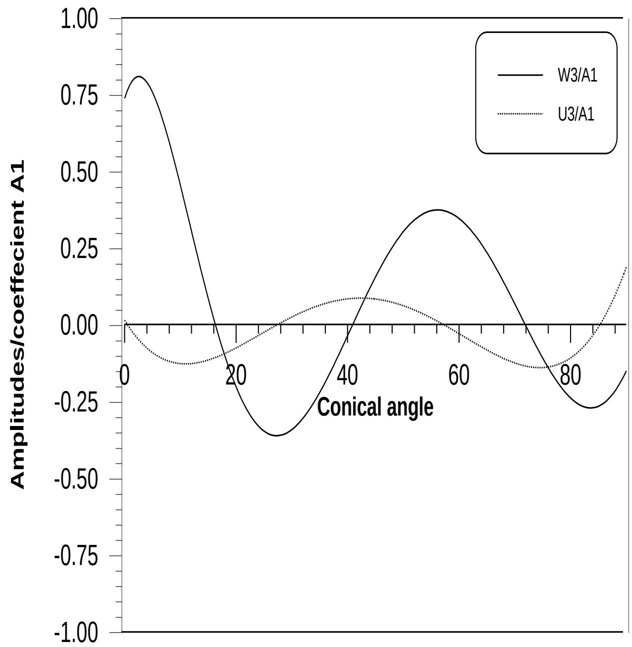
Fig.(5): Effect of eccentricity on the first membrane mode



(a) First mode



(b) Second mode



(c) Third mode

Fig. :(6) Mode shape associated with the first three natural frequency of non-shallow spheroidal shell (e=0.7) obtained by RRM

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NOMENCLATURE

A_i Arbitrary constants.

a Major semi-axis of a prolate spheroid shell.

B_i Arbitrary constants.

b Minor semi-axis of a prolate spheroid shell.

$c_{i,j}$ Element of the boundary conditions matrix.

D_b Bending stiffness ($E.h^3/12(1-\nu^2)$).

E Young's modulus of elasticity (N/ m^2).

e Ecentricity ratio($\sqrt{1-b^2}/a^2$).

h Shell thickness(mm).

M_ϕ, M_θ Moments per unit length (Nm/m).

N_ϕ, N_θ Membrane forces per unit length (N/m).

$P_n(x)$ Legendre function of the first kind.

$P_n'(x)$ First derivative of the Legendre function of the first kind.

$P_n''(x)$ Second derivative of the Legendre function of the first kind.

$Q_n(x)$ Legendre function of the second kind.

$Q_n'(x)$ First derivative of the Legendre function of the second kind.

Q_ϕ Transverse shearing force per unit length (N/m).

R_ϕ, R_θ Principal radii of curvatures of a prolate shell.

t Time (sec).

u_θ, u_θ Tangential displacement (m).

w Transverse or radial displacement (m).

$\in_\theta, \in_\theta$ Strains.

Φ' Inclination angle of a prolate spheroid.

Φ Inclination angle of a spherical shell model.

Φ_0 Opening angle of the approximate spherical shell.

λ Non-dimensional frequency parameter (($\rho/E)^{1/2}\omega.a$).

θ Angle of rotation in the meridian direction

ρ Density (kg/m ^{3}).

Ω Non-dimensional frequency parameter (($\rho/E)^{1/2}\omega.R)$.

ω Circular frequency (rad/sec).

ω_0 $(E/\rho)^{1/2}h/d$.

$\sigma_\Phi, \sigma_\theta$ Stress resultants (N/m ^{2}).

ν Poisson’s ratio.