COMPONENTS OF AR-QUIVERS FOR STRING ALGEBRAS OF TYPE $\widetilde{\mathbb{C}}$ AND A CONJECTURE BY GEISS-LECLERC-SCHRÖER

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Abstract. We study modules of certain string algebras, which are referred to as of affine type $\widetilde{\mathbb{C}}$. We introduce minimal string modules and apply them to explicitly describe components of the Auslander-Reiten quivers of the string algebras and τ-locally free modules defined by Geiss-Lerclerc-Schröer. In particular, we show that an indecomposable module is τ-locally free if and only if it is preprojective, or preinjective or regular in a tube. As an application, we prove Geiss-Leclerc-Schröer's conjecture on the correspondence between positive roots of type $\widetilde{\mathbb{C}}$ and τ-locally free modules of the corresponding string algebras. Furthermore, given a positive root α, we show that if α is real, then there is a unique τ-locally free module M (up to isomorphism) with $\underline{\text{rank}}M = \alpha$; otherwise there are families of τ-locally free modules with $\underline{\text{rank}}M = \alpha$.

1. Introduction

Given a symmetrizable Cartan matrix C with a symmetrizer D, Geiss-Leclerc-Schröer [7] construct a quiver $Q = Q(C, \Omega)$ and define a quotient path algebra $H = H(C, D, \Omega) = KQ/I$, where K is a field and I is an ideal generated by some nilpotency relations and some commutative relations. In particular, there is a loop at each vertex in Q and powers of the nilpotency relations in I encode the symmetrizer D. They then develop a sequence of work based on the representation theory of H [7, 8, 9, 10, 11], providing a uniform approach to studying connections between representation theory of simply laced and non-simply laced (or valued) quivers, and Lie theory and cluster theory. For instance, it includes a generalisation of two fundamental results in quiver representation theory, Gabriel's Theorem and Dlab-Ringel's Theorem (to be made more precise later), a construction of enveloping algebras and a generalisation of Caldero-Chapoton's formula in cluster theory.

We are interested in the correspondence between τ-locally free H-modules and positive roots of type C [7, 8], when C is a Cartan matrix of type $\widetilde{\mathbb{C}}$. Let e_i be the idempotent in H corresponding to the vertex i in Q and $H_i = e_iHe_i$. A left H-module M is said to be *locally free* if $M_i = e_iM$ is a free H_i-module for all i, and for such a module M, denote by $\underline{\text{rank}}M = (r_1, \cdots, r_n)$ the rank vector of M. That is, each r_i is the rank of the free H_i-module M_i. An indecomposable H-module M is *τ-locally free* if the AR-translations $\tau^k(M)$ for all $k \in \mathbb{Z}$ are locally free. Note that not all indecomposable locally free modules are τ-locally free. Geiss-Leclerc-Schröer [7] prove that there are only finitely many isomorphism classes of τ-locally free H-modules if and only if the Cartan matrix C is of Dynkin type. Moreover, in this case, the assignment $M \mapsto \underline{\text{rank}}M$ offers a bijection between the isomorphism classes of τ-locally free H-modules and the positive roots of type C, i.e. the positive roots of a complex Lie algebra defined by C. These results generalize Gabriel's Theorem for Dynkin quivers [5] and Dlab-Ringel's Theorem for Dynkin (valued) quivers [4] (also known as modulated graphs, see for instance [7]), in the sense that both theorems provide a one to one correspondence between the isomorphism classes of indecomposable representations of a Dynkin (valued) quiver of type C and the positive roots of type C via the map sending an indecomposable representation to its dimension vector.

For non-Dynkin symmetrizable Cartan matrices, Geiss-Leclerc-Schröer propose the following conjecture [8, Conjecture 5.3].

Conjecture 1 [Geiss-Leclerc-Schröer] *Let $H = H(C, D, \Omega)$. Then there is a bijection between the positive roots of type C and the rank vectors of τ-locally free H-modules.*

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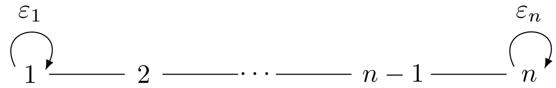
Evidence supporting the conjecture includes the following. First, when C is symmetric and D is the identity matrix, the conjecture is true by Kac's Theorem [13, 14]. Second, Geiss-Leclerc-Schröer [12] prove that there is a bijection between isomorphism classes of τ-locally free rigid H-modules and real Schur roots of Q. Note also Chen-Wang [3] work on categorification of foldings of root lattices and in the case of Dynkin type they recover Geiss-Leclerc-Schröer's result on the correspondence between τ-locally free H-modules and positive roots of type C.

In general, Conjecture 1 is still open. In this paper, we consider the affine case of type $\widetilde{\mathbb{C}}_{n-1}$, that is, the Cartan matrix C is the following $n \times n$ matrix

$$C = \begin{pmatrix} 2 & -1 & & & & & \\ -2 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -2 \\ & & & & & -1 & 2 \end{pmatrix}$$

and the symmetrizer $D = \mathrm{diag}(2, 1, 1, \cdots, 1, 1, 2)$. Then the algebra $H = H(C, D, \Omega) = KQ/I$, where Q is a quiver of type $\mathbb{A}_n$ when the two loops at 1 and n are ignored,



and I is the ideal generated by ε_1^2 and ε_n^2. In particular, H is a string algebra and we say that H is a *string algebra of type* $\widetilde{\mathbb{C}}_{n-1}$. Note that, if C is of other affine type, then $H = H(C, D, \Omega)$ is not a string algebra. We will study the representation category of H, in particular the Auslander-Reiten theory of H, using techniques from string algebras, and minimal string modules that are to be introduced later in this paper. The explicit construction of Auslander-Reiten sequences (also written as AR-sequences) for string algebras in [4] by Butler-Ringel is particularly helpful in our understanding of the Auslander-Reiten quiver (also written as AR-quiver) of H.

We define the *index* of an indecomposable H-module M to be (a, b), where a is the number of irreducible maps to M and b is the number of irreducible maps from M in the AR-quiver Γ_H of H and we say that a string module M is *minimal* if in Γ_H each irreducible map from $M(w)$ is injective and each irreducible map to $M(w)$ is surjective. Using Butler-Ringel's construction of AR-sequences, we classify all the minimal string modules. This classification then leads to explicit description of connected components of the AR-quiver of H. Consequently, we know precisely where τ-locally free modules are in the AR-quiver and so prove Conjecture 1 for the case where C is of type $\widetilde{\mathbb{C}}_{n-1}$ and D is minimal. We have the following main results.

Theorem A (Theorem 3.14) *Let $H = H(C, D, \Omega)$ be a string algebra of type $\widetilde{\mathbb{C}}_{n-1}$. The AR-quiver Γ_H of H consists of the following, which are pairwise disjoint.*

1. *One component $\mathcal{T}_{PI}$ containing all the indecomposable preprojective modules and all the indecomposable preinjective modules (up to isomorphism).*
2. *One tube of rank $n-1$.*
3. *Homogeneous tubes $\mathcal{H}_{w,S}$, where w runs through all the representatives of bands in H and S runs through the isomorphism classes of simple modules of the Laurent polynomial ring $K[T, T^{-1}]$.*
4. *Components $\mathcal{T}_\lambda$ of type $\mathbb{Z}\mathbb{A}_\infty^\infty$, where λ runs through the isomorphism classes of minimal string modules of type (2,2).*

Theorem B (Theorem 3.19) *Let $H = H(C, D, \Omega)$ be a string algebra of type $\widetilde{\mathbb{C}}_{n-1}$ and let M be an indecomposable H-module. Then M is τ-locally free if and only if one of the following is satisfied:*

1. *M is preprojective.*
2. *M is preinjective.*
3. *M is a regular module occurring in any tube.*

Theorem C (Theorem 4.9) *Let $H = H(C, D, \Omega)$ be a string algebra of type $\widetilde{\mathbb{C}}_{n-1}$. Then α is a positive root of type C if and only if there is a τ-locally free module M such that $\underline{\mathrm{rank}} M = \alpha$. Moreover,*

(1) *if α is a positive real root, then there is a unique τ-locally free H-module M (up to isomorphism) such that* $\underline{\mathrm{rank}}M = \alpha$;
(2) *if α is a positive imaginary root, then there are families of τ-locally free H-modules M such that* $\underline{\mathrm{rank}}M = \alpha$;
(3) *the modules at the bottom of the tube of rank $n-1$ are rigid.*

Corollary D (Corollary 4.10) *Let C be a Cartan matrix of type $\widetilde{\mathbb{C}}_{n-1}$ and* $D = \mathrm{diag}(2, 1, \cdots, 1, 2)$. *Then Conjecture 1 is true.*

After the paper was completed, we were informed that partial results of Theorem A were considered by C. Ricke in her thesis [15], where, relevantly, the AR-sequences connecting projective modules and injective modules and tubes are constructed, and indecomposable rigid modules are classified. Here we would like to emphasise the new ingredient in our method (i.e. minimal string modules) and a complete description of all the AR-components. In particular, we show that there are rays (corays) dividing the component containing preprojective and preinjective modules into regions. As a consequence we show that the only τ-locally free modules in the component are preprojective or preinjective. Moreover, the regular components other than tubes are of the form $\mathbb{Z}\mathbb{A}_\infty^\infty$, which are classified by minimal string modules of type $(2, 2)$ and none of which contains τ-locally free modules.

The remaining part of this paper is organized as follows. In Section 2, we recall some basic definitions on string algebras and Butler-Ringel's construction of AR-sequences. In Section 3, we develop the theory of minimal string modules to prove Theorem A and Theorem B. In Section 4, we first recall basic definitions and facts on root systems and Weyl groups, and then prove Theorem C and Corollary D.

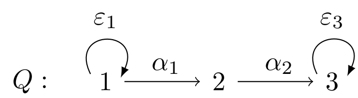
2. Basic notions and facts on string algebras

In this section, we recall the definition of string algebras and basic properties of their module categories [2]. Let K be a field and A be a finite dimensional K-algebra. Throughout this paper, all modules are finitely generated left A-modules. We denote by $S_1, S_2, \cdots, S_n$ a complete list of simple A-modules, and by $P_1, P_2, \cdots, P_n$ (resp. $I_1, I_2, \cdots, I_n$) a complete list of indecomposable projective (resp. injective) A-modules (up to isomorphism).

2.1. **String algebras.** Let $Q = (Q_0, Q_1)$ be a quiver, where Q_0 denotes the set of vertices and Q_1 denotes the set of arrows in Q. Given an arrow $\alpha \in Q_1$, its starting and ending vertices are denoted by $s(\alpha)$ and $t(\alpha)$, respectively.

Definition 2.1. A finite dimensional K-algebra $A = KQ/I$ is called a *string algebra* if the following conditions are satisfied:
(1) for any vertex $i \in Q_0$, there are at most two incoming and at most two outgoing arrows;
(2) for any arrow $\alpha \in Q_1$, there is at most one arrow β and at most one arrow γ such that $\beta\alpha \notin I$ and $\alpha\gamma \notin I$;
(3) the ideal I is generated by a set of zero relations.
In particular, a string algebra $A = KQ/I$ is called *gentle* if I is generated by paths of length 2.

Example 2.2. Let $A = KQ/I$, where



and $I = \langle \varepsilon_1^2, \varepsilon_3^2 \rangle$. Then A is a string algebra.

2.2. **Strings and bands.** Let $A = KQ/I$ be a string algebra. Given an arrow $\alpha \in Q_1$, we denote by α^{-1} the *formal inverse* of α, with $s(\alpha^{-1}) = t(\alpha)$ and $t(\alpha^{-1}) = s(\alpha)$, and write $(\alpha^{-1})^{-1} = \alpha$. A *word* $w = c_1c_2\cdots c_m$ of length $m \geq 1$ is a sequence of arrows and their formal inverses such that $s(c_i) = t(c_{i+1})$ for $1 \leq i < m$. We define $w^{-1} = (c_1c_2\cdots c_m)^{-1} = c_m^{-1}\cdots c_2^{-1}c_1^{-1}$, $s(w) = s(c_m)$ and $t(w) = t(c_1)$. A word $w = c_1c_2\cdots c_m$ of length $m \geq 1$ is called a *string* if $c_{i+1} \neq c_i^{-1}$, and no subword nor its inverse belongs to I. In addition, we associate two *trivial* strings $1_{(u,1)}$ and $1_{(u,-1)}$ of length zero for any vertex $u \in Q_0$, where $s(1_{(u,i)}) = t(1_{(u,i)}) = u$ and $(1_{(u,i)})^{-1} = 1_{(u,-i)}$ for $i = 1, -1$. A string $w = c_1c_2\cdots c_m$ is said to be *direct* if all the c_i are arrows, and *inverse* if all the c_i are inverses of arrows. By definition, a trivial string is both direct and inverse. We denote by $\mathrm{St}(A)$ the set of all strings in A.

A nontrivial string w is called a *band* if $s(w) = t(w)$ and each power w^r is a string, but w itself is not a power of a string of smaller length. We denote by $\mathrm{Ba}(A)$ the set of all bands in A.

On $\mathrm{St}(A)$, let ρ be the equivalence relation that identifies every string w with its inverse w^{-1}. On $\mathrm{Ba}(A)$, let ρ' be the equivalence relation that identifies every string $w = c_1c_2\cdots c_m$ with any cyclically permuted strings $w_{(i)} = c_ic_{i+1}\cdots c_mc_1\cdots c_{i-1}$ and their inverses $w_{(i)}^{-1}$, $1 \le i \le m$. We choose a complete set $\overline{\mathrm{St}}(A)$ of representatives of $\mathrm{St}(A)$ relative to ρ, and a complete set $\overline{\mathrm{Ba}}(A)$ of representatives of $\mathrm{Ba}(A)$ relative to ρ'.

We write $u \sim w$ if two strings (resp. bands) u and w are equivalent, and $u \not\sim w$ otherwise. Represent a string $w = \alpha_1^{\epsilon_1}\alpha_2^{\epsilon_2}\cdots\alpha_m^{\epsilon_m}$, where $\alpha_i \in Q_1$ and $\epsilon_i \in \{1, -1\}$ for all i, as a walk

$$x_1 \overset{\alpha_1}{\text{——}} x_2 \overset{\alpha_2}{\text{——}} \cdots \overset{\alpha_m}{\text{——}} x_{m+1},$$

where $x_1, x_2, \cdots, x_{m+1}$ are the vertices of Q visited by w, α_i is an arrow from x_{i+1} to x_i if $\epsilon_i = 1$, or an arrow from x_i to x_{i+1} if $\epsilon_i = -1$. This equivalence relation induces an equivalence relation on the walks. That is, the walk

$$w: \ x_1 \overset{\alpha_1}{\text{——}} x_2 \overset{\alpha_2}{\text{——}} \cdots \overset{\alpha_m}{\text{——}} x_{m+1}$$

is equivalent to the walk

$$w^{-1}: \ x_{m+1} \overset{\alpha_m}{\text{——}} x_m \overset{\alpha_{m-1}}{\text{——}} \cdots \overset{\alpha_1}{\text{——}} x_1.$$

Similarly, walks of bands are equivalent if the corresponding bands are equivalent with respect to ρ'.

2.3. **String modules and band modules.** Let $w = \alpha_1^{\epsilon_1}\alpha_2^{\epsilon_2}\cdots\alpha_m^{\epsilon_m}$ be a string with the corresponding walk

$$x_1 \overset{\alpha_1}{\text{——}} x_2 \overset{\alpha_2}{\text{——}} \cdots \overset{\alpha_{m-1}}{\text{——}} x_n \overset{\alpha_m}{\text{——}} x_{m+1}\ .$$

The *string module* defined by w is the representation $M(w) = ((V_i)_{i\in Q_0}, (\varphi_\alpha)_{\alpha\in Q_1})$, where the vector spaces

$$V_i = \begin{cases} \oplus_{x_j=i}Kx_j & \text{if } i = x_j \text{ for some } j \in \{1, 2, \cdots, m+1\}, \\ 0 & \text{otherwise,} \end{cases}$$

and the linear maps φ_α are given by

$$\varphi_\alpha(x_i) = \begin{cases} x_j & \text{if } \alpha = \alpha_i : x_i \to x_j \text{ for some } 1 \le i \le m, \\ 0 & \text{otherwise.} \end{cases}$$

Here by abuse of notation, we use x_i to denote the basis of the 1-dimensional space Kx_i. The module $M(w)$ can be unfolded as a representation $U(w)$ as follows,

$$U_{x_1} \overset{U_{\alpha_1}}{\text{——}} U_{x_2} \overset{U_{\alpha_2}}{\text{——}} \cdots \overset{U_{\alpha_{m-1}}}{\text{——}} U_{x_m} \overset{U_{\alpha_m}}{\text{——}} U_{x_{m+1}}$$

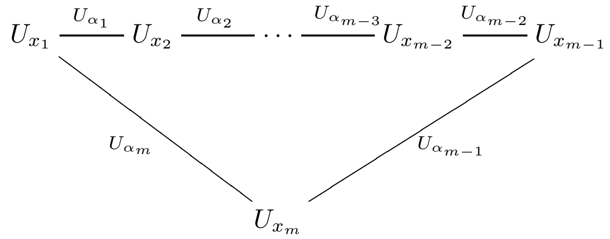
where $U_{x_i} = K$ and $U_{\alpha_j} = \mathrm{id}_K$ for all i and j.

By construction,

$$\dim_K V_i = |\{j \in \{1, 2, \cdots, m+1\} | x_j = i\}|$$

for any $i \in Q_0$, and $M(w) \cong M(w^{-1})$ as A-modules for any string w, and $M(1_{(u,t)})$ is the simple representation corresponding to the vertex u.

Next we explain the construction of a band module. Let X be a module of the Laurent polynomial ring $K[T, T^{-1}]$. Then X is determined by $s = \dim X$ and an automorphism φ of $X = K^s$. So we also write $X = (K^s, \varphi)$. Let $U(w, s, \varphi)$ be the representation associated to the module X and the band $w = \alpha_1^{\epsilon_1}\alpha_2^{\epsilon_2}\cdots\alpha_m^{\epsilon_m}$ as follows,



where $U_{x_i} = K^s$ for all $i = 1, 2, \cdots, m$ and

$$U_{\alpha_i} = \begin{cases} \varphi & \text{if } i = 1 \text{ and } \epsilon_1 = 1, \\ \varphi^{-1} & \text{if } i = 1 \text{ and } \epsilon_1 = -1, \\ \mathrm{id}_{K^s} & \text{if } 2 \le i \le m. \end{cases}$$

Now the *band module* $M(w, s, \varphi) = ((V_i)_{i \in Q_0}, (\varphi_\alpha)_{\alpha \in Q_1})$ is defined by

$$V_i = \begin{cases} \oplus_{x_j = i} U_{x_j} & \text{if } i = x_j \text{ for some } j \in \{1, 2, \cdots, m\}, \\ 0 & \text{otherwise,} \end{cases}$$

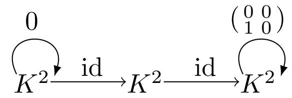
and

$$\varphi_\alpha = \begin{cases} \oplus_{\alpha_i = \alpha} U_{\alpha_i} & \text{if } \alpha = \alpha_i \text{ for some } i \in \{1, 2, \cdots, m\}, \\ 0 & \text{otherwise.} \end{cases}$$

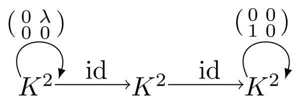
From the definition, one can check that $M(w, s, \varphi) \cong M(w^{-1}, s, \varphi^{-1})$ and $M(w, s, \varphi) \cong M(w', s, \varphi)$, where w' is equivalent to w with respect to ρ'.

Example 2.3. Let A be a string algebra as in Example 2.2.

(1) For the string $w_1 = \alpha_1^{-1}\alpha_2^{-1}\varepsilon_3\alpha_2\alpha_1$, the string module $M(w_1)$ is as follows.



(2) For the band $w_2 = \varepsilon_1\alpha_1^{-1}\alpha_2^{-1}\varepsilon_3\alpha_2\alpha_1$, the band module $M(w_2, 1, \lambda)$ is as follows, where $\lambda \neq 0$.



Denote by $\mathcal{M}$ a complete set of representatives of indecomposable $K[T, T^{-1}]$-modules.

Theorem 2.4. [2, Theorem 3.1] *Let A be a string algebra. Then the string modules $M(w)$ with $w \in \overline{\mathrm{St}}(A)$ and the band modules $M(w, s, \varphi)$ with $w \in \overline{\mathrm{Ba}}(A)$ and $(K^s, \varphi) \in \mathcal{M}$ are up to isomorphism all the indecomposable A-modules.*

2.4. **Auslaner-Reiten sequences for string algebras.** For each arrow $\alpha \in Q_1$, let

$$\alpha_- = \beta_1^{-1}\beta_2^{-1} \cdots \beta_r^{-1}$$

be the inverse string of maximal length such that $\alpha \cdot \alpha_-$ is a string, and let

$${}_-\alpha = \gamma_s^{-1} \cdots \gamma_2^{-1}\gamma_1^{-1}$$

be the inverse string of maximal length such that ${}_-\alpha \cdot \alpha$ is a string. Similarly, let

$${}_+(\alpha^{-1}) = \beta_r \cdots \beta_2\beta_1 \text{ (resp. } (\alpha^{-1})_+ = \gamma_1\gamma_2 \cdots \gamma_s)$$

be the direct string of maximal length such that ${}_+(\alpha^{-1}) \cdot \alpha^{-1}$ (resp. $\alpha^{-1} \cdot (\alpha^{-1})_+$) is a string.

Proposition 2.5. [2] *The only AR-sequences that consist of string modules and that have the middle term indecomposable are*

$$0 \to M({}_-\alpha) \to M({}_-\alpha \cdot \alpha \cdot \alpha_-) \to M(\alpha_-) \to 0,$$

where $\alpha \in Q_1$.

Next we describe the AR-sequences with the middle term decomposable. We will see shortly that in this case, the middle term in such a short exact sequence is a direct sum of two indecomposable modules.

Definition 2.6. (1) A string w is *right directly extendable* (RDE) if there is an arrow α such that $w\alpha$ is a string.

(2) A string w is *right inversely extendable* (RIE) if there is an arrow β such that $w\beta^{-1}$ is a string.

(3) A string w is *left directly extendable* (LDE) if there is an arrow α such that αw is a string.

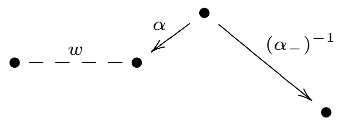
(4) A string w is *left inversely extendable* (LIE) if there is an arrow β such that $\beta^{-1}w$ is a string.

Remark 2.7. Comparing with the terminology in [2], we have the following.

(1) A string w is not RDE if and only if w starts on a peak.
(2) A string w is not RIE if and only if w starts in a deep.
(3) A string w is not LDE if and only if w ends in a deep.
(4) A string w is not LIE if and only if w ends on a peak.

If w is RDE, then there exists an arrow α such that $w\alpha$ is a string. Let

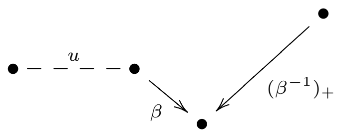
$$w_h := w \cdot \alpha \cdot \alpha_- =$$



We say w_h is *obtained from w by adding a hook from the right.* There is a canonical embedding $i : M(w) \to M(w_h)$.

If $w = u \cdot \beta^{-1} \cdot (\beta^{-1})_+$ for some string u and some arrow β,

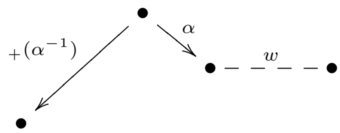
$$w = u \cdot \beta^{-1} \cdot (\beta^{-1})_+ =$$



then we say u is *obtained from w by deleting a cohook from the right.* In this case, u is RIE and there is a canonical projection $p : M(w) \to M(u)$. The string w can be understood as being obtained from u by adding a cohook from the right and so we also write $w = u_c$.

If w is LIE, then there exists an arrow α such that $\alpha^{-1}w$ is a string. Let

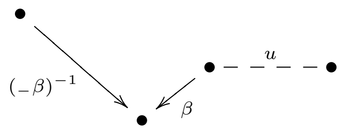
$${}_hw := {}_+(\alpha^{-1}) \cdot \alpha^{-1} \cdot w =$$



We say ${}_hw$ is *obtained from w by adding a hook from the left.* There is a canonical embedding $i : M(w) \to M({}_hw)$.

If $w = {}_-\beta \cdot \beta \cdot u$ for some string u and some arrow β,

$$w = {}_-\beta \cdot \beta \cdot u =$$



then we say u is *obtained from w by deleting a cohook from the left.* In this case, u is LDE and there is a canonical projection $p : M(w) \to M(u)$. Similar to u_c above, the string w can be understood as being obtained from u by adding a cohook from the left and so we also write $w = {}_cu$.

Proposition 2.8. [2] *Let w be a string such that $M(w)$ is not injective and $w \not\sim {}_-\alpha$ for any $\alpha \in Q_1$.*

(1) *If w is RDE and LIE, then the following*

$$0 \to M(w) \xrightarrow{(i\ i)} M({}_hw) \oplus M(w_h) \xrightarrow{\binom{i}{-i}} M({}_hw_h) \to 0$$

is an AR-sequence where ${}_hw_h = {}_h(w_h) = ({}_hw)_h$.

(2) *If w is RDE but not LIE, then $w = {}_cu$ for some string u and the following*

$$0 \to M(w) \xrightarrow{(p\ i)} M(u) \oplus M(w_h) \xrightarrow{\binom{i}{-p}} M(u_h) \to 0$$

is an AR-sequence.

(3) *If w is LIE but not RDE, then $w = u_c$ for some string u and the following*

$$0 \to M(w) \xrightarrow{(i\ p)} M({}_hw) \oplus M(u) \xrightarrow{\binom{p}{-i}} M({}_hu) \to 0$$

is an AR-sequence.

(4) *If w is neither RDE nor LIE, then $w = {}_c u_c = {}_c(u_c) = ({}_c u)_c$ for some string u and the following*

$$0 \to M(w) \xrightarrow{(p\ p)} M(u_c) \oplus M({}_c u) \xrightarrow{\binom{p}{-p}} M(u) \to 0$$

is an AR-sequence.

Theorem 2.9. [2] *Let A be a string algebra. The Auslaner-Reiten sequences in A-*mod *are those described in Propositions 2.5 and 2.8, together with those of the form*

$$0 \to M(w,s,\varphi) \to M(w,2s,\varphi') \to M(w,s,\varphi) \to 0$$

*where $w \in \overline{\mathrm{Ba}}(A)$, $(K^s,\varphi) \in \mathcal{M}$ and φ' is determined by the following AR-sequence in $K[T,T^{-1}]$-*mod,

$$0 \to (K^s,\varphi) \to (K^{2s},\varphi') \to (K^s,\varphi) \to 0.$$

3. The Auslaner-Reiten quivers and τ-locally free modules of string algebras of type $\widetilde{\mathbb{C}}$

In this section, we introduce the notion of minimal string modules to study the AR-quivers of string algebras H of type $\widetilde{\mathbb{C}}_{n-1}$. We will explicitly describe all the connected components of the AR-quiver and τ-locally free H-modules.

3.1. **Minimal string modules.** In this subsection, A can be any string algebra.

Definition 3.1. A string A-module $M(w)$ is called *minimal* if in the AR-quiver Γ_A of A, each irreducible map $M(w) \to M(u)$ is injective and each irreducible map $M(v) \to M(w)$ is surjective.

We will see that minimal string modules play an important role in determining the connected components of the AR-quiver Γ_A.

Proposition 3.2. *There exists at least one minimal string module for each connected component of Γ_A containing string modules.*

Proof. Assume that $\mathcal{T}$ is a connected component containing a string module $M(w)$. Theorem 2.9 implies that all modules in $\mathcal{T}$ are string modules, as band modules are contained in homogeneous tubes and any module in a homogeneous tube is a band module. If $M(w)$ is not minimal, then by definition there exists an irreducible surjection $f_1 : M(w) \to M(w_1)$ or an irreducible injection $g_1 : M(w_1) \to M(w)$ for some string w_1. In either case $\dim M(w_1) < \dim M(w)$. If $M(w_1)$ is not minimal, then repeat the same procedure to find a string module with smaller dimension. This procedure will terminate eventually. Then we obtain a minimal string module in the component. □

For a string x, denote by $[x]$ the equivalence class of x. Recall that for indecomposable left A-modules M and N the quotient

$$\mathrm{Irr}(M,N) = \mathrm{rad}_A(M,N)/\mathrm{rad}_A^2(M,N)$$

is the space of irreducible morphisms.

Definition 3.3. Let w be a string.

(1) The dimension $\mathrm{I_l}(w) = \Sigma_{[u]}\dim_k \mathrm{Irr}(M(u),M(w))$ is called *the left index* of w.

(2) The dimension $\mathrm{I_r}(w) = \Sigma_{[u]}\dim_k \mathrm{Irr}(M(w),M(u))$ is called *the right index* of w.

(3) The pair $\mathrm{I}(w) = (\mathrm{I_l}(w), \mathrm{I_r}(w))$ is called *the index* of w. We say w is of *type* (a,b) if $\mathrm{I_l}(w) = a$ and $\mathrm{I_r}(w) = b$. In this case, we also say that the string module $M(w)$ is of *type* (a,b).

Recall that we write $u \sim w$ if the two strings u and w are equivalent and $u \nsim w$ otherwise.

Lemma 3.4. *Let w be a string. Then*

(1) $\mathrm{I}(w) \in \{(0,1),(1,0),(1,1),(0,2),(2,0),(1,2),(2,1),(2,2)\}$.

(2) $\mathrm{I_l}(w) = 0$ *if and only if $M(w)$ is a simple projective module. Dually,* $\mathrm{I_r}(w) = 0$ *if and only if $M(w)$ is a simple injective module.*

(3) $\mathrm{I_l}(w) = 1$ *if and only if $M(w)$ is either a projective module such that* $\mathrm{rad}M(w)$ *is indecomposable or $w \sim \alpha_-$ for some arrow $\alpha \in Q_1$. Dually,* $\mathrm{I_r}(w) = 1$ *if and only if $M(w)$ is either an injective module such that $M(w)/\mathrm{soc}M(w)$ is indecomposable or $w \sim {}_-\beta$ for some arrow $\beta \in Q_1$.*

Proof. (1) follows from Theorem 2.9, (2) and (3) follow from the general Auslander-Reiten theory [1] and Proposition 2.5. □

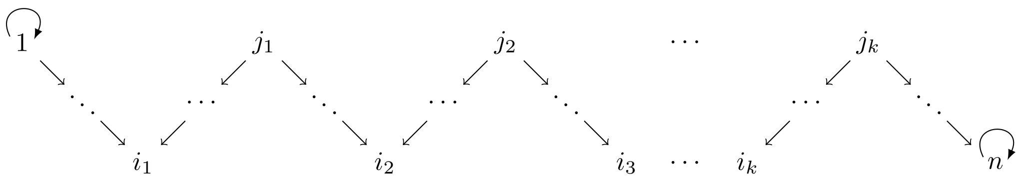


Fig 1. Q is of type (3-1)

Lemma 3.5. *An indecomposable projective A-module P is minimal if and only if P is simple. Dually, an indecomposable injective A-module I is minimal if and only if I is simple.*

Proof. We only prove the first assertion, the second one follows by duality. If P is simple, then P is minimal by definition. Next assume that P is not simple. Then $\mathrm{rad}P \neq 0$ and the embedding $\mathrm{rad}P \to P$ is a right almost split map. So the restriction of the embedding to any indecomposable summand is irreducible. This implies that P is not minimal. Therefore if P is minimal, then it is simple. □

Lemma 3.6. *Let w be a string.*

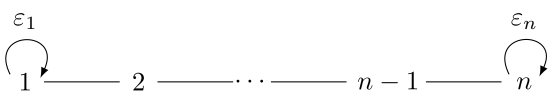
(1) *The string module $M(w)$ is minimal of type $(1,1)$ if and only if $w \sim \alpha_-$ for some $\alpha \in Q_1$ and $w \sim {}_-\beta$ for some $\beta \in Q_1$.*
(2) *The string module $M(w)$ is minimal of type $(1,2)$ if and only if $w \sim \alpha_-$ for some $\alpha \in Q_1$ but $w \not\sim {}_-\beta$ for any $\beta \in Q_1$, and w is RDE and LIE.*
(3) *The string module $M(w)$ is minimal of type $(2,1)$ if and only if $w \sim {}_-\alpha$ for some $\alpha \in Q_1$ but $w \not\sim \beta_-$ for any $\beta \in Q_1$, and w is RIE and LDE.*
(4) *The string module $M(w)$ is minimal of type $(2,2)$ if and only if $w \not\sim {}_-\alpha$ and $w \not\sim \alpha_-$ for any $\alpha \in Q_1$, and w is RDE, RIE, LDE and LIE.*

Proof. (1) follows from Lemma 3.4 and Lemma 3.5.

Now we prove (2). If $M(w)$ is minimal of type $(1,2)$, then $M(w)$ is neither projective nor injective by Lemma 3.5 and Lemma 3.4 (2). Since $\mathrm{I_l}(w) = 1$ and $\mathrm{I_r}(w) = 2$, we have $w \sim \alpha_-$ for some arrow $\alpha \in Q_1$ but $w \not\sim {}_-\beta$ for any $\beta \in Q_1$ by Lemma 3.4 (3). Note that w is minimal, w is RDE and LIE by Proposition 2.8. For the converse, first note that $M(w)$ is not injective since w is RDE. By the assumption that $w \not\sim {}_-\beta$ for any $\beta \in Q_1$, we can apply Theorem 2.9 to deduce that $\mathrm{I_r}(w) = 2$. Furthermore, w is minimal. That is $M(w)$ is minimal of type (2,1), as claimed.

(3) and (4) can be similarly proved, we skip the details. □

3.2. **String algebras of type $\widetilde{\mathbb{C}}$.** *For the remaining part of this paper we assume that $H = H(C,\Omega,D)$ with C the Cartan matrix of type $\widetilde{\mathbb{C}}_{n-1}$ and D the minimal symmetrizer* $\mathrm{diag}(2,1,\ldots,1,2)$*, unless otherwise stated.* That is, H is the quotient path algebra KQ/I, where Q is a quiver of type $\mathbb{A}_n$ when the loops at 1 and n are removed,



and $I = < \varepsilon_1^2, \varepsilon_n^2 >$. Then H is a string algebra. Moreover it is a gentle algebra. We say H is a *string algebra of type* $\widetilde{\mathbb{C}}_{n-1}$. In this subsection, we will first describe minimal string H-modules in a more concrete way, using Lemma 3.6 and Proposition 2.8, and then construct explicitly connected components of the AR-quiver of H.

Let Q^0 be the quiver obtained from Q by deleting the two loops ε_1 and ε_n. Then Q^0 is a quiver of type $\mathbb{A}_n$. We will describe connected components of the AR-quiver of H. The orientations of the arrows in Q^0 incident at 1 and n are particularly relevant to the description of the component containing the indecomposable projective A-modules and the proof. There are four possibilities, (Fig. 1), (Fig. 2) and their opposite quivers. By duality, we only need to consider the two cases, (Fig. 1) and (Fig. 2), where the difference is that both 1 and n are sources in Q^0 in (Fig. 2), but only 1 is a source in (Fig. 1).

By the definition of Q, there is at most one arrow between two vertices i, j and we denote the arrow by α_{ji} if there is one from i to j. As the way the vertices are labelled, $|i - j| = 1$ if there is an arrow between two distinct vertices i and j. Recall that a vertex i is *admissible* if it is a sink or a source.

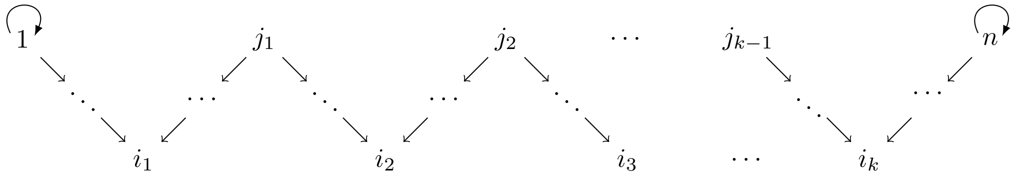


Fig 2. Q is of type (3-2)

Proposition 3.7. *The following are the minimal string modules of H up to isomorphism.*

(1) *Type (0,2): S_i with i a sink in Q.*
(2) *Type (2,0): S_i with i a source in Q.*
(3) *Type (1,1): $M(\alpha_-)$, where α is an arrow in Q^0.*
(4) *Type (1,2): $M((\varepsilon_1)_-)$ and $M((\varepsilon_n)_-)$.*
(5) *Type (2,1): $M(_-(\varepsilon_1))$ and $M(_-(\varepsilon_n))$.*
(6) *Type (2,2): $M(w)$, where $w = c_1 \dots c_m$ is a string with $\{s(w), t(w)\} \subseteq \{1, n\}$ and neither c_1 nor c_m is a loop or the formal inverse of a loop.*

Proof. Observe that the only simple projective or injective modules are those corresponding to sinks or sources in Q, which can only occur in the middle of the quiver. So (1) and (2) are true. (6) follows from Lemma 3.6 (4). To prove (3), (4) and (5), we compute $M(\alpha_-)$ and $M(_-\alpha)$ for all $\alpha \in Q_1$.

Case I: Q is of type (3-1) as in (Fig. 1). Set $j_0 = 1$ and $i_{k+1} = n$. Then we have the following:

(a) $M((\alpha_{p+1,p})_-) = M(_-(\alpha_{p,p-1})) = S_p$ for $j_r < p < i_{r+1}$ and $0 \leq r \leq k$.
(b) $M((\alpha_{q-1,q})_-) = M(_-(\alpha_{q,q+1})) = S_q$ for $i_r < q < j_r$ and $1 \leq r \leq k$.
(c) $M((\alpha_{21})_-) = M(_-(\alpha_{i_1,i_1+1}))$.
(d) $M((\alpha_{j_r+1,j_r})_-) = M(_-(\alpha_{i_r,i_r-1}))$ for $1 \leq r \leq k$.
(e) $M((\alpha_{j_r-1,j_r})_-) = M(_-(\alpha_{i_{r+1},i_{r+1}+1}))$ for $1 \leq r \leq k-1$.
(f) $M((\alpha_{j_k-1,j_k})_-) = M(_-(\alpha_{n,n-1}))$.
(g) $M(_-(\varepsilon_1)) = S_1$ and $M((\varepsilon_1)_-) = \begin{cases} M(\alpha_{21}^{-1} \cdots \alpha_{i_1,i_1-1}^{-1}) & \text{if } i_1 \neq n, \\ M(\alpha_{21}^{-1} \cdots \alpha_{i_1,i_1-1}^{-1}\varepsilon_n^{-1}) & \text{if } i_1 = n. \end{cases}$
(h) $M((\varepsilon_n)_-) = S_n$ and $M(_-(\varepsilon_n)) = \begin{cases} M(\alpha_{j_k+1,j_k}^{-1} \cdots \alpha_{n,n-1}^{-1}) & \text{if } j_k \neq 1, \\ M(\varepsilon_1^{-1}\alpha_{j_k+1,j_k}^{-1} \cdots \alpha_{n,n-1}^{-1}) & \text{if } j_k = 1. \end{cases}$

By Lemma 3.6 (1), cases (a)-(f) provide all the possible minimal string modules of type $(1,1)$. So (3) holds.

By (a)-(h), $(\varepsilon_1)_-$ and $(\varepsilon_n)_-$ are both RDE and LIE, but not of the form $_-\alpha$ for any $\alpha \in Q_1$. Since $M((\varepsilon_1)_-)$ and $M((\varepsilon_n)_-) = S_n$ are not injective, they are minimal string modules of type $(1,2)$ by Lemma 3.6 (2). Moreover from the computation list, they are the only two such modules. So (4) is true. Similarly, $M(_-(\varepsilon_1)) = S_1$ and $M(_-(\varepsilon_n))$ are the only two minimal string modules of type $(2,1)$. So (5) is true.

Case II: Q is of type (3-2) as in (Fig. 2). Set $j_0 = 1$ and $j_k = n$. The difference between the two types of Q is that n is a sink in Q of type (3-1), while it is a source in Q of type (3-2). Similar computation shows that (3), (4) and (5) hold. This completes the proof. □

Denote the Auslander-Reiten translation for H by τ. The following result gives an explicit description of the τ-orbit at the bottom of the rank $n-1$ tube (see [15, Lemma 4.5.10] for an alternative account).

Proposition 3.8. *There are $n-1$ minimal string modules of type (1,1) (up to isomorphism) and they form the τ-orbit at the bottom of a tube of rank $n-1$. In particular, for an arrow $\alpha : i \to j$ in Q^0, we have the following, depending on the properties of i and j in Q^0.*

(1) *Both vertices i and j are non-admissible. Then*

$$\tau S_i = S_j.$$

(2) *The vertex i is non-admissible and j is a sink. Then*

$$\tau S_i = M(w_1),$$

where w_1 is the direct path of maximal length terminating at j and satisfying that $w_1^{-1}\alpha$ is a string. Note that by the definition of Q, such a path w uniquely exists.

(3) *The vertex i is a source and j is non-admissible. Then*

$$\tau^{-1}S_j = M(w_2),$$

where w_2 is the direct path of maximal length starting from i and satisfying that αw_2^{-1} is a string. Again, such a path w uniquely exists.

(4) *The vertex i is a source and j is a sink. Then*

$$\tau(M(w_2)) = M(w_1),$$

where w_1 and w_2 are paths satisfying the conditions on w_1 in (2) and the conditions on w_2 in (3), respectively.

Proof. First, the claims in (1)-(4) are true, since the two modules in each case are $M(\alpha_-)$ and $M({}_-\alpha)$, respectively, for the arrow α in Q^0. By Proposition 3.7, there are exactly $n-1$ minimal string modules of type $(1,1)$, one for each arrow in Q^0. Next we prove that the $n-1$ minimal string modules form a τ-orbit.

We connect two copies of Q^0 by ε_1 and ε_n, where ε_1 goes from vertex 1 in the first copy to the 1 in the second copy and ε_n goes from vertex n in the second copy to the n in the first copy. Denote the new quiver by $\tilde{Q}$ (see Example 3.9 for an illustration). Observe that each arrow in Q^0 appears twice in $\tilde{Q}$, but in opposite directions, one goes anti-clockwise and the other one goes clockwise. So there are exactly $n-1$ anti-clockwise arrows in $\tilde{Q}\backslash\{\varepsilon_1,\ \varepsilon_n\}$ and each arrow in Q^0 appears exactly once among the $n-1$ anti-clockwise arrows.

The computation (a) - (f) in the proof of Proposition 3.7 can be interpreted as follows. For any anti-clockwise arrow γ in $\tilde{Q}\backslash\{\varepsilon_1, \varepsilon_n\}$,

$$M({}_-\gamma) = M(\beta_-),$$

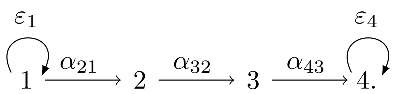
where β is the next anti-clockwise arrow in $\tilde{Q}\backslash\{\varepsilon_1,\ \varepsilon_n\}$ after γ when one walks along $\tilde{Q}$ anti-clockwise. So

$$\tau(M(\gamma_-)) = M({}_-\gamma) = M(\beta_-).$$

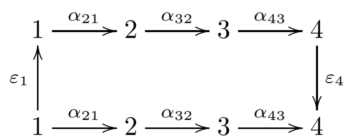
Continuing in this fashion, the τ-orbit reaches all the $n-1$ minimal string modules of type $(1,1)$ and stays within these modules. Therefore the $n-1$ minimal string modules of type $(1,1)$ form a τ-orbit, and the τ-orbit is at the bottom of the tube, because these string modules are all of type $(1,1)$. This completes the proof. □

An easy consequence of Proposition 3.8 is that the modules at the bottom of the rank $n-1$ tube are rigid. The rigidity of these modules is stated in [15, Proposition 4.5.18].

Example 3.9. (1) Let Q be the quiver:



The quiver $\tilde{Q}$ constructed in the proof of Proposition 3.8 is as follows, where the first copy of Q is at the bottom,



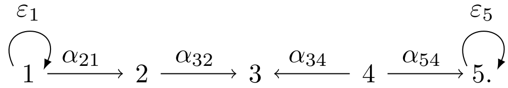
and the modules at the bottom of the tube of rank 3 are

$$M(\varepsilon_4\alpha_{43}\alpha_{32}\alpha_{21}\varepsilon_1) \qquad S_3 \qquad S_2 \qquad M(\varepsilon_4\alpha_{43}\alpha_{32}\alpha_{21}\varepsilon_1).$$

In the same order, these modules are

$$M((\alpha_{21})_-) \qquad M((\alpha_{43})_-) \qquad M((\alpha_{32})_-) \qquad M((\alpha_{21})_-).$$

(2) Let Q be the quiver:



Then the quiver $\tilde{Q}$ is

$$\begin{array}{ccccccccc}
1 & \xrightarrow{\alpha_{21}} & 2 & \xrightarrow{\alpha_{32}} & 3 & \xleftarrow{\alpha_{34}} & 4 & \xrightarrow{\alpha_{54}} & 5 \\
\uparrow{\scriptstyle \varepsilon_1} & & & & & & & & \downarrow{\scriptstyle \varepsilon_5} \\
1 & \xrightarrow{\alpha_{21}} & 2 & \xrightarrow{\alpha_{32}} & 3 & \xleftarrow{\alpha_{34}} & 4 & \xrightarrow{\alpha_{54}} & 5
\end{array}$$

and the modules at the bottom of the tube of rank 4 are:

$$M(\alpha_{32}\alpha_{21}\varepsilon_1) \qquad M(\varepsilon_5\alpha_{54}) \qquad M(\alpha_{34}) \qquad S_2 \qquad M(\alpha_{32}\alpha_{21}\varepsilon_1).$$

In the same order, these modules are

$$M((\alpha_{21})_-) \qquad M((\alpha_{34})_-) \qquad M((\alpha_{54})_-) \qquad M((\alpha_{32})_-) \qquad M((\alpha_{21})_-).$$

Each arrow in Q^0 appears exactly once.

Note that in each case, a module is the τ-translation of the module on its immediate right.

Lemma 3.10. *Let w be a string.*

(1) *If w is RDE, then so is w_h. Thus there exists the following infinite ray:*

$$M(w \to w_{h^\bullet}): \quad M(w) \to M(w_h) \to M(w_{h^2}) \to \cdots$$

where $w_{h^i} = (w_{h^{i-1}})_h$ for $i \geq 2$.

(2) *If w is LIE, then so is ${}_hw$. Thus there exists the following infinite ray:*

$$M(w \to {}_{h^\bullet}w): \quad M(w) \to M({}_hw) \to M({}_{h^2}w) \to \cdots$$

where ${}_{h^i}w = {}_h({}_{h^{i-1}}w)$ for $i \geq 2$.

(3) *If w is RIE, then so is w_c. Thus there exists the following infinite coray:*

$$M(w_{c^\bullet} \to w): \quad \cdots \to M(w_{c^2}) \to M(w_c) \to M(w)$$

where $w_{c^i} = (w_{c^{i-1}})_c$ for $i \geq 2$.

(4) *If w is LDE, then so is ${}_cw$. Thus there exists the following infinite coray:*

$$M({}_{c^\bullet}w \to w): \quad \cdots \to M({}_{c^2}w) \to M({}_cw) \to M(w)$$

where ${}_{c^i}w = {}_c({}_{c^{i-1}}w)$ for $i \geq 2$.

Moreover, the map at each step in the rays and corays is irreducible.

Proof. First by the construction of the AR-sequences in Propositions 2.5 and 2.8, the map at each step in the rays and corays is part of an AR-sequence and so it is irreducible.

(1) Suppose that w is RDE. Then there exists $\alpha \in Q_1$ such that $w\alpha$ is a string. We have

$$w_h = \begin{cases} w\alpha & \text{if } s(\alpha) \text{ is non-admissible in } Q^0, \\ & \\ w\alpha w' & \text{if } s(\alpha) \text{ is a source in } Q^0 \end{cases}$$

where $w\alpha w'$ is a string, w' is an inverse string of maximal length. In particular, $s(w')$ is a sink in Q^0. In either case, w_h is again RDE. This proves (1). Similarly, (2) is true.

(3) Suppose that w is RIE. Then there exists $\alpha \in Q_1$ such that $w\alpha^{-1}$ is a string. Similar to (1), we have

$$w_c = \begin{cases} w\alpha^{-1} & \text{if } t(\alpha) \text{ is non-admissible in } Q^0, \\ & \\ w\alpha^{-1}w' & \text{if } t(\alpha) \text{ is a sink in } Q^0 \end{cases}$$

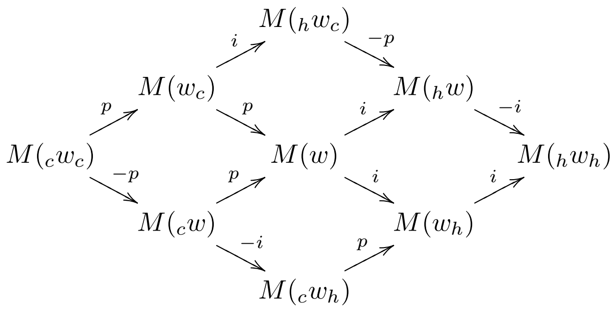
where $w\alpha^{-1}w'$ is a string, w' is a direct string of maximal length. In particular, $s(w')$ is a source in Q^0. In either case, w_c is again RIE. This proves (3). Similarly, (4) is true. □

Remark 3.11. Lemma 3.10 is not true in general. For instance, the linear quiver of type $\mathbb{A}_n$ is a string algebra, but there is no infinite ray or coray.

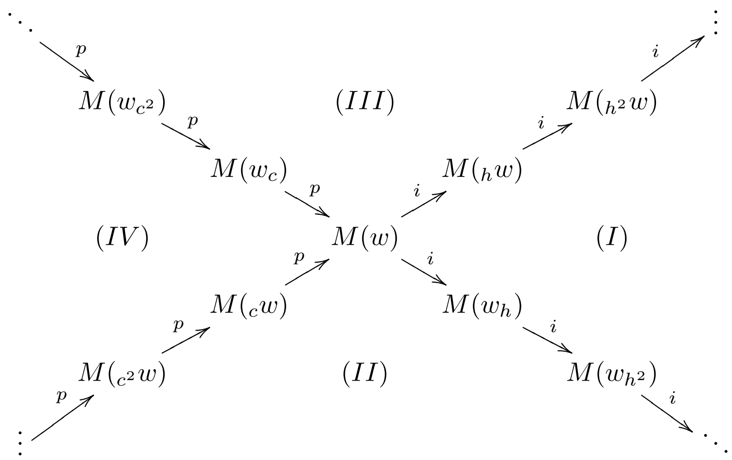
Proposition 3.12. *There is a bijection between isomorphism classes of minimal string modules of type (2,2) and connected components of Γ_H of type $\mathbb{Z}\mathbb{A}_\infty^\infty$.*

Proof. Assume that $\mathcal{T}$ is a connected component of Γ_H of type $\mathbb{Z}\mathbb{A}_\infty^\infty$. By Proposition 3.2, there is a minimal string module $M(w)$ occurring in $\mathcal{T}$ and so $M(w)$ is of type (2,2).

Conversely, assume that $M(w)$ is a minimal string module of type (2,2). Then the AR sequences containing $M(w)$ are as follows.



The connected component $\mathcal{T}_w$ containing $M(w)$ is divided into four regions as follows by the two rays $M(w \to w_{h^\bullet}), M(w \to {}_{h^\bullet}w)$ and the two corays $M(w_{c^\bullet} \to w), M({}_{c^\bullet}w \to w)$.



By induction, we see that the AR-sequences in region (I), (II), (III) and (IV) are those in Proposition 2.8 (1), (2), (3) and (4), respectively. In particular, $\dim M(w) < \dim M$ for any $M \in \mathcal{T}_w \backslash \{M(w)\}$ and so minimal string modules of type (2, 2) (up to isomorphism) are in one to one correspondence with components of type $\mathbb{Z}\mathbb{A}_\infty^\infty$. □

Remark 3.13. Geiss [6] describes modules of minimal dimension in a component of type $\mathbb{Z}\mathbb{A}_\infty^\infty$. Our minimal string modules are defined differently (see Definition 3.1) and are defined for any component. The proof of Proposition 3.12 shows that for any minimal string w of type $(2,2)$, $\dim M(w) < \dim M$ for any $M \in \mathcal{T}_w \backslash \{M(w)\}$. Therefore the minimal string modules $M(w)$ of type $(2,2)$ coincide with those described in [6, Proposition 3].

Denote by $\mathcal{S}$ a complete set of representatives of simple $K[T, T^{-1}]$-modules.

Theorem 3.14. *The AR-quiver Γ_H of H consists of the following.*

1. *One component $\mathcal{T}_{PI}$ containing all the indecomposable preprojective modules and all the indecomposable preinjective modules (up to isomorphism).*
2. *One tube of rank $n-1$, where the sum of the dimension vectors of the indecomposable modules at the bottom of the tube is $d = (d_i)$ with $d_i = 2$ for all i.(Note: we will see later if we take the sum of the rank vectors instead, then the sum is exactly δ, the minimal positive imaginary root of type C).*
3. *Homogeneous tubes $\mathcal{H}_{w,S}$, where $w \in \overline{\mathrm{Ba}}(H)$ and $S \in \mathcal{S}$ is a simple module of the Laurent polynomial ring $K[T, T^{-1}]$.*
4. *Components $\mathcal{T}_\lambda$ of type $\mathbb{Z}\mathbb{A}_\infty^\infty$, where λ runs through all the isomorphism classes of minimal string modules of type (2,2).*

Proof. Recall that for any indecomposable projective module P and any indecomposable injective module I, the natural embedding $\mathrm{rad}P \to P$ and the natural projection $I \to I/\mathrm{soc}I$ are almost split maps, see [1] for more details. We compute radicals of the indecomposable projective modules and quotients by socles of the indecomposable injective modules. By duality, we may assume that Q is of type (3-1) or (3-2) as in (Fig. 1) or (Fig. 2).

(1) $\operatorname{rad} P_1 = P_2 \oplus M((\varepsilon_1)_-)$.

(2) For $1 < i < n$: $\operatorname{rad} P_i = \begin{cases} P_j & \text{if } i \text{ is not a source and } i \to j, \\ P_{i-1} \oplus P_{i+1} & \text{if } i \text{ is a source.} \end{cases}$

(3) $\operatorname{rad} P_n = \begin{cases} S_n = M((\varepsilon_n)_-) & \text{if } Q \text{ is of type (3-1),} \\ P_{n-1} \oplus M((\varepsilon_n)_-) & \text{if } Q \text{ is of type (3-2).} \end{cases}$

(4) $I_1/\operatorname{soc} I_1 = S_1 = M({}_-(\varepsilon_1))$.

(5) For $1 < i < n$: $I_i/\operatorname{soc} I_i = \begin{cases} I_j & \text{if } i \text{ is not a sink and } j \to i, \\ I_{i-1} \oplus I_{i+1} & \text{if } i \text{ is a sink.} \end{cases}$

(6) $I_n/\operatorname{soc} I_n = \begin{cases} I_{n-1} \oplus M({}_-(\varepsilon_n)) & \text{if } Q \text{ is of type (3-1),} \\ S_n = M({}_-(\varepsilon_n)) & \text{if } Q \text{ is of type (3-2).} \end{cases}$

So in the AR-quiver, the indecomposable projective modules are in one slice, connected by irreducible maps and the same for the indecomposable injective modules. Note that the orientation of the arrow between vertices $n-1$ and n are different in the two quivers (Fig. 1) and (Fig. 2) and so strings $(\varepsilon_n)_-$ and ${}_-(\varepsilon_n)$ are different for the two quivers. However, in either case, the AR-sequences

$$0 \to M({}_-(\varepsilon_1)) \to M({}_-(\varepsilon_1)\varepsilon_1(\varepsilon_1)_-) \to M((\varepsilon_1)_-) \to 0$$

and

$$0 \to M({}_-(\varepsilon_n)) \to M({}_-(\varepsilon_n)\varepsilon_n(\varepsilon_n)_-) \to M((\varepsilon_n)_-) \to 0$$

connect the slice of injective modules and the slice of projective modules in the AR-quiver. In particular, the indecomposable projective and the indecomposable injective modules are in one component, denoted by $\mathcal{T}_{PI}$, and this component contains all the minimal string modules of type (0,2), (2,0), (1,2) and (2,1). See (Fig. 3) for the case Q of type (3-1), where $M_1 = M((\varepsilon_1)_-)$, $N_1 = M({}_-(\varepsilon_1)\varepsilon_1(\varepsilon_1)_-)$, $M_n = M({}_-(\varepsilon_n))$ and $N_n = M({}_-(\varepsilon_n)\varepsilon_n(\varepsilon_n)_-)$.

By Proposition 3.8, all minimal string modules of type (1,1) form the τ-orbit at the bottom of the tube of rank $n-1$. By construction, each vertex appears exactly twice in the walks corresponding to the minimal strings. So the sum of the dimension vectors of the minimal string modules has 2 at all entries.

By Proposition 3.12, the isomorphism classes of minimal string modules of type (2,2) are in one-to-one correspondence with connected components of type $\mathbb{Z}\mathbb{A}_\infty^\infty$. There are no other connected components containing string modules, following Proposition 3.2.

Finally, by Theorem 2.4, it remains to consider components consisting of band modules. We know from Theorem 2.9 that an indecomposable module is a band module if and only if it is contained in a homogeneous tube. Each homogenous tube is uniquely determined by a band module $M(w, m, \varphi)$ for $w \in \overline{\mathrm{Ba}}(H)$ and $S = (K^m, \varphi) \in \mathcal{S}$ is a simple module over $K[T, T^{-1}]$. So the connected components of the AR-quiver of H are as described in the theorem. □

3.3. **τ-locally free modules.** Let $e_1, e_2, \cdots, e_n$ be the idempotents in H corresponding to the vertices of Q and let $H_i = e_i H e_i$ for $1 \le i \le n$. Then

$$H_i \cong \begin{cases} K[\varepsilon_i]/(\varepsilon_i^2) & \text{if } i = 1, n, \\ K & \text{if } 2 \le i \le n-1. \end{cases}$$

Definition 3.15. A left H-module M is called *locally free* if $M_i = e_i M$ is a free H_i-module for each $i \in Q_0$. An indecomposable locally free H-module M is called *τ-locally free*, if $\tau^k(M)$ is locally free for all $k \in \mathbb{Z}$.

Lemma 3.16. *The following are true.*

(1) *If $M(w)$ is a minimal string module of type (1,2), then the modules $M(w_{h^i})$ are not locally free, where $i \ge 0$.*

(2) *If $M(w)$ is a minimal string module of type (2,1), then the modules $M({}_{c^i}w)$ are not locally free, where $i \ge 0$.*

(3) *If $M(w)$ is a minimal string module of type (2,2), then $M(w)$ is not locally free. Moreover, none of the modules $M(w_{h^i})$, $M({}_{h^i}w)$ $M(w_{c^i})$ and $M({}_{c^i}w)$ is locally free, where $i \ge 1$.*

Proof. (1) If $M(w)$ is a minimal string module of type (1,2), then $w = (\varepsilon_1)_-$ or $w = (\varepsilon_n)_-$, by Proposition 3.7. Without loss of generality, we assume that $w = (\varepsilon_n)_-$. If n is a sink in Q^0, then $w = 1_{(n,t)}$ and all the other $w_{h^\bullet}$ have the form:

$$n \longleftarrow n-1 \longrightarrow \bullet \longrightarrow \cdots$$

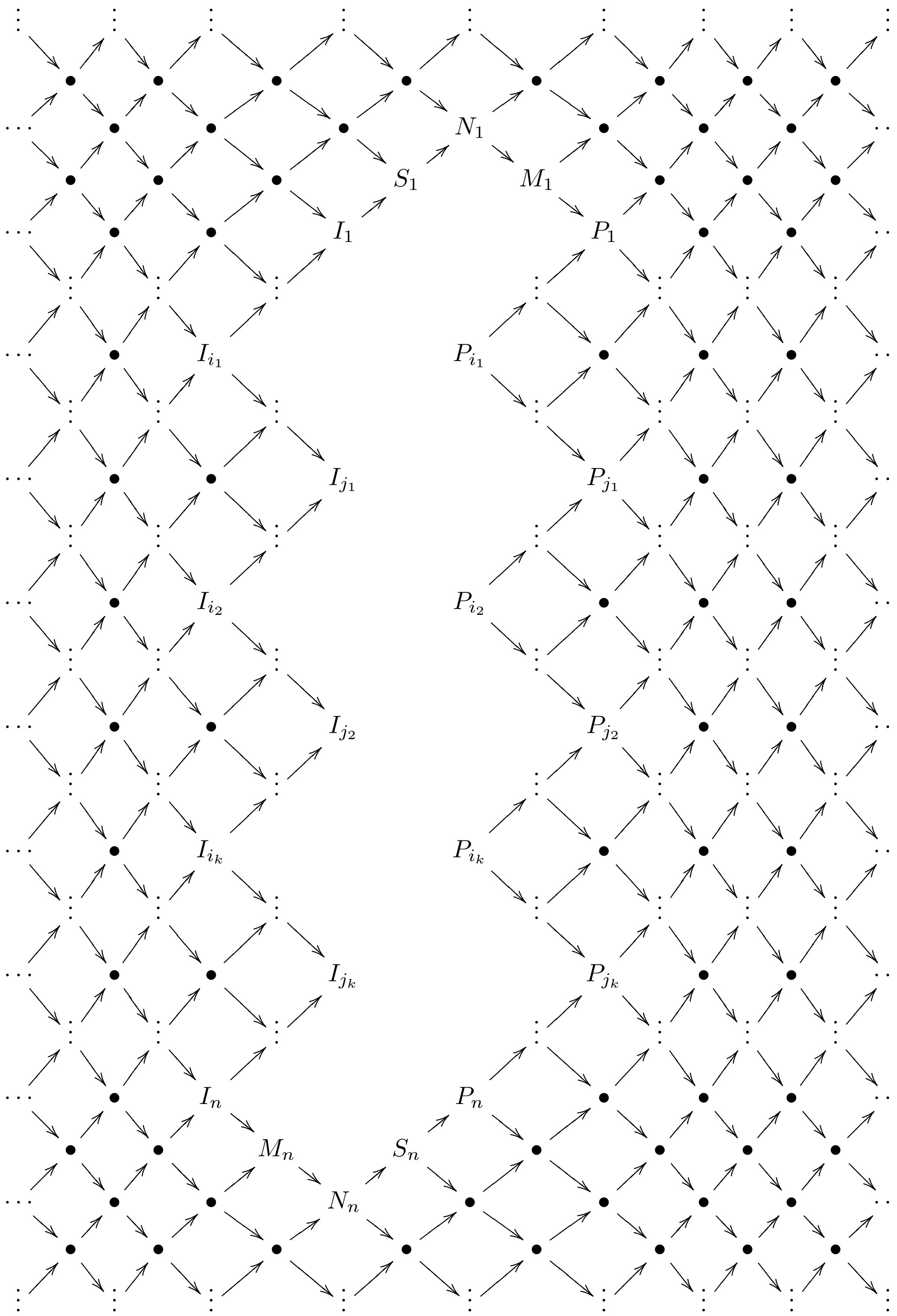


Fig 3. The connected component $\mathcal{T}_{PI}$ for Q of type (3-1)

that is, it ends with the arrow $\alpha_{n,n-1}$. Therefore none of $M(w_{h^\bullet})$ is locally free. Similarly, when n is a source in Q^0, $w_{h^\bullet}$ have the form:

$$n \longrightarrow n-1 \text{ ——— } \bullet \text{ ——— } \cdots$$

and so none of $M(w_{h^\bullet})$ is locally free either.

Similarly, (2) holds.

(3) If $M(w)$ is a minimal string module of type (2,2), then by Proposition 3.7, the string w is one of the following form

$$1 \text{ ——— } 2 \text{ ——— } \cdots \text{ ——— } n-1 \text{ ——— } n \ ,$$

$$1 \text{ ——— } 2 \text{ ——— } \cdots \text{ ——— } 2 \text{ ——— } 1\ ,$$
$$n \text{ ——— } n-1 \text{ ——— } \cdots \text{ ——— } n-1 \text{ ——— } n\ .$$

So by similar arguments as in (1), $M(w)$ is not locally free and none of the modules $M(w_{h^i})$, $M({}_{h^i}w)$ (resp. $M(w_{c^i})$, $M({}_{c^i}w)$) obtained by repeatedly adding hooks (resp. cohooks) from either the right or the left (but not both) is locally free. This completes the proof. □

Let w_0 be the shortest walk with $s(w_0) = 1$ and $t(w_0) = n$, consisting of all the arrows in Q^0. We also denote the corresponding string starting from 1 and terminating at n by w_0. Following the definition of a band, we have following.

Lemma 3.17. *Any band w is equivalent to a band of the standard form*
$$w_0^{-1}\varepsilon_n^{\pm}w_0\varepsilon_1^{\pm}\ldots\ w_0^{-1}\varepsilon_n^{\pm}w_0\varepsilon_1^{\pm},$$
whose starting vertex and terminating vertex are both 1. In particular, each time the walk of w reaches vertices 1 and n in the middle of the walk (i.e. different from $s(w)$ and $t(w)$), it goes via the loops at these vertices.

Example 3.18. The following are all bands of the standard form:
$$w_0^{-1}\varepsilon_n w_0\varepsilon_1,\ w_0^{-1}\varepsilon_n w_0\varepsilon_1^{-1},\ w_0^{-1}\varepsilon_n w_0\varepsilon_1^{-1}w_0^{-1}\varepsilon_n w_0\varepsilon_1,$$
where the last band is a composition of the first two.

Theorem 3.19. *Let M be an indecomposable H-module. Then M is τ-locally free if and only if one of the following is satisfied.*

(1) *M is preprojective.*
(2) *M is preinjective.*
(3) *M is a regular module occurring in any tube.*

Proof. Any preprojective module $\tau^i P_j$ and any preinjective module $\tau^s I_t$ are rigid, and so they are τ-locally free by [7, Proposition 11.6].

Observe that the modules at the bottom of the tube of rank $n-1$ (see Proposition 3.8) are locally free and the other modules in the tube have a filtration by these modules and so are locally free as well. Therefore they are all τ-locally free.

By Lemma 3.17, an indecomposable band module is locally free and thus τ-locally free, as such a module is in a homogeneous tube, i.e. a tube of rank 1. Consequently, any indecomposable module in a homogeneous tube is τ-locally free. Therefore the modules described in (1) - (3) are all τ-locally free.

Next we show that there is no other τ-locally free modules. First consider modules in any component $\mathcal{T}_w$ of type $\mathbb{Z}\mathbb{A}_\infty^\infty$, where w is the minimal string of type $(2,2)$ that determines the component. By Lemma 3.16, modules in the rays and corays that divides $\mathcal{T}_w$ into 4 regions in the proof of Proposition 3.12 are not locally free. Therefore any τ-orbit in $\mathcal{T}_w$ contains modules that are not locally free and so there is no τ-locally free module in $\mathcal{T}_w$.

By Theorem 3.14, it remains to show that modules other than the preprojective and preinjective modules in the component $\mathcal{T}_{PI}$ are not τ-locally free. Observe that the orbits of the other modules meet either the rays or the corays containing S_1 and S_n, respectively. As S_1 and S_n are not locally free modules, modules in those rays/corays are not locally free by Lemma 3.16. Therefore the modules in $\mathcal{T}_{PI}$ that are neither preprojective nor preinjective are not τ-locally free. This completes the proof. □

4. An application to the conjecture by Geiss-Lercler-Schröer

In this section, we apply Theorem 3.19 to prove Conjecture 1 in the case where the Cartan matrix C is of type $\widetilde{\mathbb{C}}_{n-1}$ and the symmetrizer $D = \mathrm{diag}(2, 1, 1, \cdots, 1, 1, 2)$.

4.1. **Roots and Coxeter transformations.** In this subsection C can be any symmetrizable Cartan $n \times n$ matrix of affine type and D can be any symmetrizer of C. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be a list of positive simple roots of type C. For $1 \le i, j \le n$, define
$$s_i(\alpha_j) = \alpha_j - c_{ij}\alpha_i.$$
This yields a reflection $s_i : \mathbb{Z}^n \to \mathbb{Z}^n$ on the root lattice $\mathbb{Z}^n = \sum_{i=1}^n \mathbb{Z}\alpha_i$, where α_i is identified with the ith standard basis vector of $\mathbb{Z}^n$. The *Weyl group* W is the subgroup of $\mathrm{Aut}(\mathbb{Z}^n)$ generated by $s_1, s_2, \cdots, s_n$. Denote by
$$\Delta_{\mathrm{re}} = \cup_{i=1}^n W(\alpha_i)$$

the set of *real roots*, and by

$$\Delta_{\rm im} = \mathbb{Z}\delta$$

the set of *imaginary roots*, where δ is the unique minimal positive imaginary root determined by the Cartan matrix C. For instance, in the case of type $\widetilde{\mathbb{C}}_{n-1}$,

$$\delta = \alpha_1 + 2\Sigma_{i=2}^{n-1}\alpha_i + \alpha_n = (1, 2, \cdots, 2, 1).$$

The set of *roots* determined by C is

$$\Delta = \Delta_{\rm re} \cup \Delta_{\rm im}$$

and with the set of *positive roots*

$$\Delta^+ = \Delta \cap \mathbb{N}^n = \Delta^+_{\rm re} \cup \Delta^+_{\rm im},$$

where $\Delta^+_{\rm re} = \Delta_{\rm re} \cap \mathbb{N}^n$ and $\Delta^+_{\rm im} = \Delta_{\rm im} \cap \mathbb{N}^n$.

An *orientation* of C is a subset $\Omega \subset \{1, 2, \cdots, n\} \times \{1, 2, \cdots, n\}$ such that the following hold:

(1) $\{(i, j), (j, i)\} \cap \Omega \neq \emptyset$ if and only if $c_{ij} < 0$;

(2) For each sequence $((i_1, i_2), (i_2, i_3), \cdots, (i_t, i_{t+1}))$ with $t \geq 1$ and $(i_s, i_{s+1}) \in \Omega$ for all $1 \leq s \leq t$, we have $i_1 \neq i_{t+1}$.

Let $Q = Q(C, \Omega)$ be the quiver with vertices $Q_0 = \{1, \ldots, n\}$ and arrows

$$Q_1 = \{\alpha^g_{ji} : i \to j \mid (j, i) \in \Omega \text{ and } 1 \leq g \leq \gcd\{|c_{ij}|,\ |c_{ji}|\}\} \cup \{\varepsilon_i : i \to i \mid i \in Q_0\}.$$

Let $Q^0 = Q^0(C, \Omega)$ be the quiver obtained from Q with the loops ε_i removed.

For an orientation Ω of C and an admissible vertex i in $Q^0(C, \Omega)$, let

$$s_i(\Omega) = \{(r, s) \in \Omega \mid i \notin \{r, s\}\} \cup \{(s, r) \mid i \in \{r, s\}, (r, s) \in \Omega\}.$$

Then $s_i(\Omega)$ is again an orientation of C. A sequence $\mathbf{i} = (i_1, i_2, \cdots, i_n)$ is called a $+$*-admissible sequence* for (C, Ω) if $\{i_1, i_2, \cdots, i_n\} = \{1, 2, \cdots, n\}$, i_1 is a sink in $Q^0(C, \Omega)$ and i_k is a sink in $Q^0(C, s_{i_{k-1}} \cdots s_{i_1}(\Omega))$ for $2 \leq k \leq n$. For such a sequence $\mathbf{i}$, define

$$\beta_{\mathbf{i},k} = \begin{cases} \alpha_{i_1} & \text{if } k = 1, \\ s_{i_1}s_{i_2}\cdots s_{i_{k-1}}(\alpha_{i_k}) & \text{if } 2 \leq k \leq n. \end{cases}$$

Similarly, define

$$\gamma_{\mathbf{i},k} = \begin{cases} \alpha_{i_n} & \text{if } k = n, \\ s_{i_n}\cdots s_{i_{k+1}}(\alpha_{i_k}) & \text{if } 1 \leq k \leq n-1. \end{cases}$$

Let $c_{\mathbf{i}} = s_{i_n}s_{i_{n-1}}\cdots s_{i_1} : \mathbb{Z}^n \to \mathbb{Z}^n$. Then $c^{-1}_{\mathbf{i}} = s_{i_1}s_{i_2}\cdots s_{i_n} : \mathbb{Z}^n \to \mathbb{Z}^n$. These are two Coxeter transformations associated to $\mathbf{i}$.

For a $+$-admissible sequence $\mathbf{i} = (i_1, i_2, \cdots, i_n)$ for $(C,\ \Omega)$, the rotated sequence

$$\mathbf{i}' = (i_2, i_3 \cdots, i_n, i_1)$$

is also a $+$-admissible sequence for $(C, s_{i_1}(\Omega))$, and $c_{\mathbf{i}'} = s_{i_1}s_{i_n}\cdots s_{i_3}s_{i_2}$ and $c^{-1}_{\mathbf{i}'} = s_{i_2}s_{i_3}\cdots s_{i_n}s_{i_1}$ are the Coxeter transformations associated to $\mathbf{i}'$.

Similarly, a $-$*-admissible sequence* can be defined using sources. In fact, the sequence $\mathbf{i} = (i_1, i_2, \cdots, i_n)$ is $+$-admissible if and only if its reverse sequence $\mathbf{i}^{-1} = (i_n, i_{n-1}, \cdots, i_1)$ is $-$-admissible. We have, $c^{-1}_{\mathbf{i}} = c_{\mathbf{i}^{-1}}$. Similar to $+$-admissible sequences, a rotated sequence of a $-$-admissible sequence is also $-$-admissible. We call both a $+$-admissible sequence and a $-$-admissible sequence an *admissible sequence*.

For a $-$-admissible sequence $\mathbf{i} = (i_1, i_2, \ldots, i_n)$, define

$$\gamma_{\mathbf{i},k} = \begin{cases} \alpha_{i_1} & \text{if } k = 1, \\ s_{i_1}s_{i_2}\cdots s_{i_{k-1}}(\alpha_{i_k}) & \text{if } 2 \leq k \leq n, \end{cases}$$

and

$$\beta_{\mathbf{i},k} = \begin{cases} \alpha_{i_n} & \text{if } k = n, \\ s_{i_n}\cdots s_{i_{k+1}}(\alpha_{i_k}) & \text{if } 1 \leq k \leq n-1. \end{cases}$$

Lemma 4.1. *Let $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ be an admissible sequence and $\mathbf{i}' = (i_2, i_3, \ldots, i_n, i_1)$. Then the reflection s_{i_1} induces a bijection between $\{c^{-r}_{\mathbf{i}}(\beta_{\mathbf{i},k})|r \in \mathbb{Z}_{\geq 0}, 1 \leq k \leq n\} \cup \{c^s_{\mathbf{i}}(\gamma_{\mathbf{i},k})|s \in \mathbb{Z}_{\geq 0}, 1 \leq k \leq n\} \setminus \{\alpha_{i_1}\}$ and $\{c^{-r}_{\mathbf{i}'}(\beta_{\mathbf{i}',k})|r \in \mathbb{Z}_{\geq 0}, 1 \leq k \leq n\} \cup \{c^s_{\mathbf{i}'}(\gamma_{\mathbf{i}',k})|s \in \mathbb{Z}_{\geq 0}, 1 \leq k \leq n\} \setminus \{\alpha_{i_1}\}$.*

Proof. First consider the case when $\mathbf{i}$ is $+$-admissible. Note that $s_{i_1}c_{\mathbf{i}'} = c_{\mathbf{i}}s_{i_1}$, $s_{i_1}c_{\mathbf{i}}^{-1} = c_{\mathbf{i}'}^{-1}s_{i_1}$ and $\beta_{\mathbf{i},1} = \gamma_{\mathbf{i}',n} = \alpha_{i_1}$ by definition. The lemma follows from the following calculation

$$s_{i_1}(c_{\mathbf{i}}^{-r}(\beta_{\mathbf{i},k})) = \begin{cases} c_{\mathbf{i}'}^{-r}(\beta_{\mathbf{i}',k-1}) & \text{if } 2 \le k \le n, r \ge 0, \\ c_{\mathbf{i}'}^{-r+1}(\beta_{\mathbf{i}',n}) & \text{if } k = 1, r > 0 \end{cases}$$

and

$$s_{i_1}(c_{\mathbf{i}}^{s}(\gamma_{\mathbf{i},k})) = \begin{cases} c_{\mathbf{i}'}^{s}(\gamma_{\mathbf{i}',k-1}) & \text{if } 2 \le k \le n, \\ c_{\mathbf{i}'}^{s+1}(\gamma_{\mathbf{i}',n}) & \text{if } k = 1 \end{cases}$$

for each $s \ge 0$.

When $\mathbf{i}$ is $-$-admissible, the proof can be similarly done. We skip the details. □

4.2. **Geiss-Leclerc-Schröer's Conjecture.** In this subsection, we will prove Conjecture 1 for the case where C is of type $\widetilde{\mathbb{C}}_{n-1}$ and the symmetrizer D is minimal, that is, $D = \mathrm{diag}(2, 1, \ldots, 2, 1)$.

For a locally free H-module M, denote by r_i the rank of the free H_i-module M_i, where $i \in Q_0$. We call

$$\underline{\mathrm{rank}}M := (r_1, \cdots, r_n)$$

the *rank vector* of M.

Below we recall a few results from [7], which are important to prove the main result Theorem 4.9 in this section.

Lemma 4.2. [7, Proposition 11.5] *Let $c = c_{\mathbf{i}}$ for some $+$-admissible sequence $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ and M be a τ-locally free H-module. If $\tau^k(M) \neq 0$, then*

$$\underline{\mathrm{rank}}\tau^k(M) = c^k(\underline{\mathrm{rank}}M).$$

Lemma 4.3. [7, Lemmas 2.1, 3.2 and 3.3] *Let C be a symmetrizable Cartan matrix that is not of Dynkin type and let $\mathbf{i} = (i_1, i_2, \cdots, i_n)$ be an admissible sequence. Then*

$$\underline{\mathrm{rank}}\tau^{-r}(P_{i_k}) = c_{\mathbf{i}}^{-r}(\beta_{\mathbf{i},k})$$

and

$$\underline{\mathrm{rank}}\tau^{s}(I_{i_k}) = c_{\mathbf{i}}^{s}(\gamma_{\mathbf{i},k}),$$

where $r, s \ge 0$ and $1 \le k \le n$. Moreover these rank vectors are pairwise distinct positive real roots.

Note that a representation of $Q = Q(C, \Omega)$ can be naturally viewed as a representation of a modulated graph $\mathcal{M}(C, D)$ and vice versa. The representation categories of Q and $\mathcal{M}(C, D)$ are equivalent. For a sink (resp. a source) in the modulated graph, one can define a reflection functor F_i^+ (resp. F_i^-) on the representations of the modulated graph, in a similar way as reflection functors defined for (simply-laced) quivers. When i is admissible, we write the reflection functor by F_i which should be interpreted as F_i^+ when i is a sink and F_i^- otherwise.

Lemma 4.4. [7, Proposition 9.4] *Let $H = H(C, D, \Omega)$ and $H' = H(C, D, s_i\Omega)$, where i is admissible in $Q^0(C, \Omega)$. If M is an indecomposable locally free H-module and is not isomorphic to S_i, then $F_i(M)$ is indecomposable and*

$$\underline{\mathrm{rank}}F_i(M) = s_i(\underline{\mathrm{rank}}M).$$

Proposition 4.5. [7, Proposition 9.6] *Let M be a rigid τ-locally free H-module and let i be admissible in Q^0. Then $F_i(M)$ is also rigid and τ-locally free.*

Proposition 4.6. [4, Proposition 1.9] *Let $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ be a $+$-admissible sequence with respect to the orientation Ω. The set of positive roots determined by the Cartan matrix C is the disjoint union of the following.*

(1) $\{c_{\mathbf{i}}^{-r}(\beta_{\mathbf{i},k}) \mid r \in \mathbb{Z}_{\ge 0}, 1 \le k \le n\}$.
(2) $\{c_{\mathbf{i}}^{s}(\gamma_{\mathbf{i},k}) \mid s \in \mathbb{Z}_{\ge 0}, 1 \le k \le n\}$.
(3) $\{x + r\delta \mid x = 0$ or a positive root that is $< \delta$ and can be deduced from a certain list of roots; $r \in \mathbb{Z}_{\ge 0}$ and $r \neq 0$ when $x = 0\}$.

Remark 4.7. (1) By Proposition 3.8, we know the indecomposable modules at the bottom of the tube of rank $n-1$. Their rank vectors are pairwise distinct and are exactly those in the list of roots in Proposition 4.6 (3) when the orientation Ω is linear, i.e. $\Omega = \{(2,1), (3,2), \ldots, (n, n-1)\}$. These rank vectors are $(*)$: the simple roots α_i for $1 < i < n$ and $\sum_{i=1}^n \alpha_i$. In this case the roots x in Proposition 4.6 (3) are sums of the form

$$\sum_{i \le t \le i+j} c_{\mathbf{i}}^t \alpha_2$$

for some i, j with $0 \le i < n-1$ and $0 \le j < n-2$ (see the discussion between Lemma 1.8 and Proposition 1.9 in [4]), where $\mathbf{i} = (n, n-1, \dots, 2, 1)$. In fact in the sum, α_2 can be replaced by any root in the list $(*)$.

(2) Our main result of this section below, Theorem 4.9, largely follows from Theorem 3.19 and Proposition 4.6 when Ω is linear. However, when it is not linear, Dlab-Ringel do not explain further how to deduce x from list $(*)$ of roots in the paper [4]. In our proof to Theorem 4.9, we will deal with the quiver Q with nonlinear orientation separately, using reflection functors.

By Lemma 3.17, any band w is equivalent to a band of the form

$$w_0^{-1}\varepsilon_n^{\pm} w_0 \varepsilon_1^{\pm} \dots w_0^{-1}\varepsilon_n^{\pm} w_0 \varepsilon_1^{\pm} \quad (**).$$

We define the *delta-length* of w by the number m of w_0 appearing in the band $(**)$, denoted by $\mathrm{dl}(w) = m$. For instance,

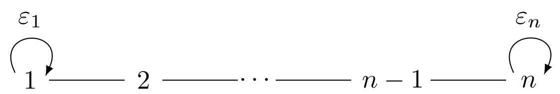
$$\mathrm{dl}(w_0^{-1}\varepsilon_n w_0 \varepsilon_1) = 1$$

and

$$\mathrm{dl}(w_0^{-1}\varepsilon_n w_0 \varepsilon_1^{-1} w_0^{-1} \varepsilon_n w_0 \varepsilon_1) = 2.$$

If $\mathrm{dl}(w) = r$, $S = (K^s, \varphi)$ is a simple representation of $K[T, T^{-1}]$, then the band module $M(w, s, \varphi)$ has rank vector $sr\delta$.

Note that when C is of type $\tilde{\mathbb{C}}_{n-1}$ and D is minimal, the quiver $Q = Q(C, \Omega)$ constructed in [7] is exactly the quiver we have in Section 3.2,



and the algebra $H = H(C, D, \Omega) = KQ/I$, where I is generated by ε_i^2 for $i = 1, n$. We restate Conjecture 1 for this case as follows.

Conjecture 4.8. *Let C be a Cartan matrix of type $\tilde{\mathbb{C}}_{n-1}$ and let D be a minimal symmetrizer of C. Then There is a bijection between positive roots of type C and rank vectors of τ-locally free H-modules.*

Theorem 4.9. *Let $H = H(C, D, \Omega)$ be a string algebra of type $\tilde{\mathbb{C}}_{n-1}$. Then α is a positive root if and only if there is a τ-locally free module M such that $\underline{\mathrm{rank}} M = \alpha$. Moreover,*

(1) *if α is a positive real root, then there is a unique τ-locally free H-module M (up to isomorphism) such that $\underline{\mathrm{rank}} M = \alpha$.*
(2) *if $\alpha = m\delta$ is a positive imaginary root, then all the following modules have rank vector α.*
   (a) *The modules in level $m(n-1)$ in the tube of rank $n-1$.*
   (b) *The modules in level r of the homogeneous tubes $\mathcal{H}_{w,S}$, where $w \in \overline{\mathrm{Ba}}(H)$ with $\mathrm{dl}(w) = t$, $S = (K^s, \varphi) \in \mathcal{S}$ is a simple $K[T, T^{-1}]$-module such that $r = \frac{m}{st}$. In particular, $r = m$ when $\mathrm{dl}(w) = 1$ and $s = 1$.*
(3) *the modules at the bottom of the tube of rank $n-1$ are rigid.*

Proof. Case I: the orientation Ω is linear. We first explain that (3) is true. By Proposition 3.8, the modules at the bottom of the tube of rank $n-1$ are the simples S_i $(1 < i < n)$ and $M((\alpha_{21})_-)$. The simples are rigid since there is no loops at vertices $2, \dots, n-1$, and $M((\alpha_{21})_-)$ is rigid, by the homological interpretation of the Ringel Form defined for Q in [7, Proposition 4.1].

Next by Lemma 4.3 and Remark 4.7, the roots in Proposition 4.6 (1) are the rank vectors of indecomposable preprojective modules; the roots in Proposition 4.6 (2) are the rank vectors of indecomposable preinjective modules; the roots in Proposition 4.6 (3) are the rank vectors of indecomposable modules in tubes. Therefore the theorem follows from Theorem 3.19 and the descriptions of tubes in Theorem 3.14.

Observation (†): for a τ-locally free module M, $\underline{\mathrm{rank}} M$ is an imaginary root if and only if M is in a homogeneous tube or in levels $r(n-1)$ $(r \in \mathbb{N})$ in the tube of rank $n-1$.

Case II: the general case. First note that any quiver L' of type $\mathbb{A}_n$ can be obtained by applying a sequence of admissible reflections $s_{i_1}, \dots, s_{i_m}$ on the linear quiver L of type $\mathbb{A}_n$, where i_1 is admissible in L, i_t is admissible in $s_{i_{t-1}} \dots s_{i_1}(L)$ for $t > 1$ and $L' = s_{i_m} \dots s_{i_1}(L)$. Assume that the theorem holds for an orientation Ω. Let i be an admissible vertex in $Q^0(C, \Omega)$. By induction, we only need to show that the theorem holds for the orientation $s_i(\Omega)$.

Let M be an H-module at the bottom of the tube of rank $n-1$. By the induction hypothesis, M is rigid and τ-locally free. As M is in a tube, M is not a simple module associated to an admissible vertex and so $F_i(M) \neq 0$. Furthermore, it is indecomposable by Lemma 4.4 and it is a rigid τ-locally free H'-module by Proposition 4.5, where $H' = H(C, D, s_i(\Omega))$. Without loss of generality, we assume that i is a sink. We choose a $+$-admissible sequence $\mathbf{i} = (i_1, i_2, \dots, i_n)$ with $i_1 = i$. By Lemma 4.4, $\underline{\mathrm{rank}} F_i^+(M) = s_i(\underline{\mathrm{rank}} M)$, which is not a root as those listed in Lemma 4.1. Note also

$$\Sigma_{j=0}^{n-2} c_{\mathbf{i}}^j (\underline{\mathrm{rank}} F_i^+(M)) = \delta,$$

which is obtained by applying s_i to $\Sigma_{j=0}^{n-2} c_{\mathbf{i}}^j ((\underline{\mathrm{rank}} M) = \delta$. Therefore, $F_i^+(M)$ is an H'-module at the bottom of the tube of rank $n-1$. Consequently, (3) holds and the reflection s_i induces a bijection between the rank vectors of the τ-locally free H-module in the tube of rank $n-1$ and the rank vectors of the τ-locally free H'-module in the tube of rank $n-1$. Therefore, together with Lemma 4.1,

$$\begin{aligned} & \{\underline{\mathrm{rank}} M \mid M \text{ is a } \tau\text{-locally free } H'\text{-module}\} \\ = \; & \{s_i(\underline{\mathrm{rank}} M) \mid M \text{ is a } \tau\text{-locally free } H\text{-module such that } \underline{\mathrm{rank}} M \neq \alpha_i\} \cup \{\alpha_i\}. \end{aligned}$$

The latter is exactly the set of positive roots by the induction hypothesis and the fact that s_i permutes $\Delta^+ \setminus \{\alpha_i\}$. Therefore α is a positive root if and only if $\alpha = \underline{\mathrm{rank}} M$ for some τ-locally free H'-module M. The remaining parts of the theorem, (1) and (2), follow from Theorems 3.14, 3.19 and the observation (†). Therefore, the theorem holds for $s_i(\Omega)$. This completes the proof. □

Corollary 4.10. *Conjecture 4.8 is true.*

Following Theorem 4.9, we can now enhance Proposition 4.6 as follows.

Corollary 4.11. *(cf.* [4, Proposition 1.9]*) Let C be the Cartan matrix of type $\widetilde{\mathbb{C}}_{n-1}$, D the minimal symmetrizer and let $\mathbf{i} = (i_1, i_2, \cdots, i_n)$ be a $+$-admissible sequence for (C, Ω). Then*

$$\Delta^+(C) = \{c_{\mathbf{i}}^{-r}(\beta_{\mathbf{i},k}) \mid r \in \mathbb{Z}_{\geq 0}, 1 \leq k \leq n\} \cup \{c_{\mathbf{i}}^{s}(\gamma_{\mathbf{i},k}) \mid s \in \mathbb{Z}_{\geq 0}, 1 \leq k \leq n\} \cup$$

$$\{(\sum_{p \leq j \leq p+q} c_{\mathbf{i}}^j(\alpha)) + m\delta \mid 0 \leq p < n-1,\ 0 \leq q < n-2 \text{ and } m \in \mathbb{Z}_{\geq 0}\} \cup \mathbb{Z}_{>0}\delta,$$

where $\alpha = \alpha_1 + \alpha_2$ (or any other $\alpha_i + \alpha_{i+1}$) if Q^0 is alternating, i.e. each vertex is admissible, and otherwise α can be any simple root α_l that is associated to a non-admissible vertex l.

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References

[1] M. Auslander, I. Reiten, S. O. Smalø, *Representation theory of Artin algebras.* Cambridge University Press, 1995.
[2] M.C.R. Butler, C.M. Ringel, Auslander-Reiten sequences with few middle terms and applications to string algebras. Comm. Algebra 15 (1987), no. 1-2, 145-179.
[3] X.W Chen, R. Wang, Skew group categories, algebras associated to Cartan matrices and folding of root lattices, arXiv: 2102. 08007.
[4] V. Dlab, C.M. Ringel, Indecomposable representations of graphs and algebras. In: Memoirs of the American Mathematical Society 6, vol. 173 (1976), pp. v+57.
[5] P. Gabriel, Unzerlegbare Darstellungen. I. (German) Manuscripta Math. 6 (1972), 71-103; correction, ibid. 6 (1972), 309.
[6] C. Geiß, On components of type $\mathbb{Z}\mathbb{A}_\infty^\infty$ for string algebras, Comm. Algebra, 26(3) (1998), 749-758.
[7] C. Geiß, B. Leclerc, J. Schröer, Quivers with relations for symmetrizable Cartan matrices I: Foundations. Invent. Math. 209 (2017), 61-158.
[8] C. Geiß, B. Leclerc, J. Schröer, Quivers with relations for symmetrizable Cartan matrices II: Change of symmetrizer. Int. Math. Res. Not. 9(2018), 2866-2898.
[9] C. Geiß, B. Leclerc, J. Schröer, Quivers with relations for symmetrizable Cartan matrices III: Convolution algebras. Represent. Theory 20 (2016), 375-413.
[10] C. Geiß, B. Leclerc, J. Schröer, Quivers with relations for symmetrizable Cartan matrices IV: crystal graphs and semicanonical functions. Selecta Math. (N.S.) 24 (2018), no. 4, 3283-3348.
[11] C. Geiß, B. Leclerc, J. Schröer, Quivers with relations for symmetrizable Cartan matrices V: Caldero-Chapoton formulas. Proc. Lond. Math. Soc. (3) 117 (2018), no. 1, 125-148.

[12] C. Geiß, B. Leclerc, J. Schröer, Rigid modules and Schur roots. Math. Z. (2020), 295, 1245-1277.
[13] V. Kac, Infinite root systems, representations of graphs and invariant theory, Invent. Math. 56 (1980), 57-92.
[14] V. Kac, Infinite root systems, representations of graphs and invariant theory. II., J. Algebra 78 (1982), no 1, 141-162.
[15] C. Ricke, On tau-Tilting Theory and Perpendicular Categories. PhD Thesis, University of Bonn (2016).

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