



Interpreting systems of continuity equations in spaces of probability measures through PDE duality

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Abstract

We introduce a notion of duality solution for a single or a system of transport equations in spaces of probability measures reminiscent of the viscosity solution notion for nonlinear parabolic equations. Our notion of solution by duality is, under suitable assumptions, equivalent to gradient flow solutions in case the single/system of equations has this structure. In contrast, we can deal with a quite general system of nonlinear non-local, diffusive or not, system of PDEs without any variational structure.

Keywords Continuity equation · Wasserstein spaces · Duality · Viscosity solution

Mathematics Subject Classification 35F55 · 35G55 · 35L65 · 35D40

1 Introduction

Many PDE modelling instances of applied analysis lead to transport equations for a *density function* ρ of the form $\partial_t \rho = \nabla \cdot (\rho \mathbf{v})$, where $t > 0$, $x \in \mathbb{R}^d$, and \mathbf{v} is the *velocity field*. In many of these applications, there is a ubiquitous choice of convolutional-type velocities depending on the density well motivated by applications in mathematical biology, social sciences and neural networks, see for instance [4, 18, 21, 36, 43, 44, 48]. Moreover, in many of these applications, \mathbf{v} is given by a function of ρ itself, and it may happen that the density ρ is not an integrable function, but rather a measure. Since the mathematical treatment of these problems is rather tricky, and indeed many “non-physical” solutions may appear, the natural way to approach these problems is the so-called vanishing-viscosity method. With this method linear diffusion $D\Delta\rho$ is included to the right-hand side, leading to the problem

$$\partial_t \rho = \nabla \cdot (\rho \mathbf{v}[\rho]) + D\Delta\rho.$$

The “physical” or *entropic* solution is the one obtained in the limit $D \searrow 0$. In this work, we deal with systems of such equations, where several *species* inhabit a domain. We will

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consider that the evolution of these species is coupled only by the velocity field. We are interested in systems of n aggregation–diffusion equations of the form

$$\partial_t \mu_t^i = \nabla \cdot (\mu_t^i \mathbf{K}_t^i[\boldsymbol{\mu}_t]) + D^i \Delta \mu_t^i, \quad i = 1, \dots, n \text{ and } (t, x) \in (0, \infty) \times \mathbb{R}^d \quad (1.1)$$

where $D^i \geq 0$ and \mathbf{K}_t^i are velocity-fields depending non-locally on the full vector of densities $\boldsymbol{\mu} = (\mu^1, \dots, \mu^n)$ and possibly in the spatial or time variables, under some hypotheses described below. In fact, we are to work with a generalisation allowing \mathbf{K}_t to depend on the previous time states see (P). Notice that we cover the cases with and without diffusion ($D_i > 0$ and $D_i = 0$) in a unified framework. We will introduce a new notion of solution, which we call dual-viscosity solution, with well-posedness, and that is able to detect this vanishing-viscosity limit.

A particularly interesting case is when the problem (1.1) has a 2-Wasserstein gradient-flow structure. For example, in the scalar setting ($n = 1$) without diffusion ($D = 0$) and with an interaction potential energy functional of the form

$$\mathbf{K}[\mu] = \nabla \frac{\delta \mathcal{F}}{\delta \rho}[\mu], \quad \mathcal{F}[\mu] = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W(x - y) d\mu(x) d\mu(y),$$

the classical gradient flow theory developed in [2, 3] applies when W is λ -convex and smooth, and this later generalised to different settings [6, 8, 13, 14, 26, 27, 29, 32, 33, 38] for instance. This can also be applied to (1.1) with $D > 0$ by including the Boltzmann entropy in the free energy functional, see [7, 15, 20, 22, 23, 42, 49, 51]. We refer to [19] for a recent survey of results in aggregation–diffusion equations. We show in Sect. 6 that our new notion of dual viscosity solutions is equivalent, in some examples, to the notion of gradient flow solutions.

In many situations, a theory of entropy solutions in the sense of Kruzkov (see [16, 40]) is possible, when we have initial data and solutions in $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$. For example, the work [30] deals with $d = n = 1, D = 0$ and $K = \theta(\mu)W' * \mu$ where $\theta(M) = 0$ for some $M > 0$, and W very regular. Then, clearly initial data $0 \leq \rho_0 \leq M$ lead to solutions $0 \leq \rho_t \leq M$. The authors prove uniqueness of entropy solutions in the sense of Kruzkov in the space $L^\infty(0, T; L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d))$, by using the continuous dependence results in [39] and a fixed point argument. This requires fairly strict hypothesis on the regularity of the initial data (namely bounded variation). They also introduce an interesting constructive method based on sums of characteristics of domains evolving in time. A recent extension of [30] to related problems can be seen in [35].

However, in the diffusion-less regime $D = 0$, these models based on transport equations can describe particle systems described by Dirac deltas, which interact through an ODE. The idea of working with Dirac deltas leads to numerical methods for general integrable data by approximating the initial datum by a sum of Diracs and regularizing the entropy [17, 24, 25]. Kruzkov’s notion of entropy solutions cannot be extended to measures. Furthermore, for less regular W (or $\mathbf{K}[\mu] = \nabla V(x)$ known), the finite-time formation of Dirac deltas even for smooth initial data is not known. The paper [32] studies uniqueness of distributional solutions for $n = 2$, when $D^i = 0$,

$$\mathbf{K}^i[\boldsymbol{\mu}] = \nabla \mathcal{H}_i * \mu^i + \nabla \mathcal{K}_i * \mu^j, \quad \{j\} = \{1, 2\} \setminus \{i\} \quad (1.2)$$

where $\mathcal{H}_i, \mathcal{K}_i \in \mathcal{W}^{2,\infty}$. The authors indicate that any distributional solution is a push-forward solution (pointing to [2, Chapter 8]) of the flow that solves

$$\frac{dX_t^i}{dt} = \mathbf{K}_t^i[\boldsymbol{\mu}_t](X_t^i), \quad X_0^i(y) = y. \quad (1.3)$$

This push-forward structure is not available for problems with diffusion. The main goal of this work is to construct a notion of well-posed solutions that works for the case

$$K^i[\mu] = \nabla \sum_{j=1}^n W_{ij} * \mu^j, \tag{1.4}$$

$d, n \geq 1, D \geq 0$, and measure initial data. We will work in 1-Wasserstein space, and recover continuous dependence estimates by discussing the “dual” problem, for which we use viscosity solutions by studying the corresponding Lipschitz estimates. We show also how our results can be adapted to different spaces (e.g., \dot{H}^{-1} and H^1).

So far, the velocity field depends only on t and μ_t . We can consider a generalisation of K , denoted \mathfrak{K} , that depends also on the past. We denote this extended notation by $\mathfrak{K}[\mu]_t$. By saying that \mathfrak{K} depends only on the past we mean that $0 \leq t \leq s \leq T$ and $\mu \in C([0, T]; \mathcal{P}_1(\mathbb{R}^d)^n)$ then

$$(\mathfrak{K}^i[\mu]_{|[0,s]})_t = (\mathfrak{K}^i[\mu]_{|[0,t]})_t. \tag{H}$$

To simplify the notation, when it will not lead to confusion, we simply write $\mathfrak{K}[\mu]_t$. For example, we cover the case

$$\mathfrak{K}[\mu]_t = \int_0^t \nabla W_s * \mu_s ds,$$

where W_t are a family of convolution kernels. Hence, we extend the (1.1) to the more general

$$\partial_t \mu_t^i = \nabla \cdot (\mu_t^i (\mathfrak{K}^i[\mu])_t) + D^i \Delta \mu_t^i, \quad i = 1, \dots, n \text{ and } (t, x) \in (0, \infty) \times \mathbb{R}^d. \tag{P}$$

This covers the previous setting by defining

$$(\mathfrak{K}[\mu])_t^i = K_t^i[\mu_t].$$

The structure of the paper is as follows. Section 2 is devoted to the main results and methods. We begin in Sect. 2.1 by motivating and introducing a new notion of solution in 1-Wasserstein space, motivated by the duality with viscosity solution of the dual problem, which we call *dual viscosity solution*. In Sect. 2.2 we present well-posedness results for K regular enough. In Sect. 2.3 we present a result of stability under passage to the limit. In Sect. 2.4 we discuss a limit case of the regularity assumptions of the well-posedness theory, given by the Newtonian potential. We close the presentation of the main results with Sect. 2.5 on open problems our current results do not cover, mainly non-linear diffusion and non-linear mobility. The full proofs are postponed to the next sections. First, in Sect. 3 we prove well-posedness and estimates for the dual problem of (1.1), where we assume $K[\mu]$ is replaced by a known field. Then, in Sect. 4 we prove well-posedness of (1.1) when $K[\mu]$ is again replaced by a known field. In Sect. 5 we prove the results stated in Sect. 2.2 by a fixed-point argument based on the previous estimates.

We conclude the paper with two sections on comments and extensions of our main results. First, in Sect. 6, we show that, where a gradient-flow structure is available, our solutions coincide with the steepest descent solutions, under some assumptions. In Sect. 7 we extend our main results from the 1-Wasserstein framework to the negative homogeneous Sobolev space \dot{H}^{-1} . We recall that \dot{H}^{-1} is related to the 2-Wasserstein space (see below).

2 Main results and methods

2.1 A new notion of solution via duality

Considering first the case $n = 1$, let $E_t = -\mathfrak{K}[\mu]_t$, so that we are looking at the problem

$$\partial_t \mu_t = -\nabla \cdot (E_t \mu_t) + D \Delta \mu_t. \tag{P_E}$$

Furthermore, if we naively assume the equation to be satisfied pointwise, we can multiply by a test function φ and assume we can integrate by parts integrate by parts, we recover

$$\int_{\mathbb{R}^d} \varphi_T d\mu_T + \int_0^T \int_{\mathbb{R}^d} \left(-\frac{\partial \varphi_t}{\partial t} - E_t \nabla \varphi_t - D \Delta \varphi_t \right) d\mu_t dt = \int_{\mathbb{R}^d} \varphi_0 d\mu_0. \tag{2.1}$$

To cancel the double integral, we take $\varphi_t = \psi_{T-t}$ and ψ_s a solution of

$$\frac{\partial \psi_s}{\partial s} = E_{T-s} \nabla \psi_s + D \Delta \psi_s, \quad s \in [0, T], \tag{P_E^*}$$

and ψ_s Lipschitz in x . Using $\varphi_t = \psi_{T-t}$ we reduce (2.1) to the equivalent formulation

$$\int_{\mathbb{R}^d} \psi_0 d\mu_T = \int_{\mathbb{R}^d} \psi_T d\mu_0. \tag{D_E}$$

This is an interesting formulation because it will allow us to exploit the duality properties of the spaces for μ and ψ .

Hence, we are exploiting the well-known duality between the continuity problem (P_E) and its dual (P_E^*).

When $D = 0$, this connects with the push-forward formulation for first order problems. Problem (P_E^*) can be solved by a generalised characteristics field X_t (e.g., [34, Sect. 3.2, p.97]), and (D_E) means precisely that μ_T is the push-forward by the same field (see, e.g., [2, Sect. 5.2, p.118]).

Notice that in the general setting we have n equations, and ψ depends on T . Hence, we denote our test functions $\psi^{T,i}$. We consider the system of n dual problems

$$\begin{cases} \frac{\partial \psi_s^{T,i}}{\partial s} = -(\mathfrak{K}^i[\mu]_{[0,T-s]})_{T-s} \nabla \psi_s^{T,i} + D^i \Delta \psi_s^{T,i} & \text{for all } s \in [0, T], y \in \mathbb{R}^d \\ \psi_0^{T,i} = \psi_0^i. \end{cases} \tag{P_i^*}$$

We also recover n duality conditions

$$\int_{\mathbb{R}^d} \psi_0^{T,i} d\mu_T^i = \int_{\mathbb{R}^d} \psi_T^{T,i} d\mu_0^i, \quad i = 1, \dots, n. \tag{D_i}$$

Remark 2.1 Notice that we have not introduced a condition as $|x| \rightarrow \infty$ for ψ . In the formal derivation of the notion of solution, we have required that we can formally integrate by parts. This is also the standard argument to define distributional solutions. As with distributional solutions, we set (D_i) as our definition and prove that this is mathematically sound in terms well-posedness. We do not claim the PDE is satisfied in any stronger sense. We will make a comment on gradient flow solutions. Furthermore, we will not rigorously integrate by parts at any point in this manuscript. We will actually discuss in detail the precise admissible initial data ψ_0 for (P_E^*). When we study solutions in 1-Wasserstein space we will only assume that ψ are Lipschitz (but not bounded). The uniqueness of solutions of (P_E^*) in that setting is

discussed in Sect. 3. In Sect. 7 we extend the results to negative Sobolev spaces, and thus we will use some decay (see Remark 7.3).

Remark 2.2 Notice that the problems for the different ψ^i are de-coupled from each other.

It is well-known that classical solutions of (P_E^*) may not exist, specially in the case $D = 0$. The approach developed by Crandall, Ishii and Lions to deal with limit is the notion of viscosity solution (see, e.g., [28]), which we introduce below. Since the theory of viscosity solutions is usually well-posed for Lipschitz functions ψ , a natural metric for μ is the 1-Wasserstein distance (see, e.g., [50, p. 207]). We recall that the space $\mathcal{P}_1(\mathbb{R}^d)$ is the space of the probability measures μ such that

$$\int_{\mathbb{R}^d} |x| d\mu(x) < \infty.$$

The distance d_1 that makes it a complete metric space is usually constructed by a Kantorovich optimal transport problem. The reason we can exploit the duality between (P_E) and (P_E^*) is the following well-known duality characterisation (see [50, Theorem 1.14]): if $\mu, \hat{\mu} \in \mathcal{P}_1(\mathbb{R}^d)$ then

$$d_1(\mu, \hat{\mu}) = \sup \left\{ \int_{\mathbb{R}^d} \psi d(\mu - \hat{\mu}) : \psi \in C(\mathbb{R}^d) \text{ such that } \text{Lip}(\psi) \leq 1 \right\}, \tag{2.2}$$

where, here and below

$$\text{Lip}(\psi) = \sup_{x \neq y} \frac{|\psi(x) - \psi(y)|}{|x - y|}.$$

We recall the notion of viscosity solution.

Definition 2.3 We say that $\psi \in C([0, T] \times \mathbb{R}^d)$ is a viscosity sub-solution of

$$\partial_s \psi_s = E_{T-s} \nabla \psi_s + D \Delta \psi_s \tag{2.3}$$

if, for every $z_0 = (s_0, x_0) \in [0, T] \times \mathbb{R}^d$ and U neighbourhood of z_0 and $\varphi \in C^2(U)$ touching ψ from above (i.e., $\varphi \geq \psi$ on U and $\varphi(z_0) = \psi(z_0)$) we have that

$$\frac{\partial \varphi}{\partial s}(z_0) \leq E_{T-s_0}(x_0) \nabla \varphi(z_0) + D \Delta \varphi(z_0).$$

Conversely, we say that $\psi \in C([0, T] \times \mathbb{R}^d)$ is a super-solution if the inequalities above are reversed for functions touching from below. We say that $\psi \in C([0, T] \times \mathbb{R}^d)$ is a viscosity solution if it is both a viscosity sub and super solution.

Thus, we introduce the following notion of solution of (1.1):

Definition 2.4 We say that $(\mu, \{\Psi^T\}_{T \geq 0})$ is an entropy pair if:

1. For every $T \geq 0$,

$$\begin{aligned} & \Psi^T : \{ \psi_0 : \nabla \psi_0 \in L^\infty(\mathbb{R}^d) \} \\ & \longrightarrow \left\{ (\psi^1, \dots, \psi^n) \in C([0, T] \times \mathbb{R}^d)^n : \frac{\psi^i}{1 + |x|} \in L^\infty((0, T) \times \mathbb{R}^d) \right\} \end{aligned}$$

is a linear map with the following property: for every ψ_0 and $i = 1, \dots, n$ we have $\psi^{T,i} = \Psi^{T,i}[\psi_0]$ is a viscosity solution of (P_i^*) .

2. For each $i = 1, \dots, n$ and $T \geq 0$, $\mu_T^i \in \mathcal{P}_1(\mathbb{R}^d)$ and satisfies the duality condition (D_i) .

Remark 2.5 Notice that in our definition we are not assuming that ψ_0 are bounded or decay at infinity, only that they are Lipschitz continuous. This means that they are bounded by $C(1 + |x|)$, so they can be integrated against functions in $\mathcal{P}_1(\mathbb{R}^d)$.

For convenience, we will simply denote $\Psi = \{\Psi^T\}_{T \geq 0}$.

Definition 2.6 We say that μ is a dual viscosity solution if there exists Ψ so that (μ, Ψ) is an entropy pair.

These definitions can be extended to other norms where a duality characterisation exists. The 2-Wasserstein distance, which is natural for many problems because of the gradient-flow structure, does not have any known such characterisation. However, other spaces like \dot{H}^{-1} do. We present extension of our results to this setting in Sect. 7.

Remark 2.7 Notice that this duality characterisation is somewhat in the spirit of the Benamou–Brenier formula for the 2-Wasserstein distance between two measures, where the optimality conditions involve viscosity solutions of a Hamilton–Jacobi equation.

Remark 2.8 Notice that we have not requested the time continuity of μ , so the notion of initial trace is not clear. However, it will follow from the continuity of ψ^T . Under some assumptions on \mathfrak{K} we will show that $\mu \in C([0, T]; \mathcal{P}_1(\mathbb{R}^d))$. However, in more general settings, the definition guarantees that the initial trace is satisfied in the weak- \star sense. Notice that if $\psi_0 \in C_c$ and ψ_t is continuous in time then

$$\int_{\mathbb{R}^d} \psi_0 d\mu_t = \int_{\mathbb{R}^d} \psi_t d\mu_0 \rightarrow \int_{\mathbb{R}^d} \psi_0 d\mu_0.$$

Remark 2.9 (General linear diffusions). Our definitions, methods, and results can be extended to the case where $D\Delta u$ is replaced by any other linear operator that commutes with ∂_x , and that satisfies the maximum principle, e.g., the fractional Laplacian $D(-\Delta)^s$.

2.2 Well-posedness when K is regularising

So far, $E_t = -K_t[\mu_t]$, i.e., we assume that at every time t the convection comes from a function $K_t : \mu \rightarrow E$. This covers, for example, non-local operators in space. However, our arguments extend to much broader arguments, where K could be non-local in time as well. We will consider the more general case of

$$\mathfrak{K} : C([0, t]; X) \rightarrow C([0, t]; Y), \quad \forall t \in [0, T].$$

The results in this section correspond to $X = \mathcal{P}_1(\mathbb{R}^d)^n$ and Y the space

$$Lip_0(\mathbb{R}^d; \mathbb{R}^d) = \left\{ E \in C_{loc}(\mathbb{R}^d; \mathbb{R}^d) : |\nabla E| \in L^\infty(\mathbb{R}^d) \right\} \tag{2.4}$$

However, the scheme of the proof is general and can be extended to other X, Y . In Sect. 7 we work on the negative homogeneous Sobolev space.

Notice that $Lip_0(\mathbb{R}^d; \mathbb{R}^d)$ is very similar $\dot{W}^{1,\infty}$, but we avoid this terminology to endow it with the following special norm

$$\|E\|_{Lip_0} = \|\nabla E\|_{L^\infty} + \left\| \frac{E}{1 + |x|} \right\|_{L^\infty}.$$

The reason for the Lip_0 nomenclature is that

$$|\mathbf{E}(0)| + \|\nabla \mathbf{E}\|_{L^\infty} \leq \|\mathbf{E}\|_{Lip_0} \leq 2(|\mathbf{E}(0)| + \|\nabla \mathbf{E}\|_{L^\infty}),$$

so it guarantees the bound of $\mathbf{E}(0)$. It is not difficult to see that this is a Banach space. In this setting, we prove the following well-posedness result

Theorem 2.10 *Let $D \geq 0, T > 0$, and assume that for all $t > 0$*

$$\mathfrak{K}^i : C([0, t]; \mathcal{P}_1(\mathbb{R}^d)^n) \rightarrow C([0, t]; Lip_0(\mathbb{R}^d; \mathbb{R}^d)), \tag{2.5}$$

with the property (H), maps bounded sets into bounded sets, and satisfies the following Lipschitz condition

$$\sup_{\substack{t \in [0, T] \\ i=1, \dots, n}} \left\| \frac{\mathfrak{K}^i[\boldsymbol{\mu}]_t - \mathfrak{K}^i[\widehat{\boldsymbol{\mu}}]_t}{1 + |x|} \right\|_{L^\infty(\mathbb{R}^d)} \leq L \sup_{\substack{t \in [0, T] \\ i=1, \dots, n}} d_1(\mu_t^i, \widehat{\mu}_t^i). \tag{2.6}$$

Then, for each $\boldsymbol{\mu}_0 \in \mathcal{P}_1(\mathbb{R}^d)^n$ there exists a unique dual viscosity solution of (P), $\mu \in C([0, T]; \mathcal{P}_1(\mathbb{R}^d)^n)$, and it depends continuously on the initial datum with respect to the d_1 distance. This solution μ also satisfies (P) in distributional sense. Furthermore, if $\mathfrak{K}^i[\boldsymbol{\mu}]_t = K^i[\boldsymbol{\mu}_t]$, then the map $S_T : \boldsymbol{\mu}_0 \in \mathcal{P}_1(\mathbb{R}^d)^n \mapsto \boldsymbol{\mu}_T \in \mathcal{P}_1(\mathbb{R}^d)^n$ is a continuous semigroup.

Going back to (2.2) we have that

$$\left| \int_{\mathbb{R}^d} \psi d(\mu - \widehat{\mu}) \right| \leq d_1(\mu, \widehat{\mu}) \text{Lip}(\psi), \quad \forall \psi \text{ such that } \text{Lip}(\psi) < \infty.$$

Thus, we can use the duality relation (D_i) to get the following upper bound

$$\begin{aligned} d_1(\mu_T, \widehat{\mu}_T) &= \sup_{\text{Lip}(\psi_0) \leq 1} \int_{\mathbb{R}^d} \psi_0 d(\mu_T - \widehat{\mu}_T) \\ &= \sup_{\text{Lip}(\psi_0) \leq 1} \left(\int_{\mathbb{R}^d} \psi_T^T d\mu_0 - \int_{\mathbb{R}^d} \widehat{\psi}_T^T d\widehat{\mu}_0 \right) \\ &= \sup_{\text{Lip}(\psi_0) \leq 1} \left(\int_{\mathbb{R}^d} \psi_T^T d(\mu_0 - \widehat{\mu}_0) + \int_{\mathbb{R}^d} (\psi_T^T - \widehat{\psi}_T^T) d\widehat{\mu}_0 \right) \\ &\leq d_1(\mu_0, \widehat{\mu}_0) \sup_{\text{Lip}(\psi_0) \leq 1} \text{Lip}(\psi_T^T) + \sup_{\text{Lip}(\psi_0) \leq 1} \int_{\mathbb{R}^d} (\psi_T^T - \widehat{\psi}_T^T) d\widehat{\mu}_0. \end{aligned} \tag{2.7}$$

To exploit this fact we take advantage of the following, rather obvious, estimate

$$\left| \int \psi(x) d\mu(x) \right| \leq \left\| \frac{\psi}{1 + |x|} \right\|_{L^\infty} \left(1 + \int_{\mathbb{R}^d} |x| d\mu(x) \right).$$

In fact, we show that, on the second supremum in (2.7), we can focus on functions such that $\psi_0(0) = 0$. The reason for this election will be made clearer later.

Lemma 2.11 *Let μ and $\widehat{\mu}$ two dual viscosity solutions of (P_E) such that $\mathbf{E}, \widehat{\mathbf{E}} \in C([0, T]; Lip_0(\mathbb{R}^d, \mathbb{R}^d))$. Then, letting ψ^T and $\widehat{\psi}^T$ be the corresponding solutions of the dual problems (P_E^{*}), we have that*

$$d_1(\mu_T, \widehat{\mu}_T) \leq d_1(\mu_0, \widehat{\mu}_0) \sup_{\text{Lip}(\psi_0) \leq 1} \text{Lip}(\psi_T^T)$$

$$+ \int_{\mathbb{R}^d} (1 + |x|) d\widehat{\mu}_0(x) \sup_{\substack{\text{Lip}(\psi_0) \leq 1 \\ \psi_0(0) = 0}} \left\| \frac{\psi_T^T - \widehat{\psi}_T^T}{1 + |x|} \right\|_{L^\infty}. \tag{2.8}$$

Proof Notice that $\psi^T = \Psi^T[\psi_0]$, by linearity

$$\Psi^T[\psi_0] = \Psi^T[\psi_0 - \psi_0(0)] + \psi_0(0)\Psi^T[1]$$

By the uniqueness in Proposition 3.1, which we will prove below, since the problem does not contain a zero-order term $\Psi^T[1] = 1 = \widehat{\Psi}^T[1]$. Therefore, we can write

$$\Psi^T[\psi_0] - \widehat{\Psi}^T[\psi_0] = \Psi^T[\psi_0 - \psi_0(0)] - \widehat{\Psi}^T[\psi_0 - \psi_0(0)].$$

Thus, we can take the second supremum of (2.7) exclusively on functions such that $\psi_0(0) = 0$. □

Remark 2.12 We recall a basic property of the 1-Wasserstein distance. It turns out that the first moment is precisely the distance to the Dirac delta, i.e.,

$$\int_{\mathbb{R}^d} |x| d\mu(x) = d_1(\mu, \delta_0).$$

Hence, our fixed point argument will be performed in the balls

$$B_{\mathcal{P}_1}(R) = \left\{ \mu \in \mathcal{P}_1(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x| d\mu(x) \leq R \right\}. \tag{2.9}$$

This explains that it is natural to work with the metric of $(1 + |x|)L^\infty$.

We will devote Sect. 3 to the estimates of ψ that allow us to control each term of (2.8). The estimates are recovered by standard, albeit involved, arguments.

The first term in (2.8), which corresponds to continuous dependence respect to the initial datum, requires only the estimation of a coefficient. If \mathfrak{K} is well-behaved, we expect to obtain an estimate of the type

$$\sup_{\text{Lip}(\psi_0) \leq 1} \text{Lip}(\psi_T^T) \leq C(T, \mathfrak{K}, \mu_0).$$

For the well-posedness theorem $\widehat{\mu}_0 = \mu_0$, we can avoid the first term in (2.8) and this constant is not relevant.

The second term in (2.8), which represents continuous dependence on \mathbf{E} for the dual problem, is harder to estimate. We prove first that even if $\psi_0 \notin L^\infty$ but is Lipschitz and $\mathbf{E}, \widehat{\mathbf{E}}$ are bounded, then $\psi_s^T - \widehat{\psi}_s^T \in L^\infty(\mathbb{R}^d)$. A second, more elaborate, argument will allow us to work of the case where \mathbf{E} grows no-more-than linearly at infinity. If $(\mathbf{P}_{\mathbf{E}}^*)$ has good continuous dependence on \mathbf{E} , we can expect

$$\left\| \frac{\psi_T^T - \widehat{\psi}_T^T}{1 + |x|} \right\|_{L^\infty} \leq C\omega(T) \|\mathfrak{K}[\mu] - \mathfrak{K}[\widehat{\mu}]\|,$$

where we leave the second norm unspecified for now. So, if \mathfrak{K} is Lipschitz in this suitable norm, then we will be able to use Banach’s fixed point theorem.

Remark 2.13 Karlsen and Risebro [39] prove that if $\rho_0, \widehat{\rho}_0 \in L^1 \cap L^\infty \cap BV$ and $\rho, \widehat{\rho} \in L^\infty([0, T]; L^1)$ are entropy solutions of

$$\partial_t \rho_t + \nabla \cdot (f(\rho)\mathbf{E}(x)) = \Delta A(\rho), \quad \partial_t \widehat{\rho} + \nabla \cdot (\widehat{f}(\widehat{\rho})\widehat{\mathbf{E}}(x)) = \Delta A(\widehat{\rho})$$

where $E, \widehat{E} : \mathbb{R}^d \rightarrow \mathbb{R}^d$, and we have an a priori estimates $\rho, \widehat{\rho} : \mathbb{R}^d \rightarrow I \subset \mathbb{R}$ with I bounded and

$$[\rho_t]_{BV(\mathbb{R}^d)}, [\widehat{\rho}_t]_{BV(\mathbb{R}^d)} \leq C\rho, \quad \forall t \in (0, T).$$

Then,

$$\|\rho_t - \widehat{\rho}_t\|_{L^1} \leq \|\rho_0 - \widehat{\rho}_0\|_{L^1} + Ct \left(\|E - \widehat{E}\|_{L^\infty} + [E - \widehat{E}]_{BV} + \|f - \widehat{f}\|_{W^{1,\infty}(I;\mathbb{R})} \right)$$

where $\rho, \widehat{\rho} : \mathbb{R}^d \rightarrow I \subset \mathbb{R}$ and C depends on C_ρ and the norms of $f, \widehat{f} \in W^{1,\infty}$, $E, \widehat{E} \in L^\infty \cap BV$. In [30] the authors use the fixed-point argument of entropy solutions only to prove uniqueness. Existence is done by proving the convergence of the particle systems, which is their main aim. They use [39] which deals directly with the conservation law in L^1 , we obtain simple estimates for the Hamilton–Jacobi dual problem in L^∞ using the maximum principle. Our duality argument is not directly extensible to $L^1(\mathbb{R}^d)$. Applying the duality characterisation of L^1 and (D_i) we can compute

$$\begin{aligned} \|\mu_T - \widehat{\mu}_T\|_{L^1} &= \sup_{|\psi_0| \leq 1} \int_{\mathbb{R}^d} \psi_0 d(\mu - \widehat{\mu}) \leq \sup_{|\psi_0| \leq 1} \left(\int_{\mathbb{R}^d} \psi_0^T d(\mu_0 - \widehat{\mu}_0) \right. \\ &\quad \left. + \int_{\mathbb{R}^d} (\psi_0^T - \widehat{\psi}_0^T) d\widehat{\mu}_0 \right) \end{aligned}$$

Due to the maximum principle we will show that $|\psi_s^T| \leq 1$ and hence

$$\|\mu_T - \widehat{\mu}_T\|_{L^1} \leq \|\mu_0 - \widehat{\mu}_0\|_{L^1} + \|\widehat{\mu}_0\|_{L^1} \sup_{|\psi_0| \leq 1} |\psi_0^T - \widehat{\psi}_0^T|.$$

There is no apparent way to bound the second supremum with a quantity related only to $|E - \widehat{E}|$. A quantity depending on $|\nabla \psi_0|$ appears, and there is no available bound of its supremum over functions such that $|\psi_0| \leq 1$.

2.3 Stability theorem

We conclude the main results by stating a stability theorem. This allows to prove existence of solutions by approximation, in more general settings that the well-posedness theory. We will use it below for several applications.

Theorem 2.14 *Let (μ^k, Ψ^k) be a sequence of entropy pairs in the sense of Definition 2.4 corresponding to some operators \mathfrak{K}^k under the assumptions of Theorem 2.10. Assume that*

1. $D^k \rightarrow D^\infty$.
2. For every $t > 0$, $\mu_t^k \xrightarrow{*} \mu_t^\infty$ in $\mathcal{M}(\mathbb{R}^d)$.
3. For every $\psi_0 \in C_c^\infty(\mathbb{R}^d)$ and $T > 0$ we have $\Psi^{T,k}[\psi_0] \rightarrow \Psi^{T,\infty}[\psi_0]$ uniformly over compacts of $[0, T] \times \mathbb{R}^d$. This allows to pass to the limit in estimate (3.3) below.
4. $\mathfrak{K}^k[\mu^k] \rightarrow \mathfrak{K}^\infty[\mu^\infty]$ uniformly over compacts of $[0, T] \times \mathbb{R}^d$.
5. For every $\psi_0 \in C_c^\infty(\mathbb{R}^d)$ we have that

$$\int_{\mathbb{R}^d} \Psi^{T,k}[\psi_0]_T d\mu_0^k \rightarrow \int_{\mathbb{R}^d} \Psi^{T,\infty}[\psi_0]_T d\mu_0^\infty.$$

Then $(\mu^\infty, \Psi^\infty)$ is an entropy pair of the limit problem.

The result follows directly from the definition and the following classical result of stability of viscosity solutions (see, e.g., [28]).

Theorem 2.15 *Let $D^k \geq 0$, $\psi^k \in C([0, T] \times \mathbb{R}^d)$, $E^k \in C([0, T] \times \mathbb{R}^d)^d$ be viscosity solutions of*

$$\partial_s \psi_s^k = E_{T-s}^k \nabla \psi_s^k + D^k \Delta \psi_s^k$$

and assume that $D^k \rightarrow D^\infty$ and $\psi^k \rightarrow \psi^\infty$, $E^k \rightarrow E^\infty$ uniformly over compacts of $[0, T] \times \mathbb{R}^d$. Then, ψ^∞ is a viscosity solution of

$$\partial_s \psi_s^\infty = E_{T-s}^\infty \nabla \psi_s^\infty + D^\infty \Delta \psi_s^\infty.$$

2.4 The limits of the theory: the diffusive Newtonian potential in $d = 1$

Let us consider the case $n = d = 1$ with K coming from the gradient-flow structure (1.4) and the Newtonian potential $W(x) = -|x|$. Then $\nabla W(x) = \text{sign}(x)$. This W falls outside the theory developed in Theorem 2.10, but can be approximated by solutions in this framework. It was proved in [11] that for $\mu_0 = \delta_0$ then the gradient flow solution is not given by particles but rather

$$\mu_t = \frac{1}{2t} \chi_{[-t,t]}.$$

Naturally, any smooth approximation of W (for example $W_k = -(x^2 + \frac{1}{k})^{\frac{1}{2}}$) yields $\mu_t^k = \delta_0$. However, solving with $\mu_0^m = \frac{1}{2m} \chi_{[-m,m]}$ gives the expected reasonable solution. We therefore have the following diagram

$$\begin{array}{ccc} \mu^{k,m} & \xrightarrow{m \rightarrow 0} & \mu^k = \delta_0 \\ \downarrow k \rightarrow \infty & & \downarrow k \rightarrow \infty \\ \mu^m & \xrightarrow{m \rightarrow 0} & \mu \neq \delta_0. \end{array}$$

Clearly, $\mu_t^k \rightarrow \delta_0$ in any conceivable topology. Checking whether δ_0 is or is not a distributional solution for W is not totally trivial. Notice that sign (as a distributional derivative of $|x|$) is not pointwise defined at 0, so the meaning of $\int \varphi \text{sign} * \delta_0 d\delta_0$ is not completely clear. With the choice $\text{sign}(0) = 0$ one could be convinced that $\mu_t = \delta_0$ is a distributional solution. But this seems arbitrary.

In fact, if one takes the approximating entropy pairs (μ^k, Ψ^k) , it is not difficult to show that a discontinuity appears in $\psi_s^{k,T}$ as $k \rightarrow \infty$. Hence, we do not have uniform convergence of $\psi^{k,T}$. To show this fact, let $E^k = -\partial_x(W_k * \mu^k)$. Since $\mu^k = \delta_0$, we have that

$$E_t^k(x) = -\partial_x W_k = \frac{x}{(x^2 + \frac{1}{k})^{\frac{1}{2}}}.$$

Since E^k does not depend on t , $\psi^{k,T}$ does not depend on T . Therefore, we drop the T to simplify the notation. We have to solve the equation

$$\partial_t \psi_t^k(y) = \frac{y}{(y^2 + \frac{1}{k})^{\frac{1}{2}}} \partial_y \psi_t^k.$$

This equation can be solved by characteristics $\psi_t^k(Y_t^k(Y_0)) = \psi_0(Y_0)$ given by

$$\partial_t Y_t^k = -\frac{Y_t^k}{(Y_t^2 + \frac{1}{k})^{\frac{1}{2}}}$$

As $k \rightarrow \infty$ we recover $Y_t^k(y_0) \rightarrow Y_t(y_0) = y_0 - \text{sign}(y_0)t$ or, equivalently, that

$$\psi_t^k(y) \rightarrow \psi_t(y) = \psi_0(y + \text{sign}(y)t).$$

Therefore, $\psi_t(0^-) = \psi_0(-t)$ and $\psi_t(0^+) = \psi_0(t)$. Thus, ψ_t is not continuous in general. So we cannot pass to the limit $\mu^k \rightarrow \mu$ in terms of entropy pairs.

The gradient flow solution is recovered by the entropy pairs if one approximates δ_0 by $\mu_0^m = \frac{1}{2m} \chi_{[-m,m]}$ and passes to the limit. It is also not difficult to show that the entropy pairs of twice regularised problem, $(\mu^{k,m}, \Psi^{k,m})$ do converge in the sense of entropy pairs as $k \rightarrow \infty$ to (μ^m, Ψ^m) . And then we can also pass to the limit in the sense of entropy pairs as $m \rightarrow \infty$.

Thus, the stable semigroup solutions are entropy pairs, and this notion of solution is powerful enough to discard the wrong order of the limits.

2.5 Open problems

We finally briefly discuss the key difficulties for some related problems that our duality theory cannot cover.

2.5.1 Nonlinear diffusion

We could deal, in general, with problems of the form

$$\partial_t \mu_t^i = \nabla \cdot (\mu^i \mathcal{R}^i[\mu]) + \nabla \cdot (\mathbf{M}_t^i[\mu_t^i] \nabla \mu_t^i). \tag{2.10}$$

Going back to (2.10), the dual problem can be written in the form

$$\frac{\partial \psi_s}{\partial s} = \mathbf{E}_{T-s} \nabla \psi_s + \nabla \cdot (\mathbf{A}_s \nabla \psi_s).$$

We have not been able to estimate $\|\nabla(\psi - \bar{\psi})\|_{L^\infty}$ or $\|\psi - \bar{\psi}\|_{H^1}$ with respect to $E - \bar{E}$ and $\mathbf{A} - \bar{\mathbf{A}}$ using the techniques below. The main difficulty is that letting $v = \frac{\partial \psi}{\partial x_i}$ and $\bar{v} = \frac{\partial \bar{\psi}}{\partial x_i}$, the reasonable extension of (3.11) contains a term $-\nabla \cdot ((\mathbf{A} - \bar{\mathbf{A}}) \nabla \bar{v})$. Controlling these terms requires estimates of $D^2 \psi$, which are not in principle present for the initial data we discuss. However, smarter estimates and choices of space for μ and ψ may lead to a successful extension of our results.

2.5.2 Coupling in the dual problem

Another possible extension is the analysis of more general systems of the form

$$\frac{\partial \mu^i}{\partial t} = \sum_{j=1}^n \nabla \cdot (\mu^j \mathcal{R}^{ij}[\mu]) = \sum_{j,k=1}^n \frac{\partial}{\partial x_k} (\mu^j \mathcal{R}^{ijk}), \quad i = 1, \dots, n. \tag{2.11}$$

The dual problem is now not decoupled, and we have to solve it a system

$$\frac{\partial}{\partial t} \psi_s^T = \nabla \psi_s^T \cdot \mathcal{R}^*[\mu_{T-s}].$$

This kind of problems does not have a comparison principle in general, for example

$$\begin{cases} \partial_t u = \partial_x v, \\ \partial_t v = \partial_x u \end{cases}$$

leads to the wave equation $\partial_{tt}u = \partial_t(\partial_x v) = \partial_x \partial_t v = \partial_{xx}u$. Some examples of equations resembling this structure appear in mathematical biology models, e.g., [47].

2.5.3 Problems with saturation

Some authors have studied the case $\mathfrak{K}_i[\boldsymbol{\mu}] = \frac{\theta(\mu)}{\mu} H_i[\boldsymbol{\mu}]$ where $\theta(0) = \theta(M) = 0$. This problem was studied for $n = 1$ and $D = 0$ by [30] using a fixed point argument with the estimates by [39]. In this setting the solutions with initial datum $0 \leq \rho_0 \leq M$ remain bounded. Therefore, we expect bounded (even continuous) solutions ρ . So the natural duality would be ψ integrable. We point the reader to [1], where the authors study the duality between L^∞ estimates for entropy solutions of conservation laws and L^1 bounds for viscosity solutions of Hamilton–Jacobi equations.

3 Viscosity solutions of (P_E^*)

The aim of this section is to prove the following existence, uniqueness, regularity and continuous dependence result for (P_E^*) which is one of the key tools in this paper. The usefulness of the estimates obtained is clear going back to Lemma 2.11.

Proposition 3.1 *Let $D \geq 0$, $E \in C([0, T]; \text{Lip}_0(\mathbb{R}^d))$ and ψ_0 be such that $\nabla \psi_0 \in L^\infty(\mathbb{R}^d)$. Then, there exists a unique viscosity solution of (P_E^*) such that $\frac{\psi}{1+|x|} \in L^\infty((0, T) \times \mathbb{R}^d)$. Furthermore, it satisfies the following estimates*

$$\|\psi_s\|_{L^\infty} \leq \|\psi_0\|_{L^\infty}, \tag{3.1}$$

$$\|\nabla \|\psi_s\|_{L^\infty}\| \leq \|\nabla \|\psi_0\|_{L^\infty}\| \exp\left(C \int_0^T \|\nabla E_{T-s}\|_{L^\infty} d\sigma\right) \tag{3.2}$$

$$\left\| \frac{\psi_s}{1+|x|} \right\|_{L^\infty} \leq C \left\| \frac{\psi_0}{1+|x|} \right\|_{L^\infty} \exp\left(DT + \int_0^T \left\| \frac{E_{T-\sigma}}{1+|x|} \right\|_{L^\infty} d\sigma\right), \tag{3.3}$$

where $C = C(d)$ depends only on the dimension, and $\|\nabla \|\psi_s\|_{L^\infty}\| = \sup_i \|\frac{\partial \psi}{\partial x_i}\|_{L^\infty}$. If $\psi_0 \geq 0$ then $\psi \geq 0$. Moreover, we obtain the following time-regularity estimate

$$\left\| \frac{\psi_{s+h} - \psi_s}{1+|x|} \right\|_{L^\infty} \leq C \left(h + D^{\frac{1}{2}} \left(h^{\frac{1}{2}} + h^{\frac{3}{2}} \right) + \sup_{\sigma \in [0, s]} \left\| \frac{E_{T-(\sigma+h)} - E_{T-\sigma}}{1+|x|} \right\|_{L^\infty} \right), \tag{3.4}$$

where $C = C(T, D, \|\psi_0\|_{\text{Lip}_0}, \sup_\sigma \|E_\sigma\|_{\text{Lip}_0})$. Lastly, given $\widehat{\psi}_0 = \psi_0$ and \widehat{E} in the same hypotheses above, there exists a corresponding solution of (2.3), denoted by $\widehat{\psi}$, and we have the continuous dependence estimate

$$\left\| \frac{\psi_s - \widehat{\psi}_s}{1+|x|} \right\|_{L^\infty} \leq C \int_0^T \left\| \frac{E_{T-\sigma} - \widehat{E}_{T-\sigma}}{1+|x|} \right\|_{L^\infty} d\sigma \tag{3.5}$$

where $C = C(d, D, \|\psi_0\|_{\text{Lip}_0}, \int_0^T \|E_{T-\sigma}\|_{\text{Lip}_0})$ is monotone non-decreasing in each entry.

We will prove this result in Sect. 3.2. The uniqueness of viscosity solutions leads to the following

Corollary 3.2 Assume that $\mathfrak{K} : C([0, T]; \mathcal{P}_1(\mathbb{R}^d)) \rightarrow C([0, T]; Lip_0(\mathbb{R}^d; \mathbb{R}^d))$, and let (μ, Ψ) and $(\mu, \widehat{\Psi})$ be entropy pairs of (1.1), then $\Psi = \widehat{\Psi}$.

There are some immediate consequences that come from the linearity of equation.

Remark 3.3 Notice that the properties above imply a comparison principle. Given two solutions, we have that $\psi_0 \leq \widehat{\psi}_0$, then $\widetilde{\psi}_0 = \widehat{\psi}_0 - \psi_0 \geq 0$. Then, by linearity, uniqueness, and preservation of positivity $\widehat{\psi} - \psi = \widetilde{\psi} \geq 0$.

Notice that the generic initial datum $Lip(\psi_0) \leq 1$, need not be bounded. This can initially seem like a problem for uniqueness, since we cannot prescribe conditions at infinity. However, following Aronson [5, Theorem 2 and 3], existence and uniqueness are obtained under the assumption that $e^{-\lambda|x|^2} u_t \in L^2((0, T) \times \mathbb{R}^d)$ if the coefficients are bounded. We tackle this issue by studying weighted versions of our solutions $v_s(x) = \psi_s(x)/\eta(x)$, which solve the following problem

$$\partial_s v = v \frac{E_{T-s} \cdot \nabla \eta + D \Delta \eta}{\eta} + \nabla v \cdot \frac{E_{T-s} + 2D \nabla \eta}{\eta} + D \Delta v. \tag{3.6}$$

Notice that, if $\eta(x) = (1 + |x|^2)^{k/2}$ with $k \geq 1$, then $\nabla \eta / \eta \sim |x|^{-1}$ at infinity. If E is Lipschitz, then all the coefficients of the equation above are bounded. And, if $k > 1$, then $|v_0(x)| \leq C(1 + |x|)^{1-k} \rightarrow 0$ as $|x| \rightarrow \infty$.

Remark 3.4 A similar argument can be adapted to initial data which are weighted with respect to $1 + |x|^p$ for any $p \geq 1$, if this is satisfied by the initial datum. However, this escapes the interest of this work.

3.1 A priori estimates in L^∞

3.1.1 General linear problem

For the dual problem we are interested in the existence and uniqueness of the linear parabolic problem in non-divergence form

$$\begin{cases} \frac{\partial u_s}{\partial s} = f_s + a_s u_s + \mathbf{b}_s \cdot \nabla u_s + D \Delta u_s & \text{for all } s > 0, x \in \mathbb{R}^d \\ u_s \rightarrow 0 & \text{as } |x| \rightarrow \infty \text{ for all } s > 0. \end{cases} \tag{3.7}$$

The theory of existence and uniqueness of classical solutions dates back to [41]. When the coefficients are smooth, the linear problem can be rewritten in divergence form as

$$\partial_s u_s = f_s + (a_s - \nabla \cdot \mathbf{b}_s) u_s + \nabla \cdot (u_s \mathbf{b}_s + D \nabla u_s). \tag{3.8}$$

We focus on obtaining suitable a priori estimates assuming that the data and solutions are smooth enough, and these estimates pass to the limit to the unique viscosity solution by approximation of the coefficients.

By studying (3.7) at the point of maximum/minimum, we formally have that

$$\frac{d}{ds} \|u_s\|_{L^\infty} \leq \|f_s\|_{L^\infty} + \|a_s\|_{L^\infty} \|u_s\|_{L^\infty}.$$

This intuition can be made precise by the following result

Lemma 3.5 *If $a, f \in C(\mathbb{R}^d)$ and bounded, $\mathbf{b} \in C(\mathbb{R}^d)^d$, and u is a classical solution of (3.7) such that $\sup_{(0,T) \times \partial B_R} |u| \rightarrow 0$ as $R \rightarrow \infty$ for each T , then*

$$\|u_s\|_{L^\infty} \leq \|u_0\|_{L^\infty} \exp\left(\int_0^s \|a_\sigma\|_{L^\infty} d\sigma\right) + \int_0^s \|f_\sigma\|_{L^\infty} \exp\left(\int_\sigma^s \|a_\kappa\|_{L^\infty} d\kappa\right) d\sigma. \tag{3.9}$$

Proof Define

$$\bar{u}_s(x) = \|u_0\|_{L^\infty} \exp\left(\int_0^s \|a_\sigma\|_{L^\infty} d\sigma\right) + \int_0^s \|f_\sigma\|_{L^\infty} \exp\left(\int_\sigma^s \|a_\kappa\|_{L^\infty} d\kappa\right) d\sigma.$$

This is a classical super-solution of (3.7) and $u_0 \leq \bar{u}_0$. Using the weak maximum principle on a ball B_R and the initial condition we get

$$\min_{(0,T) \times B_R} (\bar{u} - u) = \min_{\substack{\{0\} \times B_R \\ \cup (0,T) \times \partial B_R}} (\bar{u} - u) \geq \min \left\{ 0, \min_{(0,T) \times \partial B_R} (\bar{u} - u) \right\}.$$

As $R \rightarrow \infty$ this tends to 0 by hypothesis. We conclude that $u \leq \bar{u}$. Similarly, we prove that $-u \leq \bar{u}$. □

Now we compute estimates on the solution formally assuming that $\psi \in C([0, T]; C_0^1(\mathbb{R}^d))$, and that \mathbf{E} is smooth and satisfies all necessary bounds. Later, we will justify this formal computations.

3.1.2 L^∞ estimates of $\psi^T, \nabla \psi^T$ and $\psi^T / (1 + |\mathbf{x}|)$

Going to (\mathbf{P}_E^*) when we have suitable decay of ψ we deduce (3.1) by Lemma 3.5. When we define $U_s^i = \frac{\partial \psi_s^T}{\partial x_i}$, we recover that

$$\begin{aligned} \frac{\partial U_s^i}{\partial s} &= \nabla \psi_s \cdot \frac{\partial \mathbf{E}_{T-s}}{\partial x_i} + \nabla U_s^i \cdot \mathbf{E}_{T-s} + D \Delta U_s^i \\ &= \sum_{j=1}^d U_s^j \frac{\partial E_{T-s}^j}{\partial x_i} + \nabla U_s^i \cdot \mathbf{E}_{T-s} + D \Delta U_s^i. \end{aligned}$$

Since this is a system, we cannot directly apply Lemma 3.5. Nevertheless, we compute

$$\begin{aligned} \frac{1}{p} \frac{d}{ds} \int_{\mathbb{R}^d} |U_s^i|^p &= \int_{\mathbb{R}^d} |U_s^i|^{p-2} U_s^i \left(\sum_{j=1}^d U_s^j \frac{\partial E_{T-s}^j}{\partial x_i} + \nabla U_s^i \cdot \mathbf{E}_{T-s} + D \Delta U_s^i \right) \\ &= \sum_{j=1}^d \int |U_s^i|^{p-2} U_s^i U_s^j \frac{\partial E_{T-s}^j}{\partial x_i} + \frac{1}{p} \int \nabla |U_s^i|^p \cdot \mathbf{E}_{T-s} \\ &\quad + D \int |U_s^i|^{p-2} U_s^i \Delta U_s^i \\ &= \|U_s^i\|_{L^p}^{p-1} \sum_{j=1}^d \|U_s^j\|_{L^p} \left\| \frac{\partial E_{T-s}^j}{\partial x_i} \right\|_{L^\infty} - \frac{1}{p} \int_{\mathbb{R}^d} |U_s^i|^p \nabla \cdot \mathbf{E}_{T-s} \\ &\quad - (p-1)D \int_{\mathbb{R}^d} |U_s^i|^{p-1} |\nabla U_s^i|^2 \end{aligned}$$

$$\leq \|U_s^i\|_{L^p}^{p-1} \sum_{j=1}^d \|U_s^j\|_{L^p} \left\| \frac{\partial E_{T-s}^j}{\partial x_i} \right\|_{L^\infty} + \frac{1}{p} \|U_s^i\|_{L^p}^p \|\nabla \cdot E_{T-s}\|_{L^\infty}$$

Using the norm equivalence of norms of \mathbb{R}^d , we have

$$\frac{d}{ds} \sum_{i=1}^d \|U_s^i\|_{L^p}^p \leq \left(\sum_{i=1}^d \|U_s^i\|_{L^p}^p \right) \left(pC(d) \sup_i \sum_{j=1}^d \left\| \frac{\partial E_{T-s}^j}{\partial x_i} \right\|_{L^\infty} + \|\nabla \cdot E_{T-s}\|_{L^\infty} \right)$$

Eventually, we have that

$$\begin{aligned} \left(\sum_{i=1}^d \|U_s^i\|_{L^p}^p \right)^{\frac{1}{p}} &\leq \left(\sum_{i=1}^d \|U_0^i\|_{L^p}^p \right)^{\frac{1}{p}} \exp \left(C(d) \int_0^s \sup_i \sum_{j=1}^d \left\| \frac{\partial E_{T-\sigma}^j}{\partial x_i} \right\|_{L^\infty} d\sigma \right. \\ &\quad \left. + \frac{1}{p} \int_0^s \|\nabla \cdot E_{T-\sigma}\|_{L^\infty} d\sigma \right). \end{aligned} \tag{3.10}$$

As $p \rightarrow \infty$, we obtain (3.2). Take $\eta(x) = (1 + |x|^2)^{\frac{1}{2}}$. Recalling (3.6) and applying Lemma 3.5, we deduce

$$\left\| \frac{\psi_T^T}{1 + |x|} \right\|_{L^\infty} \leq C \left\| \frac{\psi_0}{1 + |x|} \right\|_{L^\infty} \exp \left(D\|\Delta\eta\|_{L^\infty}T + \int_0^T \left\| \frac{\nabla \cdot E_{T-\sigma}}{1 + |x|} \right\|_{L^\infty} d\sigma \right).$$

Since η is regular we eventually deduce (3.3) where C is such that $C^{-1} \leq \frac{\eta(x)}{1+|x|} \leq C$, and it does not depend even on the dimension. For the continuous dependence, we write (dropping s and $T - s$ from the subindex for convenience)

$$\begin{aligned} \partial_s(v - \widehat{v}) &= v \frac{E - \widehat{E}}{\eta} \cdot \nabla \eta + \nabla v \cdot \frac{E - \widehat{E}}{\eta} + (v - \widehat{v}) \frac{\widehat{v} \cdot \nabla \eta + D\Delta\eta}{\eta} \\ &\quad + \nabla(v - \widehat{v}) \cdot \frac{\widehat{E} + 2D\nabla\eta}{\eta} + D\Delta(v - \widehat{v}). \end{aligned} \tag{3.11}$$

Then we have

$$|a_s| = \left| \frac{\widehat{v} \cdot \nabla \eta}{\eta} \right| \leq C(1 + |\widehat{v}|)$$

and

$$|f_s| = \left| \frac{v}{\eta} ((E - \widehat{E}) \cdot \nabla \eta) + \nabla v \cdot \frac{E - \widehat{E}}{\eta} \right| \leq \left| \frac{E - \widehat{E}}{\eta} \right| (|v| + |\nabla v|).$$

Notice that

$$\nabla v = \nabla \frac{\varphi}{\eta} = \frac{1}{\eta^2} (\eta \nabla \psi - \psi \nabla \eta) = \frac{1}{\eta} \nabla \psi + \frac{v}{\eta} \nabla \eta.$$

Hence, we can deduce using Lemma 3.5 that

$$\|v_s - \widehat{v}_s\|_{L^\infty(\mathbb{R}^d)} \leq C_1 \|v_0 - \widehat{v}_0\|_{L^\infty} + C_2 \int_0^T \left\| \frac{E_{T-\sigma} - \widehat{E}_{T-\sigma}}{1 + |x|} \right\|_{L^\infty} d\sigma \tag{3.12}$$

where

$$C_1 = C(T, D, \|\psi/(1 + |x|)\|_{L^\infty}, \|E\|_{L^\infty}),$$

$$C_2 = \sup_{s \in [0, T]} (\|v_s\|_{L^\infty} \|\nabla \eta\|_{L^\infty} + \|\nabla v_s\|_{L^\infty}), \tag{3.13}$$

which can be estimated using (3.2) and (3.3).

Eventually, since $\psi_0^T - \underline{\psi}_0^T = 0$, we recover (3.5). Notice that C_1 cannot be uniformly bounded over the set $\text{Lip}(\psi_0) \leq 1$, where we can bound C_2 . Here is where the assumptions that \mathfrak{K}_i satisfies (2.5) and (2.6) will come into play in Sect. 5.

3.1.3 Time continuity

Taking $v_s = \psi_s/\eta^{(1)}$ and $\widehat{v}_s = \psi_{s+h}/\eta^{(1)}$ where $\eta^{(1)} = (1 + |x|^2)^{\frac{1}{2}}$, we similarly deduce going back to (3.12) that

$$\left\| \frac{\psi_{s+h} - \psi_s}{1 + |x|} \right\|_{L^\infty} \leq C_1 \left\| \frac{\psi_h - \psi_0}{1 + |x|} \right\|_{L^\infty} + C_2 \sup_{\sigma \in [0, T]} \left\| \frac{E_{T-(\sigma+h)} - E_{T-\sigma}}{1 + |x|} \right\|_{L^\infty} \tag{3.14}$$

Therefore, ψ inherits the time continuity of E . Then the only remaining difficulty is the time continuity at 0.

For $D > 0$ we use Duhamel’s formula for the heat equation $u_t - D\Delta u = f$, where we denote the heat kernel K_D . Notice that $K_D(t, z) = K_1(Dt, z)$. For the first term we have that

$$\begin{aligned} \frac{\psi_s(x) - \psi_0(x)}{1 + |x|} &= \frac{1}{1 + |x|} \left(\int_{\mathbb{R}^d} K_D(s, x - y) \psi_0(y) dy - \psi_0(x) \right. \\ &\quad \left. + \int_0^s \int_{\mathbb{R}^d} K_D(s - \sigma, x - y) E_{T-\sigma}(y) \cdot \nabla \psi_\sigma(y) dy d\sigma \right) \\ &= \int_{\mathbb{R}^d} K_D(t, x - y) \frac{\psi_0(y) - \psi_0(x)}{1 + |x|} dy \\ &\quad + \int_0^s \int_{\mathbb{R}^d} K_D(s - \sigma, x - y) \frac{1 + |y|}{1 + |x|} \frac{E_{T-\sigma}(y)}{1 + |y|} \cdot \nabla \psi_\sigma(y) dy d\sigma. \end{aligned}$$

We estimate as follows

$$\begin{aligned} \left| \int_{\mathbb{R}^d} K_D(s, x - y) \frac{\psi_0(y) - \psi_0(x)}{1 + |x|} dy \right| &\leq \|\nabla \psi_0\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} K_D(s, x - y) |x - y| dy \\ &= \|\nabla \psi_0\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} |z| \frac{e^{-\frac{|z|^2}{4Ds}}}{(4\pi Ds)^{\frac{d}{2}}} dz \\ &= C \|\nabla \psi_0\|_{L^\infty(\mathbb{R}^d)} (4Ds)^{\frac{1}{2}} \int_{\mathbb{R}^d} |z| \frac{e^{-|z|^2}}{\pi^{\frac{d}{2}}} dz. \end{aligned}$$

Now since $E(y)/(1 + |y|)$ and $\nabla \psi$ are bounded, it leaves to integrate

$$\begin{aligned} \int_0^s \int_{\mathbb{R}^d} K_D(\sigma, x - y) \frac{1 + |y|}{1 + |x|} dy d\sigma &= \int_0^s \int_{\mathbb{R}^d} \frac{1}{(4\pi D\sigma)^{\frac{d}{2}}} e^{-\frac{|z|^2}{4D\sigma}} \frac{1 + |x| + |z|}{1 + |x|} dz d\sigma \\ &= \int_0^s \int_{\mathbb{R}^d} \frac{1}{(4\pi D\sigma)^{\frac{d}{2}}} e^{-\frac{|z|^2}{4D\sigma}} (1 + |z|) dz d\sigma \\ &= \int_0^s \int_{\mathbb{R}^d} \frac{1}{(4\pi D\sigma)^{\frac{d}{2}}} e^{-\frac{|z|^2}{4D\sigma}} dz d\sigma \\ &\quad + \int_0^s \int_{\mathbb{R}^d} \frac{1}{(4\pi D\sigma)^{\frac{d}{2}}} e^{-\frac{|z|^2}{4\sigma}} |z| dz d\sigma \end{aligned}$$

$$\begin{aligned}
 &= \int_0^s \int_{\mathbb{R}^d} \frac{1}{\pi^{\frac{d}{2}}} e^{-|w|^2} dw d\sigma \\
 &\quad + \int_0^s \int_{\mathbb{R}^d} (2D\sigma)^{\frac{1}{2}} \frac{1}{\pi^{\frac{d}{2}}} e^{-|w|^2} |w| dw d\sigma \\
 &\leq s + CD^{\frac{1}{2}} s^{\frac{3}{2}}.
 \end{aligned}$$

Eventually, we recover using (3.2) that

$$\left\| \frac{\psi_s(x) - \psi_0(x)}{1 + |x|} \right\|_{L^\infty} \leq C(d) \left((Ds)^{\frac{1}{2}} + (s + D^{\frac{1}{2}} s^{\frac{3}{2}}) \sup_{[0,s]} \left\| \frac{E_{T-\sigma}}{1 + |x|} \right\|_{L^\infty} \right) \|\nabla \psi_0\|_{L^\infty} \tag{3.15}$$

This result can also be deduced for $D = 0$ without involving any convolution. Also, it is recovered as a limit $D \searrow 0$. Joining (3.14) and (3.15) we recover (3.4).

Remark 3.6 (Time continuity at $s = 0$ if $\Delta \psi_0$ is bounded or $D = 0$). To estimate the time derivative at time 0, we consider the candidate sub and super-solutions

$$\underline{\psi}_s = \psi_0 - C_0 s, \quad \overline{\psi}_s = \psi_0 + C_0 s.$$

We have that

$$\begin{aligned}
 \partial_s \underline{\psi}_s - E_{T-s} \cdot \nabla \underline{\psi}_s - D \Delta \underline{\psi}_s &= -C_0 - E_{T-s} \cdot \nabla \psi_0 - D \Delta \psi_0, \\
 \partial_s \overline{\psi}_s - E_{T-s} \cdot \nabla \overline{\psi}_s - D \Delta \overline{\psi}_s &= C_0 - E_{T-s} \cdot \nabla \psi_0 - D \Delta \psi_0.
 \end{aligned}$$

So we need

$$C_0 \geq \sup_{[0,T] \times \mathbb{R}^d} |E_{T-s} \cdot \nabla \psi_0 + D \Delta \psi_0|.$$

Then

$$\left\| \frac{\psi_s - \psi_0}{s} \right\|_{L^\infty(\mathbb{R}^d)} \leq \|E_{T-s} \cdot \nabla \psi_0 + D \Delta \psi_0\|_{L^\infty([0,T] \times \mathbb{R}^d)}.$$

As $s \rightarrow 0$, we get an estimate of the time derivative at $s = 0$. A similar computation can be done for $\psi/(1 + |x|)$. This kind of result is useful for the case $D = 0$, since even as $D_k \searrow 0$, one can take approximate initial data $\psi_0^{(k)}$ so that the constant is uniformly bounded.

3.2 Proof of Proposition 3.1

First we prove uniqueness. Take $v^{(2)} = \psi/\eta^{(2)}$ where $\eta^{(k)}(x) = (1 + |x|^2)^{k/2}$. We observe that if $\frac{\psi_s}{1+|x|} \in L^\infty$ then $v_0^{(2)} \in L^\infty(\mathbb{R}^d)$, with decay $1/|x|$ at infinity. By dividing viscosity test functions by $\eta^{(2)}$, we observe that $v^{(2)}$ is a viscosity solution of (3.6). Notice since $v^{(2)}$ remains bounded, $\sup_{(0,T) \times \mathbb{R}^d \setminus B_R} \psi/(1 + |x|)$ is bounded by $R^{-\frac{1}{2}}$. The uniqueness of ψ follows from the uniqueness of $v^{(2)}$ (see, e.g., [45, Theorem 3.1]).

Let us now construct the solution and prove its properties. If $D > 0$, $\psi_0 \in W_c^{2,\infty}(\mathbb{R}^d)$, and E is very regular, then existence is simple by standard arguments. For the problem in \mathbb{R}^d , regularity and decays as $|x| \rightarrow \infty$ for compactly supported initial data follow as in [34] and hence all the estimates above are justified. If $\psi_0 \geq 0$ we can proceed like in Lemma 3.5 to prove $\psi \geq 0$.

Now we consider the general setting, and we argue by approximation. Consider an approximating sequence $0 < D^{(k)} \rightarrow D$, satisfying

$$W_c^{2,\infty}(\mathbb{R}^d) \ni \frac{\psi_0^{(k)}}{(1+|x|^2)^{\frac{1}{2}}} \rightarrow \frac{\psi_0}{(1+|x|^2)^{\frac{1}{2}}} \quad \text{in } L^\infty(\mathbb{R}^d),$$

$$W^{2,\infty}([0, T] \times \mathbb{R}^d)^d \ni \mathbf{E}^{(k)} \rightarrow \mathbf{E} \quad \text{in } C((0, T) \times \mathbb{R}^d)^d.$$

We have the uniform continuity estimates (3.2) and (3.4). Due to the Ascoli–Arzelá theorem and a diagonal argument, $\psi^{(k)} \rightarrow \psi$ uniformly over compacts of $[0, T] \times \mathbb{R}^d$. Applying Lemma 2.15, the limit is a viscosity solution with the limit coefficients. By the uniqueness above this is the solution we are studying. All estimates are stable by convergence uniformly over compact sets. \square

4 Dual-viscosity solutions of problem (P_E)

Now we focus on showing well-posedness and estimates for (P_E). We construct the solutions as the duals of those in Sect. 3.

Proposition 4.1 *For every $D, T \geq 0$ and $\mathbf{E} \in C([0, T]; \text{Lip}_0(\mathbb{R}^d, \mathbb{R}^d))$, and $\mu_0 \in \mathcal{P}_1(\mathbb{R}^d)$ there exists exactly one dual viscosity solution $\mu \in C([0, T]; \mathcal{P}_1(\mathbb{R}^d))$ of (P_E). It is also a distributional solution. Furthermore, the map*

$$S_T : (\mu_0, \mathbf{E}) \in \mathcal{P}_1(\mathbb{R}^d) \times C([0, T]; \text{Lip}_0(\mathbb{R}^d, \mathbb{R}^d)) \mapsto \mu_T \in \mathcal{P}_1(\mathbb{R}^d)$$

is continuous with the following estimate

$$d_1(S_T[\mu_0, \mathbf{E}], S_T[\widehat{\mu}_0, \widehat{\mathbf{E}}]) \leq C \left(d_1(\mu_0, \widehat{\mu}_0) + \int_0^T \left\| \frac{\mathbf{E}_\sigma - \widehat{\mathbf{E}}_\sigma}{1 + |x|} \right\|_{L^\infty} d\sigma \right) \tag{4.1}$$

where $C = C \left(d, D, \int_0^T \|\nabla \mathbf{E}_\sigma\|_{L^\infty} d\sigma, \int_{\mathbb{R}^d} |x| d\mu_0 \right)$

depends monotonically on each entry. The semigroup property holds in the sense that

$$S_{\widehat{\tau}+t}[\mu_0, \mathbf{E}] = S_{\widehat{\tau}} \left[S_t \left[\mu_0, \mathbf{E}|_{[0,t]} \right], \mathbf{E}|_{[t,\widehat{\tau}+t]}(\cdot - t) \right].$$

Therefore, we have constructed a continuous flow

$$S : (\mu_0, \mathbf{E}) \in \mathcal{P}_1(\mathbb{R}^d) \times C([0, \infty); \text{Lip}_0(\mathbb{R}^d, \mathbb{R}^d)) \mapsto \mu \in C([0, \infty); \mathcal{P}_1(\mathbb{R}^d)).$$

Proof of Proposition 4.1 We begin by proving uniqueness and continuous dependence. For $D \geq 0$ and any two weak dual viscosity solutions μ and $\widehat{\mu}$ corresponding to (μ_0, \mathbf{E}) and $(\widehat{\mu}_0, \widehat{\mathbf{E}})$ we have, applying Lemma 2.11, (3.3), and (3.5), that

$$d_1(\mu_T, \widehat{\mu}_T) \leq d_1(\mu_0, \widehat{\mu}_0) \sup_{\text{Lip}(\psi_0) \leq 1} \text{Lip}(\psi_T^T)$$

$$+ \left(1 + \int_{\mathbb{R}^d} |x| d\widehat{\mu}_0(x) \right) \sup_{\substack{\text{Lip}(\psi_0) \leq 1 \\ \psi_0(0)=0}} \left\| \frac{\psi_T^T - \widehat{\psi}_T^T}{1 + |x|} \right\|_{L^\infty}$$

$$\leq C(d) d_1(\mu_0, \widehat{\mu}_0) \exp \left(\int_0^T (D + \|\nabla \mathbf{E}_\sigma\|_{L^\infty}) d\sigma \right)$$

$$\begin{aligned}
 &+ C \left(d, D, \int_0^T \|\nabla \mathbf{E}_{T-\sigma}\|_{L^\infty} \right) \left(1 + \int_{\mathbb{R}^d} |x| d\widehat{\mu}_0(x) \right) \\
 &\int_0^T \left\| \frac{\mathbf{E}_{T-\sigma} - \widehat{\mathbf{E}}_{T-\sigma}}{1 + |x|} \right\|_{L^\infty} d\sigma.
 \end{aligned}$$

The second constant obtained from (3.13) by taking the supremum when $\text{Lip}(\psi_0) \leq 1$. Eventually, we can simplify this expression to (4.1). Therefore, we have uniqueness of dual viscosity solutions for any $D \geq 0$.

For $D > 0$, $\mu_0 \in H^1(\mathbb{R}^d)$ and $\mathbf{E}_t \in C([0, T]; W^{2,\infty}(\mathbb{R}^d; \mathbb{R}^d))$, the theory of existence and regularity of weak solutions is nowadays well known (see, e.g., [9, 10] and the references therein). The dual problem is decoupled from (\mathbf{P}_E) , and we have already constructed the unique viscosity solutions of the dual problem (see Proposition 3.1), that are also weak solutions when \mathbf{E} is regular. For regular enough datum ψ_0 , we can use ψ as test function in the weak formulation of (\mathbf{P}_E) to deduce (\mathbf{D}_i) . Hence, any weak solution of (\mathbf{P}_E) is a dual viscosity solution.

Let us now show the time continuity with respect to the $\mathcal{P}_1(\mathbb{R}^d)$ distance in space. We take ψ_0 such that $\text{Lip}(\psi_0) \leq 1$, and we estimate

$$\left| \int_{\mathbb{R}^d} \psi_0 d(\mu_{t+h} - \mu_t) \right| \leq \left| \int_{\mathbb{R}^d} (\psi_{t+h}^{t+h} - \psi_t^t) d\mu_0 \right| \leq \int_{\mathbb{R}^d} (1 + |x|) d\mu_0 \left\| \frac{\psi_{t+h}^{t+h} - \psi_t^t}{1 + |x|} \right\|_{L^\infty(\mathbb{R}^d)}.$$

This last quantity is controlled by continuous dependence on \mathbf{E} and time continuity of (\mathbf{P}_E^*) . First, letting $\widehat{\psi}_s = \psi_s^{t+h}$ with $\psi_0^{t+h} = \psi_0$, which corresponds to $\widehat{\mathbf{E}}_s = \mathbf{E}_{t+h-s}$, we have that

$$\left\| \frac{\psi_t^{t+h} - \psi_t^t}{1 + |x|} \right\|_{L^\infty(\mathbb{R}^d)} \leq C \sup_{\sigma \in [0, t]} \|\mathbf{E}_{t+h-\sigma} - \mathbf{E}_{t-\sigma}\|_{\text{Lip}_0}.$$

The right-hand side of this equation is a modulus of continuity, which we denote ω_E . Now we use the time continuity of (\mathbf{P}_E^*) given by (3.4) to deduce that

$$\left\| \frac{\psi_{t+h}^{t+h} - \psi_t^{t+h}}{1 + |x|} \right\|_{L^\infty(\mathbb{R}^d)} \leq \omega_D(h),$$

where C and ω_D are given by the right-hand side of (3.4). The constants are uniform for ψ_0 with $\text{Lip}(\psi_0) \leq 1$. Eventually, taking the supremum on ψ_0 and applying (2.2) we deduce

$$d_1(\mu_{t+h}, \mu_t) \leq \omega_D(h) + \omega_E(h).$$

Let us now consider the general case for $D \geq 0$, μ_0 and \mathbf{E}_0 . The uniform estimates (4.1) shows that, if $0 < D^n \rightarrow D, H^1(\mathbb{R}^d) \ni \mu_0^n \rightarrow \mu_0$ in $\mathcal{P}_1(\mathbb{R}^d)$ and $C([0, T]; W^{2,\infty}(\mathbb{R}^d; \mathbb{R}^d)) \ni \mathbf{E}^n \rightarrow \mathbf{E}$ in $C([0, T]; \text{Lip}_0(\mathbb{R}^d, \mathbb{R}^d))$, then the sequence μ^n is Cauchy in the metric space $C([0, T]; \mathcal{P}_1(\mathbb{R}^d))$. Since this is a Banach space, the sequence μ^n converges to a unique limit μ . Due to the stability of the dual problem, we can pass to the limit in (\mathbf{D}_i) to check that μ is a dual viscosity solution. We have already shown the uniqueness at the beginning of the proof. Due to the approximation, the solution constructed is also a distributional solution.

The semigroup property holds in the regular setting, so we can pass to the limit. This completes the proof. □

5 Existence and uniqueness for (1.1). Proof of Theorem 2.10

We prove the existence by using Banach’s fixed-point contraction theorem. We construct the map

$$\mathcal{T}_t : C([0, t]; \mathcal{P}_1(\mathbb{R}^d)^n) \longrightarrow \mathcal{P}_1(\mathbb{R}^d)^n \tag{5.1}$$

by

$$\mathcal{T}_t[\boldsymbol{\mu}] = \begin{pmatrix} S_t[\mu_0^1, \mathcal{R}^1[\boldsymbol{\mu}]] \\ \vdots \\ S_t[\mu_0^n, \mathcal{R}^n[\boldsymbol{\mu}]] \end{pmatrix}.$$

We must work on the bounded cubes $Q(R) = B_{\mathcal{P}_1}(R)^n$ where we recall the definition of ball in Wasserstein space given by (2.9). Hence, we take

$$R > \max_{i=1, \dots, n} d_1(\mu_0^i, \delta_0).$$

To get a very rough estimate of $d_1(\mathcal{T}_t[\boldsymbol{\mu}], \delta_0)$ we first indicate that when $\mathbf{E} = 0$ we recover the heat kernel at time Dt

$$S_t[\delta_0, 0](x) = H_{Dt}(x) = \frac{1}{(4\pi Dt)^{-\frac{d}{2}}} \exp\left(-\frac{|x|^2}{4Dt}\right).$$

Hence, using the triangle inequality

$$d_1(\delta_0, S_T[\mu_0, \mathbf{E}]) \leq d_1(\delta_0, S_T[\delta_0, \mathbf{0}]) + d_1(S_T[\delta_0, \mathbf{0}], S_T[\mu_0, \mathbf{E}]).$$

For the first term, we simply write

$$d_1(\delta_0, S_T[\delta_0, \mathbf{0}]) = d_1(\delta_0, H_{Dt}) \leq \omega(Dt).$$

To get a uniform constant in (4.1) define

$$C_R(T) = \max_{\substack{t \in [0, T] \\ i=1, \dots, n \\ \boldsymbol{\mu} \in C([0, T]; Q(R))}} \|\nabla \mathcal{R}^i[\boldsymbol{\mu}]_t\|_{L^\infty(\mathbb{R}^d)} < \infty \tag{5.2}$$

by the assumptions. As a consequence, for any $\boldsymbol{\mu}$ such that $d_1(\mu_0^i, \delta_0) \leq R$ for all $i = 1, \dots, n$ and $t \leq T_1$, then $d_1(\delta_0, \mathcal{T}_t[\boldsymbol{\mu}]^i) \leq R$ for all $i = 1, \dots, n$ and $t \leq T_1$, i.e.,

$$\mathcal{T} : C([0, T_1]; Q(R)) \longmapsto C([0, T_1]; Q(R)).$$

Notice that, since we are constructing solutions using S_T , the constructed solution is also a distributional solution.

Applying again (4.1) we have that for every $i = 1, \dots, n$

$$\begin{aligned} \sup_{t \in [0, T]} d_1(\mathcal{T}_t^i[\boldsymbol{\mu}], \mathcal{T}_t^i[\widehat{\boldsymbol{\mu}}]) &= \sup_{t \in [0, T]} d_1\left(S_t\left[\mu_0^i, \mathcal{R}^i[\boldsymbol{\mu}]\right], S_t\left[\mu_0^i, \mathcal{R}^i[\widehat{\boldsymbol{\mu}}]\right]\right) \\ &\leq C(T_1, R) \int_0^T \left\| \frac{\mathcal{R}^i[\boldsymbol{\mu}]_\sigma - \mathcal{R}^i[\widehat{\boldsymbol{\mu}}]_\sigma}{1 + |x|} \right\|_{L^\infty} d\sigma \\ &\leq C(T_1, R) T L \sup_{\substack{\sigma \in [0, T] \\ j=1, \dots, n}} d_1(\mu_\sigma^j, \widehat{\mu}_\sigma^j), \end{aligned}$$

where $C(T_1, R)$ is recovered again through (4.1). Given that R is fixed, we can find $T_2 < T_1$ so that the map \mathcal{T}_{T_2} is contracting with the norm

$$d_1^{n, T_2}(\mu, \widehat{\mu}) = \sup_{\substack{t \in [0, T_2] \\ i=1, \dots, n}} d_1(\mu_t^i, \widehat{\mu}_t^i).$$

Therefore, we can apply Banach’s fixed-point theorem to proof existence and uniqueness for short times. Since \mathfrak{K} is uniformly Lipschitz, we can extend the existence time to infinity applying the classical argument. To prove the continuous dependence on μ_0 , we apply (2.8), (3.5) and (2.6). This completes the proof of existence and uniqueness. \square

Remark 5.1 Following the non-explicit constant in (3.5), it would be possible to recover quantitative dependence estimates.

Remark 5.2 (Numerical analysis when $D = 0$: the particle method). Our notion of solution justifies the convergence of the particle method when $D = 0$. The aim of the particle method is to consider an approximation of the initial datum given by finitely many isolated particles

$$\mu_0^{i, N} = \sum_{j=1}^N a^{ijN} \delta_{X_0^{ijN}}.$$

Then, it is not difficult to see that, for $D = 0$ the solution is given by particles

$$\mu_t^{i, N} = \sum_{j=1}^N a^{ijN} \delta_{X_t^{ijN}}.$$

The evolution of these particles is given by a system of ODEs for the particles

$$\partial_t X_t^{ijN} = - \sum_j a^{ijN} \mathbf{K}_t^i[\mu_t^N](X_t^{ijN})$$

This system is well posed. Due to the continuous dependence

$$\sup_{t \in [0, T]} d_1(\mu_t, \mu_t^N) \leq C(T, \mu_0) d_1(\mu_0, \mu_0^N)$$

It is easy to see that the finite combinations of Dirac deltas is dense in 1-Wasserstein distance, and estimates for convergence are well known (see, e.g., [37]).

6 Gradient flows of convex interaction potentials and $D = 0$

The aim of this section is to give an application of our results to the classical aggregation equation

$$\partial_t \mu_t = \nabla \cdot (\mu_t \nabla W * \mu_t). \tag{6.1}$$

This problem falls into the well-posedness theory developed in Theorem 2.10 provided that $D^2W \in L^\infty(\mathbb{R}^d)^{d \times d}$ for the $\mathcal{P}_1(\mathbb{R}^d)$ theory. In the context of gradient-flow solutions (see, e.g., [2]) we are able to weaken the hypothesis on \mathfrak{K} .

Theorem 6.1 *Let $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ be compactly supported. Assume W is convex and that $W \in C_{loc}^{1,s}(\mathbb{R}^d)$. Then, the gradient flow solution $\mu \in C([0, T]; \mathcal{P}_2(\mathbb{R}^d))$ is a dual-viscosity solution.*

We can approximate W by a sequence of convex functions W^k , so that $\nabla W^k \rightarrow \nabla W$ uniformly over compacts but satisfies $D^2W^k \in L^\infty(\mathbb{R}^d)$. We construct μ_t^k through Theorem 2.10. Since μ_0 is compactly supported and W^k is convex, if we pick A_0 the convex envelope of $\text{supp } \mu_0$, then

$$\text{supp } \mu_t^k \subset A_0, \quad \forall t > 0.$$

This is easy to see, for example, looking at the solution by characteristics. We set ourselves in the hypothesis of Theorem 2.14 by proving below that

1. $\psi^{k,T}$ for every $\psi_0 \in C_c^\infty(\mathbb{R}^d)$ are uniformly equicontinuous.
2. $\mu^k \rightarrow \mu$ in $C([0, T]; \mathcal{P}_2(\mathbb{R}^d))$.
3. $W^k * \mu^k \rightarrow W * \mu$ uniformly over compacts.

Remark 6.2 We point that the result in Theorem 6.1 can be extended from (6.1) to a system of equations coming from the gradient flow of interaction potentials between the different components. To fix ideas, we can treat systems of two species like

$$\begin{cases} \partial_t \rho_1 = \nabla \cdot (\rho_1 \nabla (H_1 * \rho_1 + K * \rho_2)) \\ \partial_t \rho_2 = \nabla \cdot (\rho_2 \nabla (H_2 * \rho_2 + K * \rho_1)) \end{cases}$$

See, e.g., [31].

6.1 Regularity in space

Constructing solutions by characteristics it is well known that

$$(E_{T-s}(x) - E_{T-s}(y)) \cdot (x - y) \geq 0,$$

then characteristics grow apart. Then, the solution of (2.3) satisfies

$$\|\nabla \psi_s\|_{L^\infty(\mathbb{R}^d)} \leq \|\nabla \psi_0\|_{L^\infty(\mathbb{R}^d)}. \tag{6.2}$$

We check that this holds for $E = \nabla W * \mu$. Since W is convex, then

$$(\nabla W(x) - \nabla W(y)) \cdot (x - y) = (x - y) \cdot D^2W(\xi(x, y))(x - y) \geq 0.$$

Then, the convolution with a non-negative measure is also convex

$$\begin{aligned} (\nabla W * \mu(x) - \nabla W * \mu(y)) \cdot (x - y) &= \int_{\mathbb{R}^d} (\nabla W(x - z) - \nabla W(y - z)) \\ &\quad \cdot ((x - z) - (y - z)) d\mu_0(z) \\ &\geq 0, \end{aligned}$$

and so (6.2) holds.

6.2 Regularity in time

First, we look at the evolution of the support. When E_{T-s} is locally bounded, we can construct a super-solutions by characteristics. We define

$$A(s_0, s_1) = \bigcup_{s \in [s_0, s_1]} \text{supp } \psi_s.$$

If we assume that $A(s, 0) \subset B(x_0, R_s)$ then

$$\partial_s R_s \leq \sup_{B(x_0, R_s)} |E_{T-s}|.$$

So we end up with an estimate

$$A(s, 0) \subset \text{supp } \psi_0 + B_{R_s}. \tag{6.3}$$

Assume that $\psi_0 \in C_c^\infty(\mathbb{R}^d)$ with $\|\nabla \psi_0\|_{L^\infty(\mathbb{R}^d)} \leq 1$. Take $s_0 < s$. Consider

$$\underline{\psi}_s = \psi_{s_0} - C_0(s - s_0), \quad \overline{\psi}_s = \psi_{s_0} + C_0(s - s_0).$$

We have that

$$\partial_s \underline{\psi}_s - E_{T-s} \cdot \nabla \underline{\psi}_s = -C_0 - E_{T-s} \cdot \nabla \psi_{s_0}, \quad \partial_s \overline{\psi}_s - E_{T-s} \cdot \nabla \overline{\psi}_s = C_0 - E_{T-s} \cdot \nabla \psi_{s_0}.$$

They are a sub and super-solution in $[s_0, s_1]$ if

$$C_0 = \|\nabla \psi_{s_0}\|_{L^\infty} \sup_{[s_0, s_1] \times A(s_0, s_1)} |E|.$$

Hence, we deduce that

$$\left\| \frac{\psi_{s_1} - \psi_{s_0}}{s_1 - s_0} \right\|_{L^\infty(\mathbb{R}^d)} \leq \|\nabla \psi_{s_0}\|_{L^\infty} \sup_{[s_0, s_1] \times A(s_0, s_1)} |E|. \tag{6.4}$$

This implies that a uniform bound on the time continuity based on T , local bounds of E and the support of ψ_0 .

6.3 Convergence of the convolution

Following [2, Theorem 11.2.1] the Γ -convergence of uniformly λ -convex interaction free energy functionals is sufficient so that

$$\sup_{[0, T]} d_2(\mu_t^k, \mu_t) \rightarrow 0, \quad \text{as } k \rightarrow \infty. \tag{6.5}$$

In particular, if $W \in C^s$ is convex and $W^k(x)(1 + |x|)^{-2} \rightarrow W(x)(1 + |x|)^{-2}$ uniformly, (6.5) holds. It follows that the sequence μ^k is uniformly continuous, i.e., the function

$$\omega(h) = \sup_k \sup_{t \in [0, T-h]} d_2(\mu_{t+h}^k, \mu_t^k)$$

is a modulus of continuity.

On the other hand, for the convergence of $\nabla W^k * \mu^k$ we prove the following result.

Lemma 6.3 *Assume that*

1. $\nabla W \in C_{loc}^s$
2. $\nabla W^k \rightarrow \nabla W$ uniformly over compacts of \mathbb{R}^d .
3. $\mu_t^k, \mu_t \in \mathcal{P}(\mathbb{R}^d)$, and, for every $t > 0$, $\mu_t^k \rightharpoonup \mu_t$ weak- \star in $\mathcal{M}(\mathbb{R}^d)$
4. There exists $A_0 \subset \mathbb{R}^d$ convex bounded such that for all $t > 0$, $\text{supp } \mu_t^k \subset A_0$
5. $\mu^k \in C([0, T]; \mathcal{P}_2(\mathbb{R}^d))$ are uniformly continuous.

Then

$$\nabla W^k * \mu^k \rightarrow \nabla W * \mu \text{ uniformly over compacts of } [0, T] \times \mathbb{R}^d.$$

Proof We use the intermediate element $\nabla W * \mu^k$. First, for $A \subset \mathbb{R}^d$ compact

$$\sup_{x \in A} \left| \int_{\mathbb{R}^d} (\nabla W^k(x - z) - \nabla W(x - z)) d\mu_s^k \right| \leq \sup_{A+A_0} |\nabla W^k - \nabla W|.$$

Hence, we have that

$$\sup_{\substack{t \in [0, T] \\ x \in A}} \left| \nabla W^k * \mu_t^k(x) - \nabla W * \mu_t^k(x) \right| \leq \sup_{A+A_0} |\nabla W^k - \nabla W|. \tag{6.6}$$

Due to weak- \star convergence, if $\nabla W \in C_{loc}^s$ then

$$\nabla W * \mu_t^k(x) \rightarrow \nabla W * \mu_t(x) \text{ for each } (t, x) \in [0, T] \times \mathbb{R}^d$$

Now we prove uniform continuity. First in space. Let $A \subset \mathbb{R}^d$ be compact. Take

$$C_K = \sup_{\substack{x, y \in A-A_0 \\ x \neq y}} \frac{|\nabla W(x) - \nabla W(y)|}{|x - y|^s}$$

Now, for $x, y \in A$ we can compute

$$\left| \nabla W * \mu^k(x) - \nabla W * \mu^k(y) \right| \leq \int_{A_0} |\nabla W(x - z) - \nabla W(y - z)| d\mu^k(z) \leq C_A |x - y|^s.$$

So $\nabla W * \mu^k$ is uniformly continuous in x over compacts of $[0, T] \times \mathbb{R}^d$.

Lastly, let π be optimal plan between μ_t^k and μ_τ^k . Due to assumption 4, the optimal plan can be selected so that $\text{supp } \pi \subset A_0 \times A_0$. Let $z \in A$. We have that

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} \nabla W(z - y) d\mu_t^k(x) - \int_{\mathbb{R}^d} \nabla W(z - y) d\mu_\tau^k(y) \right| \\ &= \left| \iint (\nabla W(z - x) - \nabla W(z - y)) d\pi(x, y) \right| \\ &\leq C_A \iint |x - y|^s d\pi(x, y) \\ &\leq C_A \left(\iint |x - y|^2 d\pi(x, y) \right)^{\frac{s}{2}} \\ &= C_A d_2(\mu_t^k, \mu_\tau^k)^s \leq C_A \omega(h)^s. \end{aligned}$$

Hence, we have the uniform estimate of continuity in time

$$\sup_{x \in K} |\nabla W * \mu_t^k - \nabla W * \mu_\tau^k| \leq C_A \omega(|t - \tau|)^s.$$

And we finally deduce that

$$\sup_{\substack{t, \tau \in [0, T] \\ x, y \in K}} |\nabla W * \mu_t^k(x) - \nabla W * \mu_\tau^k(y)| \leq C_A (|x - y| + \omega(|t - \tau|)^s).$$

In particular, by the Ascoli–Arzelá theorem, there is a subsequence converging uniformly in $[0, T]$. Since we have characterised the point-wise limit, every convergent subsequence does so to $\nabla W * \mu$. Hence, the whole sequence converges uniformly over compacts, i.e.,

$$\sup_{\substack{t \in [0, T] \\ x \in A}} |\nabla W * \mu_t^k(x) - \nabla W * \mu_t(x)| \rightarrow 0, \quad \text{as } k \rightarrow \infty. \tag{6.7}$$

Using the triangular inequality, (6.6), and (6.7) the result is proven. □

Proof of Theorem 6.1 When W^k is C^2 , we can construct a unique classical solution as the push-forward of regular characteristics. This solution is well-known to be the gradient flow solution. By construction, it coincides with the unique dual viscosity solution that exists by Theorem 2.10.

First, we showed in (6.5) that μ^k converges to the gradient flow solution in $C([0, T]; \mathcal{P}_2(\mathbb{R}^d))$, and let us denote it by $\bar{\mu}$. Now we apply Theorem 2.14 where the hypothesis have been check in (6.4) (using (6.3)), and Lemma 6.3 to show that $\bar{\mu}$ is a dual viscosity solution. This completes the proof. □

7 A \dot{H}^{-1} and 2-Wasserstein theory when $D > 0$

7.1 Notion of solution and well-posedness theorem

Many of cases of (1.1) studied in the literature are 2-Wasserstein gradient flow. Our situation is more general. Unfortunately, the 2-Wasserstein distance, d_2 , does not have a duality characterisation similar to d_1 . It is known (see, e.g., [46]) that it can be one-side compared with the $\dot{H}^{-1}(\mathbb{R}^d)$

$$d_2(\mu, \hat{\mu}) \leq 2[\mu - \hat{\mu}]_{\dot{H}^{-1}(\mathbb{R}^d)}, \tag{7.1}$$

where, for $\mu \in D'(\mathbb{R}^d)$ we define the norm

$$\|\mu\|_{\dot{H}^{-1}(\mathbb{R}^d)} = \sup_{\substack{f \in C_c^\infty(\mathbb{R}^d) \\ \|\nabla f\|_{L^2} \leq 1}} |\mu(f)|.$$

The converse inequality to (7.1) only holds for absolutely continuous measures, and the constant depends strongly on the uniform continuity.

We consider the Sobolev semi-norm $[f]_{H^1} = \|\nabla f\|_{L^2}$. The space $(C_c^\infty(\mathbb{R}^d), [\cdot]_{H^1})$ is a normed space. Notice that, if $[f]_{H^1} = 0$ then f is constant. But since it has compact supported, the value of the constant is 0. This allows to define the dual space

$$\dot{H}^{-1}(\mathbb{R}^d) = (C_c^\infty(\mathbb{R}^d), [\cdot]_{H^1(\mathbb{R}^d)})'.$$

Since it is the dual of a normed space, $\dot{H}^{-1}(\mathbb{R}^d)$ is a Banach space.

Remark 7.1 The space $\dot{H}^{-1}(\mathbb{R}^d)$ is defined as the completion of $(C_c(\mathbb{R}^d), [\cdot]_{H^1(\mathbb{R}^d)})$, which is easy to see is not complete itself. This completion can be complicated (see, e.g., [12]). Hence, with our construction $\dot{H}^{-1}(\mathbb{R}^d)$ is not the dual of $\dot{H}^{-1}(\mathbb{R}^d)$.

Similarly to above, we define

Definition 7.2 We say that $(\mu, \{\Psi^T\}_{T \geq 0})$ is an \dot{H}^{-1} entropy pair if:

1. For every $T \geq 0$,

$$\Psi^T : X = \{\psi_0 \in C_c(\mathbb{R}^d) : \nabla \psi_0 \in L^2(\mathbb{R}^d)\} \longrightarrow C([0, T]; X)^n$$

is a linear map with the following property: for every ψ_0 and $i = 1, \dots, n$ we have $\Psi^{T,i}[\psi_0]$ is a viscosity solution of (P_i^*) .

2. For each i and $T \geq 0$, $\mu_T^i \in \dot{H}^{-1}(\mathbb{R}^d)$ and satisfies the duality condition (D_i)

Remark 7.3 Notice that in this section we require that ψ_0 is compactly supported, and thus the unique viscosity solutions will satisfy $\psi(x) \rightarrow 0$ as $|x| \rightarrow \infty$.

The main result of this section is

Theorem 7.4 Let $D > 0$, $\mu_0 \in \dot{H}^{-1}(\mathbb{R}^d)$ and assume that

$$\mathfrak{R}^i : C([0, T], \dot{H}^{-1}(\mathbb{R}^d)) \rightarrow C([0, T], W^{1,\infty}(\mathbb{R}^d)) \tag{7.2}$$

is Lipschitz with data in $\dot{H}^{-1}(\mathbb{R}^d)$ in the sense that, for any μ and $\widehat{\mu}$ in $C([0, T]; \dot{H}^{-1}(\mathbb{R}^d))$

$$\sup_{\substack{t \in [0, T] \\ i=1, \dots, n}} \left\| \mathfrak{R}^i[\mu]_t - \mathfrak{R}^i[\widehat{\mu}]_t \right\|_{W^{1,\infty}(\mathbb{R}^d)} \leq L \sup_{\substack{t \in [0, T] \\ i=1, \dots, n}} \|\mu_t^i - \widehat{\mu}_t^i\|_{\dot{H}^{-1}(\mathbb{R}^d)}. \tag{7.3}$$

Then there exists exactly one dual viscosity solution $\mu \in C([0, T], \dot{H}^{-1}(\mathbb{R}^d)^n)$.

Furthermore, if \mathfrak{R}^i is autonomous (i.e., $\mathfrak{R}^i[\mu]_t = \mathbf{K}^i[\mu_t]$), then the map $S_T : \mu_0 \in \dot{H}^{-1}(\mathbb{R}^d)^n \mapsto \mu_T \in \dot{H}^{-1}(\mathbb{R}^d)^n$ is a continuous semigroup.

We point that we can only get the $\dot{H}^{-1}(\mathbb{R}^d)$ theory when diffusion is present (i.e., $D > 0$). When $D = 0$ we cannot get an estimate of $\|\nabla \psi_s\|_{L^2}$.

Proof of Theorem 7.4 The proof follows the blueprint of the proof of Theorem 2.10 using a fixed-point argument. We need an adapted version of (2.8) given by

$$\begin{aligned} \|\mu_T - \widehat{\mu}_T\|_{\dot{H}^{-1}(\mathbb{R}^d)} &\leq \|\mu_0 - \widehat{\mu}_0\|_{\dot{H}^{-1}(\mathbb{R}^d)} \sup_{\substack{\psi_0 \in C_c^\infty(\mathbb{R}^d) \\ \|\nabla \psi_0\|_{L^2(\mathbb{R}^d)} \leq 1}} \|\nabla \psi_T^T\|_{L^2(\mathbb{R}^d)} \\ &\quad + \|\widehat{\mu}_0\|_{\dot{H}^{-1}(\mathbb{R}^d)} \sup_{\substack{\psi_0 \in C_c^\infty(\mathbb{R}^d) \\ \|\nabla \psi_0\|_{L^2(\mathbb{R}^d)} \leq 1}} \left\| \nabla(\psi_T^T - \widehat{\psi}_T^T) \right\|_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{7.4}$$

This is shown by a suitable modification of the argument in (2.7). Lastly, we use the bounds and continuous dependence proved below in Propositions 7.6 and 7.9. □

Lastly, due to (7.1) we have the following partial result in 2-Wasserstein space.

Corollary 7.5 Let $D > 0$, $\mu_0 \in \dot{H}^{-1}(\mathbb{R}^d)^n \cap \mathcal{P}(\mathbb{R}^d)^n$, then the unique solution constructed in Theorem 7.4 is $C([0, T]; \mathcal{P}_2(\mathbb{R}^d)^n)$.

Proof Due to the previous theorem, we only need to show that probability distributions stay probability distributions. The non-negativity is trivial, since by Proposition 3.1 the test functions preserve the non-negativity. Hence, we get

$$\int_{\mathbb{R}^d} \psi_0 d\mu_T = \int_{\mathbb{R}^d} \psi_T d\mu_0 \geq 0, \quad \text{for all } 0 \leq \psi_0 \in C_c^\infty(\mathbb{R}^d).$$

Thus $\mu_T \geq 0$ for all $T \geq 0$.

We now prove the conservation of total mass. Consider a sequence of $0 \leq \psi_0^{(k)} \in X$ point-wise increasing k and converging uniformly over compacts to 1. Then, by the comparison principle, $0 \leq \psi^{(k)} \leq 1$ are point-wise increasing in k . By Dini’s theorem, functions $\psi^{(k)}$ converges uniformly over compacts. Let $\psi^{(\infty)}$ be its limit. By stability of viscosity solutions $\psi^{(\infty)}$ solves the same equation. By the uniform convergence $\psi_0^{(\infty)} = \lim_n \psi_0^{(k)} = 1$. Then $\psi^{(\infty)} = 1$. By definition

$$\int_{\mathbb{R}^d} \psi_0^{(k)} d\mu_T = \int_{\mathbb{R}^d} \psi_T^{(k)} d\mu_0.$$

We can apply the monotone convergence theorem on both sides to deduce

$$\int_{\mathbb{R}^d} d\mu_T = \int_{\mathbb{R}^d} d\mu_0 = 1.$$

7.2 Study of (P_E^*)

To deal with the $\dot{H}^{-1}(\mathbb{R}^d)$ estimates, we must get $L^2(\mathbb{R}^d)$ of $\nabla\psi$.

Proposition 7.6 *Under the hypothesis of Proposition 3.1 together with $\psi_0 \in C_c(\mathbb{R}^d)$, then the unique viscosity solution of (2.3) constructed in Proposition 3.1 also satisfies*

$$\|\nabla\psi_s\|_{L^2(\mathbb{R}^d)} \leq \|\nabla\psi_0\|_{L^2(\mathbb{R}^d)} \exp\left(C(d) \int_0^s \|\nabla E_{T-\sigma}\|_{L^\infty(\mathbb{R}^d)} d\sigma\right). \tag{7.5}$$

Lastly, given $\widehat{\psi}_0 = \psi_0$ and \widehat{E} in the same hypotheses above, there exists a corresponding solution of (2.3), denoted by $\widehat{\psi}$, and we have the continuous dependence estimate

$$\|\nabla(\psi_s - \widehat{\psi}_s)\|_{L^2(\mathbb{R}^d)}^2 \leq C_1 e^{C_2 s} \left(1 + \frac{1}{D}\right) \int_0^s \|E_{T-\sigma} - \widehat{E}_{T-\sigma}\|_{W^{1,\infty}(\mathbb{R}^d)} d\sigma. \tag{7.6}$$

Remark 7.7 Notice that for $\nabla\psi_s \in L^2(\mathbb{R}^d)$ we do not use that $E \in L^\infty(\mathbb{R}^d)$, or the ellipticity constant.

Proof We begin with the $L^2(\mathbb{R}^d)$ estimates for smooth $\psi_0 \in C_c^\infty(\mathbb{R}^d)$ and $E \in C_c^\infty([0, T] \times \mathbb{R}^d)$, where the solutions are classical and differentiable. For our duality characterisation in $\dot{H}^{-1}(\mathbb{R}^d)$ we do not need $L^2(\mathbb{R}^d)$ estimates on ψ , only on $\nabla\psi$. Notice that (7.5) is simply (3.10) when $p = 2$.

For the continuous dependence, we note that

$$\begin{aligned} \partial_s(U_s^i - \widehat{U}_s^i) &= \nabla(U_s^i - \widehat{U}_s^i)E_{T-s} + \nabla(\psi_s - \widehat{\psi}_s) \frac{\partial E}{\partial x_i} \\ &\quad + \nabla\widehat{U}_s^i(E_{T-s} - \widehat{E}_{T-s}) + \nabla\widehat{\psi}_s \frac{\partial}{\partial x_i}(E_{T-s} - \widehat{E}_{T-s}) \\ &\quad + D\Delta(U_s^i - \widehat{U}_s^i). \end{aligned}$$

Multiplying by $U_s^i - \widehat{U}_s^i$ and integrating, we get

$$\partial_s \|U_s^i - \widehat{U}_s^i\|_{L^2(\mathbb{R}^d)}^2 = I_1 + \dots + I_5.$$

We estimate term by term the integrals $I_i, i = 1, \dots, 5$, as

$$\begin{aligned} |I_1| &= \frac{1}{2} \left| \int (U_s^i - \widehat{U}_s^i)^2 \nabla \cdot E_{T-s} \right| \leq \frac{1}{2} \|U_s^i - \widehat{U}_s^i\|_{L^2(\mathbb{R}^d)}^2 \|\nabla \cdot E_{T-s}\|_{L^\infty(\mathbb{R}^d)}, \\ |I_2| &\leq \sum_j \left| \int (U_s^j - \widehat{U}_s^j) \frac{\partial E^j}{\partial x_i} (U_s^i - \widehat{U}_s^i) \right| \leq \|U_s^i - \widehat{U}_s^i\|_{L^2} \sum_j \|U_s^j - \widehat{U}_s^j\|_{L^2(\mathbb{R}^d)} \left\| \frac{\partial E^j}{\partial x_i} \right\|_{L^\infty}, \\ |I_4| &\leq \sum_j \left| \int \widehat{U}_s^j \frac{\partial}{\partial x_i} (E_{T-s}^j - \widehat{E}_{T-s}^j) (U_s^i - \widehat{U}_s^i) \right| \\ &\leq \|U_s^i - \widehat{U}_s^i\|_{L^2(\mathbb{R}^d)} \sum_j \left\| \frac{\partial}{\partial x_i} (E_{T-s}^j - \widehat{E}_{T-s}^j) \right\|_{L^\infty} \|\widehat{U}_s^j\|_{L^2(\mathbb{R}^d)} \end{aligned}$$

$$\leq \frac{1}{2} \|U_s^i - \widehat{U}_s^i\|_{L^2(\mathbb{R}^d)}^2 + \frac{1}{2} \left(\sum_j \left\| \frac{\partial}{\partial x_i} (\mathbf{E}_{T-s}^j - \widehat{\mathbf{E}}_{T-s}^j) \right\|_{L^\infty(\mathbb{R}^d)} \|\widehat{U}_s^j\|_{L^2(\mathbb{R}^d)} \right)^2,$$

and

$$I_5 = -D \int |\nabla(U_s^i - \widehat{U}_s^i)|^2.$$

There is only one problematic term that requires the use of the ellipticity condition

$$\begin{aligned} |I_3| &\leq \left| \int U_s^i (U_s^i - \widehat{U}_s^i) \nabla \cdot (\mathbf{E}_{T-s} - \widehat{\mathbf{E}}_{T-s}) \right| + \left| \int U_s^i (\mathbf{E}_{T-s} - \widehat{\mathbf{E}}_{T-s}) \nabla (U_s^i - \widehat{U}_s^i) \right| \\ &\leq \frac{1}{2} \|U_s^i - \widehat{U}_s^i\|_{L^2(\mathbb{R}^d)}^2 + \frac{1}{2} \|U_s^i\|_{L^2(\mathbb{R}^d)}^2 \|\nabla \cdot (\mathbf{E}_{T-s} - \widehat{\mathbf{E}}_{T-s})\|_{L^\infty(\mathbb{R}^d)}^2 \\ &\quad + \frac{1}{4D} \|U_s^i\|_{L^2(\mathbb{R}^d)}^2 \|\mathbf{E}_{T-s} - \widehat{\mathbf{E}}_{T-s}\|_{L^\infty(\mathbb{R}^d)}^2 + D \|\nabla(U_s^i - \widehat{U}_s^i)\|_{L^2(\mathbb{R}^d)}^2. \end{aligned} \tag{7.7}$$

Notice that I_5 cancels out the last term in $|I_3|$. Arguing as above we recover

$$\begin{aligned} \partial_s \|\nabla(\psi_s - \widehat{\psi}_s)\|_{L^2(\mathbb{R}^d)}^2 &\leq C(d) \left((1 + \|\nabla \mathbf{E}_{T-s}\|_{L^\infty(\mathbb{R}^d)}) \|\nabla(\psi_s^T - \widehat{\psi}_s^T)\|_{L^2(\mathbb{R}^d)}^2 \right. \\ &\quad \left. + \left(1 + \frac{1}{D}\right) \left(\|\nabla \psi_s\|_{L^2(\mathbb{R}^d)}^2 + \|\nabla \widehat{\psi}_s\|_{L^2(\mathbb{R}^d)}^2 \right) \|\mathbf{E}_{T-s} \right. \\ &\quad \left. - \widehat{\mathbf{E}}_{T-s}\|_{W^{1,\infty}(\mathbb{R}^d)}^2 \right). \end{aligned}$$

Eventually, since $\psi_0^T - \widehat{\psi}_0^T = 0$ we recover (7.6) where the constants depend on $d, \|\nabla \psi_0\|_{L^2}, \|\nabla \widehat{\psi}_0\|_{L^2}$ and $\int_0^T \|\mathbf{E}_{T-\sigma}\|_{W^{1,\infty}} d\sigma$. This completes the L^2 estimates.

When ψ_0 is a general initial datum in $C_c(\mathbb{R}^d)$ we proceed by an approximation argument as in Proposition 3.1. As for the $L^\infty(\mathbb{R}^d)$ estimates, we can first assume that $\mathbf{E} \in C_c^\infty([0, T] \times \mathbb{R}^d)$ and $\psi_0 \in C_c^\infty(\mathbb{R}^d)$ are smooth, recover the L^2 estimates for the gradient, and then pass to the limit. \square

Remark 7.8 Notice that in (7.7) we use strongly the fact that $D > 0$, and the estimates are not uniform as $D \searrow 0$.

7.3 Study of (P_E)

Similarly, applying (7.4), (7.5) and (7.6), we deduce that

Proposition 7.9 *For every $D, T > 0, \mathbf{E} \in C([0, T]; W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d))$ and $\mu_0 \in \dot{H}^{-1}(\mathbb{R}^d)$ there exists exactly one $\dot{H}^{-1}(\mathbb{R}^d)$ dual viscosity solution $\mu \in C([0, T]; \dot{H}^{-1}(\mathbb{R}^d))$ of (P_E) . Furthermore, the map*

$$S_T : \dot{H}^{-1}(\mathbb{R}^d) \times C([0, T]; W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)) \longmapsto \mu_T \in \dot{H}^{-1}(\mathbb{R}^d)$$

is continuous with the following estimate

$$\begin{aligned} \|S_T[\mathbf{E}, \mu_0] - S_T[\widehat{\mathbf{E}}, \widehat{\mu}_0]\|_{\dot{H}^{-1}} &\leq C \left(d, \frac{1}{D}, T, \int_0^T \|\nabla \mathbf{E}_{T-\sigma}\|_{L^\infty} d\sigma \right) \\ &\quad \int_0^T \|\mathbf{E}_\sigma - \widehat{\mathbf{E}}_\sigma\|_{W^{1,\infty}} d\sigma, \end{aligned} \tag{7.8}$$

where C depends monotonically on each entry. Lastly, the semigroup property holds, i.e.,

$$\mathcal{S}_{\hat{t}+t}[\mu_0, \mathbf{E}] = \mathcal{S}_{\hat{t}}\left[\mathcal{S}_t\left[\mu_0, \mathbf{E}|_{[0,t]}\right], \mathbf{E}|_{[t,t+\hat{t}]}\right].$$

Remark 7.10 (Solutions by characteristics when $D = 0$). For $D = 0$ our notion of solution is $\mu_t = X_t \# \mu_0$ where X_t is the unique solution of the flow equation

$$\begin{cases} \frac{\partial X_t}{\partial t} = -\mathbf{E}_t(X_t), \\ X_0(x) = x. \end{cases} \quad (7.9)$$

A unique solution of this pointwise-decoupled problem exists via the Picard-Lindelöf theorem. Since we have forwards and backwards uniqueness, for each $t > 0$ we know that X_t is a bijection. It is easy to show that the unique viscosity solution is given by

$$\psi_s^T = \psi_0 \circ X_T^{-1} \circ X_{T-s}$$

And the duality condition (D_i) is trivially satisfied. The problem is that we do not have suitable H^1 in this theory.

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