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Individually Rational Rules for the Division Problem when the Number of Units to be Allotted is Endogenous*

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Abstract: We study individually rational rules to be used to allot, among a group of agents, a perfectly divisible good that is freely available only in whole units. A rule is individually rational if, at each preference profile, each agent finds that her allotment is at least as good as any whole unit of the good. We study and characterize two individually rational and efficient families of rules, whenever agents' preferences are symmetric single-peaked on the set of possible allotments. Rules in the two families are in addition envy-free, but they differ on wether envy-freeness is considered on losses or on awards. Our main result states that (i) the family of constrained equal losses rules coincides with the class of all individually rational and efficient rules that satisfy justified envy-freeness on losses and (ii) the family of constrained equal awards rules coincides with the class of all individually rational and efficient rules that satisfy envy-freeness on awards.

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Keywords: Division problem; Single-peaked preferences; Individual rationality; Efficiency; Strategy-proofness; Envy-freeness.

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1 Introduction

Consider the allotment problem faced by a group of agents who may share a homogeneous and perfectly divisible good, available only in whole units. Examples of this kind of good are shares representing ownership of a company, bonds issued by a company to finance its business operations, treasury bills issued by the government to finance its short term needs, or any type of financial assets with (potentially large) face values or tickets of a lottery. The good could also be workers, with a fixed working day schedule, to be shared among departments or divisions of a big institution or company, with a fixed salary budget. Agents' risk attitudes, wealth or labor requirements and salary budgets induce single-peaked preferences on their potential allotments of the good, the set of non-negative real numbers. A solution of the problem is a rule that selects, for each profile (of agents' preferences), a non-negative integer number of units of the good to be allotted and a vector of allotments (a list of non-negative real numbers, one for each agent) whose sum is equal to this integer. Observe that although the good is only available in integer amounts agents' allotments are allowed to take non-integer values; yet, their sum has to be an integer. Namely, in the above examples agents are able to share a financial asset or a lottery ticket by getting portions of it or time of a worker. But, for most profiles, the sum of agents' best allotments will be either larger or smaller than any integer number and hence, an endogenous rationing problem emerges, positive or negative depending on whether the chosen integer is smaller or larger than the sum of agents' best allotments. Sprumont (1991) studied the problem when the amount of the good to be allotted is fixed. He characterized the uniform rule as the unique efficient, strategy-proof and anonymous rule on the domain of single-peaked preferences. The present paper can be seen as an extension of Sprumont (1991)'s paper to a setting where the amount to be allotted of a divisible good has to be an integer, which may depend on agents' preferences.

We are interested in situations where the good is freely available to agents, but only in whole units. Hence, an agent will not accept a proposal of an allotment that is strictly worse than any integer amount of the good. For an agent with a (continuous) single-peaked preference, the set of allotments that are at least as good as any integer amount of the good (the set of individually rational allotments) is a closed interval that contains the best allotment, that we call the peak, and at least one of the two extremes of the interval is an integer. If preferences are symmetric, the peak is at the midpoint of the interval.

Our main concern then is to identify rules that select, for each profile of agents' symmetric single-peaked preferences, a vector of individually rational allotments. We call such rules individually rational. But since the set of individually rational rules is extremely large, and some of them are arbitrary and non-interesting, we would like to focus further on rules that are also efficient, strategy-proof, and satisfy some minimal fairness requirement. A rule is efficient if it selects, at each profile, a Pareto optimal vector of allotments: no other choices of (i) integer unit of the good to be allotted or (ii) vector of allotments, or (iii) both, can make all agents better off, and at least one of them strictly better off. We characterize the class of all individually rational and efficient rules on the domain of symmetric single-peaked preferences by means of three properties. First, the allotted amount of the good is the closest integer to the sum of agents' peaks. Second, all agents are rationed in the same direction: all receive more than their peaks, if the integer to be allotted is larger than the sum of the peaks, or all receive less, otherwise. Third, the rule selects a vector of allotments that belong to the agents' individually rational intervals. A rule is strategy-proof if it induces, at each profile, truth-telling as a weakly dominant strategy in its associated direct revelation game. Our fairness requirements will be related to two alternative and well-known notions of envy-freeness, that we will adapt to our setting (justified envy-freeness on losses and envy-freeness on awards).[1]

We show that there is no rule that is individually rational, efficient and strategy-proof on the domain of symmetric single-peaked preferences. We then proceed by studying separately two subclasses of rules on the symmetric single-peaked domain; those that are individually rational and efficient and those that are individually rational and strategy-proof. For the first subclass, we identify the family of the constrained equal losses rules and the family of the constrained equal awards rules as the unique families of rules that, in addition of being individually rational and efficient, satisfy also either justified envy-freeness on losses or envy-freeness on awards, respectively. These rules divide the efficient integer amount of the good in such a way that all agents experience either equal losses or equal gains, subject to the constraint that all allotments have to be individually rational. Specifically, a constrained equal losses rule, evaluated at a profile, selects first the efficient number of integer units (if there are two, it selects one of them). Then, to allot this integer amount it proceeds with the rationing from the vector of peaks, by either reducing or increasing the allotment of each agent (depending on whether the sum of the peaks is larger or smaller than the integer amount to be allotted) until the total amount is allotted. However, it makes sure that the extremes of agents' individually rational intervals are not overcome by excluding any agent from the rationing process as soon as one of the extremes of the agent's individually rational interval is reached, and it continues with the rest. A constrained equal awards rule is defined similarly but instead it uses, as the starting vector of the rationing process, either the vector of lower bounds or the vector of upper bounds of the individually rational intervals, depending on whether the sum of the peaks is larger or smaller than the integer amount to be allotted, but makes sure that no agent's peak is overcome by excluding her from the rationing process as soon as her peak is reached, and it continues with the rest.

For the subclass of individually rational and strategy-proof rules, we show in contrast

[1] See Section 3 for their definitions and justifications, and Thomson (2010) for a survey on envy-freeness.

that although there are many rules satisfying the two properties simultaneously, they are not very interesting; for instance, none of them is unanimous. A rule is unanimous if, whenever the sum of the peaks is an integer, the rule selects this integer and it allots it according to the agents' peaks. We show then that individual rationality and strategy-proofness are indeed incompatible with unanimity.

Before finishing this Introduction we mention some of the most related papers to ours. As we have already said, Sprumont (1991) proposed the division problem of a fixed amount of a good among a group of agents with single-peaked preferences on their potential allotments and provided two characterizations of the uniform rule, using strategy-proofness, efficiency and either anonymity or envy-freeness. Then, a very large literature followed Sprumont (1991) by taking at least two different paths. The first contains papers providing alternative characterizations of the uniform rule. See for instance Ching (1994), Sönmez (1994) and Thomson (1994a, 1994b, 1995 and 1997). The second group of papers proposed alternative rules when the problem is modified by introducing additional features or considering alternative domains of agents' preferences, or both. For instance, Ching (1992) extended the characterization of Sprumont (using envy-freeness) to the domain of single-plateaued preference profiles and Bergantiños, Massó and Neme (2012a, 2012b and 2015), Manjunath (2012) and Kim, Bergantiños and Chun (2015) studied alternative ways of introducing individual rationality in the division problem. But in contrast with the present paper they assume that the quantity of the good to be allotted is fixed. Adachi (2010), Amorós (2002), Anno and Sasaki (2013), Cho and Thomson (2013), Erlanson and Flores-Szwagrzak (2015) and Morimoto, Serizawa and Ching (2013) contain the multi-dimensional analysis of the division problem when several commodities have to be allotted among the same group of agents, but again the quantities of the goods to be allotted are fixed.

The paper is organized as follows. The next section presents the problem, preliminary notation and basic definitions. Section 3 contains the definitions of the properties of the rules that we will be concerned with. Section 4 describes the rules and states a preliminary result. Section 5 contains the main results of the paper for symmetric single-peaked preferences.

2 The problem

We study situations where each agent of a finite set $N = \{1, \dots, n\}$ wants an amount of a perfectly divisible good that can only be obtained in integer units, but arbitrary portions of each unit can be freely allotted. We assume that $n \geq 2$ and denote by $x_i \geq 0$ the total amount of the good allotted to agent $i \in N$. Since all units of the good are alike, the amount x_i may come from different units. We assume that there is no limit on the (integer) number of units that can be allotted. Hence, and once N is fixed, the set of *feasible (vector of) allotments* is

$$FA = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^N_+ \mid \sum_{i\in N} x_i \in \mathbb{N}_0\},$$

where $\mathbb{R}_+ = [0, +\infty)$ is the set of non-negative real numbers and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ is the set of non-negative integers.[2]

Each agent i has a preference relation $\succeq_i$, defined on the set of potential allotments, which is a complete and transitive binary relation on $\mathbb{R}_+$. That is, for all $x_i, y_i, z_i \in \mathbb{R}_+$, either $x_i \succeq_i y_i$ or $y_i \succeq_i x_i$, and $x_i \succeq_i y_i$ and $y_i \succeq_i z_i$ imply $x_i \succeq_i z_i$; note that reflexivity ($x_i \succeq_i x_i$ for all $x_i \in \mathbb{R}_+$) is implied by completeness. Given $\succeq_i$, let $\succ_i$ be the antisymmetric binary relation on $\mathbb{R}_+$ induced by $\succeq_i$ (*i.e.*, for all $x_i, y_i \in \mathbb{R}_+$, $x_i \succ_i y_i$ if and only if $y_i \succeq x_i$ does not hold) and let $\sim_i$ be the indifference relation on $\mathbb{R}_+$ induced by $\succeq_i$ (*i.e.*, for all $x_i, y_i \in \mathbb{R}_+$, $x_i \sim_i y_i$ if and only if $x_i \succeq_i y_i$ and $y_i \succeq_i x_i$). We assume that $\succeq_i$ is continuous (*i.e.*, for each $x_i \in \mathbb{R}_+$ the sets $\{y_i \in \mathbb{R}_+ \mid y_i \succeq_i x_i\}$ and $\{y_i \in \mathbb{R}_+ \mid x_i \succeq_i y_i\}$ are closed) and that $\succeq_i$ is *single-peaked* on $\mathbb{R}_+$; namely, there exists a unique $p_i \in \mathbb{R}_+$, the *peak* of $\succeq_i$, such that $p_i \succ_i x_i$ for all $x_i \in \mathbb{R}_+ \backslash \{p_i\}$ and $x_i \succ_i y_i$ holds for any pair of allotments $x_i, y_i \in \mathbb{R}_+$ such that either $y_i < x_i \leq p_i$ or $p_i \leq x_i < y_i$. For each $i \in N$, let $\succeq_i^{p_i}$ be an agent i's single-peaked preference such that $p_i \in \mathbb{R}_+$ is the peak of $\succeq_i^{p_i}$. We say that agent i's single-peaked preference $\succeq_i$ is *symmetric* on $\mathbb{R}_+$ if for all $z_i \in [0, p_i]$, $(p_i - z_i) \sim_i (p_i + z_i)$; that is, for all $x_i, y_i \in \mathbb{R}_+$, $x_i \succeq_i y_i$ if and only if $|p_i - x_i| \leq |p_i - y_i|$. Notice two things. First, the peak of a symmetric single-peaked preference conveys all information about the whole preference. Thus, we will often identify a symmetric single-peaked preference $\succeq_i$ with its peak p_i. Second, for each $x \in \mathbb{R}_+$, there exists a unique integer $k_x \in \mathbb{N}_0$ such that $k_x \leq x < k_x + 1$. Hence, the following notation is well-defined:

$$\begin{aligned}
\lfloor x \rfloor &= k_x \\
\lceil x \rceil &= k_x + 1 \\
[x] &= \begin{cases} k_x & \text{when } x \leq k_x + 0.5 \\ k_x + 1 & \text{when } x > k_x + 0.5. \end{cases}
\end{aligned}$$

For each $p = (p_i)_{i\in N}$, we denote $\left\lfloor \sum_{i\in N} p_i \right\rfloor$ by $p^* \in \mathbb{N}_0$; namely,

$$p^* \leq \sum_{i\in N} p_i < p^* + 1.$$

A (division) *problem* is a pair $(N, \succeq)$ where N is the set of agents and $\succeq = (\succeq_1, \ldots, \succeq_n)$ is a profile of single-peaked preferences on $\mathbb{R}_+$, one for each agent in N. Since the set N

[2] Since no confusion can arise with negative integers, we will refer to the set of non-negative integers $\mathbb{N}_0$ as the set of integers.

will remain fixed we often write $\succeq$ instead of $(N, \succeq)$ and refer to $\succeq$ as a problem and as a profile, interchangeably. To emphasize agent i's preference $\succeq_i$ in the profile $\succeq$ we often write it as $(\succeq_i, \succeq_{-i})$.

We denote by $\mathcal{P}$ the set of all problems where agents' preferences are single-peaked and by $\mathcal{P}^S$ the set of all problems where agents' preferences are symmetric single-peaked.

Since preferences are idiosyncratic, they have to be elicited. A *rule* on $\mathcal{P}$ is a function f assigning to each problem $\succeq \in \mathcal{P}$ a feasible allotment $f(\succeq) = (f_1(\succeq), \ldots, f_n(\succeq)) \in FA$. We will also consider rules defined only on $\mathcal{P}^S$. Any rule on $\mathcal{P}$ can be restricted to operate only on $\mathcal{P}^S$.

To study rules on $\mathcal{P}^S$ selecting individually rational allotments, the following intervals will play a critical role. Fix a problem $\succeq \in \mathcal{P}^S$, with its vector of peaks $(p_1, \ldots, p_n)$. For each $i \in N$, define the associated closed interval

$$[l_i(p_i), u_i(p_i)] = \begin{cases} [\lfloor p_i \rfloor, p_i + (p_i - \lfloor p_i \rfloor)] & \text{if } p_i \leq \lfloor p_i \rfloor + 0.5 \\ [p_i - (\lceil p_i \rceil - p_i), \lceil p_i \rceil] & \text{if } p_i \geq \lfloor p_i \rfloor + 0.5. \end{cases}$$

When no confusion arises we write l_i instead of $l_i(p_i)$ and u_i instead of $u_i(p_i)$.

Allotments outside the interval $[l_i, u_i]$ are strictly worse than some integer allotment (either than $\lfloor p_i \rfloor$ or than $\lceil p_i \rceil$), and they will not be acceptable to i, if agent i has free access to any integer amount of the good. Since each interval $[l_i, u_i]$ depends only on p_i, we call it the *individually rational interval of* p_i (Proposition 2 will show the exact relationship between individually rational rules on $\mathcal{P}^S$ and the individually rational intervals). Given $p_i \in \mathbb{R}_+$, $[l_i, u_i]$ can be seen as the unique interval with the properties that p_i is equidistant to the two extremes (i.e. $p_i = \frac{l_i + u_i}{2}$), at least one of the two extremes is an integer, and its length is at most one. For instance, the individually rational interval of $p_i = 1.8$ is $[1.6, 2]$ and of $p_i = 2.3$ is $[2, 2.6]$.

3 Properties of rules

We now describe possible properties that a rule f on $\mathcal{P}$ (or on $\mathcal{P}^S$) may satisfy. Again, the properties defined on $\mathcal{P}$ can be straightforwardly extended to $\mathcal{P}^S$ by restricting their definitions to the set of problems in $\mathcal{P}^S$.

We start with the property of individual rationality, the one that we found more basic for the class of problems we are interested in, which is the main focus of this paper. Since we are assuming that all integer units of the good are freely available, even for a single agent, a rule is individually rational if each agent considers her allotment at least as good as any integer number of units of the good.

Individual rationality. For all $\succeq \in \mathcal{P}$, $i \in N$ and $k \in \mathbb{N}_0$, $f_i(\succeq) \succeq_i k$.

The next two properties are also appealing. Efficiency says that, for each problem, the vector of allotments selected by the rule is Pareto undominated in the set of feasible allotments, while a rule is strategy-proof if agents can never obtain a strictly better allotment by misrepresenting their preferences.

Efficiency. For all $\succeq \in \mathcal{P}$, there does not exist $y \in FA$ such that $y_i \succeq_i f_i(\succeq)$ for all $i \in N$ and $y_j \succ_j f_j(\succeq)$ for at least one $j \in N$.

Strategy-proofness. For all $\succeq \in \mathcal{P}$, $i \in N$ and single-peaked preference $\succeq_i'$,

$$f_i(\succeq) \succeq_i f_i(\succeq_i', \succeq_{-i}).$$

We say that agent i *manipulates* f *at* $\succeq$ *via* $\succeq_i'$ if $f_i(\succeq_i', \succeq_{-i}) \succ_i f_i(\succeq)$.

We will also consider other desirable properties of rules. Participation says that agents will not have interest in obtaining integer units of the good in addition to their received allotments. To define it formally, we need some additional notation. For each $k \in \mathbb{N}_0$ and $\succeq_i$ with peak p_i such that $k \leq p_i$, let $\succeq_i^{p_i-k}$ be the single-peaked preference on $\mathbb{R}_+$ obtained from $\succeq_i$ by shifting it downwards in k units; namely, for each pair $x_i, y_i \in \mathbb{R}_+$, $x_i \succeq_i^{p_i-k} y_i$ if and only if $k + x_i \succeq_i k + y_i$.

Participation. For all $\succeq \in \mathcal{P}$, $i \in N$ and $k \in \mathbb{N}_0$ such that $k \leq p_i$,

$$f_i(\succeq) \sim_i k + f_i(\succeq_i^{p_i-k}, \succeq_{-i}).$$

If we interpret participation as a strategic property, then we could require $f_i(\succeq) \succeq_i k + f_i(\succeq_i^{p_i-k}, \succeq_{-i})$. It is easy to check that our results also hold with this new formulation.

Unanimity says that the rule selects the profile of peaks whenever it is a feasible vector of allotments. Equal treatment of equals says that agents with the same preferences receive equal allotments.

Unanimity. For all $\succeq \in \mathcal{P}$ such that $\sum_{j \in N} p_j \in \mathbb{N}_0$, $f_i(\succeq) = p_i$ for all $i \in N$.

Equal treatment of equals. For all $\succeq \in \mathcal{P}$ and $i, j \in N$ such that $\succeq_i = \succeq_j$, $f_i(\succeq) = f_j(\succeq)$.

Envy-freeness says that the rule selects a vector of allotments with the property that no agent would strictly prefer the allotment of another agent.

Envy-freeness. For all $\succeq \in \mathcal{P}$ and $i, j \in N$, $f_i(\succeq) \succeq_i f_j(\succeq)$.

The next three properties are alternative versions of envy-freeness, adapted to our context when agents have symmetric single-peaked preferences and they have free access to any integer amount of the good. Given that, each agent is willing to accept a non-integer allotment proposed by the rule insofar as her participation in the problem helps her to circumvent the integer restriction. Hence, envy-freeness may take as reference, not the absolute amounts received but instead, how other agents are treated with respect to their peaks or to their individually rational intervals. The emphasis is then on the losses or the awards that agents' allotments represent with respect to their peaks or to the extremes of their individually rational intervals, respectively. First, envy-freeness on losses says that each agent prefers her loss (with respect to her peak) to the loss of any other agent.

Envy-freeness on losses. For all $\succeq \in \mathcal{P}^S$ and $i, j \in N$, $f_i(\succeq) \succeq_i \max\{p_i + (f_j(\succeq) - p_j), 0\}$.[3]

Second, justified envy-freeness on losses qualifies the previous property by requiring that each agent i prefers her loss (*i.e.*, $f_i(\succeq) - p_i$) to the loss of any other agent j (*i.e.*, $f_j(\succeq) - p_j$), only if j's allotment is strictly preferred by j to any integer. Since agents can obtain freely any integer number of units of the good, it may be understood that it is not legitimate for i to express envy of another agent j who is receiving an allotment that j considers indifferent to an integer because it is as if the rule would not allot to j any amount. Hence, i's envy towards j is only justified if j strictly prefers her allotment to any integer amount.

Justified envy-freeness on losses. For all $\succeq \in \mathcal{P}^S$ and $i, j \in N$ such that $f_j(\succeq) \succ_j k$ for all $k \in \mathbb{N}_0$, $f_i(\succeq) \succeq_i \max\{p_i + (f_j(\succeq) - p_j), 0\}$.

Envy-freeness on awards roughly says that each agent prefers her award, with respect to her individually rational allotment, to any amount between her award and the award of any other agent. To state it formally, let f be a rule on $\mathcal{P}^S$. Define, for each $\succeq \in \mathcal{P}^S$ and $i \in N$, the award of i (at $(\succeq, f)$) with respect to i's individually rational interval as

$$a_i(\succeq, f) = \begin{cases} f_i(\succeq) - l_i & \text{if } f_i(\succeq) \leq p_i \\ u_i - f_i(\succeq) & \text{if } f_i(\succeq) > p_i. \end{cases}$$

When no confusion arises we write a_i instead of $a_i(\succeq, f)$.

Envy-freeness on awards. For all $\succeq \in \mathcal{P}^S$ and $i, j \in N$,

$$x \in [\min\{a_i(\succeq, f), a_j(\succeq, f)\}, \max\{a_i(\succeq, f), a_j(\succeq, f)\}]$$

implies $f_i(\succeq) \succeq_i l_i + x$.[4]

To see why envy-freeness on awards is a desirable property consider for example the case where $a_i = f_i(\succeq) - l_i$, $a_j = f_j(\succeq) - l_j$ and $a_i < x < a_j$. If $l_i + x \succ_i f_i(\succeq)$, i may argue that the non-integer amount received by j was too large and that there is a compromise,

[3]Note that $f_i(\succeq) = p_i + (f_i(\succeq) - p_i)$ always holds; hence, the condition in the definition is trivially satisfied whenever $i = j$. Since $p_i + (f_j(\succeq) - p_j) < 0$ may hold, we consider the max because preferences are only defined over non negative allotments.

[4]For all such x, $f_i(\succeq) \succeq_i l_i + x$ is equivalent to $f_i(\succeq) \succeq_i u_i - x$ since $\succeq_i$ is symmetric single-peaked and, by the definition of the extremes of the individually rational interval, $p_i = \frac{l_i + u_i}{2}$.

$x \in [a_i, a_j]$, that may be used to solve the integer problem in a more fair way. Example 1 might also help to better understand this property.

Example 1 Consider the problem $(N, \succeq) \in \mathcal{P}^S$ where $N = \{1, 2, 3\}$, $p = (0.1, 0.6, 0.6)$ and assume the rule f is such that $f(\succeq) = (0, 0.5, 0.5)$. Agent 1 is not envying agent 2 since $0 \succ_1 0.5$. Note that $l_1 = 0$, $l_2 = 0.2$, $a_1 = 0$, and $a_2 = 0.3$. Hence,

$$[\min\{a_1, a_2\}, \max\{a_1, a_2\}] = [0, 0.3].$$

By setting $x = 0.3$ we have that $f_1(\succeq) = 0 \succeq_1 0.3 = l_1 + x$. Nevertheless, by setting $x = 0.1$ we have that $f_1(\succeq) = 0 \prec_1 0.1 = l_1 + x$, and so f would not satisfy envy-freeness on awards. In this case agent 1 can argue that agent 2 is receiving at $f(\succeq)$ (compared with the individually rational points $l_2 = 0.2$ and $l_1 = 0$) more than her ($a_2 = 0.3$ versus $a_1 = 0$). □

Again, envy-freeness is based on absolute references: it requires comparisons of allotments directly. In contrast, our two notions of envy-freeness are relative: they disregard the integer amounts allotted to the agents and compare (using losses or awards as references) only those fractions received away from the peaks or the relevant extremes of the individually rational intervals.

Finally, group rationality is an extension of individual rationality to groups of agents. It says that each subset of agents receives a total allotment that is (in aggregate terms) "at least as good as" any other total allotment they could receive only by themselves.

Group rationality. For all $\succeq \in \mathcal{P}^S$, $S \subset N$ and $k \in \mathbb{N}_0$,

$$\left|\sum_{i\in S} p_i - \sum_{i\in S} f_i(\succeq)\right| \leq \left|\sum_{i\in S} p_i - k\right|.$$

Remark 1 The following statements hold.[5]

(R1.1) If f is efficient on $\mathcal{P}$, then f is unanimous.

(R1.2) If f is envy-free on losses on $\mathcal{P}^S$, then f satisfies justified envy-freeness on losses on $\mathcal{P}^S$.

(R1.3) If f is group rational on $\mathcal{P}^S$, then f is individually rational on $\mathcal{P}^S$.

4 Rules

In this section we adapt, to our setting with endogenous integer amounts, fair and well-known rules that have already been used to solve the division problem with a fixed amount. Since our main results will be relative to symmetric single-peaked preferences, we restrict the rules we consider in the next two sections to operate on $\mathcal{P}^S$. This is important because

[5] The proofs are immediate.

the rules will allot the integer amount that is closest to the sum of the peaks, which is always the efficient amount only if single-peaked preferences are symmetric. Since at profiles where $\sum_{j\in N} p_j = p^* + 0.5$, p^* and $p^* + 1$ are both at the same distance of 0.5 from $\sum_{j\in N} p_j$, many rules will share the same principles but they will be different only to the extent that they select the smaller or the larger closest integer at those profiles. Hence, we will be defining classes of rules. Although we will be interested only in their constrained versions (to ensure that they are individually rational) we also present their unconstrained versions for further reference and because they may help the reader to understand the constrained ones. We start with the class of equal losses rules. At any profile p, an equal losses rule selects the feasible vector of allotments by the following egalitarian procedure. Start from the vector of peaks p and, if this is an unfeasible vector of allotments, decrease or increase all agents' allotments in the same amount until the closest integer $\left\lfloor \sum_{j\in N} p_j \right\rfloor$ or $\left\lceil \sum_{j\in N} p_j \right\rceil$ respectively is allotted, stopping the decrease (if this is the case) of any agent's allotment, as soon as the zero allotment is reached.

Equal losses. We say that f is an *equal losses* rule if, for all $\succeq \in \mathcal{P}^S$,

$$f(\succeq) = \begin{cases} (p_i - \min\{\alpha, p_i\})_{i\in N} & \text{if } \sum_{j\in N} p_j < p^* + 0.5 \\ (p_i + \alpha)_{i\in N} & \text{if } \sum_{j\in N} p_j > p^* + 0.5 \\ (p_i - \min\{\alpha, p_i\})_{i\in N} \text{ or } (p_i + \alpha)_{i\in N} & \text{if } \sum_{j\in N} p_j = p^* + 0.5, \end{cases}$$

where α is the unique real number for which $\sum_{j\in N} (p_j - \min\{\alpha, p_j\}) = p^*$ or $\sum_{j\in N} (p_j + \alpha) = p^* + 1$ holds.[6]

Denote by F^{EL} the set of all equal losses rules. Figure 1 represents a rule $f^{EL} \in F^{EL}$ at profiles $\succeq$, $\succeq'$ and $\bar{\succeq}$, where $[p_1 + p_2] = [p'_1 + p'_2] < p_1 + p_2 = p'_1 + p'_2 < p^* + 0.5 = p'^* + 0.5$ and $[\bar{p}_1 + \bar{p}_2] > \bar{p}_1 + \bar{p}_2 > \bar{p}^* + 0.5$.

[6] Corollary 1 below (that follows from Proposition 1) will establish the existence of such unique real number α, as well as the existence of the real numbers $\widehat{\alpha}$, β, and $\widehat{\beta}$, used to define the other three rules below.

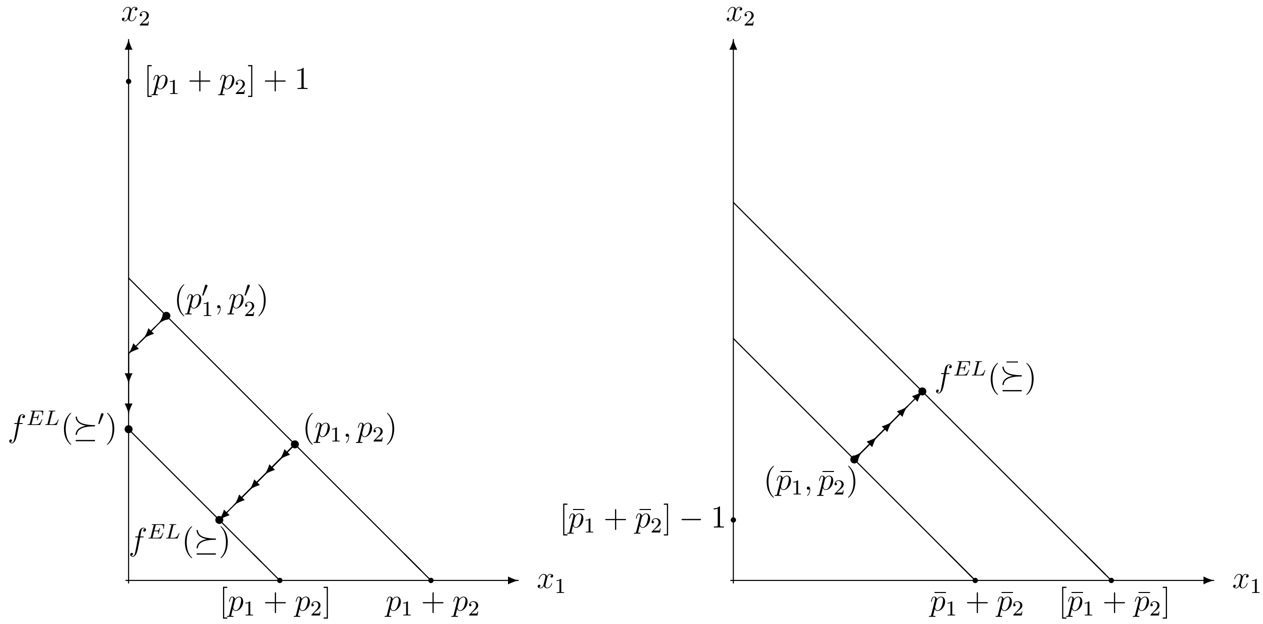

Fig. 1 An equal losses rule f^{EL}

A constrained equal losses rule proceeds by following the same egalitarian procedure but now the increase or decrease of the allotment of agent i, starting from p_i, stops as soon as i's allotment is equal to the relevant extreme of i's individually rational interval.

Constrained equal losses. We say that f is a *constrained equal losses* rule if, for all $\succeq \in \mathcal{P}^S$,

$$f(\succeq) = \begin{cases} (p_i - \min\{\widehat{\alpha}, p_i - l_i\})_{i\in N} & \text{if } \sum_{j\in N} p_j < p^* + 0.5 \\ (p_i + \min\{\widehat{\alpha}, u_i - p_i\})_{i\in N} & \text{if } \sum_{j\in N} p_j > p^* + 0.5 \\ (p_i - \min\{\widehat{\alpha}, p_i - l_i\})_{i\in N} \text{ or } (p_i + \min\{\widehat{\alpha}, u_i - p_i\})_{i\in N} & \text{if } \sum_{j\in N} p_j = p^* + 0.5, \end{cases}$$

where $\widehat{\alpha}$ is the unique real number for which it holds that $\sum_{j\in N}(p_j - \min\{\widehat{\alpha}, p_j - l_j\}) = p^*$ or $\sum_{j\in N}(p_j + \min\{\widehat{\alpha}, u_j - p_j\}) = p^* + 1$.

Denote by F^{CEL} the set of all constrained equal losses rules.

Observe that for any pair $f, f' \in F^{CEL}$, $f(\succeq) = f'(\succeq)$ for all $\succeq \in \mathcal{P}^S$ except for those profiles $\succeq$ for which $\sum_{j\in N} p_j = p^* + 0.5$. But in this case, for all $i \in N$, $f_i(\succeq) \sim_i f_i'(\succeq)$. To see that, assume $\succeq$ is such that $\sum_{j\in N} p_j = p^* + 0.5$. If $f(\succeq) = (p_i - \min\{\widehat{\alpha}, p_i - l_i\})_{i\in N}$ then

$$\begin{aligned} p^* &= \textstyle\sum_{j\in N}(p_j - \min\{\widehat{\alpha}, p_j - l_j\}) = \sum_{j\in N} p_j - \sum_{j\in N}\min\{\widehat{\alpha}, p_j - l_j\} \\ &= p^* + 0.5 - \textstyle\sum_{j\in N}\min\{\widehat{\alpha}, p_j - l_j\}, \end{aligned}$$

which implies $\sum_{j\in N}\min\{\widehat{\alpha}, p_j - l_j\} = 0.5$. If $f(\succeq) = (p_i + \min\{\widehat{\delta}, u_i - p_i\})_{i\in N}$ then

$$\begin{aligned} p^* + 1 &= \textstyle\sum_{j\in N}(p_j + \min\{\widehat{\delta}, u_j - p_j\}) = \sum_{j\in N} p_j + \sum_{j\in N}\min\{\widehat{\delta}, u_j - p_j\} \\ &= p^* + 0.5 + \textstyle\sum_{j\in N}\min\{\widehat{\delta}, u_j - p_j\}, \end{aligned}$$

which implies that $\sum_{j\in N}\min\{\widehat{\delta},u_j-p_j\}=0.5$. Since $p_j-l_j=u_j-p_j$ for all $j\in N$, we deduce that $\widehat{\alpha}=\widehat{\delta}$. Hence, for all $i\in N$,

$$p_i-\min\{\widehat{\alpha},p_i-l_i\}\sim_i p_i+\min\{\widehat{\alpha},u_i-p_i\}.$$

Thus, for any pair $f,f'\in F^{CEL}$, any profile $\succeq\in\mathcal{P}^S$ and any $i\in N$,

$$f_i(\succeq)\sim_i f_i'(\succeq). \tag{1}$$

Figure 2 represents a rule $f^{CEL}\in F^{CEL}$ at profiles $\succeq$ and $\bar{\succeq}$, where $[p_1+p_2]<p_1+p_2<p^*+0.5<\bar{p}_1+\bar{p}_2<[\bar{p}_1+\bar{p}_2]$.

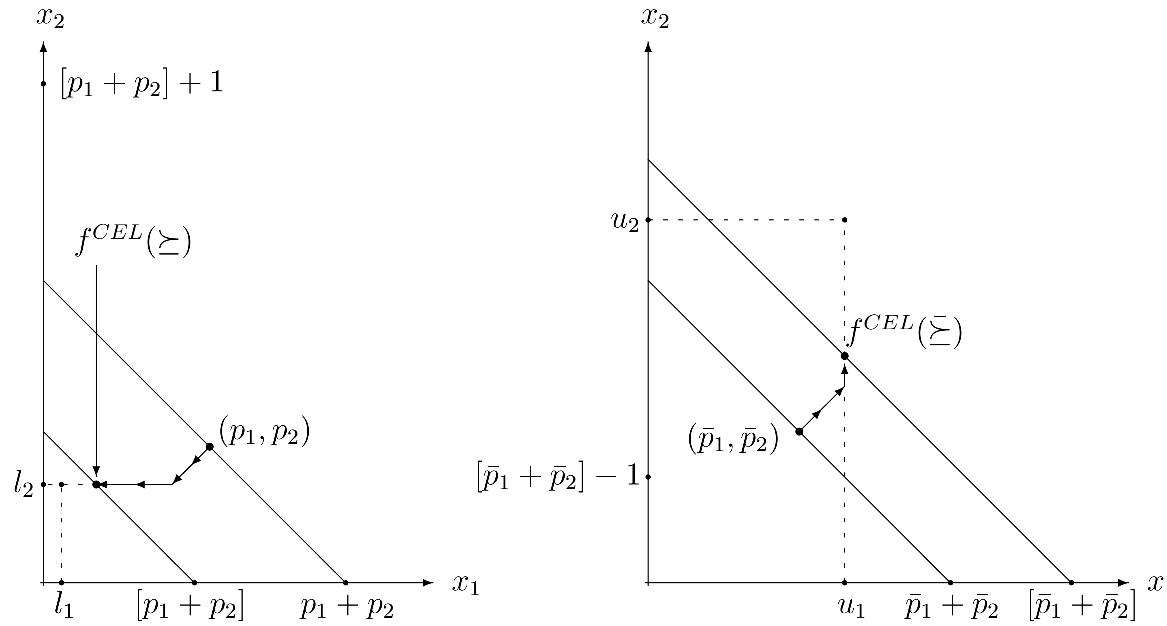

Fig. 2 A constrained equal losses rule f^{CEL}

An equal awards rule follows the same egalitarian procedure used to define equal losses rules, but instead of starting from the vector of peaks, it starts from the vector of relevant extremes of the individually rational intervals and it increases (or decreases) all agents' allotments in the same amount until the integer number of units is allotted, making sure that no agent receives a negative allotment.

Equal awards. We say that f is an *equal awards* rule if, for all $\succeq\in\mathcal{P}^S$,

$$f(\succeq)=\begin{cases}(l_i+\beta)_{i\in N} & \text{if } \sum_{j\in N}p_j<p^*+0.5\\ (u_i-\min\{\beta,u_i\})_{i\in N} & \text{if } \sum_{j\in N}p_j>p^*+0.5\\ (l_i+\beta)_{i\in N} \text{ or } (u_i-\min\{\beta,u_i\})_{i\in N} & \text{if } \sum_{j\in N}p_j=p^*+0.5,\end{cases}$$

where β is the unique real number for which $\sum_{j\in N}(l_j+\beta)=p^*$ or $\sum_{j\in N}(u_j-\min\{\beta,u_j\})=p^*+1$ holds.

Denote by F^{EA} the set of all equal awards rules. Figure 3 represents a rule $f^{EA} \in F^{EA}$ at profiles $\succeq$, $\succeq'$ and $\bar{\succeq}$, where $p^* + 0.5 > p_1 + p_2 > [p_1 + p_2]$, $p'^* + 0.5 < p'_1 + p'_2 < [p'_1 + p'_2]$, $\bar{p}^* + 0.5 < \bar{p}_1 + \bar{p}_2 < [\bar{p}_1 + \bar{p}_2]$ and $[p'_1 + p'_2] = [\bar{p}_1 + \bar{p}_2]$.

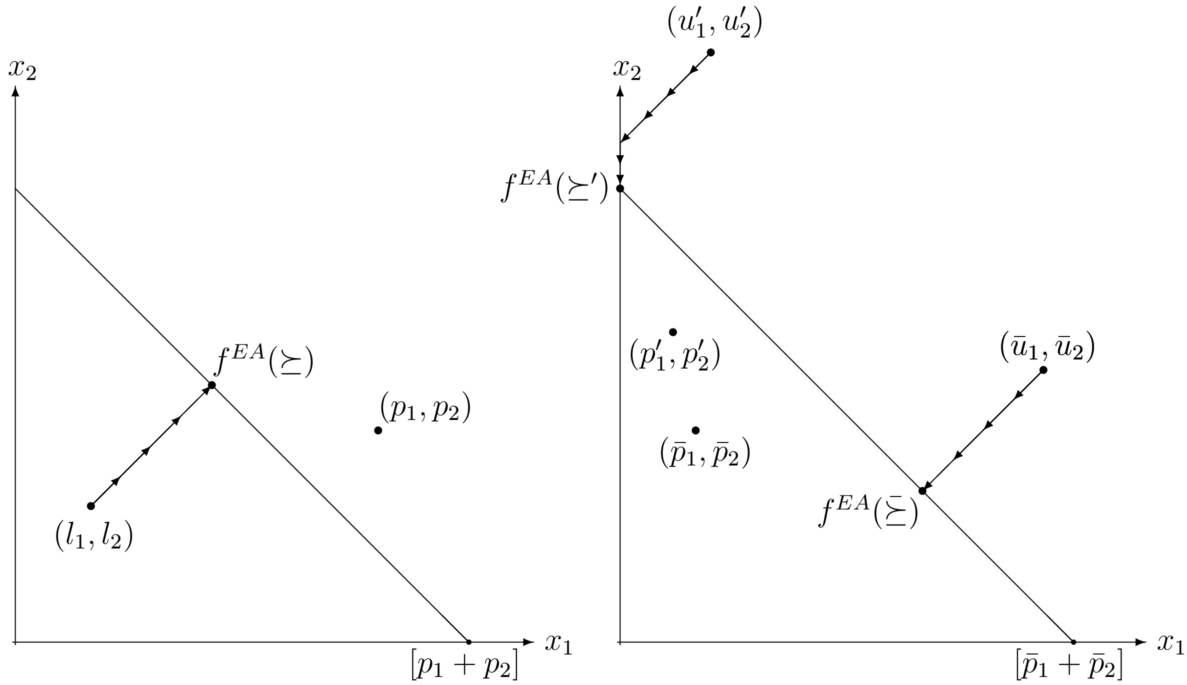

Fig. 3 An equal awards rule f^{EA}

A constrained equal awards rule proceeds by following the same egalitarian procedure but now the increase or decrease of the allotment of each agent i, starting from the relevant extreme of i's individually rational interval, stops as soon as i's allotment is equal to p_i.

Constrained equal awards. We say that f is a *constrained equal awards* rule if, for all $\succeq \in \mathcal{P}^S$,

$$f(\succeq) = \begin{cases} (l_i + \min\{\widehat{\beta}, p_i - l_i\})_{i \in N} & \text{if } \sum_{j \in N} p_j < p^* + 0.5 \\ (u_i - \min\{\widehat{\beta}, u_i - p_i\})_{i \in N} & \text{if } \sum_{j \in N} p_j > p^* + 0.5 \\ (l_i + \min\{\widehat{\beta}, p_i - l_i\})_{i \in N} \text{ or } (u_i - \min\{\widehat{\beta}, u_i - p_i\})_{i \in N} & \text{if } \sum_{j \in N} p_j = p^* + 0.5, \end{cases}$$

where $\widehat{\beta}$ is the unique real number for which $\sum_{j \in N}(l_j + \min\{\widehat{\beta}, p_j - l_j\}) = p^*$ or $\sum_{j \in N}(u_j - \min\{\widehat{\beta}, u_j - p_j\}) = p^* + 1$.

Denote by F^{CEA} the set of all constrained equal awards rules.

Observe that for any pair $f, f' \in F^{CEA}$, $f(\succeq) = f'(\succeq)$ for all $\succeq \in \mathcal{P}^S$ except for those profiles $\succeq$ for which $\sum_{j \in N} p_j = p^* + 0.5$. But in this case, for all $i \in N$, $f_i(\succeq) \sim_i f'_i(\succeq)$. To see that, assume $\succeq$ is such that $\sum_{j \in N} p_j = p^* + 0.5$. If $f(\succeq) = (l_i + \min\{\widehat{\beta}, p_i - l_i\})_{i \in N}$

then

$$\begin{aligned}
p^* &= \textstyle\sum_{j\in N}(l_j+\min\{\widehat{\beta},p_j-l_j\}) = \sum_{j\in N} l_j + \sum_{j\in N}\min\{\widehat{\beta},p_j-l_j\} \\
&= \textstyle\sum_{j\in N} p_j - \sum_{j\in N}(p_j-l_j) + \sum_{j\in N}\min\{\widehat{\beta},p_j-l_j\} \\
&= \textstyle p^*+0.5 - \sum_{j\in N}(p_j-l_j) + \sum_{j\in N}\min\{\widehat{\beta},p_j-l_j\},
\end{aligned}$$

which implies $\sum_{j\in N}\min\{\widehat{\beta},p_j-l_j\} = \sum_{j\in N}(p_j-l_j) - 0.5$. If $f(\succsim) = (u_i - \min\{\widehat{\delta}, u_i - p_i\})_{i\in N}$, then

$$\begin{aligned}
p^*+1 &= \textstyle\sum_{j\in N}(u_j-\min\{\widehat{\delta},u_j-p_j\}) = \sum_{j\in N} u_j - \sum_{j\in N}\min\{\widehat{\delta},u_j-p_j\} \\
&= \textstyle\sum_{j\in N} p_j + \sum_{j\in N}(u_j-p_j) - \sum_{j\in N}\min\{\widehat{\delta},u_j-p_j\} \\
&= \textstyle p^*+0.5 + \sum_{j\in N}(u_j-p_j) - \sum_{j\in N}\min\{\widehat{\delta},u_j-p_j\},
\end{aligned}$$

which implies that $\sum_{j\in N}\min\{\widehat{\delta},u_j-p_j\} = \sum_{j\in N}(u_j-p_j) - 0.5$. Since $p_j - l_j = u_j - p_j$ for all $j \in N$, we deduce that $\widehat{\beta} = \widehat{\delta}$. Hence, for all $i \in N$,

$$l_i + \min\{\widehat{\beta}, p_i - l_i\} \sim_i u_i - \min\{\widehat{\beta}, u_i - p_i\}.$$

Thus, for any pair $f, f' \in F^{CEA}$, any profile $\succsim \in \mathcal{P}^S$ and any $i \in N$,

$$f_i(\succsim) \sim_i f'_i(\succsim). \tag{2}$$

Figure 4 represents a rule $f^{CEA} \in F^{CEA}$ at profiles $\succsim$ and $\bar{\succsim}$, where $[p_1+p_2] < p_1+p_2 < p^*+0.5$ and $[\bar{p}_1+\bar{p}_2] > \bar{p}_1+\bar{p}_2 > \bar{p}^*+0.5$.

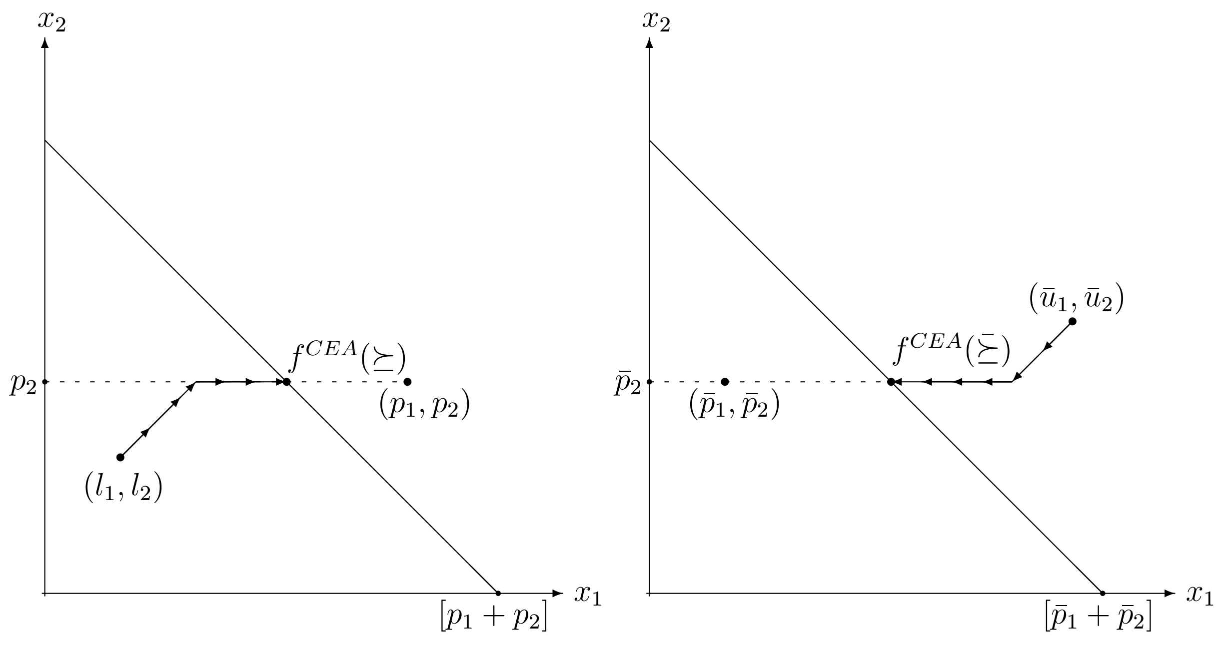

Fig. 4 A constrained equal awards rule f^{CEA}

The existence of the unique numbers α, $\widehat{\alpha}$, β and $\widehat{\beta}$ in each of the above definitions is guaranteed by Proposition 1 below.

Proposition 1 *For each $\succeq \in \mathcal{P}^S$, the appropriate statement below holds.*

(P1.1) *If* $\sum_{j\in N} p_j \leq p^* + 0.5$ *then* $\sum_{j\in N} l_j \leq p^*$.

(P1.2) *If* $\sum_{j\in N} p_j \geq p^* + 0.5$ *then* $\sum_{j\in N} u_j \geq p^* + 1$.

Proof Let $\succeq \in \mathcal{P}^S$ be arbitrary. For each $i \in N$ there exists $k_i \in \mathbb{N}_0$ such that $k_i \leq l_i \leq p_i < k_i + 1$. We define

$$t = \sum_{j:p_j \leq k_j + 0.5} k_j + \sum_{j:p_j > k_j+0.5} (k_j + 1)\,. \tag{3}$$

Notice that if $p_j \leq k_j + 0.5$, then $l_j = k_j$ and $u_j = p_j + (p_j - k_j) = 2p_j - k_j$. Similarly, if $p_j > k_j + 0.5$, then $l_j = p_j - (k_j + 1 - p_j) = 2p_j - (k + 1)$ and $u_j = k_j + 1$. Hence,

$$t = \sum_{j:p_j \leq k_j+0.5} l_j + \sum_{j:p_j > k_j+0.5} u_j. \tag{4}$$

Since $l_j \leq u_j$ for all $j \in N$,

$$\sum_{j\in N} l_j \leq t \leq \sum_{j\in N} u_j. \tag{5}$$

Making some computations we can prove that

$$\sum_{j\in N} l_j \leq 2\sum_{j\in N} p_j - t \leq \sum_{j\in N} u_j. \tag{6}$$

To prove (P1.1) we distinguish two cases ($t \leq p^*$ and $t > p^*$). In both cases it is not difficult to prove that $\sum_{j\in N} l_j \leq p^*$.

To prove (P1.2) we also distinguish two cases ($p^* + 1 \leq t$ and $p^* + 1 > t$). In both cases it is not difficult to prove that $\sum_{j\in N} u_j \geq p^* + 1$. ■

Proposition 1 implies that the real numbers α, $\widehat{\alpha}$, β and $\widehat{\beta}$ used to define the four families of rules do exist and they are unique, and hence the rules are well-defined. To see that, observe that any $f^{EL} \in F^{EL}$ and $f^{CEL} \in F^{CEL}$ start allotting the good from p in a continuous and egalitarian (or constrained egalitarian) way until the full amount is allotted. On the other hand, any $f^{EA} \in F^{EA}$ and $f^{CEA} \in F^{CEA}$ start allotting the good from the vector of relevant extremes of the individually rational intervals in a continuous and egalitarian (or constrained egalitarian) way until the full amount is allotted. Proposition 1 guarantees that the direction of the allotment process goes in the right direction to reach the full amount, from either one of the two starting vectors. So, Corollary 1 holds.

Corollary 1 *The real numbers α, $\widehat{\alpha}$, β and $\widehat{\beta}$, used to define respectively the families of rules F^{EL}, F^{CEL}, F^{EA} and F^{CEA} do exist and they are unique.*

5 Results for symmetric single-peaked preferences

5.1 Individual rationality and basic impossibilities

In the next proposition we present some results related with the properties of rules, whenever they operate on problems where agents' preferences are symmetric single-peaked. The first result characterizes individually rational rules by stating that a rule is individually rational if and only if, for all profiles, the rule selects a vector of allotments that belong to the individually rational intervals of their associated peaks. The second result characterizes individually rational and efficient rules. We also show that some basic incompatibilities among properties of rules hold, even when agents' preferences are restricted to be symmetric single-peaked.

Proposition 2 *The following statements hold.*

(P2.1) *A rule f on $\mathcal{P}^S$ is individually rational if and only if, for all $\succeq \in \mathcal{P}^S$ and $i \in N$, $f_i(\succeq) \in [l_i, u_i]$.*

(P2.2) *A rule f on $\mathcal{P}^S$ is individually rational and efficient if and only if, for all $\succeq \in \mathcal{P}^S$, three conditions hold:*

(E2.1) $\sum_{j\in N} f_j(\succeq) = \begin{cases} p^* & \text{if } \sum_{j\in N} p_j < p^* + 0.5 \\ p^*+1 & \text{if } \sum_{j\in N} p_j > p^* + 0.5 \\ p^* \text{ or } p^*+1 & \text{if } \sum_{j\in N} p_j = p^* + 0.5. \end{cases}$

(E2.2) $f_i(\succeq) \leq p_i$ *for all* $i \in N$ or $f_i(\succeq) \geq p_i$ *for all* $i \in N$.

(E2.3) $f_i(\succeq) \in [l_i, u_i]$ *for all* $i \in N$.

(P2.3) *There is no rule on $\mathcal{P}^S$ satisfying group rationality and efficiency.*

(P2.4) *There is no rule on $\mathcal{P}^S$ satisfying individual rationality, efficiency and strategy-proofness.*

(P2.5) *There is no rule on $\mathcal{P}^S$ satisfying individual rationality and envy-freeness on losses.*

(P2.6) *There is no rule on $\mathcal{P}^S$ satisfying individual rationality, efficiency, and envy-freeness.*

Proof

(P2.1) It is obvious.

(P2.2) Let f be an individually rational and efficient rule on $\mathcal{P}^S$. By (P2.1) f satisfies (E2.3).

We now prove that f satisfies (E2.2). Suppose not. Then, there exist $i, j \in N$ such that $f_i(\succeq) > p_i$ and $f_j(\succeq) < p_j$. Let ε be such that $0 < \varepsilon < \min\{f_i(\succeq) - p_i, p_j - f_j(\succeq)\}$. Then, by single-peakedness, the feasible vector of allotments $(f_i(\succeq) - \varepsilon, f_j(\succeq) + \varepsilon, (f_k(\succeq))_{k\in N\setminus\{i,j\}})$ Pareto dominates $f(\succeq)$. Hence, f is not efficient. This proves (E2.2).

We now prove that f satisfies (E2.1). We first show that for all $\succeq \in \mathcal{P}^S$,

$$\sum_{j\in N} f_j(\succeq) \in \{p^*, p^*+1\}. \tag{7}$$

Suppose that $\sum_{j\in N} f_j(\succeq) < p^*$. By (E2.2) for all $i \in N$, $f_i(\succeq) \leq p_i$ and there exists $j \in N$ such that $f_j(\succeq) < p_j$. Let $y \in FA$ be such that for all $i \in N$, $f_i(\succeq) \leq y_i \leq p_i$, $f_j(\succeq) < y_j \leq p_j$ and $\sum_{j\in N} y_j = p^*$. Since by single-peakedness $y_i \succeq_i f_i(\succeq)$ for all $i \in N$ and $y_j \succ_j f_j(\succeq)$, y Pareto dominates $f(\succeq)$, a contradiction with the efficiency of f. If $\sum_{j\in N} f_j(\succeq) > p^*+1$ the proof proceeds similarly.

We now distinguish among three cases.

<u>Case 1</u>: $\sum_{j\in N} p_j = p^* + x$ with $x < 0.5$. To obtain a contradiction, suppose that $\sum_{j\in N} f_j(\succeq) = p^*+1$. By (E2.2), for all $i \in N$, $f_i(\succeq) \geq p_i$. By individual rationality, for all $i \in N$, $f_i(\succeq) \leq u_i$. Hence, $p_i - (f_i(\succeq) - p_i) \geq l_i$ for all $i \in N$, which means that $(2p_j - f_j(\succeq))_{j\in N} \in FA$. Notice that $f_i(\succeq) \sim_i (2p_i - f_i(\succeq))$ for all $i \in N$. Now,

$$\begin{aligned}\textstyle\sum_{j\in N}(2p_j - f_j(\succeq)) &= 2\textstyle\sum_{j\in N} p_j - \sum_{j\in N} f_j(\succeq)\\ &< 2(p^*+x) - p^* - 1 = p^* + 2x - 1 < p^*.\end{aligned}$$

Let $y \in FA$ be such that, for all $i \in N$, $2p_i - f_i(\succeq) \leq y_i \leq p_i$ and $\sum_{j\in N} y_j = p^*$. By single-peakedness, $y_i \succeq_i 2p_i - f_i(\succeq) \sim_i f_i(\succeq)$ and since $\sum_{j\in N} y_j = p^* > \sum_{j\in N}(2p_j - f_j(\succeq))$ there exists $j \in N$ such that $2p_j - f_j(\succeq) < y_j$ and so $y_j \succ_j 2p_j - f_j(\succeq) \sim_j f_j(\succeq)$, a contradiction with the efficiency of f.

<u>Case 2</u>: $\sum_{j\in N} p_j = p^* + x$ with $x > 0.5$. similarly to Case 1 we can obtain a contradiction.

<u>Case 3</u>: $\sum_{i\in N} p_i = p^* + x$ with $x = 0.5$. By (7), it follows immediately.

We now prove the reciprocal. Let f be a rule satisfying (E2.1), (E2.2) and (E2.3). By (P2.1) and (E2.3) we conclude that f is individually rational. We now show that f is efficient. By (E2.1), it is enough to consider two cases.

<u>Case 1</u>: $\sum_{j\in N} f_j(\succeq) = p^*$. By (E2.2), $f_i(\succeq) \leq p_i$ for all $i \in N$. Suppose f is not efficient. Then, there exists $y \in FA$ that Pareto dominates $f(\succeq)$. Since preferences are symmetric single-peaked,

$$\begin{aligned} &y_i \in [f_i(\succeq), p_i + (p_i - f_i(\succeq))] && \text{for all } i \in N, \text{ and}\\ &y_{j'} \in (f_{j'}(\succeq), p_{j'} + (p_{j'} - f_{j'}(\succeq))) && \text{for some } j' \in N.\end{aligned}$$

By (E2.1) and our assumption,

$$\sum_{j\in N} f_j(\succeq) = p^* \leq \sum_{j\in N} p_j \leq p^* + 0.5.$$

Hence,

$$\begin{aligned} p^* &= \textstyle\sum_{j\in N} f_j(\succeq) < \sum_{j\in N} y_j < \sum_{j\in N}(p_j + (p_j - f_j(\succeq)))\\ &= \textstyle\sum_{j\in N} p_j + \sum_{j\in N} p_j - p^* \leq \sum_{j\in N} p_j + 0.5 \leq p^* + 1,\end{aligned}$$

Thus, $p^* < \sum_{j\in N} y_j < p^* + 1$. Since $\sum_{j\in N} y_j \in \mathbb{N}_0$, we have a contradiction.

Case 2: $\sum_{j\in N} f_j(\succeq) = p^* + 1$. We can find a contradiction similarly to Case 1.

(P2.3) Assume f satisfies group rationality and efficiency on $\mathcal{P}^S$. Consider the problem $(N, \succeq) \in \mathcal{P}^S$ where $N = \{1, 2, 3\}$ and $p = (0.8, 0.4, 0.4)$. By (R1.3), f is individually rational on $\mathcal{P}^S$. By efficiency, individual rationality and (P2.2), $\sum_{i\in N} f_i(\succeq) = 2$ and $f_i(\succeq) \geq p_i$ for all $i \in N$.

To apply the property of group rationality Table 1 indicates for each subset of agents with cardinality two the aggregate loss, assuming the best integer amount is allotted (*i.e.*, for each $S \subset N$ with $|S| = 2$, $\min_{k\in\mathbb{N}_0} \left|\sum_{j\in S} p_j - k\right|$).

S	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$
$\min_{k\in\mathbb{N}_0} \left\|\sum_{j\in S} p_j - k\right\|$	0.2	0.2	0.2

Table 1

Observe that $0.4 = \left|\sum_{j\in N} p_j - \sum_{j\in N} f_j(\succeq)\right| = \sum_{j\in N} (f_j(\succeq) - p_j)$. Suppose first that $f_i(\succeq) - p_i = \frac{0.4}{3}$ for all $i \in N$. Then, for any $S \subsetneq N$ with two agents,

$$\left|\sum_{j\in S} p_j - \sum_{j\in S} f_j(\succeq)\right| = \frac{0.8}{3} > 0.2 = \min_{k\in\mathbb{N}_0} \left|\sum_{j\in S} p_j - k\right|.$$

Hence, f does not satisfy group rationality. Suppose now that there exists $i \in N$ such that $(f_i(\succeq) - p_i) < \frac{0.4}{3}$. Then, by setting $S = N \backslash \{i\}$,

$$\left|\sum_{j\in S} p_j - \sum_{j\in S} f_j(\succeq)\right| > \frac{0.8}{3} > 0.2 = \min_{k\in\mathbb{N}_0} \left|\sum_{j\in S} p_j - k\right|,$$

again a contradiction with group rationality of f.

(P2.4) Since efficiency implies unanimity, (P2.4) is a consequence of Theorem 2 below.

(P2.5) Assume f satisfies individual rationality and envy-freeness on losses on $\mathcal{P}^S$. Consider the problem $(N, \succeq) \in \mathcal{P}^S$ where $N = \{1, 2\}$ and $p = (1, 0.7)$. By individual rationality, $f_1(\succeq) = 1$. Thus, $f_2(\succeq) \in \{0, 1, 2, \ldots\}$ which means that agent 2 envies the zero loss $(f_1(\succeq) - p_1 = 0)$ of agent 1.

(P2.6) Assume f satisfies individual rationality, efficiency, and envy-freeness on $\mathcal{P}^S$. Consider the problem $(N, \succeq) \in \mathcal{P}^S$ where $N = \{1, 2\}$ and $p = (0.2, 0.35)$. By individual rationality, $0 \leq f_1(\succeq) \leq 0.4$ and $0 \leq f_2(\succeq) \leq 0.7$. By efficiency and (P2.2) in Proposition 2, $f_1(\succeq) + f_2(\succeq) = 1$. Thus, $0.3 \leq f_1(\succeq) \leq 0.4$ and $0.6 \leq f_2(\succeq) \leq 0.7$. Then, $f_1(\succeq) \succ_2 f_2(\succeq)$, which contradicts envy-freeness. ■

Our main objective in this paper is to identify individually rational rules to be used to solve the division problem when the integer number of units is endogenous and agents' preferences are symmetric single-peaked. Part (P2.1) in Proposition 2 characterizes the class of all individually rational rules. Since this class is large, it is natural to ask whether individual rationality is compatible with other additional properties. Efficiency and strategy-proofness emerge as two of the most basic and desirable properties. However, (P2.4) in Proposition 2 says that no rule satisfies individual rationality, efficiency and strategy-proofness simultaneously. In the next two subsections we study rules that are individually rational and efficient (Subsection 5.2) and rules that are individually rational and strategy-proof (Subsection 5.3). For the first case, we identify the family of constrained equal losses rules and the family of constrained equal awards rules as the unique ones that in addition of being individually rational and efficient satisfy also either justified envy-freeness on losses or envy-freeness on awards, respectively (Theorem 1). In contrast, in Subsection 5.3 we first show that although there are individually rational and strategy-proof rules, they are not very interesting. For instance, we show in Theorem 2 that individually rationality and strategy-proofness are indeed incompatible with unanimity.

5.2 Individual rationality and efficiency

Let $\succeq \in \mathcal{P}^S$ be a problem. Denote by $IRE(\succeq)$ the set of feasible vectors of allotments satisfying individual rationality and efficiency. It is easy to see that, by using similar arguments to the ones used to check that (P2.1) and (P2.2) in Proposition 2 hold, this set can be written as

$$IRE(\succeq) = \{x \in \mathbb{R}^N_+ \mid \begin{array}{l} \sum_{j\in N} x_j \in \{p^*, p^*+1\} \text{ and, for all } i \in N, \\ l_i \leq x_i \leq p_i \text{ when } \sum_{j\in N} x_j = p^* \text{ and} \\ p_i \leq x_i \leq u_i \text{ when } \sum_{j\in N} x_j = p^*+1\}. \end{array}$$

By Proposition 1, the set $IRE(\succeq)$ is non-empty. Hence, a rule f satisfies individual rationality and efficiency if and only if, for each $\succeq \in \mathcal{P}^S$, $f(\succeq) \in IRE(\succeq)$.

However, individual rationality and efficiency are properties of rules that apply only to each problem separately. They do not impose conditions on how the rule should behave across problems. Thus, and given two different criteria compatible with individual rationality and efficiency, a rule can choose, in an arbitrary way, at problem $\succeq$ an allocation in $IRE(\succeq)$, following one criterion, while choosing at problem $\succeq'$ an allocation in $IRE(\succeq')$, following the other criterion. For instance the rule f that selects $f \in F^{CEL}(\succeq)$ when p^* is odd and $f \in F^{CEA}(\succeq)$ when p^* is even satisfies individual rationality and efficiency.[7] Thus, it seems appropriate to require that the rule satisfies an additional property in order to eliminate this kind of arbitrariness. We will focus on two alternative properties related to envy-freeness: justified envy-freeness on losses and envy-freeness on awards. But then,

[7] Proposition 3 below will guarantee it.

the consequence of requiring that rules (in addition of being individually rational and efficient) satisfy either one of these two forms of non-envyness is that only one family of rules is left, either the family of constrained equal losses rules or the family of constrained equal awards rules, respectively. Theorem 1, the main result of the paper, characterizes axiomatically the two families on the domain of symmetric single-peaked preferences.

Theorem 1 *The following two characterizations hold.*

(T1.1) *A rule f on $\mathcal{P}^S$ satisfies individual rationality, efficiency, and justified envy-freeness on losses if and only if f is a constrained equal losses rule.*

(T1.2) *A rule f on $\mathcal{P}^S$ satisfies individual rationality, efficiency, and envy-freeness on awards if and only if f is a constrained equal awards rule.*

In bankruptcy problems, introduced in O'Neill (1982), two of the most relevant rules are the constrained equal awards rule and the constrained equal losses rule. The rules we consider are inspired in such bankruptcy rules. Nevertheless bankruptcy problems and the problems considered in this paper are quite different. For instance, in bankruptcy problems the claim of each agent is a fixed amount that is known for everybody. Nevertheless in our problems the amount that each agents wants is only known for each agent. Then, instead of a claim, each agent has a preference over the amount to receive. Besides our characterization results involve properties quite different from the ones used in bankruptcy problems for characterizing such rules. For instance, in the survey of Thomson (2003) on bankruptcy problems, several characterizations of the constrained equal awards (or losses) rule are mentioned in Theorems 4 and 5. No property used in our Theorem 1 appears in Theorems 4 and 5 of Thomson (2003).

Before proving Theorem 1, we provide in Proposition 3 preliminary results on the two families of rules that will be useful along the proof of Theorem 1 and in the sequel.

Proposition 3

(P3.1) *Let f be a constrained equal losses rule on $\mathcal{P}^S$. Then, f satisfies individual rationality, efficiency, justified envy-freeness on losses, participation, unanimity and equal treatment of equals.*

(P3.2) *Let f be a constrained equal losses rule on $\mathcal{P}^S$. Then, f does not satisfy strategy-proofness, group rationality, envy-freeness, envy-freeness on losses, and envy-freeness on awards.*

(P3.3) *Let f be a constrained equal awards rule on $\mathcal{P}^S$. Then, f satisfies individual rationality, efficiency, envy-freeness on awards, participation, unanimity and equal treatment of equals.*

(P3.4) *Let f be a constrained equal awards rule on $\mathcal{P}^S$. Then, f does not satisfy strategy-proofness, group rationality, envy-freeness, envy-freeness on losses, and justified envy-freeness on losses.*

Proof of Proposition 3

(P3.1) That f satisfies *unanimity* and *equal treatment of equals* follows directly from its definition. Now, we show that f satisfies the other properties.

Individual rationality. By its definition, for all $\succeq \in \mathcal{P}^S$ and $i \in N$, $f_i(\succeq) \in [l_i, u_i]$. By (P2.1) in Proposition 2, f is individually rational.

Efficiency. By its definition, f satisfies conditions (E2.1), (E2.2) and (E2.3) in Proposition 2. Hence, by (P2.2), f is efficient.

Justified envy-freeness on losses. Let $j \in N$ be such that

$$f_j(\succeq) \succ_j k \text{ for all } k \in \mathbb{N}_0. \tag{8}$$

We want to show that for all $i \in N$, $f_i(\succeq) \succeq_i \max\{p_i + (f_j(\succeq) - p_j), 0\}$.

We distinguish among three cases.

<u>Case 1</u>: $\sum_{j\in N} p_j < p^* + 0.5$. By definition, $f_j(\succeq) = p_j - \min\{\widehat{\alpha}, p_j - l_j\}$ for all $j \in N$. If $p_j - l_j \le \widehat{\alpha}$, then $f_j(\succeq) = l_j$, which contradicts (8) because $f_j(\succeq) \sim_j l_j \sim u_j$ and either l_j or u_j is an integer. Hence,

$$f_j(\succeq) = p_j - \widehat{\alpha}. \tag{9}$$

Let $i \in N$ be arbitrary. We distinguish between two cases. First, $\widehat{\alpha} \le p_i - l_i$. Then, by (9), $f_i(\succeq) = p_i - \widehat{\alpha} = p_i + (f_j(\succeq) - p_j)$, which means that $f_i(\succeq) = \max\{p_i + (f_j(\succeq) - p_j), 0\}$. Hence, $f_i(\succeq) \succeq_i \max\{p_i + (f_j(\succeq) - p_j), 0\}$. Second, $\widehat{\alpha} > p_i - l_i$. Then, by definition, $f_i(\succeq) = l_i$. Since, by (9),

$$p_i + (f_j(\succeq) - p_j) = p_i - \widehat{\alpha} < l_i \le p_i,$$

single-peakedness implies that $f_i(\succeq) \succeq_i \max\{p_i + (f_j(\succeq) - p_j), 0\}$.

<u>Case 2</u>: $\sum_{j\in N} p_j > p^* + 0.5$. It could be made in a similar way to Case 1.

<u>Case 3</u>: $\sum_{j\in N} p_j = p^* + 0.5$. Two cases are possible, $\sum_{j\in N} f_j(\succeq) = p^*$ or $\sum_{j\in N} f_j(\succeq) = p^* + 1$. The former is similar to Case 1 and the latter is similar to Case 2.

Participation. Let $\succeq \in \mathcal{P}^S$, $i \in N$ and $k \in \mathbb{N}_0$ be such that $k \le p_i$. We want to show that $f_i(\succeq) \sim_i k + f_i(\succeq_i^{p_i - k}, \succeq_{-i})$. Set $\succeq' = (\succeq_i^{p_i-k}, \succeq_{-i})$ and $p' = (p_i - k, (p_j)_{j\in N\setminus\{i\}})$. We distinguish between two cases.

<u>Case 1</u>: $\sum_{j\in N} f_i(\succeq) = p^*$. Since (as we have already proved) f is individually rational and efficient, we can use (P2.2) and assert that $\sum_{j\in N} p_j \le p^* + 0.5$. Then, $f_i(\succeq) = p_i - \min\{\widehat{\alpha}, p_i - l_i\}$ where $\widehat{\alpha}$ satisfies $\sum_{j\in N} f_j(\succeq) = p^*$. Since $p'_i = p_i - k$ and k is an integer, $p'^* = p^* - k$. We distinguish between two subcases.

Subcase 1: $\sum_{j\in N} p'_j < p'^* + 0.5$. Now, $f_i(\succeq') = p'_i - \min\{\widehat{\alpha}', p'_i - l'_i\}$ where $\widehat{\alpha}'$ satisfies $\sum_{j\in N} f_j(\succeq') = p'^*$. Since $l'_i = l_i - k$ and $l'_j = l_j$ for all $j \in N\backslash\{i\}$, we deduce that $\widehat{\alpha}' = \widehat{\alpha}$. Then,

$$
\begin{aligned}
f_i(\succeq') &= p_i - k - \min\{\widehat{\alpha}, p_i - k - (l_i - k)\} \\
&= p_i - \min\{\widehat{\alpha}, p_i - l_i\} - k = f_i(\succeq) - k,
\end{aligned}
$$

which implies that $f_i(\succeq) \sim_i k + f_i(\succeq')$.

Subcase 2: $\sum_{j\in N} p'_j = p'^* + 0.5$. Again two subcases are possible. First, $\sum_{j\in N} f_j(\succeq') = p'^*$. Then, using the same argument to the one used in Subcase 1, $f_i(\succeq) \sim_i k + f_i(\succeq')$ holds. Second, $\sum_{j\in N} f_j(\succeq') = p'^* + 1$. Then, consider any $\widehat{f} \in F^{CEL}$ with $\sum_{j\in N} \widehat{f}(\succeq') = p'^*$. By (1), $\widehat{f}_i(\succeq') \sim_i f_i(\succeq')$ and, by an argument similar to the one used in the first subcase, we conclude that $f_i(\succeq) \sim_i k + f_i(\succeq')$.

Case 2: $\sum_{j\in N} f_i(\succeq) = p^* + 1$. It can be proved similarly to Case 1 distinguishing two subcases, $\sum_{j\in N} p'_j > p'^* + 0.5$ and $\sum_{j\in N} p'_j = p'^* + 0.5$.

(P3.2) We show that f does not satisfy the following properties on $\mathcal{P}^S$.

Strategy-proofness. Consider the problems $(N, \succeq)$ and $(N, \succeq')$ where $N = \{1, 2\}$, $p = (0.4, 0.8)$ and $p' = (0.4, 0.9)$. Then, $f(\succeq) = (0.3, 0.7)$ and $f(\succeq') = (0.2, 0.8)$. Since $0.8 \succ_2 0.7$, f does not satisfy strategy-proofness.

Group rationality. It follows from (P3.1) and (P2.4).

Envy-freeness. Consider the problem $(N, \succeq)$ where $N = \{1, 2\}$ and $p = (0.40, 0.46)$. Then, $f(\succeq) = (0.47, 0.53)$, which contradicts envy-freeness because agent 2 strictly prefers 0.47 to 0.53.

Envy-freeness on losses. If follows from (P3.1) and (P2.5).

Envy-freeness on awards. Consider the problem $(N, \succeq)$ where $N = \{1, 2\}$ and $p = (0.4, 0.46)$. Then, $f(\succeq) = (0.47, 0.53)$. Therefore, $a_1 = 0.8 - 0.47 = 0.33$ and $a_2 = 0.92 - 0.53 = 0.39$. For $0.38 \in [0.33, 0.39]$, we have that $f_1(\succeq) = 0.47 \prec_1 0.38$.

(P3.3) That f satisfies *unanimity* and *equal treatment of equals* follows directly from its definition. Now, we show that f satisfies the other properties. *Individual rationality* and *efficiency* could be proved similarly to (P3.1).

Envy-freeness on awards. We distinguish among three cases

Case 1: $\sum_{j\in N} p_j < p^* + 0.5$. By definition, $f_i(\succeq) \le p_i$ for all $i \in N$. Suppose that f does not satisfy envy-freeness on awards. Then, there exist $i, j \in N$ and

$$x \in [\min\{a_i, a_j\}, \max\{a_i, a_j\}]$$

such that

$$l_i + x \succ_i f_i(\succeq). \tag{10}$$

Hence, $f_i(\succeq)$ is not the peak of $\succeq_i$ and so $f_i(\succeq) < p_i$. Since $f_i(\succeq) = l_i + \min\{\widehat{\beta}, p_i - l_i\}$, $\widehat{\beta} < p_i - l_i$ and hence

$$f_i(\succeq) = l_i + \widehat{\beta}. \tag{11}$$

Thus, $a_i = \widehat{\beta}$. We distinguish between two subcases.

Subcase 1: $\min\{\widehat{\beta}, p_j - l_j\} = \widehat{\beta}$. Since $a_j = f_j(\succeq) - l_j = \widehat{\beta}$, it must be the case that $x = \widehat{\beta}$. Hence, by (10),

$$l_i + \widehat{\beta} = l_i + x \succ_i f_i(\succeq) = l_i + \widehat{\beta},$$

which is a contradiction.

Subcase 2: $\min\{\widehat{\beta}, p_j - l_j\} = p_j - l_j < \widehat{\beta}$. By definition, $f_j(\succeq) = p_j$ and $a_j = f_j(\succeq) - l_j = p_j - l_j$. Thus, $x \in [p_j - l_j, \widehat{\beta}]$ and

$$l_i + x \leq l_i + \widehat{\beta} = f_i(\succeq) \leq p_i,$$

where the equality follows from (11). By single-peakedness, $f_i(\succeq) \succeq_i l_i + x$, a contradiction with (10).

Case 2: $\sum_{j\in N} p_j > p^* + 0.5$. It could be proved in a similar way to Case 1 by distinguishing between two subcases, $\min\{\widehat{\beta}, u_j - p_j\} = \widehat{\beta}$ and $\min\{\widehat{\beta}, u_j - p_j\} = u_j - p_j < \widehat{\beta}$.

Case 3: $\sum_{j\in N} p_j = p^* + 0.5$. Two subcases are possible, $\sum_{j\in N} f_j(\succeq) = p^*$ (similar to Case 1) and $\sum_{j\in N} f_j(\succeq) = p^* + 1$ (similar to Case 2).

Participation. Let $\succeq \in \mathcal{P}^S$, $i \in N$ and $k \in \mathbb{N}_0$ be such that $k \leq p_i$. We want to show that $f_i(\succeq) \sim_i k + f_i(\succeq_i^{p_i - k}, \succeq_{-i})$. Set $\succeq' = (\succeq_i^{p_i-k}, \succeq_{-i})$ and $p' = (p_i - k, (p_j)_{j\in N\setminus\{i\}})$. We distinguish between two cases.

Case 1: $\sum_{j\in N} f_i(\succeq) = p^*$. Since (as we have already proved) f is individually rational and efficient, we can use (P2.2) and assert that $\sum_{j\in N} p_j \leq p^* + 0.5$. Then, $f_i(\succeq) = p_i - \min\{\widehat{\beta}, p_i - l_i\}$ where $\widehat{\beta}$ satisfies $\sum_{j\in N} f_j(\succeq) = p^*$. Since $p'_i = p_i - k$ and k is an integer, $p'^* = p^* - k$. We distinguish between two subcases.

Subcase 1: $\sum_{j\in N} p'_j < p'^* + 0.5$. Now, $f_i(\succeq') = l'_i + \min\{\widehat{\beta}', p'_i - l'_i\}$ where $\widehat{\beta}'$ satisfies $\sum_{j\in N} f_j(\succeq') = p'^*$. Since $l'_i = l_i - k$ and $l'_j = l_j$ for all $j \in N\setminus\{i\}$, we deduce that $\widehat{\beta}' = \widehat{\beta}$. Then,

$$\begin{aligned} f_i(\succeq') &= l_i - k + \min\{\widehat{\beta}, p_i - k - (l_i - k)\} \\ &= l_i + \min\{\widehat{\beta}, p_i - l_i\} - k = f_i(\succeq) - k, \end{aligned}$$

which implies that $f_i(\succeq) \sim_i k + f_i(\succeq')$.

Subcase 2: $\sum_{j\in N} p'_j = p'^* + 0.5$. Again two cases are possible. First, $\sum_{j\in N} f_j(\succeq') = p'^*$. Then, using the same argument to the one used in Subcase 1, $f_i(\succeq) \sim_i k + f_i(\succeq')$ holds. Second, $\sum_{j\in N} f_j(\succeq') = p'^* + 1$. Consider any $\widehat{f} \in F^{CEA}$ with $\sum_{j\in N} \widehat{f}_j(\succeq') = p'^*$. By (7),

$\widehat{f}_i(\succeq') \sim_i f_i(\succeq')$ and, by an argument similar to the one used in the first subcase, we conclude that $f_i(\succeq) \sim_i k + f_i(\succeq')$.

Case 2: $\sum_{j\in N} f_i(\succeq) = p^* + 1$. It can be proved similarly to Case 1 by distinguishing between two subcases, $\sum_{j\in N} p'_j > p'^* + 0.5$ and $\sum_{j\in N} p'_j = p'^* + 0.5$.

(P3.4) We show that f does not satisfy the following properties on $\mathcal{P}^S$.

Strategy-proofness. Consider the problems $(N, \succeq)$ and $(N, \succeq')$ where $N = \{1, 2\}$, $p = (0.4, 0.8)$ and $p' = (0.6, 0.8)$. Then, $f(\succeq) = (0.2, 0.8)$ and $f(\succeq') = (0.3, 0.7)$. Since $0.3 \succ_1 0.2$, f does not satisfy strategy-proofness.

Group rationality. It follows from (P3.3) and (P2.4).

Envy-freeness. Consider the problem $(N, \succeq)$ where $N = \{1, 2\}$ and $p = (0.6, 0.8)$. Then, $f(\succeq) = (0.3, 0.7)$, which means that f is not envy-free because agent 1 strictly prefers 0.7 to 0.3.

Envy-freeness on losses. It follows from (P3.3) and (P2.5).

Justified envy-freeness on losses. Consider the problem $(N, \succeq)$ where $N = \{1, 2\}$ and $p = (0.6, 0.8)$. Then, $f(\succeq) = (0.3, 0.7)$, which means that f does not satisfy justified envy-freeness on losses because agent 1 strictly prefers $0.6 + (0.7 - 0.8) = 0.5$ to 0.3. ■

Proof of Theorem 1

(T1.1) Let f be a constrained equal losses rule. By Proposition 3, f satisfies individual rationality, efficiency and justified envy-freeness on losses.

Let f be a rule satisfying individual rationality, efficiency, and justified envy-freeness on losses. Let $\succeq \in \mathcal{P}^S$ be a problem. By (7), it is sufficient to distinguish between two cases.

Case 1: $\sum_{j\in N} f_j(\succeq) = p^*$. By (E2.2) in (P2.2) of Proposition 2, for all $i \in N$,

$$f_i(\succeq) \leq p_i. \tag{12}$$

By (P2.1) in Proposition 2, $f_i(\succeq) \geq l_i$ for all $i \in N$. By (12), for each $i \in N$, $f_i(\succeq) = p_i - x_i$, where $x_i \geq 0$. By individual rationality, $x_i \leq p_i - l_i$. Assume first that $x_i = x$ for all $i \in N$. Then, setting $\widehat{\alpha} = x$, we have $f_i(\succeq) = p_i - \widehat{\alpha}$ and $\widehat{\alpha} \leq p_i - l_i$ for all $i \in N$. Hence, for all $i \in N$, $f_i(\succeq) = p_i - \min\{\widehat{\alpha}, p_i - l_i\}$. Thus, at profile $\succeq$, f coincides with a constrained equal losses rule. Assume now that $x_j < x_i$ for some pair $i, j \in N$. By single peakedness, $p_i - x_j \succ_i p_i - x_i$. Since

$$f_i(\succeq) = p_i - x_i \prec_i p_i - x_j = p_i + (f_j(\succeq) - p_j)$$

holds, by justified envy-freeness on losses, there must exist $y_j \in \mathbb{N}_0$ such that $f_j(\succeq) \preceq_j y_j$. By individual rationality,

$$f_j(\succeq) = l_j. \tag{13}$$

Let S be the set of agents with the largest loss from the peak. Namely, $S = \{i' \in N \mid x_{i'} \geq x_{j'}$ for all $j' \in N\}$. Since N is finite, $S \neq \emptyset$. Moreover, our assumption that $x_j < x_i$ for some pair $i, j \in N$ implies $S \subsetneq N$. For each $\widehat{j} \in S$, set $\widehat{\alpha} = x_{\widehat{j}}$ and observe that $f_{\widehat{j}}(\succeq) = p_{\widehat{j}} - \widehat{\alpha} \geq l_{\widehat{j}}$. Hence, $f_{\widehat{j}}(\succeq) = p_{\widehat{j}} - \min\{\widehat{\alpha}, p_{\widehat{j}} - l_{\widehat{j}}\}$. For each $j' \notin S$, there exists $i' \in S$ such that $x_{j'} < x_{i'}$. By (13), $f_{j'}(\succeq) = l_{j'}$. Since $f_{j'}(\succeq) = l_{j'} = p_{j'} - x_{j'}$ and $\widehat{\alpha} > x_{j'} = p_{j'} - l_{j'}$, $f_{j'}(\succeq) = p_{j'} - \min\{\widehat{\alpha}, p_{j'} - l_{j'}\}$. Thus, at profile $\succeq$, f coincides with a constrained equal losses rule.

Case 2: $\sum_{j\in N} f_j(\succeq) = p^* + 1$. It can be proved in a similar way to Case 1.

(T1.2) Let f be a constrained equal awards rule**.** By Proposition 3, f satisfies individual rationality, efficiency and envy-freeness on awards.

Let f be a rule satisfying individual rationality, efficiency, and envy-freeness on awards. Let $\succeq \in \mathcal{P}^S$ be a problem. By (7), it is sufficient to distinguish between two cases.

Case 1: $\sum_{j\in N} f_j(\succeq) = p^*$. By (12), for each $i \in N$, $f_i(\succeq) = l_i + a_i$, where $0 \leq a_i \leq p_i - l_i$. We first prove that if $a_i < a_j$ for some pair $i, j \in N$, then $a_i = p_i - l_i$. Assume not; then, there exist $i, j \in N$ such that $a_i < a_j$ and $a_i < p_i - l_i$. Let $x \in \mathbb{R}_+$ be such that $x \in (a_i, \min\{a_j, p_i - l_i\}]$. Since $f_i(\succeq) = l_i + a_i < l_i + x \leq p_i$, single-peakedness implies that $l_i + x \succ_i f_i(\succeq)$ where $x \in (a_i, a_j]$, contradicting envy-freeness on awards. Let S be the set of agents with the largest award from the peak. Namely, $S = \{i' \in N \mid a_{i'} \geq a_{j'}$ for all $j' \in N\}$. Since N is finite, $S \neq \emptyset$. We consider two subcases.

Subcase 1: $S = N$. Then, there exists a such that $a \in [0, p_i - l_i]$ and $f_i(\succeq) = l_i + a$ for all $i \in N$. Set $\widehat{\beta} = a$. Hence, $f_i(\succeq) = l_i + \min\{\widehat{\beta}, p_i - l_i\}$. Thus, at profile $\succeq$, f coincides with a constrained equal awards rule.

Subcase 2: $S \subsetneq N$. Then, for all $j, j' \in S$, $a_j = a_{j'}$. Set $\widehat{\beta} = a_j$ with $j \in S$. For each $i \in S$, $f_i(\succeq) = l_i + \widehat{\beta} \leq p_i$ and so $f_i(\succeq) = l_i + \min\{\widehat{\beta}, p_i - l_i\}$. For each $i \notin S$ there exists $j \in S$ such that $a_j > a_i$. Then, $a_i = p_i - l_i$. Since $p_i - l_i = a_i < a_j = \widehat{\beta}$, $f_i(\succeq) = l_i + a_i = l_i + \min\{\widehat{\beta}, p_i - l_i\}$. Thus, at profile $\succeq$, f coincides with a constrained equal awards rule.

Case 2: $\sum_{j\in N} f_j(\succeq) = p^* + 1$. It can be proved in a similar way to Case 1 by considering two subcases, $S = N$ and $S \subsetneq N$. ■

Remark 2 The two sets of properties used in the two characterizations of Theorem 1 are independent.

(R2.1) The rule f defined by assigning to each agent $i \in N$ her most preferred integer, satisfies individual rationality and justified envy-freeness on losses but it is not efficient.

(R2.2) Any rule $f \in F^{EL}$ satisfies efficiency and justified envy-freeness on losses but is not individually rational.

(R2.3) Any rule $f \in F^{CEA}$ satisfies individual rationality and efficiency but it does not satisfy justified envy-freeness on losses.

(R2.4) The rule f defined in (R2.1) satisfies individual rationality and envy-freeness on awards but it is not efficient.

(R2.5) Any rule $f \in F^{EA}$ satisfies efficiency and envy-freeness on awards but it is not individually rational.

(R2.6) Any rule $f \in F^{CEL}$ satisfies individual rationality and efficiency but it is not envy-freeness on awards.

5.3 Individual rationality and strategy-proofness

We now study the set of rules satisfying individual rationality and strategy-proofness on the set of symmetric single-peaked preferences. There are many rules satisfying both properties. For instance, the rule that selects $f(\succeq) = ([p_i])_{i \in N}$ for all $\succeq \in \mathcal{P}^S$ is individually rational and strategy-proof. But there are many more, yet some of them are very difficult to justify as reasonable solutions to the problem. Consider the following family of rules. For each vector $x \in \mathbb{R}_+^N$ satisfying $\sum_{i \in N} x_i \in \mathbb{N}_0$, define f^x as the rule that when x is at least as good as $([p_i])_{i \in N}$ for each $i \in N$, f^x selects x. Otherwise f^x selects $([p_i])_{i \in N}$. Formally, fix $x \in \mathbb{R}_+^N$ satisfying $\sum_{i \in N} x_i \in \mathbb{N}_0$. For each problem $\succeq \in \mathcal{P}^S$, set

$$f^x(\succeq) = \begin{cases} x & \text{if } x_i \succeq_i [p_i] \text{ for all } i \in N \\ ([p_i])_{i \in N} & \text{otherwise.} \end{cases}$$

It is easy to see that each rule in the family $\{f^x \mid x \in \mathbb{R}_+^N \text{ and } \sum_{i \in N} x_i \in \mathbb{N}_0\}$ is individually rational and strategy-proof. However, this family contains many arbitrary and non-interesting rules. Thus, we ask whether it is possible to identify a subset of individually rational and strategy-proof rules satisfying additionally a basic, weak and desirable property. We interpret Theorem 2 below as giving a negative answer to this question: individual rationality and strategy-proofness are not compatible even with unanimity, a very weak form of efficiency.

Theorem 2 *There is no rule on $\mathcal{P}^S$ satisfying individual rationality, strategy-proofness and unanimity.*

Proof To obtain a contradiction, assume that f is a rule satisfying individual rationality, strategy-proofness and unanimity.

Consider the problem $(N, \succeq) \in \mathcal{P}^S$ where $N = \{1, 2\}$ and $p = (0.2, 0.8)$. By unanimity, $f(0.2, 0.8) = (0.2, 0.8)$. The proof consists in to analyze several cases. We only explain the statement of each case because the proof is relatively easy.

Claim: $f_2(0.2, 0.5) = 0.8$.

Proof: $f_2(0.2, 0.5) > 0.8$ is not possible.

Suppose $f_2(0.2, 0.5) < 0.8$. Thus, $f(0.2, 0.5) = (0.2 + x, 0.8 - x)$. Hence, $0 < x \leq 0.2$. Let $y > 0$ be such that $0.2 - x < y < 0.2$. Thus, $f_1(y, 0.5) \leq 0.2 + x$. To show that indeed $f_1(y, 0.5) = 0.2 + x$ we distinguish between two cases.

Case 1: $0.2 - x < f_1(y, 0.5) < 0.2 + x$.

Case 2: $f_1(y, 0.5) \leq 0.2 - x$. We consider two subcases, $f_1(y, 0.5) + f_2(y, 0.5) = 1$ and $f_1(y, 0.5) + f_2(y, 0.5) = 0$.

We show now that $f_1(0.2 - x, 0.5) = 0.2 + x$. $f_1(0.2 - x, 0.5) > 0.2 + x$ is not possible. Suppose $z := f_1(0.2 - x, 0.5) < 0.2 + x$.We prove that $z = y$ or $z > y$ are not possible. Let $z < y$. We prove that $x = 0.2$ is not possible. Assume now that $x < 0.2$. We distinguish between two cases, $2y - (0.2 + x) < z < y$ and $z \leq 2y - (0.2 + x)$. In each one we can obtain a contradiction.

Hence, $f_1(0.2-x, 0.5) = 0.2+x$. Now, by individual rationality of agent 1, $|0.2 - x - 0| \geq |0.2 - x - 0.2 - x|$, so $0.2 - x \geq 2x$, or equivalently, $x \leq \frac{0.2}{3}$.

Consider now the profile $(0.2 - x, 0.5)$ instead of $(0.2, 0.5)$. Similarly, we can prove that $x \leq \frac{0.2}{7}$.

Since $x > 0$ is fixed, repeating this process several times we will eventually find a contradiction with individual rationality of agent 1. Then, $f(0.2, 0.5) = (0.2, 0.8)$, which proves the claim. $\square$

Consider now the profile $(0.2, 0.39)$. If we consider the following cases: $f_1(0.2, 0.39) + f_2(0.2, 0.39) \geq 2$, $f_1(0.2, 0.39) + f_2(0.2, 0.39) = 1$, and $f_1(0.2, 0.39) + f_2(0.2, 0.39) = 0$, we can obtain a contradiction in each of them. Then, there does not exist a rule satisfying simultaneously the properties of individual rationality, strategy-proofness and unanimity. $\blacksquare$

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