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PORT-HAMILTONIAN SYSTEMS THEORY AND MONOTONICITY*

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Abstract. The relationship of the theory of port-Hamiltonian systems with the mathematical concept of monotonicity is explored. The earlier introduced notion of incrementally port-Hamiltonian systems is extended to systems defined with respect to maximal cyclically monotone relations, together with their generating convex functions. This gives rise to interesting subclasses of incrementally port-Hamiltonian systems, with examples stemming from physical systems modeling as well as from convex optimization. Furthermore, it is shown how cyclical monotonicity for Dirac structures is equivalent to separability. An in-depth treatment is given of the composition of maximal monotone and maximal cyclically monotone relations, where in the latter case the resulting maximal cyclically monotone relation is shown to be computable through the use of generating functions. The results on compositionality are employed for steady-state analysis and for a convex optimization approach to the computation of the equilibria of interconnected incrementally port-Hamiltonian systems. Finally, the relation to incremental and differential passivity is discussed, and it is shown how incrementally port-Hamiltonian systems with strictly convex Hamiltonians are equilibrium independent passive.

Key words. port-Hamiltonian systems, monotonicity, convex functions, Dirac structures

MSC codes. 93A30, 70H05, 93C15, 34C12, 47H05, 52A41, 26A51

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1. Introduction. Port-based modeling of physical systems leads to their description as *port-Hamiltonian systems*. Such models have turned out to be powerful for purposes of analysis, simulation, and control; see, e.g., [1, 2, 3]. On the other hand, the mathematical notion of *monotonicity* has become more and more important—in several areas and from multiple points of view. With regard to the current paper the following two aspects of monotonicity are most relevant. First, monotonicity has been a key concept in the study of nonlinear electrical circuits and general nonlinear network dynamics; see, e.g., the recent paper [4] for a historical context and references. From a systems and control point of view this view on monotonicity is intimately related to notions of incremental passivity [5]. Second, monotonicity is a key concept in convex optimization (see, e.g., [6] and the references therein), as well as in nonlinear (convex) analysis (see, e.g., [7, 8, 9]) and in the study of evolution inclusions (see, e.g., [10] and the references therein).

The present paper takes a closer look at the connections between port-Hamiltonian systems and monotonicity, and explores overarching notions. Already in our paper [11], partially inspired by [12], we defined a new class of dynamical systems with inputs and outputs, coined as *incrementally port-Hamiltonian systems*. This was done by replacing the composition of the Dirac structure and the energy-dissipating relation in the standard definition of port-Hamiltonian systems by a general (maximal) monotone relation. Furthermore, it was shown in [11] how monotone relations share the compositionality property of Dirac structures, and sufficient conditions for the composition of two maximal monotone relations to be again maximal monotone were

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given. Moreover, connections between incrementally port-Hamiltonian systems and properties of incremental and differential passivity were briefly discussed.

In the current paper this line of investigation is continued in several directions. First, we pay special attention to incrementally port-Hamiltonian systems defined with respect to a (maximal) *cyclically* monotone relation. Maximal cyclically monotone relations are of special interest because they correspond to convex functions, in the sense that any maximal cyclically monotone relation is given as the subdifferential of an extended convex function, called the generating function of the relation. Precise conditions for a Dirac structure and an energy-dissipating relation to be cyclically monotone are provided. When applied to the composition of a Dirac structure and an energy-dissipating relation, this yields conditions for a port-Hamiltonian system to be also incrementally port-Hamiltonian. Second, a full-fledged theory of composition of maximal monotone relations is developed. In particular, we show how, under mild technical conditions, the composition of two maximal monotone relations is again a maximal monotone relation. Furthermore, we show how the composition of two maximal cyclically monotone relations is again maximal cyclically monotone and how the composition can be computed via their generating functions.

While convincing examples of incrementally port-Hamiltonian systems in physical systems modeling are abundant, they also naturally show up in convex optimization. In fact, examples are continuous-time gradient algorithms for convex functions, and primal-dual gradient algorithms in case of minimization under affine constraints. Furthermore, it is shown how the equilibrium of interconnections of incrementally port-Hamiltonian systems defined by maximal cyclically monotone relations can be computed by convex optimization, thereby extending the innovative work [13]. Another connection with convex analysis appears if we assume the Hamiltonian function of the incrementally port-Hamiltonian system to be convex. This is shown to lead to *shifted passivity* [2] of steady states for constant inputs, and in particular to (maximal) *equilibrium independent passivity* [13, 14].

Finally, we discuss the relation of the notion of incrementally port-Hamiltonian systems to incremental and differential *passivity*. Indeed, as already noted in [11], (maximal) monotonicity in the definition of an incrementally port-Hamiltonian system does not always correspond to incremental passivity (which can be regarded as the monotonicity of its input-output map). In fact, incrementally port-Hamiltonian systems will be shown to be incrementally and differentially passive in case the Hamiltonian is nonnegative and *quadratic*. For nonquadratic Hamiltonians the dynamical properties of incrementally port-Hamiltonian systems remain elusive, which appears to be related to fundamental issues in the notion of incremental passivity.

The organization of the paper is as follows. In section 2, we briefly review the concepts of Dirac structures and standard port-Hamiltonian systems. Section 3 provides the necessary preliminaries on monotone relations. This is followed by the definitions of incrementally port-Hamiltonian systems and their subclasses in section 4. The definitions are illustrated on a number of examples, both from the physical system domain and from convex optimization. Section 5 deals with the monotonicity properties of Dirac structures and of energy-dissipating relations. In section 6, we prove that under mild technical conditions the composition of two (maximal) (cyclically) monotone relations is (maximal) (cyclically) monotone, and thus the power-conserving interconnection of incrementally port-Hamiltonian systems is again an incrementally port-Hamiltonian system. Furthermore, by applying the results to the composition of Dirac structures and energy-dissipating relations, it follows under which conditions port-Hamiltonian systems are also incrementally port-Hamiltonian. Section 7 deals with the structure of the set of steady states of incrementally port-Hamiltonian systems and with the computation of equilibria of interconnected maximal cyclically monotone port-Hamiltonian systems via convex optimization, making crucial use of the results on composition. Finally, section 8 investigates the relations of the notion of incrementally port-Hamiltonian system with passivity. It is shown that if the Hamiltonian is strictly convex, then incrementally port-Hamiltonian systems are (maximal) equilibrium independent passive. Furthermore, if the Hamiltonian is quadratic-affine and nonnegative, then incremental passivity and differential passivity result. The conclusions and outlook are in section 9.

2. Recap of port-Hamiltonian systems on linear state spaces.

In order to motivate the definition of incrementally port-Hamiltonian systems we first review the definition of "ordinary" port-Hamiltonian systems; cf. [1, 2, 3] for more details and further ramifications.

Underlying the definition of a port-Hamiltonian system is the geometric notion of a *Dirac structure*, which relates the power variables of all the constitutive elements of the system in a power-conserving manner. Since incrementally port-Hamiltonian systems will be defined on *linear*[1] spaces we restrict as well attention to port-Hamiltonian systems on linear state spaces, and correspondingly to *constant* Dirac structures on linear spaces.[2]

Power variables (such as voltages and currents, or forces and velocities), appear in conjugated pairs, whose products have physical dimension of power. In particular, let $\mathcal{F}$ be a finite-dimensional linear space and $\mathcal{E} := \mathcal{F}^*$ be its dual space. We call $\mathcal{F}$ the space of *flow* variables and $\mathcal{E}$ the space of *effort* variables. The duality product for the pair $(\mathcal{E}, \mathcal{F})$, denoted by $\langle \cdot \mid \cdot \rangle$, is given as

$$\langle e \mid f \rangle = e^T f \in \mathbb{R}$$

for $e \in \mathcal{E}$ and $f \in \mathcal{F}$, and is the *power* associated to the pair (f, e). Furthermore on $\mathcal{F} \times \mathcal{E}$ an *indefinite bilinear form* is defined as

$$\langle\langle (f_1, e_1), (f_2, e_2) \rangle\rangle = \langle e_1 \mid f_2 \rangle + \langle e_2 \mid f_1 \rangle,$$

where $(f_i, e_i) \in \mathcal{F} \times \mathcal{E}$ with $i \in \{1, 2\}$. For any subspace $\mathcal{S} \subset \mathcal{F} \times \mathcal{E}$, we denote its orthogonal companion with respect to this indefinite bilinear form by $\mathcal{S}^{\perp\!\!\perp}$.

Throughout the paper, we will work with various spaces of flow/effort variables. By convention, if $\mathcal{F}_\bullet$ denotes a certain space of flow variables, then $\mathcal{E}_\bullet := \mathcal{F}_\bullet^*$ will denote the corresponding space of effort variables.

Definition 2.1 (see [16]). *Let $\mathcal{F}$ be a linear space. A subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ is a constant Dirac structure on $\mathcal{F}$ if $\mathcal{D} = \mathcal{D}^{\perp\!\!\perp}$.*

From now on in this paper a Dirac structure will simply refer to a *constant* Dirac structure on a linear space.

Remark 2.2. An equivalent definition is the following [1, 15, 16]. A Dirac structure is any subspace $\mathcal{D}$ with the property

$$\langle e \mid f \rangle = 0 \text{ for all } (f, e) \in \mathcal{D}, \tag{2.1}$$

[1]In fact, everything can be naturally extended from *linear* to *affine* state spaces. Note that the tangent space to a point in an affine state space is a linear space, which is independent of the base point taken in the affine space.

[2]For the extension to port-Hamiltonian systems on manifolds, and the corresponding notions of Dirac on manifolds, we refer the reader to, e.g., [2, 15, 16].

which is *maximal* with respect to this property. (That is, there does not exist a subspace $\mathcal{D}'$ with $\mathcal{D} \subsetneq \mathcal{D}'$ such that $\langle e \mid f \rangle = 0$ for all $(f,e) \in \mathcal{D}'$.)

In the finite-dimensional case (as will be the case throughout this paper) the maximal dimension of any subspace $\mathcal{D}$ satisfying (2.1) equals $\dim \mathcal{F} = \dim \mathcal{E}$. Thus, equivalently, a Dirac structure is any subspace $\mathcal{D}$ satisfying (2.1) together with

$$\dim \mathcal{D} = \dim \mathcal{F}.$$

The definition of a *port-Hamiltonian system* on a linear space contains the following ingredients (see, e.g., [2, 3, 15, 17]). First there is a Dirac structure $\mathcal{D}$ defined on the space of all flow and effort variables, that is,

$$\mathcal{D} \subset \mathcal{F}_x \times \mathcal{F}_P \times \mathcal{F}_R \times \mathcal{E}_x \times \mathcal{E}_P \times \mathcal{E}_R. \tag{2.2}$$

Here $(f_x, e_x) \in \mathcal{F}_x \times \mathcal{E}_x$ are the flow and effort variables linking to the *energy-storing* elements, $(f_R, e_R) \in \mathcal{F}_R \times \mathcal{E}_R$ are the flow and effort variables linking to *energy-dissipating* elements, and finally $(f_P, e_P) \in \mathcal{F}_P \times \mathcal{E}_P$ are the flow and effort *port* variables (e.g., inputs and outputs). The port-Hamiltonian system is defined by specifying, next to its Dirac structure $\mathcal{D}$, the constitutive relations of the energy-dissipating elements and of the energy-storing elements. An *energy-dissipating relation* is any subset $\mathcal{R} \subset \mathcal{F}_R \times \mathcal{E}_R$ with the property

$$\langle e_R \mid f_R \rangle \geqslant 0 \text{ for all } (f_R, e_R) \in \mathcal{R}. \tag{2.3}$$

Finally, the constitutive relations of the *energy-storing* elements are specified by a *Hamiltonian* $H : \mathcal{X} \to \mathbb{R}$, where the state space $\mathcal{X}$ is equal to $\mathcal{F}_x$. Thus the total energy while at state x is given as $H(x)$. This defines the following constitutive relations between the state variables x and the flow and effort vectors (f_x, e_x) of the energy-storing elements:[3]

$$\dot{x} = -f_x \quad \text{and} \quad e_x = \frac{\partial H}{\partial x}(x). \tag{2.4}$$

DEFINITION 2.3. *Consider a Dirac structure* (2.2)*, a Hamiltonian* $H : \mathcal{X} \to \mathbb{R}$*, and an energy-dissipating relation* $\mathcal{R} \subset \mathcal{F}_R \times \mathcal{E}_R$ *as above. Then the dynamics of the corresponding port-Hamiltonian system on* $\mathcal{X}$ *is given as*

$$\left(-\dot{x}(t), f_P(t), -f_R(t), \frac{\partial H}{\partial x}\big(x(t)\big), e_P(t), e_R(t)\right) \in \mathcal{D}, \tag{2.5a}$$

$$(f_R(t), e_R(t)) \in \mathcal{R} \tag{2.5b}$$

at (almost) all time instants t.

Equation (2.4) immediately implies the energy balance $\frac{d}{dt}H = \frac{\partial H}{\partial x^T}(x)\dot{x} = -e_x^T f_x$. Furthermore, the *composition*

$$\begin{aligned} \mathcal{D} \rightleftarrows \mathcal{R} := \{&(f_x, f_P, e_x, e_P) \in \mathcal{F}_x \times \mathcal{F}_P \times \mathcal{E}_x \times \mathcal{E}_P \mid \\ &\exists (f_R, e_R) \in \mathcal{R} \text{ s.t. } (f_x, f_P, -f_R, e_x, e_P, e_R) \in \mathcal{D}\} \end{aligned} \tag{2.6}$$

satisfies by the power-conserving property (2.1) of the Dirac structure and by (2.3)

$$e_x^T f_x + e_P^T f_P = e_R^T f_R \geqslant 0 \tag{2.7}$$

[3]Throughout this paper the vector $\frac{\partial H}{\partial x}(x)$ denotes the *column* vector of partial derivatives; the corresponding *row* vector is denoted as $\frac{\partial H}{\partial x^T}(x)$.

for all $(f_x, f_P, e_x, e_P) \in \mathcal{D} \rightleftarrows \mathcal{R}$. Taken together this implies

$$\frac{d}{dt} H(x(t)) \leqslant e_P^T(t) f_P(t), \tag{2.8}$$

showing *cyclo-passivity*[4] of any port-Hamiltonian system, and *passivity* if $H : \mathcal{X} \to \mathbb{R}^+$ [2].

The basic idea in the definition of an *incrementally port-Hamiltonian system*, as first introduced in [11], is to replace the composition $\mathcal{D} \rightleftarrows \mathcal{R}$ of a Dirac structure $\mathcal{D}$ and an energy-dissipating relation $\mathcal{R}$ by a *monotone relation* $\mathcal{M}$.

3. (Maximal) (cyclically) monotone relations.

In this section we provide the necessary preliminaries about monotone relations; see [9] for further discussion.

DEFINITION 3.1. *A relation $\mathcal{M} \subset \mathcal{F} \times \mathcal{E}$ is said to be*

- *monotone if*

$$\langle e_1 - e_2 \mid f_1 - f_2 \rangle \geqslant 0$$

  for all $(f_i, e_i) \in \mathcal{M}$ with $i \in \{1, 2\}$;
- *cyclically monotone if*

$$\langle e_0 \mid f_0 - f_1 \rangle + \langle e_1 \mid f_1 - f_2 \rangle + \ldots + \langle e_{m-1} \mid f_{m-1} - f_m \rangle + \langle e_m \mid f_m - f_0 \rangle \geqslant 0$$

  for all $m \geqslant 1$ and $(f_i, e_i) \in \mathcal{M}$ with $i \in \{0, 1, \ldots, m\}$.

Since $\langle e_0 \mid f_0 - f_1 \rangle + \langle e_1 \mid f_1 - f_0 \rangle = \langle e_0 - e_1 \mid f_0 - f_1 \rangle$ for all e_0, f_0, e_1, f_1, every cyclically monotone relation is automatically monotone.

A simple example of a monotone relation $\mathcal{M} \subset \mathbb{R} \times \mathbb{R}$ is the graph of a monotone (i.e., nondecreasing), possibly discontinuous, function. For example, the graph of the discontinuous function $\theta : \mathbb{R} \to \mathbb{R}$ given by

$$\theta(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x \geqslant 0 \end{cases} \tag{3.1}$$

is a monotone relation. This example motivates the following strengthened definition of a *maximal* monotone relation.

DEFINITION 3.2. *A relation $\mathcal{M} \subset \mathcal{F} \times \mathcal{E}$ is called maximal (cyclically) monotone if it is (cyclically) monotone and the implication*

$$\mathcal{M}' \text{ is (cyclically) monotone and } \mathcal{M} \subset \mathcal{M}' \quad \Longrightarrow \quad \mathcal{M} = \mathcal{M}'$$

holds.

The graph of the discontinuous function θ in (3.1) is monotone, but not maximal monotone. In fact, its graph can be enlarged so as to obtain the following maximal monotone relation:

$$\mathcal{M} = \left\{ (x, y) \mid y \in \begin{cases} \{-1\} & \text{if } x < 0, \\ [-1, 1] & \text{if } x = 0, \\ \{1\} & \text{if } x > 0 \end{cases} \right\}. \tag{3.2}$$

[4] Cyclo-passivity is a weakened form of passivity where the storage function, in this case denoted by H, is not required to be nonnegative [2]. (NB: "cyclo" has nothing to do with "cyclically.")

Note that the function θ in (3.1) can be regarded as the description of a relay, while its closure given by the maximal monotone relation $\mathcal{M}$ defined in (3.2) defines, for example, an ideal Coulomb friction characteristic.

A few well-known facts are noteworthy. For *continuous* functions, monotonicity of the graph implies maximal monotonicity (see, e.g., [9]). Also, every maximal monotone relation on $\mathbb{R} \times \mathbb{R}$ is maximal cyclically monotone. (Hence the above Coulomb friction characteristic in (3.2) is maximal cyclically monotone.) In higher dimensions, however, not every maximal monotone relation enjoys the cyclical monotonicity property. Indeed, for example the relation given by

$$\left\{ \left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} -y \\ x \end{bmatrix} \right) \mid x, y \in \mathbb{R} \right\} \subset \mathbb{R}^2 \times \mathbb{R}^2$$

is maximal monotone but not cyclically monotone. More generally, later on (Proposition 5.4) we will see that Dirac structures are maximal monotone, but not cyclically monotone if they are the graph of a nonzero map.

The importance of maximal *cyclically* monotone relations $\mathcal{M}$ lies in the fact that they correspond to extended real-valued *convex functions*. This will be briefly reviewed next; for more details we refer the reader to [9]. Let $\phi : \mathcal{F} \to (-\infty, +\infty]$ be a proper[5] convex function. Its *effective domain* is defined by

$$\text{dom } \phi := \{ f \in \mathcal{F} \mid \phi(f) < +\infty \},$$

its *subdifferential of ϕ at f by*

$$\partial\phi(f) := \begin{cases} \{ e \in \mathcal{E} \mid \phi(\bar{f}) \geqslant \phi(f) + \langle e \mid \bar{f} - f \rangle \ \forall \bar{f} \in \mathcal{F} \} & \text{if } f \in \text{dom } (\phi), \\ \varnothing & \text{otherwise,} \end{cases} \tag{3.3}$$

and its *conjugate* $\phi^\star : \mathcal{E} \to (-\infty, +\infty]$ by

$$\phi^\star(e) := \sup \{ \langle e \mid f \rangle - \phi(f) \mid f \in \mathcal{F} \}. \tag{3.4}$$

The conjugate $\phi^\star$ is also convex, while if ϕ is lower semicontinuous, then $\phi = (\phi^\star)^\star$ and

$$e \in \partial\phi(f) \iff f \in \partial\phi^\star(e). \tag{3.5}$$

It turns out (see [9, Thm. 12.25]) that a relation $\mathcal{M} \subset \mathcal{F} \times \mathcal{E}$ is maximal cyclically monotone if and only if there exists a proper lower semicontinuous convex function ϕ such that

$$\mathcal{M} = \{ (f, e) \mid e \in \partial\phi(f) \} = \{ (f, e) \mid f \in \partial\phi^\star(e) \}. \tag{3.6}$$

In this case, we say that $\mathcal{M}$ is *generated by* ϕ, or that ϕ is a *generating function* of the relation $\mathcal{M}$. Note that ϕ is determined by $\mathcal{M}$ uniquely up to an additive constant.

As an example, consider the relation given by (3.2). One easily verifies that $\mathcal{M}$ is generated by the convex function $\phi(x)$ given by $x \mapsto |x|$. Furthermore,

$$\phi^\star(y) = \begin{cases} 0 & \text{if } y \in [-1, 1], \\ +\infty & \text{if } y \notin [-1, 1]. \end{cases}$$

[5] A convex function is called proper if it never takes on the value $-\infty$ and also is not identically equal to $+\infty$.

For later purposes we also need the following extension of (3.6); cf. [9, Example 12.27]. Consider a function $\psi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, written as $\psi(y,z), y \in \mathbb{R}^n, z \in \mathbb{R}^m$. Suppose that for every $z \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, the functions $\psi(\cdot, z)$ and $\psi(y, \cdot)$ are proper lower semicontinuous extended real-valued convex and concave, respectively, functions. Let ∂_y and ∂_z denote the *partial* subdifferentials (as in (3.3)) with respect to y and z, for fixed z, respectively, fixed y. Then, the relation

$$\mathcal{M} := \{(y,z,y^*,z^*) \mid y^* \in \partial_y \psi, z^* \in \partial_z(-\psi)\} \tag{3.7}$$

is maximal monotone.

4. Incrementally port-Hamiltonian systems. In [11] we introduced incrementally port-Hamiltonian systems by replacing in the definition of port-Hamiltonian systems the composition $\mathcal{D} \rightleftarrows \mathcal{R}$ by a (maximal) monotone relation $\mathcal{M}$. The definition of [11] is now extended as follows.

Definition 4.1. *Consider a (maximal) (cyclically) monotone relation*

$$\mathcal{M} \subset \mathcal{F}_x \times \mathcal{F}_P \times \mathcal{E}_x \times \mathcal{E}_P,$$

and a Hamiltonian $H : \mathcal{F}_x \to \mathbb{R}$. *Then the dynamics of the corresponding (maximal) (cyclically) monotone port-Hamiltonian system is given by the requirement*

$$\left(-\dot{x}(t), f_P(t), \frac{\partial H}{\partial x}(x(t)), e_P(t)\right) \in \mathcal{M} \tag{4.1}$$

for (almost) all time instants t.

Remark 4.2. Throughout this paper we use the terminology *incrementally port-Hamiltonian system* as shorthand for *all* systems defined with respect to monotone relations $\mathcal{M}$. Whenever we need to be more precise about the properties of the monotone relation $\mathcal{M}$ we will refer to the system as a *(maximal) (cyclically) monotone port-Hamiltonian system*.

Since $\langle e_x^1 - e_x^2 \mid f_x^1 - f_x^2\rangle + \langle e_P^1 - e_P^2 \mid f_P^1 - f_P^2\rangle \geqslant 0$ for all $(f_x^i, f_P^i, e_x^i, e_P^i) \in \mathcal{M}, i = 1,2$, it follows by substituting $f_x^i = -\dot{x}_i, e_x^i = \frac{\partial H}{\partial x}(x_i)$ that the dynamics of any incrementally port-Hamiltonian system satisfies the *incremental dissipation inequality*

$$\left\langle \frac{\partial H}{\partial x}(x_1) - \frac{\partial H}{\partial x}(x_2) \mid \right\rangle \dot{x}_1 - \dot{x}_2 \leqslant \langle e_P^1 - e_P^2 \mid f_P^1 - f_P^2\rangle \tag{4.2}$$

for all quadruples $(x_i, \dot{x}_i, f_P^i, e_P^i)$ satisfying $\left(-\dot{x}_i, f_P^i,, \frac{\partial H}{\partial x}(x_i), e_P^i\right) \in \mathcal{M}$, $i = 1,2$. The consequences of this dynamical inequality, and especially the relation with equilibrium independent, incremental, and differential passivity, will be discussed in section 8.

Remark 4.3. Note that in case of *linear* monotone relations and *linear* energy-dissipating relations the notions of port-Hamiltonian and incrementally port-Hamiltonian systems basically coincide. For more information regarding *linear* maximal monotone relations, see [18, 19].

4.1. Subclasses of incrementally port-Hamiltonian systems. An appealing subclass of maximal cyclically monotone port-Hamiltonian systems is defined as follows. Consider any Hamiltonian $H : \mathcal{X} \to \mathbb{R}$ on the linear state space $\mathcal{X}$, and any proper *convex* function $K : \mathcal{X}^* \times \mathcal{Y}^* \to \mathbb{R}$, with $\mathcal{Y}$ the linear space of outputs and $\mathcal{U} := \mathcal{Y}$ the linear space of inputs. Denote $\nabla H(x) := \frac{\partial H}{\partial x}(x)$. Then the system

$$\begin{aligned} &\dot{x} \in -\partial_e K(\nabla H(x), u), \quad x \in \mathcal{X},\ e = \nabla H(x) \in \mathcal{X}^*,\ u \in \mathcal{Y}^*, \\ &y \in \partial_u K(\nabla H(x), u), \quad y \in \mathcal{Y}, \end{aligned} \tag{4.3}$$

is a maximal cyclically monotone port-Hamiltonian system, defined with respect to the maximal cyclically monotone relation $\mathcal{M} \subset \mathcal{X}^* \times \mathcal{Y}^* \times \mathcal{X} \times \mathcal{Y}$ given as $\mathcal{M} = \text{graph}\ (\partial K)$.

Of course, in case K is differentiable, (4.3) reduces to the ordinary input-state-output system

$$\begin{aligned} \dot{x} &= -\frac{\partial K}{\partial x}(\nabla H(x), u), \\ y &= \frac{\partial K}{\partial u}(\nabla H(x), u). \end{aligned} \tag{4.4}$$

A special case occurs if the convex function $K(e,u)$ is of the form

$$K(e,u) = P(e) - e^T B u + \frac{1}{2} u^T D u, \tag{4.5}$$

where P is a differentiable convex function of e, B is an $n \times m$ matrix, and the matrix $D = D^T > 0$ is large enough such that K is convex. This yields the system class

$$\begin{aligned} \dot{x} &= -\frac{\partial P}{\partial e}(\nabla H(x)) + Bu, \\ y &= -B^T \nabla H(x) + Du. \end{aligned} \tag{4.6}$$

With the help of (3.7) we can furthermore consider the case (4.5) for $D = 0$, i.e., $K(e,u) = P(e) - e^T B u$. Indeed, since P is convex, K is convex in e for fixed u, as well as concave in u for fixed e, and thus the relation

$$\mathcal{M} = \left\{ (e,u,f,y) \mid f = \frac{\partial P}{\partial e}(e) - Bu, y = B^T e \right\} \tag{4.7}$$

is maximal monotone. Hence

$$\begin{aligned} \dot{x} &= -\frac{\partial P}{\partial e}(\nabla H(x)) + Bu, \\ y &= B^T \nabla H(x) \end{aligned} \tag{4.8}$$

is a maximal monotone port-Hamiltonian system (although not cyclically monotone). Note that the (multidimensional) *nonlinear integrator*

$$\dot{x} = u, \quad y = \nabla H(x) \tag{4.9}$$

is an example of this.

Finally an *extended* version of (4.8) can be defined as

$$\begin{aligned} \dot{x} &= J\nabla H(x) - \frac{\partial P}{\partial e}(\nabla H(x)) + Bu, \\ y &= B^T \nabla H(x), \end{aligned} \tag{4.10}$$

where J is a skew-symmetric matrix, and P a convex function as above. It will follow from Proposition 5.4 that if $J \neq 0$, then the underlying maximal monotone relation is *not* derivable from a convex function, and the system is maximal monotone port-Hamiltonian but *not* cyclically monotone port-Hamiltonian.

We will see in the next two subsections that systems of the forms (4.3), (4.4), (4.5), (4.6), (4.8) arise naturally, both in physical systems modeling and in convex optimization.

4.2. Examples from physical systems. Incrementally port-Hamiltonian systems are ubiquitous in physical systems modeling, as illustrated by the following examples.

Example 4.4 (mechanical systems with friction). Consider a mechanical system subject to friction. The friction characteristic is given by a relation between the power variables f_R (velocity) and e_R (friction force). In the case of a scalar friction characteristic $e_R = R(f_R)$ the system is port-Hamiltonian if the graph of the function $R : \mathbb{R} \to \mathbb{R}$ is contained in the first and the third quadrant. On the other hand, it is maximal cyclically monotone port-Hamiltonian if the function R is monotonically nondecreasing and moreover continuous, or otherwise the graph of R is extended by the interval between the left- and right-limit values at its discontinuities. (A typical example of the latter is Coulomb friction as mentioned above.)

Mechanical systems *without potential energy* and with friction derivable from a Rayleigh function are given in the form (4.8), with x the vector of momenta, $e = M^{-1}p$ the vector of velocities (derivable from the quadratic Hamiltonian $H(x) = \frac{1}{2}x^T M^{-1} x$ with mass matrix M), and Rayleigh dissipation function P.

Example 4.5 (systems with constant sources). Physical systems containing nonzero constant *sources* are generally incrementally port-Hamiltonian, but need not be port-Hamiltonian. In order to obtain a port-Hamiltonian formulation the constant forcing needs to be incorporated into an adapted Hamiltonian. However, this is not always possible as already illustrated by the nonlinear integrator $\dot{x} = u + d, y = \frac{dH}{dx}(x)$ for constant d (for instance, a mass moving under the influence of a control force u and a constant external force d). Precise conditions when this *is* possible can be found in [2, pp. 138–139].

Example 4.6 (Van der Pol oscillator). Consider an electrical LC circuit with (possible nonlinear) capacitors and inductors, together with a single conductor with current $f_R = I$ and voltage $e_R = V$. In case of a linear conductor $I = GV, G > 0$, the system is both port-Hamiltonian and maximal monotone port-Hamiltonian. In case of a nonlinear conductor $I = \Phi(V)$, the system is port-Hamiltonian if and only if the graph of the function Φ is in the first and the third quadrant, while it is maximal monotone port-Hamiltonian if G is monotonically nondecreasing and continuous, or otherwise the graph of G is extended by the interval between the left- and right-limit values at its discontinuities. For example, the conductor characteristic $I = \Phi(V) := \gamma V^3 - \alpha V$, with $\alpha, \gamma > 0$, defines a system which is port-Hamiltonian but *not* monotone port-Hamiltonian, since the function Φ is not monotone. On the other hand, by adding a constant source voltage V_0 and constant source current I_0 with V_0, I_0 such that the resulting *tunnel diode* characteristic

$$I = \Phi(V - V_0) + I_0$$

passes through the origin of the (I, V)-plane, we obtain the Van der Pol oscillator. This system is *not* port-Hamiltonian, since close to the origin the tunnel diode characteristic is in the second and the fourth quadrant. Furthermore, the Van der Pol oscillator is also not incrementally port-Hamiltonian since the tunnel diode characteristic is not monotone.

Example 4.7 (nonlinear RC electrical circuits with terminals). Another physical example of the form (4.3) is the following. Consider an RC electrical circuit, with nonlinear conductors at the edges and grounded nonlinear capacitors at part of the nodes, while the remaining nodes are the boundary nodes (terminals). Let the circuit graph be defined by an incidence matrix D, which is decomposed, according to the splitting of the capacitor ("c") and boundary nodes ("b"), as

$$D = \begin{bmatrix} D_c \\ D_b \end{bmatrix}. \tag{4.11}$$

Furthermore, let the conductors at the edges be given as $I_j = G_j(V_j)$, where I_j, V_j is the current through, respectively, voltage across, the jth edge, $j = 1, \dots, m$. Assume that the conductance functions G_j are all monotone (although not necessarily in the first and the third quadrant). This means that there exist convex functions $\widehat{K}_j$ such that $G_j(V_j) = \frac{d\widehat{K}_j}{dV_j}(V_j)$ (for simplicity, assuming that the functions G_j are continuous and that $\widehat{K}_j$ are differentiable). Define the convex functions

$$\widehat{K}(V_1, \dots, V_m) := \sum_{j=1}^{m} \widehat{K}_j(V_j), \quad K(\psi) := \widehat{K}(D^T \psi), \tag{4.12}$$

where ψ is the vector of nodal voltage potentials. (Recall that by Kirchhoff's voltage law $V = D^T\psi$.) Then $\frac{\partial K}{\partial \psi}$ is the vector of nodal currents (entering the circuit at the nodes), which can be split into the nodal currents I_c at the capacitor nodes and the nodal currents I_b at the boundary nodes. Denoting the vector of charges of the grounded capacitors by Q, it follows by Kirchhoff's current laws that $\dot{Q} = -I_c$. Furthermore it can be checked that

$$\frac{\partial K}{\partial \psi} = D \frac{\partial \hat{K}}{\partial V}(D^T \psi).$$

Hence the dynamics of the nonlinear RC circuit is given by

$$\begin{aligned} \dot{Q} &= -D_c \frac{\partial \hat{K}}{\partial V}(D^T \psi), \\ I_b &= D_b \frac{\partial \hat{K}}{\partial V}(D^T \psi). \end{aligned} \tag{4.13}$$

According to the splitting of the nodes in capacitor and boundary nodes write $\psi = \left[\begin{smallmatrix} \psi_c \\ \psi_b \end{smallmatrix}\right]$. Then by specifying the nonlinear grounded capacitors by a Hamiltonian function $H(Q)$, it follows that $\psi_c = \frac{\partial H}{\partial Q}(Q)$, while the remaining nodal voltage potentials ψ_b can be considered to be the inputs to the system. Hence (4.13) is a maximal cyclically monotone port-Hamiltonian system of the form (4.3), with state vector Q, input vector ψ_b, and outputs I_b. The generating function of its maximal cyclically monotone relation is given by the convex function $K(\psi_c, \psi_b)$.

Same equations hold for a mechanical network with masses at the nodes and nonlinear dampers at the edges. In this case the state vector Q should be replaced by the vector of momenta of the masses, while ψ_b and I_b are replaced by input forces applied to the boundary masses and velocities of these boundary masses. The difference is that in this case the Hamiltonian is necessarily quadratic (kinetic energy).

4.3. Examples from convex optimization. It is well known that convex optimization algorithms can be reformulated as zero-finding problems of (cyclically) monotone relations (see, e.g., [20]). An example of a maximal cyclically monotone port-Hamiltonian system in the form (4.4) that is *not* stemming from physical systems modeling, but instead from *optimization*, is the following.

Example 4.8 (gradient algorithm in continuous time). Consider the problem of minimizing a *strongly* convex function $P : \mathbb{R}^n \to \mathbb{R}$, that is, the function $x \mapsto P(x) -$

$\frac{\sigma}{2}\|x\|^2$ is convex for some $\sigma > 0$. Suppose that P is twice differentiable. Then, the Hessian of P satisfies

$$\nabla^2 P(x) \geqslant \sigma I \tag{4.14}$$

for all $x \in \mathbb{R}^n$. The gradient algorithm in continuous time is given as

$$\tau \dot{q} = -\frac{\partial P}{\partial q}(q), \tag{4.15}$$

where τ is a positive definite matrix determining the time-scales of the algorithm. This defines a maximal cyclically monotone port-Hamiltonian system with state vector $x := \tau q$, quadratic Hamiltonian $H(x) = \frac{1}{2}x^T \tau^{-1} x$, and maximal cyclically monotone relation

$$\mathcal{M} = \left\{ (f, e) \mid f = \frac{\partial P}{\partial e}(e) \right\}, \tag{4.16}$$

where $e = \frac{\partial H}{\partial x}(x) = \tau^{-1}x = q$. This can be extended to include inputs and outputs (e.g., if the gradient algorithm is carried out in a distributed fashion) by considering the function $K(q,u) = P(q) - q^T Bu + \frac{1}{2}u^T Du$ with $D = D^T > 0$. Thanks to strong convexity of P (4.14), K is convex for D large enough. The resulting maximal cyclically monotone port-Hamiltonian system is given as

$$\begin{aligned} \dot{x} &= -\frac{\partial P}{\partial q}(q) + Bu, \quad q = \tau^{-1}x, \\ y &= -B^T q + Du, \end{aligned} \tag{4.17}$$

with maximal cyclically monotone relation

$$\mathcal{M} = \left\{ (f, q, y, u) \mid f = \frac{\partial P}{\partial q}(q) - Bu,\ y = -B^T q + Du \right\}. \tag{4.18}$$

In case $D = 0$ (or any D such that K is convex in q and concave in u) we can alternatively consider the maximal monotone port-Hamiltonian system (cf. (4.8))

$$\begin{aligned} \dot{x} &= -\frac{\partial P}{\partial q}(q) + Bu, \quad q = \tau^{-1}x, \\ y &= B^T q. \end{aligned} \tag{4.19}$$

Example 4.9 (primal-dual gradient algorithm [21]). Consider the *constrained* optimization problem

$$\min_{q;\, Aq=b} P(q), \tag{4.20}$$

where $P : \mathbb{R}^n \to \mathbb{R}$ is a convex function, and $Aq = b$ are affine constraints for some $k \times n$ matrix A and vector $b \in \mathbb{R}^k$. The resulting Lagrangian function is defined as

$$L(q, \lambda) := P(q) + \lambda^T (Aq - b), \quad \lambda \in \mathbb{R}^k, \tag{4.21}$$

which is convex in q and concave in λ. The *primal-dual gradient* algorithm for solving the optimization problem in continuous time is given as

$$\begin{aligned} \tau_q \dot{q} &= -\frac{\partial L}{\partial q}(q, \lambda) = -\frac{\partial P}{\partial q}(q) - A^T \lambda, \\ \tau_\lambda \dot{\lambda} &= \frac{\partial L}{\partial \lambda}(q, \lambda) = Aq - b, \end{aligned} \tag{4.22}$$

where τ_q, τ_λ are positive-definite matrices determining the time-scales of the algorithm. Again, an input vector $Bu \in \mathbb{R}^n$ with conjugate output can be added in order to represent possible interaction with other algorithms or dynamics. Consider thereto the function $K(q,u) = P(q) - q^T Bu$, which is convex in q and concave in u. Furthermore, consider the skew-symmetric matrix $J = \begin{bmatrix} 0 & -A^T \\ A & 0 \end{bmatrix}$. Together this defines a maximal monotone port-Hamiltonian system (4.10), with state vector $x = (x_q, x_\lambda) := (\tau_q q, \tau_\lambda \lambda)$, quadratic Hamiltonian

$$H(x) = \frac{1}{2} x_q^T \tau_q^{-1} x_q + \frac{1}{2} x_\lambda \tau_\lambda^{-1} x_\lambda, \tag{4.23}$$

and maximal monotone relation

$$\mathcal{M} = \left\{ (f,e,y,u) \mid f = \begin{bmatrix} 0 & A^T \\ -A & 0 \end{bmatrix} e - \begin{bmatrix} \frac{\partial P}{\partial q}(q) \\ b \end{bmatrix} - \begin{bmatrix} B \\ 0 \end{bmatrix} u, y = \begin{bmatrix} B^T & 0 \end{bmatrix} e \right\}, \tag{4.24}$$

where

$$e = \nabla H(x) = \begin{bmatrix} \tau_q^{-1} x_q \\ \tau_\lambda^{-1} x_\lambda \end{bmatrix} = \begin{bmatrix} q \\ \lambda \end{bmatrix}. \tag{4.25}$$

For an application, see the optimization of social welfare in a dynamic pricing algorithm for power networks in [22].

5. Monotonicity of Dirac structures and of energy-dissipating relations. In this section we investigate the (maximal) (cyclical) monotonicity of two building blocks of the definition of a port-Hamiltonian system: Dirac structure and energy-dissipating relation. First we show that every Dirac structure is a maximal monotone relation, which moreover is maximal cyclically monotone if and only if it belongs to a special subclass of Dirac structures. This special subclass is defined and characterized as follows [23].

DEFINITION 5.1. *A Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ is separable if*

$$\langle e_a \mid f_b \rangle = 0 \quad \text{for all } (f_a, e_a), (f_b, e_b) \in \mathcal{D}. \tag{5.1}$$

Separable Dirac structures have the following simple geometric characterization [23].

PROPOSITION 5.2. *Any separable Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ can be written as*

$$\mathcal{D} = \mathcal{K} \times \mathcal{K}^\perp \tag{5.2}$$

for some subspace $\mathcal{K} \subset \mathcal{F}$, where $\mathcal{K}^\perp = \{ e \in \mathcal{E} \mid \langle e \mid f \rangle = 0 \, \forall f \in \mathcal{K} \}$. Conversely, any subspace $\mathcal{D}$ as in (5.2) *for some $\mathcal{K} \subset \mathcal{F}$ is a separable Dirac structure.*

Remark 5.3. A typical example of a separable Dirac structure is provided by Kirchhoff's current and voltage laws of an electrical circuit. Indeed, take, e.g., $\mathcal{F}$ to be the space of currents, and $\mathcal{K}$ the space of currents satisfying Kirchhoff's current laws. Then $\mathcal{E} = \mathcal{F}^*$ is the space of voltages, and $\mathcal{K}^\perp$ defines Kirchhoff's voltage laws. Moreover, $\langle e_a \mid f_b \rangle = 0$ for all $(f_a, e_a), (f_b, e_b) \in \mathcal{K} \times \mathcal{K}^\perp$ expresses Tellegen's law.

PROPOSITION 5.4. *Every Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ is maximal monotone. It is maximal cyclically monotone if and only if $\mathcal{D}$ is separable. If $\mathcal{D}$ is the graph of a mapping $J : \mathcal{E} \to \mathcal{F}$ or $J : \mathcal{F} \to \mathcal{E}$, then $\mathcal{D}$ is cyclically monotone if and only if $J = 0$.*

Proof. Let $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ be a Dirac structure. Let $(f_i, e_i) \in \mathcal{D}$ with $i = 1, 2$. Since $\langle e \mid f \rangle = 0$ for all $(f, e) \in \mathcal{D}$ due to (2.1) in Remark 2.2, we obtain by linearity

$$\langle e_1 - e_2 \mid f_1 - f_2 \rangle = 0.$$

Therefore, $\mathcal{D}$ is monotone on $\mathcal{F} \times \mathcal{E}$. Let $\mathcal{D}'$ be a monotone relation on $\mathcal{F} \times \mathcal{E}$ such that $\mathcal{D} \subseteq \mathcal{D}'$. Let $(f', e') \in \mathcal{D}'$ and $(f, e) \in \mathcal{D}$. Since $\mathcal{D}'$ is monotone, $\mathcal{D}$ is a subspace, and $\mathcal{D} \subseteq \mathcal{D}'$, we have

$$0 \leqslant \langle e' - \alpha e \mid f' - \alpha f \rangle = \langle e' \mid f' \rangle - \alpha(\langle e' \mid f \rangle + \langle e \mid f' \rangle)$$

for any $\alpha \in \mathbb{R}$. This means that $\langle e' \mid f \rangle + \langle e \mid f' \rangle = 0$, and hence $(f', e') \in \mathcal{D}^{\perp\!\!\perp} = \mathcal{D}$. Therefore, we see that $\mathcal{D}' \subseteq \mathcal{D}$, and thus $\mathcal{D}' = \mathcal{D}$. Consequently, $\mathcal{D}$ is maximal monotone.

Next, let $\mathcal{D}$ be separable, i.e., $\langle e_a \mid f_b \rangle = 0$ for all $(f_a, e_a), (f_b, e_b) \in \mathcal{D}$. Then it immediately follows from Definition 3.1 that $\mathcal{D}$ is cyclically monotone. Conversely, let $\mathcal{D}$ be cyclically monotone. Then take any $(f_i, e_i) \in \mathcal{D}$ with $i \in \{0, 1, 2\}$. It follows from Definition 3.1 that

$$\langle e_0 \mid f_0 - f_1 \rangle + \langle e_1 \mid f_1 - f_2 \rangle + \langle e_2 \mid f_2 - f_0 \rangle \geqslant 0.$$

Since $\langle e \mid f \rangle = 0$ for all $(f, e) \in \mathcal{D}$ due to Remark 2.2, we see that

$$\langle e_0 \mid -f_1 \rangle + \langle e_1 \mid -f_2 \rangle + \langle e_2 \mid -f_0 \rangle \geqslant 0. \tag{5.3}$$

As $\mathcal{D}$ is a subspace, $(-f_0, -e_0) \in \mathcal{D}$. Therefore, we see from (5.3) that

$$\langle -e_0 \mid -f_1 \rangle + \langle e_1 \mid -f_2 \rangle + \langle e_2 \mid f_0 \rangle \geqslant 0.$$

By summing this inequality and (5.3), we obtain $\langle e_1 \mid -f_2 \rangle \geqslant 0$. By using the fact that $\mathcal{D}$ is a subspace, we see that $\langle e_1 \mid f_2 \rangle = 0$, and thus $\mathcal{D}$ is separable.

Finally, let $\mathcal{D}$ be the graph of a mapping $J : \mathcal{E} \to \mathcal{F}$. Since $\mathcal{D}$ is a Dirac structure, necessarily J is skew-symmetric. Take again any $(f_i, e_i) \in \mathcal{D}$ with $i \in \{0, 1, 2\}$, where now $f_i = Je_i$. Then if $\mathcal{D}$ is cyclically monotone,

$$\langle e_0 \mid J(e_0 - e_1) \rangle + \langle e_1 \mid J(e_1 - e_2) \rangle + \langle e_2 \mid J(e_2 - e_0) \rangle \geqslant 0.$$

Using $\langle e_i \mid Je_i \rangle = 0$ by skew-symmetry of J this yields $\langle e_1 \mid Je_2 \rangle \geqslant 0$ for all e_1, e_2, which clearly implies $J = 0$. The proof for the case $J : \mathcal{F} \to \mathcal{E}$ follows the same line of reasoning. ∎

Remark 5.5. As mentioned before, a typical example of a separable Dirac structure is provided by Kirchhoff's current and voltage laws. In particular, it follows that for any electrical circuit there exists a convex function specifying Kirchhoff's current and voltage laws. Indeed, let the circuit graph be given by its incidence matrix D. Identify as above $\mathcal{F}$ with the set of currents $f = I$ through the edges, and $\mathcal{E} = \mathcal{F}^*$ with the set of voltages $e = V$ across the edges. Then Kirchhoff's current laws are given as $DI = 0$ and Kirchhoff's voltage laws as $V \in \operatorname{im} D^T$. The convex function generating the resulting separable Dirac structure is given by (see the proof of the subsequent Proposition 5.6 for similar arguments)

$$\phi(f) = \begin{cases} 0 & \text{if } f \in \ker D, \\ +\infty & \text{otherwise.} \end{cases} \tag{5.4}$$

An arbitrary energy-dissipating relation need *not* be a (maximal) monotone relation, as was illustrated by some of the examples in the previous section. A *special* type of energy-dissipating relation that *is* a maximal cyclically monotone relation is that of a *linear* energy-dissipating relation which is of *maximal dimension*. Such an energy-dissipating relation in the port-variables $(f,e) \in \mathcal{F} \times \mathcal{E}$ can be represented as a subspace

$$\mathcal{R} = \{(f,e) \in \mathcal{F} \times \mathcal{E} \mid R_f f - R_e e = 0\}, \tag{5.5}$$

where the matrices R_f, R_e satisfy the property

$$R_f R_e^T = R_e R_f^T \geqslant 0, \tag{5.6}$$

together with the dimensionality condition

$$\operatorname{rank} \begin{bmatrix} R_f & R_e \end{bmatrix} = \dim \mathcal{F}. \tag{5.7}$$

First, this is seen to define an energy-dissipating relation as follows. By the dimensionality condition (5.7) and the equality in (5.6) we can equivalently rewrite the kernel representation (5.5) as an image representation

$$f = R_e^T \lambda, \quad e = R_f^T \lambda. \tag{5.8}$$

That is, any pair (f,e) satisfying (5.5) also satisfies (5.8) for some λ, and conversely, every (f,e) satisfying (5.8) for some λ also satisfies (5.5). Hence by (5.6) for all (f,e) satisfying (5.5),

$$e^T f = \left(R_f^T \lambda\right)^T R_e^T \lambda = \lambda^T R_f R_e^T \lambda \geqslant 0. \tag{5.9}$$

A subspace $\mathcal{R} \subset \mathcal{F} \times \mathcal{E}$ as in (5.5) where R_f, R_e satisfy (5.6) and (5.7) is called a *linear resistive structure*. A linear resistive structure can be regarded as a geometric object having properties which are opposite to those of a Dirac structure, in the sense that a Dirac structure can be regarded as the generalization of a *skew-symmetric* map, while a linear resistive relation is the generalization of a positive semidefinite *symmetric* map. (Geometrically $\mathcal{R}$ defines a Lagrangian subspace of the linear space $\mathcal{F} \times \mathcal{E}$.)

It turns out that every linear resistive structure $\mathcal{R} \subset \mathcal{F} \times \mathcal{E}$ is maximal cyclically monotone. To elaborate further, note that there exists $R = R^T \geqslant 0$ such that

$$R_e R R_e^T = R_e R_f^T \tag{5.10}$$

due to [24, Thm. 2.5]. In general, R is not unique, but the matrix RR_e^T does not depend on the choice of $R = R^T \geqslant 0$ satisfying (5.10). Now, define the extended real-valued convex function

$$\phi_{\mathcal{R}}(f) = \begin{cases} \frac{1}{2} f^T R f & \text{if } f \in \operatorname{im} R_e^T, \\ +\infty & \text{otherwise.} \end{cases} \tag{5.11}$$

With these preparations, we obtain

Proposition 5.6. *Let $\mathcal{R} \subset \mathcal{F} \times \mathcal{E}$ be a linear resistive structure. Then $\mathcal{R}$ is generated by $\phi_{\mathcal{R}}$ and hence is maximal cyclically monotone.*

Proof. Clearly, $\phi_{\mathcal{R}}$ is a proper lower semicontinuous function with dom $\phi = \text{im } R_e^T$ and

$$\partial\phi_{\mathcal{R}}(f) = \begin{cases} Rf + \ker R_e & \text{if } f \in \text{im } R_e^T, \\ \varnothing & \text{otherwise.} \end{cases} \tag{5.12}$$

Now, we claim that $\mathcal{R}$ is generated by $\phi_{\mathcal{R}}$ and hence maximal cyclically monotone. To verify this claim, one needs to show that

$$\mathcal{R} = \{(f,e) \mid e \in \partial\phi_{\mathcal{R}}(f)\}. \tag{5.13}$$

To see this, first let $(f,e) \in \mathcal{R}$. Then, we see from (5.8) that $f = R_e^T\lambda$ and $e = R_f^T\lambda$ for some λ. Note that

$$\text{im } (RR_e^T - R_f^T) \subseteq \ker R_e \tag{5.14}$$

due to (5.10). As such there must exist $\mu \in \ker R_e$ such that $R_f^T\lambda = RR_e^T\lambda + \mu$. Therefore, it follows from (5.12) that $e \in \partial\phi_{\mathcal{R}}(f)$. This proves that

$$\mathcal{R} \subseteq \{(f,e) \mid e \in \partial\phi_{\mathcal{R}}(f)\}. \tag{5.15}$$

To see that the reverse inclusion also holds, let (f,e) be such that $e \in \partial\phi_{\mathcal{R}}(f)$. From (5.12), we see that there exist λ and $\mu \in \ker R_e$ such that $f = R_e^T\lambda$ and $e = RR_e^T\lambda + \mu$. Since $\ker R_e \subseteq R_f^T \ker R_e^T$ due to (5.5) and (5.8), it follows from (5.14) that $e = R_f^T(\lambda + \theta)$, where $\theta \in \ker R_e^T$. Note that $f = R_e^T\lambda = R_e^T(\lambda + \theta)$. Consequently, we see that

$$\{(f,e) \mid e \in \partial\phi_{\mathcal{R}}(f)\} \subseteq \mathcal{R}, \tag{5.16}$$

which, together with (5.15), proves (5.13). ◻

Of course, apart from linear resistive structures, there are also *nonlinear* energy-dissipating relations that are maximal monotone or even maximal cyclically monotone. In the latter case, the energy-dissipating relation $\mathcal{R}$ is given as

$$\mathcal{R} = \{(f,e) \mid e \in \partial P(f)\} \tag{5.17}$$

for some extended convex function $P(f)$. In a mechanical system context such functions are often referred to as Rayleigh dissipation functions.

One of the consequences of Theorem 6.2 in the next section will be that the *composition* of a Dirac structure with a linear resistive structure (or, under technical conditions, with a maximal monotone energy-dissipating relation) is maximal monotone, and consequently the corresponding port-Hamiltonian system is a maximal monotone port-Hamiltonian system as well; cf. Proposition 6.4.

6. Composition of monotone relations. A cornerstone of port-Hamiltonian systems theory is the fact that the power-conserving interconnection of port-Hamiltonian systems defines again a port-Hamiltonian system. This in turn is based on the fact that the *composition* of Dirac structures is again a Dirac structure. In this section we will show that the same property holds for incrementally port-Hamiltonian systems. This follows from the corresponding compositionality property of (maximal) (cyclically) monotone relations.

Let us start by considering two monotone relations $\mathcal{M}_a \subset \mathcal{F}_a \times \mathcal{F} \times \mathcal{E}_a \times \mathcal{E}$ and $\mathcal{M}_b \subset \mathcal{F}_b \times \mathcal{F} \times \mathcal{E}_b \times \mathcal{E}$. Define the *composition* of $\mathcal{M}_a$ and $\mathcal{M}_b$, denoted as $\mathcal{M}_a \overset{\mathcal{F}\times\mathcal{E}}{\leftrightarrows} \mathcal{M}_b$, as before, by

$$\mathcal{M}_a \overset{\mathcal{F}\times\mathcal{E}}{\rightleftarrows} \mathcal{M}_b := \{(f_a, f_b, e_a, e_b) \in \mathcal{F}_a \times \mathcal{F}_b \times \mathcal{E}_a \times \mathcal{E}_b \mid \exists (f, e) \in \mathcal{F} \times \mathcal{E} \text{ s.t. } (f_a, f, e_a, e) \in \mathcal{M}_a, (f_b, -f, e_b, e) \in \mathcal{M}_b\}. \tag{6.1}$$

Thus the composition of $\mathcal{M}_a$ and $\mathcal{M}_b$ is obtained by imposing the interconnection constraints

$$f_1 = -f_2, \quad e_1 = e_2 \tag{6.2}$$

on the vectors $(f_a, f_1, e_a, e_1) \in \mathcal{M}_a$ and $(f_b, f_2, e_b, e_2) \in \mathcal{M}_b$ and looking at the resulting vectors $(f_a, f_b, e_a, e_b) \in \mathcal{F}_a \times \mathcal{F}_b \times \mathcal{E}_a \times \mathcal{E}_b$.

Whenever interconnection flow and effort spaces $\mathcal{F}$ and $\mathcal{E}$ are clear from the context, we will simply write $\mathcal{M}_a \leftrightarrows \mathcal{M}_b$. The following result is straightforward.

PROPOSITION 6.1. *Let $\mathcal{M}_a \subset \mathcal{F}_a \times \mathcal{F} \times \mathcal{E}_a \times \mathcal{E}$ and $\mathcal{M}_b \subset \mathcal{F}_b \times \mathcal{F} \times \mathcal{E}_b \times \mathcal{E}$ be (cyclically) monotone relations. Then, $\mathcal{M}_a \rightleftarrows \mathcal{M}_b \subset \mathcal{F}_a \times \mathcal{F}_b \times \mathcal{E}_a \times \mathcal{E}_b$ is (cyclically) monotone.*

Proof. Suppose that both $\mathcal{M}_a$ and $\mathcal{M}_b$ are monotone relations. Let

$$(f_a, f_b, e_a, e_b), (\bar{f}_a, \bar{f}_b, \bar{e}_a, \bar{e}_b) \in \mathcal{M}_a \rightleftarrows \mathcal{M}_b.$$

Then, there exist $(f, e), (\bar{f}, \bar{e}) \in \mathcal{F} \times \mathcal{E}$ such that $(f_a, f, e_a, e), (\bar{f}_a, \bar{f}, \bar{e}_a, \bar{e}) \in \mathcal{M}_a$ and $(f_b, -f, e_b, e), (\bar{f}_b, -\bar{f}, \bar{e}_b, \bar{e}) \in \mathcal{M}_b$. From monotonicity of $\mathcal{M}_a$ and $\mathcal{M}_b$, we have

$$\left\langle \begin{bmatrix} e_a - \bar{e}_a \\ e - \bar{e} \end{bmatrix} \mid \begin{bmatrix} f_a - \bar{f}_a \\ f - \bar{f} \end{bmatrix} \right\rangle \geqslant 0 \quad \text{and} \quad \left\langle \begin{bmatrix} e_b - \bar{e}_b \\ e - \bar{e} \end{bmatrix} \mid \begin{bmatrix} f_b - \bar{f}_b \\ -f + \bar{f} \end{bmatrix} \right\rangle \geqslant 0.$$

By adding these left-hand sides of these inequalities, we obtain

$$\left\langle \begin{bmatrix} e_a - \bar{e}_a \\ e_b - \bar{e}_b \end{bmatrix} \mid \begin{bmatrix} f_a - \bar{f}_a \\ f_b - \bar{f}_b \end{bmatrix} \right\rangle \geqslant 0.$$

This means that $\mathcal{M}_a \rightleftarrows \mathcal{M}_b$ is monotone. The cyclical monotone case follows in a similar fashion. ∎

Also the composition of two *maximal* monotone relations turns out to be maximal monotone, provided certain (mild) regularity conditions are met. To elaborate on this, we first introduce some nomenclature and review some known facts about maximal monotone relations.

For a set $S \in \mathcal{F}$, cl S denotes its *closure*. The *relative interior* of a convex set $C \subseteq \mathcal{F}$ is denoted by rint C. A set $S \subseteq \mathcal{F}$ is said to be *nearly convex* if there exists a convex set $C \subseteq \mathcal{F}$ such that $C \subseteq S \subseteq \text{cl } C$. For a nearly convex set S, in general, there can be multiple convex sets C satisfying $C \subseteq S \subseteq \text{cl } C$. For any such set C, however, we have that cl $C =$ cl S. As such, cl S is convex if S is nearly convex. Based on this observation, one can extend the notion of relative interior to nearly convex sets by defining rint $S =$ rint (cl S).

Let $S \subseteq \mathcal{F}_1 \times \mathcal{F}_2 \times \mathcal{E}_1 \times \mathcal{E}_2$. The projection of S on $\mathcal{F}_1 \times \mathcal{F}_2$, denoted by $\Pi(S, \mathcal{F}_1 \times \mathcal{F}_2)$, is defined as

$$\Pi(S, \mathcal{F}_1 \times \mathcal{F}_2) := \{(f_1, f_2) \mid \exists (e_1, e_2) \in \mathcal{E}_1 \times \mathcal{E}_2 \text{ s.t. } (f_1, f_2, e_1, e_2) \in S\}.$$

We define projections of S on $\mathcal{E}_1 \times \mathcal{E}_2$, $\mathcal{F}_1 \times \mathcal{E}_2$, $\mathcal{F}_1$, and $\mathcal{E}_2$ in a similar fashion.

Let $S \subseteq \mathcal{F} \times \mathcal{G}$ be a nearly convex set. Then, both $\Pi(S, \mathcal{F})$ and $\Pi(S, \mathcal{G})$ are nearly convex sets. Furthermore, one can show that

$$\text{rint } S = \{(f,g) \mid f \in \text{rint } \Pi(S,\mathcal{F}) \text{ and } g \in \text{rint } \Pi(S \cap (\{f\} \times \mathcal{G}), \mathcal{G})\}. \tag{6.3}$$

Let $\mathcal{M} \subset \mathcal{F} \times \mathcal{E}$ be a maximal monotone relation. Then, the projections $\Pi(\mathcal{M}, \mathcal{F})$ and $\Pi(\mathcal{M}, \mathcal{E})$ are nearly convex sets [9, Thm. 12.41].

Let $L : \mathcal{G} \to \mathcal{H}$ be a linear map, and let $L^* : \mathcal{H}^* \to \mathcal{G}^*$ denote its adjoint. For maximal monotone relations $\mathcal{M} \subseteq \mathcal{H} \times \mathcal{H}^*$ and $\mathcal{N} \subseteq \mathcal{G} \times \mathcal{G}^*$, define $\mathcal{M}_L \subseteq \mathcal{G} \times \mathcal{G}^*$ and ${}_L\mathcal{N} \subseteq \mathcal{H} \times \mathcal{H}^*$ by

$$\mathcal{M}_L = \{(g, L^*h^*) \mid (Lg, h^*) \in \mathcal{M}\},$$
$${}_L\mathcal{N} = \{(Lg, h^*) \mid (g, L^*h^*) \in \mathcal{N}\}.$$

From [9, Thm. 12.43], we know that $\mathcal{M}_L$ is maximal monotone if

$$\text{im } L \cap \text{rint } \Pi(\mathcal{M}, \mathcal{H}) \neq \varnothing, \tag{6.4}$$

and that ${}_L\mathcal{N}$ is maximal monotone if

$$\text{im } L^* \cap \text{rint } \Pi(\mathcal{N}, \mathcal{G}^*) \neq \varnothing. \tag{6.5}$$

Furthermore, if $\mathcal{M}$ is generated by $\phi : \mathcal{H} \to (-\infty, +\infty]$ and

$$\text{im } L \cap \text{rint dom } (\phi) \neq \varnothing, \tag{6.6}$$

then $\mathcal{M}_L$ is generated by $\phi \circ L$ given by $h \mapsto \phi(Lh)$. Dually, if $\mathcal{N}$ is generated by $\psi : \mathcal{G} \to (-\infty, +\infty]$ and

$$\text{im } L^* \cap \text{rint dom } (\psi^\star) \neq \varnothing, \tag{6.7}$$

then ${}_L\mathcal{N}$ is generated by the function $(\psi^\star \circ L^*)^\star$.

THEOREM 6.2. *Let $\mathcal{M}_a \subset \mathcal{F}_a \times \mathcal{F} \times \mathcal{E}_a \times \mathcal{E}$ and $\mathcal{M}_b \subset \mathcal{F}_b \times \mathcal{F} \times \mathcal{E}_b \times \mathcal{E}$ be maximal monotone relations. Let*

$$C_f = \{(f_1, f_2) \mid f_1 \in \Pi(\mathcal{M}_a, \mathcal{F}) \textit{ and } f_2 \in \Pi(\mathcal{M}_b, \mathcal{F})\}$$

and

$$C_e = \{(e_1, e_2) \mid \exists f \textit{ s.t. } (f, e_1) \in \Pi(\mathcal{M}_a, \mathcal{F} \times \mathcal{E}) \textit{ and } (-f, e_2) \in \Pi(\mathcal{M}_b, \mathcal{F} \times \mathcal{E})\}.$$

Suppose that there exists $(\bar{f}, \bar{e}) \in \mathcal{F} \times \mathcal{E}$ such that
(i) $(\bar{f}, -\bar{f}) \in \text{rint } C_f$ *and*
(ii) $(\bar{e}, \bar{e}) \in \text{rint } C_e$.
Then, $\mathcal{M}_a \rightleftarrows \mathcal{M}_b \subset \mathcal{F}_a \times \mathcal{F}_b \times \mathcal{E}_a \times \mathcal{E}_b$ is a maximal monotone relation.

Proof. First, we give an alternative characterization of $\mathcal{M}_a \rightleftarrows \mathcal{M}_b$. Let $\mathcal{M} \subset \mathcal{F}_a \times \mathcal{F} \times \mathcal{F}_b \times \mathcal{F} \times \mathcal{E}_a \times \mathcal{E} \times \mathcal{E}_b \times \mathcal{E}$ be defined by

$$\mathcal{M} := \{(f_a, f_1, f_b, f_2, e_a, e_1, e_b, e_2) \mid (f_a, f_1, e_a, e_1) \in \mathcal{M}_a \text{ and } (f_b, f_2, e_b, e_2) \in \mathcal{M}_b\}.$$

Since $\mathcal{M}_a$ and $\mathcal{M}_b$ are both maximal monotone, so is $\mathcal{M}$. Let $A : \mathcal{F}_a \times \mathcal{F} \times \mathcal{F}_b \to \mathcal{F}_a \times \mathcal{F} \times \mathcal{F}_b \times \mathcal{F}$ be the linear map given by

$$(f_a, f, f_b) \mapsto (f_a, f, f_b, -f),$$

and let $B:\mathcal{F}_a\times\mathcal{F}\times\mathcal{F}_b\to\mathcal{F}_a\times\mathcal{F}_b$ be the linear map given by

$$(f_a,f,f_b)\mapsto(f_a,f_b).$$

Note that $A^*:\mathcal{E}_a\times\mathcal{E}\times\mathcal{E}_b\times\mathcal{E}\to\mathcal{E}_a\times\mathcal{E}\times\mathcal{E}_b$ is given by

$$(e_a,e_1,e_b,e_2)\mapsto(e_a,e_1-e_2,e_b)$$

and $B^*:\mathcal{E}_a\times\mathcal{E}_b\to\mathcal{E}_a\times\mathcal{E}\times\mathcal{E}_b$ is given by

$$(e_a,e_b)\mapsto(e_a,0,e_b).$$

Now, we claim that

$$\mathcal{M}_a\rightleftarrows\mathcal{M}_b={}_B(\mathcal{M}_A).$$

To see this, note that

$$\begin{aligned}\mathcal{M}_A&=\{(f_a,f,f_b,e_a,e_1-e_2,e_b)\mid(f_a,f,f_b,-f,e_a,e_1,e_b,e_2)\in\mathcal{M}\}\\&=\{(f_a,f,f_b,e_a,e_1-e_2,e_b)\mid(f_a,f,e_a,e_1)\in\mathcal{M}_a\\&\qquad\text{and }(f_b,-f,e_b,e_2)\in\mathcal{M}_b\}\end{aligned}$$

and

$$\begin{aligned}{}_B(\mathcal{M}_A)&=\{(f_a,f_b,e_a,e_b)\mid\exists f\in\mathcal{F}\text{ s.t. }(f_a,f,f_b,e_a,0,e_b)\in\mathcal{M}_a\}\\&=\{(f_a,f_b,e_a,e_b)\mid\exists(f,e)\in\mathcal{F}\times\mathcal{E}\text{ s.t. }(f_a,f,e_a,e)\in\mathcal{M}_a\\&\qquad\text{and }(f_b,-f,e_b,e)\in\mathcal{M}_b\}\\&=\mathcal{M}_a\rightleftarrows\mathcal{M}_b.\end{aligned}\tag{6.8}$$

where $A:\mathcal{F}_a\times\mathcal{V}_a\times\mathcal{F}_b\times\mathcal{V}_b\to\mathcal{F}_a\times\mathcal{V}_a\times\mathcal{V}_c\times\mathcal{V}_d\times\mathcal{F}_b\times\mathcal{V}_b$ is the linear map given by

$$(f_a,v_a,f_b,v_b)\mapsto(f_a,v_a,v_a,v_b,f_b,v_b).$$

Since $\mathcal{M}$ is maximal monotone, we see from (6.4) that $\mathcal{M}_A$ is maximal monotone if

$$\text{im }A\cap\text{rint }\Pi(\mathcal{M},\mathcal{F}_a\times\mathcal{F}\times\mathcal{F}_b\times\mathcal{F})\neq\varnothing.\tag{6.9}$$

From (6.3), it follows that

$$\begin{aligned}\text{rint }S_A=\{(f_a,f_1,f_b,f_2)\mid&(f_1,f_2)\in\text{rint }\Pi(S_A,\mathcal{F}\times\mathcal{F})\\&\text{and }(f_a,f_b)\in\text{rint }\Pi(S_A\cap(\mathcal{F}_a\times\{f_1\}\times\mathcal{F}_b\times\{f_2\}),\mathcal{F}_a\times\mathcal{F}_b)\},\end{aligned}$$

where $S_A=\Pi(\mathcal{M},\mathcal{F}_a\times\mathcal{F}\times\mathcal{F}_b\times\mathcal{F})$. By observing that $\Pi(S_A,\mathcal{F}\times\mathcal{F})=\Pi(\mathcal{M},\mathcal{F}\times\mathcal{F})=C_f$, we see that the condition (6.9) is equivalent to the existence of $\bar f\in\mathcal{F}$ such that $(\bar f,-\bar f)\in\text{rint }C_f$. Therefore, $\mathcal{M}_A$ is maximal monotone due to (i). As such, it follows from (6.5) that ${}_B(\mathcal{M}_A)$, and thus $\mathcal{M}_a\rightleftarrows\mathcal{M}_b$, is maximal monotone if

$$\text{im }B^*\cap\text{rint }\Pi(\mathcal{M}_A,\mathcal{E}_a\times\mathcal{E}\times\mathcal{E}_b)\neq\varnothing.\tag{6.10}$$

To verify this condition, let $S_B=\Pi(\mathcal{M}_A,\mathcal{E}_a\times\mathcal{E}\times\mathcal{E}_b)$, and note that

$$\begin{aligned}\text{rint }S_B=\{(e_a,e,e_b)\mid&e\in\text{rint }\Pi(S_B,\mathcal{E})\\&\text{and }(e_a,e_b)\in\text{rint }\Pi(S_B\cap(\mathcal{E}_a\times\{e\}\times\mathcal{E}_b))\}.\end{aligned}$$

Therefore, (6.10) holds if and only if $0 \in \operatorname{rint} \Pi(S_B, \mathcal{E})$. Note that

$$\Pi(S_B, \mathcal{E}) = \{e \mid \exists\, (e_1, e_2) \text{ s.t. } e = e_1 - e_2\}.$$

As such, (ii) is equivalent to $0 \in \operatorname{rint} \Pi(S_B, \mathcal{E})$ and hence (6.10). Consequently, $\mathcal{M}_a \rightleftarrows \mathcal{M}_b$ is maximal monotone. ∎

Example 6.3. Consider an incrementally port-Hamiltonian system with maximal monotone relation $\mathcal{M}$. Let us split f_P into a vector u and a vector z, and correspondingly e_P into a vector y and w. Then "terminate" the equally dimensioned vectors z and w on a maximal monotone relation $\mathcal{M}'$ by imposing the constraint $(-z, w) \in \mathcal{M}'$. Under the assumptions specified in Theorem 6.2 the resulting system is again an incrementally port-Hamiltonian system with respect to the maximal monotone relation that is determined by $\mathcal{M}$ and $\mathcal{M}'$. This is very similar to the setting in [12], where the interconnection of a linear passive system to a maximal monotone relation was considered.

Of course, by repeated application of Theorem 6.2 it can be shown that the interconnection of *multiple* maximal monotone relations is also maximal monotone. Furthermore, Theorem 6.2 can be extended to certain interconnections other than the standard interconnection $f_1 = -f_2$, $e_1 = e_2$ in (6.2). Especially the extension to the "feedback" interconnection

$$f_1 = e_2 \quad \text{and} \quad e_1 = -f_2 \tag{6.11}$$

of maximal monotone relations $\mathcal{M}_a$ and $\mathcal{M}_b$ is of interest. Denote the variables associated with $\mathcal{M}_a$ by f_a, e_a, f_1, e_1, and those of $\mathcal{M}_b$ by f_b, e_b, f_2, e_2. Then the interconnection (6.11) can be realized by first defining the *auxiliary system* ("symplectic gyrator")

$$f_1' = -e_2',\; e_1' = f_2', \tag{6.12}$$

which obviously defines a maximal monotone relation (in fact, a Dirac structure), and then interconnecting this auxiliary system to $\mathcal{M}_a$ and $\mathcal{M}_b$ by the standard interconnections

$$f_1 = -f_1',\; e_1 = e_1', \quad -f_2 = f_2,\; e_2' = e_2.$$

Hence maximal monotonicity is also preserved under the "feedback" interconnection (6.11).

Finally, an important consequence of Theorem 6.2 is that a port-Hamiltonian system with Dirac structure $\mathcal{D}$ and energy-dissipating relation $\mathcal{R}$ is maximal monotone port-Hamiltonian provided that $\mathcal{D} \rightleftarrows \mathcal{R}$ is maximal monotone. In particular we have the following result.

PROPOSITION 6.4. *Let $\mathcal{R} \subset \mathcal{F} \times \mathcal{E}$ be a linear resistive structure and $\mathcal{D} \subset \mathcal{F}' \times \mathcal{F} \times \mathcal{E}' \times \mathcal{E}$ be a Dirac structure. Then, the composition $\mathcal{D} \rightleftarrows \mathcal{R}$ is maximal monotone. Therefore port-Hamiltonian systems with linear resistive structures are maximal monotone.*

Proof. Note first that $\mathcal{R}$ is maximal monotone. Since both $\mathcal{D}$ and $\mathcal{R}$ are subspaces, the sets $\Pi(\mathcal{D}, \mathcal{F})$, $\Pi(\mathcal{R}, \mathcal{F})$, $\Pi(\mathcal{D}, \mathcal{F} \times \mathcal{E})$, and $\Pi(\mathcal{R}, \mathcal{F} \times \mathcal{E}) = \mathcal{R}$ are all subspaces. As such, the conditions (i) and (ii) of Theorem 6.2 are trivially satisfied by the choices $\bar{f} = 0 = \bar{e}$. Consequently, the composition $\mathcal{D} \rightleftarrows \mathcal{R}$ is maximal monotone. ∎

Remark 6.5. Monotonicity is closely related to *contraction*. Consider a monotone relation $\mathcal{M}\subset\mathcal{F}\times\mathcal{E}$ in the vectors $f\in\mathcal{F}$ and $e\in\mathcal{E}$. Define now the *scattering* vectors v,z by

$$f=\frac{1}{\sqrt{2}}(v-z),\ e=\frac{1}{\sqrt{2}}(v+z).$$

Then the monotonicity property $\langle e_1-e_2\,|\,f_1-f_2\rangle\geqslant 0$ is immediately seen to be equivalent to the *contraction* property

$$\|z_1-z_2\|^2\leqslant\|v_1-v_2\|^2.$$

Furthermore, the interconnection constraints $f_1=-f_2, e_1=e_2$ of two monotone relations translate into the constraints $v_2=z_1, v_1=z_2$ for the corresponding scattering vectors, and the resulting composition into the *Redheffer star product* of the two contractions; cf. [25] for similar developments in the context of composition of Dirac structures.

Not only is maximal monotonicity preserved under composition by Theorem 6.2, but the same holds for maximal *cyclical* monotonicity, as stated in the following theorem. Furthermore, the generating function of the composition can be computed on the basis of the generating functions of the constitutive maximal cyclically monotone relations.

THEOREM 6.6. *Let $\mathcal{M}_a\subset\mathcal{F}_a\times\mathcal{F}\times\mathcal{E}_a\times\mathcal{E}$ and $\mathcal{M}_b\subset\mathcal{F}_b\times\mathcal{F}\times\mathcal{E}_b\times\mathcal{E}$ be maximal cyclically monotone relations that are generated by proper lower semicontinuous convex functions $\phi_a:\mathcal{F}_a\times\mathcal{F}\to(-\infty,+\infty]$ and $\phi_b:\mathcal{F}_b\times\mathcal{F}\to(-\infty,+\infty]$, respectively. Let*

$$C_f=\{(f_1,f_2)\,|\,f_1\in\Pi(\text{dom }\phi_a,\mathcal{F})\ \textit{and}\ f_2\in\Pi(\text{dom }\phi_a,\mathcal{F})\}$$

and

$$C_e=\{(e_1,e_2)\,|\,\exists\, f\ \text{s.t.}\ (f,e_1)\in\text{dom }\phi_a\ \textit{and}\ (-f,e_2)\in\text{dom }\phi_b\}.$$

Suppose that there exists $(\bar{f},\bar{e})\in\mathcal{F}\times\mathcal{E}$ such that
(i) $(\bar{f},-\bar{f})\in\text{rint }C_f$ *and*
(ii) $(\bar{e},\bar{e})\in\text{rint }C_e$.
Then, $\mathcal{M}_a\rightleftarrows\mathcal{M}_b\subset\mathcal{F}_a\times\mathcal{F}_b\times\mathcal{E}_a\times\mathcal{E}_b$ is a maximal cyclically monotone relation that is generated by $\theta^\star:\mathcal{F}_a\times\mathcal{F}_b\to(-\infty,+\infty]$, where $\theta:\mathcal{E}_a\times\mathcal{E}_b\to(-\infty,+\infty]$ is given by

$$\theta(e_a,e_b)=\phi^\star(e_a,0,e_b)$$

and $\phi:\mathcal{F}_a\times\mathcal{F}\times\mathcal{F}_b\to(-\infty,+\infty]$ is given by

$$\phi(f_a,f,f_b)=\phi_a(f_a,f)+\phi_b(f_b,-f).$$

Proof. Let $\mathcal{M}$, A, B, $\mathcal{M}_A$, and ${}_B(\mathcal{M}_A)$ be as in the proof of Theorem 6.2. Note that $\mathcal{M}$ is generated by the proper lower semicontinuous convex function $\phi_{ab}:\mathcal{F}_a\times\mathcal{F}\times\mathcal{F}_b\times\mathcal{F}\to(-\infty,+\infty]$ given by

$$\phi_{ab}(f_a,f_1,f_b,f_2)=\phi_a(f_a,f_1)+\phi_b(f_b,f_2).$$

For a proper convex function $\Psi : \mathcal{G} \to (-\infty, +\infty]$ and a linear map $L : \mathcal{H} \to \mathcal{G}$, let $\psi \circ L : \mathcal{H} \to (-\infty, +\infty]$ denote the function given by $h \mapsto \psi(Lh)$. It follows from the definition of $\mathcal{M}_A$ that $(f_a, f, f_b, e_a, e, e_b)$ if and only if

$$(e_a, e, e_b) \in A^* \partial\phi_{ab}\big(A(f_a, f, f_b)\big). \tag{6.13}$$

Arguments similar to those employed in the proof Theorem 6.2 show that (i) is equivalent to

$$\text{im } A \cap \text{rint dom } \phi_{ab} \neq \varnothing.$$

Then, it follows from [26, Prop. 5.4.5] that $A^*\partial\phi_{ab}(Ax) = \partial(\phi_{ab} \circ A)(x)$ for all $x \in \mathcal{F}_a \times \mathcal{F} \times \mathcal{F}_b$. Since ϕ_{ab} is lower semicontinuous, so is $\phi_{ab} \circ A$. As such, we see from (6.13) that $\mathcal{M}_A$ is maximal cyclically monotone and generated by $\phi = \phi_{ab} \circ A$. Now, it follows from (6.8), the definition of ${}_B(\mathcal{M}_A)$, and (3.5) that $(f_a, f_b, e_a, e_b) \in \mathcal{M}_a \rightleftarrows \mathcal{M}_b$ if and only if

$$(f_a, f_b) \in B\partial\phi^\star\big(B^*(e_a, e_b)\big) \tag{6.14}$$

One can show that (ii) is equivalent to

$$\text{im } B^* \cap \text{rint dom } \phi^\star \neq \varnothing$$

by employing arguments similar to those in the proof of Theorem 6.2. Then, it follows from [26, Prop. 5.4.5] that $B\partial\phi^\star(B^*y) = \partial(\phi^\star \circ B^*)(y)$ for all $y \in \mathcal{F}_a \times \mathcal{F}_b$. Since $\phi^\star$ is lower semicontinuous, so is $\phi^\star \circ B^*$. Consequently, (6.14) and (3.5) imply that $\mathcal{M}_a \rightleftarrows \mathcal{M}_b$ is maximal cyclically monotone and generated by $(\phi^\star \circ B^*)^\star$. Since $\theta = \phi^\star \circ B^*$, this concludes the proof. ∎

Example 6.7. Consider two gradient algorithms in continuous time with external forcing terms, as given in (4.17):

$$\begin{aligned} \tau_i \dot{q}_i &= -\frac{\partial P_i}{\partial q_i}(q_i) + B_i u_i, \\ y_i &= -B_i^T q_i + D_i u_i, \qquad i = 1, 2. \end{aligned} \tag{6.15}$$

Consider the *coupled* gradient algorithm that results from the interconnection $y_1 = y_2, u_1 = -u_2 := u$. According to Theorem 6.6 the interconnection is again a maximal cyclically monotone port-Hamiltonian system, whose generating function $K(q_1, q_2)$ can be computed on the basis of the generating functions

$$K_i(q_i, u_i) = P_i(q_i) - q_i^T B_i u_i + \frac{1}{2} u_i^T D_i u_i, \quad i = 1, 2,$$

of the two gradient algorithms. In fact, following Theorem 6.6 we consider

$$\begin{aligned} \phi(q_1, u, q_2) &:= K_1(q_1, u) + K_2(q_2, -u) \\ &= P_1(q_1) + P_2(q_2) - q_1^T B_1 u + q_2^T B_2 u + \frac{1}{2} u_1^T D_1 u_1 + \frac{1}{2} u_2^T D_2 u_2, \end{aligned}$$

compute its conjugate, substitute $y = 0$ (with y the vector dual to u), and then take the conjugate $K(q_1, q_2)$ of the resulting convex function. A direct computation yields

$$K(q_1, q_2) = P_1(q_1) + P_2(q_2) - \frac{1}{2}\left(q_1^T B_1 - q_2^T B_2\right)(D_1 + D_2)^{-1}\left(B_1^T q_1 - B_2^T q_2\right).$$

Hence the coupling of the two gradient algorithms computes the minimum of $K(q_1, q_2)$.

7. Steady-state analysis of incrementally port-Hamiltonian systems. In this section we utilize the theory from the previous section to analyze the set of *steady states* of an incrementally port-Hamiltonian system, as well as the equilibria of the *interconnection* of incrementally port-Hamiltonian systems. For simplicity of exposition, we will denote throughout this section $\mathcal{Y} := \mathcal{F}_P, \mathcal{U} := \mathcal{E}_P$, and correspondingly set $y = f_P, u = e_P$.

7.1. The steady-state input-output relation. First recall the notion of steady-state input-output relation of an input-state-output system

$$\Sigma: \begin{array}{rcll} \dot{x} &=& f(x,u), & x \in \mathbb{R}^n, u \in \mathbb{R}^m, \\ y &=& h(x,u), & y \in \mathbb{R}^m. \end{array} \tag{7.1}$$

Consider any constant input vector $\bar{u}$ for which there exists an $\bar{x} \in \mathbb{R}^n$ with $0 = f(\bar{x},\bar{u})$, and denote $\bar{y} = h(\bar{x},\bar{u})$. Then the set of all such pairs $(\bar{y},\bar{u})$, i.e.,

$$\mathcal{G} = \{(\bar{y},\bar{u}) \mid \exists \bar{x},\, 0 = f(\bar{x},\bar{u}),\, \bar{y} = h(\bar{x},\bar{u})\}, \tag{7.2}$$

is called the *steady-state input-output relation* of Σ; cf. [2].

In the case of an incrementally port-Hamiltonian system more can be said about $\mathcal{G}$. First we note the following direct applications of Theorems 6.2 and 6.6.

COROLLARY 7.1. *Consider an incrementally port-Hamiltonian system with underlying maximal monotone relation* $\mathcal{M} \subset \mathcal{F} \times \mathcal{Y} \times \mathcal{E} \times \mathcal{U}$. *Assume* $\mathcal{M}$ *satisfies*

$$0 \in \operatorname{rint} \Pi(\mathcal{M}, \mathcal{F}) \tag{7.3}$$

and there exists $\bar{e}$ *such that*

$$\bar{e} \in \operatorname{rint} \{e \mid (0,e) \in \Pi(\mathcal{M}, \mathcal{F} \times \mathcal{E})\}. \tag{7.4}$$

Then

$$\mathcal{M}^s := \{(y,u) \mid \exists e \text{ such that } (0,y,e,u) \in \mathcal{M}\} \tag{7.5}$$

is a maximal monotone relation. Furthermore, if the maximal monotone relation is cyclically monotone, and thus is given as the graph of the subdifferential of some convex function $K(e,u)$*, then*

$$\mathcal{M}^s = \operatorname{graph}\, (\partial K_s), \tag{7.6}$$

where the convex function $K_s : \mathcal{U} \to \mathbb{R}$ *is given as*

$$K_s(u) = K^\star(0,u),$$

with $K^\star(f,u)$ *the partial conjugate of* $K(e,u)$ *with respect to* e*, i.e.,*

$$K^\star(f,u) := \sup_e \langle f \mid e \rangle - K(e,u). \tag{7.7}$$

Proof. First note that $\mathcal{M}^s$ is the composition of $\mathcal{M}$ with the trivial maximally monotone relation $\{(0,e) \mid e \in \mathcal{E}\}$. Thus in order to apply Theorem 6.2 we need to show that there exists $(\bar{f},\bar{e}) \in \mathcal{F} \times \mathcal{E}$ such that (following the notation of Theorem 6.2)

$$\text{(i)} \quad (\bar{f}, -\bar{f}) \in \operatorname{rint} \mathcal{D}_f,$$

$$\text{(ii)} \quad (\bar{e}, \bar{e}) \in \operatorname{rint} \mathcal{D}_e,$$

where $\mathcal{D}_f = \Pi(\mathcal{M}, \mathcal{F}) \times 0$ and $\mathcal{D}_e = \{(e_1, e_2) \mid (0, e_1) \in \Pi(\mathcal{M}, \mathcal{F} \times \mathcal{E})\}$. It is easily seen that conditions (i), (ii) reduce to (7.3), (7.4). The rest of the proof follows from Theorem 6.6. ∎

It is directly seen that the steady-state input-output relation $\mathcal{G}$ of the incrementally port-Hamiltonian system with maximal monotone relation $\mathcal{M}$ is *contained* in the maximal monotone relation $\mathcal{M}^s$. Indeed, if $(\bar{y}, \bar{u}) \in \mathcal{G}$, then there exists $\bar{x}$ such that $(0, \bar{y}, \frac{\partial H}{\partial x}(\bar{x}), \bar{u}) \in \mathcal{M}$, and thus $(\bar{y}, \bar{u}) \in \mathcal{M}^s$. Consequently, $\mathcal{G}$ is always *monotone*. However $\mathcal{M}^s$ may be *larger* than $\mathcal{G}$, since there may not exist for every e such that $(0, e, \bar{y}, \bar{u}) \in \mathcal{M}$ an $\bar{x}$ such that $e = \frac{\partial H}{\partial x}(\bar{x})$. A simple example is provided by the nonlinear integrator (cf.(4.9)) $\dot{x} = u$, $y = \frac{\partial H}{\partial x}(x)$. This is a maximal monotone port-Hamiltonian system with

$$\mathcal{M} = \{(f, y, e, u) \mid f = -u, y = e\},$$

and thus $\mathcal{M}^s = \{(y, u) \mid u = 0\}$. If H is such that the mapping $x \mapsto \frac{\partial H}{\partial x}(x)$ is *not* surjective, then $\mathcal{G}$ is strictly contained in $\mathcal{M}^s$. (The condition $\mathcal{G} = \mathcal{M}^s$ shows up in the definition of *maximal* equilibrium independent passivity as given in [13]; see section 8.1.)

7.2. Determination of the equilibria of interconnected incrementally port-Hamiltonian systems. In this subsection we analyze how the equilibria of the interconnection of maximal cyclically monotone port-Hamiltonian systems can be computed, by solving a convex optimization problem. This subsection is motivated by some of the developments in [13].

Consider k maximal monotone port-Hamiltonian systems with input and output vectors $u_i \in \mathbb{R}^{m_i}, y_i \in \mathbb{R}^{m_i}$, and maximal monotone relations $\mathcal{M}_i, i = 1, \ldots, k$. Let, as before, $\mathcal{M}_i^s \subset \mathbb{R}^{m_i} \times \mathbb{R}^{m_i}, i = 1, \ldots, k$, be maximal monotone relations. Additionally, assume that $\mathcal{M}_i^s, i = 1, \ldots, k$, are maximal *cyclically* monotone, and thus given as the graphs of subdifferentials of convex functions $K_i(u_i), i = 1, \ldots, k$.

Consider now an interconnection of the following general type. For any subset $\pi \subset \{1, \ldots, k\}$ define

$$\begin{aligned} f_i &:= u_i, \quad i \in \pi, \quad & f_i &:= y_i, \quad i \notin \pi, \\ e_i &:= y_i, \quad i \in \pi, \quad & e_i &:= u_i, \quad i \notin \pi. \end{aligned} \tag{7.8}$$

Furthermore, consider any subspace $\mathcal{C}$ of the linear space of variables $(f_1, \ldots, f_k) \in \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_k}$, and define *interconnection constraints*

$$(f_1, \ldots, f_k) \in \mathcal{C}, \quad (e_1, \ldots, e_k) \in \mathcal{C}^{\perp}. \tag{7.9}$$

Define the convex function $K : \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_k} \to \mathbb{R} \cup \{\infty\}$ given as

$$K(f_1, \ldots, f_k) := \sum_{i \in \pi} K_i(u_i) + \sum_{i \notin \pi} K_i^{\star}(y_i), \tag{7.10}$$

where as before (see (3.4)) $K_i^{\star}(y_i)$ denotes the conjugate of $K_i(u_i)$. Now consider the minimization

$$\min_{(f_1, \ldots, f_k) \in \mathcal{C}} K(f_1, \ldots, f_k), \tag{7.11}$$

and write $\mathcal{C} = \ker C$ for some constraint matrix C with rows $C_1, \ldots, C_k$. Then the minimization is equivalent to the *unconstrained* minimization

$$\min_{f,\lambda} K(f_1, \ldots, f_k) - \sum_{i=1}^{s} \lambda_i^T C_i f_i, \tag{7.12}$$

where λ is a vector of Lagrange multipliers. This yields the first-order optimality conditions

$$\begin{aligned} 0 &\in \tfrac{\partial K_i}{\partial u_i}(u_i) - C_i^T \lambda, \quad i \in \pi, \\ 0 &\in \tfrac{\partial K_i^\star}{\partial y_i}(y_i) - C_i^T \lambda, \quad i \notin \pi. \end{aligned} \tag{7.13}$$

Consider a solution $(\bar{f}_1, \ldots, \bar{f}_k) \in \mathcal{C}$ of these first-order optimality conditions. Hence there exist $\bar{e}_i = \bar{y}_i \in \frac{\partial K_i}{\partial u_i}(\bar{u}_i), i \in \pi$, and $\bar{e}_i = \bar{u}_i \in \frac{\partial K_i^\star}{\partial y_i}(\bar{y}_i), i \notin \pi$, such that $\bar{e} \in \operatorname{im} C^T$, which is nothing else than $\bar{e} \in \mathcal{C}^\perp$. This is summarized in the following theorem.

Theorem 7.2. *Consider k maximal cyclically monotone port-Hamiltonian systems with input and output variables $u_1, \ldots, u_k, y_1, \ldots, y_k$, where $u_i \in \mathbb{R}^{m_i}, y_i \in \mathbb{R}^{m_i}, i = 1, \ldots, k$. Let the maximal cyclically monotone relations $\mathcal{M}_i^s \subset \mathbb{R}^{m_i} \times \mathbb{R}^{m_i}$ be given as the graphs of subdifferentials ∂K_i for convex functions $K_i, i = 1, \ldots, k$. Furthermore, let $\pi \subset \{1, \ldots, k\}$ be an index set and consider any constraint subspace $\mathcal{C} \subset \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_k}$ leading to the interconnection*

$$(f_1, \ldots, f_k) \in \mathcal{C}, \ (e_1, \ldots, e_k) \in \mathcal{C}^\perp.$$

Then if $(\bar{f}_1, \ldots, \bar{f}_k) \in \mathcal{C}$ is a solution of the minimization

$$\min_{(f_1, \ldots, f_k) \in \mathcal{C}} K(f_1, \ldots, f_k),$$

then there exists $(\bar{e}_1, \ldots, \bar{e}_k) \in \mathcal{C}^\perp$.

Note that once we have computed $(\bar{e}_1, \ldots, \bar{e}_k)$ and there exists $(\bar{x}_1, \ldots, \bar{x}_k)$ such that $\frac{\partial H_i}{\partial x_i}(\bar{x}_i) = \bar{e}_i$, $i = 1, \ldots, k$, then $(\bar{x}_1, \ldots, \bar{x}_k)$ is an equilibrium of the interconnected system (unique if K_i are strictly convex). Furthermore, if we additionally assume that the Hamiltonians H_i are strictly convex, then this equilibrium is *stable*.

Finally, note that the interconnection constraints can be equivalently formulated as the solution of the *dual* minimization problem

$$\min_{(e_1, \ldots, e_k) \in \mathcal{C}^\perp} K^\star(e_1, \ldots, e_k), \tag{7.14}$$

where

$$K^\star(e_1, \ldots, e_k) := \sum_{i \in \pi} K_i^\star(y_i) + \sum_{i \notin \pi} K_i(u_i). \tag{7.15}$$

8. Connections with passivity notions. As already noted, any incrementally port-Hamiltonian system satisfies the incremental dissipation inequality (4.2). This suggests close links with some form of passivity. In this section it is shown how this can be substantiated under extra conditions on the Hamiltonian H, extending the discussion given before in [11].

8.1. Relation with equilibrium independent passivity. Up to now, in the definition of an incrementally port-Hamiltonian system no conditions were imposed on the Hamiltonian H. If it is assumed that H is *strictly convex* (as well as differentiable), then for every $\bar{x}$ the function $H_{\bar{x}} : \mathcal{X} \to \mathbb{R}$ defined as

$$H_{\bar{x}}(x) := H(x) - \frac{\partial H}{\partial x^T}(\bar{x})(x - \bar{x}) - H(\bar{x}) \tag{8.1}$$

(as a function of x and $\bar{x}$ called the Bregman divergence of H [27], or as a function of x alone for fixed $\bar{x}$ called the shifted Hamiltonian [2]) has a strict minimum at $\bar{x}$, and is again strictly convex. Furthermore,

$$\frac{\partial H_{\bar{x}}}{\partial x}(x) = \frac{\partial H}{\partial x}(x) - \frac{\partial H}{\partial x}(\bar{x}).$$

Hence for any $(\bar{u}, \bar{y})$ in the steady-state input-output relation of an incrementally port-Hamiltonian system one verifies

$$\frac{d}{dt} H_{\bar{x}} = \frac{\partial H_{\bar{x}}}{\partial x^T}(x)\dot{x} = \left(\frac{\partial H}{\partial x^T}(x) - \frac{\partial H}{\partial x^T}(\bar{x}) \right)(\dot{x} - 0) \leqslant (y - \bar{y})^T (u - \bar{u}), \tag{8.2}$$

implying passivity with respect to the *shifted* passivity supply rate $(y - \bar{y})^T(u - \bar{u})$. This was called shifted passivity in [2], while the property that this holds for *any* steady-state values $(\bar{u}, \bar{x}, \bar{y})$ was coined as *equilibrium independent passivity* in [14]. This is summarized as follows.

PROPOSITION 8.1. *Consider an incrementally port-Hamiltonian system with maximal monotone relation*

$$\mathcal{M} \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m,$$

with a strictly convex differentiable Hamiltonian $H : \mathbb{R}^n \to \mathbb{R}$. *Then the system is equilibrium independent passive, with static input-output relation given by the monotone relation* $\mathcal{G} \subset \mathcal{M}^s$, *and with storage functions* $H_{\bar{x}}$ *having a strict minimum at* $\bar{x}$. *If* H *is such that for every* $\bar{e}$ *there exists an* $\bar{x}$ *with* $\bar{e} = \frac{\partial H}{\partial x}(\bar{x})$, *then* $\mathcal{G} = \mathcal{M}^s$.

The case $\mathcal{G} = \mathcal{M}^s$ was called *maximal equilibrium independent passivity* in [13]. (Maximal) equilibrium independent passivity is a desirable property for showing stability of the steady-state values of a port-Hamiltonian system for different constant input values, since by (8.2) the shifted Hamiltonians can be employed as Lyapunov functions for $u = \bar{u}$.

8.2. Relation with incremental and differential passivity. In this subsection it is shown how the notion of incrementally port-Hamiltonian systems is closely related to *incremental passivity* and *differential passivity*, at least in the case when the Hamiltonian is nonnegative and *quadratic-affine*. Thus let $H(x) = \frac{1}{2}x^T Q x + Ax + c$ for some symmetric positive semidefinite matrix Q, matrix A, and constant c. In this case, the inequality (4.2) reduces to

$$\langle Q(x_1 - x_2) \mid \rangle \dot{x}_1 - \dot{x}_2 \leqslant \langle e_P^1 - e_P^2 \mid f_P^1 - f_P^2 \rangle, \tag{8.3}$$

which is equivalent to

$$\frac{d}{dt}\frac{1}{2}(x_1(t) - x_2(t))^T Q (x_1(t) - x_2(t)) \leqslant (e_P^1(t) - e_P^2(t))^T (f_P^1(t) - f_P^2(t)). \tag{8.4}$$

Recall [5, 28, 29] that a system $\dot{x} = f(x,u), y = h(x,u)$ with $x \in \mathbb{R}^n, u, y \in \mathbb{R}^m$ is called *incrementally passive* if there exists a nonnegative function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{d}{dt} V(x_1, x_2) \leqslant (u_1 - u_2)^T (y_1 - y_2) \tag{8.5}$$

for all $(x_i, u_i, y_i), i = 1, 2$, satisfying $\dot{x} = f(x,u), y = h(x,u)$. We immediately obtain the following result.

PROPOSITION 8.2. *Any incrementally port-Hamiltonian system with nonnegative quadratic-affine Hamiltonian* $H(x) = \frac{1}{2} x^T Q x + Ax + c$ *is incrementally passive.*

Proof. The function $V(x_1, x_2) = \frac{1}{2}(x_1 - x_2)^T Q (x_1 - x_2)$ defines an incremental storage function for incremental passivity. ∎

Recall furthermore from [30, 31, 32] the following definition of differential passivity.

DEFINITION 8.3. *Consider a nonlinear control system* Σ *with state space* $\mathcal{X}$*, affine in the inputs* u*, and with an equal number of outputs* y*, given as*

$$\Sigma : \begin{array}{l} \dot{x} = f(x) + \displaystyle\sum_{j=1}^m u_j g_j(x) \,, \\ y_j = H_j(x) \,, \quad j = 1, \ldots, m \,, \end{array} \tag{8.6}$$

The variational system along any input-state-output trajectory

$$t \in [0, T] \mapsto (x(t), u(t), y(t))$$

is given by the following time-varying system (cf. [33]*):*

$$\begin{array}{rcl} \dot{\delta x}(t) & = & \frac{\partial f}{\partial x}(x(t)) \delta x(t) \\ & & + \sum_{j=1}^m u_j(t) \frac{\partial g_j}{\partial x}(x(t)) \delta x(t) + \sum_{j=1}^m \delta u_j g_j(x(t)), \\ \delta y_j(t) & = & \frac{\partial H_j}{\partial x}(x(t)) \delta x(t) \,, \quad j = 1, \ldots, m \,, \end{array} \tag{8.7}$$

with state $\delta x \in^n$*, where* $\delta u = (\delta u_1, \ldots, \delta u_m)$*,* $\delta y = (\delta y_1, \ldots, \delta y_m)$ *denote the inputs and the outputs of the variational system. Then* Σ *is called differentially passive if the system together with all its variational systems is dissipative with respect to the supply rate* $\delta u^T \delta y$*, that is, if there exists a function* $P : T\mathcal{X} \to \mathbb{R}^+$ *(called the differential storage function) satisfying*

$$\frac{d}{dt} P \leqslant \delta u^T \delta y \tag{8.8}$$

for all $x, u, \delta u$*.*

Similar to incremental passivity we obtain the following proposition.

PROPOSITION 8.4. *A monotone port-Hamiltonian system with nonnegative quadratic-affine Hamiltonian* $H(x) = \frac{1}{2} x^T Q x + Ax + c$ *is differentially passive.*

Proof. Consider the *infinitesimal* version of (4.2). In fact, let (f_P^1, e_P^1, x_1) and (f_P^2, e_P^2, x_2) be two triples of system trajectories arbitrarily near each other. Taking the limit we deduce from (4.2)

$$\delta x^T \frac{\partial^2 H}{\partial x^2}(x) \delta \dot{x} \leqslant \delta e_P^T \delta f_P, \tag{8.9}$$

where δx denotes the variational state, and $\partial f_P, \partial e_P$ the variational inputs and outputs. If the Hamiltonian H is a quadratic function $H(x) = \frac{1}{2}x^TQx + Ax + c$, then the left-hand side of the inequality (8.9) is equal to $\frac{d}{dt}\frac{1}{2}\delta x^TQ\delta x$, and hence amounts to the differential dissipativity inequality

$$\frac{d}{dt}\frac{1}{2}\delta x^TQ\delta x \leqslant \delta e_P^T\delta f_P, \tag{8.10}$$

implying that the monotone port-Hamiltonian system is differentially passive, with differential storage function $\frac{1}{2}\delta x^TQ\delta x$. ∎

Remark 8.5. Note that the Hamiltonians in the examples stemming from optimization algorithms (cf. (4.17), (4.19), (4.22)) are all positive quadratic. Thus the corresponding incrementally port-Hamiltonian systems are all incrementally and differentially passive. The same holds, e.g., for the nonlinear RC circuit of Example 4.7 or mechanical systems without potential energy (Hamiltonian equal to kinetic energy).

Of course, the assumption of a quadratic-affine Hamiltonian in order to let the monotone port-Hamiltonian system be incrementally and differentially passive is restrictive. On the other hand, it is known from the literature [4, 34] that for "unconditional" incremental properties such an assumption may be necessary as well. For example, we can formulate the following simple result. Consider a scalar nonlinear integrator system (cf. (4.9))

$$\dot{x} = u, \quad y = \frac{dH}{dx}(x). \tag{8.11}$$

As noted before, this is a maximal monotone port-Hamiltonian system. In order to evaluate its incremental properties consider two copies

$$\dot{x}_1 = u_1, \dot{x}_2 = u_2, \quad y_1 = \frac{dH}{dx_1}(x_1), y_2 = \frac{dH}{dx_2}(x_2). \tag{8.12}$$

Then the system (8.11) is incrementally passive if and only if there exists $S(x_1, x_2) \geqslant 0$ satisfying

$$\frac{\partial S}{\partial x_1}(x_1)u_1 + \frac{\partial S}{\partial x_2}(x_2)u_2 \leqslant (u_1 - u_2)\left(\frac{dH}{dx_1}(x_1) - \frac{dH}{dx_2}(x_2)\right) \tag{8.13}$$

for all x_1, x_2, u_1, u_2 related by (8.12). This is equivalent to

$$\frac{\partial S}{\partial x_1}(x_1, x_2) = \frac{dH}{dx_1}(x_1) - \frac{dH}{dx_2}(x_2) = -\frac{\partial S}{\partial x_2}(x_1, x_2) \tag{8.14}$$

for all x_1, x_2. Differentiation of the first equality with respect to x_2, and of the second equality with respect to x_1, yields

$$-\frac{d^2H}{dx_2^2}(x_2) = \frac{\partial^2 S}{\partial x_1 \partial x_2}(x_1, x_2) = -\frac{d^2H}{dx_1^2}(x_1), \tag{8.15}$$

implying that $\frac{d^2H}{dx^2}(x)$ is a constant; i.e., $H(x)$ must be a quadratic-affine function $H(x) = \frac{1}{2}qx^2 + ax + c$, for some constants q, a, c. Hence the (8.11) is incrementally passive *if and only if* H is quadratic-affine with $q > 0$ (in which case the integrator is actually linear). This example is easily extendable to more general situations, basically implying that unconditional incremental passivity implies a nonnegative quadratic-affine storage function.

9. Conclusions. The notion of an incrementally port-Hamiltonian system was first introduced in [11], by replacing in the definition of port-Hamiltonian systems the composition of a Dirac structure and an energy-dissipation relation by a (maximal) monotone relation. The present paper discusses the properties of incrementally port-Hamiltonian systems in much more detail, including a wealth of examples and the formulation of specific system subclasses. In particular, the class of maximal *cyclically* monotone port-Hamiltonian systems and its connection to convex generating functions is studied. A key mathematical contribution of the present paper is the detailed treatment of composition of maximal (cyclically) monotone relations, and its implications for the interconnection of incrementally port-Hamiltonian systems. In particular, it is shown how under mild technical conditions the composition of maximal (cyclically) monotone relations defines a maximal (cyclically) monotone relation.

In addition to the abundance of physical examples, this paper relates incrementally port-Hamiltonian systems to convex optimization as well. Such relations are multifaceted—from the formulation of gradient and primal-dual gradient algorithms in continuous time as incrementally port-Hamiltonian systems to the computation of the equilibria of interconnected incrementally port-Hamiltonian systems via convex optimization. Furthermore, apart from the convex generating functions of maximal cyclically monotone relations, *another* use of convexity in this incrementally port-Hamiltonian framework is the consideration of convex Hamiltonians. The use of the shifted Hamiltonian (or Bregman divergence) of a convex function turns out to be natural in establishing equilibrium independent passivity and assessing the stability of equilibria of (interconnected) incrementally port-Hamiltonian systems. Still, many more connections between port-Hamiltonian theory and convex analysis are to be explored.

The precise dynamical properties of incrementally port-Hamiltonian systems remain somewhat elusive. The dynamical implications of the key inequality (4.2) are only fully clear if the Hamiltonian H is a quadratic-affine function. Indeed, in this case the incrementally port-Hamiltonian system with nonnegative Hamiltonian is incrementally and differentially passive. On the other hand, as shown in [34] and in the example of a scalar integrator discussed at the end of the previous section, unconditional incremental properties are typically very demanding (see also the theory of contractive systems [35]), and one could argue that the notion of incrementally port-Hamiltonian systems is less restrictive.

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