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Ketelaars, Martijn; Borm, Peter; Herings, P.J.J.

Publication date:
2024

Document Version
Early version, also known as pre-print

No. 2024-013

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By

Martijn W. Ketelaars, Peter Borm,
P. Jean-Jacques Herings

23 April 2024

ISSN 0924-7815
ISSN 2213-9532

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Martijn W. Ketelaars[1,2] Peter Borm[2] P. Jean-Jacques Herings[2]

April 22, 2024

Abstract

Clearing payments are payments between agents to settle their mutual liabilities. The interdependence between agents in financial networks complicates the analysis of clearing payments as the extent to which an agent can pay its creditors depends not only on its own assets but also on the incoming payments from the other agents. Each financial network is endowed with agent-specific claims rules that prescribe how each agent pays its creditors. Consequently, our model not only captures standard principles of bankruptcy law, such as limited liability of equity, absolute priority of debt over equity, proportionality, and priority, but also allows for more general underlying payment mechanisms. A payment matrix that contains clearing payments in accordance with claims rules is a transfer scheme, which is not necessarily uniquely determined. This article is the first to provide the complete characterization of all such transfer schemes. Our characterization relies on additional cash vectors, which summarize the payments in excess of the minimum clearing payments. The set of such vectors is shown to be homeomorphic to the set of transfer schemes. We introduce a recursive procedure to compute any additional cash vector, and thereby indirectly a corresponding transfer scheme. The characterization opens up the opportunity for a network-based axiomatic analysis of transfer rules, which prescribe clearing payments for each financial network. In fact, we show that the characterization can be used to provide new axiomatizations of transfer rules in which each agent pays its creditors in accordance with the proportional claims rule.

Keywords: transfer schemes, additional cash vectors, homeomorphism, axiomatization, proportional rule.

JEL Classification Number: D74, G10, G33,

[1]Corresponding author. Email: M.W.Ketelaars@tilburguniversity.edu
[2]Tilburg University, Department of Econometrics and Operations Research, The Netherlands.

1 Introduction

Financial globalization, spurred on by the fact that financial interconnectedness allows risk diversification, is not always to the benefit of the resilience of the financial system as a whole. A noteworthy example is the 2007-2008 financial crisis in which instability in the United States housing market led, as a result of financial interconnectedness, to financial instability in markets elsewhere in the world. Financial instability in one market spreading to other markets is a phenomenon known as *financial contagion*. Glasserman and Young (2016), Jackson and Pernoud (2021), and Elliott and Golub (2022) provide excellent surveys of recent research on financial contagion in financial networks.

Clearing payments are at the core of the global financial system, especially in times of financial distress. Clearing payments are payments between agents (e.g., financial institutions) to settle their mutual liabilities. The interdependence between agents complicates the analysis of clearing payments in a network setting because the payment of one agent to another agent is contingent on its own assets as well as the payments it receives from other agents. Understanding clearing payments is therefore fundamental to facilitate proper decision-making, both from a normative and a positive point of view. The current article is, to our knowledge, the first to provide a complete characterization of clearing payments.

The model of a financial network we consider is in line with Eisenberg and Noe (2001), a seminal article on financial contagion. A financial network is characterized by a finite set of agents that each have an estate and liabilities to the other agents. Eisenberg and Noe (2001), along with extensions of its model such as Cifuentes, Ferrucci, and Shin (2005), Elsinger, Lehar, and Summer (2006), Rogers and Veraart (2013), and Demange (2018), assume proportionality of the payments. From the outset, however, we do not restrict our analysis to proportional repayments of liabilities by the agents. Instead, we allow each financial network to be endowed with agent-specific *claims rules* that prescribe how each agent pays its creditors (cf. Csóka and Herings (2018); Ketelaars, Borm, and Herings (2023); Csóka and Herings (2024)). In particular, for each agent, its payments to its creditors follow from allocating its asset value (i.e., its initial estate plus incoming payments) in accordance with its claims rule. For example, the *proportional rule* states that the creditors of an agent should be paid in proportion to their claims on that agent.

The agent-specific claims rules ensure that the payments adhere to two standard principles of bankruptcy law, namely *limited liability of equity* and *absolute priority of debt over equity* (cf. Eisenberg and Noe (2001)). The first requires that the total payments by an agent can never exceed the amount it has at its disposal to pay its creditors. The second requires that an agent can have a strictly positive equity — available to be divided among the shareholders — only if it has paid all its debts in full.

As claims may be of different seniority, creditors in insolvency proceedings (e.g., in the United States and the European Union) are partitioned into classes of different priority and within each class the creditors are paid proportionally (see, e.g., Kaminski (2000), and Chapter 4 of Wessels and Madaus (2017)). Moulin (2000) provides an axiomatic characterization of claims rules that incorporate such a priority structure, in which the payments within each priority class can also differ from proportional payments. The claims rules that we consider capture the priority and proportionality principles of bankruptcy law, but also allow for more general underlying payment mechanisms.

Our focus of interest is on clearing payments that are in accordance with claims rules. We define a *transfer scheme* as a clearing payment matrix that respects the claims rules. Csóka and Herings (2024) formulate a sufficient condition for uniqueness of transfer schemes in financial networks. In general, the set of transfer schemes for a financial network is a non-empty complete lattice, so both a bottom transfer scheme and top transfer scheme exist. A bottom (resp. top) transfer scheme contains the minimum (resp. maximum) amount of payments required to clear the financial network. However, there can exist infinitely many transfer schemes for a financial network.

We call the agents for which the payments are not uniquely determined *irregular*. We partition the set of irregular agents into *strongly connected components*. Two distinct irregular agents are *connected* if there exists a chain of payments from one agent to the other, such that for any two adjacent irregular agents in the chain, their mutual payments are not uniquely determined. A strongly connected component is a set of irregular agents in which any two distinct agents in that set are connected, and that set is maximal with respect to this property. We show that the mutual payments that are not uniquely determined are restricted to strongly connected components. More specifically, the payments between agents in a strongly connected component and agents outside that strongly connected component are always uniquely determined.

Our characterization of transfer schemes is based on *additional cash vectors*. An additional cash vector for a financial network contains the total cash of an agent in excess of the total cash under the bottom transfer scheme for that financial network. Agents use their additional cash to pay their creditors an additional amount. Additional cash vectors are consistent in the sense that, for each agent, its total additional outgoing payments equal its total additional incoming payments. We show that, for each financial network, the set of transfer schemes is *homeomorphic* to the set of additional cash vectors. That is, we provide a continuous one-to-one correspondence between transfer schemes and additional cash vectors. We also establish that our correspondence is monotonic.

So far, the literature on financial networks has introduced procedures to determine the bottom and top transfer schemes for a financial network. For example, Eisenberg and Noe (2001) introduces the fictitious default algorithm, which computes the top transfer scheme for a financial network in which each agent uses the proportional rule and in which each estate is non-negative. In a more general setting, which allows for general agent-specific claims rules and possibly negative estates, like we do in this article, Ketelaars et al. (2023) shows that the bottom and top transfer schemes can be obtained as the limit of iterative procedures.

Computing a transfer scheme that is different from the bottom and the top transfer schemes requires more work. We introduce a recursive procedure to compute every additional cash vector for an arbitrary financial network. The set of additional cash vectors is homeomorphic to the set of transfer schemes, so the iterative procedure indirectly computes any transfer scheme as well. We take as input a vector of agent-specific *weights* that are between zero and one. The weight corresponding to an agent determines the additional cash of that agent. In each step, the additional cash of each agent determined in a previous step is fixed, and the procedure determines the corresponding minimum and maximum additional cash of the current agent under consideration, which may require an infinite number of iterations. The additional cash of this agent is then a convex combination of its minimum and

maximum additional cash. The convex combination is based on the agent-specific weight. If the agent-specific weight is zero, the additional cash of that agent equals the minimum; if it is one, it equals the maximum. The result of the iterative procedure is an additional cash vector that is based on a vector of weights. In particular, if all agent-specific weights are equal to zero (resp. one), we obtain the bottom (resp. top) transfer scheme. The set of additional cash vectors for a financial network is then characterized by considering all possible vectors of weights.

Our analysis shows that computing additional cash vectors is generally complex. Interestingly, as we will see, it is significantly simpler when each agent uses the proportional rule to pay its creditors. The literature on financial contagion emphasizes uniqueness of proportional payments, in which case only one proportional transfer scheme has to be computed. For example, conditions for uniqueness of proportional payments in a financial network are stated in Eisenberg and Noe (2001) and Glasserman and Young (2015). Nevertheless, it is quite common for proportional payments not to be uniquely determined. This occurs, for instance, when at least two agents have an estate of zero and have positive liabilities towards each other, but no liabilities to the other agents.

Even if proportional payments are not uniquely determined, we show that proportional additional cash vectors and proportional transfer schemes still have a relatively simple structure. We accomplish this by associating each strongly connected component with a component-specific weight between zero and one, which is shared among the irregular agents within that component. The *total excess* of an agent is defined as the total amount this agent could receive in excess of the bottom transfer scheme. Then, for each strongly connected component, and for each agent belonging to that component, its proportional additional cash is equal to a fraction of its total excess, and its proportional payments are given by a convex combination of its proportional bottom and top payments. The specific fraction and convex combination are determined by the weight associated with the strongly connected component, which is allowed to differ across strongly connected components.

Furthermore, the characterization of the set of transfer schemes facilitates the formulation of *axioms* that lay the foundations of clearing payments in a financial network. We introduce three axioms specific to the financial network setting: *weak convexity*, *decomposition*, and *convexity*. We show that each of three axioms, when considered individually, lead to an axiomatization of the proportional rule in financial networks. Our axioms, weak convexity, decomposition, and convexity are related to homogeneity, additivity, and linearity for functions on general vector spaces, respectively. In general, homogeneity and additivity are independent axioms, and they jointly imply linearity (see also Hamel (1905), and Torchinsky (1988)). Conversely, linearity implies both homogeneity and additivity.

Weak convexity states that any convex combination of the bottom and top transfer schemes for a financial network is also a transfer scheme for that financial network. Convexity extends this notion by requiring that any convex combination of any two transfer schemes is also a transfer scheme. Both axioms provide a remedy when one has two candidate transfer schemes for a financial network. As only one transfer scheme can be carried out, one could, as a compromise, take a payment matrix that is in between the two transfer schemes. However, in general, such a payment matrix is not a transfer scheme, unless the agents pay according to the proportional rule. Decomposition requires that the additional payments by the agents can be decomposed as follows. Consider a transfer scheme based on

a vector of weights. First, subtract from this transfer scheme the bottom transfer scheme. This constitutes an additional payment matrix that contains the payments by the agents net of the bottom transfer scheme. Second, decompose the vector of weights and construct the corresponding additional payment matrices. Decomposition states that the additional payment matrices obtained in the second part added together are equal to the additional payment matrix obtained in the first part. Decomposition thus facilitates prescribing clearing payments either in stages or at once.

Csóka and Herings (2021) provides an axiomatization of the proportional rule in financial networks using invariance to mitosis, impartiality, and continuity.[1] This axiomatization is on the class of financial networks for which the estate of each agent is strictly positive, which implies that there always exists exactly one proportional transfer scheme. Calleja and Llerena (2023) generalizes this result by considering non-negative estates. On the other hand, our characterization of transfer schemes and the axiomatizations of the proportional rule hold for any real value of the estates (i.e., negative, zero, or positive).

This article is organized as follows. Section 2 introduces additional cash vectors and shows that, for each financial network, the set of all such vectors is homeomorphic to the set of transfer schemes. Section 3 contains our characterization result via additional cash vectors that are constructed by a recursive procedure. Section 4 shows that the set of irregular agents in a financial network can be partitioned into strongly connected components. This result is then used to show that proportional clearing payments have a relatively simple structure. Section 5 provides new axiomatizations of the proportional rule in financial networks. Section 6 concludes. The proofs of the main results are given in the main text, whereas the remaining proofs are presented in the appendix.

2 Towards Characterizing Transfer Schemes

2.1 Financial Networks

A *financial network* is a pair $(E, C) \in \mathbb{R}^N \times \mathbb{R}_+^{N \times N}$ in which N is a finite set of agents, $E = (e_i)_{i \in N}$ is an *estates vector*, and $C = (c_{ij})_{i,j \in N}$ is a *claims matrix*. Each coordinate e_i of E represents the, possibly negative, estate corresponding to agent $i \in N$. The negative estate of an agent is interpreted as the amount that it uses for own consumption before it pays its creditors. The claims matrix C represents mutual liabilities between agents. Each cell c_{ij} of C represents the non-negative claim of agent $j \in N$ on agent $i \in N$. Row i in C thus captures creditors of agent i, whereas column i of C captures debtors of agent i. We assume that agents have no claim on themselves, that is, $c_{ii} = 0$ for all $i \in N$. No additional conditions are imposed on the claims matrix; in particular, there is no condition on the relation between claims c_{ij} and c_{ji} for $i \neq j$. The class of all financial networks on N is denoted by $\mathcal{F}^N$.

Each financial network on N is endowed with a vector of *claims rules* $\phi = (\varphi^i)_{i \in N}$ in

[1] The axiomatization relies also on claims boundedness, limited liability of equity, and absolute priority of debt over equity. In this article, these axioms are already embedded in the claims rules associated with a financial network.

which $\varphi^i\colon \mathbb{R} \times (\mathbb{R}_+^{N\setminus\{i\}} \times \{0\}^{\{i\}}) \to \mathbb{R}_+^N$ is the claims rule associated with agent $i \in N$ that prescribes how each creditor of agent i is to be paid. The vector of claims on agent $i \in N$ is denoted by $\overline{c}_i = (c_{ij})_{j\in N} \in \mathbb{R}_+^N$, in which $\overline{c}_{ii} = 0$ because agent i has no claim on itself. Formally, for all $i \in N$, a claims rule φ^i prescribes, for all $a \in \mathbb{R}$ and $\overline{c}_i \in (\mathbb{R}_+^{N\setminus\{i\}} \times \{0\}^{\{i\}})$, the allocation vector $\varphi^i(a, \overline{c}_i)$ that satisfies

(i) $0 \leq \varphi_j^i(a, \overline{c}_i) \leq c_{ij}$ for all $j \in N$,

(ii) $\sum_{j\in N} \varphi_j^i(a, \overline{c}_i) = \min\{\max\{0, a\}, \sum_{j\in N} c_{ij}\}$.

The set of all vectors of claims rules on N that satisfy (i) and (ii) is denoted by $\mathcal{V}^N$.

Consider $i \in N$, $a \in \mathbb{R}$, and $\overline{c}_i \in (\mathbb{R}_+^{N\setminus\{i\}} \times \{0\}^{\{i\}})$. The claim of agent i on itself is zero, so condition (i) implies that $\varphi_i^i(a, \overline{c}_i) = 0$. Conditions (i) and (ii) imply that $\varphi^i(a, \overline{c}_i) = 0^N$ in case a, the amount agent i has at its disposal to pay its creditors, is negative, and $\varphi^i(a, \overline{c}_i) = \overline{c}_i$ in case a is larger than or equal to the sum of the claims. Otherwise $0 \leq a < \sum_{j\in N} c_{ij}$, so condition (ii) is given by $\sum_{j\in N} \varphi_j^i(a, \overline{c}_i) = a$, which entails that all what agent i has at its disposal to pay its creditors is paid to its creditors.

We assume that all the claims rules satisfy *resource monotonicity*: for all $i \in N$, for all $\overline{c}_i \in (\mathbb{R}_+^{N\setminus\{i\}} \times \{0\}^{\{i\}})$, and for all $a' \in \mathbb{R}$ and $a \in \mathbb{R}$ with $a' \leq a$, it holds that $\varphi^i(a', \overline{c}_i) \leq \varphi^i(a, \overline{c}_i)$.[2] Define

$$\mathcal{R}^N = \{\phi \in \mathcal{V}^N \,|\, \text{for all } i \in N, \varphi^i \text{ satisfies resource monotonicity}\}$$

as the set of all vectors of claims rules on N for which each claims rule satisfies resource monotonicity.

We consider two claims rules in particular, namely the *proportional rule*, and the *Talmud rule* (Aumann & Maschler, 1985). Both rules satisfy resource monotonicity.

The proportional rule allocates, to each creditor, a proportion of the amount to be divided, where the proportions are determined on the basis of each creditor's claim relative to the total claims. A claims rule φ^i is the proportional rule PROP (i.e., $\varphi^i = \text{PROP}$) if, for all $a \in \mathbb{R}$ and $\overline{c}_i \in (\mathbb{R}_+^{N\setminus\{i\}} \times \{0\}^{\{i\}})$ with $0 \leq a < \sum_{k\in N} c_{ik}$, and all $j \in N$,

$$\varphi_j^i(a, \overline{c}_i) = \frac{c_{ij}}{\sum_{k\in N} c_{ik}} a.$$

If the amount to be divided is relatively small, the Talmud rule divides the estate equally among the creditors, where the amount that each creditor receives is capped at half its claim. If the amount to be divided is relatively large, all creditors receive at least half their claim, and what they receive on top of that is such that the losses (i.e., the part of the claim that each creditor does not receive) are incurred equally by the creditors, where the loss of

[2]Resource monotonicity implies another desirable property, namely resource continuity (i.e., continuity with respect to the amount to be divided among the creditors). All the claims rules that we consider in this article are thus also resource continuous.

each creditor cannot exceed half its claim. A claims rule φ^i is the Talmud rule TAL (i.e., $\varphi^i = \text{TAL}$) if, for all $a \in \mathbb{R}$ and $\overline{c}_i \in (\mathbb{R}_+^{N\setminus\{i\}} \times \{0\}^{\{i\}})$ with $0 \leq a < \sum_{k\in N} c_{ik}$, and all $j \in N$,

$$\varphi_j^i(a, \overline{c}_i) = \begin{cases} \min\{\beta, \frac{1}{2}c_{ij}\} & \text{if } a \leq \frac{1}{2}\sum_{k\in N} c_{ik}, \\ \frac{1}{2}c_{ij} + \max\{\frac{1}{2}c_{ij} - \beta, 0\} & \text{if } a > \frac{1}{2}\sum_{k\in N} c_{ik}, \end{cases}$$

where, in both cases, $\beta \geq 0$ is such that $\sum_{j\in N} \varphi_j^i(a, \overline{c}_i) = a$.

2.2 Transfer Schemes and Additional Cash Vectors

A payment matrix for a financial network on N that contains consistent transfers between agents in accordance with $\phi \in \mathcal{R}^N$ is a *transfer scheme*. For each agent, its payments to its creditors follow from allocating its estate plus incoming payments in accordance with its claims rule. Moreover, for each agent, the payment to itself is equal to zero because it has no claim on itself.

Definition 2.1. Let $(E, C) \in \mathcal{F}^N$, and let $\phi \in \mathcal{R}^N$. The payment matrix P is a *transfer scheme* for (E, C) with respect to ϕ if, for all $i, j \in N$,

$$p_{ij} = \varphi_j^i(e_i + \sum_{k\in N} p_{ki}, \overline{c}_i). \tag{2.1}$$

The set of all possible transfer schemes for (E, C) with respect to ϕ is denoted by $\mathcal{P}^\phi(E, C)$.

Payments in a transfer scheme $P \in \mathcal{P}^\phi(E, C)$ are in accordance with claims rules, which implies that *limited liability of equity* and *absolute priority of debt over equity* are satisfied. Limited liability of equity entails that the total outgoing payments of each agent cannot exceed the amount it has at its disposal. That is, for all $i \in N$,

$$\sum_{j\in N} p_{ij} \leq \max\{0, e_i + \sum_{j\in N} p_{ji}\}.$$

Absolute priority of debt over equity implies that an agent has a strictly positive equity only if it has paid off all its outstanding liabilities. Hence, each agent either settles all its claims or pays everything it has at its disposal to his creditors. That is, for all $i \in N$, either $p_{ij} = c_{ij}$ for all $j \in N$, or

$$\sum_{j\in N} p_{ij} = \max\{0, e_i + \sum_{j\in N} p_{ji}\}.$$

Ketelaars et al. (2023) shows that the set of transfer schemes for a financial network is a non-empty complete lattice with respect to the element-wise ordering $\leq$ of $\mathbb{R}^{N\times N}$. A lattice is a partially ordered set in which every pair of elements has a greatest lower bound (bottom) and a least upper bound (top) within the lattice. A lattice is complete if also every non-empty subset has a bottom and a top within the lattice.

Proposition 2.2 (Proposition 3.4 in Ketelaars et al. (2023))**.** *Let $(E, C) \in \mathcal{F}^N$, and let $\phi \in \mathcal{R}^N$. Then, the set of transfer schemes $\mathcal{P}^\phi(E, C)$ is a non-empty complete lattice.*

The set of ϕ-transfer schemes for a financial network $(E, C) \in \mathcal{F}^N$ is a non-empty complete lattice, so there exists a *bottom transfer scheme* and a *top transfer scheme* within the lattice. The bottom transfer scheme $\underline{P}^\phi(E, C) \in \mathcal{P}^\phi(E, C)$ is such that, for all $P \in \mathcal{P}^\phi(E, C)$, it holds that $\underline{P}^\phi(E, C) \leq P$; the top transfer scheme $\overline{P}^\phi(E, C) \in \mathcal{P}^\phi(E, C)$ is such that, for all $P \in \mathcal{P}^\phi(E, C)$, it holds that $\overline{P}^\phi(E, C) \geq P$.

In the remainder of this section, we provide a new equivalent representation of the set of transfer schemes that relies on the bottom and top transfer schemes and on vectors that contain a specific additional amount of cash that agents receive and use to pay their creditors. This turns out to be useful later on when we introduce a recursive procedure, which takes as input the bottom and top transfer schemes, that characterizes all transfer schemes for a financial network.

We distinguish between two types of agents in a financial network, namely those for which the payments to their creditors are the same under any two transfer schemes, and those for which this is not the case. We focus our attention on the latter type of agents because their irregular payments are the reason why transfer schemes need not necessarily be unique. Agents for which their payments under the bottom transfer scheme differ from their payments under the top transfer scheme, are called *irregular*. Agents for which this is not the case are called *regular*. The total amount each agent could receive in excess of the bottom transfer scheme is called the *total excess*.

Definition 2.3. Let $(E, C) \in \mathcal{F}^N$, and let $\phi \in \mathcal{R}^N$. The set of *irregular agents* $I^\phi(E, C)$ for (E, C) with respect to ϕ is defined by

$$I^\phi(E, C) = \{i \in N \mid \text{for some } j \in N,\ \underline{p}^\phi_{ij} < \overline{p}^\phi_{ij}\}.$$

Additionally, the vector of *total excess* $\overline{d}^\phi(E, C) \in \mathbb{R}^N$ for (E, C) with respect to ϕ is given by, for all $i \in N$, $\overline{d}^\phi_i(E, C) = \sum_{j \in N}(\overline{p}^\phi_{ji} - \underline{p}^\phi_{ji})$.

Note that there may exist two transfer schemes under which an irregular agent pays its creditors equally. In general, however, there always exist two transfer schemes under which an irregular agent pays at least one of it creditors differently.

The following example demonstrates that irregular agents appear not only when each agent uses the proportional rule, and at least two of them have an estate of zero and the others have a non-negative estate (see, e.g., Eisenberg and Noe (2001)).

Example 2.4. Consider the financial network $(E, C) \in \mathcal{F}^N$ given by $N = \{1, 2, 3, 4\}$,

$$E = \begin{pmatrix} 0 \\ 4 \\ -3 \\ 1 \end{pmatrix} \text{ and } C = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 3 & 0 \end{bmatrix}.$$

Let $\phi = (\text{TAL}, \text{TAL}, \text{TAL}, \text{TAL}) \equiv \text{TAL}$. The bottom transfer scheme and the top transfer scheme for (E, C) with respect to TAL are given by

$$\underline{P}^{\text{TAL}}(E, C) = \begin{bmatrix} 0 & \frac{1}{2} & 1\frac{1}{4} & 0 \\ 0 & 0 & 1 & 2 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 1\frac{3}{4} & 0 & 1\frac{3}{4} & 0 \end{bmatrix} \text{ and } \overline{P}^{\text{TAL}}(E, C) = \begin{bmatrix} 0 & \frac{1}{2} & 1\frac{1}{2} & 0 \\ 0 & 0 & 1 & 2 \\ 0 & \frac{1}{2} & 0 & 1 \\ 2 & 0 & 2 & 0 \end{bmatrix}, \tag{2.2}$$

respectively. It holds that $I^{\text{TAL}}(E,C) = \{1,3,4\}$ and $\overline{d}^{\text{TAL}}(E,C) = (\frac{1}{4}, 0, \frac{1}{2}, \frac{1}{2})$. $\triangle$

The following proposition states four properties regarding regular and irregular agents.

Proposition 2.5. *Let $(E,C) \in \mathcal{F}^N$, and let $\phi \in \mathcal{R}^N$.*

(i) It holds that $|I^\phi(E,C)| \neq 1$.

(ii) Let $i \in N$. Then, $i \in I^\phi(E,C)$ if and only if $\sum_{j\in N} \underline{p}^\phi_{ji} < \sum_{j\in N} \overline{p}^\phi_{ji}$.

(iii) If $i \in I^\phi(E,C)$, then, for all $P \in \mathcal{P}^\phi(E,C)$,

$$\sum_{j\in N} p_{ij} = e_i + \sum_{j\in N} p_{ji}.$$

(iv) Let $i \in N$. Then, for all $\Delta \in [0, \overline{d}^\phi_i(E,C)]$,

$$\sum_{j\in N} \underline{p}^\phi_{ij} + \Delta = \sum_{j\in N} \varphi^i_j(e_i + \sum_{k\in N} \underline{p}^\phi_{ki} + \Delta, \overline{c}_i).$$

Proposition 2.5 (i) states there can not be exactly one irregular agent in a financial network. The number of irregular agents is either equal to zero, that is, there is exactly one transfer scheme for the financial network, or the number of irregular agents is at least equal to two and at most equal to the number of agents in the financial network.

Second, the payments that irregular agents receive from the other agents are strictly smaller under the bottom transfer scheme than under the top transfer scheme. The total excess is therefore always strictly positive for an irregular agent. On the other hand, regular agents receive the same amount from the other agents under any transfer scheme. The total excess is therefore equal to zero for regular agents. Agents with a negative estate and no incoming payments and agents with an estate exceeding their total liabilities are always regular.

Third, for each irregular agent, its total outgoing payments equal its estate plus its total incoming payments.[3] It implies that the total excess of each irregular agent is alternatively given by the difference between its outgoing payments under the top transfer scheme and its outgoing payments under the bottom transfer scheme. Formally, for all $i \in I^\phi(E,C)$, it holds that $\overline{d}^\phi_i(E,C) = \sum_{j\in N}(\overline{p}^\phi_{ij} - \underline{p}^\phi_{ij})$. A total excess can as such be interpreted as a total excess of incoming payments or a total excess of outgoing payments. Additionally, Proposition 2.5 (iii) and condition (ii) of a claims rule imply that, for all $P \in \mathcal{P}^\phi(E,C)$ and all $i \in I^\phi(E,C)$, $0 \leq e_i + \sum_{j\in N} p_{ji} \leq \sum_{j\in N} c_{ij}$.

Fourth, if an agent has additional cash at its disposal and distributes this among its creditors, then this additional cash is distributed in full among its creditors.

The following definition introduces *additional cash vectors* that represent the cash of agents in excess of the cash under the bottom transfer scheme. An additional cash vector

[3]Proposition 2.5 (iii) generalizes Proposition 1 in Csóka and Herings (2024) as result (iii) allows for negative estates.

contains the additional cash that each agent has at its disposal to pay its creditors, which is bounded from below by zero and bounded from above by its total excess. An additional cash vector satisfies a consistency requirement in the sense that, for each agent, the total additional outgoing payments (i.e., the left-hand side of (2.3)) equal the total additional incoming payments (i.e., the right-hand side of (2.3)). Thus, the additional cash of an agent can be interpreted as either the total additional incoming payments or the total additional outgoing payments of this agent.

Definition 2.6. Let $(E, C) \in \mathcal{F}^N$, and let $\phi \in \mathcal{R}^N$. The vector $d \in [0^N, \overline{d}^\phi(E, C)]$ is an *additional cash vector* for (E, C) with respect to ϕ if, for all $i \in N$,

$$d_i = \sum_{j \in N} (\varphi_i^j(e_j + \sum_{k \in N} \underline{p}_{kj}^\phi + d_j, \overline{c}_j) - \underline{p}_{ji}^\phi). \tag{2.3}$$

The set of all possible additional cash vectors for (E, C) with respect to ϕ is denoted by $D^\phi(E, C)$.

Condition (2.3) essentially relates only to irregular agents because it is always satisfied for regular agents. If $i \notin I^\phi(E, C)$, then, by Proposition 2.5 (ii), agent i always gets paid the same under any transfer scheme, so $\overline{d}_i^\phi(E, C) = 0$ and $d_i = 0$; furthermore, the right-hand side of condition (2.3) denotes the total additional incoming payments of agent i that is bounded from below by zero and from above by $\overline{d}_i^\phi(E, C)$, which in this case means that it is zero as well.

For each irregular agent, condition (2.3) requires that its total additional payments flow through the financial network in such a way that this exact amount is being paid back to this irregular agent. If that were not the case, then this irregular agent could not have paid this additional amount in the first place. Note that 0^N and $\overline{d}^\phi(E, C)$ are solutions to (2.3), corresponding to the bottom and top transfer schemes, respectively.

The following example illustrates additional cash vectors with respect to the financial network of Example 2.4.

Example 2.7. According to (2.3), an additional cash vector for the financial network $(E, C) \in \mathcal{F}^N$ of Example 2.4 with respect to TAL follows from solving

$$\begin{aligned}
d_1 &= \mathrm{TAL}_1(1 + 2\tfrac{1}{2} + d_4, (3, 0, 3, 0)) - 1\tfrac{3}{4}, && \text{(Agent 1)}\\
d_3 &= \mathrm{TAL}_3(0 + 1\tfrac{3}{4} + d_1, (0, 1, 2, 0)) - 1\tfrac{1}{4} + \mathrm{TAL}_3(1 + 2\tfrac{1}{2} + d_4, (3, 0, 3, 0)) - 1\tfrac{3}{4}, && \text{(Agent 3)}\\
d_4 &= \mathrm{TAL}_4(-3 + 4 + d_3, (0, 1, 0, 2)) - \tfrac{1}{2}. && \text{(Agent 4)}
\end{aligned}$$

It follows that the set of additional cash vectors for (E, C) with respect to TAL is given by $D^{\mathrm{TAL}}(E, C) = \{(\Delta, 0, 2\Delta, 2\Delta) \mid \Delta \in [0, \frac{1}{4}]\}$. If $\Delta = \frac{1}{8}$, then $d = (\frac{1}{8}, 0, \frac{1}{4}, \frac{1}{4})$. The inflow of payments to agent 1 equals $\frac{1}{8}$, which comes from agent 4; the outflow of payments from agent 1 equals $\frac{1}{8}$ as well, which goes to agent 3. The inflow to agent 3 equals $\frac{1}{4}$, of which $\frac{1}{8}$ comes from agent 1 and $\frac{1}{8}$ comes from agent 4; the outflow from agent 3 equals $\frac{1}{4}$ as well, which goes to agent 4. The inflow to agent 4 equals $\frac{1}{4}$, which comes from agent 3; the outflow from agent 4 equals $\frac{1}{4}$ as well, of which $\frac{1}{8}$ goes to agent 1 and $\frac{1}{8}$ goes to agent 3. $\triangle$

In Example 2.7, the set of additional cash vectors has one degree of freedom. In general, however, this need not always be the case as the following example illustrates.

Example 2.8. Consider the financial network $(E, C) \in \mathcal{F}^N$ given by $N = \{1, 2, 3, 4\}$,

$$E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } C = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{bmatrix}.$$

Let $\phi = \text{TAL}$. The bottom transfer scheme and the top transfer scheme for (E, C) with respect to TAL are given by $0^{N \times N}$ and C, respectively. Therefore, $I^{\text{TAL}}(E, C) = N$ and $\overline{d}^{\text{TAL}}(E, C) = (3, 3, 3, 3)$.

Anything that agents 3 and 4 have at their disposal is paid to agents 1 and 2, respectively. Therefore, any additional cash vector has the form $d = (d_1, d_2, d_1, d_2)$. Figure 2.1 visualizes all additional cash vectors for (E, C) with respect to TAL. Any point on the solid black line or in the shaded gray area corresponds to an additional cash vector for (E, C) with respect to TAL.

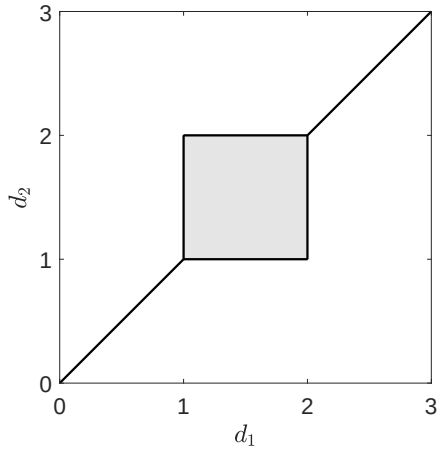


Figure 2.1: The solid black line and the shaded gray area represent the set of additional cash vectors for (E, C) with respect to TAL.

△

In the following definition, we translate an additional cash vector into a payment matrix in the sense that agents pay their creditors using the additional cash that they receive on top of what they receive under the bottom transfer scheme.

Definition 2.9. Let $(E, C) \in \mathcal{F}^N$, let $\phi \in \mathcal{R}^N$, and let $d \in D^\phi(E, C)$. Then, the function f on $D^\phi(E, C)$ produces an $N \times N$ payment matrix that is defined by setting, for all $i, j \in N$,

$$f_{ij}(d) = \varphi_j^i(e_i + \sum_{k \in N} \underline{p}_{ki}^\phi + d_i, \overline{c}_i). \tag{2.4}$$

In a similar fashion, we can define the function that translates a payment matrix into a vector in which each coordinate equals the total excess, albeit with respect to an arbitrary transfer scheme instead of the top transfer scheme. For each $(E, C) \in \mathcal{F}^N$ and $\phi \in \mathcal{R}^N$, the function g on $\mathcal{P}^\phi(E, C)$ is defined by setting, for all $P \in \mathcal{P}^\phi(E, C)$ and all $i \in N$,

$$g_i(P) = \sum_{j \in N} p_{ji} - \sum_{j \in N} \underline{p}^\phi_{ji}. \tag{2.5}$$

In this way, as the following proposition states, the function g is the inverse of f. This implies that the function f produces a transfer scheme and the function g produces an additional cash vector. The proposition additionally states that the function f and its inverse g are monotone.

Proposition 2.10. *The function g is the inverse of f, and the functions f and g are monotone.*

As a consequence of the following theorem, finding a transfer scheme boils down to finding an additional cash vector. It states that the set of additional cash vectors is *homeomorphic* to the set of transfer schemes. That is, the function f is continuous and provides a one-to-one correspondence between additional cash vectors and transfer schemes, and its inverse, given by g, is continuous as well.

Theorem 2.11. *Let $(E, C) \in \mathcal{F}^N$, and let $\phi \in \mathcal{R}^N$. Then, $D^\phi(E, C)$ and $\mathcal{P}^\phi(E, C)$ are homeomorphic through f.*

Proof. We have to show that f is bijective and continuous, and that g is continuous. Proposition 2.10 implies that f is invertible, so it is bijective. Because, for all $i \in N$, φ^i satisfies resource continuity, the function f is continuous. Clearly, g is continuous. □

The following corollary follows from Theorem 2.11 and the fact that the set of transfer schemes is a non-empty complete lattice (see Proposition 2.2). It states that the set of additional cash vectors for a financial network is a non-empty complete lattice, so there exists a bottom additional cash vector and a top additional cash vector.

Corollary 2.12. *Let $(E, C) \in \mathcal{F}^N$, and let $\phi \in \mathcal{R}^N$. Then, the set of additional cash vectors $D^\phi(E, C)$ is a non-empty complete lattice.*

For all $(E, C) \in \mathcal{F}^N$ and $\phi \in \mathcal{R}^N$, it holds that $0^N \in D^\phi(E, C)$ is the bottom additional cash vector and that $\overline{d}^\phi(E, C) \in D^\phi(E, C)$ is the top additional cash vector.

The following example builds on Example 2.7 and illustrates how additional cash vectors can be used to construct the set of transfer schemes.

Example 2.13. Recall from Example 2.7 that the set of additional cash vectors for (E, C) with respect to TAL is given by $D^{\text{TAL}}(E, C) = \{(\Delta, 0, 2\Delta, 2\Delta) \mid \Delta \in [0, \frac{1}{4}]\}$. Then, for all $d \in D^{\text{TAL}}(E, C)$,

$$\text{TAL}(0 + 1\tfrac{3}{4} + \Delta, (0, 1, 2, 0)) = (0, \tfrac{1}{2}, 1\tfrac{1}{4} + \Delta, 0), \qquad \text{(Agent 1)}$$
$$\text{TAL}(-3 + 4 + 2\Delta, (0, 1, 0, 2)) = (0, \tfrac{1}{2}, 0, \tfrac{1}{2} + 2\Delta), \qquad \text{(Agent 3)}$$

$$\mathrm{TAL}(1+2\tfrac{1}{2}+2\Delta,(3,0,3,0))=(1\tfrac{3}{4}+\Delta,0,1\tfrac{3}{4}+\Delta,0), \qquad \text{(Agent 4)}$$

such that the set of transfer schemes for (E,C) with respect to TAL comprises

$$P^{\Delta}=\begin{bmatrix} 0 & \frac{1}{2} & 1\frac{1}{4}+\Delta & 0 \\ 0 & 0 & 1 & 2 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2}+2\Delta \\ 1\frac{3}{4}+\Delta & 0 & 1\frac{3}{4}+\Delta & 0 \end{bmatrix},$$

in which $\Delta\in[0,\frac{1}{4}]$. Note that $\Delta=0$ gives the bottom transfer scheme and that $\Delta=\frac{1}{4}$ gives the top transfer scheme (see also (2.2)). $\triangle$

3 The Characterization of Transfer Schemes

In the existing literature on financial networks, there exist iterative procedures to find only the bottom and top transfer schemes. Ketelaars et al. (2023) shows that the bottom (resp. top) transfer scheme is obtained by a, possibly infinite, iterative procedure that generates a monotonically increasing (resp. decreasing) sequence of payment matrices that approaches the complete lattice of transfer schemes from below (resp. above). The iterative procedures are defined on $[0^{N\times N},C]=\{P\in\mathbb{R}^{N\times N}\mid \text{for all } i,j\in N,\ 0\leq p_{ij}\leq c_{ij}\}$, which is a complete lattice.

Proposition 3.1 (cf. Theorem 3.10 in Ketelaars et al. (2023))**.** *Let $(E,C)\in\mathcal{F}^N$, let $\phi\in\mathcal{R}^N$, and let $h\colon[0^{N\times N},C]\to[0^{N\times N},C]$ be defined by setting, for all $P\in[0^{N\times N},C]$, and for all $i,j\in N$, $h_{ij}(P)=\varphi^i_j(e_i+\sum_{k\in N}p_{ki},\overline{c}_i)$. Then,*

(i) $\underline{P}^{\phi}(E,C)=\lim\limits_{k\to\infty}P^k$, where, for all $k\in\mathbb{N}$, $P^{k+1}=h(P^k)$ with $P^1=0^{N\times N}$;

(ii) $\overline{P}^{\phi}(E,C)=\lim\limits_{k\to\infty}P^k$, where, for all $k\in\mathbb{N}$, $P^{k+1}=h(P^k)$ with $P^1=C$.

Alternatively, one can obtain the bottom and top transfer schemes by solving an appropriately defined optimization problem (see, e.g., Eisenberg and Noe (2001) and Csóka and Herings (2022)).

In this section, we introduce an iterative procedure that one can use to find any additional cash vector that differs from the zero vector and the vector of total excess. The homeomorphism between the set of additional cash vectors and the set of transfer schemes (Theorem 2.11) implies that iterative procedure can be used to find a transfer scheme that differs from the bottom and top transfer schemes.

For ease of exposition, we set $N=\{1,\ldots,n\}$, in which $n=|N|$. We take a vector $\lambda\in[0,1]^N$ of *weights* as given, in which, for each $i\in N$, λ_i is a measure of the payment by this agent, in addition to what it pays in total under the bottom transfer scheme. The procedure constructs a vector of additional cash in the following recursive way. The procedure starts with agent 1 for which λ_1 determines its additional cash, which stay constant throughout. Agent 1 essentially injects cash into the financial network, so the procedure subsequently constructs two vectors of additional cash such that, for each agent, the inflow of additional

payments equals the outflow of additional payments. The procedure determines both the minimum and maximum additional cash of the next agent, which is agent 2. These two bounds on the additional cash together with λ_2 determine the additional cash of agent 2, which stay constant throughout. Keeping the additional cash of agents 1 and 2 fixed, the procedure determines the minimum and maximum additional cash of agent 3, which are then used to construct the additional cash of agent 3. This process is repeated until the additional cash of each agent is determined, which takes exactly n steps.

Definition 3.2. Let $(E, C) \in \mathcal{F}^N$, let $\phi \in \mathcal{R}^N$, and let $\lambda \in [0,1]^N$. The vector $\theta \in \mathbb{R}^N$ is a λ-*additional cash vector* for (E, C) with respect to ϕ if there exists $\underline{\theta} = (\underline{\theta}_1, \ldots, \underline{\theta}_n)$ and $\overline{\theta} = (\overline{\theta}_1, \ldots, \overline{\theta}_n)$ such that the additional cash of agent 1 is given by

$$\theta_1 = (1 - \lambda_1)\underline{\theta}_1 + \lambda_1 \overline{\theta}_1$$

with $\underline{\theta}_1 = 0$ and $\overline{\theta}_1 = \overline{d}_1^{\phi}(E, C)$, and, recursively, for all $\ell \in \{2, \ldots, n\}$, the additional cash of agent ℓ is given by

$$\theta_\ell = (1 - \lambda_\ell)\underline{\theta}_\ell + \lambda_\ell \overline{\theta}_\ell$$

in which $\underline{\theta}_\ell$ and $\overline{\theta}_\ell$ are equal to

$$\begin{aligned} \underline{\theta}_\ell = \min_{d \in D^\phi(E,C)} \; & d_\ell \\ \text{subject to } \; & d_i = \theta_i \qquad \forall\, i < \ell, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \overline{\theta}_\ell = \max_{d \in D^\phi(E,C)} \; & d_\ell \\ \text{subject to } \; & d_i = \theta_i \qquad \forall\, i < \ell, \end{aligned} \tag{3.2}$$

respectively.

It is important to note that the additional cash of agent n follows directly from the additional cash of the other agents, irrespective of $\lambda_n \in [0,1]$. The optimization problems (3.1) and (3.2) lead to two additional cash vectors $(\theta_1, \ldots, \theta_{n-1}, \underline{\theta}_n) \in \mathcal{D}^\phi(E, C)$ and $(\theta_1, \ldots, \theta_{n-1}, \overline{\theta}_n) \in \mathcal{D}^\phi(E, C)$. As a result, we obtain that

$$\underline{\theta}_n = \sum_{j<n} (\varphi_n^j(e_j + \sum_{k \in N} \underline{p}_{kj}^\phi + \theta_j, \overline{c}_j) - \underline{p}_{jn}^\phi) = \overline{\theta}_n, \tag{3.3}$$

which implies that $\theta_n = (1 - \lambda_n)\underline{\theta}_n + \lambda_n \overline{\theta}_n = \underline{\theta}_n$. Hence, a λ-additional cash vector, if it exists, is an additional cash vector, that is, $\theta \in D^\phi(E, C)$.

We will now describe the recursive procedure of Definition 3.2 in more detail and show how to construct solutions to (3.1) and (3.2), which are in fact well-defined in each step.

Let $(E, C) \in \mathcal{F}^N$, and let $\phi \in \mathcal{R}^N$. As agent 1 is the first agent, the minimum additional cash is zero, whereas the maximum additional cash is $\overline{d}_1^{\phi}(E, C)$. The additional cash of agent 1 is given by

$$\theta_1 = (1 - \lambda_1)0 + \lambda_1 \overline{d}_1^{\phi}(E, C),$$

and stays constant throughout. If agent 1 is regular (i.e., $1 \notin I^{\phi}(E,C)$), then $\overline{d}_1^{\phi}(E,C) = 0$ and therefore $\theta_1 = 0$. Given θ_1, we construct a vector of additional cash in such a way that the additional cash of agent 2 is minimal, and that the vector of all these additional cash constitutes an additional cash vector. That is, we solve the following optimization problem:

$$\begin{aligned} &\min_{d \in D^{\phi}(E,C)} d_2 \\ &\text{subject to } d_1 = \theta_1. \end{aligned} \tag{3.4}$$

To find a solution to (3.4), we define, for all $k \in \mathbb{N}$, the vector $\underline{\gamma}^2(k) \in \mathbb{R}^{\{2,\dots,n\}}$, in which, for all $i \in \{2,\dots,n\}$,

$$\underline{\gamma}_i^2(1) = \varphi_i^1(e_1 + \sum_{h \in N} \underline{p}_{h1}^{\phi} + \theta_1, \overline{c}_1) - \underline{p}_{1i}^{\phi}), \tag{3.5}$$

and, recursively, for all $k \in \mathbb{N}$,

$$\underline{\gamma}_i^2(k+1) = \underline{\gamma}_i^2(1) + \sum_{j \geq 2} (\varphi_i^j(e_j + \sum_{h \in N} \underline{p}_{hj}^{\phi} + \underline{\gamma}_j^2(k), \overline{c}_j) - \underline{p}_{ji}^{\phi}). \tag{3.6}$$

The additional payments by agent 1 with respect θ_1 are represented by $\underline{\gamma}^2(1)$ (see (3.5)). Indeed, because $\theta_1 \in [0, \overline{d}_1^{\phi}(E,C)]$, Proposition 2.5 (iv) implies that

$$\sum_{j \geq 2} \underline{\gamma}_j^2(1) = \sum_{j \geq 2} \varphi_j^1(e_1 + \sum_{h \in N} \underline{p}_{h1}^{\phi} + \theta_1, \overline{c}_1) - \underline{p}_{1j}^{\phi}) = \theta_1. \tag{3.7}$$

Resource monotonicity of φ^1 and $\theta_1 \in [0, \overline{d}_1^{\phi}(E,C)]$ imply that, for all $i \in \{2,\dots,n\}$, $\underline{\gamma}_i^2(1) \in [0, \overline{d}_i^{\phi}(E,C)]$ as well. The additional payment by agent 1 that is given in (3.5) induces a sequence $(\underline{\gamma}^2(k))_{k \in \mathbb{N}}$. This sequence is monotonically increasing and bounded from above, so by the monotone convergence theorem for sequences it has a limit. To see this, let $i \in \{2,\dots,n\}$, and note that

$$\underline{\gamma}_i^2(1) \leq \underline{\gamma}_i^2(1) + \sum_{j \geq 2} (\varphi_i^j(e_j + \sum_{h \in N} \underline{p}_{hj}^{\phi} + \underline{\gamma}_j^2(1), \overline{c}_j) - \underline{p}_{ji}^{\phi}) = \underline{\gamma}_i^2(2),$$

in which the inequality follows from the fact that, for all $j \in \{2,\dots,n\}$, φ^j satisfies resource monotonicity and $\underline{\gamma}_j^2(1) \in [0, \overline{d}_j^{\phi}(E,C)]$. Let $k \in \mathbb{N}$ and assume that $\underline{\gamma}^2(k) \leq \underline{\gamma}^2(k+1)$. Then, by resource monotonicity of the claims rules in ϕ, it follows that

$$\begin{aligned} \underline{\gamma}_i^2(k+1) &= \underline{\gamma}_i^2(1) + \sum_{j \geq 2} (\varphi_i^j(e_j + \sum_{h \in N} \underline{p}_{hj}^{\phi} + \underline{\gamma}_j^2(k), \overline{c}_j) - \underline{p}_{ji}^{\phi}) \\ &\leq \underline{\gamma}_i^2(1) + \sum_{j \geq 2} (\varphi_i^j(e_j + \sum_{h \in N} \underline{p}_{hj}^{\phi} + \underline{\gamma}_j^2(k+1), \overline{c}_j) - \underline{p}_{ji}^{\phi}) \\ &= \underline{\gamma}_i^2(k+2), \end{aligned}$$

in which the first and second equality follow from (3.6). By induction, the sequence $(\underline{\gamma}_i^2(k))_{k \in \mathbb{N}}$ is monotonically increasing. In addition to this, because, for all $k \in \mathbb{N}$, and for all $j \in$

$\{2,\dots,n\}$, $\underline{\gamma}_j^2(k) \in [0, \overline{d}_j^{\phi}(E,C)]$, the sequence $(\underline{\gamma}_i^2(k))_{k\in\mathbb{N}}$ is bounded from above by $\overline{d}_i^{\phi}(E,C)$. We denote the limit of the sequence $(\underline{\gamma}^2(k))_{k\in\mathbb{N}}$ by $\underline{\theta}^2$, that is, for all $i \in \{2,\dots,n\}$,

$$\underline{\theta}_i^2 = \lim_{k\to\infty} \underline{\gamma}_i^2(k). \tag{3.8}$$

Combining the vector in (3.8) with the additional cash of agent 1 results in an additional cash vector. In fact, the additional cash of the other agents, which are induced by the additional payments of agent 1, are minimal. The following lemma formalizes this.

Lemma 3.3. *It holds that* $(\theta_1, \underline{\theta}_2^2, \dots, \underline{\theta}_n^2) \in D^{\phi}(E,C)$. *Furthermore, for all* $d \in D^{\phi}(E,C)$ *with* $d_1 = \theta_1$, *it holds that* $d_i \geq \underline{\theta}_i^2$ *for all* $i \in \{2,\dots,n\}$. *In particular, if* $\lambda_1 = 0$, *then* $\underline{\theta}_i^2 = 0$ *for all* $i \in \{2,\dots,n\}$.

Lemma 3.3 implies that $\underline{\theta}_2^2$ is the optimal value of the minimization problem (3.4).

Second, given θ_1, we can also determine the maximum additional cash vector in which the additional cash of agent 1 is fixed, but the additional cash of agent 2 is maximal, such that, for all agents, the inflow of additional payments equals the outflow of additional payments. This entails solving the following optimization problem:

$$\begin{aligned} &\max_{d\in D^{\phi}(E,C)} d_2 \\ &\text{subject to } \; d_1 = \theta_1. \end{aligned} \tag{3.9}$$

To find a solution to (3.9), we define, for all $k \in \mathbb{N}$, the vector $\overline{\gamma}^2(k) \in \mathbb{R}^{\{2,\dots,n\}}$, in which, for all $i \in \{2,\dots,n\}$,

$$\overline{\gamma}_i^2(1) = \overline{d}_i^{\phi}(E,C) - (\overline{p}_{1i}^{\phi} - \varphi_i^1(e_1 + \sum_{h\in N} \underline{p}_{h1}^{\phi} + \theta_1, \overline{c}_1)), \tag{3.10}$$

and, recursively, for all $k \in \mathbb{N}$,

$$\overline{\gamma}_i^2(k+1) = \overline{\gamma}_i^2(1) - \sum_{j\geq 2} (\overline{p}_{ji}^{\phi} - \varphi_i^j(e_j + \sum_{h\in N} \underline{p}_{hj}^{\phi} + \overline{\gamma}_j^2(k), \overline{c}_j)).$$

Here, for each agent $i \in \{2,\dots,n\}$, $\overline{\gamma}_i^2(1)$ denotes the excess amount this agent has at its disposal initially, net of the payment by agent 1 with respect to θ_1. In particular, although the additional payments by agent 1 to the other agents remain the same in this 'top' procedure in comparison with the previous 'bottom' procedure, the additional total amount available to the remaining agents with respect to (3.10) differs (see (3.7)) and is equal to

$$\sum_{j\geq 2} \overline{\gamma}_j^2(1) = \sum_{j\geq 2} \overline{d}_j^{\phi}(E,C) - (\overline{d}_1^{\phi}(E,C) - \theta_1),$$

in which the equality follows from Proposition 2.5 (iv). In contrast to the sequence $(\underline{\gamma}^2(k))_{k\in\mathbb{N}}$, the sequence $(\overline{\gamma}^2(k))_{k\in\mathbb{N}}$ is monotonically decreasing and bounded from below by zero — one can verify this by applying the same arguments that we used for the sequence $(\underline{\gamma}^2(k))_{k\in\mathbb{N}}$. We denote the limit of the sequence $(\overline{\gamma}^2(k))_{k\in\mathbb{N}}$ by $\overline{\theta}^2$, that is, for all $i \in \{2,\dots,n\}$,

$$\overline{\theta}_i^2 = \lim_{k\to\infty} \overline{\gamma}_i^2(k). \tag{3.11}$$

The following lemma states that the vector in (3.11) combined with the additional cash of agent 1 is an additional cash vector in which the additional cash of the agents are maximal, though they are not necessarily maximal for agent 1. The lemma is stated without an explicit proof because its proof is along the lines of the proof of Lemma 3.3.

Lemma 3.4. *It holds that* $(\theta_1, \overline{\theta}_2^2, \dots, \overline{\theta}_n^2) \in D^\phi(E, C)$. *Furthermore, for all* $d \in D^\phi(E, C)$ *with* $d_1 = \theta_1$, *it holds that* $d_i \leq \overline{\theta}_i^2$ *for all* $i \in \{2, \dots, n\}$. *In particular, if* $\lambda_1 = 1$, *then* $\overline{\theta}_i^2 = \overline{d}_i^\phi(E, C)$ *for all* $i \in \{2, \dots, n\}$.

Likewise Lemma 3.3, it follows from Lemma 3.4 that $\overline{\theta}_2^2$ is the optimal value of the maximization problem (3.9).

The following example demonstrates how a minimum and a maximum additional cash vector is constructed, provided that the additional cash of agent 1 is fixed. The example additionally shows that the minimum and maximum additional cash of the remaining agents need not necessarily coincide.

Example 3.5. Reconsider the financial network $(E, C) \in \mathcal{F}^N$ of Example 2.8 given by $N = \{1, 2, 3, 4\}$,

$$E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } C = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{bmatrix}.$$

Let $\phi = \text{TAL}$. Recall that the bottom transfer scheme and the top transfer scheme for (E, C) with respect to TAL are given by $0^{N \times N}$ and C, respectively, and that $I^{\text{TAL}}(E, C) = N$ and $\overline{d}^{\text{TAL}}(E, C) = (3, 3, 3, 3)$.

Let $\lambda_1 = \frac{1}{3}$, so $\theta_1 = \frac{2}{3}0 + \frac{1}{3}3 = 1$ and the additional payments by agent 1 to the other agents are given by (see (3.5))

$$\underline{\gamma}^2(1) = (\text{TAL}_i(1, (0, 0, 2, 1)))_{i \geq 2} = (0, \tfrac{1}{2}, \tfrac{1}{2}).$$

The excess amounts that the other agents have at their disposal with respect to θ_1, are given by (see (3.10))

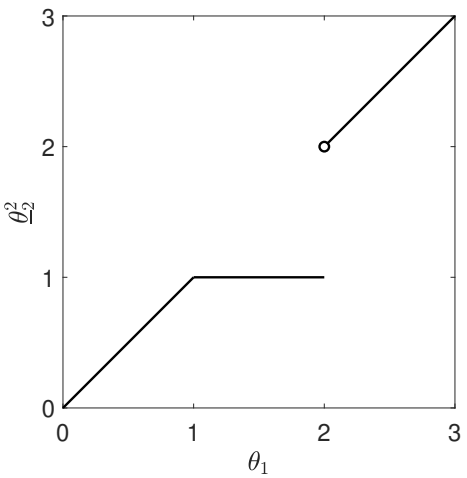
$$\overline{\gamma}^2(1) = (3, 3, 3) - ((0, 2, 1) - (\text{TAL}_i(1, (0, 0, 2, 1)))_{i \geq 2}) = (3, 1\tfrac{1}{2}, 2\tfrac{1}{2}).$$

Table 3.1 shows $\underline{\gamma}^2(k)$ and $\overline{\gamma}^2(k)$ for the next four steps. The sequences $(\underline{\gamma}^2(k))_{k \in \mathbb{N}}$ and $(\overline{\gamma}^2(k))_{k \in \mathbb{N}}$ converge to $\underline{\theta}^2 = (1, 1, 1)$ and $\overline{\theta}^2 = (2, 1, 2)$, respectively.

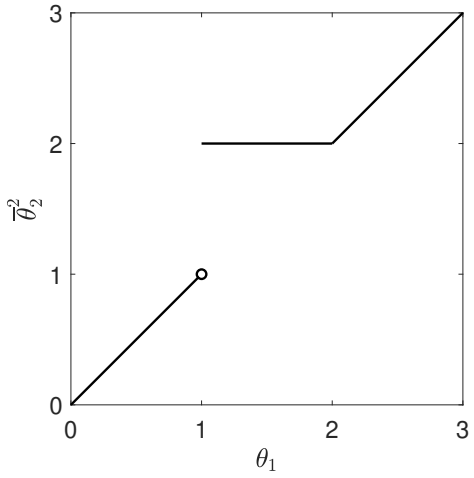
k	1	2	3	4	5	...
$\underline{\gamma}^2(k)$	$(0, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{3}{4}, \frac{3}{4})$	$(\frac{3}{4}, \frac{3}{4}, \frac{3}{4})$	$(\frac{3}{4}, \frac{7}{8}, \frac{7}{8})$	...
$\overline{\gamma}^2(k)$	$(3, 1\frac{1}{2}, 2\frac{1}{2})$	$(2\frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2})$	$(2\frac{1}{2}, 1\frac{1}{4}, 2\frac{1}{4})$	$(2\frac{1}{4}, 1\frac{1}{4}, 2\frac{1}{4})$	$(2\frac{1}{4}, 1\frac{1}{8}, 2\frac{1}{8})$	...

Table 3.1: The values of the first five elements of the sequences $(\underline{\gamma}^2(k))_{k \in \mathbb{N}}$ and $(\overline{\gamma}^2(k))_{k \in \mathbb{N}}$, generated for the financial network (E, C) with respect to the Talmud claims rule TAL.

Figure 3.1a visualizes $\underline{\theta}_2^2$, the minimum additional cash of agent 2, as θ_1, the additional cash of agent 1, varies from 0 to 3. Figure 3.1b visualizes this for the maximum additional cash of agent 2. Note that $\overline{\theta}_2^2$ is discontinuous at $\theta_1 = 1$, whereas $\underline{\theta}_2^2$ is discontinuous at $\theta_1 = 2$.



(a) The minimum additional cash of agent 2, $\underline{\theta}_2^2$, as a function of θ_1, the additional cash of agent 1.



(b) The maximum additional cash of agent 2, $\overline{\theta}_2^2$, as a function of θ_1, the additional cash of agent 1.

Figure 3.1

△

As the additional cash of agent $i \in \{2, \ldots, n\}$ is bounded from below by $\underline{\theta}_i^2$ (Lemma 3.3) and bounded from above by $\overline{\theta}_i^2$ (Lemma 3.4), we thus also find that $\underline{\theta}_i^2$ is at most equal to $\overline{\theta}_i^2$.

Corollary 3.6. *For all $i \in \{2, \ldots, n\}$, it holds that $\underline{\theta}_i^2 \leq \overline{\theta}_i^2$.*

Now, we fix the additional cash of agent 2:

$$\theta_2 = (1 - \lambda_2)\underline{\theta}_2^2 + \lambda_2 \overline{\theta}_2^2,$$

which is a convex combination of its minimum additional cash and maximum additional cash. Moreover, by combining the results of Lemmas 3.3 and 3.4 and Corollary 3.6, for any $d \in D^{\phi}(E, C)$, we can set

$$\lambda_2 = \begin{cases} \dfrac{d_2 - \underline{\theta}_2^2}{\overline{\theta}_2^2 - \underline{\theta}_2^2} & \text{if } \underline{\theta}_2^2 < \overline{\theta}_2^2 \\ 0 & \text{if } \underline{\theta}_2^2 = \overline{\theta}_2^2, \end{cases}$$

such that $\lambda_2 \in [0, 1]$ and $\theta_2 = d_2$. The choice of $\lambda_2 = 0$ in case $\underline{\theta}_2^2 = \overline{\theta}_2^2$ is arbitrary.

Given the additional cash of agents 1 and 2, we can subsequently determine the minimum and maximum additional cash of agent 3 in order to construct the additional cash of agent 3. By repeating this recursive process, the additional cash of each agent can be constructed, which takes exactly n steps. This leads to the following definition.

Definition 3.7. Let $(E, C) \in \mathcal{F}^N$, let $\phi \in \mathcal{R}^N$, and let $\lambda \in [0,1]^N$. The vector $\theta(\lambda) \in \mathbb{R}^N$ is defined by setting

$$\theta_1(\lambda) = (1 - \lambda_1)\underline{\theta}_1^1 + \lambda_1 \overline{\theta}_1^1$$

with $\underline{\theta}^1 = 0^N$ and $\overline{\theta}^1 = \overline{d}^\phi(E, C)$, and, recursively, for all $\ell \in \{2, \ldots, n\}$,

$$\theta_\ell(\lambda) = (1 - \lambda_\ell)\underline{\theta}_\ell^\ell + \lambda_\ell \overline{\theta}_\ell^\ell,$$

in which $\underline{\theta}^\ell, \overline{\theta}^\ell \in \mathbb{R}^{\{\ell,\ldots,n\}}$ are, for all $i \in \{\ell, \ldots, n\}$, given by

$$\underline{\theta}_i^\ell = \lim_{k\to\infty} \underline{\gamma}_i^\ell(k) \text{ and } \overline{\theta}_i^\ell = \lim_{k\to\infty} \overline{\gamma}_i^\ell(k), \tag{3.12}$$

respectively, in which,

$$\underline{\gamma}_i^\ell(1) = \sum_{j<\ell} (\varphi_i^j(e_j + \sum_{h\in N} \underline{p}_{hj}^\phi + \theta_j(\lambda), \overline{c}_j) - \underline{p}_{ji}^\phi) \tag{3.13}$$

and

$$\overline{\gamma}_i^\ell(1) = \overline{d}_i^\phi(E, C) - \sum_{j<\ell} (\overline{p}_{ji}^\phi - \varphi_i^j(e_j + \sum_{h\in N} \underline{p}_{hj}^\phi + \theta_j(\lambda), \overline{c}_j)),$$

and, recursively, for all $k \in \mathbb{N}$,

$$\underline{\gamma}_i^\ell(k+1) = \underline{\gamma}_i^\ell(1) + \sum_{j\geq\ell} (\varphi_i^j(e_j + \sum_{h\in N} \underline{p}_{hj}^\phi + \underline{\gamma}_j^\ell(k), \overline{c}_j) - \underline{p}_{ji}^\phi), \tag{3.14}$$

and

$$\overline{\gamma}_i^\ell(k+1) = \overline{\gamma}_i^\ell(1) - \sum_{j\geq\ell} (\overline{p}_{ji}^\phi - \varphi_i^j(e_j + \sum_{h\in N} \underline{p}_{hj}^\phi + \overline{\gamma}_j^\ell(k), \overline{c}_j)). \tag{3.15}$$

By applying the same arguments that we used for the sequence $(\underline{\gamma}^2(k))_{k\in\mathbb{N}}$, it follows that, in each step, the sequences given by (3.14) and (3.15) are monotonically increasing and monotonically decreasing, respectively. As both sequences are bounded from below by zero and from above by the total excess, their limits, which are given in (3.12), exist.

The vector that is the result of the recursive procedure given in Definition 3.7 is a λ-additional cash vector; in fact, it is the only one.

Proposition 3.8. *Let $(E, C) \in \mathcal{F}^N$, let $\phi \in \mathcal{R}^N$, and let $\lambda \in [0,1]^N$. Then, the vector $\theta(\lambda)$ is the only λ-additional cash vector for (E, C) with respect to ϕ.*

Using the following three lemmas, we are able to characterize the set of additional cash vectors via λ-additional cash vectors that follow from the recursive procedure given in Definition 3.7.

The additional cash of exactly one agent is determined in each step of the iterative procedure given in Definition 3.7. As a result, the range of additional cash available to subsequent agents, which is always well-defined, becomes smaller. The following lemma formalizes this observation.

Lemma 3.9. *Let $(E, C) \in \mathcal{F}^N$, let $\phi \in \mathcal{R}^N$, and let $\lambda \in [0,1]^N$. Then, for all $\ell \in \{2, \ldots, n-1\}$ and all $i \in \{\ell+1, \ldots, n\}$, it holds that $\underline{\theta}_i^\ell \leq \underline{\theta}_i^{\ell+1}$ and $\overline{\theta}_i^\ell \geq \overline{\theta}_i^{\ell+1}$. Moreover, for all $\ell \in \{2, \ldots, n\}$, it holds that $\underline{\theta}^\ell \leq \overline{\theta}^\ell$.*

The following lemma generalizes Lemma 3.3 and Lemma 3.4 in the sense that in each step of the iterative procedure given in Definition 3.7 two additional cash vectors are obtained.

Lemma 3.10. *Let $(E, C) \in \mathcal{F}^N$, let $\phi \in \mathcal{R}^N$, and let $\lambda \in [0,1]^N$. Then, for all $\ell \in \{2, \ldots, n\}$, it holds that*

(i) $(\theta_1(\lambda), \ldots, \theta_{\ell-1}(\lambda), \underline{\theta}_\ell^\ell, \ldots, \underline{\theta}_n^\ell) \in D^\phi(E, C)$

(ii) $(\theta_1(\lambda), \ldots, \theta_{\ell-1}(\lambda), \overline{\theta}_\ell^\ell, \ldots, \overline{\theta}_n^\ell) \in D^\phi(E, C)$.

As the following lemma implies, the additional cash vectors that are constructed in each step of the iterative procedure given in Definition 3.7 are minimal and maximal. Moreover, if the additional cash of each agent equals the minimum amount (i.e., $\lambda = 0^N$), then the iterative procedure yields the zero vector. On the other hand, if the additional cash of each agent equals the maximum amount (i.e., $\lambda = 1^N$), then the iterative procedures yields the total excess vector.

Lemma 3.11. *Let $(E, C) \in \mathcal{F}^N$, let $\phi \in \mathcal{R}^N$, and let $\lambda \in [0,1]^N$. Then, for all $\ell \in \{2, \ldots, n\}$, and all $d \in D^\phi(E, C)$ with $d_i = \theta_i(\lambda)$ for all $i \in \{1, \ldots, \ell-1\}$, it holds that $d_i \geq \underline{\theta}_i^\ell$ and $d_i \leq \overline{\theta}_i^\ell$ for all $i \in \{\ell, \ldots, n\}$; in particular, if, for all $i \in \{1, \ldots, \ell-1\}$, $\lambda_i = 0$ ($\lambda_i = 1$), then $\underline{\theta}_i^\ell = 0$ ($\overline{\theta}_i^\ell = \overline{d}_i^\phi(E, C)$) for all $i \in \{\ell, \ldots, n\}$.*

More importantly, Lemma 3.10 implies that the vector of additional cash that is the result of the iterative procedure is an additional cash vector — after all, if the additional cash of the first $n-1$ agents are known, then so is the additional cash of agent n. And, Lemma 3.9 and Lemma 3.11 imply that, for any $d \in D^\phi(E, C)$, we can set, for each agent $\ell \in \{1, \ldots, n\}$,

$$\lambda_\ell = \begin{cases} \dfrac{d_\ell - \underline{\theta}_\ell^\ell}{\overline{\theta}_\ell^\ell - \underline{\theta}_\ell^\ell} & \text{if } \underline{\theta}_\ell^\ell < \overline{\theta}_\ell^\ell \\ 0 & \text{if } \underline{\theta}_\ell^\ell = \overline{\theta}_\ell^\ell, \end{cases} \tag{3.16}$$

to obtain $\theta_\ell(\lambda) = d_\ell$. Combining Lemma 3.9, Lemma 3.10 and Lemma 3.11 therefore yields the following characterization of the set of additional cash vectors.

Theorem 3.12. *Let $(E, C) \in \mathcal{F}^N$, and let $\phi \in \mathcal{R}^N$. Then,*

$$D^\phi(E, C) = \{\theta(\lambda) \mid \lambda \in [0, 1]^N\}.$$

Proof. First, let $\lambda \in [0, 1]^N$. From Lemma 3.10 it follows that $(\theta_1(\lambda), \ldots, \theta_{n-1}(\lambda), \underline{\theta}^n_n) \in D^\phi(E, C)$ and $(\theta_1(\lambda), \ldots, \theta_{n-1}(\lambda), \overline{\theta}^n_n) \in D^\phi(E, C)$. Then, by (3.3), $\underline{\theta}^n_n = \overline{\theta}^n_n$, which implies that $\theta_n(\lambda) = (1 - \lambda_n)\underline{\theta}^n_n + \lambda_n \overline{\theta}^n_n = \underline{\theta}^n_n$. Hence, we have that $\theta(\lambda) \in D^\phi(E, C)$.

Second, let $d \in D^\phi(E, C)$. For each $\ell \in \{1, \ldots, n\}$, set λ_ℓ according to (3.16). If $\overline{d}^\phi_1(E, C) > 0$, then $\lambda_1 = (d_1/\overline{d}^\phi_1(E, C)) \in [0, 1]$ such that $\theta_1(\lambda) = \lambda_1 \overline{d}^\phi_1(E, C) = d_1$. Otherwise $\overline{d}^\phi_1(E, C) = 0$, so $d_1 = 0$ and $\theta_1(\lambda) = 0$. Next, let $\ell \in \{2, \ldots, n\}$ and assume that, for all $i \in \{1, \ldots, \ell - 1\}$, $\theta_i(\lambda) = d_i$. From $\underline{\theta}^\ell_\ell \leq \overline{\theta}^\ell_\ell$ (Lemma 3.9) and $\underline{\theta}^\ell_\ell \leq d_\ell \leq \overline{\theta}^\ell_\ell$ (Lemma 3.11) it follows that $\lambda_\ell \in [0, 1]$ such that

$$\theta_\ell(\lambda) = \underline{\theta}^\ell_\ell + \lambda_\ell(\overline{\theta}^\ell_\ell - \underline{\theta}^\ell_\ell) = \underline{\theta}^\ell_\ell + d_\ell - \underline{\theta}^\ell_\ell = d_\ell.$$

By induction, for all $i \in \{1, \ldots, n\}$, it holds that $\lambda_i \in [0, 1]$ and $\theta_i(\lambda) = d_i$. □

Because the iterative procedure given in Definition 3.7 can be used to obtain any additional cash vector, and because each additional cash vector corresponds to exactly one transfer scheme, the iterative procedure can also be used to obtain any transfer scheme. To emphasize that, for each $\lambda \in [0, 1]^N$, the vector $\theta(\lambda)$ corresponds to an additional cash vector for a financial network $(E, C) \in \mathcal{F}^N$ with respect to $\phi \in \mathcal{R}^N$, we henceforth denote $\theta(\lambda)$ by $\theta^{\phi,\lambda}(E, C)$. Hence, for all $(E, C) \in \mathcal{F}^N$ and for all $\phi \in \mathcal{R}^N$,

$$\mathcal{P}^\phi(E, C) = \{f(\theta^{\phi,\lambda}(E, C)) \mid \lambda \in [0, 1]^N\}.$$

Correspondingly, we define a (ϕ, λ)-*based transfer rule* $\tau^{\phi,\lambda}$ on $\mathcal{F}^N$ that assigns to each financial network exactly one ϕ-based transfer scheme with respect to λ, which indicates the extent to which agents pay in addition to what they pay under the ϕ-based bottom transfer scheme.

Definition 3.13. Let $\phi \in \mathcal{R}^N$, and let $\lambda \in [0, 1]^N$. The (ϕ, λ)-*based transfer rule* $\tau^{\phi,\lambda} \colon \mathcal{F}^N \to \mathbb{R}^{N \times N}$ is, for all $(E, C) \in \mathcal{F}^N$, given by

$$\tau^{\phi,\lambda}(E, C) = f(\theta^{\phi,\lambda}(E, C)).$$

Denote the *set of transfer rules* by $\mathcal{T}^\phi = \{(\lambda, \tau^{\phi,\lambda}) \mid \lambda \in [0, 1]^N\}$.

For each vector of agent-specific claims rules $\phi \in \mathcal{R}^N$, the set of transfer rules $\mathcal{T}^\phi$ consists of pairs of weight vectors and associated transfer rules that prescribe, for each financial network, a transfer scheme based on ϕ and $\lambda \in [0, 1]^N$.

Given $\phi \in \mathcal{R}^N$, two distinct weight vectors $\lambda, \lambda' \in [0, 1]^N$ with $\lambda_i \neq \lambda'_i$ for some $i \in \{1, \ldots, n - 1\}$, lead to two distinct transfer rules $\tau^{\phi,\lambda}$ and $\tau^{\phi,\lambda'}$.[4] However, two weight vectors $\lambda, \lambda' \in [0, 1]^N$ that are equal except for the last coordinate, lead to identical transfer rules $\tau^{\phi,\lambda}$ and $\tau^{\phi,\lambda'}$ because λ_n can be chosen arbitrarily in the construction of a λ-additional cash vector.

[4]To illustrate this, consider the financial network $(E, C) \in \mathcal{F}^N$ given by $N = \{1, 2\}$,

$$E = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Let $\phi \in \mathcal{R}^N$ and $\lambda, \lambda' \in [0, 1]^N$ with $\lambda_1 \neq \lambda'_1$. Then, $\tau^{\phi,\lambda}(E, C) = \lambda_1 C$ and $\tau^{\phi,\lambda'}(E, C) = \lambda'_1 C$.

4 Strongly Connected Components in Financial Networks

In this section, we show the general result that the set of irregular agents can be partitioned into strongly connected components for which the payments between any pair of strongly connected components are uniquely determined. We then use this result to show that the set of PROP-based transfer schemes has a relatively simple structure.

Let $(E, C) \in \mathcal{F}^N$, and let $\phi \in \mathcal{R}^N$. We define the *directed graph* $G^\phi(E, C) = (I^\phi(E, C), A)$ that consists of the irregular agents in (E, C) with respect to ϕ and in which there is an arc from $i \in I^\phi(E, C)$ to $j \in I^\phi(E, C)$ if agent i pays agent j differently with respect to the bottom and top transfer schemes; that is, the set of arcs is given by

$$A = \{(i, j) \in I^\phi(E, C) \times I^\phi(E, C) \,|\, \underline{p}_{ij}^\phi < \overline{p}_{ij}^\phi\}.$$

Note that, for all $i \in I^\phi(E, C)$, $(i, i) \notin A$ because $c_{ii} = 0$, which implies that $\underline{p}_{ii}^\phi = 0 = \overline{p}_{ii}^\phi$.

An irregular agent $i \in I^\phi(E, C)$ is *connected* to another irregular agent $j \in I^\phi(E, C)$ in $G^\phi(E, C)$ if, for some $k \geq 2$, there is a *directed path* of agents $(i_1, \ldots, i_k)$ such that $i_1 = i$ and $i_k = j$ and, for all $\ell \in \{1, \ldots, k-1\}$, $(i_\ell, i_{\ell+1}) \in A$. A set of irregular agents $S \subseteq I^\phi(E, C)$ is a *strongly connected component* in $G^\phi(E, C)$ if any two distinct irregular agents in S are connected, and S is maximal with respect to this property. For each $i \in I^\phi(E, C)$, let $S(i)$ denote the strongly connected component to which i belongs. Then, the collection of strongly connected components $\mathcal{C}^\phi(E, C) = \{S(i) \,|\, i \in I^\phi(E, C)\}$ forms a partition of $I^\phi(E, C)$.

The following theorem implies that the payments between two distinct strongly connected components are uniquely determined. The payments to and by regular agents are always uniquely determined. Therefore, in particular, the payments between agents in a strongly connected component and agents outside that strongly connected component, are uniquely determined.

Theorem 4.1. *Let $(E, C) \in \mathcal{F}^N$, and let $\phi \in \mathcal{R}^N$. Then, for all $S \in \mathcal{C}^\phi(E, C)$, for all $i \in S$, and for all $j \in I^\phi(E, C) \setminus S$, it holds that $\underline{p}_{ij}^\phi = \overline{p}_{ij}^\phi$.*

Proof. Assume that $|\mathcal{C}^\phi(E, C)| \geq 2$. Define the following subcollection of strongly connected components:

$$\mathcal{S} = \{S \in \mathcal{C}^\phi(E, C) \,|\, \text{for all } i \in S, \text{ for all } j \in I^\phi(E, C) \setminus S, \, \underline{p}_{ij}^\phi = \overline{p}_{ij}^\phi\}.$$

If $S \in \mathcal{S}$, then there is no arc from S to a different strongly connected component $S' \in \mathcal{C}^\phi(E, C)$. On the other hand, if $S \in \mathcal{C}^\phi(E, C) \setminus \mathcal{S}$, then there is an arc from S to a different strongly connected component $S' \in \mathcal{C}^\phi(E, C)$.

It suffices to prove that $\mathcal{S} = \mathcal{C}^\phi(E, C)$. To this end, we will prove that, for all $S \in \mathcal{S}$,

$$\sum_{j \in I^\phi(E,C) \setminus S} \sum_{i \in S} (\varphi_i^j(e_j + \sum_{k \in N} \underline{p}_{kj}^\phi + d_j, \overline{c}_j) - \underline{p}_{ji}^\phi) = 0, \tag{4.1}$$

which implies that there is also no arc from a different strongly connected component $S' \in \mathcal{C}^\phi(E, C)$ to S. And, for all $S \in \mathcal{C}^\phi(E, C) \setminus \mathcal{S}$,

$$\sum_{j \in I^\phi(E,C) \setminus S} \sum_{i \in S} (\varphi_i^j(e_j + \sum_{k \in N} \underline{p}_{kj}^\phi + d_j, \overline{c}_j) - \underline{p}_{ji}^\phi) > 0, \tag{4.2}$$

which implies that there exist an arc from a different strongly connected component $S' \in \mathcal{C}^\phi(E,C)$ to S.

Combining (4.1) and (4.2) gives the desired result. To see this, suppose that $\mathcal{S} \neq \mathcal{C}^\phi(E,C)$. It follows from (4.1) that each $S \in \mathcal{C}^\phi(E,C) \setminus \mathcal{S}$ must have an outgoing arc that points to a different $S' \in \mathcal{C}^\phi(E,C) \setminus \mathcal{S}$. Additionally, from (4.2) it follows that each $S \in \mathcal{C}^\phi(E,C) \setminus \mathcal{S}$ must have an incoming arc that comes from a different $S' \in \mathcal{C}^\phi(E,C) \setminus \mathcal{S}$. As a consequence, at least two strongly connected components $S, S' \in \mathcal{C}^\phi(E,C) \setminus \mathcal{S}$ are connected in the sense that for all $i \in S$ and all $j \in S'$, i is connected to j and j is connected to i. However, this is a contradiction because it entails that neither S nor S' is a strongly connected component.

Let $S \in \mathcal{C}^\phi(E,C)$. For all $j \in I^\phi(E,C) \setminus S$, condition (2.3) of an additional cash vector can be restricted to irregular agents, so it can be written as

$$d_j = \sum_{i \in S} (\varphi_j^i(e_i + \sum_{k \in N} \underline{p}_{ki}^\phi + d_i, \overline{c}_i) - \underline{p}_{ij}^\phi) + \sum_{i \in I^\phi(E,C) \setminus S} (\varphi_j^i(e_i + \sum_{k \in N} \underline{p}_{ki}^\phi + d_i, \overline{c}_i) - \underline{p}_{ij}^\phi). \quad (4.3)$$

Similarly, for all $j \in I^\phi(E,C) \setminus S$, Proposition 2.5 (iv) is given by

$$d_j = \sum_{i \in S} (\varphi_i^j(e_j + \sum_{k \in N} \underline{p}_{kj}^\phi + d_j, \overline{c}_j) - \underline{p}_{ji}^\phi) + \sum_{i \in I^\phi(E,C) \setminus S} (\varphi_i^j(e_j + \sum_{k \in N} \underline{p}_{kj}^\phi + d_j, \overline{c}_j) - \underline{p}_{ji}^\phi). \quad (4.4)$$

Using (4.3) and (4.4), we find that

$$\begin{aligned}
&\sum_{i \in S} \sum_{j \in I^\phi(E,C) \setminus S} (\varphi_j^i(e_i + \sum_{k \in N} \underline{p}_{ki}^\phi + d_i, \overline{c}_i) - \underline{p}_{ij}^\phi) \\
&\quad = \sum_{j \in I^\phi(E,C) \setminus S} d_j - \sum_{j \in I^\phi(E,C) \setminus S} \sum_{i \in I^\phi(E,C) \setminus S} (\varphi_j^i(e_i + \sum_{k \in N} \underline{p}_{ki}^\phi + d_i, \overline{c}_i) - \underline{p}_{ij}^\phi) \\
&\quad = \sum_{j \in I^\phi(E,C) \setminus S} \sum_{i \in S} (\varphi_i^j(e_j + \sum_{k \in N} \underline{p}_{kj}^\phi + d_j, \overline{c}_j) - \underline{p}_{ji}^\phi) \\
&\qquad + \sum_{j \in I^\phi(E,C) \setminus S} \sum_{i \in I^\phi(E,C) \setminus S} (\varphi_i^j(e_j + \sum_{k \in N} \underline{p}_{kj}^\phi + d_j, \overline{c}_j) - \underline{p}_{ji}^\phi) \\
&\qquad - \sum_{j \in I^\phi(E,C) \setminus S} \sum_{i \in I^\phi(E,C) \setminus S} (\varphi_j^i(e_i + \sum_{k \in N} \underline{p}_{ki}^\phi + d_i, \overline{c}_i) - \underline{p}_{ij}^\phi) \\
&\quad = \sum_{j \in I^\phi(E,C) \setminus S} \sum_{i \in S} (\varphi_i^j(e_j + \sum_{k \in N} \underline{p}_{kj}^\phi + d_j, \overline{c}_j) - \underline{p}_{ji}^\phi).
\end{aligned}$$

The first equality follows from (4.3); the second equality follows from (4.4). If $S \in \mathcal{S}$, we obtain (4.1); if $S \in \mathcal{C}^\phi(E,C) \setminus \mathcal{S}$, we obtain (4.2). □

Using Theorem 4.1, we will show that the set of additional cash vectors with respect to the proportional rule has a relatively simple structure. For each financial network and each vector of claims rules, the set of irregular agents can be partitioned into strongly connected components in which the payments between any two strongly connected components are uniquely determined. With each strongly connected component we associate a specific weight that is common between all agents in that strongly connected component.

Definition 4.2. Let $(E, C) \in \mathcal{F}^N$, and let $\phi \in \mathcal{R}^N$. Then, $\lambda \in [0,1]^N$ is a *component-based weight vector* for (E, C) with respect to ϕ if, for each $S \in \mathcal{S}^\phi(E, C)$, there exists a $\lambda_S \in [0,1]$ such that, for all $i \in S$, $\lambda_i = \lambda_S$. The set of all possible component-based weight vectors for (E, C) with respect to ϕ is denoted by $\Lambda^\phi(E, C)$.

Let $(E, C) \in \mathcal{F}^N$, and let $\phi = (\text{PROP})_{i \in N} \equiv \text{PROP}$. For each $\lambda \in \Lambda^\phi(E, C)$, define the vector of additional cash $d(\lambda) \in \mathbb{R}^N$ as follows. The additional cash of a regular agent is set to zero, thus, for all $i \notin I^\phi(E, C)$, $d_i(\lambda) = 0$. On the other hand, for all $S \in \mathcal{C}^\phi(E, C)$ and all $i \in S$, the additional cash of irregular agent i is determined by the weight λ_S that is associated with the strongly connected component to which agent i belongs, thus $d_i(\lambda) = \lambda_S \overline{d}_i^\phi(E, C)$.

Indeed, as the following theorem states, $d(\lambda)$ is an additional cash vector for (E, C) with respect to PROP. Furthermore, any additional cash vector for $(E, C) \in \mathcal{F}^N$ with respect to PROP can be written as $d(\lambda)$ for some $\lambda \in \Lambda^{\text{PROP}}(E, C)$.

Theorem 4.3. *Let $(E, C) \in \mathcal{F}^N$. Then, $D^{\text{PROP}}(E, C) = \{d(\lambda) \mid \lambda \in \Lambda^{\text{PROP}}(E, C)\}$.*

Proof. For ease of exposition, set $\phi = \text{PROP}$. Theorem 4.1 implies that, for each $i \in I^\phi(E, C)$, condition (2.3) of an additional cash vector can be restricted to the strongly connected component $S \in \mathcal{C}^\phi(E, C)$ to which irregular agent i belongs. That is, $d \in [0, \overline{d}^\phi(E, C)]$ is an additional cash vector for (E, C) with respect to ϕ if, for all $i \notin I^\phi(E, C)$, $d_i = 0$, and, for all $S \in \mathcal{C}^\phi(E, C)$ and all $i \in S$,

$$d_i = \sum_{j \in S} (\varphi_i^j(e_j + \sum_{k \in N} \underline{p}_{kj}^\phi + d_j, \overline{c}_j) - \underline{p}_{ji}^\phi).$$

We need thus only focus on the irregular agents and the strongly connected components to which they belong.

For all $S \in \mathcal{C}^\phi(E, C)$ and all $i \in S$, we have $c_{ij} \geq \overline{p}_{ij}^\phi > \underline{p}_{ij}^\phi \geq 0$ for some $j \in S$, so also $\sum_{k \in N} c_{ik} > 0$; furthermore, Proposition 2.5 (iii) implies that, for all $P \in \mathcal{P}^\phi(E, C)$, $0 \leq e_i + \sum_{k \in N} p_{ki} \leq \sum_{k \in N} c_{ik}$. Consequently, when $\phi = \text{PROP}$, we have $d \in D^\phi(E, C)$ if, for all $S \in \mathcal{C}^\phi(E, C)$ and all $i \in S$,

$$\begin{aligned} d_i &= \sum_{j \in S} (\varphi_i^j(e_j + \sum_{k \in N} \underline{p}_{kj}^\phi + d_j, \overline{c}_j) - \underline{p}_{ji}^\phi) \\ &= \sum_{j \in S} \big((e_j + \sum_{k \in N} \underline{p}_{kj}^\phi + d_j) \frac{c_{ji}}{\sum_{k \in N} c_{jk}} - (e_j + \sum_{k \in N} \underline{p}_{kj}^\phi) \frac{c_{ji}}{\sum_{k \in N} c_{jk}}\big) \\ &= \sum_{j \in S} d_j \frac{c_{ji}}{\sum_{k \in N} c_{jk}}. \end{aligned}$$

First, let $\lambda \in \Lambda^\phi(E, C)$. Then, for all $S \in \mathcal{C}^\phi(E, C)$ and all $i \in S$,

$$d_i(\lambda) = \lambda_S d_i^\phi(E, C) = \sum_{j \in S} \lambda_S \overline{d}_j^\phi(E, C) \frac{c_{ji}}{\sum_{k \in N} c_{jk}} = \sum_{j \in S} \lambda_S d_j(\lambda) \frac{c_{ji}}{\sum_{k \in N} c_{jk}},$$

in which the second equality follows from $\overline{d}^{\phi}(E,C) \in D^{\phi}(E,C)$. Thus, $d(\lambda) \in D^{\phi}(E,C)$.

Second, let $d \in D^{\phi}(E,C)$ and let $\lambda \in [0,1]^N$. Let $S \in \mathcal{C}^{\phi}(E,C)$ and, for all $i \in S$, set $\lambda_i = d_i/\overline{d}_i^{\phi}(E,C)$ such that $d_i = \lambda_i \overline{d}_i^{\phi}(E,C)$. We will prove that there exists a $\lambda_S \in [0,1]$ such that, for all $i \in S$, $\lambda_i = \lambda_S$. This implies that, for all $i \in N$, $d_i = d_i(\lambda)$.

Because $d \in D^{\phi}(E,C)$, for all $i \in S$, it holds that

$$\lambda_i \overline{d}_i^{\phi}(E,C) = \sum_{j \in S} \lambda_j \overline{d}_j^{\phi}(E,C) \frac{c_{ji}}{\sum_{k \in N} c_{jk}}. \tag{4.5}$$

In addition to this, because $\overline{d}^{\phi}(E,C) \in D^{\phi}(E,C)$, for all $i \in S$, it holds that

$$\lambda_i \overline{d}_i^{\phi}(E,C) = \sum_{j \in S} \lambda_i \overline{d}_j^{\phi}(E,C) \frac{c_{ji}}{\sum_{k \in N} c_{jk}}. \tag{4.6}$$

Combining (4.5) and (4.6) implies that, for all $i \in S$,

$$\sum_{j \in S} (\lambda_j - \lambda_i) \overline{d}_j^{\phi}(E,C) \frac{c_{ji}}{\sum_{k \in N} c_{jk}} = 0. \tag{4.7}$$

Without loss of generality, let $S = \{i_1, \ldots, i_s\}$ with $s = |S|$ be such that $\lambda_{i_1} \geq \cdots \geq \lambda_{i_s}$. Let $\ell \in \{1, \ldots, s-1\}$ and assume that $\lambda_{i_1} = \cdots = \lambda_{i_\ell}$. We will show that $\lambda_{i_\ell} = \lambda_{i_{\ell+1}}$. If $\lambda_{i_1} = \cdots = \lambda_{i_\ell}$, then, for all $i \in \{i_1, \ldots, i_\ell\}$, (4.7) simplifies to

$$\sum_{j \in \{i_{\ell+1}, \ldots, i_s\}} (\lambda_j - \lambda_i) \overline{d}_j^{\phi}(E,C) \frac{c_{ji}}{\sum_{k \in N} c_{jk}} = 0. \tag{4.8}$$

As a result, we obtain

$$\begin{aligned}
&(\lambda_{i_{\ell+1}} - \lambda_{i_\ell}) \sum_{j \in \{i_{\ell+1}, \ldots, i_s\}} \overline{d}_j^{\phi}(E,C) \frac{\sum_{i \in \{i_1, \ldots, i_\ell\}} c_{ji}}{\sum_{k \in N} c_{jk}} \\
&\quad = (\lambda_{i_{\ell+1}} - \lambda_{i_\ell}) \sum_{i \in \{i_1, \ldots, i_\ell\}} \sum_{j \in \{i_{\ell+1}, \ldots, i_s\}} \overline{d}_j^{\phi}(E,C) \frac{c_{ji}}{\sum_{k \in N} c_{jk}} \\
&\quad = \sum_{i \in \{i_1, \ldots, i_\ell\}} (\lambda_{i_{\ell+1}} - \lambda_i) \sum_{j \in \{i_{\ell+1}, \ldots, i_s\}} \overline{d}_j^{\phi}(E,C) \frac{c_{ji}}{\sum_{k \in N} c_{jk}} \\
&\quad \geq \sum_{i \in \{i_1, \ldots, i_\ell\}} \sum_{j \in \{i_{\ell+1}, \ldots, i_s\}} (\lambda_j - \lambda_i) \overline{d}_j^{\phi}(E,C) \frac{c_{ji}}{\sum_{k \in N} c_{jk}} \\
&\quad = 0.
\end{aligned}$$

The second equality follows from $\lambda_{i_1} = \cdots = \lambda_{i_\ell}$; the inequality follows from $\lambda_{i_{\ell+1}} \geq \cdots \geq \lambda_{i_s}$; the last equality follows from (4.8).

Because $S \in \mathcal{C}^{\phi}(E,C)$, agents i_ℓ and $i_{\ell+1}$ are connected, and thus by Theorem 4.1, $c_{ji} \geq \overline{p}_{ji}^{\phi} > \underline{p}_{ji}^{\phi} \geq 0$ for some $j \in \{i_{\ell+1}, \ldots, i_s\}$ and $i \in \{i_1, \ldots, i_\ell\}$. Consequently, it follows that $\lambda_{i_{\ell+1}} - \lambda_{i_\ell} \geq 0$, which, in turn, implies that $\lambda_{i_\ell} = \lambda_{i_{\ell+1}}$. Hence, by induction, $\lambda_{i_1} = \cdots = \lambda_{i_s}$. In particular, if we set $\lambda_S = \lambda_{i_1} = \cdots = \lambda_{i_s}$, then $\lambda_S \in [0,1]$. □

Theorem 4.3 that characterizes the set of additional cash vectors with respect to the proportional rule implies that the iterative procedure as given in Definition 3.7 simplifies considerably when $\phi = \text{PROP}$. Let $(E, C) \in \mathcal{F}^N$, and let $\lambda \in [0,1]^N$. Consider $S \in \mathcal{C}^\phi(E, C)$ and let $\ell = \min\{j \mid j \in S\}$ be the first agent in the strongly connected component S that is selected in the iterative procedure. Theorem 4.3 implies that, for all $i \in S$, $\theta_i^{\phi,\lambda}(E, C) = \lambda_S \overline{d}_i^\phi(E, C)$, in which $\lambda_S = \lambda_\ell$. The weight $\lambda_\ell \in [0,1]$ of agent $\ell \in S$ thus determines the additional cash of the other agents in the strongly connected component S. In fact, only the bottom and top PROP-based transfer schemes have to be determined to construct the set of PROP-based additional cash vectors.

The set of additional cash vectors is homeomorphic to the set of transfer schemes (Theorem 2.11), which means that Theorem 4.3 reveals the structure of the set of transfer schemes with respect to the proportional rule. Let $(E, C) \in \mathcal{F}^N$. For each $\lambda \in \Lambda^{\text{PROP}}(E, C)$, the payment matrix $f(d(\lambda))$ is a transfer scheme for (E, C) with respect to PROP. More specifically, for all $S \in \mathcal{C}^{\text{PROP}}(E, C)$ and all $i \in S$, the payment by agent i to agent $j \in N$ is given by a convex combination of the bottom PROP-based payment and the top PROP-based payment with respect to $\lambda_S \in [0,1]$, that is,

$$f_{ij}(d(\lambda)) = (1 - \lambda_S)\underline{p}_{ij}^{\text{PROP}} + \lambda_S \overline{p}_{ij}^{\text{PROP}}. \tag{4.9}$$

All mutual payments between agents in a strongly connected component and agents outside that strongly connected component are uniquely determined, which is not the case for mutual payments inside the strongly connected component (Theorem 4.1). In the latter case, the convex combination in (4.9) is therefore nontrivial. Theorem 2.11 and Theorem 4.3 imply the following corollary.

Corollary 4.4. *Let $(E, C) \in \mathcal{F}^N$. Then, $\mathcal{P}^{\text{PROP}}(E, C) = \{f(d(\lambda)) \mid \lambda \in \Lambda^{\text{PROP}}(E, C)\}$.*

5 Axiomatizations of the Proportional Rule

This section demonstrates that Theorem 3.12 on the characterization of additional cash vectors lends itself to an axiomatic analysis of proportional clearing payments.

We will introduce axioms for any set of transfer rules $\mathcal{T}^\phi$. Recall from Definition 3.13 that a set of transfer rules $\mathcal{T}^\phi$ comprises pairs of weight vectors and associated transfer rules. A transfer rule $\tau^{\phi,\lambda}$ prescribes, for each financial network $(E, C) \in \mathcal{F}^N$, a transfer scheme based on $\lambda \in [0,1]^N$ in which each agent $i \in N$ uses its claims rule φ^i to pay its creditors. A *proportional transfer rule* $\tau^{\text{PROP},\lambda}$ is a specific type of transfer rule in which each agent $i \in N$ uses the proportional rule (i.e., $\varphi^i = \text{PROP}$). The set $\mathcal{T}^{\text{PROP}}$ contains such proportional transfer rules.

Our axioms are *weak convexity*, *convexity*, and *decomposition*, which are related to *homogeneity*, *linearity*, and *additivity* for mappings on general vector spaces, respectively. Homogeneity and additivity are independent axioms in general. There exist mappings that are homogeneous but not additive, and vice versa. A mapping is linear if and only if it is homogeneous and additive.

For example, let $M = \{1, 2\}$ and $c > 0$ and consider the mapping $g\colon [0,1]^M \to \mathbb{R}$ in which, for all $\lambda \in [0,1]^M$, $g(\lambda) = \lambda_1 c$ if $\lambda_1 = \lambda_2$, and $g(\lambda) = 0$ otherwise. The mapping g is

homogeneous if, for all $\mu \in [0,1]$ and all $\lambda \in [0,1]^M$, $g(\mu\lambda) = \mu g(\lambda)$. Clearly, the mapping g is homogeneous. The mapping g is additive if, for all $\lambda, \lambda' \in [0,1]^M$ with $\lambda + \lambda' \in [0,1]^M$, $g(\lambda+\lambda') = g(\lambda)+g(\lambda')$. Consider $\lambda = (0, 1/2)$ and $\lambda' = (1/2, 0)$. Then, $g(\lambda+\lambda') = (1/2)c \neq 0 = g(\lambda) + g(\lambda')$, illustrating that g is not additive.

Mappings that are additive but not homogeneous exist, too, and such mappings must be discontinuous. Exemplifying existence of such mappings requires the axiom of choice, which goes beyond the scope of this article. We refer the interested reader to Hamel (1905), and also to page 20 of Torchinsky (1988).

Interestingly, we will see that weak convexity, convexity, and decomposition are equivalent in the financial network setting because all three characterize the set of proportional transfer rules. In fact, weak convexity, being the axiom imposing the least restrictive conditions on the clearing payments, suffices to characterize the proportional rule in financial networks.

Weak convexity entails that any convex combination of the bottom and top transfer schemes is also a transfer scheme.

Definition 5.1. Let $\phi \in \mathcal{R}^N$, and let $\sigma^0 = (0^N, \tau^{\phi,0^N}) \in \mathcal{T}^\phi$ and $\sigma^1 = (1^N, \tau^{\phi,1^N}) \in \mathcal{T}^\phi$. The set of transfer rules $\mathcal{T}^\phi$ satisfies *weak convexity* if, for all $\mu \in [0,1]$, $(1-\mu)\sigma^0 + \mu\sigma^1 = (\mu^N, (1-\mu)\tau^{\phi,0^N} + \mu\tau^{\phi,1^N}) \in \mathcal{T}^\phi$.

That is, weak convexity requires that a convex combination of the transfer rules $\tau^{\phi,0^N}$ and $\tau^{\phi,1^N}$ with respect to $\mu \in [0,1]$ equals the transfer rule τ^{ϕ,μ^N}.

In the unilateral *claims problem* setting, in which an amount is to be divided among a group of claimants with respect to a claims vector, weak convexity as in Definition 5.1 has no bite because it is satisfied by any claims rule. The claims rule φ^i of agent $i \in N$ prescribes a unique allocation vector for each claims problem. In the financial network setting, however, multiple transfer schemes can exist for a financial network and there is a choice to be made.

Theorem 5.2 states that only the set of proportional transfer rules is weakly convex.

Theorem 5.2. *Let $\phi \in \mathcal{R}^N$. Then, $\mathcal{T}^\phi$ satisfies weak convexity if and only if $\phi = \mathrm{PROP}$.*

Proof. First, assume that $\mathcal{T}^\phi$ satisfies weak convexity. Consider $(E, C) \in \mathcal{F}^N$ with $E = 0^N$ and $C > 0$, in which C is a symmetric claims matrix. Then, it holds that $\tau^{\phi,0^N}(E,C) = \underline{P}^\phi(E,C) = 0^{N\times N}$ and $\tau^{\phi,1^N}(E,C) = \overline{P}^\phi(E,C) = C$.

Let $i \in N$. Weak convexity of $\mathcal{T}^\phi$ implies that, for all $\mu \in [0,1]$, $\tau^{\phi,\mu^N}(E,C) = (1-\mu)\tau^{\phi,0^N}(E,C) + \mu\tau^{\phi,1^N}(E,C) = (1-\mu)0^{N\times N} + \mu C$ and thus

$$\mu\overline{c}_i = \varphi^i(\mu \sum_{k\in N} c_{ki}, \overline{c}_i), \tag{5.1}$$

in which the equality follows from $\tau^{\phi,\mu^N}(E,C) = \mu C \in \mathcal{P}^\phi(E,C)$. The claims matrix C is symmetric, so $\sum_{k\in N} c_{ki} = \sum_{k\in N} c_{ik}$. Let $\mu \in [0,1)$ and set $a = \mu \sum_{k\in N} c_{ki}$. Then, $a \in [0, \sum_{k\in N} c_{ik})$ and

$$\varphi^i(a, \overline{c}_i) = \mu\overline{c}_i = \mu \sum_{k\in N} c_{ki} \frac{\overline{c}_i}{\sum_{k\in N} c_{ik}} = \frac{\overline{c}_i}{\sum_{k\in N} c_{ik}} a = \mathrm{PROP}(a, \overline{c}_i),$$

in which the first equality follows from (5.1) and the second equality follows from $\sum_{k\in N} c_{ki} = \sum_{k\in N} c_{ik}$. Hence, we establish that $\varphi^i = \text{PROP}$.

Second, assume that $\phi = \text{PROP}$. Let $\mu \in [0,1]$, and let $(E,C) \in \mathcal{F}^N$. To prove that $\mathcal{T}^{\text{PROP}}$ satisfies weak convexity, it suffices to show that

$$\tau^{\text{PROP},\mu^N}(E,C) = (1-\mu)\underline{P}^{\text{PROP}}(E,C) + \mu\overline{P}^{\text{PROP}}(E,C). \tag{5.2}$$

It holds that $\mu^N \in \Lambda^{\text{PROP}}(E,C)$, so (4.9) and Corollary 4.4 imply that (5.2) holds. □

Convexity generalizes weak convexity by requiring that any convex combination of any two transfer rules must also be a transfer rule.

Definition 5.3. Let $\phi \in \mathcal{R}^N$. The set of transfer rules $\mathcal{T}^\phi$ satisfies *convexity* if, for all $\sigma = (\lambda, \tau^{\phi,\lambda}) \in \mathcal{T}^\phi$ and $\sigma' = (\lambda', \tau^{\phi,\lambda'}) \in \mathcal{T}^\phi$, and for all $\mu \in [0,1]$, it holds that $(1-\mu)\sigma + \mu\sigma' = ((1-\mu)\lambda + \mu\lambda', (1-\mu)\tau^{\phi,\lambda} + \mu\tau^{\phi,\lambda'}) \in \mathcal{T}^\phi$.

That is, convexity requires that a convex combination of the transfer rules $\tau^{\phi,\lambda}$ and $\tau^{\phi,\lambda'}$ with respect to $\mu \in [0,1]$ equals the transfer rule $\tau^{\phi,(1-\mu)\lambda+\mu\lambda'}$. In essence, convexity means that any convex combination of any two transfer schemes is also a transfer scheme.

In the financial network setting, only the set of proportional transfer rules is weakly convex. In fact, as the following proposition states, it is also the only one to be convex.

Proposition 5.4. *Let $\phi \in \mathcal{R}^N$. Then, $\mathcal{T}^\phi$ satisfies convexity if and only if $\phi = \text{PROP}$.*

We conclude the section by introducing decomposition. Let $\phi \in \mathcal{R}^N$, and define the set $\widetilde{\mathcal{T}}^\phi = \mathcal{T}^\phi - \{(0^N, \tau^{\phi,0^N})\}$. A pair $(\lambda, \tilde{\tau}^{\phi,\lambda}) \in \widetilde{\mathcal{T}}^\phi$ prescribes, for each financial network $(E,C) \in \mathcal{F}^N$, the additional payment matrix $\tilde{\tau}^{\phi,\lambda}(E,C) = \tau^{\phi,\lambda}(E,C) - \tau^{\phi,0^N}(E,C)$ that contains payments net of the bottom transfer scheme $\underline{P}^\phi(E,C)$. Weak convexity can be expressed in terms of $\widetilde{\mathcal{T}}^\phi$ by requiring that, for all $\mu \in [0,1]$, $(\mu^N, \mu\tilde{\tau}^{\phi,1^N}) \in \widetilde{\mathcal{T}}^\phi$.

Decomposition says that the ϕ-based additional payments by the agents can be decomposed as follows. If we have the pair $(\lambda+\lambda', \tilde{\tau}^{\phi,\lambda+\lambda'}) \in \widetilde{\mathcal{T}}^\phi$ with $\lambda + \lambda' \in [0,1]^N$, then decomposition implies that the additional payment matrices prescribed by $\tilde{\tau}^{\phi,\lambda+\lambda'}$ can be decomposed into the payment matrices prescribed by $\tilde{\tau}^{\phi,\lambda}$ and $\tilde{\tau}^{\phi,\lambda'}$.

Definition 5.5. Let $\phi \in \mathcal{R}^N$. The set $\widetilde{\mathcal{T}}^\phi$ satisfies *decomposition* if, for all $\sigma = (\lambda, \tilde{\tau}^{\phi,\lambda}) \in \widetilde{\mathcal{T}}^\phi$, and $\sigma' = (\lambda', \tilde{\tau}^{\phi,\lambda'}) \in \widetilde{\mathcal{T}}^\phi$ with $\lambda+\lambda' \in [0,1]^N$, it holds that $\sigma+\sigma' = (\lambda+\lambda', \tilde{\tau}^{\phi,\lambda}+\tilde{\tau}^{\phi,\lambda'}) \in \widetilde{\mathcal{T}}^\phi$.

Also decomposition has no bite in the unilateral context of claims problems and claims rules because the allocation vector prescribed by each claims rule is uniquely determined, which implies that any claims rule satisfies decomposition.

Decomposition entails that there exist a way to prescribe the payments between agents in stages. For example, to prescribe the top transfer scheme for a financial network (i.e., $\lambda+\lambda' = 1^N$), one can first prescribe a transfer scheme with respect to $\lambda = (1/m)^N$ with $m \in \mathbb{N}$ and subsequently construct the additional payment matrix with respect to $\lambda' = (1-1/m)^N$ to prescribe the remaining transfers. That is, decomposition of $\widetilde{\mathcal{T}}^\phi$ implies that

$$\tau^{\phi,1^N} = \tau^{\phi,(1/m)^N} + (\tau^{\phi,(1-1/m)^N} - \tau^{\phi,0^N}). \tag{5.3}$$

In fact, as the following proposition states, decomposition implies weak convexity. Crucial in proving this result is that the agent-specific claims rules satisfy resource monotonicity, and, by extension, also resource continuity.

Proposition 5.6. *Let $\phi \in \mathcal{R}^N$. If $\widetilde{\mathcal{T}}^{\phi}$ satisfies decomposition, then $\mathcal{T}^{\phi}$ satisfies weak convexity.*

In addition to Proposition 5.6, the set $\widetilde{\mathcal{T}}^{\text{PROP}}$ satisfies decomposition. This leads to the following result that the proportional rule can also be axiomatized using decomposition.

Proposition 5.7. *Let $\phi \in \mathcal{R}^N$. Then, $\widetilde{\mathcal{T}}^{\phi}$ satisfies decomposition if and only if $\phi = \text{PROP}$.*

6 Concluding Remarks

This article analyzes financial networks that consist of agents that have assets and are connected to each other by mutual liabilities. We study transfer schemes, which are payment matrices that contain clearing payments in accordance with claims rules. A transfer scheme prescribes, for each financial network, payments between agents to settle their mutual liabilities. This article is the first to fully characterize such transfer schemes. In fact, our characterization result, which relies on additional cash vectors, encompasses a wide variety of claims rules that dictate how agents should pay their creditors, including the proportional rule that is commonly used in practice.

We introduce a recursive procedure that computes all transfer schemes for an arbitrary financial network. Using this complete characterization, we show that transfer schemes in which each agent pays its creditors in accordance with the proportional rule, have a relatively simple structure. When the mutual payments of an agent are not uniquely determined, we speak of an irregular agent, whereas we speak of regular agent otherwise. Payments between irregular agents in any proportional transfer scheme can be written as a component-based convex combination of their proportional bottom and top payments. Irregular agents belonging to the same strongly connected component have the same weight that determines the convex combination, but the weights may vary across strongly connected components.

Our characterization additionally facilitates a general network-based axiomatic analysis of transfer rules, which associate with each financial network a transfer scheme. We introduce three axioms, namely weak convexity, convexity, and decomposition, that each individually provide an axiomatization of transfer rules that are based on proportional claims rules. To formulate different axioms specific to the financial network setting, one can take inspiration from axiomatizations of claims rules for a unilateral bankruptcy situation. Thomson (2019) provides an extensive survey of the literature on axiomatizations of claims rules. We leave a study in this vein as future research.

Finally, we want to stress that Demange (2023) and Csóka and Herings (2023) focus on kinds of proportionality that are different from ours. Demange (2023) provides an axiomatization of the constrained proportional rule in financial networks, which is not a claims rule because payments can be larger than the claims to prevent bankruptcy of some agents. Csóka and Herings (2023) provides an axiomatization of the pairwise netting proportional

rule in financial networks in which agents first settle their bilateral claims after which the proportional rule is applied with respect to the reduced claims. The two consecutive steps of the pairwise netting proportional rule can not be captured by a single claims rule.

A Proofs

Payments in accordance with a transfer scheme give rise to a redistribution of the estates vector, which we call a *transfer allocation*.

Definition A.1. Let $(E, C) \in \mathcal{F}^N$, let $\phi \in \mathcal{R}^N$, and let $P \in \mathcal{P}^\phi(E, C)$. The vector $a(P) \in \mathbb{R}^N$ is the *transfer allocation* corresponding to P with respect to ϕ if, for all $i \in N$,

$$a_i(P) = e_i + \sum_{j \in N} p_{ji} - \sum_{j \in N} p_{ij}. \tag{A.1}$$

If an agent has a positive transfer allocation, then it has paid off all its debts; if an agent has a negative transfer allocation, then it has not paid any of its creditors. This is what the following lemma formalizes.

Lemma A.2 (Lemma 5.2 in Ketelaars et al. (2023))**.** *Let $(E, C) \in \mathcal{F}^N$, let $\phi \in \mathcal{R}^N$, let $P \in \mathcal{P}^\phi(E, C)$, and let $i \in N$. Then,*

(i) if $a_i(P) > 0$, then $p_{ij} = c_{ij}$ for all $j \in N$;

(ii) if $a_i(P) < 0$, then $p_{ij} = 0$ for all $j \in N$.

As the following lemma states, any two transfer schemes lead to the same transfer allocation.

Lemma A.3 (Proposition 5.3 in Ketelaars et al. (2023))**.** *Let $(E, C) \in \mathcal{F}^N$, let $\phi \in \mathcal{R}^N$, and let $P, P' \in \mathcal{P}^\phi(E, C)$. Then, $a(P) = a(P')$.*

Both Lemma A.2 and Lemma A.3 are used in the proof of Proposition 2.5.

Proof of Proposition 2.5. (i). Suppose that $|I^\phi(E, C)| = 1$. Let $i \in I^\phi(E, C)$. Then, there exists a $j \in N$ with $j \neq i$ such that

$$\underline{p}^\phi_{ij} = \varphi^i_j(e_i + \sum_{k \in N} \underline{p}^\phi_{ki}, \overline{c}_i) < \varphi^i_j(e_i + \sum_{k \in N} \overline{p}^\phi_{ki}, \overline{c}_i) = \overline{p}^\phi_{ij}.$$

By resource monotonicity of φ^i, it holds that

$$e_i + \sum_{k \in N} \underline{p}^\phi_{ki} < e_i + \sum_{k \in N} \overline{p}^\phi_{ki},$$

or, equivalently,

$$\sum_{k \in N} \underline{p}^\phi_{ki} < \sum_{k \in N} \overline{p}^\phi_{ki}.$$

Therefore, because $\overline{p}^\phi_{ii} = \underline{p}^\phi_{ii} = 0$, there must exist an $\ell \in N$ with $\ell \neq i$ such that $\underline{p}^\phi_{\ell i} < \overline{p}^\phi_{\ell i}$, that is, $\ell \in I^\phi(E, C)$. This contradicts $|I^\phi(E, C)| = 1$.

(ii). Let $i \in N$. First, assume that $i \in I^{\phi}(E,C)$. Suppose that $\sum_{k\in N} \underline{p}^{\phi}_{ki} = \sum_{k\in N} \overline{p}^{\phi}_{ki}$. Then,

$$e_i + \sum_{k\in N} \underline{p}^{\phi}_{ki} = e_i + \sum_{k\in N} \overline{p}^{\phi}_{ki},$$

such that, for all $j \in N \setminus \{i\}$,

$$\underline{p}^{\phi}_{ij} = \varphi^i_j(e_i + \sum_{k\in N} \underline{p}^{\phi}_{ki}, \overline{c}_i) = \varphi^i_j(e_i + \sum_{k\in N} \overline{p}^{\phi}_{ki}, \overline{c}_i) = \overline{p}^{\phi}_{ij},$$

which contradicts $i \in I^{\phi}(E,C)$. Hence, it must hold that $\sum_{k\in N} \underline{p}^{\phi}_{ki} < \sum_{k\in N} \overline{p}^{\phi}_{ki}$.

Second, assume that $\sum_{k\in N} \underline{p}^{\phi}_{ki} < \sum_{k\in N} \overline{p}^{\phi}_{ki}$. Resource monotonicity of φ^i implies that there exists a $j \in N \setminus \{i\}$ such that

$$\underline{p}^{\phi}_{ij} = \varphi^i_j(e_i + \sum_{k\in N} \underline{p}^{\phi}_{ki}, \overline{c}_i) < \varphi^i_j(e_i + \sum_{k\in N} \overline{p}^{\phi}_{ki}, \overline{c}_i) = \overline{p}^{\phi}_{ij},$$

which implies that $i \in I^{\phi}(E,C)$.

(iii). Let $i \in I^{\phi}(E,C)$, and let $P \in \mathcal{P}^{\phi}(E,C)$. First, suppose that $P \in \mathcal{P}^{\phi}(E,C)$ is such that $a_i(P) > 0$. Then, by Lemma A.3, $a_i(P) = a_i(\underline{P}^{\phi}(E,C)) = a_i(\overline{P}^{\phi}(E,C)) > 0$. From Lemma A.2 it consequently follows that $\underline{p}^{\phi}_{ij} = \overline{p}^{\phi}_{ij} = c_{ij}$ for all $j \in N$, which contradicts the assumption that $i \in I^{\phi}(E,C)$.

Second, suppose that $P \in \mathcal{P}^{\phi}(E,C)$ is such that $a_i(P) < 0$. Then, by applying similar arguments as for the case in which $a_i(P) > 0$, it follows that $\underline{p}^{\phi}_{ij} = \overline{p}^{\phi}_{ij} = 0$ for all $j \in N$, which contradicts the assumption that $i \in I^{\phi}(E,C)$.

Hence, it must hold that $a_i(P) = 0$, which is equivalent to $\sum_{j\in N} p_{ij} = e_i + \sum_{j\in N} p_{ji}$ (see (A.1)).

(iv). Let $i \in N$, and let $\Delta \in [0, \overline{d}^{\phi}_i(E,C)]$. If $i \notin I^{\phi}(E,C)$, then Proposition 2.5 (ii) implies that $\overline{d}^{\phi}_i(E,C) = 0$ and $\Delta = 0$. The fact that $\underline{P}^{\phi}(E,C) \in \mathcal{P}^{\phi}(E,C)$ implies that

$$\sum_{j\in N} \underline{p}^{\phi}_{ij} + \Delta = \sum_{j\in N} \varphi^i_j(e_i + \sum_{k\in N} \underline{p}^{\phi}_{ki} + \Delta, \overline{c}_i).$$

Assume that $i \in I^{\phi}(E,C)$. We distinguish between two cases with respect to Δ.

First, let Δ be such that

$$e_i + \sum_{k\in N} \underline{p}^{\phi}_{ki} + \Delta < \sum_{j\in N} c_{ij}.$$

Then, by condition (ii) of claims rule φ^i,

$$\sum_{j\in N} \varphi^i_j(e_i + \sum_{k\in N} \underline{p}^{\phi}_{ki} + \Delta, \overline{c}_i) = e_i + \sum_{k\in N} \underline{p}^{\phi}_{ki} + \Delta. \tag{A.2}$$

Furthermore, from Proposition 2.5 (iii) it follows that

$$\sum_{j\in N} \underline{p}^{\phi}_{ij} = e_i + \sum_{k\in N} \underline{p}^{\phi}_{ki}. \tag{A.3}$$

Therefore, (A.2) and (A.3) imply that

$$\sum_{j\in N} \underline{p}^{\phi}_{ij} + \Delta = \sum_{j\in N} \varphi^i_j(e_i + \sum_{k\in N} \underline{p}^{\phi}_{ki} + \Delta, \overline{c}_i).$$

Second, let Δ be such that

$$e_i + \sum_{k\in N} \underline{p}^{\phi}_{ki} + \Delta \geq \sum_{j\in N} c_{ij}. \tag{A.4}$$

We will first show that (A.4) implies that $\Delta = \sum_{j\in N}(\overline{p}^{\phi}_{ji} - \underline{p}^{\phi}_{ji})$. If (A.4) holds, then condition (ii) of claims rule φ^i implies that

$$\sum_{j\in N} \varphi^i_j(e_i + \sum_{k\in N} \underline{p}^{\phi}_{ki} + \Delta, \overline{c}_i) = \sum_{j\in N} c_{ij}. \tag{A.5}$$

Moreover, if (A.4) holds, then, because $\Delta \leq \sum_{j\in N}(\overline{p}^{\phi}_{ji} - \underline{p}^{\phi}_{ji})$, it must also hold that

$$e_i + \sum_{k\in N} \overline{p}^{\phi}_{ki} \geq \sum_{j\in N} c_{ij},$$

and conditions (i) and (ii) of claims rule φ^i imply that $\overline{p}^{\phi}_{ij} = c_{ij}$ for all $j \in N$. Moreover, from Proposition 2.5 (iii) it follows that

$$\sum_{j\in N} c_{ij} = \sum_{j\in N} \overline{p}^{\phi}_{ij} = e_i + \sum_{k\in N} \overline{p}^{\phi}_{ki}. \tag{A.6}$$

Combining (A.4) and (A.6) gives

$$\sum_{j\in N} c_{ij} \leq e_i + \sum_{k\in N} \underline{p}^{\phi}_{ki} + \Delta \leq e_i + \sum_{k\in N} \overline{p}^{\phi}_{ki} = \sum_{j\in N} c_{ij},$$

from which it follows that $e_i + \sum_{k\in N} \underline{p}^{\phi}_{ki} + \Delta = e_i + \sum_{k\in N} \overline{p}^{\phi}_{ki}$ so that

$$\Delta = \sum_{j\in N}(\overline{p}^{\phi}_{ji} - \underline{p}^{\phi}_{ji}). \tag{A.7}$$

Lastly, Proposition 2.5 (iii) implies that

$$\sum_{j\in N} \underline{p}^{\phi}_{ij} = e_i + \sum_{k\in N} \underline{p}^{\phi}_{ki}. \tag{A.8}$$

Therefore,

$$\begin{aligned}\sum_{j\in N}\underline{p}^{\phi}_{ij}+\Delta &= e_i+\sum_{k\in N}\underline{p}^{\phi}_{ki}+\Delta\\ &= e_i+\sum_{k\in N}\underline{p}^{\phi}_{ki}+\sum_{k\in N}(\overline{p}^{\phi}_{ki}-\underline{p}^{\phi}_{ki})\\ &= e_i+\sum_{k\in N}\overline{p}^{\phi}_{ki}\\ &= \sum_{j\in N}c_{ij}\\ &= \sum_{j\in N}\varphi^i_j(e_i+\sum_{k\in N}\underline{p}^{\phi}_{ki}+\Delta,\overline{c}_i).\end{aligned}$$

The first equality follows from (A.8); the second equality follows from (A.7); the fourth equality follows from (A.6); the last equality follows from (A.5). □

Proof of Proposition 2.10. Let $(E,C)\in\mathcal{F}^N$, and let $\phi\in\mathcal{R}^N$. To show that g is the inverse of f, we first show that, for all $d\in D^{\phi}(E,C)$, $f(d)\in\mathcal{P}^{\phi}(E,C)$, and that, for all $P\in\mathcal{P}^{\phi}(E,C)$, $g(P)\in D^{\phi}(E,C)$. Subsequently, we show that, for all $d\in D^{\phi}(E,C)$, $g(f(d))=d$, and that, for all $P\in\mathcal{P}^{\phi}(E,C)$, $f(g(P))=P$. Finally, we show that f and g are monotone.

Let $d\in D^{\phi}(E,C)$. We will show that $f(d)\in\mathcal{P}^{\phi}(E,C)$. By (2.1), it suffices to show that, for all $i,j\in N$, it holds that

$$f_{ij}(d)=\varphi^i_j(e_i+\sum_{k\in N}f_{ki}(d),\overline{c}_i).$$

Let $i\in N$. Then, for all $j\in N$,

$$\begin{aligned}f_{ij}(d) &= \varphi^i_j(e_i+\sum_{k\in N}\underline{p}^{\phi}_{ki}+d_i,\overline{c}_i)\\ &= \varphi^i_j(e_i+\sum_{k\in N}\varphi^k_i(e_k+\sum_{h\in N}\underline{p}^{\phi}_{hk}+d_k,\overline{c}_k),\overline{c}_i)\\ &= \varphi^i_j(e_i+\sum_{k\in N}f_{ki}(d),\overline{c}_i),\end{aligned}$$

in which the first equality follows from (2.4), the second equality follows from $d\in D^{\phi}(E,C)$, and the third equality follows from (2.4).

Let $P\in\mathcal{P}^{\phi}(E,C)$. As $P\in\mathcal{P}^{\phi}(E,C)$, it holds that, for all $i,j\in N$,

$$p_{ij}=\varphi^i_j(e_i+\sum_{k\in N}p_{ki},\overline{c}_i).$$

Then, for all $i,j\in N$, it also holds that

$$p_{ij}=\varphi^i_j(e_i+\sum_{k\in N}\underline{p}^{\phi}_{ki}+g_i(P),\overline{c}_i). \tag{A.9}$$

From $P \geq \underline{P}^{\phi}(E,C)$ it follows that $g_i(P) \geq 0$ for all $i \in N$; from $P \leq \overline{P}^{\phi}(E,C)$ it follows that $g_i(P) \leq \overline{d}_i^{\phi}(E,C)$ for all $i \in N$. For all $i \in N$, (2.5) and (A.9) imply that

$$g_i(P) = \sum_{j \in N} (\varphi_i^j(e_j + \sum_{k \in N} \underline{p}_{kj}^{\phi} + g_j(P), \overline{c}_j) - \underline{p}_{ji}^{\phi}).$$

Therefore, it holds that $g(P) \in D^{\phi}(E,C)$.

Let $d \in D^{\phi}(E,C)$. Then, for all $i \in N$,

$$g_i(f(d)) = \sum_{j \in N} f_{ji}(d) - \sum_{j \in N} \underline{p}_{ji}^{\phi} = \sum_{j \in N} (\varphi_i^j(e_j + \sum_{k \in N} \underline{p}_{kj}^{\phi} + d_j, \overline{c}_j) - \underline{p}_{ji}^{\phi}) = d_i,$$

which implies that $g(f(d)) = d$.

Let $P \in \mathcal{P}^{\phi}(E,C)$. Then, for all $i,j \in N$, it holds that

$$f_{ij}(g(P)) = \varphi_j^i(e_i + \sum_{k \in N} \underline{p}_{ki}^{\phi} + g_i(P), \overline{c}_i) = \varphi_j^i(e_i + \sum_{k \in N} p_{ki}, \overline{c}_i) = p_{ij},$$

which implies that $f(g(P)) = P$.

Let $d, d' \in D^{\phi}(E,C)$ such that $d \leq d'$. Then, for all $i,j \in N$,

$$f_{ij}(d) = \varphi_j^i(e_i + \sum_{k \in N} \underline{p}_{ki}^{\phi} + d_i, \overline{c}_i) \leq \varphi_j^i(e_i + \sum_{k \in N} \underline{p}_{ki}^{\phi} + d'_i, \overline{c}_i) = f_{ij}(d'),$$

in which the inequality follows from resource monotonicity of φ^i. Hence, $f(d) \leq f(d')$, so f is monotone.

Let $P, P' \in \mathcal{P}^{\phi}(E,C)$ such that $P \leq P'$. Then, for all $i \in N$, $g_i(P) = \sum_{j \in N} p_{ji} - \sum_{j \in N} \underline{p}_{ji}^{\phi} \leq \sum_{j \in N} p'_{ji} - \sum_{j \in N} \underline{p}_{ji}^{\phi} = g_i(P')$. Hence, g is monotone. □

Proof of Lemma 3.3. To show that $(\theta_1, \underline{\theta}_2^2, \ldots, \underline{\theta}_n^2) \in D^{\phi}(E,C)$, we have to show that,

$$\theta_1 = \sum_{j \geq 2} (\varphi_1^j(e_j + \sum_{h \in N} \underline{p}_{hj}^{\phi} + \underline{\theta}_j^2, \overline{c}_j) - \underline{p}_{j1}^{\phi}),$$

and, for all $i \in \{2, \ldots, n\}$,

$$\underline{\theta}_i^2 = \underline{\gamma}_i^2(1) + \sum_{j \geq 2} (\varphi_i^j(e_j + \sum_{h \in N} \underline{p}_{hj}^{\phi} + \underline{\theta}_j^2, \overline{c}_j) - \underline{p}_{ji}^{\phi}), \tag{A.10}$$

in which $\underline{\gamma}_i^2(1)$ is given by (3.5). Let $i \in \{2, \ldots, n\}$. Then,

$$\begin{aligned} \underline{\theta}_i^2 &= \lim_{k \to \infty} \underline{\gamma}_i^2(k) \\ &= \underline{\gamma}_i^2(1) + \lim_{k \to \infty} \sum_{j \geq 2} (\varphi_i^j(e_j + \sum_{h \in N} \underline{p}_{hj}^{\phi} + \underline{\gamma}_j^2(k), \overline{c}_j) - \underline{p}_{ji}^{\phi}) \end{aligned}$$

$$= \underline{\gamma}_i^2(1) + \sum_{j\geq 2}(\varphi_i^j(e_j + \sum_{h\in N}\underline{p}_{hj}^{\phi} + \lim_{k\to\infty}\underline{\gamma}_j^2(k), \overline{c}_j) - \underline{p}_{ji}^{\phi})$$
$$= \underline{\gamma}_i^2(1) + \sum_{j\geq 2}(\varphi_i^j(e_j + \sum_{h\in N}\underline{p}_{hj}^{\phi} + \underline{\theta}_j^2, \overline{c}_j) - \underline{p}_{ji}^{\phi}).$$

The first equality follows from (3.8); the second equality follows from (3.6); the third equality follows from the fact that, for all $j \in \{2,\ldots,n\}$, φ^j satisfies resource continuity and $\lim_{k\to\infty}\varphi_i^j(e_j + \sum_{h\in N}\underline{p}_{hj}^{\phi} + \underline{\gamma}_j^2(k), \overline{c}_j)$ exists; the last equality follows from (3.8). Hence, (A.10) holds. Finally, for agent 1 it holds that

$$\begin{aligned}
\theta_1 &= \sum_{j\geq 2}(\varphi_j^1(e_1 + \sum_{h\in N}\underline{p}_{h1}^{\phi} + \theta_1, \overline{c}_1) - \underline{p}_{1j}^{\phi}) \\
&= \sum_{j\geq 2}\underline{\gamma}_j^2(1) \\
&= \sum_{j\geq 2}(\underline{\theta}_j^2 - \sum_{k\geq 2}(\varphi_j^k(e_k + \sum_{h\in N}\underline{p}_{hk}^{\phi} + \underline{\theta}_k^2, \overline{c}_k) - \underline{p}_{kj}^{\phi})) \\
&= \sum_{j\geq 2}\sum_{k\geq 1}(\varphi_k^j(e_j + \sum_{h\in N}\underline{p}_{hj}^{\phi} + \underline{\theta}_j^2, \overline{c}_j) - \underline{p}_{jk}^{\phi}) \\
&\quad - \sum_{j\geq 2}\sum_{k\geq 2}(\varphi_j^k(e_k + \sum_{h\in N}\underline{p}_{hk}^{\phi} + \underline{\theta}_k^2, \overline{c}_k) - \underline{p}_{kj}^{\phi}) \\
&= \sum_{j\geq 2}(\varphi_1^j(e_j + \sum_{h\in N}\underline{p}_{hj}^{\phi} + \underline{\theta}_j^2, \overline{c}_j) - \underline{p}_{j1}^{\phi}).
\end{aligned}$$

The first equality follows from Proposition 2.5 (iv); the second equality follows from (3.5); the third equality follows from (A.10); the fourth equality follows from Proposition 2.5 (iv).

Let $d \in D^{\phi}(E,C)$ with $d_1 = \theta_1$. Let $i \in \{2,\ldots,n\}$. The proof that $d_i \geq \underline{\theta}_i^2$ follows by contradiction. Assume that $d_i < \underline{\theta}_i^2$. Note that

$$\begin{aligned}
d_i &= \sum_{j\in N}(\varphi_i^j(e_j + \sum_{h\in N}\underline{p}_{hj}^{\phi} + d_j, \overline{c}_j) - \underline{p}_{ji}^{\phi}) \\
&\geq \varphi_i^1(e_1 + \sum_{h\in N}\underline{p}_{h1}^{\phi} + \theta_1, \overline{c}_1) - \underline{p}_{1i}^{\phi}) \\
&= \underline{\gamma}_i^2(1),
\end{aligned}$$

in which the first equality follows from $d \in D^{\phi}(E,C)$, the inequality follows from $d_1 = \theta_1$ and resource monotonicity of φ^j for all $j \in \{2,\ldots,n\}$, and the last equality follows from (3.5). Therefore, because $d_i < \underline{\theta}_i^2$, there must exist a $K_i \in \mathbb{N}$ such that

$$\underline{\gamma}_i^2(1) \leq \cdots \leq \underline{\gamma}_i^2(K_i) \leq d_i < \underline{\gamma}_i^2(K_i+1) \leq \cdots \leq \underline{\theta}_i^2.$$

Consequently, using $d \in D^{\phi}(E,C)$ and (3.6),

$$d_i = \underline{\gamma}_i^2(1) + \sum_{j\geq 2}(\varphi_i^j(e_j + \sum_{h\in N}\underline{p}_{hj}^{\phi} + d_j, \overline{c}_j) - \underline{p}_{ji}^{\phi})$$

$$
\begin{aligned}
&< \underline{\gamma}_i^2(1) + \sum_{j\geq 2}(\varphi_i^j(e_j + \sum_{h\in N}\underline{p}_{hj}^{\phi} + \underline{\gamma}_j^2(K_i), \overline{c}_j) - \underline{p}_{ji}^{\phi}) \\
&= \underline{\gamma}_i^2(K_i + 1),
\end{aligned}
$$

which implies that

$$
\varphi_i^j(e_j + \sum_{h\in N}\underline{p}_{hj}^{\phi} + d_j, \overline{c}_j) < \varphi_i^j(e_j + \sum_{h\in N}\underline{p}_{hj}^{\phi} + \underline{\gamma}_j^2(K_i), \overline{c}_j)
$$

for some $j \in \{2, \ldots, n\} \setminus \{i\}$. In fact, resource monotonicity of the claims rules in ϕ implies that $d_j < \underline{\gamma}_j^2(K_i)$ for some $j \in \{2, \ldots, n\} \setminus \{i\}$. Let $i_1 \in \{2, \ldots, n\} \setminus \{i\}$ be such that $d_{i_1} < \underline{\gamma}_{i_1}^2(K_i)$. Then, there exists a $K_{i_1} \in \mathbb{N}$ such that

$$
\underline{\gamma}_{i_1}^2(1) \leq \cdots \leq \underline{\gamma}_{i_1}^2(K_{i_1}) \leq d_{i_1} < \underline{\gamma}_{i_1}^2(K_{i_1} + 1) \leq \cdots \leq \underline{\theta}_{i_1}^2.
$$

Note that $\underline{\gamma}_{i_1}^2(K_{i_1}) < \underline{\gamma}_{i_1}^2(K_i)$; hence $K_{i_1} < K_i$ and $d_i \geq \underline{\gamma}_i^2(K_i) \geq \underline{\gamma}_i^2(K_{i_1})$. Subsequently, in a similar fashion as for agent i, $d_{i_1} < \underline{\gamma}_{i_1}^2(K_{i_1} + 1)$ implies that

$$
\varphi_{i_1}^j(e_j + \sum_{h\in N}\underline{p}_{hj}^{\phi} + d_j, \overline{c}_j) < \varphi_{i_1}^j(e_j + \sum_{h\in N}\underline{p}_{hj}^{\phi} + \underline{\gamma}_j^2(K_{i_1}), \overline{c}_j)
$$

for some $j \in \{2, \ldots, n\} \setminus \{i_1\}$. Resource monotonicity of the claims rules in ϕ implies that there exists a $j \in \{2, \ldots, n\} \setminus \{i_1, i\}$ such that $d_j < \underline{\gamma}_j^2(K_{i_1})$. Let $i_2 \in \{2, \ldots, n\} \setminus \{i_1, i\}$ be such that $d_{i_2} < \underline{\gamma}_j^2(K_{i_1})$. Note that $i_2 \notin \{i_1, i\}$ because $d_i \geq \underline{\gamma}_i^2(K_{i_1})$ and agent i_1 can not make a payment to itself, that is, we must be able to select an agent that is different from agents i_1 and i. By repeatedly applying the same arguments, we eventually run out of agents to select because the number of agents is finite. This thus leads to a contradiction, so we can assert that $d_i \geq \underline{\theta}_i^2$.

If $\lambda_1 = 0$, then $\theta_1 = 0$. If we consider $0^N \in D^{\phi}(E, C)$, then, for all $i \in \{2, \ldots, n\}$, we must have that $0 = d_i \geq \underline{\theta}_i^2 \geq 0$, which implies that $\underline{\theta}_i^2 = 0$. □

Proof of Lemma 3.9. Let $\ell \in \{2, \ldots, n-1\}$. We prove that, for all $i \in \{\ell + 1, \ldots, n\}$, $\underline{\theta}_i^{\ell} \leq \underline{\theta}_i^{\ell+1}$. The proof for $\overline{\theta}_i^{\ell} \geq \overline{\theta}_i^{\ell+1}$ follows by applying similar arguments. It suffices to show that, for all $i \in \{\ell + 1, \ldots, n\}$ and all $k \in \mathbb{N}$, $\underline{\gamma}_i^{\ell}(k) \leq \underline{\gamma}_i^{\ell+1}(k)$.

First, for all $i \in \{\ell + 1, \ldots, n\}$,

$$
\begin{aligned}
\underline{\gamma}_i^{\ell}(1) &= \sum_{j<\ell}(\varphi_i^j(e_j + \sum_{h\in N}\underline{p}_{hj}^{\phi} + \theta_j(\lambda), \overline{c}_j) - \underline{p}_{ji}^{\phi}) \\
&\leq \sum_{j<\ell+1}(\varphi_i^j(e_j + \sum_{h\in N}\underline{p}_{hj}^{\phi} + \theta_j(\lambda), \overline{c}_j) - \underline{p}_{ji}^{\phi}) \\
&= \underline{\gamma}_i^{\ell+1}(1),
\end{aligned}
$$

in which the inequality follows from resource monotonicity of φ^{ℓ} and $\theta_{\ell}(\lambda) \in [0, \overline{d}_{\ell}^{\phi}(E, C)]$.

Second, let $k \in \mathbb{N}$ and let $i \in \{\ell+1, \ldots, n\}$. Assume that, for all $j \in \{\ell+1, \ldots, n\}$, $\underline{\gamma}_j^\ell(k) \leq \underline{\gamma}_j^{\ell+1}(k)$. Then,

$$\begin{aligned}
\underline{\gamma}_i^\ell(k+1) &= \underline{\gamma}_i^\ell(1) + \sum_{j \geq \ell} (\varphi_i^j(e_j + \sum_{h \in N} \underline{p}_{hj}^\phi + \underline{\gamma}_j^\ell(k), \overline{c}_j) - \underline{p}_{ji}^\phi) \\
&= \sum_{j < \ell} (\varphi_i^j(e_j + \sum_{h \in N} \underline{p}_{hj}^\phi + \theta_j(\lambda), \overline{c}_j) - \underline{p}_{ji}^\phi) \\
&\quad + (\varphi_i^\ell(e_\ell + \sum_{h \in N} \underline{p}_{h\ell}^\phi + \underline{\gamma}_\ell^\ell(k), \overline{c}_\ell) - \underline{p}_{\ell i}^\phi) \\
&\quad + \sum_{j \geq \ell+1} (\varphi_i^j(e_j + \sum_{h \in N} \underline{p}_{hj}^\phi + \underline{\gamma}_j^\ell(k), \overline{c}_j) - \underline{p}_{ji}^\phi) \\
&\leq \sum_{j < \ell+1} (\varphi_i^j(e_j + \sum_{h \in N} \underline{p}_{hj}^\phi + \theta_j(\lambda), \overline{c}_j) - \underline{p}_{ji}^\phi) \\
&\quad + \sum_{j \geq \ell+1} (\varphi_i^j(e_j + \sum_{h \in N} \underline{p}_{hj}^\phi + \underline{\gamma}_j^{\ell+1}(k), \overline{c}_j) - \underline{p}_{ji}^\phi) \\
&= \underline{\gamma}_i^{\ell+1}(k+1),
\end{aligned}$$

in which the inequality follows from resource monotonicity of φ^ℓ and $\underline{\gamma}_\ell^\ell(k) \leq \underline{\theta}_\ell^\ell \leq \theta_\ell(\lambda)$, and resource monotonicity of φ^j and $\underline{\gamma}_j^\ell(k) \leq \underline{\gamma}_j^{\ell+1}(k)$ for all $j \in \{\ell+1, \ldots, n\}$.

Therefore, by induction, for all $i \in \{\ell+1, \ldots, n\}$ and all $k \in \mathbb{N}$, $\underline{\gamma}_i^\ell(k) \leq \underline{\gamma}_i^{\ell+1}(k)$.

Now, let $\ell \in \{2, \ldots, n\}$. To show that $\underline{\theta}^\ell \leq \overline{\theta}^\ell$, it suffices to show that, for all $k \in \mathbb{N}$, $\overline{\gamma}^\ell(k) - \underline{\gamma}^\ell(k) \geq 0$. Let $i \in \{\ell, \ldots, n\}$. Then,

$$\overline{\gamma}_i^\ell(1) - \underline{\gamma}_i^\ell(1) = \sum_{j \geq \ell} (\overline{p}_{ji}^\phi - \underline{p}_{ji}^\phi) \geq 0.$$

Let $k \in \mathbb{N}$, and assume that $\overline{\gamma}^\ell(k) - \underline{\gamma}^\ell(k) \geq 0$. Because, for all $j \in \{\ell, \ldots, n\}$, φ^j satisfies resource monotonicity, it follows that

$$\begin{aligned}
\overline{\gamma}_i^\ell(k+1) - \underline{\gamma}_i^\ell(k+1) &= \sum_{j \geq \ell} (\varphi_i^j(e_j + \sum_{h \in N} \underline{p}_{hj}^\phi + \overline{\gamma}_j^\ell(k), \overline{c}_j) - \varphi_i^j(e_j + \sum_{h \in N} \underline{p}_{hj}^\phi + \underline{\gamma}_j^\ell(k), \overline{c}_j)) \\
&\geq \sum_{j \geq \ell} (\varphi_i^j(e_j + \sum_{h \in N} \underline{p}_{hj}^\phi + \underline{\gamma}_j^\ell(k), \overline{c}_j) - \varphi_i^j(e_j + \sum_{h \in N} \underline{p}_{hj}^\phi + \underline{\gamma}_j^\ell(k), \overline{c}_j)) \\
&= 0.
\end{aligned}$$

Hence, by induction, for all $k \in \mathbb{N}$, $\overline{\gamma}_i^\ell(k) - \underline{\gamma}_i^\ell(k) \geq 0$, which implies that $\underline{\theta}_i^\ell \leq \overline{\theta}_i^\ell$. □

Proof of Lemma 3.10. For all $\ell \in \{2, \ldots, n\}$, let

$$\begin{aligned}
\underline{\alpha}^\ell &= (\theta_1(\lambda), \ldots, \theta_{\ell-1}(\lambda), \underline{\theta}_\ell^\ell, \ldots, \underline{\theta}_n^\ell), \\
\overline{\alpha}^\ell &= (\theta_1(\lambda), \ldots, \theta_{\ell-1}(\lambda), \overline{\theta}_\ell^\ell, \ldots, \overline{\theta}_n^\ell).
\end{aligned}$$

By Lemma 3.3 and Lemma 3.4, we know that $\underline{\alpha}^2, \overline{\alpha}^2 \in D^\phi(E, C)$. Let $q \in \{2, \ldots, n-1\}$. Assume that $\underline{\alpha}^q, \overline{\alpha}^q \in D^\phi(E, C)$. We show that $\underline{\alpha}^{q+1} \in D^\phi(E, C)$. The proof for $\overline{\alpha}^{q+1} \in D^\phi(E, C)$ follows by applying similar arguments.

For notational convenience, we set, for all $j \in N$,

$$\xi_j = e_j + \sum_{h \in N} \underline{p}^\phi_{hj}.$$

First, let $i \in \{q+1, \ldots, n\}$. Then,

$$\begin{aligned}
\underline{\theta}^{q+1}_i &= \lim_{k \to \infty} \underline{\gamma}^{q+1}_i(k) \\
&= \underline{\gamma}^{q+1}_i(1) + \lim_{k \to \infty} \sum_{j \geq q+1} (\varphi^j_i(\xi_j + \underline{\gamma}^{q+1}_j(k), \overline{c}_j) - \underline{p}^\phi_{ji}) \\
&= \underline{\gamma}^{q+1}_i(1) + \sum_{j \geq q+1} (\varphi^j_i(\xi_j + \lim_{k \to \infty} \underline{\gamma}^{q+1}_j(k), \overline{c}_j) - \underline{p}^\phi_{ji}) \\
&= \underline{\gamma}^{q+1}_i(1) + \sum_{j \geq q+1} (\varphi^j_i(\xi_j + \underline{\theta}^{q+1}_j, \overline{c}_j) - \underline{p}^\phi_{ji}).
\end{aligned}$$

The first equality follows from (3.12); the second equality follows from (3.14); the third equality follows from the fact that, for all $j \in \{q+1, \ldots, n\}$, φ^j satisfies resource continuity and $\lim_{k \to \infty} \varphi^j_i(\xi_j + \underline{\gamma}^{q+1}_j(k), \overline{c}_j)$ exists; the last equality follows from (3.12). Hence, for all $i \in \{q+1, \ldots, n\}$,

$$\underline{\theta}^{q+1}_i = \sum_{j < q+1} (\varphi^j_i(\xi_j + \theta_j(\lambda), \overline{c}_j) - \underline{p}^\phi_{ji}) + \sum_{j \geq q+1} (\varphi^j_i(\xi_j + \underline{\theta}^{q+1}_j, \overline{c}_j) - \underline{p}^\phi_{ji}). \tag{A.11}$$

What remains to be shown is that, for all $i \in \{1, \ldots, q\}$,

$$\theta_i(\lambda) = \sum_{j < q+1} (\varphi^j_i(\xi_j + \theta_j(\lambda), \overline{c}_j) - \underline{p}^\phi_{ji}) + \sum_{j \geq q+1} (\varphi^j_i(\xi_j + \underline{\theta}^{q+1}_j, \overline{c}_j) - \underline{p}^\phi_{ji}).$$

Let $i \in \{1, \ldots, q-1\}$. The fact that $\underline{\alpha}^q, \overline{\alpha}^q \in D^\phi(E, C)$ implies that

$$\begin{aligned}
\theta_i(\lambda) &= \sum_{j < q} (\varphi^j_i(\xi_j + \theta_j(\lambda), \overline{c}_j) - \underline{p}^\phi_{ji}) + \sum_{j \geq q} (\varphi^j_i(\xi_j + \underline{\theta}^q_j, \overline{c}_j) - \underline{p}^\phi_{ji}) \\
&= \sum_{j < q} (\varphi^j_i(\xi_j + \theta_j(\lambda), \overline{c}_j) - \underline{p}^\phi_{ji}) + \sum_{j \geq q} (\varphi^j_i(\xi_j + \overline{\theta}^q_j, \overline{c}_j) - \underline{p}^\phi_{ji}) \\
&= \theta_i(\lambda).
\end{aligned}$$

Hence, it must hold that

$$\sum_{j \geq q} \varphi^j_i(\xi_j + \underline{\theta}^q_j, \overline{c}_j) = \sum_{j \geq q} \varphi^j_i(\xi_j + \overline{\theta}^q_j, \overline{c}_j). \tag{A.12}$$

Recall from Lemma 3.9 that $\underline{\theta}^q \leq \overline{\theta}^q$. As a consequence, because, for all $j \in \{q, \ldots, n\}$, φ^j satisfies resource monotonicity, (A.12) implies that, for all $j \in \{q, \ldots, n\}$,

$$\varphi^j_i(\xi_j + \underline{\theta}^q_j, \overline{c}_j) = \varphi^j_i(\xi_j + \overline{\theta}^q_j, \overline{c}_j). \tag{A.13}$$

For all $j \in \{q+1, \ldots, n\}$, Lemma 3.9 implies that $\underline{\theta}_j^q \leq \underline{\theta}_j^{q+1} \leq \overline{\theta}_j^{q+1} \leq \overline{\theta}_j^q$, so resource monotonicity of φ^j and (A.13) imply that

$$\varphi_i^j(\xi_j + \underline{\theta}_j^q, \overline{c}_j) = \varphi_i^j(\xi_j + \underline{\theta}_j^{q+1}, \overline{c}_j). \tag{A.14}$$

In particular, for agent q, $\underline{\theta}_q^q \leq \theta_q(\lambda) \leq \overline{\theta}_q^q$, so resource monotonicity of φ^q and (A.13) imply that

$$\varphi_i^q(\xi_q + \underline{\theta}_q^q, \overline{c}_q) = \varphi_i^q(\xi_q + \theta_q(\lambda), \overline{c}_q). \tag{A.15}$$

Using $\underline{\alpha}^q \in D^\phi(E, C)$, (A.14) and (A.15), we obtain that

$$\begin{aligned}
\theta_i(\lambda) &= \sum_{j<q} (\varphi_i^j(\xi_j + \theta_j(\lambda), \overline{c}_j) - \underline{p}_{ji}^\phi) + \sum_{j \geq q} (\varphi_i^j(\xi_j + \underline{\theta}_j^q, \overline{c}_j) - \underline{p}_{ji}^\phi) \\
&= \sum_{j<q+1} (\varphi_i^j(\xi_j + \theta_j(\lambda), \overline{c}_j) - \underline{p}_{ji}^\phi) + \sum_{j \geq q+1} (\varphi_i^j(\xi_j + \underline{\theta}_j^{q+1}, \overline{c}_j) - \underline{p}_{ji}^\phi).
\end{aligned}$$

Hence, for all $i \in \{1, \ldots, q-1\}$,

$$\theta_i(\lambda) = \sum_{j<q+1} (\varphi_i^j(\xi_j + \theta_j(\lambda), \overline{c}_j) - \underline{p}_{ji}^\phi) + \sum_{j \geq q+1} (\varphi_i^j(\xi_j + \underline{\theta}_j^{q+1}, \overline{c}_j) - \underline{p}_{ji}^\phi) \tag{A.16}$$

Finally, for agent q it holds that

$$\begin{aligned}
\theta_q(\lambda) &= \sum_{j<q} (\varphi_j^q(\xi_q + \theta_q(\lambda), \overline{c}_q) - \underline{p}_{qj}^\phi) + \sum_{j \geq q+1} (\varphi_j^q(\xi_q + \theta_q(\lambda), \overline{c}_q) - \underline{p}_{qj}^\phi) \\
&= \sum_{j<q} \theta_j(\lambda) - \sum_{j<q} \sum_{k<q} (\varphi_j^k(\xi_k + \theta_k(\lambda), \overline{c}_k) - \underline{p}_{kj}^\phi) - \sum_{j<q} \sum_{k \geq q+1} (\varphi_j^k(\xi_k + \underline{\theta}_k^{q+1}, \overline{c}_k) - \underline{p}_{kj}^\phi) \\
&\quad + \sum_{j \geq q+1} \underline{\theta}_j^{q+1} - \sum_{j \geq q+1} \sum_{k<q} (\varphi_j^k(\xi_k + \theta_k(\lambda), \overline{c}_k) - \underline{p}_{kj}^\phi) \\
&\quad - \sum_{j \geq q+1} \sum_{k \geq q+1} (\varphi_j^k(\xi_k + \underline{\theta}_k^{q+1}, \overline{c}_k) - \underline{p}_{kj}^\phi) \\
&= \sum_{j<q} \sum_{k<q+1} (\varphi_k^j(\xi_j + \theta_j(\lambda), \overline{c}_j) - \underline{p}_{jk}^\phi) + \sum_{j<q} \sum_{k \geq q+1} (\varphi_k^j(\xi_j + \theta_j(\lambda), \overline{c}_j) - \underline{p}_{jk}^\phi) \\
&\quad - \sum_{j<q} \sum_{k<q} (\varphi_j^k(\xi_k + \theta_k(\lambda), \overline{c}_k) - \underline{p}_{kj}^\phi) - \sum_{j<q} \sum_{k \geq q+1} (\varphi_j^k(\xi_k + \underline{\theta}_k^{q+1}, \overline{c}_k) - \underline{p}_{kj}^\phi) \\
&\quad + \sum_{j \geq q+1} \sum_{k<q+1} (\varphi_k^j(\xi_j + \underline{\theta}_j^{q+1}, \overline{c}_j) - \underline{p}_{jk}^\phi) + \sum_{j \geq q+1} \sum_{k \geq q+1} (\varphi_k^j(\xi_j + \underline{\theta}_j^{q+1}, \overline{c}_j) - \underline{p}_{jk}^\phi) \\
&\quad - \sum_{j \geq q+1} \sum_{k<q} (\varphi_j^k(\xi_k + \theta_k(\lambda), \overline{c}_k) - \underline{p}_{kj}^\phi) - \sum_{j \geq q+1} \sum_{k \geq q+1} (\varphi_j^k(\xi_k + \underline{\theta}_k^{q+1}, \overline{c}_k) - \underline{p}_{kj}^\phi) \\
&= \sum_{j<q} (\varphi_q^j(\xi_j + \theta_j(\lambda), \overline{c}_j) - \underline{p}_{jq}^\phi) + \sum_{j \geq q+1} (\varphi_q^j(\xi_j + \underline{\theta}_j^{q+1}, \overline{c}_j) - \underline{p}_{jq}^\phi) \\
&\quad + \sum_{j<q} \sum_{k<q} (\varphi_k^j(\xi_j + \theta_j(\lambda), \overline{c}_j) - \underline{p}_{jk}^\phi) - \sum_{j<q} \sum_{k<q} (\varphi_j^k(\xi_k + \theta_k(\lambda), \overline{c}_k) - \underline{p}_{kj}^\phi)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j<q}\sum_{k\geq q+1}(\varphi_k^j(\xi_j+\theta_j(\lambda),\overline{c}_j)-\underline{p}_{jk}^{\phi})-\sum_{j\geq q+1}\sum_{k<q}(\varphi_j^k(\xi_k+\theta_k(\lambda),\overline{c}_k)-\underline{p}_{kj}^{\phi})\\
&+\sum_{j\geq q+1}\sum_{k<q}(\varphi_k^j(\xi_j+\underline{\theta}_j^{q+1},\overline{c}_j)-\underline{p}_{jk}^{\phi})-\sum_{j<q}\sum_{k\geq q+1}(\varphi_j^k(\xi_k+\underline{\theta}_k^{q+1},\overline{c}_k)-\underline{p}_{kj}^{\phi})\\
&+\sum_{j\geq q+1}\sum_{k\geq q+1}(\varphi_k^j(\xi_j+\underline{\theta}_j^{q+1},\overline{c}_j)-\underline{p}_{jk}^{\phi})-\sum_{j\geq q+1}\sum_{k\geq q+1}(\varphi_j^k(\xi_k+\underline{\theta}_k^{q+1},\overline{c}_k)-\underline{p}_{kj}^{\phi})\\
=&\sum_{j<q+1}(\varphi_q^j(\xi_j+\theta_j(\lambda),\overline{c}_j)-\underline{p}_{jq}^{\phi})+\sum_{j\geq q+1}(\varphi_q^j(\xi_j+\underline{\theta}_j^{q+1},\overline{c}_j)-\underline{p}_{jq}^{\phi}).
\end{aligned}
$$

The first equality follows from Proposition 2.5 (iv); the second equality follows from (A.16) and (A.11); the third equality follows from Proposition 2.5 (iv); in the fourth equality, we rearrange the summations such that each row between the fourth equality and the fifth equality, except the first one, equals zero; the fifth equality follows from the fact that the payment of agent q to itself is zero. □

Proof of Lemma 3.11. By Lemma 3.3 and Lemma 3.4, we know that, for all $d\in D^{\phi}(E,C)$ with $d_1=\theta_1(\lambda)$, it holds that $d_i\in[\underline{\theta}_i^2,\overline{\theta}_i^2]$ for all $i\in\{2,\ldots,n\}$.

Let $\ell\in\{3,\ldots,n\}$ and $d\in D^{\phi}(E,C)$ with, for all $i\in\{1,\ldots,\ell-1\}$, $d_i=\theta_i(\lambda)$. We need to show that, for all $i\in\{\ell,\ldots,n\}$, it holds that $d_i\geq\underline{\theta}_i^{\ell}$ and $d_i\leq\overline{\theta}_i^{\ell}$. In particular, we will show that, for all $i\in\{\ell,\ldots,n\}$, we have $d_i\geq\underline{\theta}_i^{\ell}$. The proof for $d_i\leq\overline{\theta}_i^{\ell}$ follows by applying similar arguments.

Note that, for all $i\in\{\ell,\ldots,n\}$,

$$
\begin{aligned}
d_i&=\sum_{j<\ell}(\varphi_i^j(e_j+\sum_{h\in N}\underline{p}_{hj}^{\phi}+d_j,\overline{c}_j)-\underline{p}_{ji}^{\phi})+\sum_{j\geq\ell}(\varphi_i^j(e_j+\sum_{h\in N}\underline{p}_{hj}^{\phi}+d_j,\overline{c}_j)-\underline{p}_{ji}^{\phi})\\
&=\sum_{j<\ell}(\varphi_i^j(e_j+\sum_{h\in N}\underline{p}_{hj}^{\phi}+\theta_j(\lambda),\overline{c}_j)-\underline{p}_{ji}^{\phi})+\sum_{j\geq\ell}(\varphi_i^j(e_j+\sum_{h\in N}\underline{p}_{hj}^{\phi}+d_j,\overline{c}_j)-\underline{p}_{ji}^{\phi})\\
&=\underline{\gamma}_i^{\ell}(1)+\sum_{j\geq\ell}(\varphi_i^j(e_j+\sum_{h\in N}\underline{p}_{hj}^{\phi}+d_j,\overline{c}_j)-\underline{p}_{ji}^{\phi})\\
&\geq\underline{\gamma}_i^{\ell}(1),
\end{aligned}
$$

in which the first equality follows from $d\in D^{\phi}(E,C)$, the second equality follows from $\theta_j(\lambda)=d_j$ for all $j\in\{1,\ldots,\ell-1\}$, the third equality follows from (3.13), and the inequality follows from resource monotonicity of the claims rules in ϕ. Hence, if there exists an $i\in\{\ell,\ldots,n\}$ such that $d_i<\underline{\theta}_i^{\ell}$, then there must exist a $K_i\in\mathbb{N}$ such that

$$
\underline{\gamma}_i^{\ell}(1)\leq\cdots\leq\underline{\gamma}_i^{\ell}(K_i)\leq d_i<\underline{\gamma}_i^{\ell}(K_i+1)\leq\cdots\leq\underline{\theta}_i^{\ell}.
$$

In the proof of Lemma 3.3, we have shown that this leads to a contradiction. Therefore, for all $i\in\{\ell,\ldots,n\}$, it holds that $d_i\geq\underline{\theta}_i^{\ell}$.

Consider $d=0^N\in D^{\phi}(E,C)$. Let $\ell\in\{2,\ldots,n\}$. Assume that, for all $i\in\{1,\ldots,\ell-1\}$, $\lambda_i=0$. If $\lambda_1=0$, then $\theta_1(\lambda)=\underline{\theta}_1^1=0$, and, by Lemma 3.3, $\underline{\theta}_2^2=0$. Consequently, $\lambda_2=0$ implies that $\theta_2(\lambda)=\underline{\theta}_2^2=0$. We now have that $d_1=0=\theta_1(\lambda)$ and $d_2=0=\theta_2(\lambda)$, so

$0 = d_3 \geq \underline{\theta}_3^3 \geq 0$, which gives $\underline{\theta}_3^3 = 0$ and, as a result of $\lambda_3 = 0$, $\theta_3(\lambda) = \underline{\theta}_3^3 = 0$. By induction, we obtain that, for all $i \in \{1, \ldots, \ell - 1\}$, $\theta_i(\lambda) = 0 = d_i$. Hence, for all $i \in \{\ell, \ldots, n\}$, $0 = d_i \geq \underline{\theta}_i^\ell \geq 0$, which implies that $\underline{\theta}_i^\ell = 0$. If we consider $\overline{d}^\phi(E, C) \in D^\phi(E, C)$ instead, then, by applying similar arguments, we find that $\lambda_i = 1$ for all $i \in \{1, \ldots, \ell - 1\}$ implies that $\overline{\theta}_i^\ell = \overline{d}_i^\phi(E, C)$ for all $i \in \{\ell, \ldots, n\}$. □

Proof of Proposition 3.8. The proof of Proposition 3.8 follows from Lemma 3.10 and Lemma 3.11, which is why it is given after the proofs of these two respective lemmas.

Set $\underline{\theta} = (\underline{\theta}_1^1, \ldots, \underline{\theta}_n^n)$ and $\overline{\theta} = (\overline{\theta}_1^1, \ldots, \overline{\theta}_n^n)$.

We have to show that, for all $\ell \in \{2, \ldots, n\}$, $\underline{\theta}_\ell^\ell$ and $\overline{\theta}_\ell^\ell$ are given by (3.1) and (3.2), respectively. Let $\ell \in \{2, \ldots, n\}$. Then, $\theta_\ell(\lambda) = (1-\lambda_\ell)\underline{\theta}_\ell^\ell + \lambda_\ell \overline{\theta}_\ell^\ell$, and, by Lemma 3.10, we have that $(\theta_1(\lambda), \ldots, \theta_{\ell-1}(\lambda), \underline{\theta}_\ell^\ell, \ldots, \underline{\theta}_n^\ell) \in D^\phi(E, C)$ and $(\theta_1(\lambda), \ldots, \theta_{\ell-1}(\lambda), \overline{\theta}_\ell^\ell, \ldots, \overline{\theta}_n^\ell) \in D^\phi(E, C)$. Lemma 3.11 implies that, for all $d \in D^\phi(E, C)$ with $d_i = \theta_i(\lambda)$ for all $i \in \{1, \ldots, \ell - 1\}$, it holds that $d_\ell \geq \underline{\theta}_\ell^\ell$ and $d_\ell \leq \overline{\theta}_\ell^\ell$.

Assume that $d' \in \mathbb{R}^N$ is a λ-additional cash vector for (E, C) with respect to ϕ. We will show that $\theta(\lambda) = d'$. First, $\theta_1(\lambda) = \lambda_1 \overline{d}_1^\phi(E, C) = d'_1$. Second, let $\ell \in \{2, \ldots, n\}$ and assume that, for all $i \in \{1, \ldots, \ell - 1\}$, $\theta_i(\lambda) = d'_i$. We will show that $\theta_\ell(\lambda) = d'_\ell$. Consider $d \in D^\phi(E, C)$ with $d_i = d'_i$ for all $i \in \{1, \ldots, \ell - 1\}$. Because d' is a λ-additional cash vector for (E, C) with respect to ϕ, we know that $d'_\ell = (1 - \lambda_\ell)\underline{d}'_\ell + \lambda_\ell \overline{d}'_\ell$ in which $\underline{d}'_\ell \leq d_\ell$ and $\overline{d}'_\ell \geq d_\ell$. In particular, as $(\theta_1(\lambda), \ldots, \theta_{\ell-1}(\lambda), \underline{\theta}_\ell^\ell, \ldots, \underline{\theta}_n^\ell) \in D^\phi(E, C)$ and $(\theta_1(\lambda), \ldots, \theta_{\ell-1}(\lambda), \overline{\theta}_\ell^\ell, \ldots, \overline{\theta}_n^\ell) \in D^\phi(E, C)$ (see Lemma 3.10), it follows that $\underline{d}'_\ell \leq \underline{\theta}_\ell$ and $\overline{d}'_\ell \geq \overline{\theta}_\ell$. Lemma 3.11 implies that $\underline{d}'_\ell \geq \underline{\theta}_\ell$ and $\overline{d}'_\ell \leq \overline{\theta}_\ell$. Hence, we find that $\underline{d}'_\ell = \underline{\theta}_\ell$ and $\overline{d}'_\ell = \overline{\theta}_\ell$, so $\theta_\ell(\lambda) = d'_\ell$. By induction, for all $i \in \{1, \ldots, n\}$, $\theta_i(\lambda) = d'_i$ and therefore $\theta(\lambda) = d'$. □

Proof of Proposition 5.6. Assume that $\widetilde{\mathcal{T}}^\phi$ satisfies decomposition. Let $m \in \mathbb{N}$ and set $\lambda = (1/m)^N$ and $\lambda' = (1 - 1/m)^N$ such that $\lambda + \lambda' = 1^N$. Let $(\lambda, \tilde{\tau}^{\phi,\lambda}), (\lambda', \tilde{\tau}^{\phi,\lambda'}) \in \widetilde{\mathcal{T}}^\phi$. Let $(E, C) \in \mathcal{F}^N$. Decomposition of $\widetilde{\mathcal{T}}^\phi$ implies that

$$\begin{aligned}\tau^{\phi,1^N}(E, C) &= \tau^{\phi,(1/m)^N}(E, C) + \tau^{\phi,(1-1/m)^N}(E, C) - \tau^{\phi,0^N}(E, C) \\ &= m\tau^{\phi,(1/m)^N}(E, C) - (m - 1)\tau^{\phi,0^N}(E, C).\end{aligned}$$

By using the fact that $\tau^{\phi,0^N}(E, C) = \underline{P}^\phi(E, C)$ and $\tau^{\phi,1^N}(E, C) = \overline{P}^\phi(E, C)$, we find that

$$\tau^{\phi,(1/m)^N}(E, C) = (1 - \frac{1}{m})\underline{P}^\phi(E, C) + \frac{1}{m}\overline{P}^\phi(E, C).$$

Furthermore, if $\lambda = (m'/m)^N$, in which $m', m \in \mathbb{N}$ with $m' \leq m$, then decomposition of $\widetilde{\mathcal{T}}^\phi$ implies that

$$\begin{aligned}\tau^{\phi,(m'/m)^N}(E, C) &= m'\tau^{\phi,(1/m)^N}(E, C) - (m' - 1)\underline{P}^\phi(E, C) \\ &= (1 - \frac{m'}{m})\underline{P}^\phi(E, C) + \frac{m'}{m}\overline{P}^\phi(E, C).\end{aligned} \tag{A.17}$$

Let $\mu \in [0,1]$ and consider the sequences $(m'_\ell)_{\ell\in\mathbb{N}}$ and $(m_\ell)_{\ell\in\mathbb{N}}$ in which, for all $\ell \in \mathbb{N}$, $m'_\ell, m_\ell \in \mathbb{N}$ with $m'_\ell \leq m_\ell$ such that $\lim_{\ell\to\infty}(m'_\ell/m_\ell) = \mu$. Then, for all $i, j \in N$,

$$\begin{aligned}
\tau_{ij}^{\phi,\mu^N}(E,C) &= \varphi_j^i(e_i + \sum_{k\in N} \underline{p}_{ki}^{\phi} + \underline{\theta}_i^i + \mu(\overline{\theta}_i^i - \underline{\theta}_i^i), \overline{c}_i) \\
&= \varphi_j^i(e_i + \sum_{k\in N} \underline{p}_{ki}^{\phi} + \underline{\theta}_i^i + \lim_{\ell\to\infty} \frac{m'_\ell}{m_\ell}(\overline{\theta}_i^i - \underline{\theta}_i^i), \overline{c}_i) \\
&= \lim_{\ell\to\infty} \varphi_j^i(e_i + \sum_{k\in N} \underline{p}_{ki}^{\phi} + \underline{\theta}_i^i + \frac{m'_\ell}{m_\ell}(\overline{\theta}_i^i - \underline{\theta}_i^i), \overline{c}_i) \\
&= \lim_{\ell\to\infty} \tau_{ij}^{\phi,(m'_\ell/m_\ell)^N}(E,C) \\
&= \lim_{\ell\to\infty} \big((1 - \frac{m'_\ell}{m_\ell})\underline{p}_{ij}^{\phi} + \frac{m'_\ell}{m_\ell}\overline{p}_{ij}^{\phi}\big) \\
&= (1-\mu)\underline{p}_{ij}^{\phi} + \mu\overline{p}_{ij}^{\phi} \\
&= (1-\mu)\tau_{ij}^{\phi,0^N}(E,C) + \mu\tau_{ij}^{\phi,1^N}(E,C),
\end{aligned}$$

in which the first equality follows from the definition of τ^{ϕ,μ^N}, the third equality follows from resource continuity of φ^i, the fourth equality follows from the definition of $\tau^{\phi,(m_\ell/m'_\ell)^N}$, and the fifth equality follows from (A.17). □

Proof of Proposition 5.7. First, assume that $\widetilde{\mathcal{T}}^{\phi}$ satisfies decomposition. Proposition 5.6 implies that $\mathcal{T}^{\phi}$ satisfies weak convexity, and Theorem 5.2 implies that $\phi = \text{PROP}$.

Second, assume that $\phi = \text{PROP}$. To prove that $\widetilde{\mathcal{T}}^{\text{PROP}}$ satisfies decomposition, it suffices to show that

$$(\tau^{\text{PROP},\lambda+\lambda'} - \tau^{\text{PROP},0^N}) = (\tau^{\text{PROP},\lambda} - \tau^{\text{PROP},0^N}) + (\tau^{\text{PROP},\lambda'} - \tau^{\text{PROP},0^N}). \tag{A.18}$$

Let $(\lambda, \tilde{\tau}^{\text{PROP},\lambda}), (\lambda', \tilde{\tau}^{\text{PROP},\lambda'}) \in \widetilde{\mathcal{T}}^{\text{PROP}}$ with $\lambda + \lambda' \leq 1^N$. Let $(E,C) \in \mathcal{F}^N$. For all $i \notin I^{\text{PROP}}(E,C)$, the payments are uniquely determined, which implies that (A.18) is satisfied with respect to such mutual payments. Thus, let $S \in \mathcal{C}^{\text{PROP}}(E,C)$, $i \in S$, and $j \in N$. Corollary 4.4 and (4.9) imply that there exists a $(\lambda_S + \lambda'_S) \in [0,1]$ such that the left-hand side of (A.18) boils down to

$$(\lambda_S + \lambda'_S)(\overline{p}_{ij}^{\text{PROP}} - \underline{p}_{ij}^{\text{PROP}}). \tag{A.19}$$

In a similar fashion, we obtain that the right-hand side of (A.18) boils down to

$$\lambda_S(\overline{p}_{ij}^{\text{PROP}} - \underline{p}_{ij}^{\text{PROP}}) + \lambda'_S(\overline{p}_{ij}^{\text{PROP}} - \underline{p}_{ij}^{\text{PROP}}). \tag{A.20}$$

Clearly, (A.19) equals (A.20), so $\widetilde{\mathcal{T}}^{\text{PROP}}$ satisfies decomposition. □

Proof of Proposition 5.4. First, assume that $\mathcal{T}^{\phi}$ satisfies convexity. Consider $\sigma = (0^N, \tau^{\phi,0^N}) \in \mathcal{T}^{\phi}$ and $\sigma' = (1^N, \tau^{\phi,1^N}) \in \mathcal{T}^{\phi}$. Then, by convexity, for all $\mu \in [0,1]$, $(1-\mu)\sigma + \mu\sigma' \in \mathcal{T}^{\phi}$.

Because $\mathcal{T}^\phi$ satisfies weak convexity, Theorem 5.2 implies that $\phi = \text{PROP}$.

Second, assume that $\phi = \text{PROP}$. Let $\sigma = (\lambda, \tau^{\phi,\lambda}) \in \mathcal{T}^\phi$ and $\sigma' = (\lambda', \tau^{\phi,\lambda'}) \in \mathcal{T}^\phi$, let $\mu \in [0,1]$, and let $(E,C) \in \mathcal{F}^N$. To prove that $(1-\mu)\sigma + \mu\sigma' \in \mathcal{T}^\phi$, we have to show that $(1-\mu)\tau^{\phi,\lambda}(E,C) + \mu\tau^{\phi,\lambda'}(E,C) = \tau^{\phi,(1-\mu)\lambda+\mu\lambda'}(E,C)$.

Let $i \notin I^\phi(E,C)$. Then, for all $\lambda'' \in [0,1]^N$, it holds that $\theta_i^{\phi,\lambda''}(E,C) = 0$ which implies that, for all $j \in N$,

$$(1-\mu)\tau_{ij}^{\phi,\lambda}(E,C) + \mu\tau_{ij}^{\phi,\lambda'}(E,C) = (1-\mu)\underline{p}_{ij}^\phi + \mu\underline{p}_{ij}^\phi = \underline{p}_{ij}^\phi = \tau_{ij}^{\phi,(1-\mu)\lambda+\mu\lambda'}(E,C).$$

Let $i \in I^\phi(E,C)$, and let $S \in \mathcal{C}^\phi(E,C)$ with $S \ni i$. In addition, let $\ell = \min\{j \mid j \in S\}$ be the first agent in the strongly connected component S that is selected in the iterative procedure. Theorem 4.3 implies that, for all $\lambda'' \in [0,1]^N$, $\theta_i^{\phi,\lambda''}(E,C) = \lambda''_S \overline{d}_i^\phi(E,C)$, in which $\lambda''_S = \lambda''_\ell$; correspondingly, Corollary 4.4 implies that, for all $j \in N$, $\tau_{ij}^{\phi,\lambda''}(E,C) = (1-\lambda''_S)\underline{p}_{ij}^\phi + \lambda''_S\overline{p}_{ij}^\phi$. Consequently, if we denote λ_ℓ and λ'_ℓ by λ_S and λ'_S, respectively, we obtain, for all $j \in N$,

$$\begin{aligned}
(1-\mu)\tau_{ij}^{\phi,\lambda}(E,C) + \mu\tau_{ij}^{\phi,\lambda'}(E,C) &= (1-\mu)(1-\lambda_S)\underline{p}_{ij}^\phi + (1-\mu)\lambda_S\overline{p}_{ij}^\phi \\
&\quad + \mu(1-\lambda'_S)\underline{p}_{ij}^\phi + \mu\lambda'_S\overline{p}_{ij}^\phi \\
&= (1-(1-\mu)\lambda_S - \mu\lambda'_S)\underline{p}_{ij}^\phi + ((1-\mu)\lambda_S + \mu\lambda'_S)\overline{p}_{ij}^\phi \\
&= \tau_{ij}^{\phi,(1-\mu)\lambda+\mu\lambda'}(E,C).
\end{aligned}$$

□

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