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Publication date:
2024

Document Version
Early version, also known as pre-print

Citation for published version (APA):
Ketelaars, M. (2024). *On the Compatibility of Composition Axioms in Financial Networks*. (CentER Discussion Paper; Vol. 2024-007). CentER, Center for Economic Research.

CentER

Discussion Paper

No. 2024-007

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25 March 2024

ISSN 0924-7815
ISSN 2213-9532

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On the Compatibility of Composition Axioms in Financial Networks

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March 21, 2024

Abstract

This article introduces composition down and composition up in financial networks, and analyzes their compatibility. The starting point for the analysis is the outcome that is prescribed by a rule. For example, a transfer rule prescribes, for each financial network, a clearing payment matrix that contains mutual payments between the agents to settle their mutual liabilities. However, as it turns out, the assets of some of the agents in the financial network are either undervalued or overvalued, making the proposed outcome infeasible to carry out. In such cases, one has the option to either apply the rule to the new situation, or reapply the rule to an appropriately adjusted situation that honors the initial outcome. A composition axiom requires that both options are equivalent to all the agents. In the context of financial networks, there can exist various adequate versions of the two composition axioms. Interestingly, the adequate versions of the two composition axioms that are compatible for allocation rules are not compatible for transfer rules in the sense that no transfer rule can satisfy them simultaneously. Nevertheless, we show that there exist alternative adequate versions of the two composition axioms that are compatible with respect to transfer rules. In fact, we show that the transfer rules that either always prescribe the bottom payment matrix, or the top payment matrix, satisfy the two composition axioms simultaneously.

Keywords: composition down, composition up, claims rules, allocation rules, transfer rules.

JEL Classification: D74, G10, G33.

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1 Introduction

A financial network consists of a finite number of interconnected agents in which each agent may own individual assets and may be in debt to the other agents (cf. Eisenberg and Noe (2001)). Agents in the network are inherently at risk if an agent defaults on its debt obligations as this can cause financial instability elsewhere in the network that, in the worst case, may lead to a collapse of the network. Glasserman and Young (2016), Jackson and Pernoud (2021), and Elliott and Golub (2022) are excellent surveys of recent literature on this phenomenon, which is called *financial contagion*.

We assume that each financial network is endowed with agent-specific *claims rules* that stipulate how the agents pay their creditors. More specifically, each agent takes its initial individual assets as given and subsequently receives payments from the other agents. Its claims rule then prescribes how this total amount should be divided among its creditors. A well-known claims rule is the *proportional rule* that states that an agent should pay its creditors proportionally to their claims. In fact, also other claims rules such as the *constrained equal awards rule*, or claims rules that incorporate seniority among the creditors (see, e.g., Moulin (2000)) are included in our analysis. We refer the reader to Thomson (2019) for an extensive survey of the literature on claims rules.

A clearing payment matrix that contains the payments of the agents in accordance with claims rules is called a *transfer scheme*. Clearing payments are payments made by the agents to settle their mutual liabilities. Thus, a transfer scheme leads to a possibly partial settlement of claims and, consequently, to a redistribution of the assets, referred to as a *transfer allocation*. The set of transfer schemes for a financial network is a non-empty complete lattice, which means that both a bottom transfer scheme and a top transfer scheme exist. The bottom transfer scheme for a financial network contains the minimal clearing payments, whereas the top transfer scheme contains the maximal clearing payments. The bottom and top transfer schemes for a financial network do not always coincide, so multiple transfer schemes can exist. Despite the multiplicity of transfer schemes, any two transfer schemes for a financial network result in the same transfer allocation. Accordingly, an *allocation rule* prescribes, for each financial network, the transfer allocation that is based on a transfer scheme for that network. Moreover, a *transfer rule* prescribes, for each financial network, exactly one transfer scheme for that network. In this article, we study the invariance of these two types of rules with respect to changes in the available assets of the agents.

Consider an allocation rule that has prescribed how to redistribute the assets based on a transfer scheme, but now assume that the assets of some of the agents are either undervalued or overvalued, which makes the proposed redistribution infeasible to carry out. *Composition axioms* stipulate that we can either apply the allocation rule to the new situation, or reapply the allocation rule to an appropriately adjusted situation that honors the initial decision. Although both approaches are valid, it could be the case that the former is the favorable option for one group of agents, whereas the latter is the favorable option for another group of agents. A composition axiom requires that all the agents are indifferent between the two approaches, for example in relation to the transfer allocation, the transfer scheme, or both. This article introduces composition axioms in financial networks, and analyzes their *compatibility* with respect to allocation rules and transfer rules.

Our point of departure is invariance of a solution to a *claims problem*, which can be seen as a special case of a financial network in the sense that exactly one agent is in debt to the remaining agents. In a claims problem, an amount, which we call an *estate*, is to be divided among a group of claimants that each have a claim on the estate. A claims rule offers a solution to each claims problem in the form of a division of the estate among the group of claimants. Suppose that a division of the estate has been carried out in accordance with a claims rule, but upon reevaluation of the estate, it turns out that the estate is either smaller or larger. In case the estate is smaller, it is impossible to still honor the promise to all the claimants; in case the estate is larger, there still is a surplus estate to be divided among the claimants. The estate can change for several reasons, for instance, the estate can depreciate or appreciate, or part of the estate can become inaccessible or new assets can become available. The *composition down* principle requires invariance with respect to downward changes in the estate, whereas the *composition up* principle requires invariance with respect to upward changes in the estate.[1]

This article introduces adequate extensions of composition down and composition up to be applicable to both allocation rules and transfer rules. An extension of a property for a rule is said to be *adequate* if the individual claims rules on which the rule is based satisfy the property if the rule satisfies its extension. For instance, an adequate extension of composition down for transfer rules is such that, if a transfer rule satisfies composition down, then each agent-specific claims rule, on which the transfer rule is based, satisfies composition down as well. Nevertheless, as an extension need not necessarily be done in a unique way, one has to be careful. In fact, we show that the extensions of the two composition axioms for transfer rules must differ from the extensions for allocation rules. Although the extensions of composition down and composition up for allocation rules are compatible, extending them adequately in a similar way for transfer rules means that they become *incompatible* in the sense that no transfer rule can satisfy both axioms simultaneously.

Nonetheless, we show that there exist alternative adequate extensions of composition down and composition up for transfer rules that are compatible. Specifically, provided that each agent-specific claims rule satisfies composition down and composition up, such as the constrained equal awards claims rule, the corresponding transfer rule consistently prescribing the bottom transfer scheme for a financial network also satisfies composition down and composition up. This transfer rule is called the *bottom transfer rule*. The same holds true for the *top transfer rule*, which for each financial network prescribes the top transfer scheme.

We show that composition down and composition up carry over to the corresponding allocation rule. That is, if each claims rule on which the allocation rule is based satisfies the property, then so does the corresponding allocation rule itself. Interestingly, this is not the case for transfer rules in general. For a transfer rule there is sometimes a choice to be made among possibly infinitely many transfer schemes. Then, even though the claims rules on which the transfer rule is based satisfy a property, the corresponding transfer rule need not satisfy it.

Groote Schaarsberg, Reijnierse, and Borm (2018) extends concede-and-divide of claims rules and consistency of claims rules to the financial network setting. Although its extensions

[1] We adopt the terminology of Thomson (2019). In the literature on claims problems, composition down is also referred to as path independence (Moulin, 1987; Herrero & Villar, 2001) or upper composition (Moulin, 2000); furthermore, composition up is also referred to as composition (Young, 1988; Herrero & Villar, 2001) or lower composition (Moulin, 2000).

allow for allocation rules that do not rely on underlying claims rules, the application is only with respect to claims-rule-based allocation rules in which each agent uses the same claims rule. Csóka and Herings (2021) extends impartiality, or "equal treatment of equals," of claims rules to the financial network setting, though only with respect to the transfers between the agents.[2] Moreover, Csóka and Herings (2021) shows that, in regard to the mutual payments in a financial network, the straightforward extension of nonmanipulability in claims problems, also known as strategy-proofness (see also O'Neill (1982)), to the financial network setting is incompatible with the other axioms in the characterization of the proportional rule in financial networks. Nevertheless, a weaker, though more appropriate, form is compatible. However, Csóka and Herings (2021) restricts the analysis to the mutual payments between the agents, and does not consider compatibility with respect to the resulting allocation. Ketelaars, Borm, and Herings (2023) extends self-duality of claims rules to the financial network setting, showing that self-duality of transfer rules is a stronger requirement than self-duality of allocation rules.

This article is organized as follows. Section 2 discusses financial networks and claims problems. It shows how claims problems can be associated with financial networks and that allocation rules generalize claims rules. Section 3 introduces composition down and composition up for allocation rules, whereas Section 4 introduces these two axioms for transfer rules. Section 4 demonstrates that one has to be careful when formulating adequate extensions of axioms as they need not be compatible; however, the section also shows that compatible extensions exist with respect to the bottom transfer rule and the top transfer rule. Section 5 concludes.

2 Financial Networks and Claims Problems

The pair $(E, C) \in \mathbb{R}^N \times \mathbb{R}_+^{N \times N}$ represents a *financial network* in which N is a finite set of agents, $E = (e_i)_{i \in N}$ is an *estates vector*, and $C = (c_{ij})_{i,j \in N}$ is a *claims matrix*. The estate e_i belonging to agent $i \in N$ could be negative, in which case it represents the amount that the agent uses for own consumption before it pays its creditors. The claims matrix C contains linkages between the agents in the form of mutual claims. For all $i, j \in N$ with $i \neq j$, the cell c_{ij} of C represents the non-negative amount agent i is in debt to agent j. Agents in a financial network can thus be debtors and creditors simultaneously. It is assumed that the agents have no claim on themselves, that is, for all $i \in N$, we assume that $c_{ii} = 0$. There are no further conditions imposed on the claims matrix; in particular, for agents $i, j \in N$ with $i \neq j$ there is no condition on the relation between claims c_{ij} and c_{ji}. The vector containing the debts of agent $i \in N$ towards the other agents is denoted by $\bar{c}_i = (c_{ij})_{j \in N \setminus \{i\}} \in \mathbb{R}^{N \setminus \{i\}}$. The class of all financial networks on N is denoted by $\mathcal{F}^N$.

We will frequently consider a specific type of financial network that can be interpreted as a *claims problem*. A claims problem is represented by a pair $(e, c) \in \mathbb{R} \times \mathbb{R}_+^M$ in which M is a finite set of *claimants*, e is an *estate*, which could be negative, and $c = (c_i)_{i \in M}$ is a vector containing non-negative claims on the estate. The class of all claims problems on M is denoted by $\mathcal{C}^M$.[3]

[2] A different extension of impartiality, called net impartiality, is introduced in Csóka and Herings (2023).

[3] In the literature on claims problems (cf. Thomson (2019)), the estate is assumed to be non-negative and

Let $i \in N$ and consider $(e, c) \in \mathcal{C}^{N\setminus\{i\}}$. A financial network that corresponds to (e, c) is given by $(E, C) \in \mathcal{F}^N$ in which $e_i = e$, $\bar{c}_i = c$, and for all $j \in N \setminus \{i\}$, $e_j = 0$, and $\bar{c}_j = 0^{N\setminus\{j\}}$; for instance, if $N = \{1, 2, \ldots, n\}$ and $i = 1$, then (E, C) is given by

$$E = \begin{pmatrix} e \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ and } C = \begin{bmatrix} 0 & c_2 & \ldots & c_n \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}. \tag{2.1}$$

For each $i \in N$, and for all $(e, c) \in \mathcal{C}^{N\setminus\{i\}}$, define

$$\mathcal{F}^{N,i}(e, c) = \{(E, C) \in \mathcal{F}^N \mid e_i = e, \bar{c}_i = c, \text{ for all } j \in N \setminus \{i\}, e_j = 0, \bar{c}_j = 0^{N\setminus\{j\}}\}$$

as the set, which consists of exactly one element, of financial networks induced by the claims problem $(e, c) \in \mathcal{C}^{N\setminus\{i\}}$. Correspondingly, the collection of all financial networks in which agent $i \in N$ is the only agent to have a possibly non-zero estate and possibly debts towards the other agents is denoted by

$$\mathcal{F}^{N,i} = \bigcup_{(e,c)\in\mathcal{C}^{N\setminus\{i\}}} \mathcal{F}^{N,i}(e, c).$$

Claims rules, which prescribe, for each claims problem, a division of the estate among the claimants, will form the basis for the mutual payments in financial networks. A *claims rule* $\varphi\colon \mathcal{C}^M \to \mathbb{R}^M$ prescribes, for all $(e, c) \in \mathcal{C}^M$, the allocation vector $\varphi(e, c)$ satisfying

(i) $0 \leq \varphi_i(e, c) \leq c_i$ for all $i \in M$,

(ii) $\sum_{i\in M} \varphi_i(e, c) = \min\{\max\{0, e\}, \sum_{i\in M} c_i\}$.

Consider the claims problem $(e, c) \in \mathcal{C}^M$. If the estate is non-positive (i.e., $e \leq 0$), then conditions (i) and (ii) imply that $\varphi(e, c) = 0^M$. If the estate is at least the total of the claims (i.e., $e \geq \sum_{i\in M} c_i$), then conditions (i) and (ii) imply that $\varphi(e, c) = c$. Finally, if $0 \leq e \leq \sum_{i\in M} c_i$, then condition (ii) is given by $\sum_{i\in M} \varphi_i(e, c) = e$, so the estate is distributed in full among the claimants.

In the examples in this article, we consider one claims rule in particular, namely the *constrained equal awards rule*. It divides the estate as equally as possible among the claimants, provided that no claimant receives more than it claims. A claims rule φ on $\mathcal{C}^M$ is the constrained equal awards rule CEA if, for all $(e, c) \in \mathcal{C}^M$ with $0 \leq e \leq \sum_{i\in M} c_i$, and for all $i \in M$, $\varphi_i(e, c) = \max\{\lambda, c_i\}$, in which $\lambda \geq 0$ is such that $\sum_{j\in M} \varphi_j(e, c) = e$.

Each financial network on N is endowed with a vector of claims rules on N, which is denoted by $\phi = (\varphi^i)_{i\in N}$, in which, for all $i \in N$, $\varphi^i\colon \mathcal{C}^{N\setminus\{i\}} \to \mathbb{R}^{N\setminus\{i\}}$ is the claims rule associated with agent i that stipulates how it should pay its creditors. We denote by $\mathcal{V}^N$ the set of all vectors of claims rules on N that satisfy conditions (i) and (ii).

at most equal to the sum of the claims. In this article, we consider a more general class of claims problems in which the estate can be strictly negative or strictly larger than the sum of the claims.

We assume throughout that a claims rule φ on $\mathcal{C}^M$ satisfies *estate monotonicity*: for all $(e', c) \in \mathcal{C}^M$ and $(e, c) \in \mathcal{C}^M$ with $e' \leq e$, it holds that $\varphi(e', c) \leq \varphi(e, c)$. The set of all vectors of claims rules on N that satisfy estate monotonicity is denoted by

$$\mathcal{R}^N = \{\phi \in \mathcal{V}^N \mid \text{for all } i \in N, \varphi^i \text{ satisfies estate monotonicity}\}.$$

The constrained equal awards rule CEA satisfies estate monotonicity.

Each agent $i \in N$ uses its agent-specific claims rule φ^i to pay its creditors by allocating its estate plus incoming payments from its debtors. The payment matrix that contains all such transfers constitutes a *transfer scheme*. As each agent has no debt towards itself, the payment to itself is zero.

Definition 2.1. Let $\phi \in \mathcal{R}^N$, and let $(E, C) \in \mathcal{F}^N$. The payment matrix P is a *transfer scheme* for (E, C) with respect to ϕ if, for all $i \in N$, $p_{ii} = 0$, and for all $j \in N \setminus \{i\}$,

$$p_{ij} = \varphi^i_j(e_i + \sum_{k \in N} p_{ki}, \overline{c}_i).$$

The set of all possible transfer schemes for (E, C) with respect to ϕ is denoted by $\mathcal{P}^\phi(E, C)$.

The following example illustrates transfer schemes for a financial network with respect to the constrained equal awards rule.

Example 2.2. Consider the financial network $(E, C) \in \mathcal{F}^N$ given by $N = \{1, 2, 3, 4\}$,

$$E = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 3 \end{pmatrix} \text{ and } C = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 3 & 1 & 0 \end{bmatrix}.$$

Let $\phi = (\text{CEA}, \text{CEA}, \text{CEA}, \text{CEA}) \equiv \text{CEA}$.

Let $\lambda \in [0, 1]$. The payment matrix

$$P^\lambda = \begin{bmatrix} 0 & 1 & 1+\lambda & 1 \\ 0 & 0 & 1 & 1 \\ 1+\lambda & 1 & 0 & 1 \\ 1 & 3 & 1 & 0 \end{bmatrix}$$

is a transfer scheme for (E, C) with respect to CEA because, for all $i \in N$, $p^\lambda_{ii} = 0$, and

$$(1, 1+\lambda, 1) = \text{CEA}(1 + 2 + \lambda, (1, 2, 1)) \qquad \text{(Agent 1)}$$
$$(0, 1, 1) = \text{CEA}(3 + 5, (0, 1, 1)) \qquad \text{(Agent 2)}$$
$$(1+\lambda, 1, 1) = \text{CEA}(0 + 3 + \lambda, (2, 1, 1)) \qquad \text{(Agent 3)}$$
$$(1, 3, 1) = \text{CEA}(3 + 3, (1, 3, 1)). \qquad \text{(Agent 4)}$$

△

The set of transfer schemes for a financial network $(E, C) \in \mathcal{F}^N$ with respect to $\phi \in \mathcal{R}^N$ is a non-empty *complete lattice* with respect to the element-wise ordering $\leq$ of $\mathbb{R}^{N\times N}$.[4] A lattice is a partially ordered set in which every pair of elements has a greatest lower bound, which we call a *bottom*, and a least upper bound, which we call a *top*, within the lattice. A lattice is complete if also every non-empty subset has a bottom and a top within the lattice. As a consequence, there exists a *bottom transfer scheme* $\underline{P}^\phi(E, C) \in \mathcal{P}^\phi(E, C)$ and a *top transfer scheme* $\overline{P}^\phi(E, C) \in \mathcal{P}^\phi(E, C)$ such that, for all $P \in \mathcal{P}^\phi(E, C)$, it holds that $\underline{P}^\phi(E, C) \leq P \leq \overline{P}^\phi(E, C)$.

In situations in which for some agents the estate turns out to be larger than expected, whereas it remains unchanged for the other agents, we consider the following lemma. It shows that in these cases the bottom and top transfer schemes shift upward accordingly.

Lemma 2.3. *Let $\phi \in \mathcal{R}^N$, and let $(E', C) \in \mathcal{F}^N$ and $(E, C) \in \mathcal{F}^N$ with $E' \leq E$. Then, $\underline{P}^\phi(E', C) \leq \underline{P}^\phi(E, C)$ and $\overline{P}^\phi(E', C) \leq \overline{P}^\phi(E, C)$.*

Proof. The set of transfer schemes for (E, C) with respect to ϕ is a complete lattice and each coordinate of ϕ satisfies estate monotonicity. The result then follows from Theorem 3 in Milgrom and Roberts (1994). □

The bottom and top transfer scheme are not equal in general, so there can exist distinct transfer schemes for a financial network (see Example 2.2). A transfer scheme that is used to settle the mutual claims of the agents results in a redistributed estates vector that is called a *transfer allocation*. Although infinitely many transfer schemes can exist, any two transfer schemes for a financial network lead to the same transfer allocation.[5] A transfer allocation thus depends on the underlying vector of claims rules and not on a specific corresponding transfer scheme. Correspondingly, given $\phi \in \mathcal{R}^N$, we define the *allocation rule* μ^ϕ that prescribes, for each financial network, the transfer allocation that is based on any transfer scheme for that financial network with respect to ϕ.

Definition 2.4. Let $\phi \in \mathcal{R}^N$. The *allocation rule* μ^ϕ on $\mathcal{F}^N$ is, for all $(E, C) \in \mathcal{F}^N$, and all $i \in N$, defined by

$$\mu_i^\phi(E, C) = e_i + \sum_{j\in N} p_{ji} - \sum_{j\in N} p_{ij},$$

in which $P \in \mathcal{P}^\phi(E, C)$.

Indeed, an allocation rule μ^ϕ provides a redistribution of the estates vector because, for all $(E, C) \in \mathcal{F}^N$ and $P \in \mathcal{P}^\phi(E, C)$,

$$\sum_{i\in N} \mu_i^\phi(E, C) = \sum_{i\in N} e_i + \sum_{i\in N}\sum_{j\in N}(p_{ji} - p_{ij}) = \sum_{i\in N} e_i.$$

Although in general there can exist infinitely many transfer schemes for a financial network, this is not the case for a financial network that corresponds to a claims problem. For

[4]See, for example, Proposition 3.4 in Ketelaars et al. (2023). The set of transfer schemes for a financial network could be empty if the claims rule of an agent does not satisfy estate monotonicity.

[5]See, for example, Proposition 5.3 in Ketelaars et al. (2023)

example, for all $\phi \in \mathcal{R}^{\{1,2,\dots,n\}}$ there exists exactly one transfer scheme for (2.1), which is given by

$$P = \begin{bmatrix} 0 & \varphi^1_2(e,c) & \dots & \varphi^1_n(e,c) \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

As a result, the allocation prescribed by an allocation rule μ^ϕ coincides with the division prescribed by a claims rule of an agent. The following lemma formalizes this.

Lemma 2.5. *Let $i \in N$, let $\phi \in \mathcal{R}^N$, and let $(E,C) \in \mathcal{F}^{N,i}$. Then, $\mathcal{P}^\phi(E,C) = \{P\}$ in which $p_{jk} = 0$ for all $j \in N \setminus \{i\}$ and $k \in N$, $p_{ii} = 0$, and $p_{ij} = \varphi^i_j(e,c)$ for all $j \in N \setminus \{i\}$. Furthermore, for all $j \in N \setminus \{i\}$, $\mu^\phi_j(E,C) = \varphi^i_j(e,c)$.*

Proof. Let P be a payment matrix for which, for all $j \in N$, $p_{jj} = 0$. For P to be a transfer scheme for (E,C) with respect to ϕ, we require that, for all $j \in N$ and all $k \in N \setminus \{j\}$,

$$p_{jk} = \varphi^j_k(e_j + \sum_{h \in N} p_{hj}, \bar{c}_j).$$

From $(E,C) \in \mathcal{F}^{N,i}$ it follows that, for all $j \in N \setminus \{i\}$ and all $k \in N \setminus \{j\}$, $c_{jk} = 0$, and therefore also

$$\varphi^j_k(e_j + \sum_{h \in N} p_{hj}, \bar{c}_j) = 0,$$

which in turn implies that $p_{jk} = 0$. Then, for agent i, for all $k \in N \setminus \{i\}$,

$$\varphi^i_k(e_i + \sum_{h \in N} p_{hi}, \bar{c}_i) = \varphi^i_k(e,c),$$

in which the equality follows from $e_i = e$ and $\bar{c}_i = c$; hence, we must have that $p_{ik} = \varphi^i_k(e,c)$. Consequently, for all $j \in N \setminus \{i\}$,

$$\mu^\phi_j(E,C) = e_j + \sum_{k \in N} p_{kj} - \sum_{k \in N} p_{jk} = p_{ij} = \varphi^i_j(e,c).$$

□

3 Composition of Allocation Rules

Composition down and *composition up* are two invariance properties in the sense that a claims rule that satisfies these properties is invariant with respect to a change in the available estate. Composition down pertains to situations in which the estate is overestimated, whereas composition up pertains to situations in which the estate is underestimated.

3.1 Composition Down

Regarding composition down, suppose that a claims rule has prescribed a division of the estate among the claimants, but it turns out that the estate is smaller than expected. Composition down entails that there are two equivalent ways of dealing with such a situation. One way to resolve the issue is to cancel the initial division and apply the claims rule to the new situation. On the other hand, as the initial promise could not be honored, we can also, for each claimant, replace its claim on the estate with the amount it should have received initially. And, subsequently, use the claims rule to divide the actual, smaller, estate among the claimants, but now with respect to the revised claims. A claims rule satisfies composition down if it offers the exact same solution in both scenarios.

Definition 3.1. A claims rule φ on $\mathcal{C}^M$ satisfies *composition down* if, for all $(e', c) \in \mathcal{C}^M$ and $(e, c) \in \mathcal{C}^M$ with $e' \geq e$, it holds that $\varphi(e, c) = \varphi(e, \varphi(e', c))$.

Composition down implies estate monotonicity.[6]

Composition down for allocation rules is formulated as follows. Suppose that some agents have overestimated their estate, whereas the estates of the other agents are correctly evaluated, and that an allocation has been prescribed to the agents with respect to a transfer scheme that is based on this overestimated estates vector. Composition down for an allocation rule μ^ϕ implies that there are two equivalent ways of dealing with such a situation: provide an allocation of the actual smaller estates vector either with respect to the claims matrix or with respect to the, infeasible, transfer scheme that is based on the larger estates vector. As the allocation prescribed by an allocation rule μ^ϕ depends only on the vector of claims rules ϕ and not on the underlying transfer scheme, composition down requires invariance of an allocation rule μ^ϕ with respect to all transfer schemes that are based on the larger estates vector.

Definition 3.2. An allocation rule μ^ϕ on $\mathcal{F}^N$ satisfies *composition down* if, for all $(E', C) \in \mathcal{F}^N$ and $(E, C) \in \mathcal{F}^N$ with $E' \geq E$, and for all $P' \in \mathcal{P}^\phi(E', C)$, it holds that

$$\mu^\phi(E, C) = \mu^\phi(E, P').$$

The following theorem implies that Definition 3.2 is an adequate extension of the definition of composition down for claims rules (see Definition 3.1) because each claims rule in ϕ satisfies composition down if μ^ϕ satisfies composition down.[7] In fact, the reverse also holds, that is, composition down of the claims rules in ϕ carries over to composition down of the corresponding allocation rule μ^ϕ.

Theorem 3.3. *Let $\phi \in \mathcal{R}^N$. The allocation rule μ^ϕ satisfies composition down if and only if, for all $i \in N$, φ^i satisfies composition down.*

[6]Let φ on $\mathcal{C}^M$ satisfy composition down, and let $(e', c) \in \mathcal{C}^M$ and $(e, c) \in \mathcal{C}^M$ with $e' \leq e$. Then, $\varphi(e', c) = \varphi(e', \varphi(e, c)) \leq \varphi(e, c)$, in which the equality follows from composition down of φ and the inequality follows from condition (i) of the claims rule φ.

[7]In general, there may be multiple ways to adequately extend a property to the financial network setting. For example, considering Definition 3.2, one can formulate a 'weaker' form of composition down in the sense that there exists a $P' \in \mathcal{P}^\phi(E', C)$ such that the invariance requirement holds.

Proof. First, assume that μ^ϕ satisfies composition down. Let $i \in N$, and let $(E', C) \in \mathcal{F}^{N,i}$ and $(E, C) \in \mathcal{F}^{N,i}$ with $E' \geq E$. Lemma 2.5 implies that $\mathcal{P}^\phi(E', C) = \{P'\}$ in which $p'_{jk} = 0$ for all $j \in N \setminus \{i\}$ and $k \in N$, $p'_{ii} = 0$, and $p'_{ij} = \varphi^i_j(e', c)$ for all $j \in N \setminus \{i\}$. It therefore holds that $(E, P') \in \mathcal{F}^{N,i}$ and, by Lemma 2.5, for all $j \in N \setminus \{i\}$, $\mu^\phi_j(E, P') = \varphi^i_j(e, \varphi^i(e', c))$. Hence, for all $j \in N \setminus \{i\}$,

$$\varphi^i_j(e, c) = \mu^\phi_j(E, C) = \mu^\phi_j(E, P') = \varphi^i_j(e, \varphi^i(e', c)),$$

in which the first equality and third equality follow from Lemma 2.5, and the second equality follows from composition down of μ^ϕ. Thus, φ^i satisfies composition down.

Second, assume that, for all $i \in N$, φ^i satisfies composition down. Let $(E', C) \in \mathcal{F}^N$ and $(E, C) \in \mathcal{F}^N$ with $E' \geq E$, and let $P' \in \mathcal{P}^\phi(E', C)$. We will show that $\underline{P}^\phi(E, C) \in \mathcal{P}^\phi(E, P')$, such that, by definition of the allocation rule μ^ϕ, $\mu^\phi(E, C) = \mu^\phi(E, P')$. Let $i \in N$. Then, $\underline{p}^\phi_{ii} = 0$, and, for all $j \in N \setminus \{i\}$,

$$\underline{p}^\phi_{ij} = \varphi^i_j(e_i + \sum_{k \in N} \underline{p}^\phi_{ki}, \bar{c}_i) = \varphi^i_j(e_i + \sum_{k \in N} \underline{p}^\phi_{ki}, \varphi^i(e'_i + \sum_{k \in N} p'_{ki}, \bar{c}_i)),$$

in which the first equality follows from $\underline{P}^\phi(E, C) \in \mathcal{P}^\phi(E, C)$, and the second equality follows from composition down of φ^i, $E' \geq E$, and the fact that, by Lemma 2.3, $P' \geq \underline{P}^\phi(E', C) \geq \underline{P}^\phi(E, C)$. □

Hence, an allocation rule μ^ϕ does not satisfy composition down if the claims rule of at least one agent does not satisfy composition down.

3.2 Composition Up

Composition up provides an invariance requirement for cases in which one has divided the estate in accordance with a claims rule but later learns that the estate is larger. First, we can nullify the initial division and simply divide the actual, larger, estate among the claimants in accordance with the claims rule. Alternatively, we take the initial division of the, smaller, estate as given and add to this the division of the surplus estate in accordance with the claims rule. To do so, the claims of the claimants on the surplus estate are scaled down appropriately by subtracting the initial amount they received from their initial claim. If a claims rule satisfies composition up, then both approaches lead to the same outcome.

Definition 3.4. A claims rule φ on $\mathcal{C}^M$ satisfies *composition up* if, for all $(e', c) \in \mathcal{C}^M$ and $(e, c) \in \mathcal{C}^M$ with $0 \leq e' \leq e$, it holds that $\varphi(e, c) = \varphi(e', c) + \varphi(e - e', c - \varphi(e', c))$.

Composition up implies estate monotonicity.[8]

In Definition 3.4, there is no composition up requirement if the initial, underestimated, estate is negative. Consider $(e', c) \in \mathcal{C}^M$ and $(e, c) \in \mathcal{C}^M$ with $e' \leq e$, so e' is the initial,

[8] Indeed, if a claims rule φ on $\mathcal{C}^M$ satisfies composition up, then, for all $(e', c) \in \mathcal{C}^M$ and $(e, c) \in \mathcal{C}^M$ with $0 \leq e' \leq e$, it holds that $\varphi(e, c) = \varphi(e', c) + \varphi(e - e', c - \varphi(e', c)) \geq \varphi(e', c)$, in which the inequality follows from condition (i) of the claims rule φ. Additionally, for all $(e', c) \in \mathcal{C}^M$ and $(e, c) \in \mathcal{C}^M$ with $e' \leq e$, in which $e' < 0$, $\varphi(e, c) \geq 0 = \varphi(e', c)$.

underestimated, estate and e is the true estate. If $e' < 0$, then nothing has been allocated to the claimants to begin with, that is, $\varphi(e', c) = 0$. There can hence be no dispute between the claimants regarding the division of the true estate e as any positive surplus estate is simply equal to e itself. More specifically, if $e > 0$, the additional estate that has become available to be allocated among the claimants is equal to e, which results in an allocation vector $\varphi(e, c)$. We thus only impose a composition up requirement on a claims rule if the initial, underestimated, estate is non-negative.

Composition up for allocation rules is based on composition up for claims rules. Suppose that some agents in a financial network have underestimated their estate, whereas the estates of the other agents are correctly evaluated, and that an allocation has been prescribed to the agents with respect to a transfer scheme that is based on this underestimated estates vector. If an allocation rule μ^ϕ satisfies composition up, then one can either allocate the true estates vector, or allocate the *surplus estates vector* with respect to any corresponding *residual claims matrix* and add this to the initial allocation. The surplus estates vector is equal to the actual larger estates vector minus the underestimated estates vector. A residual claims matrix contains the remaining mutual claims of the agents after agents have paid each other according to a transfer scheme with respect to the smaller estates vector.

Definition 3.5. An allocation rule μ^ϕ on $\mathcal{F}^N$ satisfies *composition up* if, for all $(E', C) \in \mathcal{F}^N$ and $(E, C) \in \mathcal{F}^N$ with $0 \leq E' \leq E$, and for all $P' \in \mathcal{P}^\phi(E', C)$, it holds that

$$\mu^\phi(E, C) = \mu^\phi(E', C) + \mu^\phi(E - E', C - P').$$

As the following theorem implies, also Definition 3.5 is an adequate extension of the definition of composition up for claims rules (see Definition 3.4). In addition to this, composition up of the claims rules in ϕ is inherited by the corresponding allocation rule μ^ϕ.

Theorem 3.6. *Let $\phi \in \mathcal{R}^N$. The allocation rule μ^ϕ satisfies composition up if and only if, for all $i \in N$, φ^i satisfies composition up.*

Proof. First, assume that μ^ϕ satisfies composition up. Let $i \in N$ and let $(E', C) \in \mathcal{F}^{N,i}$ and $(E, C) \in \mathcal{F}^{N,i}$ with $0 \leq E' \leq E$. Lemma 2.5 implies that $\mathcal{P}^\phi(E', C) = \{P'\}$ in which $p'_{jk} = 0$ for all $j \in N \setminus \{i\}$ and $k \in N$, $p'_{ii} = 0$, and $p'_{ij} = \varphi^i_j(e', c)$ for all $j \in N \setminus \{i\}$. It therefore holds that $(E - E', C - P') \in \mathcal{F}^{N,i}$ and, by Lemma 2.5, for all $j \in N \setminus \{i\}$, $\mu^\phi_j(E - E', C - P') = \varphi^i_j(e - e', c - \varphi^i(e', c))$. Hence, for all $j \in N \setminus \{i\}$,

$$\varphi^i_j(e, c) = \mu^\phi_j(E, C) = \mu^\phi_j(E', C) + \mu^\phi_j(E - E', C - P') = \varphi^i_j(e', c) + \varphi^i_j(e - e', c - \varphi^i(e', c)),$$

in which the first equality and third equality follow from Lemma 2.5, and the second equality follows from composition up of μ^ϕ. Thus, φ^i satisfies composition up.

Second, assume that, for all $i \in N$, φ^i satisfies composition up. Let $(E', C) \in \mathcal{F}^N$ and $(E, C) \in \mathcal{F}^N$ with $0 \leq E' \leq E$, and let $P' \in \mathcal{P}^\phi(E', C)$. We will show that $(\overline{P}^\phi(E, C) - P') \in \mathcal{P}^\phi(E - E', C - P')$, which implies that, for all $i \in N$,

$$\mu^\phi_i(E - E', C - P') = (e_i - e'_i) + \sum_{j \in N} (\overline{p}^\phi_{ji} - p'_{ji}) - \sum_{j \in N} (\overline{p}^\phi_{ij} - p'_{ij})$$

$$= e_i + \sum_{j \in N} \overline{p}^{\phi}_{ji} - \sum_{j \in N} \overline{p}^{\phi}_{ij} - (e'_i + \sum_{j \in N} p'_{ji} - \sum_{j \in N} p'_{ij})$$
$$= \mu^{\phi}_i(E, C) - \mu^{\phi}_i(E', C).$$

Let $i \in N$. Set $e' = e'_i + \sum_{k \in N} p'_{ki}$ and $e = e_i + \sum_{k \in N} \overline{p}^{\phi}_{ki}$. Then, $e' \leq e$ because $E' \leq E$ and, by Lemma 2.3, $P' \leq \overline{P}^{\phi}(E', C) \leq \overline{P}^{\phi}(E, C)$. In addition, $e' \geq 0$ because $E' \geq 0$ and $P' \geq 0$. Hence, $\overline{p}^{\phi}_{ii} - p'_{ii} = 0$, and, for all $j \in N \setminus \{i\}$,

$$\overline{p}^{\phi}_{ij} - p'_{ij} = \varphi^i_j(e, \overline{c}_i) - \varphi^i_j(e', \overline{c}_i) = \varphi^i_j(e - e', \overline{c}_i - \varphi^i(e', \overline{c}_i)),$$

where the first equality follows from $\overline{P}^{\phi}(E, C) \in \mathcal{P}^{\phi}(E, C)$ and $P' \in \mathcal{P}^{\phi}(E', C)$, and the second equality follows from composition up of φ^i. □

4 Composition of Transfer Rules

The inheritance of the properties in the previous section with respect to allocation rules did not cause any problems in the sense that an allocation rule μ^{ϕ} satisfies the property if and only if each claims rule in ϕ satisfies it. This section introduces adequate extensions of composition down and composition up, though now for the *transfer rules* underlying an allocation rule μ^{ϕ}. However, we demonstrate that one needs to be careful in doing so because two adequate extensions need not be compatible in the sense that no transfer rule can satisfy them simultaneously, despite the fact that the claims rules in ϕ in fact do satisfy them simultaneously.

A transfer rule τ^{ϕ} prescribes, for each financial network, exactly one transfer scheme with respect to ϕ.

Definition 4.1. Let $\phi \in \mathcal{R}^N$. A *transfer rule* τ^{ϕ} on $\mathcal{F}^N$ assigns to each financial network $(E, C) \in \mathcal{F}^N$ exactly one transfer scheme $P \in \mathcal{P}^{\phi}(E, C)$.

Indeed, since any transfer rule τ^{ϕ} prescribes a transfer scheme, it forms the basis for the allocation prescribed by the corresponding allocation rule μ^{ϕ}. For all $(E, C) \in \mathcal{F}^N$, and all $\tau^{\phi}(E, C)$, it holds that, for all $i \in N$,

$$\mu^{\phi}_i(E, C) = e_i + \sum_{j \in N} \tau^{\phi}_{ji}(E, C) - \sum_{j \in N} \tau^{\phi}_{ij}(E, C). \tag{4.1}$$

Lemma 2.5 states that there exists exactly one transfer scheme for a financial network that corresponds to a claims problem. As a consequence, a transfer rule τ^{ϕ} must prescribe that specific transfer scheme. More specifically, consider $i \in N$, $\phi \in \mathcal{R}^N$, and $(E, C) \in \mathcal{F}^{N,i}$. Then, Lemma 2.5 implies that, for all $j \in N \setminus \{i\}$, $\tau^{\phi}_{ij}(E, C) = \varphi^i_j(e, c)$.

We consider two transfer rules in particular, namely one that always prescribes the bottom transfer scheme and one that always prescribes the top transfer scheme.

Definition 4.2. A transfer rule $\underline{\tau}^{\phi}$ is the *bottom transfer rule* if, for all $(E, C) \in \mathcal{F}^N$, $\underline{\tau}^{\phi}(E, C) = \underline{P}^{\phi}(E, C)$. A transfer rule $\overline{\tau}^{\phi}$ is the *top transfer rule* if, for all $(E, C) \in \mathcal{F}^N$, $\overline{\tau}^{\phi}(E, C) = \overline{P}^{\phi}(E, C)$.

The following lemma states that the bottom and top transfer schemes can be obtained as the limit of iterative procedures defined on a complete lattice with respect to a mapping. More specifically, let $\phi \in \mathcal{R}^N$, let $(E, C) \in \mathcal{F}^N$, and consider the complete lattice $[0^{N\times N}, C] = \{P \in \mathbb{R}^{N\times N} \,|\, \text{for all } i, j \in N,\ 0 \leq p_{ij} \leq c_{ij}\}$. Define the function $f(\cdot\,; E, C)\colon [0^{N\times N}, C] \to [0^{N\times N}, C]$ by setting, for all $P \in [0^{N\times N}, C]$, for all $i \in N$, $f_{ii}(P; E, C) = 0$, and, for all $j \in N \setminus \{i\}$,[9]

$$f_{ij}(P; E, C) = \varphi^i_j(e_i + \sum_{k\in N} p_{ki}, \bar{c}_i).$$

Starting with $P^1 = 0^{N\times N}$ and iteratively applying the function f gives a sequence of payment matrices that is monotonically increasing (i.e., $P^1 \leq P^2 \leq \dots$) and equals the bottom transfer scheme in the limit; starting with $P^1 = C$ gives a sequence of payment matrices that is monotonically decreasing and equals the top transfer scheme in the limit.

Lemma 4.3 (cf. Theorem 3.10 in Ketelaars et al. (2023))**.** *Let $\phi \in \mathcal{R}^N$, and let $(E, C) \in \mathcal{F}^N$. Then,*

(i) $\underline{P}^\phi(E, C) = \lim_{k\to\infty} P^k$, *where, for all* $k \in \mathbb{N}$, $P^{k+1} = f(P^k; E, C)$ *with* $P^1 = 0^{N\times N}$;

(ii) $\overline{P}^\phi(E, C) = \lim_{k\to\infty} P^k$, *where, for all* $k \in \mathbb{N}$, $P^{k+1} = f(P^k; E, C)$ *with* $P^1 = C$.

4.1 Not All Adequate Extensions are Compatible

It is natural to define composition down and composition up for transfer rules in a way that is analogous to their definitions for allocation rules.

Strong composition down requires invariance of the prescribed payments between the agents, irrespective of the initial, infeasible, choice of transfer scheme that is based on the larger estates vector.

Definition 4.4. A transfer rule τ^ϕ on $\mathcal{F}^N$ satisfies *strong composition down* if, for all $(E', C) \in \mathcal{F}^N$ and $(E, C) \in \mathcal{F}^N$ with $E' \geq E$, and for all $P' \in \mathcal{P}^\phi(E', C)$, it holds that

$$\tau^\phi(E, C) = \tau^\phi(E, P').$$

Strong composition up is formulated likewise.

Definition 4.5. A transfer rule τ^ϕ on $\mathcal{F}^N$ satisfies *strong composition up* if, for all $(E', C) \in \mathcal{F}^N$ and $(E, C) \in \mathcal{F}^N$ with $0 \leq E' \leq E$, and for all $P' \in \mathcal{P}^\phi(E', C)$, it holds that

$$\tau^\phi(E, C) = P' + \tau^\phi(E - E', C - P').$$

The following proposition states that Definition 4.4 and Definition 4.5 are adequate extensions of composition down and composition up for claims rules, respectively.

[9] The function f explicitly incorporates (E, C) in the notation because it improves clarity when we use f with respect to distinct financial networks.

Proposition 4.6. *Let $\phi \in \mathcal{R}^N$. It holds that:*

(i) if the transfer rule τ^ϕ satisfies strong composition down, then, for all $i \in N$, φ^i satisfies composition down;

(ii) if the transfer rule τ^ϕ satisfies strong composition up, then, for all $i \in N$, φ^i satisfies composition up.

Proof. (i). Assume that τ^ϕ satisfies strong composition down. Let $(E', C) \in \mathcal{F}^N$ and $(E, C) \in \mathcal{F}^N$ with $E' \geq E$, and let $P' \in \mathcal{P}^\phi(E', C)$. Then, for all $i \in N$,

$$\begin{aligned}
\mu_i^\phi(E, C) &= e_i + \sum_{j \in N} \tau_{ji}^\phi(E, C) - \sum_{j \in N} \tau_{ij}^\phi(E, C) \\
&= e_i + \sum_{j \in N} \tau_{ji}^\phi(E, P') - \sum_{j \in N} \tau_{ij}^\phi(E, P') \\
&= \mu_i^\phi(E, P'),
\end{aligned}$$

in which the first equality and third equality follow from (4.1), and the second equality follows from strong composition down of τ^ϕ. Hence, μ^ϕ satisfies composition down and, by Theorem 3.3, for all $i \in N$, φ^i satisfies composition down.

(ii). Assume that τ^ϕ satisfies strong composition up. Let $(E', C) \in \mathcal{F}^N$ and $(E, C) \in \mathcal{F}^N$ with $0 \leq E' \leq E$, and let $P' \in \mathcal{P}^\phi(E', C)$. Then, for all $i \in N$,

$$\begin{aligned}
\mu_i^\phi(E, C) &= e_i + \sum_{j \in N} \tau_{ji}^\phi(E, C) - \sum_{j \in N} \tau_{ij}^\phi(E, C) \\
&= e_i + \sum_{j \in N} (p'_{ji} + \tau_{ji}^\phi(E - E', C - P')) - \sum_{j \in N} (p'_{ij} + \tau_{ij}^\phi(E - E', C - P')) \\
&= e'_i + \sum_{j \in N} p'_{ji} - \sum_{j \in N} p'_{ij} \\
&\quad + (e_i - e'_i) + \sum_{j \in N} \tau_{ji}^\phi(E - E', C - P') - \sum_{j \in N} \tau_{ij}^\phi(E - E', C - P') \\
&= \mu_i^\phi(E', C) + \mu_i^\phi(E - E', C - P'),
\end{aligned}$$

in which the first equality and fourth equality follow from (4.1), and the second equality follows from strong composition up of τ^ϕ. Hence, μ^ϕ satisfies composition up and, by Theorem 3.6, for all $i \in N$, φ^i satisfies composition up. □

The follow theorem states that the bottom transfer rule $\underline{\tau}^\phi$ satisfies strong composition down if the claims rules in ϕ satisfy composition down.

Theorem 4.7. *Let $\phi = (\varphi^i)_{i \in N}$ be a vector of claims rules such that, for all $i \in N$, φ^i satisfies composition down. Then, $\underline{\tau}^\phi$ satisfies strong composition down.*

Proof. Let $(E', C) \in \mathcal{F}^N$ and $(E, C) \in \mathcal{F}^N$ with $E' \geq E$, and let $P' \in \mathcal{P}^\phi(E', C)$. To show that $\underline{\tau}^\phi(E, C) = \underline{\tau}^\phi(E, P')$, it suffices to prove that $\underline{P}^\phi(E, C) = \underline{P}^\phi(E, P')$.

Lemma 4.3 states that

$$\underline{P}^{\phi}(E,C)=\lim_{k\to\infty}P^k,$$

where, for all $k\in\mathbb{N}$, $P^{k+1}=f(P^k;E,C)$ with $P^1=0^{N\times N}$. It furthermore holds that

$$\underline{P}^{\phi}(E,P')=\lim_{k\to\infty}Q^k,$$

where, for all $k\in\mathbb{N}$, $Q^{k+1}=f(Q^k;E,P')$ with $Q^1=0^{N\times N}$. We will show that, for all $k\in\mathbb{N}$, $P^k=Q^k$, from which it follows that $\underline{P}^{\phi}(E,C)=\underline{P}^{\phi}(E,P')$. Clearly, $P^1=0^{N\times N}=Q^1$. Let $k\in\mathbb{N}$ and assume that $P^k=Q^k$. Let $i\in N$. Then, $p_{ii}^{k+1}=f_{ii}(P^k;E,C)=0=f_{ii}(Q^k;E,P)=q_{ii}^{k+1}$, and, for all $j\in N\setminus\{i\}$,

$$\begin{aligned}
p_{ij}^{k+1}&=f_{ij}(P^k;E,C)\\
&=\varphi_j^i(e_i+\sum_{h\in N}p_{hi}^k,\overline{c}_i)\\
&=\varphi_j^i(e_i+\sum_{h\in N}p_{hi}^k,\varphi^i(e_i'+\sum_{h\in N}p_{hi}',\overline{c}_i))\\
&=\varphi_j^i(e_i+\sum_{h\in N}q_{hi}^k,\varphi^i(e_i'+\sum_{h\in N}p_{hi}',\overline{c}_i))\\
&=f_{ij}(Q^k;E,P')\\
&=q_{ij}^{k+1}.
\end{aligned}$$

The third equality follows from composition down of φ^i, $E'\geq E$, and the fact that, by Lemma 2.3, $P'\geq\underline{P}^{\phi}(E',C)\geq\underline{P}^{\phi}(E,C)\geq P^k$; the fourth equality follows from $P^k=Q^k$. Hence, by induction, it holds that, for all $k\in\mathbb{N}$, $P^k=Q^k$. □

On the other hand, as the following theorem states, the top transfer rule $\overline{\tau}^{\phi}$ satisfies strong composition up if the claims rules in ϕ satisfy composition up.

Theorem 4.8. *Let $\phi=(\varphi^i)_{i\in N}$ be a vector of claims rules such that, for all $i\in N$, φ^i satisfies composition up. Then, $\overline{\tau}^{\phi}$ satisfies strong composition up.*

Proof. Let $(E',C)\in\mathcal{F}^N$ and $(E,C)\in\mathcal{F}^N$ with $0\leq E'\leq E$, and let $P'\in\mathcal{P}^{\phi}(E',C)$. To show that

$$\overline{\tau}^{\phi}(E,C)=P'+\overline{\tau}^{\phi}(E-E',C-P'),$$

it suffices to prove that

$$\overline{P}^{\phi}(E,C)-P'=\overline{P}^{\phi}(E-E',C-P'). \tag{4.2}$$

Lemma 4.3 states that

$$\overline{P}^{\phi}(E,C)-P'=\lim_{k\to\infty}P^k-P'$$

where, for all $k \in \mathbb{N}$, $P^{k+1} = f(P^k; E, C)$ with $P^1 = C$. Additionally, we have

$$\underline{P}^{\phi}(E - E', C - P') = \lim_{k \to \infty} Q^k,$$

where, for all $k \in \mathbb{N}$, $Q^{k+1} = f(Q^k; E - E', C - P')$ with $Q^1 = C - P'$. We will show that, for all $k \in \mathbb{N}$, $P^k - P' = Q^k$, from which it follows that (4.2) holds. Clearly, $P^1 - P' = C - P' = Q^1$. Let $k \in \mathbb{N}$ and assume that $P^k - P' = Q^k$. Let $i \in N$. Then, $p_{ii}^{k+1} - p'_{ii} = f_{ii}(P^k; E, C) - p'_{ii} = 0 - 0 = 0 = f_{ii}(Q^k; E - E', C - P') = q_{ii}^{k+1}$, and, for all $j \in N \setminus \{i\}$,

$$\begin{aligned}
p_{ij}^{k+1} - p'_{ij} &= f_{ij}(P^k; E, C) - p'_{ij} \\
&= \varphi_j^i(e_i + \sum_{h \in N} p_{hi}^k, \bar{c}_i) - \varphi_j^i(e'_i + \sum_{h \in N} p'_{hi}, \bar{c}_i) \\
&= \varphi_j^i((e_i - e'_i) + \sum_{h \in N} (p_{hi}^k - p'_{hi}), \bar{c}_i - \varphi^i(e'_i + \sum_{h \in N} p'_{hi}, \bar{c}_i)) \\
&= \varphi_j^i((e_i - e'_i) + \sum_{h \in N} q_{hi}^k, \bar{c}_i - \varphi^i(e'_i + \sum_{h \in N} p'_{hi}, \bar{c}_i)) \\
&= f_{ij}(Q^k; E - E', C - P') \\
&= q_{ij}^{k+1}.
\end{aligned}$$

The second equality follows from $P' \in \mathcal{P}^{\phi}(E', C)$; the third equality follows from composition up of φ^i, $E' \leq E$, and the fact that, by Lemma 2.3, $P' \leq \overline{P}^{\phi}(E', C) \leq \overline{P}^{\phi}(E, C) \leq P^k$; the fourth equality follows from $P^k - P' = Q^k$. Hence, by induction, for all $k \in \mathbb{N}$, $P^k - P' = Q^k$. □

Theorem 4.7 states that the bottom transfer rule $\underline{\tau}^{\phi}$ is always invariant with respect to downward changes in the estates vector, whereas Theorem 4.8 states that the top transfer rule $\overline{\tau}^{\phi}$ is always invariant with respect to upward changes in the estates vector. These two results suggest that strong composition down and strong composition up are incompatible properties for transfer rules in the sense that, although all the claims rules in ϕ satisfy composition down and composition up, no transfer rule τ^{ϕ} can satisfy the extended versions of these properties simultaneously. This is indeed what the following example confirms.

Example 4.9. Consider the financial network $(E, C) \in \mathcal{F}^N$ given by $N = \{1, 2, 3\}$,

$$E = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Moreover, consider $(E', C) \in \mathcal{F}^N$ and $(E'', C) \in \mathcal{F}^N$ with $E' = (0, 0, 2)$ and $E'' = (0, 0, 0)$ (i.e., $E'' \leq E \leq E'$). It holds that $\mathcal{P}^{\phi}(E, C) = \mathcal{P}^{\phi}(E', C) = \mathcal{P}^{\phi}(E'', C) = \{\lambda C \mid \lambda \in [0, 1]\}$.

Let $\phi \in \mathcal{R}^N$. If a transfer rule τ^{ϕ} satisfies strong composition down and strong composition up, then, for all $P' \in \mathcal{P}^{\phi}(E', C)$,

$$\tau^{\phi}(E, C) = \tau^{\phi}(E, P') \tag{4.3}$$

and, for all $P'' \in \mathcal{P}^\phi(E'', C)$,

$$\tau^\phi(E, C) = P'' + \tau^\phi(E - E'', C - P''). \tag{4.4}$$

Suppose that $\tau^\phi(E, C) = \lambda C$ for some $\lambda \in (0, 1]$. Then, $P' = 0^{N\times N}$ gives $\mathcal{P}^\phi(E, P') = \{P'\}$ and (4.3) implies that $\tau^\phi(E, C) = \tau^\phi(E, P') = 0^{N\times N}$, which contradicts $\lambda \in (0, 1]$. Hence, if τ^ϕ satisfies strong composition down, we must have that $\tau^\phi(E, C) = 0^{N\times N} = \underline{P}^\phi(E, C)$.

Suppose that $\tau^\phi(E, C) = \lambda C$ for some $\lambda \in [0, 1)$. Then, $P'' = C$ gives $\mathcal{P}^\phi(E - E'', C - P'') = \{0^{N\times N}\}$ and (4.4) implies that $\tau^\phi(E, C) = P'' + 0^{N\times N} = C$, which contradicts $\lambda \in [0, 1)$. Hence, if τ^ϕ satisfies strong composition up, we must have that $\tau^\phi(E, C) = C = \overline{P}^\phi(E, C)$.

However, because $\underline{P}^\phi(E, C)$ differs from $\overline{P}^\phi(E, C)$, τ^ϕ can not satisfy strong composition down and strong composition up simultaneously. △

4.2 Compatible Extensions of Composition Axioms

The previous section shows that adequate extensions of axioms for claims rules can be incompatible in the context of transfer rules. Fortunately, there exist adequate extensions of composition down and composition up that are compatible with respect to transfer rules, which we will introduce in this section.

A transfer rule τ^ϕ provides a specification of the mutual payments for each financial network, which is why composition down for a transfer rule τ^ϕ now requires invariance of the mutual payments with respect to the transfer scheme prescribed by τ^ϕ only. We thus take the transfer scheme in accordance with τ^ϕ as a substitute for the claims matrix, and not any transfer scheme as was the case under strong composition down.

Definition 4.10. A transfer rule τ^ϕ on $\mathcal{F}^N$ satisfies *composition down* if, for all $(E', C) \in \mathcal{F}^N$ and $(E, C) \in \mathcal{F}^N$ with $E' \geq E$, it holds that

$$\tau^\phi(E, C) = \tau^\phi(E, \tau^\phi(E', C)).$$

Clearly, strong composition down of τ^ϕ implies composition down of τ^ϕ.

In regard to composition up for a transfer rule τ^ϕ, we take the transfer scheme prescribed by τ^ϕ with respect to the smaller estates vector as given.

Definition 4.11. A transfer rule τ^ϕ on $\mathcal{F}^N$ satisfies *composition up* if, for all $(E', C) \in \mathcal{F}^N$ and $(E, C) \in \mathcal{F}^N$ with $0 \leq E' \leq E$, it holds that

$$\tau^\phi(E, C) = \tau^\phi(E', C) + \tau^\phi(E - E', C - \tau^\phi(E', C)).$$

Clearly, strong composition up of τ^ϕ implies composition up of τ^ϕ.

Both Definition 4.10 and Definition 4.11 adequately extend the respective axioms, as the following proposition implies.

Proposition 4.12. *Let $\phi \in \mathcal{R}^N$. It holds that:*

(i) if the transfer rule τ^ϕ satisfies composition down, then, for all $i \in N$, φ^i satisfies composition down;

(ii) if the transfer rule τ^ϕ satisfies composition up, then, for all $i \in N$, φ^i satisfies composition up.

Proof. (i). Assume that τ^ϕ satisfies composition down. Let $i \in N$, and let $(E', C) \in \mathcal{F}^{N,i}$ and $(E, C) \in \mathcal{F}^{N,i}$ with $E' \geq E$. Lemma 2.5 implies that $\mathcal{P}^\phi(E', C) = \{P'\}$ in which $p'_{jk} = 0$ for all $j \in N \setminus \{i\}$ and $k \in N$, $p'_{ii} = 0$, and $p'_{ij} = \varphi^i_j(e', c)$ for all $j \in N \setminus \{i\}$. It therefore holds that $\tau^\phi(E', C) = P'$ and $(E, \tau^\phi(E', C)) \in \mathcal{F}^{N,i}$ and, by Lemma 2.5, for all $j \in N \setminus \{i\}$, $\tau^\phi_{ij}(E, \tau^\phi(E', C)) = \varphi^i_j(e, \varphi^i(e', c))$. Hence, for all $j \in N \setminus \{i\}$,

$$\varphi^i_j(e, c) = \tau^\phi_{ij}(E, C) = \tau^\phi_{ij}(E, \tau^\phi(E', C)) = \varphi^i_j(e, \varphi^i(e', c)),$$

in which the first equality and third equality follow from Lemma 2.5, and the second equality follows from composition down of τ^ϕ. Thus, φ^i satisfies composition down.

(ii) Assume that τ^ϕ satisfies composition up. Let $i \in N$, and let $(E', C) \in \mathcal{F}^{N,i}$ and $(E, C) \in \mathcal{F}^{N,i}$ with $0 \leq E' \leq E$. Lemma 2.5 implies that $\mathcal{P}^\phi(E', C) = \{P'\}$ in which $p'_{jk} = 0$ for all $j \in N \setminus \{i\}$ and $k \in N$, $p'_{ii} = 0$, and $p'_{ij} = \varphi^i_j(e', c)$ for all $j \in N \setminus \{i\}$. It therefore holds that $\tau^\phi(E', C) = P'$ and $(E - E', C - \tau^\phi(E', C)) \in \mathcal{F}^{N,i}$ and, by Lemma 2.5, for all $j \in N \setminus \{i\}$, $\tau^\phi_{ij}(E - E', C - \tau^\phi(E', C)) = \varphi^i_j(e - e', c - \varphi^i(e', c))$. Hence, for all $j \in N \setminus \{i\}$,

$$\begin{aligned}\varphi^i_j(e, c) &= \tau^\phi_{ij}(E, C)\\ &= \tau^\phi_{ij}(E', C) + \tau^\phi_{ij}(E - E', C - \tau^\phi(E', C))\\ &= \varphi^i_j(e', c) + \varphi^i_j(e - e', c - \varphi^i(e', c)),\end{aligned}$$

in which the first equality and third equality follow from Lemma 2.5, and the second equality follows from composition up of τ^ϕ. Thus, φ^i satisfies composition up. □

It is not necessarily true that a transfer rule τ^ϕ satisfies composition down (resp. composition up) if each claims rule in ϕ satisfies composition down (resp. composition up). The following example illustrates this.

Example 4.13. Reconsider the financial network $(E, C) \in \mathcal{F}^N$ of Example 2.2 given by $N = \{1, 2, 3, 4\}$,

$$E = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 3 \end{pmatrix} \text{ and } C = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 3 & 1 & 0 \end{bmatrix}.$$

Let $\phi = \text{CEA}$. The constrained equal awards rule satisfies both composition down and composition up.

The set of transfer schemes for (E, C) with respect to CEA is given by

$$\mathcal{P}^{\text{CEA}}(E, C) = \{P^\lambda \,|\, \lambda \in [0, 1]\}, \tag{4.5}$$

in which, for all $\lambda \in [0, 1]$, P^λ is given by

$$P^\lambda = \begin{bmatrix} 0 & 1 & 1+\lambda & 1 \\ 0 & 0 & 1 & 1 \\ 1+\lambda & 1 & 0 & 1 \\ 1 & 3 & 1 & 0 \end{bmatrix}.$$

Here, $\lambda = 0$ corresponds to the bottom transfer scheme for (E, C), and $\lambda = 1$ corresponds to the top transfer scheme for (E, C), which is equal to C. Suppose that $\tau^{\text{CEA}}(E, C) = P^\lambda$ for some $\lambda \in [0, 1]$.

Consider $(E', C) \in \mathcal{F}^N$ with $E' = (1, 3, 0, 4)$, that is, $E' \geq E$. Assume that the initial mutual payments with respect to E' are given by $\tau^{\text{CEA}}(E', C) = \underline{P}^{\text{CEA}}(E', C)$, where the bottom transfer scheme for (E', C) with respect to CEA is given by

$$\underline{P}^{\text{CEA}}(E', C) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 3 & 1 & 0 \end{bmatrix}.$$

The set of transfer schemes for $(E, \underline{P}^{\text{CEA}}(E', C))$ with respect to CEA is given by

$$\mathcal{P}^{\text{CEA}}(E, \underline{P}^{\text{CEA}}(E', C)) = \{P^0\}, \tag{4.6}$$

which implies that $\tau^{\text{CEA}}(E', \tau^{\text{CEA}}(E', C)) = P^0$. Hence, if $\lambda \neq 0$, then τ^{CEA} does not satisfy composition down.

Now, consider $(E'', C) \in \mathcal{F}^N$ with $E'' = (1, 2, 0, 0)$, that is, $E'' \leq E$. Assume that the initial mutual payments with respect to E'' are in accordance with $\tau^{\text{CEA}}(E'', C) = \overline{P}^{\text{CEA}}(E'', C)$, where the top transfer scheme for (E'', C) with respect to CEA is given by

$$\overline{P}^{\text{CEA}}(E'', C) = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}. \tag{4.7}$$

The surplus estates vector is equal to

$$E - E'' = (1, 3, 0, 3) - (1, 2, 0, 0) = (0, 1, 0, 3),$$

and the residual claims matrix with respect to $\overline{P}^{\text{CEA}}(E'', C)$ is given by

$$C - \overline{P}^{\text{CEA}}(E'', C) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}.$$

Then, the set of CEA-transfer schemes for $(E - E'', C - \overline{P}^{\text{CEA}}(E'', C))$ is given by

$$\mathcal{P}^{\text{CEA}}(E - E'', C - \overline{P}^{\text{CEA}}(E'', C)) = \{C - \overline{P}^{\text{CEA}}(E'', C)\}. \tag{4.8}$$

Hence, we have

$$\tau^{\text{CEA}}(E'', C) + \tau^{\text{CEA}}(E - E'', C - \tau^{\text{CEA}}(E'', C)) = P^1.$$

If $\lambda \neq 1$, then τ^{CEA} does not satisfy composition up. △

In the remainder of this section, we show that the bottom and top transfer rules satisfy composition down and composition up simultaneously if the underlying claims rules do so, too.

The following lemma describes what happens to the set of transfer schemes in case some agents have overestimated their estate, but the other agents have correctly evaluated their estate. First, if one takes a transfer scheme that is based on a larger estates vector as a substitute for the claims matrix, then the corresponding set of transfer schemes is a subset of the set of transfer schemes for the financial network with respect to the original claims matrix. This is also what Example 4.13 has demonstrated: considering (4.5) and (4.6), we see that[10]

$$\mathcal{P}^{\text{CEA}}(E, \underline{P}^{\text{CEA}}(E', C)) = \{P^0\} \subset \mathcal{P}^{\text{CEA}}(E, C).$$

Second, if one takes an appropriately large transfer scheme as a substitute for the claims matrix, then composition down of the individual claims rules implies equivalence of the set of transfer schemes. In particular, one can take the top transfer scheme as a substitute for the claims matrix, and we obtain result (iii) in the following lemma.

Lemma 4.14. *Let $\phi = (\varphi^i)_{i\in N}$ be a vector of claims rules such that, for all $i \in N$, φ^i satisfies composition down, and let $(E', C) \in \mathcal{F}^N$ and $(E, C) \in \mathcal{F}^N$ with $E' \geq E$. Then,*

(i) for all $P' \in \mathcal{P}^\phi(E', C)$, $\mathcal{P}^\phi(E, P') \subseteq \mathcal{P}^\phi(E, C)$;

(ii) for all $P' \in \mathcal{P}^\phi(E', C)$ such that $P' \geq \overline{P}^\phi(E, C)$, $\mathcal{P}^\phi(E, P') = \mathcal{P}^\phi(E, C)$;

(iii) $\mathcal{P}^\phi(E, \overline{P}^\phi(E', C)) = \mathcal{P}^\phi(E, C)$.

Proof. (i). Let $P' \in \mathcal{P}^\phi(E', C)$ and let $P \in \mathcal{P}^\phi(E, P')$. We will show that $P \in \mathcal{P}^\phi(E, C)$.

For all $i, j \in N$ with $i \neq j$, it holds that

$$p_{ij} = \varphi^i_j(e_i + \sum_{k\in N} p_{ki}, \varphi^i(e'_i + \sum_{k\in N} p'_{ki}, \bar{c}_i)) \leq \varphi^i_j(e'_i + \sum_{k\in N} p'_{ki}, \bar{c}_i) = p'_{ij}.$$

The first equality follows from $P \in \mathcal{P}^\phi(E, P')$ and $P' \in \mathcal{P}^\phi(E', C)$; the inequality follows from the condition (i) of the claims rule φ^i; the second equality follows from $P' \in \mathcal{P}^\phi(E', C)$. Therefore, for all $i \in N$, it holds that

$$e_i + \sum_{k\in N} p_{ki} \leq e'_i + \sum_{k\in N} p'_{ki}. \tag{4.9}$$

[10]Here, for any two sets S and T, the notation $S \subset T$ means that S is a proper subset of T. The notation $S \subseteq T$ means that $S = T$ is allowed as well.

Let $i \in N$. Then, $p_{ii} = 0$, and, for all $j \in N \setminus \{i\}$,

$$p_{ij} = \varphi^i_j(e_i + \sum_{k \in N} p_{ki}, \varphi^i(e'_i + \sum_{k \in N} p'_{ki}, \bar{c}_i)) = \varphi^i_j(e_i + \sum_{k \in N} p_{ki}, \bar{c}_i),$$

in which the first equality follows from $P \in \mathcal{P}^\phi(E, P')$ and $P' \in \mathcal{P}^\phi(E', C)$, and the second equality follows from composition down of φ^i and inequality (4.9).

(ii). Let $P' \in \mathcal{P}^\phi(E', C)$ such that $P' \geq \overline{P}^\phi(E, C)$. From Lemma 4.14 (i) it follows that $\mathcal{P}^\phi(E, P') \subseteq \mathcal{P}^\phi(E, C)$. Let $P \in \mathcal{P}^\phi(E, C)$. It suffices to prove that $P \in \mathcal{P}^\phi(E, P')$.

Let $i \in N$. Then, $p_{ii} = 0$, and for all $j \in N \setminus \{i\}$,

$$p_{ij} = \varphi^i_j(e_i + \sum_{k \in N} p_{ki}, \bar{c}_i) = \varphi^i_j(e_i + \sum_{k \in N} p_{ki}, \varphi^i(e'_i + \sum_{k \in N} p'_{ki}, \bar{c}_i)),$$

in which the first equality follows from $P \in \mathcal{P}^\phi(E, C)$, and the second equality follows from composition down of φ^i, $P' \in \mathcal{P}^\phi(E', C)$, and the fact that $E' \geq E$ and $P' \geq \overline{P}^\phi(E, C) \geq P$.

(iii). In particular, if $P' = \overline{P}^\phi(E', C)$, then, by Lemma 2.3, $\overline{P}^\phi(E', C) \geq \overline{P}^\phi(E, C)$, and Lemma 4.14 (ii) implies that $\mathcal{P}^\phi(E, \overline{P}^\phi(E', C)) = \mathcal{P}^\phi(E, C)$. □

Provided that the claims rules in ϕ satisfy composition down, Lemma 4.14 (iii) states that the set of transfer schemes with respect to the top transfer scheme based on the larger estates vector and the set of transfer schemes with respect to the original claims matrix, coincide. As a consequence, the top transfer rule $\overline{\tau}^\phi$ satisfies composition down.

Theorem 4.15. *Let $\phi = (\varphi^i)_{i \in N}$ be a vector of claims rules such that, for all $i \in N$, φ^i satisfies composition down. Then, $\overline{\tau}^\phi$ satisfies composition down.*

Proof. Let $(E', C) \in \mathcal{F}^N$ and $(E, C) \in \mathcal{F}^N$ with $E' \geq E$. Lemma 4.14 (iii) implies that $\mathcal{P}^\phi(E, C) = \mathcal{P}^\phi(E, \overline{P}^\phi(E', C))$, so

$$\overline{\tau}^\phi(E, C) = \overline{P}^\phi(E, C) = \overline{P}^\phi(E, \overline{P}^\phi(E', C)) = \overline{\tau}^\phi(E, \overline{\tau}^\phi(E', C)).$$

□

Theorem 4.7 and Theorem 4.15 assert that the bottom transfer rule $\underline{\tau}^\phi$ and the top transfer rule $\overline{\tau}^\phi$ satisfy composition down if the claims rules in ϕ satisfy composition down.

We now turn our attention to composition up. Lemma 4.16 (i) states that a transfer scheme for the actual financial network with a larger estates vector can be obtained if we have a transfer scheme with respect to the smaller estates vector, and, subsequently, add to this a transfer scheme with respect to the surplus estates vector and the residual claims matrix.[11] As an illustration, Example 4.13 shows that (see (4.7) and (4.8)),

$$\{\overline{P}^{\text{CEA}}(E'', C)\} + \mathcal{P}^{\text{CEA}}(E - E'', C - \overline{P}^{\text{CEA}}(E'', C)) = \{P^1\} \subset \mathcal{P}^{\text{CEA}}(E, C).$$

[11] The notation $\{P'\} + \mathcal{P}^\phi(E - E', C - P')$ means that the matrix P' is added to each element of the set $\mathcal{P}^\phi(E - E', C - P')$. Formally, $\{P'\} + \mathcal{P}^\phi(E - E', C - P') = \{P' + P'' \mid P'' \in \mathcal{P}^\phi(E - E', C - P')\}$.

Lemma 4.16 (ii) states that the set of transfer schemes with respect to the new situation coincides with the set of payment matrices that consists of an appropriately small transfer scheme plus any transfer scheme that corresponds with the surplus estates vector and the residual claims matrix. Finally, one obtains equivalence of the sets of payment matrices if one constructs the residual claims matrix on the basis of the bottom transfer scheme, as per Lemma 4.16 (iii).

Lemma 4.16. *Let $\phi = (\varphi^i)_{i\in N}$ be a vector of claims rules such that, for all $i \in N$, φ^i satisfies composition up, and let $(E', C) \in \mathcal{L}^N$ and $(E, C) \in \mathcal{L}^N$ with $0 \leq E' \leq E$. Then,*

(i) for all $P' \in \mathcal{P}^\phi(E', C)$, $\{P'\} + \mathcal{P}^\phi(E - E', C - P') \subseteq \mathcal{P}^\phi(E, C)$;

(ii) for all $P' \in \mathcal{P}^\phi(E', C)$ such that $P' \leq \underline{P}^\phi(E, C)$, $\{P'\}+\mathcal{P}^\phi(E-E', C-P') = \mathcal{P}^\phi(E, C)$;

(iii) $\{\underline{P}^\phi(E', C)\} + \mathcal{P}^\phi(E - E', C - \underline{P}^\phi(E', C)) = \mathcal{P}^\phi(E, C)$.

Proof. (i). Let $P' \in \mathcal{P}^\phi(E', C)$ and let $P'' \in \mathcal{P}^\phi(E - E', C - P')$. We will show that $(P' + P'') \in \mathcal{P}^\phi(E, C)$.

For all $i \in N$, it holds that

$$e_i + \sum_{k\in N}(p'_{ki} + p''_{ki}) = e'_i + \sum_{k\in N} p'_{ki} + (e_i - e'_i) + \sum_{k\in N} p''_{ki}. \tag{4.10}$$

Let $i \in N$, then $p'_{ii} + p''_{ii} = 0$, and, for all $j \in N \setminus \{i\}$,

$$\begin{aligned}\varphi^i_j(e_i + \sum_{k\in N}(p'_{ki} + p''_{ki}), \bar{c}_i) &= \varphi^i_j(e'_i + \sum_{k\in N} p'_{ki}, \bar{c}_i)\\ &\quad + \varphi^i_j((e_i - e'_i) + \sum_{k\in N} p''_{ki}, \bar{c}_i - \varphi^i(e'_i + \sum_{k\in N} p'_{ki}, \bar{c}_i))\\ &= p'_{ij} + p''_{ij},\end{aligned}$$

in which the first equality follows from composition up of φ^i and (4.10), and the second equality follows from $P' \in \mathcal{P}^\phi(E', C)$ and $P'' \in \mathcal{P}^\phi(E - E', C - P')$.

(ii). Let $P' \in \mathcal{P}^\phi(E', C)$ such that $P' \leq \underline{P}^\phi(E, C)$. From Lemma 4.16 (i) it follows that

$$\{P'\} + \mathcal{P}^\phi(E - E', C - P')\} \subseteq \mathcal{P}^\phi(E, C).$$

Let $P \in \mathcal{P}^\phi(E, C)$. It suffices to prove that $(P - P') \in \mathcal{P}^\phi(E - E', C - P')$. Note that $P' \leq \underline{P}^\phi(E, C) \leq P$. Let $i \in N$. Then, $p_{ii} - p'_{ii} = 0$, and, for all $j \in N \setminus \{i\}$,

$$\begin{aligned}p_{ij} - p'_{ij} &= \varphi^i_j(e_i + \sum_{k\in N} p_{ki}, \bar{c}_i) - \varphi^i_j(e'_i + \sum_{k\in N} p'_{ki}, \bar{c}_i)\\ &= \varphi^i_j((e_i - e'_i) + \sum_{k\in N}(p_{ki} - p'_{ki}), \bar{c}_i - \varphi^i(e'_i + \sum_{k\in N} p'_{ki}, \bar{c}_i)),\end{aligned}$$

in which the first equality follows from $P \in \mathcal{P}^\phi(E, C)$ and $P' \in \mathcal{P}^\phi(E', C)$, and the second equality follows from composition up of φ^i and the fact that $E' \leq E$ and $P' \leq P$.

(iii). In particular, if $P' = \underline{P}^\phi(E', C)$, then, by Lemma 2.3, $\underline{P}^\phi(E', C) \leq \underline{P}^\phi(E, C)$, and Lemma 4.16 (ii) implies that $\{\underline{P}^\phi(E', C)\} + \mathcal{P}^\phi(E - E', C - \underline{P}^\phi(E', C)) = \mathcal{P}^\phi(E, C)$. □

Lemma 4.16 (iii) provides an equivalence relation with respect the set of transfer schemes by using the bottom transfer scheme to construct the residual claims matrix. More specifically, if the claims rules in ϕ satisfy composition up, then the payment matrix that is constructed by adding the bottom transfer scheme to any transfer scheme corresponding to the surplus estates vector and the residual claims matrix based on the bottom transfer scheme, is a transfer scheme for the actual financial network. Conversely, any transfer scheme for the actual financial network can be decomposed into the bottom transfer scheme based on the smaller estates vector and a transfer scheme corresponding to the surplus estates vector and the residual claims matrix based on the bottom transfer scheme.

This leads to the result that the bottom transfer rule $\underline{\tau}^\phi$ satisfies composition up.

Theorem 4.17. *Let $\phi = (\varphi^i)_{i \in N}$ be a vector of claims rules such that, for all $i \in N$, φ^i satisfies composition up. Then, $\underline{\tau}^\phi$ satisfies composition up.*

Proof. Let $(E', C) \in \mathcal{L}^N$ and $(E, C) \in \mathcal{L}^N$ with $0 \leq E' \leq E$. Then, by Lemma 4.16 (iii), $\mathcal{P}^\phi(E, C) = \{\underline{P}^\phi(E', C)\} + \mathcal{P}^\phi(E - E', C - \underline{P}^\phi(E', C))$, which implies that

$$\begin{aligned} \underline{\tau}^\phi(E, C) &= \underline{P}^\phi(E, C) \\ &= \underline{P}^\phi(E', C) + \underline{P}^\phi(E - E', C - \underline{P}^\phi(E', C)) \\ &= \underline{\tau}^\phi(E', C) + \underline{\tau}^\phi(E - E', C - \underline{\tau}^\phi(E', C)). \end{aligned}$$

□

Hence, if the claims rules in ϕ satisfy composition up, then not only the top transfer rule $\overline{\tau}^\phi$ satisfies composition up (see Theorem 4.8), but also the bottom transfer rule $\underline{\tau}^\phi$ satisfies composition up (see Theorem 4.17).

In other words, to establish compatibility of transfer rules with respect to composition down and composition up, it is required that the set of transfer schemes is invariant with respect to a change in the available estate. Indeed, if the claims rules in ϕ satisfy both composition down and composition up, then it follows from Theorem 4.7 and Theorem 4.17 that the bottom transfer rule $\underline{\tau}^\phi$ satisfies composition down and composition up simultaneously. The same holds true in regard to the top transfer rule $\overline{\tau}^\phi$ (see Theorem 4.8 and Theorem 4.15).

5 Concluding Remarks

In a financial network, a claims-rule-based transfer rule prescribes a transfer scheme that contains the payments made by the agents to settle their mutual liabilities. A claims-rule-based allocation rule prescribes a reallocation of the total of the estates in accordance with a transfer rule. Although a transfer scheme need not necessarily be uniquely determined, any resulting reallocation of the total of the estates is uniquely determined. This article analyzes invariance of claims-rule-based transfer rules and claims-rule-based allocation rules with respect to a change in the estate. A claims problem is a special type of financial network in which exactly one agent is in debt to the remaining agents. A claims rule provides for each claims problem a division of the estate among the group of claimants. In the financial network setting, both a claims-rule-based transfer rule and a claims-rule-based allocation rule generalize a claims rule. In the literature on claims problems, composition down of a claims rule entails invariance of downward changes in the estate, whereas composition up of a claims rule entails invariance of upward changes in the estate.

In this article, we introduce adequate extensions of composition axioms in such a way that they are compatible properties with respect to claims-rule-based transfer rules and claims-rule-based allocation rules. To achieve this, however, the extension for claims-rule-based allocation rules must differ from the extension for claims-rule-based transfer rules. We show that an allocation rule satisfies composition down and composition up if and only if each claims rule on which the allocation rule is based satisfies the two properties. Furthermore, we show that the bottom transfer rule and top transfer rule satisfy composition down and composition up if and only if each claims rule on which the rules are based satisfies both properties. The bottom transfer rule always selects the bottom transfer scheme; the top transfer rule always selects the top transfer scheme. In general, however, it may be the case a transfer rule does not satisfy composition down and composition up despite the fact that all the claims rule on which it is based satisfy both properties. The reason for this is that transfer schemes for a financial network are not always uniquely determined, so a transfer rule has to select exactly one transfer scheme (e.g., the bottom or the top transfer scheme, or one in between). This freedom of choice makes that the composition axioms are not always inherited by the corresponding transfer rule.

Ketelaars and Borm (2021) introduces several claims-rule-based decentralized clearing mechanisms that do not prescribe a transfer scheme in general. Nevertheless, if the claims rules on which a decentralized clearing mechanism is based satisfy composition up, then the payment matrix that the decentralized clearing mechanism prescribes equals the bottom transfer scheme. Consequently, any claims-rule-based decentralized clearing mechanism meeting this criterion effectively equals the bottom transfer rule. From Theorem 4.17 it follows that any such claims-rule-based decentralized clearing mechanism satisfies composition up. However, Example 4.9 demonstrates that any such claims-rule-based decentralized clearing mechanism can not satisfy strong composition up. Interestingly, if each claims rule also satisfies composition down, then, by Theorem 4.7, any corresponding decentralized clearing mechanism satisfies strong composition down.

The extensions of composition down and composition up in this article are formulated for claims-rule-based allocation rules and transfer rules. Consistent with the approach outlined in this article, it is nonetheless straightforward to formulate adequate extensions of these composition axioms for transfer rules and allocation rules that do not explicitly rely on an underlying vector of claims rules. In the same vein, one can formulate composition down and composition up for transfer rules that select payment matrices which are not always a transfer scheme, as is the case for claims-rule-based decentralized clearing mechanisms.

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