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The parton-level structure of e^+e^- to 2 jets at N^{3}LO

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Abstract: We investigate the quantum chromodynamics (QCD) corrections to hadronic final states in electron-positron collisions at $\mathcal{O}(\alpha_s^3)$ in the strong coupling constant α_s. Namely, we analytically compute the total cross section for this process by separately integrating the tree-level five-parton, the one-loop four-parton, the two-loop three-parton, and the three-loop two-parton matrix elements over the respective phase space. All the contributions to the calculation are treated in a common framework whereby phase space integrals are expressed as physical cuts of the four-loop two-point function. We check the cancellation of infrared poles at all colour levels and we reproduce the known result for the R-ratio at order α_s^3.

Keywords: Higher-Order Perturbative Calculations

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Contents

1 Introduction

Hadronic final states at electron-positron colliders constitute the cleanest environment for testing quantum chromodynamics (QCD). Historically, measurements of the R-ratio, defined as the ratio of the cross sections for hadron production and muon production in e^+e^- collisions mediated by photon exchange,

$$R = \frac{\sigma(e^+e^- \to \gamma^* \to \text{hadrons})}{\sigma(e^+e^- \to \gamma^* \to \mu^+\mu^-)} = N \sum_q e_q^2 \Big(1 + \mathcal{O}(\alpha_s)\Big) , \tag{1.1}$$

with N the number of colours and e_q the electric charge of the quark, supplied the first experimental evidence of the fractional charge of quarks and for the existence of the colour quantum number. From a theoretical point of view, this process provides the ideal playground for studying the dynamics of strong interactions, given the colourless nature of the initial state and the absence of QCD vertices at Born level.

The total cross section can be written as a perturbative expansion in terms of the QCD strong coupling constant α_s,

$$\sigma(e^+e^- \to \gamma^* \to \text{hadrons}) = \sigma^{(0)} + \left(\frac{\alpha_s}{2\pi}\right) \sigma^{(1)} + \left(\frac{\alpha_s}{2\pi}\right)^2 \sigma^{(2)} + \left(\frac{\alpha_s}{2\pi}\right)^3 \sigma^{(3)} + \mathcal{O}(\alpha_s^4) , \tag{1.2}$$

with $\sigma^{(0)}$ the Born-level or leading-order (LO) cross section, and $\sigma^{(k)}$ denoting the N^{k}LO correction to the cross section. The N^{3}LO [1, 2] and the N^{4}LO corrections [3–7] are known.

Generally speaking, evaluating the terms in (1.2) beyond Born level requires the calculation of Feynman diagrams with additional real and virtual particles. The emission of real particles causes singular behaviour of the matrix element in the collinear and/or soft limit, while the exchange of virtual particles leads to a divergent loop integration. The real and virtual divergences can, however, be consistently treated separately after the introduction of a regulator. The most common choice is dimensional regularisation where phase space integrals as well as loop integrals are performed in $d = 4 - 2\epsilon$ dimensions. This leads to the presence of explicit $1/\epsilon^n$ poles in the intermediate steps of the calculation. Nonetheless, the KLN theorem [8, 9] guarantees cancellation of infrared (IR) poles for sufficiently inclusive observables — such as the total cross section — and thus one can eventually take the limit $\epsilon \to 0$.

In [10] the infrared structure of the process $e^+e^- \to 2$ jets was scrutinized up to NNLO. Namely, the tree-level four-parton, the one-loop three-parton and the two-loop two-parton matrix elements were integrated over the respective phase space and summed to obtain the NNLO contribution to the cross section,

$$\sigma^{(2)} = \int \mathrm{d}\Phi_4\, M_4^0 + \int \mathrm{d}\Phi_3\, M_3^1 + \int \mathrm{d}\Phi_2\, M_2^2\,, \tag{1.3}$$

where M_n^l denotes the l-loop squared matrix element for the decay of a virtual photon into n final state QCD particles. The phase space $\mathrm{d}\Phi_n$ is defined as

$$\mathrm{d}\Phi_n = \frac{\mathrm{d}^{d-1}p_1}{2E_1(2\pi)^{d-1}} \cdots \frac{\mathrm{d}^{d-1}p_n}{2E_n(2\pi)^{d-1}}\,(2\pi)^d\,\delta^d(q - p_1 - \ldots - p_n)\,. \tag{1.4}$$

We refer to each of the terms in (1.3) as a *layer* of the calculation. In an N^{k}LO calculation, each layer can feature infrared poles up to $1/\epsilon^{2k}$, i.e. up to $1/\epsilon^4$ at NNLO and up to $1/\epsilon^6$ at N^{3}LO, etc. The authors of [10] identified IR-singular terms in the three- and four-parton final states with the IR singularities of the two-parton final state. They further observed the cancellation of the contribution from the one-loop soft gluon current between three- and four-parton final states.

The purpose of the present work is to extend the analysis of [10] to the layers of the N^{3}LO cross section. We analogously integrate the tree-level five-parton, the one-loop four-parton, the two-loop three-parton and the three-loop two-parton matrix elements over their respective phase space:

$$\sigma^{(3)} = \int \mathrm{d}\Phi_5\, M_5^0 + \int \mathrm{d}\Phi_4\, M_4^1 + \int \mathrm{d}\Phi_3\, M_3^2 + \int \mathrm{d}\Phi_2\, M_2^3\,. \tag{1.5}$$

This work forms part of the joint effort of the community towards N^{3}LO calculations [11–24]. Several steps have been taken towards a complete description of the universal behaviour of matrix elements in unresolved configurations in N^{3}LO calculations. These scenarios include single-unresolved limits of two-loop amplitudes, double-unresolved limits of one-loop amplitudes, and triple-unresolved limits of tree-level amplitudes. Some unresolved limits can be described with the iteration of single and double unresolved structures, while others require novel computations, e.g. single collinear limits of two-loop amplitudes [25, 26]; two-loop current for the emission of a soft gluon [27–29]; triple collinear limits of one-loop amplitudes [30, 31]; double soft emission at one loop [32–34]; quadruple collinear splitting functions [35, 36]; and triple soft emission in tree-level amplitudes [37–39]. At the integrated level, the interplay of the different unresolved limits is not yet understood. The cancellation pattern between the components of (1.5) could shed more light on the way the implicit infrared divergent behaviour translates to the integrated level.

We anticipate the importance of the presented results for the development of a local subtraction scheme at N^{3}LO. The matrix elements of $\gamma^* \to q\bar{q}$ and their integrated versions have been used in the context of the antenna subtraction scheme for NNLO calculations [40]. The matrix elements themselves (with a proper normalisation) can serve as subtraction terms at the real level to remove implicit singularities due to unresolved radiation between a quark-antiquark pair of hard emitters. On the other hand, the integrals of the matrix elements over the phase space have been used to remove the explicit singularities present in virtual matrix elements. The results of the present work can therefore provide necessary ingredients for a future formulation of the antenna subtraction scheme at N^{3}LO.

The remainder of this paper is organised as follows. In section 2, we briefly describe our methods. In section 3, we present results for the integrated matrix elements in order of final-state multiplicity. In section 4, we combine the expressions to recover the known R-ratio at $\mathcal{O}(\alpha_s^3)$ and we comment on the results. We conclude in section 5 and elaborate on possible future directions.

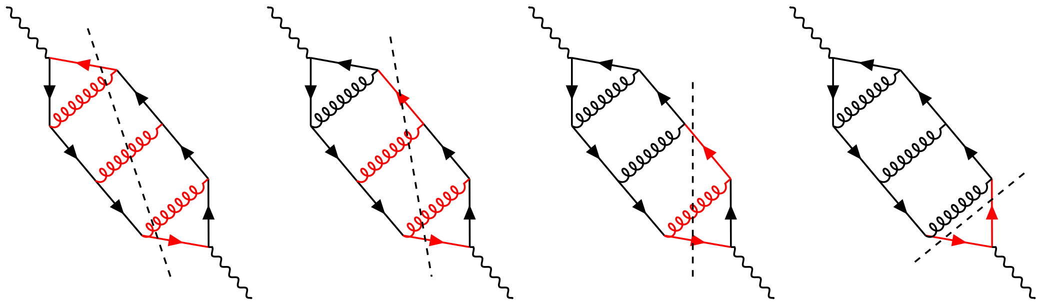
2 Method

Phase-space integrals are related to loop integrals by the reverse unitarity relation [41, 42] which reformulates the mass-shell condition for p, a final-state massless momentum, as an inverse propagator on cut (*cut-propagator*),

$$2\pi i\delta^+(p^2) = \frac{1}{p^2 - i0} - \frac{1}{p^2 + i0}\,. \tag{2.1}$$

This way, all the integrals present in the matrix elements in (1.5) can be expressed as cuts of the four-loop photon self-energy and we can leverage techniques developed for the computation of loop integrals. According to the number of propagators on cut, a given self-energy diagram contributes to a layer in (1.5). Namely, n-particle cuts represent the integration over the inclusive phase space of a matrix element for the emission of n real particles with $(5-n)$ loops. The relevant cuts of a particular diagram are depicted in figure 1.

At first, the four-loop diagrams with two external legs are generated with QGRAF [43] using a model which includes the Standard Model QCD particles and couplings, as well as a set of fields for cut-propagators, which are also allowed to couple to regular particles. However, most naive arrangements of the cut-fields within a four-loop diagram are unphysical. In fact, a diagram only contributes to the physical integrated cross section if the cuts



(a) 5 cut-propagators (b) 4 cut-propagators (c) 3 cut-propagators (d) 2 cut-propagators

Figure 1. Example of a four-loop massless photon self-energy featuring three internal gluons with different possible choices of cut propagators (in red). According to the number of propagators on cut, the same diagram can represent the integral over the phase space of: (a) a tree-level squared amplitude $\gamma \to qggg\bar{q}$, (b) the interference between a one-loop amplitude $\gamma \to qgg\bar{q}$ with its tree-level, (c) the interference between a two-loop amplitude $\gamma \to qg\bar{q}$ with its tree-level, (d) the interference between a three-loop amplitude $\gamma \to q\bar{q}$ with its tree-level.

F1		F2		F3	
k_1	$k_2 - k_3$	k_1	$k_2 - k_3$	k_1	$k_4 - k_2$
k_2	$k_2 - k_4$	k_2	$k_2 - k_3$	k_2	$k_3 - k_4$
k_3	$k_3 - k_4$	k_3	$k_3 - k_4$	k_4	$q - k_1$
k_4	$q - k_1$	k_4	$q - k_1$	$k_1 - k_2$	$q - k_2$
$k_1 - k_2$	$q - k_2$	$k_1 - k_2$	$q - k_2$	$k_1 - k_3$	$q - k_3$
$k_1 - k_3$	$q - k_3$	$k_1 - k_3$	$q - k_4$	$k_1 - k_4$	$q + k_2 - k_3$
$k_1 - k_4$	$q - k_4$	$k_1 - k_4$	$q + k_3 - k_4$	$k_2 - k_3$	$q + k_4 - k_3 - k_1$

Table 1. The three auxiliary topologies which label every diagram present in the calculation. The knowledge of cut propagators is re-inserted only after the matching of diagrams onto the topologies F1, F2 and F3. One can subsequently infer which combinations of cuts are required in each topology to accommodate every cut diagram.

divide it into exactly two connected graphs, each attached to an external photon current. Moreover, each contribution in (1.5) requires a specific number of cuts and loops on either side of the cut. Finally, only cuts which fulfill momentum conservation and do not contain a self-energy insertion on cut-propagators are retained. In this way, we had to evaluate 1592, 3114, 2556 and 1764 diagrams for the integrations of the matrix elements M_2^3, M_3^2, M_4^1 and M_5^0, respectively.

We subsequently insert the Feynman rules into the selected diagrams, compute the colour algebra, and evaluate the traces of gamma matrices. After imposing the on-shellness condition for cut propagators and representing each integral as a combination of scalar integrals, we can use Reduze2 [44] to sort the integrals into the auxiliary topologies reported in table 1.

Keeping track of which propagators are cut in any diagram, we can define for every auxiliary topology (F1, F2, F3) a set of *cut-families* which cover all the integrals appearing

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in the matrix elements. Since we re-introduced the cuts only after matching onto the topologies, we are left with redundancy in the definition of the cut-families, which is taken care of internally by REDUZE2's sector relation finder before the reduction.

The integrals appearing in the matrix elements have up to eleven propagators in the denominator and a maximum of four scalar products in the numerator, in line with the calculation of the three-loop quark form factor in [45]. For each layer, we reduce all integrals to a set of master integrals, finding 22, 27, 35 and 31 master integrals for the two-, three-, four-, and five-particle final state respectively. These master integral are in a one-to-one correspondence with those analytically computed in [46, 47]. Namely, up to trivial relabeling of loop momenta, we can directly use their results.

FORM [48, 49] together with MATHEMATICA and PYTHON scripts were used extensively throughout the calculation.

3 Results

We illustrate the general structure of the different partonic contributions and we provide explicit expressions only for the new $\mathcal{O}(\alpha_s^3)$ results. The notation of [10] is adopted throughout this section for ease of reference.

We present results for the integration of renormalised squared amplitudes. The renormalisation of ultraviolet divergences is performed in the $\overline{\text{MS}}$ scheme by means of the replacement

$$\alpha_0\, \mu_0^{2\epsilon}\, S_\epsilon = \alpha_s\, \mu^{2\epsilon} \left[1 - \frac{\beta_0}{\epsilon}\left(\frac{\alpha_s}{2\pi}\right) + \left(\frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{2\epsilon}\right)\left(\frac{\alpha_s}{2\pi}\right)^2 + \mathcal{O}(\alpha_s^3)\right], \tag{3.1}$$

with α_0 the bare coupling and μ_0^2 the mass parameter introduced in dimensional regularisation to maintain a dimensionless coupling in the bare QCD Lagrangian density. Additionally, α_s is the renormalised coupling evaluated at the renormalisation scale μ^2 and

$$\beta_0 = \frac{11C_A - 2N_F}{6}, \tag{3.2}$$

$$\beta_1 = \frac{17C_A^2 - 5C_AN_F - 3C_FN_F}{6}, \tag{3.3}$$

$$S_\epsilon = (4\pi)^\epsilon e^{-\epsilon\gamma}, \qquad \text{with Euler constant } \gamma = 0.5772\ldots \tag{3.4}$$

Henceforth, we fix the renormalisation scale to be $\mu^2 = q^2$, i.e. $\alpha_s \equiv \alpha_s(q^2)$. Explicit relations between unrenormalised and renormalised amplitudes are provided in appendix A.

3.1 Two-parton final state

The two-parton contribution to (1.5) is given by the QCD loop corrections to the $\gamma^* \to q\bar{q}$ process:

$$\begin{aligned}|\mathcal{M}\rangle_{q\bar{q}} = \sqrt{4\pi\alpha}e_q \Bigg[&|\mathcal{M}^{(0)}\rangle_{q\bar{q}} + \left(\frac{\alpha_s}{2\pi}\right)|\mathcal{M}^{(1)}\rangle_{q\bar{q}} + \left(\frac{\alpha_s}{2\pi}\right)^2|\mathcal{M}^{(2)}\rangle_{q\bar{q}} \\ &+ \left(\frac{\alpha_s}{2\pi}\right)^3|\mathcal{M}^{(3)}\rangle_{q\bar{q}} + \mathcal{O}(\alpha_s^4)\Bigg],\end{aligned} \tag{3.5}$$

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with α the electromagnetic coupling constant, e_q the quark charge, and $|\mathcal{M}^{(i)}\rangle$ the i-loop contributions to the renormalised amplitude. The squared amplitude, summed over spins, colours and quark flavours is given by:

$$\begin{aligned}\langle\mathcal{M}|\mathcal{M}\rangle_{q\bar{q}} &= \sum|\mathcal{M}(\gamma^*\to q\bar{q})|^2 \\ &= 4\pi\alpha\sum_q e_q^2\Bigg[\langle\mathcal{M}^{(0)}|\mathcal{M}^{(0)}\rangle_{q\bar{q}} + \left(\frac{\alpha_s}{2\pi}\right)2\,\mathrm{Re}\big[\langle\mathcal{M}^{(1)}|\mathcal{M}^{(0)}\rangle_{q\bar{q}}\big] \\ &\quad + \left(\frac{\alpha_s}{2\pi}\right)^2\Big(2\,\mathrm{Re}\big[\langle\mathcal{M}^{(2)}|\mathcal{M}^{(0)}\rangle_{q\bar{q}}\big] + \langle\mathcal{M}^{(1)}|\mathcal{M}^{(1)}\rangle_{q\bar{q}}\Big) \\ &\quad + \left(\frac{\alpha_s}{2\pi}\right)^3\Big(2\,\mathrm{Re}\big[\langle\mathcal{M}^{(3)}|\mathcal{M}^{(0)}\rangle_{q\bar{q}}\big] + 2\,\mathrm{Re}\big[\langle\mathcal{M}^{(2)}|\mathcal{M}^{(1)}\rangle_{q\bar{q}}\big]\Big) + \mathcal{O}(\alpha_s^4)\Bigg]. \end{aligned} \tag{3.6}$$

We define

$$\mathcal{T}_{q\bar{q}}^{(2)} = \int \mathrm{d}\Phi_2\,\langle\mathcal{M}^{(0)}|\mathcal{M}^{(0)}\rangle_{q\bar{q}}, \tag{3.7}$$

$$\mathcal{T}_{q\bar{q}}^{(4)} = \int \mathrm{d}\Phi_2\,2\,\mathrm{Re}\big[\langle\mathcal{M}^{(1)}|\mathcal{M}^{(0)}\rangle_{q\bar{q}}\big], \tag{3.8}$$

$$\mathcal{T}_{q\bar{q}}^{(6)} = \mathcal{T}_{q\bar{q}}^{(6,[2\times 0])} + \mathcal{T}_{q\bar{q}}^{(6,[1\times 1])}, \tag{3.9}$$

$$\mathcal{T}_{q\bar{q}}^{(6,[2\times 0])} = \int \mathrm{d}\Phi_2\,2\,\mathrm{Re}\big[\langle\mathcal{M}^{(2)}|\mathcal{M}^{(0)}\rangle_{q\bar{q}}\big], \tag{3.10}$$

$$\mathcal{T}_{q\bar{q}}^{(6,[1\times 1])} = \int \mathrm{d}\Phi_2\,\langle\mathcal{M}^{(1)}|\mathcal{M}^{(1)}\rangle_{q\bar{q}}, \tag{3.11}$$

$$\mathcal{T}_{q\bar{q}}^{(8)} = \mathcal{T}_{q\bar{q}}^{(8,[3\times 0])} + \mathcal{T}_{q\bar{q}}^{(8,[2\times 1])}, \tag{3.12}$$

$$\mathcal{T}_{q\bar{q}}^{(8,[3\times 0])} = \int \mathrm{d}\Phi_2\,2\,\mathrm{Re}\big[\langle\mathcal{M}^{(3)}|\mathcal{M}^{(0)}\rangle_{q\bar{q}}\big], \tag{3.13}$$

$$\mathcal{T}_{q\bar{q}}^{(8,[2\times 1])} = \int \mathrm{d}\Phi_2\,2\,\mathrm{Re}\big[\langle\mathcal{M}^{(2)}|\mathcal{M}^{(1)}\rangle_{q\bar{q}}\big]. \tag{3.14}$$

The lowest order contribution is given by

$$\mathcal{T}_{q\bar{q}}^{(2)} = 4N(1-\epsilon)q^2 P_2\,, \tag{3.15}$$

with P_2 the volume of the two-particle phase space,

$$P_2 = \int \mathrm{d}\Phi_2 = 2^{-3+2\epsilon}\,\pi^{-1+\epsilon}\,\frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}\,(q^2)^{-\epsilon}\,. \tag{3.16}$$

Note that in contrast with [10], our definition of $\mathcal{T}_{q\bar{q}}^{(2)}$ includes a factor of P_2 because we always integrate over the full phase space. The one- and two-loop contributions $\mathcal{T}_{q\bar{q}}^{(4)}$ and $\mathcal{T}_{q\bar{q}}^{(6)}$ are given in (4.6) and (4.7) of [10] and in appendix B. The colour decompositions of

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$\mathcal{T}_{q\bar{q}}^{(8,[3\times 0])}$ and $\mathcal{T}_{q\bar{q}}^{(8,[2\times 1])}$ reads

$$\begin{aligned}\mathcal{T}_{q\bar{q}}^{(8,[3\times 0])} = \left(N-\frac{1}{N}\right)\mathcal{T}_{q\bar{q}}^{(2)}\Bigg[&N^2\mathcal{T}_{q\bar{q}}^{(8,[3\times 0])}\Big|_{N^2}+\mathcal{T}_{q\bar{q}}^{(8,[3\times 0])}\Big|_{N^0}+\frac{1}{N^2}\mathcal{T}_{q\bar{q}}^{(8,[3\times 0])}\Big|_{1/N^2}\\&+N_F N\,\mathcal{T}_{q\bar{q}}^{(8,[3\times 0])}\Big|_{N_F N}+\frac{N_F}{N}\mathcal{T}_{q\bar{q}}^{(8,[3\times 0])}\Big|_{N_F/N}+N_F^2\,\mathcal{T}_{q\bar{q}}^{(8,[3\times 0])}\Big|_{N_F^2}\\&+N_{F,\gamma}\left(N-\frac{4}{N}\right)\mathcal{T}_{q\bar{q}}^{(8,[3\times 0])}\Big|_{N_{F,\gamma}}\Bigg],\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}\mathcal{T}_{q\bar{q}}^{(8,[2\times 1])} = \left(N-\frac{1}{N}\right)\mathcal{T}_{q\bar{q}}^{(2)}\Bigg[&N^2\mathcal{T}_{q\bar{q}}^{(8,[2\times 1])}\Big|_{N^2}+\mathcal{T}_{q\bar{q}}^{(8,[2\times 1])}\Big|_{N^0}+\frac{1}{N^2}\mathcal{T}_{q\bar{q}}^{(8,[2\times 1])}\Big|_{1/N^2}\\&+N_F N\,\mathcal{T}_{q\bar{q}}^{(8,[2\times 1])}\Big|_{N_F N}+\frac{N_F}{N}\mathcal{T}_{q\bar{q}}^{(8,[2\times 1])}\Big|_{N_F/N}\Bigg],\end{aligned} \tag{3.18}$$

where

$$N_{F,\gamma} = \frac{(\sum_q e_q)^2}{\sum_q e_q^2}\,. \tag{3.19}$$

The last term in (3.17) features a singlet contribution proportional to $(\sum_q e_q)^2$ which always comes with the expected quartic Casimir $d^{abc}d^{abc} = N - 4/N$. The terms in (3.17) up to finite order in ϵ are given by:

$$\begin{aligned}\mathcal{T}_{q\bar{q}}^{(8,[3\times 0])}\Big|_{N^2} =& +\frac{1}{\epsilon^6}\left(-\frac{1}{24}\right)+\frac{1}{\epsilon^5}\left(-\frac{7}{8}\right)+\frac{1}{\epsilon^4}\left(-\frac{5299}{1296}+\frac{17}{96}\pi^2\right)\\&+\frac{1}{\epsilon^3}\left(-\frac{751}{243}+\frac{2185}{1296}\pi^2-\frac{7}{12}\zeta_3\right)\\&+\frac{1}{\epsilon^2}\left(\frac{3371}{15552}+\frac{45551}{15552}\pi^2-\frac{1387}{432}\zeta_3-\frac{2299}{20736}\pi^4\right)\\&+\frac{1}{\epsilon}\left(\frac{6297767}{279936}-\frac{79421}{46656}\pi^2-\frac{523}{108}\zeta_3-\frac{72347}{103680}\pi^4+\frac{151}{54}\pi^2\zeta_3-\frac{631}{60}\zeta_5\right)\\&+\frac{80373631}{1679616}-\frac{12363151}{559872}\pi^2+\frac{5521}{972}\zeta_3-\frac{506143}{622080}\pi^4+\frac{5477}{576}\pi^2\zeta_3\\&-\frac{2191}{80}\zeta_5+\frac{208037}{3732480}\pi^6-\frac{163}{144}\zeta_3^2+\mathcal{O}(\epsilon),\end{aligned} \tag{3.20}$$

$$\begin{aligned}\mathcal{T}_{q\bar{q}}^{(8,[3\times 0])}\Big|_{N^0} =& +\frac{1}{\epsilon^6}\left(\frac{1}{12}\right)+\frac{1}{\epsilon^5}\left(\frac{17}{16}\right)+\frac{1}{\epsilon^4}\left(\frac{811}{288}-\frac{3}{8}\pi^2\right)+\frac{1}{\epsilon^3}\left(\frac{6185}{864}-\frac{499}{192}\pi^2-\frac{11}{24}\zeta_3\right)\\&+\frac{1}{\epsilon^2}\left(\frac{28075}{2592}-\frac{20135}{3456}\pi^2-\frac{95}{18}\zeta_3+\frac{1573}{5760}\pi^4\right)\\&+\frac{1}{\epsilon}\left(-\frac{13385}{15552}-\frac{97123}{10368}\pi^2-\frac{29}{32}\zeta_3+\frac{240217}{207360}\pi^4+\frac{593}{288}\pi^2\zeta_3+\frac{33}{40}\zeta_5\right)\end{aligned}$$

$$
\begin{aligned}
&-\frac{9295397}{93312}+\frac{66571}{7776}\pi^2+\frac{3919}{96}\zeta_3+\frac{2757217}{1244160}\pi^4+\frac{47}{32}\pi^2\zeta_3 \\
&-\frac{923}{180}\zeta_5-\frac{35857}{1088640}\pi^6+\frac{35}{8}\zeta_3^2+\mathcal{O}(\epsilon),
\end{aligned}
\tag{3.21}
$$

$$
\begin{aligned}
\mathcal{T}_{q\bar{q}}^{(8,[3\times0])}\Big|_{1/N^2}=&+\frac{1}{\epsilon^6}\left(-\frac{1}{24}\right)+\frac{1}{\epsilon^5}\left(-\frac{3}{16}\right)+\frac{1}{\epsilon^4}\left(-\frac{25}{32}+\frac{19}{96}\pi^2\right) \\
&+\frac{1}{\epsilon^3}\left(-\frac{83}{32}+\frac{53}{64}\pi^2+\frac{25}{24}\zeta_3\right) \\
&+\frac{1}{\epsilon^2}\left(-\frac{515}{64}+\frac{1273}{384}\pi^2+\frac{69}{16}\zeta_3-\frac{649}{3840}\pi^4\right) \\
&+\frac{1}{\epsilon}\left(-\frac{9073}{384}+\frac{4015}{384}\pi^2+\frac{2119}{96}\zeta_3-\frac{3833}{7680}\pi^4-\frac{1457}{288}\pi^2\zeta_3+\frac{161}{40}\zeta_5\right) \\
&-\frac{53675}{768}+\frac{70429}{2304}\pi^2+\frac{2669}{32}\zeta_3-\frac{67177}{46080}\pi^4-\frac{3665}{192}\pi^2\zeta_3 \\
&+\frac{2119}{80}\zeta_5-\frac{19301}{1741824}\pi^6-\frac{913}{48}\zeta_3^2+\mathcal{O}(\epsilon),
\end{aligned}
\tag{3.22}
$$

$$
\begin{aligned}
\mathcal{T}_{q\bar{q}}^{(8,[3\times0])}\Big|_{N_F N}=&+\frac{1}{\epsilon^5}\left(\frac{1}{8}\right)+\frac{1}{\epsilon^4}\left(\frac{1283}{1296}\right)+\frac{1}{\epsilon^3}\left(-\frac{253}{3888}-\frac{419}{2592}\pi^2\right) \\
&+\frac{1}{\epsilon^2}\left(-\frac{6461}{3888}-\frac{671}{15552}\pi^2+\frac{47}{216}\zeta_3\right) \\
&+\frac{1}{\epsilon}\left(-\frac{240167}{34992}+\frac{99107}{46656}\pi^2-\frac{521}{1296}\zeta_3+\frac{3941}{103680}\pi^4\right) \\
&+\frac{573737}{209952}+\frac{387179}{69984}\pi^2-\frac{4549}{1296}\zeta_3-\frac{14527}{69120}\pi^4 \\
&+\frac{41}{288}\pi^2\zeta_3-\frac{205}{72}\zeta_5+\mathcal{O}(\epsilon),
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
\mathcal{T}_{q\bar{q}}^{(8,[3\times0])}\Big|_{N_F/N}=&+\frac{1}{\epsilon^5}\left(-\frac{1}{8}\right)+\frac{1}{\epsilon^4}\left(-\frac{35}{144}\right)+\frac{1}{\epsilon^3}\left(-\frac{139}{432}+\frac{17}{96}\pi^2\right) \\
&+\frac{1}{\epsilon^2}\left(\frac{775}{1296}-\frac{137}{1728}\pi^2+\frac{55}{72}\zeta_3\right) \\
&+\frac{1}{\epsilon}\left(\frac{24761}{3888}-\frac{9859}{5184}\pi^2-\frac{469}{432}\zeta_3-\frac{577}{20736}\pi^4\right) \\
&+\frac{691883}{23328}-\frac{87853}{7776}\pi^2-\frac{21179}{1296}\zeta_3+\frac{125143}{622080}\pi^4 \\
&+\frac{85}{96}\pi^2\zeta_3+\frac{193}{72}\zeta_5+\mathcal{O}(\epsilon),
\end{aligned}
\tag{3.24}
$$

$$\mathcal{T}_{q\bar{q}}^{(8,[3\times0])}\Big|_{N_F^2} = +\frac{1}{\epsilon^4}\left(-\frac{11}{162}\right)+\frac{1}{\epsilon^3}\left(-\frac{1}{243}\right)+\frac{1}{\epsilon^2}\left(\frac{23}{324}+\frac{\pi^2}{108}\right)$$
$$+\frac{1}{\epsilon}\left(\frac{2417}{17496}-\frac{5}{324}\pi^2-\frac{\zeta_3}{81}\right)$$
$$-\frac{190931}{104976}+\frac{403}{972}\pi^2-\frac{52}{243}\zeta_3+\frac{43}{9720}\pi^4+\mathcal{O}(\epsilon), \tag{3.25}$$

$$\mathcal{T}_{q\bar{q}}^{(8,[3\times0])}\Big|_{N_{F,\gamma}} = +\frac{1}{2}+\frac{5}{24}\pi^2+\frac{7}{12}\zeta_3-\frac{\pi^4}{720}-\frac{10\zeta_5}{3}+\mathcal{O}(\epsilon), \tag{3.26}$$

The terms in (3.17) up to finite order in ϵ are given by:

$$\mathcal{T}_{q\bar{q}}^{(8,[2\times1])}\Big|_{N^2} = +\frac{1}{\epsilon^6}\left(-\frac{1}{8}\right)+\frac{1}{\epsilon^5}\left(-\frac{5}{4}\right)+\frac{1}{\epsilon^4}\left(-\frac{259}{72}+\frac{7}{96}\pi^2\right)+\frac{1}{\epsilon^3}\left(-\frac{8507}{864}+\frac{37}{96}\pi^2\right)$$
$$+\frac{1}{\epsilon^2}\left(-\frac{99269}{5184}+\frac{517}{432}\pi^2+\frac{253}{144}\zeta_3-\frac{11}{768}\pi^4\right)$$
$$+\frac{1}{\epsilon}\left(-\frac{655493}{31104}+\frac{30709}{10368}\pi^2-\frac{527}{144}\zeta_3-\frac{55}{1152}\pi^4+\frac{11}{48}\pi^2\zeta_3-\frac{33}{10}\zeta_5\right)$$
$$+\frac{8695267}{186624}+\frac{303889}{62208}\pi^2-\frac{33707}{864}\zeta_3-\frac{11953}{34560}\pi^4+\frac{2365}{1728}\pi^2\zeta_3$$
$$-\frac{3839}{240}\zeta_5-\frac{389}{193536}\pi^6+\frac{95}{16}\zeta_3^2+\mathcal{O}(\epsilon), \tag{3.27}$$

$$\mathcal{T}_{q\bar{q}}^{(8,[2\times1])}\Big|_{N^0} = +\frac{1}{\epsilon^6}\left(\frac{1}{4}\right)+\frac{1}{\epsilon^5}\left(\frac{29}{16}\right)+\frac{1}{\epsilon^4}\left(\frac{1711}{288}-\frac{\pi^2}{6}\right)+\frac{1}{\epsilon^3}\left(\frac{15311}{864}-\frac{143}{192}\pi^2-\frac{13}{8}\zeta_3\right)$$
$$+\frac{1}{\epsilon^2}\left(\frac{28153}{648}-\frac{8843}{3456}\pi^2-\frac{313}{36}\zeta_3+\frac{59}{5760}\pi^4\right)$$
$$+\frac{1}{\epsilon}\left(\frac{708011}{7776}-\frac{72937}{10368}\pi^2-\frac{8657}{288}\zeta_3+\frac{29}{23040}\pi^4+\frac{37}{32}\pi^2\zeta_3+\frac{9}{40}\zeta_5\right)$$
$$+\frac{6795617}{46656}-\frac{122945}{7776}\pi^2-\frac{89467}{864}\zeta_3-\frac{1787}{138240}\pi^4+\frac{3367}{864}\pi^2\zeta_3$$
$$+\frac{197}{60}\zeta_5+\frac{6647}{241920}\pi^6+\frac{125}{8}\zeta_3^2+\mathcal{O}(\epsilon), \tag{3.28}$$

$$\mathcal{T}_{q\bar{q}}^{(8,[2\times1])}\Big|_{1/N^2} = +\frac{1}{\epsilon^6}\left(-\frac{1}{8}\right)+\frac{1}{\epsilon^5}\left(-\frac{9}{16}\right)+\frac{1}{\epsilon^4}\left(-\frac{75}{32}+\frac{3}{32}\pi^2\right)$$
$$+\frac{1}{\epsilon^3}\left(-\frac{63}{8}+\frac{23}{64}\pi^2+\frac{13}{8}\zeta_3\right)$$
$$+\frac{1}{\epsilon^2}\left(-\frac{1555}{64}+\frac{523}{384}\pi^2+\frac{111}{16}\zeta_3+\frac{47}{11520}\pi^4\right)$$
$$+\frac{1}{\epsilon}\left(-\frac{8957}{128}+\frac{391}{96}\pi^2+\frac{1079}{32}\zeta_3+\frac{119}{2560}\pi^4-\frac{133}{96}\pi^2\zeta_3+\frac{123}{40}\zeta_5\right)$$

$$
\begin{aligned}
&-\frac{49215}{256}+\frac{2797}{256}\pi^2+\frac{2281}{16}\zeta_3+\frac{1837}{5120}\pi^4-\frac{337}{64}\pi^2\zeta_3\\
&+\frac{1017}{80}\zeta_5-\frac{24643}{967680}\pi^6-\frac{345}{16}\zeta_3^2+\mathcal{O}(\epsilon),
\end{aligned}
\tag{3.29}
$$

$$
\begin{aligned}
\mathcal{T}_{q\bar q}^{(8,[2\times1])}\Big|_{N_F N} =& +\frac{1}{\epsilon^5}\left(\frac{1}{8}\right)+\frac{1}{\epsilon^4}\left(\frac{35}{144}\right)+\frac{1}{\epsilon^3}\left(\frac{187}{432}-\frac{\pi^2}{96}\right)\\
&+\frac{1}{\epsilon^2}\left(-\frac{751}{1296}+\frac{41}{1728}\pi^2-\frac{23}{72}\zeta_3\right)\\
&+\frac{1}{\epsilon}\left(-\frac{58381}{7776}+\frac{1279}{5184}\pi^2+\frac{71}{144}\zeta_3-\frac{7}{2304}\pi^4\right)\\
&-\frac{1801117}{46656}+\frac{21899}{15552}\pi^2+\frac{2617}{432}\zeta_3\\
&+\frac{833}{69120}\pi^4-\frac{53}{864}\pi^2\zeta_3+\frac{5}{24}\zeta_5+\mathcal{O}(\epsilon),
\end{aligned}
\tag{3.30}
$$

$$
\begin{aligned}
\mathcal{T}_{q\bar q}^{(8,[2\times1])}\Big|_{N_F/N} =& +\frac{1}{\epsilon^5}\left(-\frac{1}{8}\right)+\frac{1}{\epsilon^4}\left(-\frac{35}{144}\right)+\frac{1}{\epsilon^3}\left(-\frac{187}{432}+\frac{\pi^2}{96}\right)\\
&+\frac{1}{\epsilon^2}\left(\frac{751}{1296}-\frac{41}{1728}\pi^2+\frac{23}{72}\zeta_3\right)\\
&+\frac{1}{\epsilon}\left(\frac{58381}{7776}-\frac{1279}{5184}\pi^2-\frac{71}{144}\zeta_3+\frac{7}{2304}\pi^4\right)\\
&+\frac{1801117}{46656}-\frac{21899}{15552}\pi^2-\frac{2617}{432}\zeta_3\\
&-\frac{833}{69120}\pi^4+\frac{53}{864}\pi^2\zeta_3-\frac{5}{24}\zeta_5+\mathcal{O}(\epsilon),
\end{aligned}
\tag{3.31}
$$

The presented expressions are related to the quark form factors up to three loops [45, 50, 51]. In particular

$$
\begin{aligned}
\mathcal{T}_{q\bar q}^{(8,[3\times0])} =& \frac{1}{8}\mathcal{T}_{q\bar q}^{(2)}\left\{\mathcal{F}_3^q\,\mathrm{Re}\left[\Delta(q^2)^3\right]-\frac{4\beta_0}{\epsilon}\mathcal{F}_2^q\,\mathrm{Re}\left[\Delta(q^2)^2\right]\right.\\
&\left.-\left(\frac{2\beta_1}{\epsilon}-\frac{4\beta_0^2}{\epsilon^2}\right)\mathcal{F}_1^q\,\mathrm{Re}\left[\Delta(q^2)\right]\right\}
\end{aligned}
\tag{3.32}
$$

and

$$
\mathcal{T}_{q\bar q}^{(8,[2\times1])} = \frac{1}{8}\mathcal{T}_{q\bar q}^{(2)}\left\{\mathcal{F}_1^q\mathcal{F}_2^q\,\mathrm{Re}\left[\Delta(q^2)\right]-\frac{2\beta_0}{\epsilon}(\mathcal{F}_1^q)^2\right\},
\tag{3.33}
$$

where $\mathcal{F}_1^q$, $\mathcal{F}_2^q$ and $\mathcal{F}_3^q$ are given by (2.23), (2.27) and (5.4) of [45]. The factor

$$
\Delta(q^2) = (-1-i0)^{-\epsilon}
\tag{3.34}
$$

appears in time-like kinematics. The factor $1/8$ in (3.32) and (3.33) compensates for the fact that $\mathcal{F}_1^q$, $\mathcal{F}_2^q$ and $\mathcal{F}_3^q$ refer to an expansion in $\alpha_s/(4\pi)$. Equations (3.32) and (3.33) are indeed satisfied after the insertion of the expressions from [45], which is a strong check of our results.

3.2 Three-parton final states

The three-parton configuration receives contributions from the two processes $\gamma^* \to q\bar{q}g$ and $\gamma^* \to ggg$. The three-gluon final state only arises at order $\mathcal{O}(\alpha_s^3)$ as the interference of two one-loop diagrams where three gluons are emitted from a fermionic loop attached to the photon. The amplitudes are

$$|\mathcal{M}\rangle_{q\bar{q}g} = \sqrt{4\pi\alpha}e_q \sqrt{4\pi\alpha_s}\Bigg[|\mathcal{M}^{(0)}\rangle_{q\bar{q}g} + \left(\frac{\alpha_s}{2\pi}\right)|\mathcal{M}^{(1)}\rangle_{q\bar{q}g} + \left(\frac{\alpha_s}{2\pi}\right)^2 |\mathcal{M}^{(2)}\rangle_{q\bar{q}g} + \mathcal{O}(\alpha_s^3)\Bigg] \tag{3.35}$$

and

$$|\mathcal{M}\rangle_{ggg} = \sqrt{4\pi\alpha}e_q\,(4\pi\alpha_s)^{3/2}\left[|\mathcal{M}^{(1)}\rangle_{ggg} + \mathcal{O}(\alpha_s)\right]. \tag{3.36}$$

The squared amplitudes summed over spins, colours and quark flavours, are given by

$$\begin{aligned}\langle\mathcal{M}|\mathcal{M}\rangle_{q\bar{q}g} &= \sum|\mathcal{M}(\gamma^*\to q\bar{q}g)|^2 \\ &= 4\pi\alpha\sum_q e_q^2\,8\pi^2\Bigg[\left(\frac{\alpha_s}{2\pi}\right)\langle\mathcal{M}^{(0)}|\mathcal{M}^{(0)}\rangle_{q\bar{q}g} + \left(\frac{\alpha_s}{2\pi}\right)^2 2\,\mathrm{Re}[\langle\mathcal{M}^{(0)}|\mathcal{M}^{(1)}\rangle_{q\bar{q}g}] \\ &\quad + \left(\frac{\alpha_s}{2\pi}\right)^3\left(2\,\mathrm{Re}[\langle\mathcal{M}^{(2)}|\mathcal{M}^{(0)}\rangle_{q\bar{q}g}] + \langle\mathcal{M}^{(1)}|\mathcal{M}^{(1)}\rangle_{q\bar{q}g}\right) + \mathcal{O}(\alpha_s^4)\Bigg]\end{aligned} \tag{3.37}$$

and

$$\begin{aligned}\langle\mathcal{M}|\mathcal{M}\rangle_{ggg} &= \sum|\mathcal{M}(\gamma^*\to ggg)|^2 \\ &= 4\pi\alpha\sum_q e_q^2\,512\pi^6\left[\left(\frac{\alpha_s}{2\pi}\right)^3\langle\mathcal{M}^{(1)}|\mathcal{M}^{(1)}\rangle_{ggg} + \mathcal{O}(\alpha_s^4)\right].\end{aligned} \tag{3.38}$$

The integrated matrix elements squared are defined as

$$\mathcal{T}_{q\bar{q}g}^{(4)} = 8\pi^2\int \mathrm{d}\Phi_3\,\langle\mathcal{M}^{(0)}|\mathcal{M}^{(0)}\rangle_{q\bar{q}g}\,, \tag{3.39}$$

$$\mathcal{T}_{q\bar{q}g}^{(6)} = 8\pi^2\int \mathrm{d}\Phi_3\,2\,\mathrm{Re}[\langle\mathcal{M}^{(1)}|\mathcal{M}^{(0)}\rangle_{q\bar{q}g}]\,, \tag{3.40}$$

$$\mathcal{T}_{q\bar{q}g}^{(8)} = \mathcal{T}_{q\bar{q}g}^{(8,[2\times 0])} + \mathcal{T}_{q\bar{q}g}^{(8,[1\times 1])}\,, \tag{3.41}$$

$$\mathcal{T}_{q\bar{q}g}^{(8,[2\times 0])} = 8\pi^2\int \mathrm{d}\Phi_3\,2\,\mathrm{Re}[\langle\mathcal{M}^{(2)}|\mathcal{M}^{(0)}\rangle_{q\bar{q}g}]\,, \tag{3.42}$$

$$\mathcal{T}_{q\bar{q}g}^{(8,[1\times 1])} = 8\pi^2\int \mathrm{d}\Phi_3\,\langle\mathcal{M}^{(1)}|\mathcal{M}^{(1)}\rangle_{q\bar{q}g}\,, \tag{3.43}$$

$$\mathcal{T}_{ggg}^{(8)} = 512\pi^6\int \mathrm{d}\Phi_3\,\langle\mathcal{M}^{(1)}|\mathcal{M}^{(1)}\rangle_{ggg}\,. \tag{3.44}$$

Expressions for $\mathcal{T}^{(4)}_{q\bar{q}g}$ and $\mathcal{T}^{(6)}_{q\bar{q}g}$ are given in (4.28) and (4.33) of [10] and in appendix B. For $\mathcal{T}^{(8,[2\times0])}_{q\bar{q}g}$ and $\mathcal{T}^{(8,[1\times1])}_{q\bar{q}g}$ we refer to the following decompositions:

$$\begin{aligned}\mathcal{T}^{(8,[2\times0])}_{q\bar{q}g} = \left(N-\frac{1}{N}\right)\mathcal{T}^{(2)}_{q\bar{q}}\Bigg[&N^2\mathcal{T}^{(8,[2\times0])}_{q\bar{q}g}\Big|_{N^2}+\mathcal{T}^{(8,[2\times0])}_{q\bar{q}g}\Big|_{N^0}+\frac{1}{N^2}\mathcal{T}^{(8,[2\times0])}_{q\bar{q}g}\Big|_{1/N^2}\\ &+N_F N\,\mathcal{T}^{(8,[2\times0])}_{q\bar{q}g}\Big|_{N_F N}+\frac{N_F}{N}\,\mathcal{T}^{(8,[2\times0])}_{q\bar{q}g}\Big|_{N_F/N}+N_F^2\,\mathcal{T}^{(8,[2\times0])}_{q\bar{q}g}\Big|_{N_F^2}\\ &+N_{F,\gamma}\left(N-\frac{4}{N}\right)\mathcal{T}^{(8,[2\times0])}_{q\bar{q}g}\Big|_{N_{F,\gamma}}\Bigg],\end{aligned}\tag{3.45}$$

$$\begin{aligned}\mathcal{T}^{(8,[1\times1])}_{q\bar{q}g} = \left(N-\frac{1}{N}\right)\mathcal{T}^{(2)}_{q\bar{q}}\Bigg[&N^2\mathcal{T}^{(8,[1\times1])}_{q\bar{q}g}\Big|_{N^2}+\mathcal{T}^{(8,[1\times1])}_{q\bar{q}g}\Big|_{N^0}+\frac{1}{N^2}\mathcal{T}^{(8,[1\times1])}_{q\bar{q}g}\Big|_{1/N^2}\\ &+N_F N\,\mathcal{T}^{(8,[1\times1])}_{q\bar{q}g}\Big|_{N_F N}+\frac{N_F}{N}\,\mathcal{T}^{(8,[1\times1])}_{q\bar{q}g}\Big|_{N_F/N}+N_F^2\,\mathcal{T}^{(8,[1\times1])}_{q\bar{q}g}\Big|_{N_F^2}\Bigg].\end{aligned}\tag{3.46}$$

For $\mathcal{T}^{(8,[2\times0])}_{q\bar{q}g}$, explicit expressions for the different contributions appearing in (3.45) are:

$$\begin{aligned}\mathcal{T}^{(8,[2\times0])}_{q\bar{q}g}\Big|_{N^2} = &+\frac{1}{\epsilon^6}\left(\frac{29}{72}\right)+\frac{1}{\epsilon^5}\left(\frac{128}{27}\right)+\frac{1}{\epsilon^4}\left(\frac{23689}{1296}-\frac{343}{288}\pi^2\right)\\ &+\frac{1}{\epsilon^3}\left(\frac{497425}{7776}-\frac{44041}{5184}\pi^2-\frac{557}{72}\zeta_3\right)\\ &+\frac{1}{\epsilon^2}\left(\frac{2820559}{11664}-\frac{15847}{486}\pi^2-\frac{10645}{144}\zeta_3+\frac{99149}{103680}\pi^4\right)\\ &+\frac{1}{\epsilon}\Bigg(\frac{265414681}{279936}-\frac{5932087}{46656}\pi^2-\frac{61387}{216}\zeta_3\\ &\qquad+\frac{960277}{207360}\pi^4+\frac{20885}{864}\pi^2\zeta_3-\frac{31687}{360}\zeta_5\Bigg)\\ &+\frac{206830619}{52488}-\frac{143664697}{279936}\pi^2-\frac{186071}{162}\zeta_3+\frac{750137}{38880}\pi^4\\ &+\frac{254009}{1728}\pi^2\zeta_3-\frac{178661}{240}\zeta_5-\frac{5842331}{26127360}\pi^6+\frac{14893}{144}\zeta_3^2+\mathcal{O}(\epsilon),\end{aligned}\tag{3.47}$$

$$\begin{aligned}\mathcal{T}^{(8,[2\times0])}_{q\bar{q}g}\Big|_{N^0} = &+\frac{1}{\epsilon^6}\left(-\frac{5}{8}\right)+\frac{1}{\epsilon^5}\left(-\frac{245}{48}\right)+\frac{1}{\epsilon^4}\left(-\frac{667}{36}+\frac{547}{288}\pi^2\right)\\ &+\frac{1}{\epsilon^3}\left(-\frac{30959}{432}+\frac{18439}{1728}\pi^2+\frac{135}{8}\zeta_3\right)\\ &+\frac{1}{\epsilon^2}\left(-\frac{1384741}{5184}+\frac{107647}{2592}\pi^2+\frac{14849}{144}\zeta_3-\frac{17101}{11520}\pi^4\right)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\epsilon}\Bigg(-\frac{32266075}{31104}+\frac{5124899}{31104}\pi^2+\frac{173377}{432}\zeta_3\\
&\qquad-\frac{1309403}{207360}\pi^4-\frac{16825}{288}\pi^2\zeta_3+\frac{5579}{24}\zeta_5\Bigg)\\
&-\frac{795917293}{186624}+\frac{122157485}{186624}\pi^2+\frac{290617}{162}\zeta_3-\frac{4177081}{155520}\pi^4\\
&-\frac{151987}{576}\pi^2\zeta_3+\frac{275473}{240}\zeta_5+\frac{2075989}{8709120}\pi^6-\frac{17695}{48}\zeta_3^2+\mathcal{O}(\epsilon),
\end{aligned}
\tag{3.48}
$$

$$
\begin{aligned}
\mathcal{T}_{q\bar{q}g}^{(8,[2\times 0])}\Big|_{1/N^2} =& +\frac{1}{\epsilon^6}\left(\frac{1}{4}\right)+\frac{1}{\epsilon^5}\left(\frac{9}{8}\right)+\frac{1}{\epsilon^4}\left(\frac{83}{16}-\frac{37}{48}\pi^2\right)\\
&+\frac{1}{\epsilon^3}\left(\frac{83}{4}-\frac{103}{32}\pi^2-\frac{103}{12}\zeta_3\right)\\
&+\frac{1}{\epsilon^2}\left(\frac{1301}{16}-\frac{933}{64}\pi^2-\frac{265}{8}\zeta_3+\frac{3289}{5760}\pi^4\right)\\
&+\frac{1}{\epsilon}\left(\frac{122941}{384}-\frac{2093}{36}\pi^2-\frac{2597}{16}\zeta_3+\frac{4877}{2304}\pi^4+\frac{4211}{144}\pi^2\zeta_3-\frac{6449}{60}\zeta_5\right)\\
&+\frac{83007}{64}-\frac{66487}{288}\pi^2-\frac{33883}{48}\zeta_3+\frac{68713}{7680}\pi^4+\frac{3399}{32}\pi^2\zeta_3\\
&-\frac{15757}{40}\zeta_5+\frac{14593}{622080}\pi^6+\frac{4943}{24}\zeta_3^2+\mathcal{O}(\epsilon),
\end{aligned}
\tag{3.49}
$$

$$
\begin{aligned}
\mathcal{T}_{q\bar{q}g}^{(8,[2\times 0])}\Big|_{N_F N} =& +\frac{1}{\epsilon^5}\left(-\frac{115}{216}\right)+\frac{1}{\epsilon^4}\left(-\frac{1439}{648}\right)+\frac{1}{\epsilon^3}\left(-\frac{20489}{3888}+\frac{739}{1296}\pi^2\right)\\
&+\frac{1}{\epsilon^2}\left(-\frac{189505}{11664}+\frac{5747}{3888}\pi^2+\frac{64}{9}\zeta_3\right)\\
&+\frac{1}{\epsilon}\left(-\frac{789553}{17496}+\frac{55595}{23328}\pi^2+\frac{1195}{54}\zeta_3-\frac{193}{3240}\pi^4\right)\\
&-\frac{49737247}{419904}+\frac{68021}{139968}\pi^2+\frac{31019}{648}\zeta_3+\frac{26777}{77760}\pi^4-\frac{1501}{216}\pi^2\zeta_3\\
&+\frac{190}{3}\zeta_5+\mathcal{O}(\epsilon),
\end{aligned}
\tag{3.50}
$$

$$
\begin{aligned}
\mathcal{T}_{q\bar{q}g}^{(8,[2\times 0])}\Big|_{N_F/N} =& +\frac{1}{\epsilon^5}\left(\frac{5}{12}\right)+\frac{1}{\epsilon^4}\left(\frac{71}{72}\right)+\frac{1}{\epsilon^3}\left(\frac{649}{216}-\frac{193}{432}\pi^2\right)\\
&+\frac{1}{\epsilon^2}\left(\frac{19691}{2592}-\frac{1489}{2592}\pi^2-\frac{227}{36}\zeta_3\right)\\
&+\frac{1}{\epsilon}\left(\frac{263363}{15552}-\frac{7315}{15552}\pi^2-\frac{2051}{216}\zeta_3+\frac{1661}{51840}\pi^4\right)
\end{aligned}
$$

$$
+\frac{3046295}{93312}+\frac{480743}{93312}\pi^2-\frac{27095}{1296}\zeta_3-\frac{60871}{311040}\pi^4
+\frac{775}{144}\pi^2\zeta_3-\frac{3019}{60}\zeta_5+\mathcal{O}(\epsilon), \tag{3.51}
$$

$$
\mathcal{T}_{q\bar{q}g}^{(8,[2\times0])}\Big|_{N_F^2} = +\frac{1}{\epsilon^4}\left(\frac{1}{12}\right)+\frac{1}{\epsilon^3}\left(\frac{1}{8}\right)+\frac{1}{\epsilon^2}\left(\frac{19}{48}-\frac{7}{144}\pi^2\right)
+\frac{1}{\epsilon}\left(\frac{109}{96}-\frac{7}{96}\pi^2-\frac{25}{36}\zeta_3\right)
+\frac{213}{64}-\frac{133}{576}\pi^2-\frac{25}{24}\zeta_3-\frac{71}{17280}\pi^4+\mathcal{O}(\epsilon), \tag{3.52}
$$

$$
\mathcal{T}_{q\bar{q}g}^{(8,[2\times0])}\Big|_{N_{F,\gamma}} = -\frac{15}{4}-\frac{2}{9}\pi^2+\frac{\zeta_3}{6}+\frac{\pi^4}{360}-\frac{\pi^2\zeta_3}{3}+\frac{25\zeta_5}{3}+\mathcal{O}(\epsilon). \tag{3.53}
$$

The results for $\mathcal{T}_{q\bar{q}g}^{(8,[1\times1])}$ are:

$$
\mathcal{T}_{q\bar{q}g}^{(8,[1\times1])}\Big|_{N^2} = +\frac{1}{\epsilon^6}\left(\frac{29}{72}\right)+\frac{1}{\epsilon^5}\left(\frac{71}{24}\right)+\frac{1}{\epsilon^4}\left(\frac{1879}{144}-\frac{373}{864}\pi^2\right)
+\frac{1}{\epsilon^3}\left(\frac{851}{16}-\frac{659}{192}\pi^2-\frac{685}{72}\zeta_3\right)
+\frac{1}{\epsilon^2}\left(\frac{63401}{288}-\frac{6403}{432}\pi^2-\frac{8585}{144}\zeta_3-\frac{2737}{34560}\pi^4\right)
+\frac{1}{\epsilon}\left(\frac{135041}{144}-\frac{74453}{1152}\pi^2-\frac{30355}{108}\zeta_3+\frac{809}{23040}\pi^4+\frac{9749}{864}\pi^2\zeta_3-\frac{12349}{120}\zeta_5\right)
+\frac{530995}{128}-\frac{243419}{864}\pi^2-\frac{124439}{96}\zeta_3-\frac{48167}{51840}\pi^4+\frac{42359}{576}\pi^2\zeta_3
-\frac{142601}{240}\zeta_5-\frac{94961}{967680}\pi^6+\frac{18773}{144}\zeta_3^2+\mathcal{O}(\epsilon), \tag{3.54}
$$

$$
\mathcal{T}_{q\bar{q}g}^{(8,[1\times1])}\Big|_{N^0} = +\frac{1}{\epsilon^6}\left(-\frac{5}{8}\right)+\frac{1}{\epsilon^5}\left(-\frac{179}{48}\right)+\frac{1}{\epsilon^4}\left(-\frac{769}{48}+\frac{199}{288}\pi^2\right)
+\frac{1}{\epsilon^3}\left(-\frac{6347}{96}+\frac{2375}{576}\pi^2+\frac{137}{8}\zeta_3\right)
+\frac{1}{\epsilon^2}\left(-\frac{25441}{96}+\frac{9997}{576}\pi^2+\frac{12457}{144}\zeta_3+\frac{6817}{34560}\pi^4\right)
+\frac{1}{\epsilon}\left(-\frac{207427}{192}+\frac{42641}{576}\pi^2+\frac{18713}{48}\zeta_3+\frac{37817}{69120}\pi^4-\frac{7411}{288}\pi^2\zeta_3+\frac{1903}{8}\zeta_5\right)
-\frac{1748021}{384}+\frac{358529}{1152}\pi^2+\frac{84229}{48}\zeta_3+\frac{248171}{69120}\pi^4-\frac{222493}{1728}\pi^2\zeta_3
+\frac{88921}{80}\zeta_5+\frac{607987}{2903040}\pi^6-\frac{5805}{16}\zeta_3^2+\mathcal{O}(\epsilon), \tag{3.55}
$$

$$
\mathcal{T}_{q\bar{q}g}^{(8,[1\times1])}\Big|_{1/N^2} = +\frac{1}{\epsilon^6}\left(\frac{1}{4}\right)+\frac{1}{\epsilon^5}\left(\frac{9}{8}\right)+\frac{1}{\epsilon^4}\left(\frac{83}{16}-\frac{13}{48}\pi^2\right)+\frac{1}{\epsilon^3}\left(\frac{667}{32}-\frac{35}{32}\pi^2-\frac{85}{12}\zeta_3\right)
+\frac{1}{\epsilon^2}\left(\frac{2607}{32}-\frac{959}{192}\pi^2-\frac{217}{8}\zeta_3-\frac{41}{384}\pi^4\right)
$$

$$
\begin{aligned}
&+\frac{1}{\epsilon}\left(\frac{61319}{192}-\frac{7895}{384}\pi^2-\frac{6109}{48}\zeta_3-\frac{1349}{3840}\pi^4+\frac{1505}{144}\pi^2\zeta_3-\frac{2113}{20}\zeta_5\right)\\
&+\frac{494285}{384}-\frac{32531}{384}\pi^2-\frac{53047}{96}\zeta_3-\frac{8137}{4608}\pi^4+\frac{3629}{96}\pi^2\zeta_3\\
&-\frac{7191}{20}\zeta_5-\frac{161467}{1451520}\pi^6+\frac{1265}{8}\zeta_3^2+\mathcal{O}(\epsilon),
\end{aligned}
\tag{3.56}
$$

$$
\begin{aligned}
\mathcal{T}^{(8,[1\times1])}_{q\bar{q}g}\Big|_{N_F N}=&+\frac{1}{\epsilon^5}\left(-\frac{5}{24}\right)+\frac{1}{\epsilon^4}\left(-\frac{67}{72}\right)+\frac{1}{\epsilon^3}\left(-\frac{47}{16}+\frac{13}{48}\pi^2\right)\\
&+\frac{1}{\epsilon^2}\left(-\frac{1481}{144}+\frac{107}{108}\pi^2+\frac{55}{18}\zeta_3\right)\\
&+\frac{1}{\epsilon}\left(-\frac{10385}{288}+\frac{32}{9}\pi^2+\frac{1265}{108}\zeta_3-\frac{41}{576}\pi^4\right)\\
&-\frac{8305}{64}+\frac{22223}{1728}\pi^2+\frac{1517}{36}\zeta_3-\frac{10289}{51840}\pi^4-\frac{149}{36}\pi^2\zeta_3\\
&+\frac{143}{6}\zeta_5+\mathcal{O}(\epsilon),
\end{aligned}
\tag{3.57}
$$

$$
\begin{aligned}
\mathcal{T}^{(8,[1\times1])}_{q\bar{q}g}\Big|_{N_F/N}=&+\frac{1}{\epsilon^5}\left(\frac{1}{6}\right)+\frac{1}{\epsilon^4}\left(\frac{1}{2}\right)+\frac{1}{\epsilon^3}\left(\frac{31}{16}-\frac{2}{9}\pi^2\right)+\frac{1}{\epsilon^2}\left(\frac{79}{12}-\frac{5}{8}\pi^2-\frac{53}{18}\zeta_3\right)\\
&+\frac{1}{\epsilon}\left(\frac{1069}{48}-\frac{697}{288}\pi^2-\frac{91}{12}\zeta_3+\frac{19}{432}\pi^4\right)\\
&+\frac{3653}{48}-\frac{25}{3}\pi^2-\frac{241}{8}\zeta_3+\frac{19}{144}\pi^4+\frac{493}{108}\pi^2\zeta_3-\frac{299}{10}\zeta_5+\mathcal{O}(\epsilon),
\end{aligned}
\tag{3.58}
$$

$$
\begin{aligned}
\mathcal{T}^{(8,[1\times1])}_{q\bar{q}g}\Big|_{N_F^2}=&+\frac{1}{\epsilon^4}\left(\frac{1}{36}\right)+\frac{1}{\epsilon^3}\left(\frac{1}{24}\right)+\frac{1}{\epsilon^2}\left(\frac{19}{144}-\frac{7}{432}\pi^2\right)\\
&+\frac{1}{\epsilon}\left(\frac{109}{288}-\frac{7}{288}\pi^2-\frac{25}{108}\zeta_3\right)\\
&+\frac{71}{64}-\frac{133}{1728}\pi^2-\frac{25}{72}\zeta_3-\frac{71}{51840}\pi^4+\mathcal{O}(\epsilon).
\end{aligned}
\tag{3.59}
$$

The three-gluon final state $\mathcal{T}^{(8)}_{ggg}$ is a genuine singlet contribution:

$$
\mathcal{T}^{(8)}_{ggg}=\left(N-\frac{1}{N}\right)\mathcal{T}^{(2)}_{q\bar{q}}N_{F,\gamma}\left(N-\frac{4}{N}\right)\mathcal{T}^{(8)}_{ggg}\Big|_{N_{F,\gamma}}, \tag{3.60}
$$

with

$$
\mathcal{T}^{(8)}_{ggg}\Big|_{N_{F,\gamma}}=-\frac{31}{12}+\frac{41}{144}\pi^2-\frac{8}{3}\zeta_3+\frac{7}{720}\pi^4-\frac{\pi^2\zeta_3}{6}+\frac{25}{6}\zeta_5+\mathcal{O}(\epsilon)\,. \tag{3.61}
$$

In order to compute the renormalised $\mathcal{T}^{(8)}_{q\bar{q}g}$, we needed to evaluate the master integrals $V_{5,a}$, $V_{5,b}$ and V_8 up to weight 5, in the notation of [10]. $V_{5,a}$ and $V_{5,b}$ are provided in a closed form, whereas the expansion of V_8 is truncated at weight 4 [52]. We numerically computed the master integral V_8 up to $\mathcal{O}(\epsilon^2)$ with AMFlow [53], and we reconstructed the analytic expression with the PSLQ algorithm [54]. The result reads:

$$V_8 = S_{\Gamma,2}(q^2)^{-2-2\epsilon}\Bigg[-\frac{5}{2\epsilon^4}+\frac{9\pi^2}{2\epsilon^2}+\frac{89\zeta_3}{\epsilon}+\frac{13\pi^4}{180}+\epsilon\left(-\frac{407\pi^2\zeta_3}{3}+1135\zeta_5\right) + \epsilon^2\left(\frac{1451\pi^6}{2268}-1181\zeta_3^2\right)+\mathcal{O}(\epsilon^3)\Bigg], \tag{3.62}$$

with

$$S_{\Gamma,2} = P_2\left(\frac{(4\pi)^\epsilon}{16\pi^2\Gamma(1-\epsilon)}\right)^2 . \tag{3.63}$$

As a validation of this result, we notice that the master integral denoted by VVRR_{22} in [47] is related to V_8 by a trivial multiplication with a one-loop bubble, as can be inferred by the definition of the two integrals. We find complete agreement between the two expressions.

3.3 Four-parton final states

The processes contributing to the four-parton final state are $\gamma^*\to q\bar{q}gg$, $\gamma^*\to q\bar{q}q'\bar{q}'$ and $\gamma^*\to q\bar{q}q\bar{q}$, where q' denotes a quark of a flavour different to that of q. The amplitude up to one loop is

$$|\mathcal{M}\rangle_{q\bar{q}ij} = \sqrt{4\pi\alpha}e_q\,4\pi\alpha_s\left[|\mathcal{M}^{(0)}\rangle_{q\bar{q}ij}+\left(\frac{\alpha_s}{2\pi}\right)|\mathcal{M}^{(1)}\rangle_{q\bar{q}ij}+\mathcal{O}(\alpha_s^2)\right], \tag{3.64}$$

with $ij = q'\bar{q}', q\bar{q}, gg$. The perturbative expansions of the squared amplitudes summed over spins, colours and quark flavours are

$$\begin{aligned}\langle\mathcal{M}|\mathcal{M}\rangle_{q\bar{q}ij} &= \sum|\mathcal{M}(\gamma^*\to q\bar{q}ij)|^2 \\ &= 4\pi\alpha\sum_q e_q^2\,64\pi^4\Bigg[\left(\frac{\alpha_s}{2\pi}\right)^2\langle\mathcal{M}^{(0)}|\mathcal{M}^{(0)}\rangle_{q\bar{q}ij} \\ &\qquad + \left(\frac{\alpha_s}{2\pi}\right)^3\left(2\,\mathrm{Re}\big[\langle\mathcal{M}^{(1)}|\mathcal{M}^{(0)}\rangle_{q\bar{q}ij}\big]\right)+\mathcal{O}(\alpha_s^4)\Bigg].\end{aligned} \tag{3.65}$$

We define

$$\mathcal{T}^{(6)}_{q\bar{q}ij} = 64\pi^4\int \mathrm{d}\Phi_4\,\langle\mathcal{M}^{(0)}|\mathcal{M}^{(0)}\rangle_{q\bar{q}ij}\,, \tag{3.66}$$

$$\mathcal{T}^{(8)}_{q\bar{q}ij} = 64\pi^4\int \mathrm{d}\Phi_4\,2\,\mathrm{Re}\big[\langle\mathcal{M}^{(1)}|\mathcal{M}^{(0)}\rangle_{q\bar{q}ij}\big]. \tag{3.67}$$

The expressions for $\mathcal{T}^{(6)}_{q\bar{q}ij}$ are given in equations (4.49), (4.51) and (4.53) of [10] and in appendix B. The colour decomposition of $\mathcal{T}^{(8)}_{q\bar{q}gg}$ reads

$$\begin{aligned}\mathcal{T}^{(8)}_{q\bar{q}gg} = \left(N-\frac{1}{N}\right)\mathcal{T}^{(2)}_{q\bar{q}}\Bigg[&N^2\mathcal{T}^{(8)}_{q\bar{q}gg}\Big|_{N^2}+\mathcal{T}^{(8)}_{q\bar{q}gg}\Big|_{N^0}+\frac{1}{N^2}\mathcal{T}^{(8)}_{q\bar{q}gg}\Big|_{1/N^2} \\ &+N_F N\,\mathcal{T}^{(8)}_{q\bar{q}gg}\Big|_{N_F N}+\frac{N_F}{N}\,\mathcal{T}^{(8)}_{q\bar{q}gg}\Big|_{N_F/N}+N_{F,\gamma}\left(N-\frac{4}{N}\right)\mathcal{T}^{(8)}_{q\bar{q}gg}\Big|_{N_{F,\gamma}}\Bigg].\end{aligned} \tag{3.68}$$

The results for the coefficients are

$$
\begin{aligned}
\mathcal{T}^{(8)}_{q\bar{q}gg}\Big|_{N^2} = &+\frac{1}{\epsilon^6}\left(-\frac{41}{36}\right)+\frac{1}{\epsilon^5}\left(-\frac{311}{36}\right)+\frac{1}{\epsilon^4}\left(-\frac{54325}{1296}+\frac{1151}{432}\pi^2\right)\\
&+\frac{1}{\epsilon^3}\left(-\frac{1590017}{7776}+\frac{23519}{1296}\pi^2+\frac{380}{9}\zeta_3\right)\\
&+\frac{1}{\epsilon^2}\left(-\frac{7646353}{7776}+\frac{1461895}{15552}\pi^2+\frac{66593}{216}\zeta_3-\frac{16537}{10368}\pi^4\right)\\
&+\frac{1}{\epsilon}\left(-\frac{84015367}{17496}+\frac{44125165}{93312}\pi^2+\frac{2192809}{1296}\zeta_3\right.\\
&\qquad\left.-\frac{7973}{960}\pi^4-\frac{7435}{72}\pi^2\zeta_3+\frac{40319}{90}\zeta_5\right)\\
&-\frac{20052623335}{839808}+\frac{163823405}{69984}\pi^2+\frac{22820177}{2592}\zeta_3-\frac{4992721}{124416}\pi^4\\
&-\frac{22493}{32}\pi^2\zeta_3+\frac{1335263}{360}\zeta_5+\frac{4433837}{13063680}\pi^6-\frac{64345}{72}\zeta_3^2+\mathcal{O}(\epsilon)\,,
\end{aligned}
\tag{3.69}
$$

$$
\begin{aligned}
\mathcal{T}^{(8)}_{q\bar{q}gg}\Big|_{N^0} = &+\frac{1}{\epsilon^6}\left(\frac{3}{2}\right)+\frac{1}{\epsilon^5}\left(\frac{217}{24}\right)+\frac{1}{\epsilon^4}\left(\frac{389}{9}-\frac{65}{18}\pi^2\right)\\
&+\frac{1}{\epsilon^3}\left(\frac{177335}{864}-\frac{17189}{864}\pi^2-\frac{761}{12}\zeta_3\right)\\
&+\frac{1}{\epsilon^2}\left(\frac{4999865}{5184}-\frac{128357}{1296}\pi^2-\frac{12755}{36}\zeta_3+\frac{16579}{8640}\pi^4\right)\\
&+\frac{1}{\epsilon}\left(\frac{143110091}{31104}-\frac{14956337}{31104}\pi^2-\frac{65791}{36}\zeta_3\right.\\
&\qquad\left.+\frac{310769}{34560}\pi^4+\frac{23987}{144}\pi^2\zeta_3-\frac{51389}{60}\zeta_5\right)\\
&+\frac{4179822425}{186624}-\frac{431838125}{186624}\pi^2-\frac{7979983}{864}\zeta_3+\frac{62213}{1440}\pi^4\\
&+\frac{190001}{216}\pi^2\zeta_3-\frac{856933}{180}\zeta_5-\frac{719437}{1088640}\pi^6+\frac{19345}{12}\zeta_3^2+\mathcal{O}(\epsilon)\,,
\end{aligned}
\tag{3.70}
$$

$$
\begin{aligned}
\mathcal{T}^{(8)}_{q\bar{q}gg}\Big|_{1/N^2} = &+\frac{1}{\epsilon^6}\left(-\frac{1}{2}\right)+\frac{1}{\epsilon^5}\left(-\frac{9}{4}\right)+\frac{1}{\epsilon^4}\left(-\frac{91}{8}+\frac{29}{24}\pi^2\right)\\
&+\frac{1}{\epsilon^3}\left(-\frac{1683}{32}+\frac{83}{16}\pi^2+\frac{137}{6}\zeta_3\right)\\
&+\frac{1}{\epsilon^2}\left(-\frac{1937}{8}+\frac{843}{32}\pi^2+\frac{373}{4}\zeta_3-\frac{169}{320}\pi^4\right)\\
&+\frac{1}{\epsilon}\left(-\frac{54331}{48}+\frac{142745}{1152}\pi^2+\frac{965}{2}\zeta_3-\frac{13073}{5760}\pi^4-\frac{1463}{24}\pi^2\zeta_3+\frac{10469}{30}\zeta_5\right)\\
&-\frac{2082919}{384}+\frac{168199}{288}\pi^2+\frac{222847}{96}\zeta_3-\frac{14011}{1280}\pi^4\\
&-\frac{3885}{16}\pi^2\zeta_3+\frac{56609}{40}\zeta_5+\frac{51599}{311040}\pi^6-\frac{7667}{12}\zeta_3^2+\mathcal{O}(\epsilon)\,,
\end{aligned}
\tag{3.71}
$$

$$\mathcal{T}^{(8)}_{q\bar{q}gg}\Big|_{N_F N} = +\frac{1}{\epsilon^5}\left(\frac{1}{2}\right)+\frac{1}{\epsilon^4}\left(\frac{65}{36}\right)+\frac{1}{\epsilon^3}\left(\frac{869}{108}-\frac{13}{18}\pi^2\right)+\frac{1}{\epsilon^2}\left(\frac{14459}{432}-\frac{589}{216}\pi^2-\frac{71}{6}\zeta_3\right)$$
$$+\frac{1}{\epsilon}\left(\frac{1084631}{7776}-\frac{15943}{1296}\pi^2-\frac{1327}{27}\zeta_3+\frac{373}{2160}\pi^4\right)$$
$$+\frac{27302353}{46656}-\frac{200555}{3888}\pi^2-\frac{18365}{81}\zeta_3+\frac{5207}{12960}\pi^4$$
$$+\frac{1891}{108}\pi^2\zeta_3-\frac{887}{10}\zeta_5+\mathcal{O}(\epsilon)\,, \qquad (3.72)$$

$$\mathcal{T}^{(8)}_{q\bar{q}gg}\Big|_{N_F/N} = +\frac{1}{\epsilon^5}\left(-\frac{1}{3}\right)+\frac{1}{\epsilon^4}(-1)+\frac{1}{\epsilon^3}\left(-\frac{13}{3}+\frac{\pi^2}{2}\right)+\frac{1}{\epsilon^2}\left(-\frac{845}{48}+\frac{3}{2}\pi^2+\frac{80}{9}\zeta_3\right)$$
$$+\frac{1}{\epsilon}\left(-\frac{2307}{32}+\frac{473}{72}\pi^2+\frac{80}{3}\zeta_3-\frac{17}{216}\pi^4\right)$$
$$-\frac{57305}{192}+\frac{7735}{288}\pi^2+\frac{4295}{36}\zeta_3-\frac{17}{72}\pi^4-\frac{121}{9}\pi^2\zeta_3+\frac{424}{5}\zeta_5+\mathcal{O}(\epsilon)\,, \qquad (3.73)$$

$$\mathcal{T}^{(8)}_{q\bar{q}gg}\Big|_{N_{F,\gamma}} = +\frac{139}{12}-\frac{49}{72}\pi^2-\frac{5}{4}\zeta_3-\frac{7}{720}\pi^4+\frac{2}{3}\pi^2\zeta_3-10\zeta_5+\mathcal{O}(\epsilon)\,. \qquad (3.74)$$

The colour decomposition of $\mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'}$ is given by

$$\mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'} = \left(N-\frac{1}{N}\right)\mathcal{T}^{(2)}_{q\bar{q}}\left[(N_F-1)N\,\mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'}\Big|_{N_F N}+\frac{(N_F-1)}{N}\,\mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'}\Big|_{N_F/N}\right.$$
$$\left.+(N_F-1)N_F\,\mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'}\Big|_{N_F^2}+(N_{F,\gamma}-1)\left(N-\frac{4}{N}\right)\mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'}\Big|_{N_{F,\gamma}}\right], \qquad (3.75)$$

where we have used N_F-1 and $N_{F,\gamma}-1$ because of the different flavour of the two final-state quark lines. The coefficients above read:

$$\mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'}\Big|_{N_F N} = +\frac{1}{\epsilon^5}\left(\frac{13}{108}\right)+\frac{1}{\epsilon^4}\left(\frac{679}{648}\right)+\frac{1}{\epsilon^3}\left(\frac{11249}{1944}-\frac{425}{1296}\pi^2\right)$$
$$+\frac{1}{\epsilon^2}\left(\frac{185695}{5832}-\frac{20297}{7776}\pi^2-\frac{265}{36}\zeta_3\right)$$
$$+\frac{1}{\epsilon}\left(\frac{1554361}{8748}-\frac{344569}{23328}\pi^2-\frac{1457}{24}\zeta_3+\frac{3239}{51840}\pi^4\right)$$
$$+\frac{210844381}{209952}-\frac{1432307}{17496}\pi^2-\frac{8225}{24}\zeta_3+\frac{64091}{311040}\pi^4$$
$$+\frac{8261}{432}\pi^2\zeta_3-\frac{9121}{60}\zeta_5+\mathcal{O}(\epsilon)\,, \qquad (3.76)$$

$$\mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'}\Big|_{N_F/N} = +\frac{1}{\epsilon^5}\left(-\frac{11}{108}\right)+\frac{1}{\epsilon^4}\left(-\frac{425}{648}\right)+\frac{1}{\epsilon^3}\left(-\frac{7627}{1944}+\frac{133}{432}\pi^2\right)$$
$$+\frac{1}{\epsilon^2}\left(-\frac{131273}{5832}+\frac{5047}{2592}\pi^2+\frac{889}{108}\zeta_3\right)$$
$$+\frac{1}{\epsilon}\left(-\frac{2276611}{17496}+\frac{89357}{7776}\pi^2+\frac{34279}{648}\zeta_3-\frac{125}{10368}\pi^4\right)$$

$$
\begin{aligned}
&-\frac{159423833}{209952}+\frac{1510867}{23328}\pi^2+\frac{606965}{1944}\zeta_3+\frac{17861}{311040}\pi^4 \\
&-\frac{10375}{432}\pi^2\zeta_3+\frac{34661}{180}\zeta_5+\mathcal{O}(\epsilon)\,,
\end{aligned} \tag{3.77}
$$

$$
\begin{aligned}
\mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'}\Big|_{N_F^2} =& +\frac{1}{\epsilon^4}\left(-\frac{7}{162}\right)+\frac{1}{\epsilon^3}\left(-\frac{79}{486}\right)+\frac{1}{\epsilon^2}\left(-\frac{97}{162}+\frac{\pi^2}{18}\right) \\
&+\frac{1}{\epsilon}\left(-\frac{3613}{2187}+\frac{73}{648}\pi^2+\frac{76}{81}\zeta_3\right) \\
&+\frac{4661}{26244}-\frac{523}{3888}\pi^2-\frac{247}{486}\zeta_3+\frac{41}{38880}\pi^4+\mathcal{O}(\epsilon)\,,
\end{aligned} \tag{3.78}
$$

$$
\begin{aligned}
\mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'}\Big|_{N_{F,\gamma}} =& +\frac{1}{\epsilon}\left(-\frac{7}{3}+\frac{\pi^2}{9}+\frac{\zeta_3}{2}+\frac{\pi^4}{135}\right) \\
&-\frac{2335}{72}+\frac{119\pi^2}{108}+\frac{19\zeta_3}{3}+\frac{961\pi^4}{12960}-\frac{14\pi^2\zeta_3}{9}+26\zeta_5+\mathcal{O}(\epsilon)\,.
\end{aligned} \tag{3.79}
$$

Finally, the colour decomposition of $\mathcal{T}^{(8)}_{q\bar{q}q\bar{q}}$ is

$$
\begin{aligned}
\mathcal{T}^{(8)}_{q\bar{q}q\bar{q}} =& \left(N-\frac{1}{N}\right)\mathcal{T}^{(2)}_{q\bar{q}}\Bigg[N\,\mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'}\Big|_{N_F N}+\frac{1}{N}\,\mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'}\Big|_{N_F/N}+N_F\,\mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'}\Big|_{N_F^2} \\
&+\left(N-\frac{4}{N}\right)\mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'}\Big|_{N_{F,\gamma}}+\mathcal{T}^{(8)}_{q\bar{q}q\bar{q}}\Big|_{N^0}+\frac{1}{N^2}\mathcal{T}^{(8)}_{q\bar{q}q\bar{q}}\Big|_{1/N^2}+\frac{N_F}{N}\mathcal{T}^{(8)}_{q\bar{q}q\bar{q}}\Big|_{N_F/N}\Bigg],
\end{aligned} \tag{3.80}
$$

where the first three terms are identical to those in (3.75). The factors N_F and $N_{F,\gamma}$ are correctly restored in the sum of the $q\bar{q}q'\bar{q}'$ and $q\bar{q}q\bar{q}$ sub-channels. The new terms appearing in the same-flavour case are

$$
\begin{aligned}
\mathcal{T}^{(8)}_{q\bar{q}q\bar{q}}\Big|_{N^0} =& +\frac{1}{\epsilon^3}\left(-\frac{65}{48}+\frac{5}{24}\pi^2-\frac{5}{6}\zeta_3\right)+\frac{1}{\epsilon^2}\left(-\frac{3527}{144}+\frac{275}{144}\pi^2+\frac{80}{9}\zeta_3-\frac{101}{1080}\pi^4\right) \\
&+\frac{1}{\epsilon}\left(-\frac{217805}{864}+\frac{23891}{1728}\pi^2+\frac{10573}{108}\zeta_3+\frac{281}{6480}\pi^4+\frac{169}{72}\pi^2\zeta_3-\frac{117}{2}\zeta_5\right) \\
&-\frac{325597}{162}+\frac{280337}{2592}\pi^2+\frac{97657}{162}\zeta_3+\frac{171733}{77760}\pi^4 \\
&-\frac{6265}{216}\pi^2\zeta_3+\frac{4403}{18}\zeta_5-\frac{8171}{90720}\pi^6+\frac{110}{3}\zeta_3^2+\mathcal{O}(\epsilon)\,,
\end{aligned} \tag{3.81}
$$

$$
\begin{aligned}
\mathcal{T}^{(8)}_{q\bar{q}q\bar{q}}\Big|_{1/N^2} =& +\frac{1}{\epsilon^3}\left(\frac{65}{48}-\frac{5}{24}\pi^2+\frac{5}{6}\zeta_3\right)+\frac{1}{\epsilon^2}\left(\frac{185}{8}-\frac{79}{48}\pi^2-9\zeta_3+\frac{7}{90}\pi^4\right) \\
&+\frac{1}{\epsilon}\left(\frac{45953}{192}-\frac{7435}{576}\pi^2-\frac{935}{12}\zeta_3-\frac{43}{720}\pi^4-\frac{229}{72}\pi^2\zeta_3+\frac{277}{6}\zeta_5\right) \\
&+\frac{251839}{128}-\frac{30857}{288}\pi^2-\frac{24691}{48}\zeta_3-\frac{5089}{4320}\pi^4 \\
&+\frac{281}{12}\pi^2\zeta_3-\frac{421}{2}\zeta_5-\frac{11}{336}\pi^6-76\zeta_3^2+\mathcal{O}(\epsilon)\,,
\end{aligned} \tag{3.82}
$$

$$\mathcal{T}^{(8)}_{q\bar{q}q\bar{q}}\Big|_{N_F/N} = +\frac{1}{\epsilon^2}\left(\frac{13}{72}-\frac{\pi^2}{36}+\frac{\zeta_3}{9}\right)+\frac{1}{\epsilon}\left(\frac{19}{108}+\frac{11}{216}\pi^2-\frac{28}{27}\zeta_3+\frac{17}{3240}\pi^4\right)$$
$$-\frac{3437}{162}+\frac{247}{162}\pi^2+\frac{2125}{324}\zeta_3-\frac{1249}{19440}\pi^4+\frac{8}{27}\pi^2\zeta_3-\frac{19}{9}\zeta_5+\mathcal{O}(\epsilon)\,. \tag{3.83}$$

3.4 Five-parton final states

The processes contributing to the five-parton final state are $\gamma^* \to q\bar{q}ggg$, $\gamma^* \to q\bar{q}q'\bar{q}'g$ and $\gamma^* \to q\bar{q}q\bar{q}g$, and the tree-level amplitudes read

$$|\mathcal{M}\rangle_{q\bar{q}ijk} = \sqrt{4\pi\alpha}e_q\,(4\pi\alpha_s)^{3/2}\left[|\mathcal{M}^{(0)}\rangle_{q\bar{q}ijk}+\mathcal{O}(\alpha_s)\right], \tag{3.84}$$

with $ijk = ggg, q'\bar{q}'g, q\bar{q}g$. The squared amplitude, summed over spins, colours and quark flavours is

$$\langle\mathcal{M}|\mathcal{M}\rangle_{q\bar{q}ijk} = \sum|\mathcal{M}(\gamma^*\to q\bar{q}ijk)|^2$$
$$= 4\pi\alpha\sum_q e_q^2\,512\pi^6\left[\left(\frac{\alpha_s}{2\pi}\right)^3\langle\mathcal{M}^{(0)}|\mathcal{M}^{(0)}\rangle_{q\bar{q}ijk}+\mathcal{O}(\alpha_s^4)\right]. \tag{3.85}$$

We define

$$\mathcal{T}^{(8)}_{q\bar{q}ijk} = 512\pi^8\int \mathrm{d}\Phi_5\,\langle\mathcal{M}^{(0)}|\mathcal{M}^{(0)}\rangle_{q\bar{q}ijk}\,. \tag{3.86}$$

The colour decomposition for $\mathcal{T}^{(8)}_{q\bar{q}ggg}$ reads:

$$\mathcal{T}^{(8)}_{q\bar{q}ggg} = \left(N-\frac{1}{N}\right)\mathcal{T}^{(2)}_{q\bar{q}}\left[N^2\mathcal{T}^{(8)}_{q\bar{q}ggg}\Big|_{N^2}+\mathcal{T}^{(8)}_{q\bar{q}ggg}\Big|_{N^0}+\frac{1}{N^2}\mathcal{T}^{(8)}_{q\bar{q}ggg}\Big|_{1/N^2}\right], \tag{3.87}$$

with

$$\mathcal{T}^{(8)}_{q\bar{q}ggg}\Big|_{N^2} = +\frac{1}{\epsilon^6}\left(\frac{1}{2}\right)+\frac{1}{\epsilon^5}\left(\frac{331}{108}\right)+\frac{1}{\epsilon^4}\left(\frac{11843}{648}-\frac{31}{24}\pi^2\right)$$
$$+\frac{1}{\epsilon^3}\left(\frac{259867}{2592}-\frac{10745}{1296}\pi^2-\frac{439}{18}\zeta_3\right)$$
$$+\frac{1}{\epsilon^2}\left(\frac{6302057}{11664}-\frac{394223}{7776}\pi^2-\frac{6239}{36}\zeta_3+\frac{21853}{25920}\pi^4\right)$$
$$+\frac{1}{\epsilon}\left(\frac{815913157}{279936}-\frac{26347837}{93312}\pi^2-\frac{181151}{162}\zeta_3\right.$$
$$\left.+\frac{75767}{17280}\pi^4+\frac{13993}{216}\pi^2\zeta_3-\frac{10946}{45}\zeta_5\right)$$
$$+\frac{736904809}{46656}-\frac{107045579}{69984}\pi^2-\frac{49920557}{7776}\zeta_3+\frac{7130357}{311040}\pi^4$$
$$+\frac{67895}{144}\pi^2\zeta_3-\frac{103894}{45}\zeta_5-\frac{93257}{1306368}\pi^6+\frac{7861}{12}\zeta_3^2+\mathcal{O}(\epsilon), \tag{3.88}$$

$$\mathcal{T}^{(8)}_{q\bar{q}ggg}\Big|_{N^0} = +\frac{1}{\epsilon^6}\left(-\frac{7}{12}\right)+\frac{1}{\epsilon^5}\left(-\frac{37}{12}\right)+\frac{1}{\epsilon^4}\left(-\frac{1255}{72}+\frac{25}{16}\pi^2\right)$$
$$+\frac{1}{\epsilon^3}\left(-\frac{39895}{432}+\frac{76}{9}\pi^2+\frac{63}{2}\zeta_3\right)$$

$$+\frac{1}{\epsilon^2}\left(-\frac{78673}{162}+\frac{20903}{432}\pi^2+\frac{12907}{72}\zeta_3-\frac{15811}{17280}\pi^4\right)$$
$$+\frac{1}{\epsilon}\left(-\frac{40021591}{15552}+\frac{335677}{1296}\pi^2+\frac{458257}{432}\zeta_3-\frac{15115}{3456}\pi^4-\frac{1541}{18}\pi^2\zeta_3+\frac{5774}{15}\zeta_5\right)$$
$$-\frac{320911123}{23328}+\frac{42522607}{31104}\pi^2+\frac{15074377}{2592}\zeta_3-\frac{2348393}{103680}\pi^4$$
$$-\frac{142223}{288}\pi^2\zeta_3+\frac{890561}{360}\zeta_5+\frac{916243}{4354560}\pi^6-\frac{21473}{24}\zeta_3^2+\mathcal{O}(\epsilon), \tag{3.89}$$

$$\mathcal{T}^{(8)}_{q\bar{q}ggg}\Big|_{1/N^2}=+\frac{1}{\epsilon^6}\left(\frac{1}{6}\right)+\frac{1}{\epsilon^5}\left(\frac{3}{4}\right)+\frac{1}{\epsilon^4}\left(\frac{33}{8}-\frac{11}{24}\pi^2\right)+\frac{1}{\epsilon^3}\left(\frac{687}{32}-\frac{33}{16}\pi^2-\frac{59}{6}\zeta_3\right)$$
$$+\frac{1}{\epsilon^2}\left(\frac{1787}{16}-\frac{1099}{96}\pi^2-\frac{177}{4}\zeta_3+\frac{659}{2880}\pi^4\right)$$
$$+\frac{1}{\epsilon}\left(\frac{225217}{384}-\frac{69133}{1152}\pi^2-\frac{1997}{8}\zeta_3+\frac{1973}{1920}\pi^4+\frac{1981}{72}\pi^2\zeta_3-\frac{1451}{10}\zeta_5\right)$$
$$+\frac{597437}{192}-\frac{180415}{576}\pi^2-\frac{126785}{96}\zeta_3+\frac{187531}{34560}\pi^4$$
$$+\frac{5917}{48}\pi^2\zeta_3-\frac{26103}{40}\zeta_5-\frac{197047}{2177280}\pi^6+\frac{1831}{6}\zeta_3^2+\mathcal{O}(\epsilon). \tag{3.90}$$

The colour decomposition for $\mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'g}$ reads:

$$\mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'g}=\left(N-\frac{1}{N}\right)\mathcal{T}^{(2)}_{q\bar{q}}\Bigg[+(N_F-1)N\,\mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'g}\Big|_{N_FN}+\frac{(N_F-1)}{N}\,\mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'g}\Big|_{N_F/N}$$
$$+(N_{F,\gamma}-1)\left(N-\frac{4}{N}\right)\mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'g}\Big|_{N_{F,\gamma}}\Bigg], \tag{3.91}$$

with

$$\mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'g}\Big|_{N_FN}=+\frac{1}{\epsilon^5}\left(-\frac{7}{54}\right)+\frac{1}{\epsilon^4}\left(-\frac{101}{108}\right)+\frac{1}{\epsilon^3}\left(-\frac{11651}{1944}+\frac{247}{648}\pi^2\right)$$
$$+\frac{1}{\epsilon^2}\left(-\frac{426175}{11664}+\frac{11227}{3888}\pi^2+\frac{493}{54}\zeta_3\right)$$
$$+\frac{1}{\epsilon}\left(-\frac{5169679}{23328}+\frac{437695}{23328}\pi^2+\frac{24599}{324}\zeta_3-\frac{3613}{25920}\pi^4\right)$$
$$-\frac{563396717}{419904}+\frac{15881671}{139968}\pi^2+\frac{328225}{648}\zeta_3-\frac{17281}{31104}\pi^4$$
$$-\frac{205}{8}\pi^2\zeta_3+\frac{13757}{90}\zeta_5+\mathcal{O}(\epsilon), \tag{3.92}$$

$$\mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'g}\Big|_{N_F/N}=+\frac{1}{\epsilon^5}\left(\frac{11}{108}\right)+\frac{1}{\epsilon^4}\left(\frac{425}{648}\right)+\frac{1}{\epsilon^3}\left(\frac{15821}{3888}-\frac{47}{144}\pi^2\right)$$
$$+\frac{1}{\epsilon^2}\left(\frac{573287}{23328}-\frac{1829}{864}\pi^2-\frac{979}{108}\zeta_3\right)$$
$$+\frac{1}{\epsilon}\left(\frac{20849357}{139968}-\frac{67769}{5184}\pi^2-\frac{38797}{648}\zeta_3+\frac{1777}{51840}\pi^4\right)$$

$$
\begin{aligned}
&+\frac{757887911}{839808}-\frac{2403131}{31104}\pi^2-\frac{1436257}{3888}\zeta_3+\frac{36571}{311040}\pi^4\\
&+\frac{1261}{48}\pi^2\zeta_3-\frac{34651}{180}\zeta_5+\mathcal{O}(\epsilon)\,,
\end{aligned}
\tag{3.93}
$$

$$
\begin{aligned}
\mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'g}\Big|_{N_{F,\gamma}} =& +\frac{1}{\epsilon}\left(\frac{7}{3}-\frac{\pi^2}{9}-\frac{\zeta_3}{2}-\frac{\pi^4}{135}\right)\\
&+\frac{977}{36}-\frac{299}{432}\pi^2-\frac{25}{6}\zeta_3-\frac{979}{12960}\pi^4+\frac{25}{18}\pi^2\zeta_3-\frac{151\zeta_5}{6}+\mathcal{O}(\epsilon)\,.
\end{aligned}
\tag{3.94}
$$

Finally, the colour decomposition for $\mathcal{T}^{(8)}_{q\bar{q}q\bar{q}g}$ reads:

$$
\begin{aligned}
\mathcal{T}^{(8)}_{q\bar{q}q\bar{q}g} =& \left(N-\frac{1}{N}\right)\mathcal{T}^{(2)}_{q\bar{q}}\left[N\,\mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'g}\Big|_{N_F N}+\frac{1}{N}\,\mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'g}\Big|_{N_F/N}\right.\\
&\left.+\left(N-\frac{4}{N}\right)\mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'g}\Big|_{N_{F,\gamma}}+\mathcal{T}^{(8)}_{q\bar{q}q\bar{q}g}\Big|_{N^0}+\frac{1}{N^2}\,\mathcal{T}^{(8)}_{q\bar{q}q\bar{q}g}\Big|_{1/N^2}\right],
\end{aligned}
\tag{3.95}
$$

and analogously to the four-parton final state case, the first three contributions in (3.95) are inherited from (3.91). The new terms are given by:

$$
\begin{aligned}
\mathcal{T}^{(8)}_{q\bar{q}q\bar{q}g}\Big|_{N^0} =& +\frac{1}{\epsilon^3}\left(\frac{65}{48}-\frac{5}{24}\pi^2+\frac{5}{6}\zeta_3\right)+\frac{1}{\epsilon^2}\left(\frac{47}{2}-\frac{253}{144}\pi^2-\frac{19}{2}\zeta_3+\frac{101}{1080}\pi^4\right)\\
&+\frac{1}{\epsilon}\left(\frac{48383}{192}-\frac{2761}{192}\pi^2-\frac{731}{8}\zeta_3-\frac{13}{240}\pi^4-\frac{19}{8}\pi^2\zeta_3+\frac{176}{3}\zeta_5\right)\\
&+\frac{411893}{192}-\frac{69037}{576}\pi^2-\frac{10409}{16}\zeta_3-\frac{14657}{8640}\pi^4\\
&+\frac{731}{24}\pi^2\zeta_3-\frac{1941}{8}\zeta_5+\frac{167}{1701}\pi^6-\frac{511}{12}\zeta_3^2+\mathcal{O}(\epsilon)\,,
\end{aligned}
\tag{3.96}
$$

$$
\begin{aligned}
\mathcal{T}^{(8)}_{q\bar{q}q\bar{q}g}\Big|_{1/N^2} =& +\frac{1}{\epsilon^3}\left(-\frac{65}{48}+\frac{5}{24}\pi^2-\frac{5}{6}\zeta_3\right)+\frac{1}{\epsilon^2}\left(-\frac{185}{8}+\frac{79}{48}\pi^2+9\zeta_3-\frac{7}{90}\pi^4\right)\\
&+\frac{1}{\epsilon}\left(-\frac{46055}{192}+\frac{7591}{576}\pi^2+\frac{473}{6}\zeta_3-\frac{\pi^4}{90}+\frac{27}{8}\pi^2\zeta_3-44\zeta_5\right)\\
&-\frac{379673}{192}+\frac{63601}{576}\pi^2+\frac{2185}{4}\zeta_3+\frac{133}{216}\pi^4\\
&-\frac{565}{24}\pi^2\zeta_3+\frac{1297}{8}\zeta_5+\frac{22229}{272160}\pi^6+\frac{345}{4}\zeta_3^2+\mathcal{O}(\epsilon)\,.
\end{aligned}
\tag{3.97}
$$

4 Comments on the results

4.1 Total cross section at $\mathcal{O}(\alpha_s^3)$

The natural check for our results is the complete cancellation of all infrared singularities in the total cross section at $\mathcal{O}(\alpha_s^3)$, that is, the sum of the layers (3.17), (3.18), (3.45), (3.46), (3.60), (3.68), (3.75), (3.80), (3.87), (3.91), (3.95). We achieve the cancellation of the poles and recover the $\mathrm{N^3LO}$ coefficient of the R-ratio [55]:

$$
\begin{aligned}
R\Big|_{\alpha_s^3} &= \left(\frac{\alpha_s}{2\pi}\right)^3 \frac{\mathcal{T}^{(8)}_{q\bar{q}} + \mathcal{T}^{(8)}_{q\bar{q}g} + \mathcal{T}^{(8)}_{ggg} + \mathcal{T}^{(8)}_{q\bar{q}gg} + \mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'} + \mathcal{T}^{(8)}_{q\bar{q}q\bar{q}} + \mathcal{T}^{(8)}_{q\bar{q}ggg} + \mathcal{T}^{(8)}_{q\bar{q}q'\bar{q}'g} + \mathcal{T}^{(8)}_{q\bar{q}q\bar{q}g}}{\mathcal{T}^{(2)}_{q\bar{q}}} \\
&= \left(\frac{\alpha_s}{2\pi}\right)^3 \left(N - \frac{1}{N}\right) \Bigg[N^2 \left(\frac{346201}{3456} - \frac{121}{144}\pi^2 - \frac{6761}{72}\zeta_3 + \frac{55}{3}\zeta_5\right) \\
&\qquad + \frac{323}{64} + \frac{143}{8}\zeta_3 - \frac{55}{2}\zeta_5 + \frac{1}{N^2}\left(-\frac{69}{128}\right) \\
&\qquad + N_F N \left(-\frac{62863}{1728} + \frac{11}{36}\pi^2 + \frac{1067}{36}\zeta_3 - \frac{10}{3}\zeta_5\right) \\
&\qquad + \frac{N_F}{N}\left(\frac{29}{64} + 5\zeta_5 - \frac{19}{4}\zeta_3\right) + N_F^2\left(\frac{151}{54} - \frac{1}{36}\pi^2 - \frac{19}{9}\zeta_3\right) \\
&\qquad + N_{F,\gamma}\left(N - \frac{4}{N}\right)\left(\frac{11}{24} - \zeta_3\right)\Bigg]. \qquad (4.1)
\end{aligned}
$$

4.2 Singlet contribution

The singlet contribution proportional to $(\sum_q e_q)^2$ appears for the first time at three-loop order. In the two-parton final state it arises from the three-loop quark form factor and is finite due to the absence of counterterms for the associated diagrams [51]. Analogously, the singlet term is also finite in the three-parton final state case. Due to Furry's theorem, neither the $\gamma^* \to q\bar{q}g$ nor the $\gamma^* \to ggg$ sub-processes allow for lower-loop or lower-multiplicity counterterms which could accommodate infrared singularities.

The four- and five-parton final states exhibit a ϵ^{-1} pole in the singlet term. The pole is due to a real or virtual infrared gluon and cancels in the sum of the two contributions. Such single pole proportional to the quartic Casimir is present also in the one-loop triple collinear splitting function [30]. The real emission counterpart of this singularity is found in the antisymmetric tripole contribution to the soft current for a soft gluon-quark-antiquark emission [38, 39]. The absence of a ϵ^{-2} pole is explained by the fact that the singlet part of the tree-level matrix element squared for $\gamma^* \to q\bar{q}q'\bar{q}'$ vanishes [10], $\mathcal{T}^{(6)}_{q\bar{q}q'\bar{q}'}\Big|_{N_{F,\gamma}} = 0$, as also noted in [30].

5 Conclusions and outlook

In this paper, we presented analytic results for the integration over the respective inclusive phase space of all the contributions to e^+e^- annihilation to hadrons at order $\mathcal{O}(\alpha_s^3)$. A common strategy was used for all layers of the calculation, exploiting reverse unitarity relations to compute phase space integrals. The cancellation of infrared singularities and the recovery of the known finite result at $\mathrm{N^3LO}$ provides strong checks on our calculations. We provide expressions for the results in section 3 and in appendix B in FORM format in the supplementary material.

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It will be interesting to analyze the infrared structure of our result in more detail. The poles of the purely virtual $\mathcal{O}(\alpha_s^3)$ two-parton corrections can be predicted by means of universal infrared factorization formulae [56, 57]. Some infrared singular terms in the three-, four- and five-parton final states are related to individual terms in the infrared factorization formula. Other infrared terms can be obtained from the integration of known ingredients [25–39], and they should cancel between the three-, four- and five-parton final states. We leave this investigation to future work.

Our results represent the first step towards a future extension of the antenna subtraction method at N^{3}LO. In particular, from the analytic expressions obtained here, it is possible to directly read off the integrated form of N^{3}LO quark-antiquark antenna functions, in the configuration where the hard quark-antiquark pair is emitted in the final state. We envisage future work for the calculation of analogous phase space integrals in the gluon-gluon and quark-gluon case.

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A Renormalisation of amplitudes

Expanding α_s according to (3.1), the renormalised amplitudes are

$$|\mathcal{M}^{(1)}\rangle_{q\bar{q}} = |\mathcal{M}^{(1),U}\rangle_{q\bar{q}}\,, \tag{A.1}$$

$$|\mathcal{M}^{(2)}\rangle_{q\bar{q}} = |\mathcal{M}^{(2),U}\rangle_{q\bar{q}} - \frac{\beta_0}{\epsilon}|\mathcal{M}^{(1),U}\rangle_{q\bar{q}}\,, \tag{A.2}$$

$$|\mathcal{M}^{(3)}\rangle_{q\bar{q}} = |\mathcal{M}^{(3),U}\rangle_{q\bar{q}} - \frac{2\beta_0}{\epsilon}|\mathcal{M}^{(2),U}\rangle_{q\bar{q}} + \left(\frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{2\epsilon}\right)|\mathcal{M}^{(1),U}\rangle_{q\bar{q}}\,, \tag{A.3}$$

$$|\mathcal{M}^{(1)}\rangle_{q\bar{q}g} = |\mathcal{M}^{(1),U}\rangle_{q\bar{q}g} - \frac{\beta_0}{2\epsilon}|\mathcal{M}^{(0)}\rangle_{q\bar{q}g}\,, \tag{A.4}$$

$$|\mathcal{M}^{(2)}\rangle_{q\bar{q}g} = |\mathcal{M}^{(2),U}\rangle_{q\bar{q}g} - \frac{3\beta_0}{2\epsilon}|\mathcal{M}^{(1),U}\rangle_{q\bar{q}g} + \left(\frac{3}{8}\frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{4\epsilon}\right)|\mathcal{M}^{(0)}\rangle_{q\bar{q}g}\,, \tag{A.5}$$

$$|\mathcal{M}^{(1)}\rangle_{q\bar{q}ij} = |\mathcal{M}^{(1),U}\rangle_{q\bar{q}ij} - \frac{\beta_0}{\epsilon}|\mathcal{M}^{(0)}\rangle_{q\bar{q}ij}\,, \tag{A.6}$$

where the superscript U denotes unrenormalised quantities.

B NNLO results

Here we summarize the analytic results up to order $\mathcal{O}(\alpha_s^2)$ [10], extended to weight 6. For the two-particle final state:

$$\mathcal{T}_{q\bar{q}}^{(2)} = \int \mathrm{d}\Phi_2 \, \langle \mathcal{M}^{(0)} | \mathcal{M}^{(0)} \rangle_{q\bar{q}} = 4N(1-\epsilon) q^2 P_2 \,, \tag{B.1}$$

$$\begin{aligned} \mathcal{T}_{q\bar{q}}^{(4)} &= \int \mathrm{d}\Phi_2 \, 2\,\mathrm{Re}\big[\langle \mathcal{M}^{(1)} | \mathcal{M}^{(0)} \rangle_{q\bar{q}} \big] \\ &= \left(N - \frac{1}{N}\right) \mathcal{T}_{q\bar{q}}^{(2)} \Bigg[-\frac{1}{\epsilon^2} - \frac{3}{2\epsilon} - 4 + \frac{7\pi^2}{12} + \left(-8 + \frac{7\pi^2}{8} + \frac{7}{3}\zeta_3\right)\epsilon \\ &\quad + \left(-16 + \frac{7\pi^2}{3} + \frac{7}{2}\zeta_3 - \frac{73\pi^4}{1440}\right)\epsilon^2 \\ &\quad + \epsilon^3\Bigg(-32 + \frac{14\pi^2}{3} + \frac{28\zeta_3}{3} - \frac{73\pi^4}{960} - \frac{49\zeta_3\pi^2}{36} + \frac{31\zeta_5}{5} \Bigg) \\ &\quad + \epsilon^4\Bigg(-64 + \frac{28\pi^2}{3} + \frac{56\zeta_3}{3} - \frac{73\pi^4}{360} - \frac{49\zeta_3\pi^2}{24} \\ &\qquad + \frac{93\zeta_5}{10} - \frac{437\pi^6}{120960} - \frac{49\zeta_3^2}{18} \Bigg) + \mathcal{O}(\epsilon^5) \Bigg] \,, \end{aligned} \tag{B.2}$$

$$\begin{aligned} \mathcal{T}_{q\bar{q}}^{(6,[2\times 0])} &= \int \mathrm{d}\Phi_2 \, 2\,\mathrm{Re}\big[\langle \mathcal{M}^{(2)} | \mathcal{M}^{(0)} \rangle_{q\bar{q}} \big] \\ &= \left(N - \frac{1}{N}\right) \mathcal{T}_{q\bar{q}}^{(2)} \Bigg\{ N\, \mathcal{T}_{q\bar{q}}^{(6,[2\times 0])}\Big|_N + \frac{1}{N}\, \mathcal{T}_{q\bar{q}}^{(6,[2\times 0])}\Big|_{1/N} \\ &\qquad + N_F \, \mathcal{T}_{q\bar{q}}^{(6,[2\times 0])}\Big|_{N_F} \Bigg\} \,, \end{aligned} \tag{B.3}$$

with

$$\begin{aligned} \mathcal{T}_{q\bar{q}}^{(6,[2\times 0])}\Big|_N &= \frac{1}{4\epsilon^4} + \frac{17}{8\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{433}{144} - \frac{\pi^2}{2}\right) \\ &\quad + \frac{1}{\epsilon}\left(\frac{4045}{864} - \frac{83\pi^2}{48} + \frac{7}{12}\zeta_3\right) + \left(-\frac{9083}{5184} - \frac{2153\pi^2}{864} + \frac{13}{9}\zeta_3 + \frac{263\pi^4}{1440}\right) \\ &\quad + \epsilon\left(-\frac{1244339}{31104} - \frac{1943\pi^2}{5184} + \frac{4235\zeta_3}{216} + \frac{389\pi^4}{720} - \frac{13\zeta_3\pi^2}{8} + \frac{163\zeta_5}{20} \right) \\ &\quad + \epsilon^2\Bigg(-\frac{36528395}{186624} + \frac{611833\pi^2}{31104} + \frac{109019\zeta_3}{1296} + \frac{38519\pi^4}{25920} - \frac{2087\zeta_3\pi^2}{216} \\ &\qquad + \frac{529\zeta_5}{15} - \frac{631\pi^6}{15120} - \frac{403\zeta_3^2}{36} \Bigg) + \mathcal{O}(\epsilon^3) \,, \end{aligned} \tag{B.4}$$

$$
\begin{aligned}
\mathcal{T}_{q\bar{q}}^{(6,[2\times 0])}\Big|_{1/N} &= -\frac{1}{4\epsilon^4} - \frac{3}{4\epsilon^3} + \frac{1}{\epsilon^2}\left(-\frac{41}{16} + \frac{13\pi^2}{24}\right) \\
&+ \frac{1}{\epsilon}\left(-\frac{221}{32} + \frac{3\pi^2}{2} + \frac{8}{3}\zeta_3\right) + \left(-\frac{1151}{64} + \frac{475\pi^2}{96} + \frac{29}{4}\zeta_3 - \frac{59\pi^4}{288}\right) \\
&+ \epsilon\left(-\frac{5741}{128} + \frac{813\pi^2}{64} + \frac{839\zeta_3}{24} - \frac{61\pi^4}{160} + \frac{23\zeta_5}{5} - \frac{55\zeta_3\pi^2}{9}\right) \\
&+ \epsilon^2\left(-\frac{27911}{256} + \frac{3991\pi^2}{128} + \frac{6989\zeta_3}{48} - \frac{4399\pi^4}{5760} - \frac{125\zeta_3\pi^2}{8}\right. \\
&\qquad \left. + \frac{231\zeta_5}{20} - \frac{571\pi^6}{8640} - \frac{326\zeta_3^2}{9}\right) + \mathcal{O}(\epsilon^3)\,,
\end{aligned}
\tag{B.5}
$$

$$
\begin{aligned}
\mathcal{T}_{q\bar{q}}^{(6,[2\times 0])}\Big|_{N_F} &= -\frac{1}{4\epsilon^3} - \frac{1}{9\epsilon^2} + \frac{1}{\epsilon}\left(\frac{65}{216} + \frac{\pi^2}{24}\right) + \left(\frac{4085}{1296} - \frac{91\pi^2}{216} + \frac{1}{18}\zeta_3\right) \\
&+ \epsilon\left(+\frac{108653}{7776} - \frac{2875\pi^2}{1296} - \frac{119\zeta_3}{54} + \frac{\pi^4}{720}\right) \\
&+ \epsilon^2\left(+\frac{2379989}{46656} - \frac{70855\pi^2}{7776} - \frac{3581\zeta_3}{324} + \frac{311\pi^4}{5184}\right. \\
&\qquad \left. + \frac{47\zeta_3\pi^2}{54} - \frac{59\zeta_5}{30}\right) + \mathcal{O}(\epsilon^3)\,,
\end{aligned}
\tag{B.6}
$$

and

$$
\begin{aligned}
\mathcal{T}_{q\bar{q}}^{(6,[1\times 1])} &= \int \mathrm{d}\Phi_2\, \langle \mathcal{M}^{(1)} | \mathcal{M}^{(1)} \rangle_{q\bar{q}} \\
&= \mathcal{T}_{q\bar{q}}^{(2)} \left(N - \frac{1}{N}\right)^2 \left[\frac{1}{4\epsilon^4} + \frac{3}{4\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{41}{16} - \frac{\pi^2}{24}\right)\right. \\
&+ \frac{1}{\epsilon}\left(7 - \frac{\pi^2}{8} - \frac{7}{6}\zeta_3\right) + \left(18 - \frac{41\pi^2}{96} - \frac{7}{2}\zeta_3 - \frac{7\pi^4}{480}\right) \\
&+ \epsilon\left(+44 - \frac{7\pi^2}{6} - \frac{287\zeta_3}{24} - \frac{7\pi^4}{160} + \frac{7\zeta_3\pi^2}{36} - \frac{31\zeta_5}{10}\right) \\
&+ \epsilon^2\left(+104 - 3\pi^2 - \frac{98\zeta_3}{3} - \frac{287\pi^4}{1920} + \frac{7\zeta_3\pi^2}{12}\right. \\
&\qquad \left.\left. - \frac{93\zeta_5}{10} - \frac{31\pi^6}{12096} + \frac{49\zeta_3^2}{18}\right) + \mathcal{O}(\epsilon)\right].
\end{aligned}
\tag{B.7}
$$

We notice a typo in (4.10) of [10]: the $1/\epsilon$ coefficient in the previous equation features a $-7/6\,\zeta_3$ instead of $+7/6\,\zeta_3$.

For the three-particle final state:

$$
\begin{aligned}
\mathcal{T}^{(4)}_{q\bar{q}g} &= 8\pi^2 \int \mathrm{d}\Phi_3 \langle \mathcal{M}^{(0)} | \mathcal{M}^{(0)} \rangle_{q\bar{q}g} \\
&= \left(N - \frac{1}{N}\right) \mathcal{T}^{(2)}_{q\bar{q}} \left[\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} + \frac{19}{4} - \frac{7\pi^2}{12} + \epsilon \left(\frac{109}{8} - \frac{7\pi^2}{8} - \frac{25}{3}\zeta_3 \right) \right. \\
&\quad + \epsilon^2 \left(\frac{639}{16} - \frac{133\pi^2}{48} - \frac{25}{2}\zeta_3 - \frac{71\pi^4}{1440} \right) \\
&\quad + \epsilon^3 \left(\frac{3789}{32} - \frac{763\pi^2}{96} - \frac{475\zeta_3}{12} - \frac{71\pi^4}{960} + \frac{175\pi^2\zeta_3}{36} - \frac{241\zeta_5}{5} \right) \\
&\quad + \epsilon^4 \left(\frac{22599}{64} - \frac{1491\pi^2}{64} - \frac{2725\zeta_3}{24} - \frac{1349\pi^4}{5760} + \frac{175\pi^2\zeta_3}{24} \right. \\
&\qquad \left. \left. - \frac{723\zeta_5}{10} - \frac{4027\pi^6}{120960} + \frac{625\zeta_3^2}{18} \right) + \mathcal{O}(\epsilon^3) \right],
\end{aligned} \tag{B.8}
$$

$$
\begin{aligned}
\mathcal{T}^{(6)}_{q\bar{q}g} &= 8\pi^2 \int \mathrm{d}\Phi_3 2\,\mathrm{Re}[\langle \mathcal{M}^{(1)} | \mathcal{M}^{(0)} \rangle_{q\bar{q}g}] \\
&= \left(N - \frac{1}{N}\right) \mathcal{T}^{(2)}_{q\bar{q}} \left\{ N\, \mathcal{T}^{6}_{q\bar{q}g}\Big|_N + \frac{1}{N} \mathcal{T}^{6}_{q\bar{q}g}\Big|_{1/N} + N_F\, \mathcal{T}^{6}_{q\bar{q}g}\Big|_{N_F} \right\},
\end{aligned} \tag{B.9}
$$

with

$$
\begin{aligned}
\mathcal{T}^{(6)}_{q\bar{q}g}\Big|_N &= -\frac{5}{4\epsilon^4} - \frac{67}{12\epsilon^3} + \frac{1}{\epsilon^2}\left(-\frac{141}{8} + \frac{13\pi^2}{8} \right) \\
&\quad + \frac{1}{\epsilon}\left(-\frac{1481}{24} + \frac{107\pi^2}{18} + \frac{55}{3}\zeta_3 \right) + \left(-\frac{10385}{48} + \frac{64\pi^2}{3} + \frac{1265}{18}\zeta_3 - \frac{41\pi^4}{96} \right) \\
&\quad + \epsilon \left(-\frac{24915}{32} + \frac{22223\pi^2}{288} + \frac{1517\zeta_3}{6} - \frac{10289\pi^4}{8640} - \frac{149\pi^2\zeta_3}{6} + 143\zeta_5 \right) \\
&\quad + \epsilon^2 \left(-\frac{183957}{64} + \frac{162077\pi^2}{576} + \frac{68069\zeta_3}{72} - \frac{28493\pi^4}{5760} - \frac{18017\pi^2\zeta_3}{216} \right. \\
&\qquad \left. + \frac{15521\zeta_5}{30} + \frac{5357\pi^6}{60480} - \frac{1345\zeta_3^2}{9} \right) + \mathcal{O}(\epsilon^3),
\end{aligned} \tag{B.10}
$$

$$
\begin{aligned}
\mathcal{T}^{(6)}_{q\bar{q}g}\Big|_{1/N} &= \frac{1}{\epsilon^4} + \frac{3}{\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{93}{8} - \frac{4\pi^2}{3} \right) \\
&\quad + \frac{1}{\epsilon}\left(\frac{79}{2} - \frac{15\pi^2}{4} - \frac{53}{3}\zeta_3 \right) + \left(\frac{1069}{8} - \frac{697\pi^2}{48} - \frac{91}{2}\zeta_3 + \frac{19\pi^4}{72} \right)
\end{aligned}
$$

$$+\epsilon\left(\frac{3653}{8}-50\pi^2-\frac{723\zeta_3}{4}+\frac{19\pi^4}{24}+\frac{493\pi^2\zeta_3}{18}-\frac{897\zeta_5}{5}\right)$$
$$+\epsilon^2\left(\frac{6375}{4}-\frac{8297\pi^2}{48}-\frac{3911\zeta_3}{6}+\frac{8303\pi^4}{2880}+\frac{273\pi^2\zeta_3}{4}\right.$$
$$\left.-\frac{4257\zeta_5}{10}-\frac{827\pi^6}{11340}+\frac{1931\zeta_3^2}{9}\right)+\mathcal{O}(\epsilon^3)\,, \tag{B.11}$$

$$\mathcal{T}^{(6)}_{q\bar{q}g}\Big|_{N_F}=\frac{1}{3\epsilon^3}+\frac{1}{2\epsilon^2}+\frac{1}{\epsilon}\left(\frac{19}{12}-\frac{7\pi^2}{36}\right)+\left(\frac{109}{24}-\frac{7\pi^2}{24}-\frac{25}{9}\zeta_3\right)$$
$$+\epsilon\left(\frac{213}{16}-\frac{133\pi^2}{144}-\frac{25\zeta_3}{6}-\frac{71\pi^4}{4320}\right)$$
$$+\epsilon^2\left(\frac{1263}{32}-\frac{763\pi^2}{288}-\frac{475\zeta_3}{36}-\frac{71\pi^4}{2880}+\frac{175\pi^2\zeta_3}{108}-\frac{241\zeta_5}{15}\right)+\mathcal{O}(\epsilon^3)\,. \tag{B.12}$$

We notice a typo in (4.34) of [10]: the $1/\epsilon$ coefficient in the previous equation features a $19/12$ instead of $19/2$. This typo was fixed in [40].

For the four-particle final state:

$$\mathcal{T}^{(6)}_{q\bar{q}gg}=64\pi^4\int \mathrm{d}\Phi_4\langle\mathcal{M}^{(0)}|\mathcal{M}^{(0)}\rangle_{q\bar{q}gg}$$
$$=\left(N-\frac{1}{N}\right)\mathcal{T}^{(2)}_{q\bar{q}}\left\{N\,\mathcal{T}^{(6)}_{q\bar{q}gg}\Big|_N+\frac{1}{N}\mathcal{T}^{(6)}_{q\bar{q}gg}\Big|_{1/N}\right\}, \tag{B.13}$$

with

$$\mathcal{T}^{(6)}_{q\bar{q}gg}\Big|_N=\frac{3}{4\epsilon^4}+\frac{65}{24\epsilon^3}+\frac{1}{\epsilon^2}\left(\frac{217}{18}-\frac{13\pi^2}{12}\right)$$
$$+\frac{1}{\epsilon}\left(\frac{43223}{864}-\frac{589\pi^2}{144}-\frac{71}{4}\zeta_3\right)+\left(\frac{1076717}{5184}-\frac{7955\pi^2}{432}-\frac{1327}{18}\zeta_3+\frac{373\pi^4}{1440}\right)$$
$$+\epsilon\left(\frac{26964431}{31104}-\frac{398557\pi^2}{5184}-\frac{73301\zeta_3}{216}+\frac{5207\pi^4}{8640}+\frac{1891\pi^2\zeta_3}{72}-\frac{2661\zeta_5}{20}\right)$$
$$+\epsilon^2\left(\frac{677415461}{186624}-\frac{9919003\pi^2}{31104}-\frac{1857845\zeta_3}{1296}+\frac{2975\pi^4}{1296}\right.$$
$$\left.+\frac{11969\pi^2\zeta_3}{108}-\frac{4211\zeta_5}{6}-\frac{139\pi^6}{30240}+\frac{2723\zeta_3^2}{12}\right)+\mathcal{O}(\epsilon^3)\,, \tag{B.14}$$

$$\mathcal{T}^{(6)}_{q\bar{q}gg}\Big|_{1/N}=-\frac{1}{2\epsilon^4}-\frac{3}{2\epsilon^3}+\frac{1}{\epsilon^2}\left(-\frac{13}{2}+\frac{3\pi^2}{4}\right)+\frac{1}{\epsilon}\left(-\frac{845}{32}+\frac{9\pi^2}{4}+\frac{40}{3}\zeta_3\right)$$
$$+\left(-\frac{6921}{64}+\frac{473\pi^2}{48}+40\zeta_3-\frac{17\pi^4}{144}\right)$$

$$+\epsilon\left(-\frac{57305}{128}+\frac{7735\pi^2}{192}+\frac{4295\zeta_3}{24}-\frac{17\pi^4}{48}-\frac{121\pi^2\zeta_3}{6}+\frac{636\zeta_5}{5}\right)$$

$$+\epsilon^2\bigg(-\frac{477601}{256}+\frac{21125\pi^2}{128}+\frac{35555\zeta_3}{48}-\frac{107\pi^4}{80}-\frac{121\pi^2\zeta_3}{2}$$

$$+\frac{1908\zeta_5}{5}+\frac{4763\pi^6}{90720}-\frac{3281\zeta_3^2}{18}\bigg)+\mathcal{O}(\epsilon^3)\,, \tag{B.15}$$

$$\mathcal{T}^{(6)}_{q\bar{q}q'\bar{q}'}=64\pi^4\int \mathrm{d}\Phi_4\langle\mathcal{M}^{(0)}|\mathcal{M}^{(0)}\rangle_{q\bar{q}q'\bar{q}'}$$

$$=\mathcal{T}^{(2)}_{q\bar{q}}\left(N-\frac{1}{N}\right)(N_F-1)\bigg[-\frac{1}{12\epsilon^3}-\frac{7}{18\epsilon^2}+\frac{1}{\epsilon}\left(-\frac{407}{216}+\frac{11\pi^2}{72}\right)$$

$$+\left(-\frac{11753}{1296}+\frac{77\pi^2}{108}+\frac{67}{18}\zeta_3\right)+\epsilon\left(-\frac{340475}{7776}+\frac{4369\pi^2}{1296}+\frac{469\zeta_3}{27}+\frac{137\pi^4}{4320}\right)$$

$$+\epsilon^2\bigg(-\frac{9739325}{46656}+\frac{120859\pi^2}{7776}+\frac{25811\zeta_3}{324}+\frac{959\pi^4}{6480}$$

$$-\frac{629\pi^2\zeta_3}{108}+\frac{1651\zeta_5}{30}\bigg)+\mathcal{O}(\epsilon^3)\bigg], \tag{B.16}$$

and

$$\mathcal{T}^{(6)}_{q\bar{q}q\bar{q}}=64\pi^4\int \mathrm{d}\Phi_4\langle\mathcal{M}^{(0)}|\mathcal{M}^{(0)}\rangle_{q\bar{q}q\bar{q}}$$

$$=\frac{1}{N_F-1}\mathcal{T}^{(6)}_{q\bar{q}q'\bar{q}'}+\left(N-\frac{1}{N}\right)\mathcal{T}^{(2)}_{q\bar{q}}\,\frac{1}{N}\bigg[\frac{1}{\epsilon}\left(\frac{13}{16}-\frac{\pi^2}{8}+\frac{1}{2}\zeta_3\right)$$

$$+\left(\frac{339}{32}-\frac{17\pi^2}{24}-\frac{21}{4}\zeta_3+\frac{2\pi^4}{45}\right)$$

$$+\epsilon\left(\frac{5391}{64}-\frac{133\pi^2}{32}-\frac{125\zeta_3}{4}-\frac{\pi^4}{10}-\frac{11\pi^2\zeta_3}{12}+22\zeta_5\right)$$

$$+\epsilon^2\bigg(\frac{68123}{128}-\frac{5089\pi^2}{192}-\frac{3649\zeta_3}{24}-\frac{7\pi^4}{10}+\frac{203\pi^2\zeta_3}{24}$$

$$-\frac{357\zeta_5}{4}+\frac{277\pi^6}{11340}-\frac{95\zeta_3^2}{6}\bigg)+\mathcal{O}(\epsilon^3)\bigg]. \tag{B.17}$$

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