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Asymmetric dynamic plastic response of stepped plates

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Abstract. The dynamic plastic response of circular plates to asymmetric loading is studied. An approximate theoretical procedure is developed for the evaluation of asymmetrical residual deflections. The solution technique is based on the equality of the internal dissipation and the external work, respectively. Maximal residual deflections are defined for plates of piece-wise constant thickness.

1. Introduction

The investigation on the dynamic plastic behavior of circular plates subjected to the impulsive and impact loadings got its start with papers by Hopkins and Prager [3] also by Wang and Hopkins [15]. In [3, 15] the stress-strain state of a circular plate was established in the case of material obeying the Tresca yield condition and the associated flow law. Later these solutions were extended to various external loadings, provided the behaviour of the plate remains axi-symmetric (see Jones [4], Kaliszky [5], Nurick and Martin [12], Stronge [14], Wierzbicki and Jones [16], Capurso [2], Lepik and Mróz [11], Kaliszky and Lógó [6, 7]).

An approximate method for determination of residual deflections of circular plates made of an ideal rigid-plastic material was developed by Lellep and Mürk [8, 9, 10] for plates subjected to asymmetrically distributed loadings.

In previous papers the case of impulsive loading [8], the rectangular pressure pulse [9] and the exponentially decaying pressure [10] were studied. In the current paper it is assumed that the plate is subjected to a concentrated force with the decaying magnitude of general form, provided it is applied off the centre of the plate.

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2. Problem formulation and internal dissipation

Let us study the residual deflections of a stepped circular plate of radius R subjected to the asymmetric loading. Following the previous paper [10] it is assumed that the plate is loaded by the concentrated force P applied at the point O_1 of the plate while

$$P(t) = \begin{cases} p\,g(t), & t \in [0, t_1], \\ 0, & t > t_1. \end{cases} \tag{1}$$

Here t stands for time and t_1 is a positive number whereas $g(t)$ is a continuous function of time t and p is a given constant. For the sake of simplicity we shall study the case where t_1 is relatively small in a greater detail.

Let the thickness of the plate be denoted by $h = h(r, \theta)$ where r and θ stand for polar coordinates. It is assumed that the thickness of the plate is piece-wise constant. Thus for $r \in (r_j, r_{j+1})$

$$h = h_j(\theta), \quad j = 1, \ldots, n, \tag{2}$$

and $h_j(\theta) = const$; $0 \leq \theta \leq 2\pi$. The equations

$$r = r_j(\theta), \tag{3}$$

where r_j $(j = 1, \ldots, n+1)$ define the boundaries of closed regions where $r = const$. Evidently, at the boundary of the plate $r = r_{n+1}(\theta)$, where

$$r_{n+1}(\theta) = a\cos\theta + \sqrt{R^2 - a^2\sin^2\theta}. \tag{4}$$

Here a is the distance between O_1 and the centre of the plate, and $r_{n+1} \leq R - a$.

The previous works in the analysis of rigid-plastic circular plates subjected to the dynamic loads have shown that the plates take the form of a cone if the load belongs to the class of moderate loads (see Jones [4], Kaliszky [5], Skrzypek and Hetnarski [13], Chakrabarty [1]). Thus, the transverse velocity field can be presented as (see Figure 1)

$$\dot{W}(r, \theta, t) = \dot{W}_0(t)\left(1 - \frac{r}{r_{n+1}(\theta)}\right), \tag{5}$$

provided the origin of coordinates is located at the point O_1.

In (5), W stands for the vertical displacement and dot denotes the differentiation with respect to time t; thus

$$\dot{W}_0 = \frac{dW_0}{dt}. \tag{6}$$

It is also useful to denote the differentiation with respect to θ by prime; for instance

$$r_j{}' = \frac{dr_j}{d\theta}. \tag{7}$$

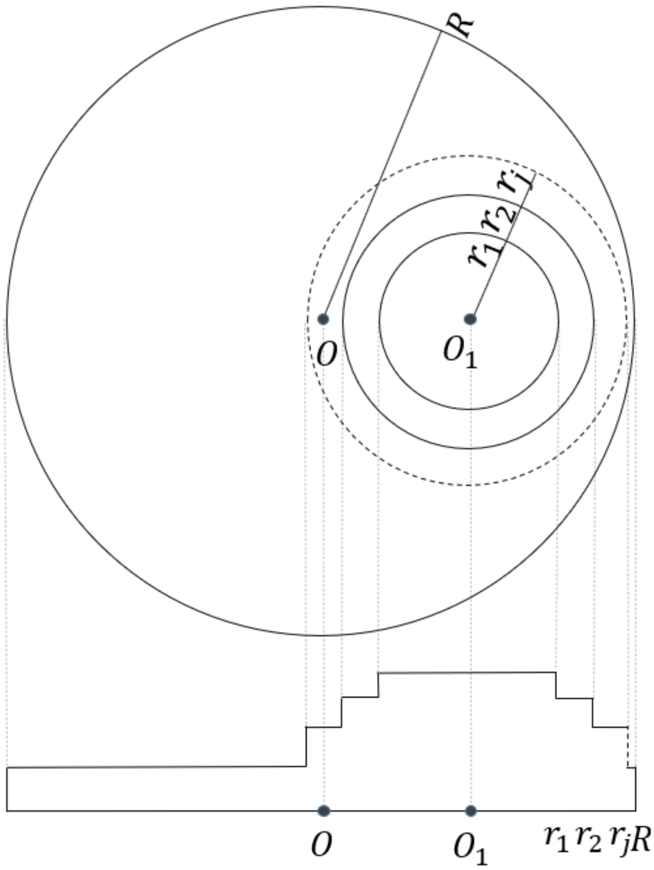

FIGURE 1. Geometry of the plate.

It was shown in the earlier papers (see Lellep and Mürk [8, 9, 10]) that the internal energy dissipation corresponding to the velocity field (5) can be calculated as

$$\dot{D}_i = \dot{W}_0(t) \sum_{j=0}^{n} M_{0j} \int_{r_j}^{r_{j+1}} dr \int_{0}^{2\pi} \frac{1}{r_{n+1}} \left(1 + \frac{(r'_{n+1})^2}{r_{n+1}^2} \right) d\theta. \tag{8}$$

Here

$$M_{0j} = \frac{\sigma_0}{4} h_j^2, \tag{9}$$

where σ_0 is the yield stress of the material.

3. Residual deflections of the plate

The power of the external forces including the concentrated force (1) and the inertial forces due to the acceleration

$$\ddot{W} = \ddot{W}_0(t)\left(1 - \frac{r}{r_{n+1}}\right), \tag{10}$$

can be calculated as (see Jones [4], Lellep and Mürk [9, 10])

$$\begin{aligned} \dot{D}_e = {} & P_0 \dot{W}_0 g(t) - \mu \dot{W}_0 \ddot{W}_0 \sum_{j=0}^{n} \int_0^{2\pi} d\theta \int_{r_j}^{r_{j+1}} h_j \left(1 - \frac{(r'_{n+1})^2}{r_{n+1}^2}\right)^2 r dr \\ & - \int_0^{2\pi} \frac{M_{0n}}{r_{n+1}} d\theta \end{aligned} \tag{11}$$

for $0 < t \leq t_1$, where μ stands for the density of the material of the plate. Taking $\dot{D}_i = \dot{A}_e$ one obtains for $0 \leq t \leq t_1$ that

$$\ddot{W}_0 = \frac{1}{A}\left(\dot{P}_0 g(t) - B\right), \tag{12}$$

where

$$A = \mu \sum_{j=0}^{n} \int_0^{2\pi} d\theta \int_{r_j}^{r_{j+1}} h_j \left(1 - \frac{r}{r_{n+1}}\right)^2 r dr \tag{13}$$

and

$$B = \int_0^{2\pi} \frac{M_{0n}}{r_{n+1}} d\theta - \sum_{j=0}^{n} M_{0j} \int_{r_j}^{r_{j+1}} d\theta \int_0^{2\pi} \left(\frac{1}{r_{n+1}} + \frac{(r'_{n+1})^2}{r_{n+1}^3}\right) dr. \tag{14}$$

It is worthwhile to mention that the relations (10)–(14) are applicable for both, simply supported and clamped plates. However, in the case of simply supported plates the last term in (11) and the first term in (14) must be omitted. Thus, in the case of simply supported plates one has to take $M_{0n} = 0$ in (11), (14) for determination of accelerations during the first stage of motion. For the second stage of motion the acceleration

$$\ddot{W}_1 = -\frac{B}{A}. \tag{15}$$

From (15) after integration one easily obtains

$$\dot{W}_1 = -\frac{B}{A}(t - t_1) + \dot{W}_0(t_1) \tag{16}$$

and

$$W_1 = -\frac{B}{2A}(t - t_1)^2 + \dot{W}_0(t_1)(t - t_1) + W_0(t_1). \tag{17}$$

The motion of the plate ceases at the moment $t = t_2$, where

$$t_2 = t_1 + \frac{A}{B}\dot{W}_0(t_1). \tag{18}$$

According to (15)–(18) the maximal residual deflection

$$W_1(t_2) = W_0(t_1)\left(1 + \frac{A}{2B} W_0(t_1)\right). \tag{19}$$

The quantities $\dot{W}_0(t_1)$ and $W_0(t_1)$ in (16)–(19) can be determined after integration of (12) making use of the initial conditions $W_0(0) = 0, \dot{W}_0(0) = 0$.

4. Numerical results

The results of calculations are presented for simply supported and clamped plates in Figures 2–8, whereas $g(t) = \cos \beta t$. In Figures 2–6 the maximal residual deflections are plotted versus the load intensity, provided the plate is exerted to the rectangular pressure loading of intensity p and of the duration t_1. Figure 2 corresponds to the plate of constant thickness, for $\beta = 0.2$, $a = 0.7R$. It can be seen from Figure 2 that the residual deflection increases

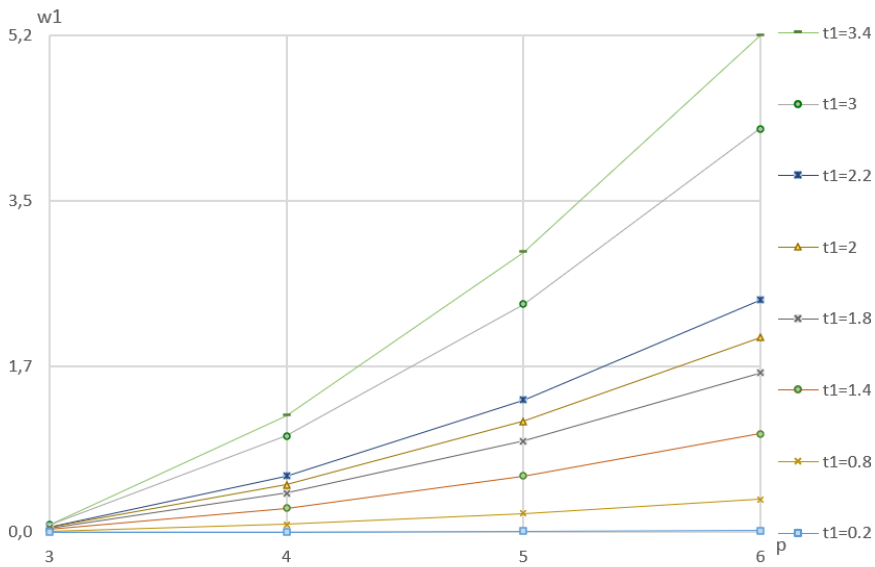

FIGURE 2. Maximal permanent deflections of the plate of constant thickness.

if the load intensity p increases. The residual deflections at the final instant t_2 are calculated numerically after the determination of quantities A and B. It is assumed that

$$h = \begin{cases} h_1, & (x, y) \in D_0, \\ h_2, & (x, y) \notin D_0, \end{cases}$$

where D_0 is a circle with radius $r_1 < R - a$ and the centre at the point O_1. Figure 3 is associated with the simply supported plate and Figure 4 corresponds to the fully clamped plate, for $\beta = 0.2$, $a = 0.7R$, $r_1 = 0.2R$

and $h_2 = 0.8h_1$. It can be seen from Figure 2 and Figure 3 that the residual

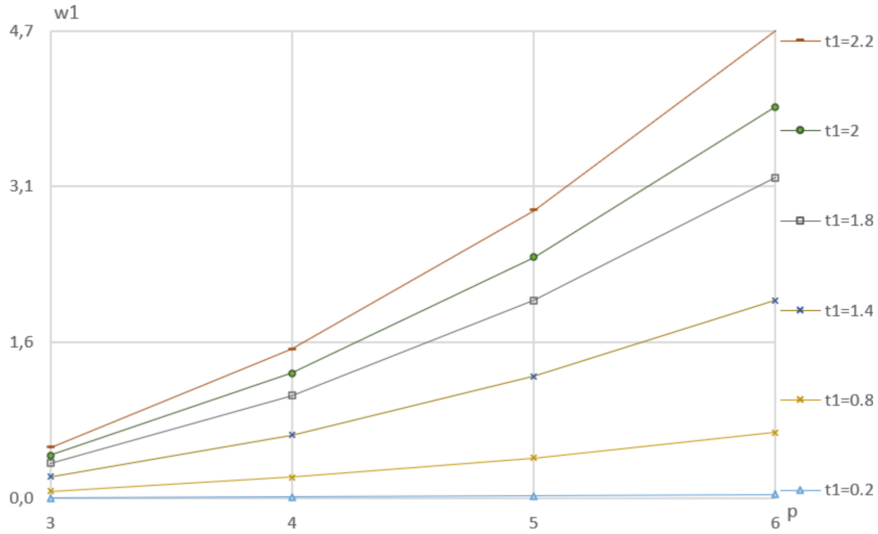

FIGURE 3. Maximal permanent deflections of the simply supported plate ($a = 0.7R$).

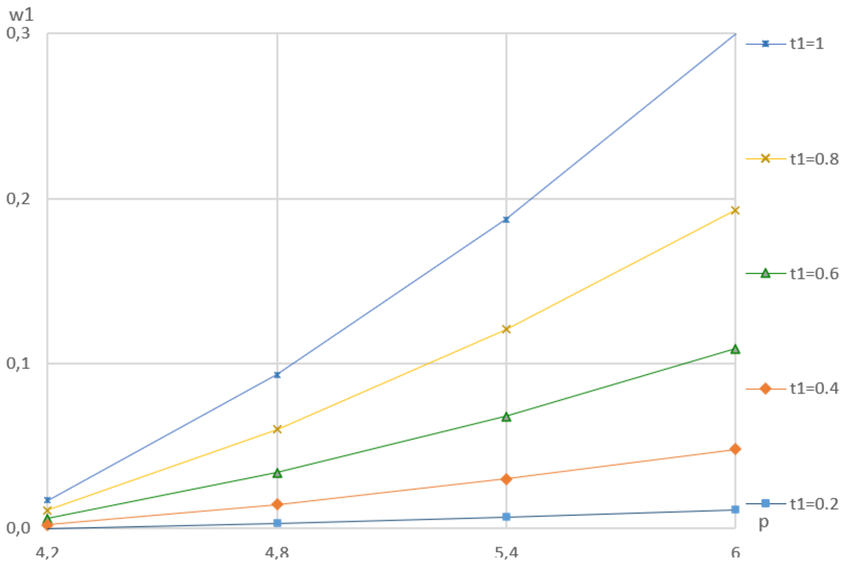

FIGURE 4. Maximal permanent deflections of the fully clamped plate ($a = 0.7R$).

deflections are smaller in the case of plate of constant thickness than for the stepped plate, as might be expected.

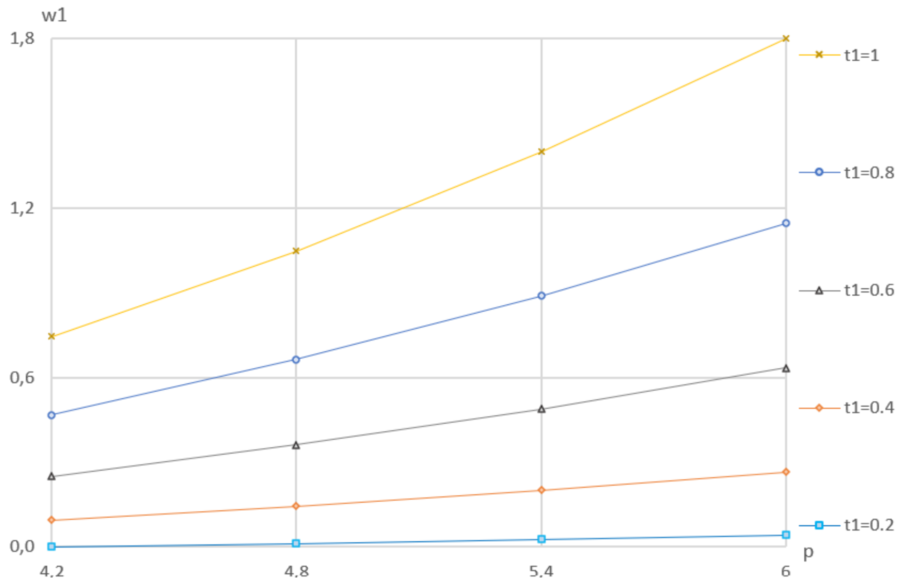

FIGURE 5. Maximal permanent deflections of the stepped plate ($r_1 = 0.2R$).

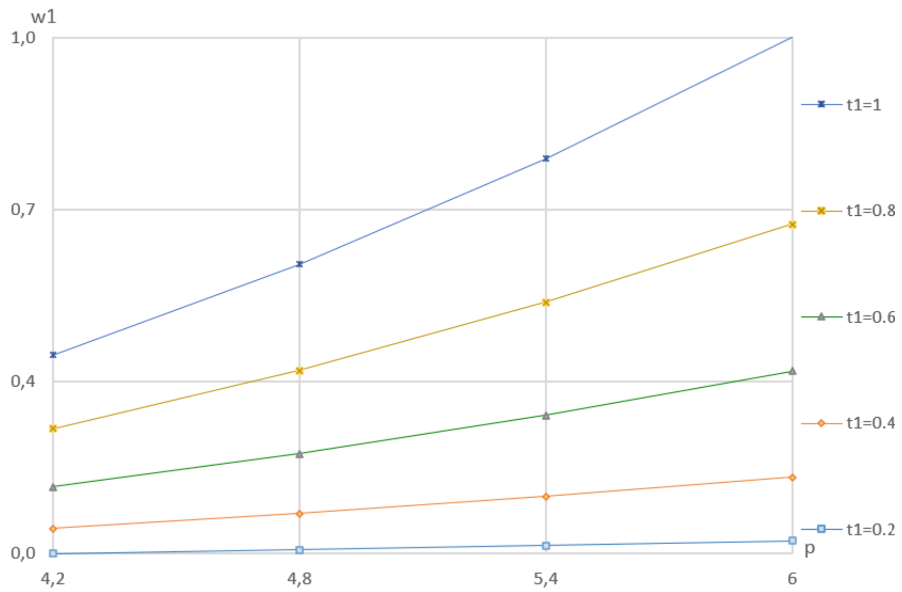

FIGURE 6. Maximal permanent deflections of the stepped plate ($r_1 = 0.8R$).

It can be seen from Figures 3–4 that the residual deflection increases if the load intensity p increases, and the residual deflections are larger for the simply supported plate.

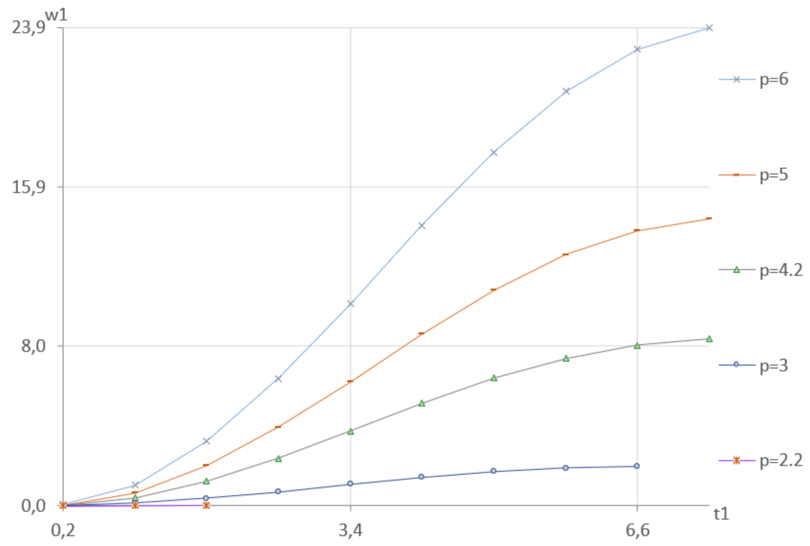

Figure 7. Maximal permanent residual deflections of simply supported plate.

Figure 5 stands for the case of simply supported stepped plate, whereas $a = 0.1R$, $\beta = 0.2$, $r_1 = 0.2R$ and $h_2 = 0.8h_1$. Comparing Figure 2 and Figure 5 we can see more larger residual deflections for the case $a = 0.1R$, as might be expected.

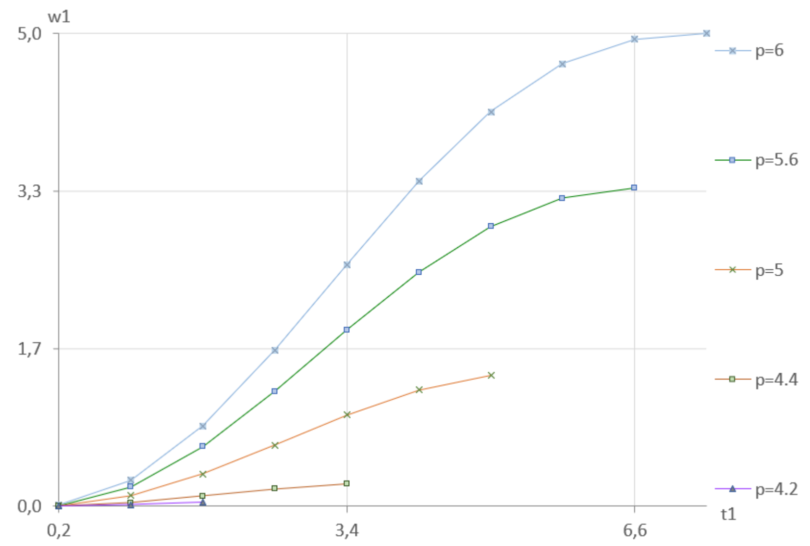

Figure 8. Maximal permanent residual deflections of clamped plate.

In Figure 6, the case of simply supported stepped plate for $a = 0.1R$, $\beta = 0.2$, $r_1 = 0.8R$ and $h_2 = 0.8h_1$ is presented. It can be seen that the residual deflections are smaller for the case of larger thickness of the plate. The results of calculations of the maximal residual deflections of the plate versus time t_1 are presented in Figure 7 and Figure 8 for $\beta = 0.2$, $h_2 = 0.8h_1$. Here the plates subjected to the concentrated force applied at the distance $a = 0.7R$ from the centre of the plate. It can be seen from the figures that the permanent deflections of clamped plates are smaller than those corresponding to simply supported plates.

5. Concluding remarks

An approximate theoretical method developed earlier is applied to the circular plate subjected to the asymmetrically loaded circular plate clamped at the edge. Numerical results are obtained for simply supported and fully clamped circular plates of piece wise constant thickness subjected to the concentrated loading intensity of which is a function of time.

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