Advancing the program packages method for positional control problems with incomplete information

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International Conference "Systems Analysis: Modeling and Control" in memory of Academician A.V. Kryazhimskiy
Russia, Moscow, 23-24 January 2024

Program packages method for solving control problems with incomplete information

«The problem of constructing optimal closed-loop control strategies under uncertainty is one of the key problems of the mathematical control theory. Its solution would give a new impetus to the theory's development and create the foundation for its new applications.»

Arkady Kryazhimskiy (2013)

- Yu. S. Osipov. *Control Packages: An Approach to Solution of Positional Control Problems with Incomplete Information.* Usp. Mat. Nauk 61:4 (2006), 25–76.
- A. V. Kryazhimskiy, Yu. S. Osipov. *Idealized Program Packages and Problems of Positional Control with Incomplete Information*. Trudy IMM UrO RAN 15:3 (2009), 139–157.

Program packages method for guidance of linear control systems

- A. V. Kryazhimskiy, Yu. S. Osipov. *On the solvability of problems of guaranteeing control for partially observable linear dynamical systems*. Proc. Steklov Inst. Math., 277 (2012), 144–159

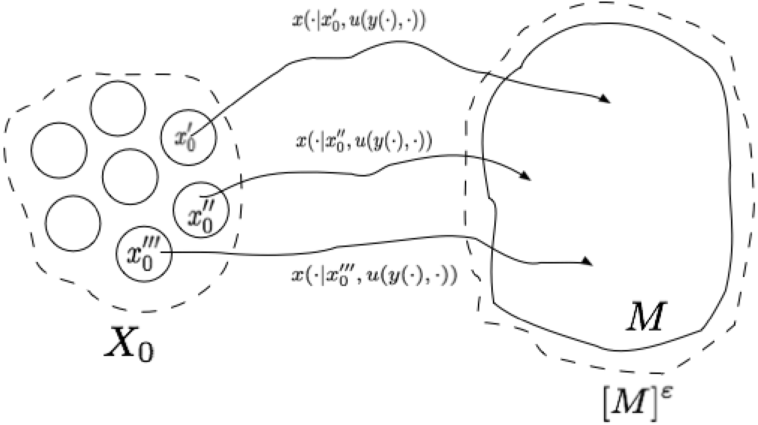
$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + c(t), t_0 \leq t \leq \vartheta \tag{1}$$

Open-loop control (program) $u(\cdot)$ is measurable, $u(t) \in P \subset \mathbb{R}^r$, P is a convex compact set
Initial states $x(t_0) = x_0 \in X_0 \subset \mathbb{R}^n$, X_0 is a **finite** set
Terminal state $x(\vartheta) \in M \subset \mathbb{R}^n$, M is a **closed and convex** set
Observed signal $y(t) = Q(t)x(t)$, $Q(\cdot) \in \mathbb{R}^{q \times n}$ is left piecewise continuous

Guidance problem statement

Based on the given arbitrary $\varepsilon > 0$ choose a closed-loop control strategy with memory, **whatever the system's initial state x_0 from the set X_0**, *the system's motion $x(\cdot)$ corresponding to the chosen closed-loop strategy and starting at the time t_0 from the state x_0 reaches the state $x(\vartheta)$ belonging to the ε-neighbourhood of the target set M at the time ϑ.*

Program packages method for guidance of linear control systems



Homogeneous signals

Homogeneous system, corresponding to (1)

$$\dot{x}(t) = A(t)x(t)$$

For each $x_0 \in X_0$ its solution is given by the Cauchy formula:

$$x(t) = F(t, t_0)x_0;\ F(t, s)\ (t, s \in [t_0, \vartheta])\ \text{is the fundamental matrix.}$$

Homogeneous signal, corresponding to an admissible initial state $x_0 \in X_0$:

$$g_{x_0}(t) = Q(t)F(t, t_0)x_0\ (t \in [t_0, \vartheta],\ x_0 \in X_0).$$

Let $G = \{g_{x_0}(\cdot) | x_0 \in X_0\}$ be the set of all homogeneous signals and let $X_0(\tau | g(\cdot))$ be the set of all admissible initial states $x_0 \in X_0$, corresponding to the homogeneous signal $g(\cdot) \in G$ *till time point* $\tau \in [t_0, \vartheta]$:

$$X_0(\tau | g(\cdot)) = \{x_0 \in X_0 : g(\cdot)|_{[t_0,\tau]} = g_{x_0}(\cdot)|_{[t_0,\tau]}\}.$$

Package guidance problem

Program package is an open-loop controls family $(u_{x_0}(\cdot))_{x_0 \in X_0}$, satisfying **non-anticipatory condition**: for any homogeneous signal $g(\cdot)$, any time $\tau \in (t_0, \vartheta]$ and any $x_0', x_0'' \in X_0(\tau | g(\cdot))$ the equality $u_{x_0'}(t) = u_{x_0''}(t)$ holds for almost all $t \in [t_0, \tau]$.

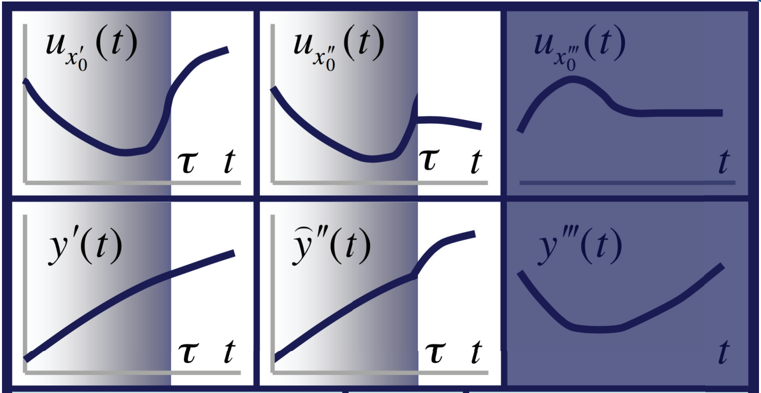
Program package $(u_{x_0}(\cdot))_{x_0 \in X_0}$ is **guiding**, if for all $x_0 \in X_0$ holds $x(\vartheta | x_0, u_{x_0}(\cdot)) \in M$.

Package guidance problem is solvable, if a guiding program package exists.

Theorem (Osipov, Kryazhimskiy, 2006)

The problem of positional guidance is solvable if and only if the problem of package guidance is solvable.

Package guidance problem



$x'(t) = x(t, x_0', u_{x_0'})$	$y'(t)$	$y'(t) = \hat{y}''(t)$ $(t \le \tau)$
$\hat{x}''(t) = x(t, x_0'', u_{x_0'})$	$\hat{y}''(t)$	$u_{x_0'}(t) = u_{x_0''}(t)$ $(t \le \tau)$

Homogeneous signals splitting

For an arbitrary homogeneous signal $g(\cdot)$ let

$$G_0(g(\cdot)) = \left\{\tilde{g}(\cdot) \in G : \lim_{\zeta \to +0} (\tilde{g}(t_0 + \zeta) - g(t_0 + \zeta)) = 0\right\}$$

be the set of **initially compatible** homogeneous signals and let

$$\tau_1(g(\cdot)) = \max\left\{\tau \in [t_0, \vartheta] : \max_{\tilde{g}(\cdot) \in G_0(g(\cdot))} \max_{t \in [t_0, \tau]} |\tilde{g}(t) - g(t)| = 0\right\}$$

be its **first splitting moment**. For each $i = 1, 2, \ldots$ let

$$G_i(g(\cdot)) = \left\{\tilde{g}(\cdot) \in G_{i-1}(g(\cdot)) : \lim_{\zeta \to +0} (\tilde{g}(\tau_i(g(\cdot)) + \zeta) - g(\tau_i(g(\cdot)) + \zeta)) = 0\right\}$$

be the set of all homogeneous signals from $G_{i-1}(g(\cdot))$ equal to $g(\cdot)$ in the right-sided neighbourhood of the time-point $\tau_i(g(\cdot))$ and let

$$\tau_{i+1}(g(\cdot)) = \max\left\{\tau \in (\tau_i(g(\cdot)), \vartheta] : \max_{\tilde{g}(\cdot) \in G_i(g(\cdot))} \max_{t \in [\tau_i(g(\cdot)), \tau]} |\tilde{g}(t) - g(t)| = 0\right\}$$

be the $(i+1)$**-th splitting moment** of the homogeneous signal $g(\cdot)$.

Initial states set clustering

Let

$$T(g(\cdot)) = \{\tau_j(g(\cdot)) : j = 1, \ldots, k_{g(\cdot)}\}$$

be the set of all splitting moments of the homogeneous signal $g(\cdot)$ and let

$$T = \bigcup_{g(\cdot) \in G} T(g(\cdot))$$

be the set of all splitting moments of all homogeneous signals. T is finite and $|T| \leq |X_0|$. Let us represent this set as $T = \{\tau_1, \ldots, \tau_K\}$, where $t_0 < \tau_1 < \ldots < \tau_K = \vartheta$.
For every $k = 1, \ldots, K$ let the set

$$\mathcal{X}_0(\tau_k) = \{X_0(\tau_k | g(\cdot)) : g(\cdot) \in G\}$$

be the **cluster position** at the time-point τ_k, and let each its element $X_{0j}(\tau_k)$, $j = 1, \ldots, J(\tau_k)$ be a **cluster of initial states** at this time-point; $J(\tau_k)$ is the number of clusters in the cluster position $\mathcal{X}_0(\tau_k)$, $k = 1, \ldots, K$.

Homogeneous signals and cluster positions

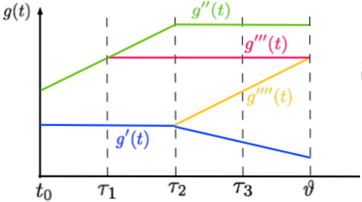


Figure: Homogeneous signals splitting

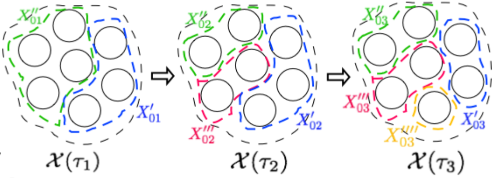


Figure: Initial states set clustering

Extended space

Let $\mathcal{R}^h$ ($h = 1, 2, \ldots$) be a finite-dimensional Euclidean space of all families $(r_{x_0})_{x_0 \in X_0}$ from $\mathbb{R}^h$ with a scalar product $\langle \cdot, \cdot \rangle_{\mathcal{R}^h}$ defined as

$$\langle r', r'' \rangle_{\mathcal{R}^h} = \langle (r'_{x_0})_{x_0 \in X_0}, (r''_{x_0})_{x_0 \in X_0} \rangle_{\mathcal{R}^h} = \sum_{x_0 \in X_0} \langle r'_{x_0}, r''_{x_0} \rangle_{\mathbb{R}^h} \quad ((r'_{x_0})_{x_0 \in X_0}, (r''_{x_0})_{x_0 \in X_0} \in \mathcal{R}^h).$$

For each non-empty set $\mathcal{E} \subset \mathcal{R}^h$ ($h = 1, 2, \ldots$) let us define its *lower* $\rho^-(\cdot|\mathcal{E}) : \mathcal{R}^h \mapsto \mathbb{R}$ and *upper* support functions $\rho^+(\cdot|\mathcal{E}) : \mathcal{R}^h \mapsto \mathbb{R}$:

$$\rho^-((l_{x_0})_{x_0 \in X_0} | \mathcal{E}) = \inf_{(e_{x_0})_{x_0 \in X_0} \in \mathcal{E}} \langle (l_{x_0})_{x_0 \in X_0}, (e_{x_0})_{x_0 \in X_0} \rangle_{\mathcal{R}^h} \quad ((l_{x_0})_{x_0 \in X_0} \in \mathcal{R}^h),$$

$$\rho^+((l_{x_0})_{x_0 \in X_0} | \mathcal{E}) = \sup_{(e_{x_0})_{x_0 \in X_0} \in \mathcal{E}} \langle (l_{x_0})_{x_0 \in X_0}, (e_{x_0})_{x_0 \in X_0} \rangle_{\mathcal{R}^h} \quad ((l_{x_0})_{x_0 \in X_0} \in \mathcal{R}^h)$$

Extended open-loop control

Let $\mathcal{P} \subset \mathcal{R}^m$ be the set of all families $(u_{x_0})_{x_0 \in X_0}$ of vectors from P.
Extended open-loop control control is a measurable function
$t \mapsto (u_{x_0}(t))_{x_0 \in X_0} : [t_0, \vartheta] \mapsto \mathcal{P}$.
Let us identify arbitrary programs family $(u_{x_0}(\cdot))_{x_0 \in X_0}$ and an extended open-loop control $t \mapsto (u_{x_0}(t))_{x_0 \in X_0}$.

For each $k = 1, \ldots, K$ let $\mathcal{P}_k$ be an **extended admissible control set** on $(\tau_{k-1}, \tau_k]$ in case $k > 1$ and on $[t_0, \tau_1]$ in case $k = 1$ as a set of all vector families $(u_{x_0})_{x_0 \in X_0} \in \mathcal{P}$ such that, for each cluster $X_{0j}(\tau_k) \in \mathcal{X}_0(\tau_k), j = 1, \ldots, J(\tau_k)$ and any $x_0', x_0'' \in X_{0j}(\tau_k)$ holds $u_{x_0'} = u_{x_0''}$.

Extended open-loop control control $(u_{x_0}(\cdot))_{x_0 \in X_0}$ is **admissible**, if for each $k = 1, \ldots, K$ holds $(u_{x_0}(t))_{x_0 \in X_0} \in \mathcal{P}_k$ for almost all $t \in (\tau_{k-1}, \tau_k]$ in case $k > 1$ and for almost all $t \in [t_0, \tau_1]$ in case $k = 1$;

Lemma

Extended open-loop control control $(u_{x_0}(\cdot))_{x_0 \in X_0}$ is a control package if and only if it is admissible.

Cluster positions and extended open-loop controls

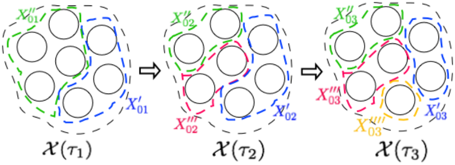


Figure: Initial states set clustering

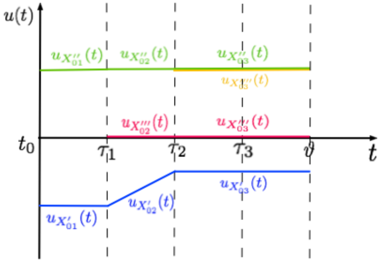


Figure: Extended open-loop control control

Extended problem of program guidance

Extended system (in the space $\mathcal{R}^n$):

$$\begin{cases} \dot{x}_{x_0}(t) = A(t)x_{x_0}(t) + B(t)u_{x_0}(t) + c(t) \\ x_{x_0}(t_0) = x_0 \end{cases}$$

$$(x_0 \in X_0)$$

Extended target set $\mathcal{M}$ is the set of all families $(x_{x_0})_{x_0 \in X_0} \in \mathcal{R}^n$ such, that $x_{x_0} \in M$ for all $x_0 \in X_0$.

An admissible extended open-loop control $(u_{x_0}(\cdot))_{x_0 \in X_0}$ is **guiding the extended system**, if $(x(\vartheta | x_0, u_{x_0}(\cdot)))_{x_0 \in X_0} \in \mathcal{M}$.

The **extended problem of open-loop guidance** is solvable, if there exists an admissible extended open-loop control which is guiding the extended system.

Attainability set of the extended system at the time ϑ:
$\mathcal{A} = \{(x(\vartheta | x_0, u_{x_0}(\cdot)))_{x_0 \in X_0} : (u_{x_0}(\cdot))_{x_0 \in X_0} \in \mathcal{U}_{ext}\}$, where $\mathcal{U}_{ext}$ is the set of all admissible extended open-loop control controls.

Solvability criterion

- A. V. Kryazhimskiy, N. V. Strelkovskii, “An open-loop criterion for the solvability of a closed-loop guidance problem with incomplete information. Linear control systems,” *Proc. Steklov Inst. Math. (Suppl.)*, 291, Suppl. 1, 113–127 (2015).

Theorem

1) The package guidance problem is solvable if and only if the extended problem of open-loop guidance is solvable. 2) An admissible extended open-loop control is a guiding program package if and only if it is guiding extended system.

Let us denote $D(t) = B^{\mathrm{T}}(t)F^{\mathrm{T}}(\vartheta, t) \quad (t \in [t_0, \vartheta])$ and set the function $p(\cdot, \cdot) : \mathbb{R}^n \times X_0 \mapsto \mathbb{R}$:

$$p(l, x_0) = \langle l, F(\vartheta, t_0)x_0 \rangle_{\mathbb{R}^n} + \left\langle l, \int_{t_0}^{\vartheta} F(\vartheta, t)c(t)dt \right\rangle_{\mathbb{R}^n} \qquad (l \in \mathbb{R}^n,\ x_0 \in X_0).$$

Let us set

$$\gamma((l_{x_0})_{x_0 \in X_0}) = \rho^- \left((l_{x_0})_{x_0 \in X_0} | \mathcal{A}\right) - \rho^+ \left((l_{x_0})_{x_0 \in X_0} | \mathcal{M}\right) =$$

$$= \sum_{x_0 \in X_0} p(l_{x_0}, x_0) - \sum_{x_0 \in X_0} \rho^+(l_{x_0} | M) + \sum_{k=1}^{K} \int_{\tau_{k-1}}^{\tau_k} \sum_{X_{0j}(\tau_k) \in \mathcal{X}_0(\tau_k)} \rho^- \left(\sum_{x_0 \in X_{0j}(\tau_k)} D(t) l_{x_0} | P \right) dt.$$

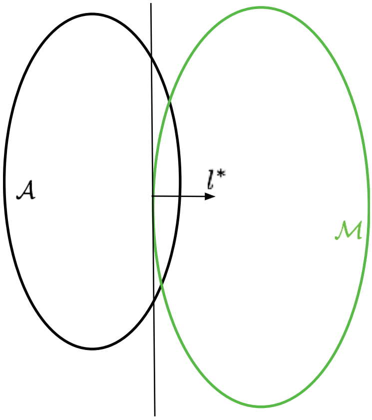
Solvability criterion

Let $\mathcal{L}$ be a compact set in $\mathcal{R}^n$, containing an image of the unit sphere $\mathcal{S}^n$ — for some positive r_1 and $r_2 \geq r_1$ for each $\ell \in \mathcal{S}^n$ there is $r \in [r_1, r_2]$, for which $r\ell \in \mathcal{L}$.

Theorem

Each of the three problems – (i) the extended open-loop control guidance problem, (ii) the package guidance problem and (iii) the guaranteed positional guidance problem – is solvable if and only if

$$\max_{(l_{x_0})_{x_0 \in X_0} \in \mathcal{L}} \gamma((l_{x_0})_{x_0 \in X_0}) \leq 0. \qquad (2)$$



Construction of the guiding program package

- N. V. Strelkovskii, "Constructing a strategy for the guaranteed positioning guidance of a linear controlled system with incomplete data," *Moscow University Computational Mathematics and Cybernetics*, 39, No. 3, 126–134 (2015).

Assuming that the solvability criterion (2) is satisfied, let us introduce the function $\hat{\gamma}(\cdot,\cdot) : \mathcal{R}^n \times [0,1] \mapsto \mathbb{R}$:

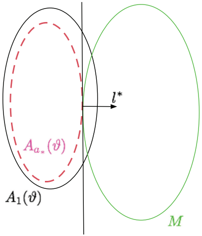
$$\begin{aligned}\hat{\gamma}((l_{x_0})_{x_0 \in X_0}, a) \quad = \quad & \sum_{x_0 \in X_0} \langle l_{x_0}, F(\vartheta, t_0) x_0 \rangle_{\mathbb{R}^n} + \left\langle l_{x_0}, \int_{t_0}^{\vartheta} F(\vartheta, t) c(t) dt \right\rangle_{\mathbb{R}^n} - \sum_{x_0 \in X_0} \rho^+(l_{x_0} | M) - \\ & - \sum_{k=1}^{K} \int_{\tau_{k-1}}^{\tau_k} \sum_{X_{0j}(\tau_k) \in \mathcal{X}_0(\tau_k)} \rho^- \left(\sum_{x_0 \in X_{0j}(\tau_k)} D(t) l_{x_0} \middle| aP \right) dt. \qquad (3)\end{aligned}$$

Program package $(u^0_{x_0}(\cdot))_{x_0 \in X_0}$ is **zero-valued**, if $u^0_{x_0}(t) = 0$ for almost all $t \in [t_0, \vartheta]$, $x_0 \in X_0$.

Lemma

If the solvability criterion (2) holds and zero-valued program package is not guiding the extended system, then exists $\mathbf{a}_*$ $\in (0,1]$ *such, that*

$$\max_{(l_{x_0})_{x_0 \in X_0} \in \mathcal{L}} \hat{\gamma}((l_{x_0})_{x_0 \in X_0}, \mathbf{a}_*) = 0. \qquad (4)$$



Guiding program package

For each program package $(u_{x_0}(\cdot))_{x_0 \in X_0}$, arbitrary cluster $X_{0j}(\tau_k) \in \mathcal{X}(\tau_k)$, $j = 1, \ldots, J(\tau_k)$, $k = 1, \ldots, K$ and arbitrary $t \in [\tau_{k-1}, \tau_k)$ let us denote $u_{X_{0j}(\tau_k)}(t)$ program values $u_{x_0}(t)$, which are equal for all $x_0 \in X_{0j}(\tau_k)$. Let $(\mathbf{l}^*_{\mathbf{x_0}})_{\mathbf{x_0} \in \mathbf{X_0}}$ be the maximizer of the left handside of (4). Cluster $X_{0j}(\tau_k)$ is **regular**, if

$$\sum_{x_0 \in X_{0j}(\tau_k)} D(t)\mathbf{l}^*_{\mathbf{x_0}} \neq 0, \ \ t \in [\tau_{k-1}, \tau_k).$$

Otherwise the cluster is **singular**.

Theorem

Let P be a strictly convex compact set, containing the zero vector; condition (4) holds and the program package $(u^*_{x_0}(\cdot))_{x_0 \in X_0}$ satisfies the condition $u^*_{x_0}(t) \in \mathbf{a}_* P \quad (x_0 \in X_0, \ t \in [t_0, \vartheta])$. Let the clusters $X_{0j}(\tau_k) \in \mathcal{X}_0(\tau_k)$, $k = 1, \ldots, K$, $j = 1, \ldots, J(\tau_k)$ be regular, and for each of them the following equality holds

$$\left\langle D(t) \sum_{x_0 \in X_{0j}(\tau_k)} \mathbf{l}^*_{\mathbf{x_0}}, u^*_{X_{0j}(\tau_k)}(t) \right\rangle_{\mathbb{R}^m} = \rho^- \left(D(t) \sum_{x_0 \in X_{0j}(\tau_k)} \mathbf{l}^*_{\mathbf{x_0}} \middle| \mathbf{a}_* P \right) \ \ (t \in [\tau_{k-1}, \tau_k)). \qquad (5)$$

Then the program package $(u^*_{x_0}(\cdot))_{x_0 \in X_0}$ is guiding.

Further method developments

- Numerical algorithm for the case of regular clusters of the initial positions set was developed using a modification of the subsequent approximations method in extended space

  S. M. Orlov, N. V. Strelkovskii, “Algorithm for Constructing a Guaranteeing Program Package in a Control Problem with Incomplete Information,” *Moscow University Computational Mathematics and Cybernetics*, 42, No. 2, 69–79 (2018).

- The case when the initial positions set has singular clusters was addressed by perturbing the original extended program guidance problem and solving it for a smoothed control set

  S. M. Orlov, N. V. Strelkovskii, “Calculation of elements of a guiding program package for singular clusters of the set of initial states in the package guidance problem,” *Proc. Steklov Inst. Math. (Suppl.)*, 308, Suppl. 1, S163–S177 (2020)

- Extension of the method to provide a solution for a closed-loop guidance problem onto one of the given convex target sets *by* a predefined time

  A. V. Kryazhimskiy, N. V. Strelkovskii, “A problem of guaranteed closed-loop guidance by a fixed time for a linear control system with incomplete information. Program solvability criterion,” *Trudy Inst. Mat. i Mekh. UrO RAN*, 20, No. 4, 168–177 (2014).

Further method developments

- Extension of the method to the case when the linear control system contains a delay

  P. G. Surkov, "The problem of closed-loop guidance by a given time for a linear control system with delay," *Proc. Steklov Inst. Math. (Suppl.)*, 296, Suppl. 1, 218–227 (2017).

- Extension of the method to the problem of guidance to a system of target sets

  V. I. Maksimov, P. G. Surkov, "On the solvability of the problem of guaranteed package guidance to a system of target sets," *Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki*, 27, No. 3, 344–354 (2017)

- Extension of the method to the problem of guaranteed control problem for a linear stochastic differential equation

  V. L. Rozenberg, "A guaranteed control problem for a linear stochastic differential equation," *Ural Math. J.*, 1, No. 1, 68–82 (2015)

- Extension of the method to the closed-loop terminal control problem

  S. M. Orlov, N. V. Strelkovskii, "Program Packages Method for Solution of a Linear Terminal Control Problem with Incomplete Information," In: *Stability, Control and Differential Games*, 213–223, Springer, (2020).

Open gaps

- Extension of the method to other classes of problems for linear systems
  - For example, time-optimal control problem
- Extension of the method to other types of control systems (non-linear)
  - For example, some classes of bilinear systems as well as other special cases

    N. L. Grigorenko, A. E. Rumyantsev, “On a class of control problems with incomplete information,” *Proc. Steklov Inst. Math.*, 291, 68-77 (2015).

Thank you for your attention!
Questions?