The Radical Complexity of Rewiring Supplier–Buyer Networks

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Abstract

This paper questions the dynamic stability of supplier–buyer networks. We investigate a simple rewiring process, by which firms change suppliers if it increases profit, within an otherwise classical production network model with market clearing, profit-maximization, and complete information. We find that these systems exhibit extremely vast sets of path-dependent and locally-stable configurations, in which firms have no interest to change suppliers. Following an external shock, the network undergoes a cascade of rewirings and reaches a new stable configuration. However the duration of those cascades quickly increases with network size, suggesting that, in real network made of millions of firm, locally-stable configurations are not likely to be ever reached. Moreover, as soon as firms have limited visibility over their supply chain, networks fail to stabilize and keep changing configurations forever. They either drift in the topological space or alternate within a reduced set of configurations. Our results are consistent with the well-known combinatorial problems marked by rugged dynamical landscapes. Because of such a radical complexity, one cannot assume that supplier-buyer networks are in equilibrium.

Keywords: input–output, supply chains, multiple equilibria, rough landscape, path dependency, limited information, rewiring

1. Introduction

Commercial relationships form the most prominent layer of interactions in the economy. Instead of debts or ownership linkages, they are grounded in the physical reality, materialize in the physical flows of goods and services, and signal technological choices. They underpin supply chains and production networks, which have grown highly complex (Colon and Hochrainer-Stigler, 2022). Through offshoring, outsourcing, and vertical specialization, production has been split into multiple stages operated by distinct and spatially scattered firms (Hummels et al., 2001; Jones et al., 2005). The resulting networks are complicated to map, and firms generally have minimal visibility over their supply chains. They usually know their direct suppliers, but often struggle to keep track of their sub-suppliers, also called tier-2 suppliers, and entities further away in the chain (Wang et al., 2015; BCI, 2018).

The network approach to economics (see e.g. Jackson (2010); Carvalho and Tahbaz-Salehi (2019)) has given new tools to appraise the complex patterns of interactions in the economy and highlighted the risks associated with such degree of interconnectedess, often called systemic risks (Colon and Hochrainer-Stigler, 2022). Disasters, bankruptcies, or failures, cascade from one firm to another and lead to economic losses in distant locations (Barrot and Sauvagnat, 2016; Norrman and Jansson, 2004; Fujiwara, 2008). Such phenomena have been illustrated by several large-scale events that sparked quick, large, and unexpected side-effects on international supply chains, namely the 2010 eruption of the Eyjafjallajökull volcano in Iceland, the 2011

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floods in Thaïland, the 2011 Tōhoku earthquake in Japan (see e.g. Carvalho et al. (2021)). Since then, businesses, especially insurers, have been looking for new models to manage systemic risks.

More fundamentally, the network approach casts a new light on the concepts of economic equilibrium and stability. Simple economic models have stressed that high degrees of interdependencies have the potential to exacerbate small perturbations (Acemoglu et al., 2012; Moran and Bouchaud, 2019; Dessertaine et al., 2022). Micro-fluctuations do not fade out as we zoom out but can turn into macro-perturbations. Market mechanisms have the potential to generate endogenous macroeconomic variability; see, for instance, Bonart et al. (2014); Mandel et al. (2015); Moran and Bouchaud (2019); Dessertaine et al. (2022), or the well-documented bullwhip effects (Lee et al., 1997, 2004; Thun and Hoenig, 2011). An amplifying factor is the scale-free nature of production networks, which has been empirically documented (Fujiwara and Aoyama, 2010; Ohnishi et al., 2010; Mizuno et al., 2014). Economies do not need external shocks—e.g., climatic turbulences or pandemics—to fluctuate Dessertaine et al. (2022).

Most of these studies consider that the network structure of commercial relationships is fixed in time and study fluctuations taking place on such fixed structures. However, production networks do evolve over time. According to Bacilieri et al. (2022), the lifetime of a supplier–buyer relationship is about two years. "Rewirings"—i.e., a change of supplier–buyer link—may be triggered by external events, such as natural disasters. Todo et al. (2013), for instance, documented how some parts of the Japanese production network changed after the 2011 Tōhoku earthquake. But often, such decisions are motivated by purely economic factors. Firms are permanently looking for better opportunities, cheaper suppliers, and higher-quality services (Fontaine et al., 2022).

How does this restructuring of commercial relationships affect macroeconomic variables? Is it primarily driven by endogenous process—i.e., the strive for profit—or by external shocks? In the absence of any exogenous perturbations or technological innovations, should we expect that the structure of supplier–buyer networks will eventually stabilize? Would this configuration be, in some sense, optimal, e.g., all agents would maximize their utility or profits, as suggested by the invisible hand metaphor? Can such a stable network emerge even when firms lack visibility over their supply chain?

The optimization problem faced by firms embedded in large networks belongs to a broad class of combinatorial optimization problems studied in statistical physics and computer science in the last few decades, e.g., the traveling salesman problem or the so-called spin-glass problem; see Mézard and Montanari (2009) for a review. Such problems are NP-hard, meaning there is no "fast" algorithm to solve them, which are therefore expected to be impossible to solve by economic agents, even aided with the best computational tools. Pictorially, such problems are characterized by a vast number of locally optimal solutions that can be very different from one another. This generically leads to a hyper-slow convergence of any kind of dynamical algorithm that aims at finding the optimal solution—in practice, these algorithms never converge as soon as the number n of individual entities is somewhat large (say a few hundred). One often speaks of "rough landscapes", which means that the cost function (here the total utility of households) is a complicated function of the dynamical variables of the system (here the structure of the input-output network), with an exponentially large (in n) number of quasi-equivalent solutions.

This rough landscape paradigm is highly relevant in many different contexts (physics, computer science, biology, ecology, etc.) but is still, in our opinion, overlooked in the economics community. Our main message here is that similar effects exist in economics, with large economic networks as a case in point: due to the inherent complexity of the problem, one cannot assume that such networks are in equilibrium. This is a predicament we would like to call "radical complexity" (Bouchaud, 2021), by analogy with Knight's "radical uncertainty". We hope that the present work will contribute to popularizing some of the ideas that have become key to understanding the behavior of complex systems (see also Sharma et al. (2022); Garnier-Brun et al. (2021) for related discussions).

Specifically, our paper examines the existence of stable structures of supplier–buyer networks made of profit-maximizing firms. We formulate a simple and reasonably general model of production networks based on Acemoglu et al. (2012). Firms have suppliers and clients, i.e., input–output links. They can, however, modify their set of suppliers to maximize profits, a feature absent from previous investigations. To remain close to classical economic hypotheses, we suppose that prices and production levels perfectly and instantaneously adjust such that supply and demand are always balanced—an obviously unrealistic assumption that prevents potentially relevant out-of-equilibrium effects to take place, see e.g. Mandel et al. (2015); Dessertaine et al. (2022). We also assume in the baseline that firms have *complete information* over the network structure, whatever its complexity, and over all other firms—a hypothesis that we then relax, with important practical consequences when the network is large.

Our investigation of the model is primarily numerical at this stage. We find that when firms have complete information over the network, small topologies (a few hundred) eventually converge toward a locally stable configuration, which is almost always different, even with similar initial conditions. Shocks, such as the removal of a firm, trigger reconfiguration waves. In larger networks, convergence to a stable configuration becomes increasingly slow. When firms have limited visibility over the network, even up to tier-2 suppliers and clients, there is no stable configuration. Networks keep on rewiring forever.

2. Model: an input–output network with a rewiring dynamics

The building block of our model is a slightly modified version of the multisectoral model initially formulated by Long and Plosser (1983), updated by Acemoglu et al. (2012), and extended by Bonart et al. (2014); Dessertaine et al. (2022). Firms form a supplier–buyer network serving a representative household. At each time step, prices and productions adjust instantaneously and simultaneously such that supply meets demand in all markets. The model is briefly described in Sec. 2.1.

The originality of our model is twofold. First, we introduce a rewiring dynamics, described in Sec. 2.2, by which firms update their supplier network to increase profits whenever possible. Consequently, the connectivity matrix, denoted by $\mathbb{M}_t$, may change over time. The time series of $\mathbb{M}_t$ is our main focus of analysis.

Next, firms' knowledge of the network and the other firms can vary. They only know their suppliers, clients, and the commercial relationships between them up to a certain tier. When assessing the profitability of replacing a supplier with a new one, they try to anticipate how firms within this sub-network would react, assuming that markets would clear and that those firms would maximize their profits; see Sec. 2.3.

The network evolves within a technologically constraint network space, explained in Sec. 2.4. We choose as starting topology a class of scale-free networks proposed by Colon and Ghil (2017) based on the empirical analysis of Fujiwara and Aoyama (2010). The campaign of simulations carried out with the model is described in Sec. 2.5.

2.1. Firms embedded in a market-balanced supplier–buyer network

The supply side of the economy consists of n firms indexed by integer $i = 1, ..., n$. Each one of them produces a specific good in quantity x_i. In this section, there is no dynamics. We drop the time subscript $(*)_t$ to lighten the notations. Two types of inputs are used for production: labor ℓ_i hired from households and a mix of goods g_{ji} purchased from other firms $j \in \mathcal{S}_i$; where $\mathcal{S}_i$ is the set of suppliers of firm i. We denote by c_i the number of suppliers, i.e., the cardinality of $\mathcal{S}_i$. The connectivity matrix $\mathbb{M}$ captures the topology of the supplier–buyer network. Element $\mathbb{M}_{ij}$ is equal to 1 if firm i is supplying firm j and to 0 otherwise. A firm cannot supply itself, such that $\mathbb{M}_{ii} = 0$.

The production process of firm i is represented by a Cobb–Douglas function[1] $f_i : (\ell_i, g_{1,i}, \dots, g_{n,i}) \mapsto x_i$, such that

$$x_i = z_i \left(\frac{\ell_i}{a_i}\right)^{a_i b_i} \prod_{j \in \mathcal{S}_i} \left(\frac{g_{ji}}{(1-a_i)w_{ji}}\right)^{(1-a_i)w_{ji}b_i}. \tag{1}$$

Parameter $z_i > 0$ describes firm i's productivity, and $b_i < 1$ represents its overall return to scale. Parameter $a_i \in [0, 1]$ describes the share of labor in firm i's input mix, and $w_{ji} \in [0, 1]$ represents the share of good j in its nonlabor input mix. The input-output coefficients w_{ji} are such that $\sum_{j \in \mathcal{S}_i} w_{ji} = 1$.

[1] While the present paper only considers a Cobb-Douglas production function, we are confident that the basic phenomenology reported below is robust against the precise specification of the production function. This should however be explored in more detail.

The $n \times n$ elements w_{ij}, also called link weights, form the so-called input–output matrix, denoted by $\mathbb{W}$. The input–output matrix $\mathbb{W}$ and the connectivity matrix $\mathbb{M}$ share the same nil elements.

The profit of firm i is

$$\mathcal{P}_i = p_i x_i - h\ell_i - \sum_{j \in \mathcal{S}_i} p_j g_{ji}, \tag{2}$$

where p_i is the price of good i and h the wage. Given their production level, firms choose the quantities of inputs that maximize their profit, i.e.,

$$\ell_i = a_i b_i \frac{p_i}{h} x_i, \tag{3}$$

$$g_{ji} = (1 - a_i) w_{ji} b_i \frac{p_i}{p_j} x_i. \tag{4}$$

Such maximization is only possible if the marginal productivity of each input has diminishing returns, i.e., if $a_i b_i < 1$ and $(1 - a_i) w_{ji} b_i < 1$.

Using equations (3) and (4) into Eq. (1) yields n conditions linking prices and productions:

$$x_i = z_i (b_i p_i x_i)^{b_i} h^{-a_i b_i} \prod_j p_j{}^{-(1-a_i) w_{ji} b_i}. \tag{5}$$

Households offer a quantity of labor that scales linearly with the number of firms, $L \times n$, where L is the average quantity of labor per firm. They maximize their utility, which derives from the consumption of goods. Using a log-utility function, households order from each firm i a quantity $\{B/(np_i)\}$ of products, where B is the households' available income. We denote by $\mathcal{D}_i^F$ this final demand directed to firm i and U the total utility, which is $-\sum_i \log(p_i)$.

We exclude banks and governments from this simplified economy. Neither savings nor investments are made. Households own all the firms, such that any profit made is returned to households as dividends. Once all business-to-business payments are made, the net profits of firms correspond to sales to the final consumer minus wage. Households thus earn exactly what they spend, such that their available income B is fixed and corresponds to their initial financial endowment. We assume that this initial endowment is proportional to the number of firms. As in Acemoglu et al. (2012) and Bonart et al. (2014), we set B to n and L to 1, such that households' budget and labor offer corresponds to the number of firms. This assumption will facilitate comparing networks of different sizes.

At equilibrium, the labor market and the n good markets clear. On the labor market, supply is n and demand is $\sum_i \ell_i$. Using Eq. (3), the condition for labor market clearing is

$$n = \sum_{i=1}^{n} a_i b_i \frac{p_i}{h} x_i. \tag{6}$$

On the market for good i, supply is x_i and demand is $\mathcal{D}_i^F + \mathcal{D}_i^I$, where $\mathcal{D}_i^I$ is the intermediary demand. We have $\mathcal{D}_i^F = 1/(np_i)$ and $\mathcal{D}_i^I = \sum_j g_{ij}$. Using Eq. (4), the n conditions for good market clearing are, for $i = 1, \ldots, n$,

$$x_i = \frac{1}{p_i} + \sum_{j=1}^{n} (1 - a_j) b_j w_{ij} \frac{p_j}{p_i} x_j. \tag{7}$$

Equations (5), (6), (7) form a system of $2n + 1$ equations which determine the equilibrium value of x_i, h, and p_i, for $i = 1, \ldots, n$. Appendix 4 provides a method to solve this system.

2.2. Firms rewire their supplier network through times

The first novelty of our model is that the network evolves over time. Firms try to increase profits by replacing existing suppliers with better ones, if available. As a consequence, the connectivity matrix $\mathbb{M}_t$ changes over time.

We suppose that there are technical constraints on the feasibility of a supplier–buyer relationship, because not any firm can supply any other. We preselect for each firm i a fixed set of possible suppliers among the other firms, denoted by $\overline{\mathcal{S}_i}$. The set of current suppliers $\mathcal{S}_{i,t}$, which is a subset of $\overline{\mathcal{S}_i}$, i.e., $\mathcal{S}_{i,t} \subset \overline{\mathcal{S}_i}$, may change through time, whereas $\overline{\mathcal{S}_i}$ does not. The input–output coefficients corresponding to these possible supplier–buyer relationships are fixed and denoted by $\overline{w}_{ji}$. They form the so-called technological matrix, $\overline{\mathbb{W}}$, of which the current input–output matrix $\mathbb{W}_t$ is a subset, i.e., $\mathbb{W}_t \subset \overline{\mathbb{W}}$. Matrix $\mathbb{W}_t$ is derived from the connectivity matrix $\mathbb{M}_t$: $\mathbb{W}_t = \mathbb{M}_t \circ \overline{\mathbb{W}}$, where $\circ$ indicates element-wise multiplication. The construction of the technological matrix $\overline{\mathbb{W}}$ and of the initial connectivity matrix $\mathbb{M}_0$ is described in Sec. 2.4.

We introduce integer c'_i, which is the number of extra suppliers of firm i, such that $\overline{c}_i := c_i + c'_i$ is the cardinality of $\overline{\mathcal{S}_i}$. Firms are only allowed to replace an existing supplier with a new one. They cannot only add a supplier or only remove one, such that c_i and c'_i are fixed over time. Parameter c'_i can be interpreted as the technological options of firm i. If $c'_i = 0$, then firm i has no choice but to keep with the same suppliers. If $\overline{c}_i = n - 1$, then firm i can use any input.

Note that the above specification may appear to be contradictory with the use of a Cobb-Douglas production function, since it may well be that in order to maximize its profits, and depending on the technological matrix, firm i should use more than c_i inputs, even if some are redundant. Implicit in our model is the existence of costs associated with maintaining commercial relations with each supplier, so that firms in fact attempt to reduce their number to the bare minimum, equal to c_i. This is a way to account for the idea of essential inputs within a Cobb-Douglas framework.

The space of possible configurations $\mathbb{M}_t$ increases very quickly with c'_i and with n: its cardinality $\mathfrak{C}$ is $\mathfrak{C} = \prod_i^n \binom{\overline{c}_i}{c_i}$. For example, take $c_i = 4$ and $c'_i = 1$ for all i, one finds $\mathfrak{C} = 5^n$. This exponential growth of the number of configurations is typical of the combinatorial optimization problems mentioned in the introduction, with their associated rough landscapes.

The dynamical rule by which the connectivity matrix gets updated is driven by the search for profit, with the constraint that c_i is fixed. At each time step t, one firm assesses whether replacing any currently used supplier with any unused one can increase its profit. For instance, if $\mathcal{S}_{i,t} = 4, 9$ and $\overline{\mathcal{S}_i} = 3, 4, 9$, firm i will assess three cases: (1) do nothing, i.e., stay with suppliers 4 and 9, (2) replace supplier 4 by supplier 3, and (3) replace supplier 9 by supplier 3. In general, there are $c_i \times (c'_i - 1)$ possible switches, plus the "do-nothing" option. We index these $c_i \times (c'_i - 1) + 1$ options by integer r. Firm i estimates its profit for each option and chooses the one that is most profitable. By doing so, the network is updated, and all prices and production levels are adjusted according to Eqs. (5), (6), and (7), and the system moves to the next time steps.

To ensure that each firm gets its turn to rewire, we organize "rounds". A round, indexed by integer q, lasts for n time steps, over which each firm gets its chance to rewire, in a random order that changes at each round. Technically, the dynamics can be described as follows.

- Start of round q, at time t. Firms are randomly ordered.
- Suppose that the first firm to rewire is firm i. Time step t unfolds as follows.
  - Firm i identifies all possible rewiring options, including the "do-nothing" option.
  - For each option r, firm i estimates the corresponding profit $\mathbb{E}_r(\mathcal{P}_{i,t+1})$ using an counterfactual version of the connectivity matrix $\mathbb{M}^{(r)}$ in which the rewiring option is implemented. For instance, suppose that firm i evaluates the rewiring r which corresponds to replacing supplier a by supplier b. Matrix $\mathbb{M}^{(r)}$ is like $\mathbb{M}_t$, except that $\mathbb{M}_{ai,r} = 0$ and $\mathbb{M}_{bi,r} = 1$.
  - Firm i chooses the most profitable option, denoted by r^*, such that $\mathbb{M}_{t+1} = \mathbb{M}^{(r^*)}$.
  - All market variables get updated to the new equilibrium associated with $\mathbb{M}_{t+1}$ according to Eqs. (5), (6), and (7) leading to new values of productions, prices, wage, goods exchanged, and labor hired.
- At time step $t + 1$, the next firm of round q gets its turn to rewire. The same process unfolds. This process repeats itself until all firms have had their chance to rewire. Round q ends.

- Round $q+1$ starts at time step $t+n$. A new random order is drawn, and the same process is implemented.

If no firm rewires during a whole round, then the network has reached a stable configuration, i.e., a fixed point in the theory of dynamical systems. No isolated rewiring from a firm can further improve its profit. This situation corresponds to a Nash equilibrium in a game theoretical setting.

2.3. Network knowledge modifies how firms assess rewiring profitability

The main originality of our work is that, contrary to classical economic assumptions, firms may only have limited knowledge about the network structure and the other firms. Such limited information affect how firms estimate profit $\mathbb{E}_r(\mathcal{P}_{i,t+1})$. We define *tiers* in the sense of supply chains; see Fig. 1. Tier 0 represents the firm itself. Tier 1 corresponds to the direct suppliers and clients of a firm, tier 2 to the suppliers and clients of tier-1 firms, etc., thereby recursively defining tier τ. We now define the *tier scenarios*. In the tier-τ scenario, firms have enough information to estimate the next-time-step wage, prices, and production levels of firms in any tiers up to tier τ. What firms know or do not know about the other firms is detailed in the remained of this section.

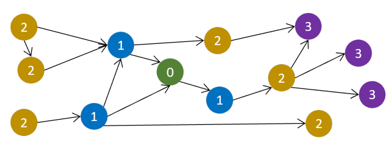

Figure 1: Firms have complete oversight up to a certain *tier*. In the network below, tier = 0 for the green firm means that it has full information only about itself and partial information about its direct suppliers and clients. It can only estimate the next-time-step wage, price, and production level for itself. When tier = 1, it has enough information to also estimate the next-time-step wage, prices, and production levels of the blue firms. When tier = 2, it does it for the blue and yellow firms. When tier = 3, it has perfect information on all the firms and is able to perfectly anticipate the general equilibrium.

Let us assume that firm i has oversight over its tier level τ and is considering the rewiring option r. The full notation of the ensemble of firms within this tier is $\mathcal{T}^{\tau}_{i,r}$, which will simply denote by $\mathcal{T}$ in the next few paragraphs. To evaluate the profitability of r, firm i will anticipate the state of the economy after rewiring, i.e., the wage $\mathbb{E}_r(h_{t+1})$, the price $\mathbb{E}_r(p_{k,t+1})$ and production $\mathbb{E}_r(x_{k,t+1})$ of the other firms $k=1,\ldots,n$. To improve readability, we will use $\mathbb{E}(p_k)$, $\mathbb{E}(x_k)$, and $\mathbb{E}(h)$ in the next few paragraphs.

- For all firms $k \in \mathcal{T}$, firm i will anticipate production $\mathbb{E}(x_k)$ and price $\mathbb{E}(p_k)$ by solving the equilibrium equations for the reduced network composed of those firms.
- Wage $\mathbb{E}(h)$ will be derived from those reduced network.
- For the other firms, firm i assumes that the production and price will not change, i.e., $\mathbb{E}(p_k) = p_{k,t}$ and $\mathbb{E}(x_k) = x_{k,t}$.

To form an expectation of price, $\mathbb{E}(p_k)$, production $\mathbb{E}(x_k)$, and wage $\mathbb{E}(h)$ of the reduced network composed of firms $k \in \mathcal{T}$, firm i needs to know:

- parameters a_k, b_k, z_k, which are assumed to be fixed in time,
- the input-output coefficients between any firm k and its suppliers, i.e., $w_{jk,t+1}$ for any supplier j, and the input-output coefficients between any firm k and its clients, i.e., $w_{kj,t+1}$ for any clients j.

Firm i also needs information for suppliers and clients of the firms of this reduced network, which may not be in the reduced network. It corresponds to firms that are in $\mathcal{T}^{(\tau+1)}_{i,r}$ but not in $\mathcal{T}^{\tau}_{i,r}$. It needs to know:

- parameters a_j, b_j, z_j of those firms j,
- their previous price $p_{j,t}$ and production $x_{j,t}$, which, as stated above, firm i assumes that they will not change.

Hence, the equation for the good market clearing of firm $k \in \mathcal{T}$, which was Eq. (7) in the baseline, becomes:

$$\mathbb{E}(x_k) = \underbrace{\frac{1}{\mathbb{E}(p_k)}}_{\text{final demand}} + \underbrace{\sum_{j \in \mathcal{S}_k \cap \mathcal{T}} (1-a_j) b_j w_{ij} \frac{\mathbb{E}(p_j)}{\mathbb{E}(p_i)} \mathbb{E}(x_j)}_{\text{demand from firms in } \mathcal{T}} + \underbrace{\sum_{j \in \mathcal{S}_k \setminus \mathcal{T}} (1-a_j) b_j w_{ij} \frac{p_{j,t}}{p_{i,t}} x_{j,t}}_{\text{demand from firms not in } \mathcal{T}} . \tag{8}$$

Similarly, Eq. (5) becomes:

$$x_k = z_k (b_k \mathbb{E}(p_k) \mathbb{E}(x_k))^{b_k} \mathbb{E}(h)^{-a_k b_k} \underbrace{\prod_{j \in \mathcal{S}_k \cap \mathcal{T}} \mathbb{E}(p_j)^{-(1-a_i) w_{ji} b_i}}_{\text{factors from suppliers in } \mathcal{T}} \underbrace{\prod_{j \in \mathcal{S}_k \setminus \mathcal{T}} p_{j,t}{}^{-(1-a_i) w_{ji} b_i}}_{\text{factors from suppliers not in } \mathcal{T}} . \tag{9}$$

For labor market clearing, firm i will estimate the new wage $E(h)$ based on the demand for labor in $\mathcal{T}$. It assumes that the total amount of labor hired by firms in $\mathcal{T}$ remains constant. Equation (6) then becomes:

$$\underbrace{\sum_{i \in \mathcal{T}} a_i b_i \frac{p_{i,t}}{h_t} x_{i,t}}_{\text{labor supply in } \mathcal{T}} = \underbrace{\sum_{i \in \mathcal{T}} a_i b_i \frac{\mathbb{E}(p_i)}{\mathbb{E}(h)} \mathbb{E}(x_i)}_{\text{labor demand in } \mathcal{T}} . \tag{10}$$

The firm i who got its turn to rewire solves these equations for each rewiring options. They form a system of $2 \times n_r + 1$ equations, where n_r is the cardinality of tier $\mathcal{T}_{i,r}^K$, i.e., for each $k \in \mathcal{T}$. Appendix 4 describes a procedure to solve those equations. It uses the resulting $\mathbb{E}(x_{i,t+1})$, $\mathbb{E}(p_{i,t+1})$, and $\mathbb{E}(h_{t+1})$ to calculate the estimated profits associated to each rewiring options r—note that we turn back to the full notations:

$$\mathbb{E}_r(\mathcal{P}_{i,t+1}) = \mathbb{E}(p_{i,t+1}) \mathbb{E}(x_{i,t+1}) - \mathbb{E}(h_{t+1}) \mathbb{E}(\ell_{i,t+1}) - \sum_{j \in \mathcal{S}_i} \mathbb{E}(p_{j,t+1}) \mathbb{E}(g_{ji,t+1}), \tag{11}$$

where,

$$\mathbb{E}(\ell_{i,t+1}) = a_i b_i \frac{\mathbb{E}(p_{i,t+1})}{\mathbb{E}(h_{t+1})} \mathbb{E}(x_{i,t+1}), \tag{12}$$

$$\mathbb{E}(g_{ji,t+1}) = (1-a_i) b_i w_{ji} \frac{\mathbb{E}(p_{i,t+1})}{\mathbb{E}(p_{j,t+1})} \mathbb{E}(x_{i,t+1}), \tag{13}$$

$\mathbb{E}(p_{j,t+1})$ being the result of solving Eqs. (8), (9), and (10) if $j \in \mathcal{T}_{i,r}^{\tau}$, or is $p_{j,t}$ otherwise.

Note that depending on the size of the network, when τ is high enough, the tier-τ scenario corresponds to the baseline case in which all firms have complete information on the network. One exception to this perfect foresight is the full technology matrix $\overline{\mathbb{W}}$, which remains unknown to firms in whatever scenarios. Firms do not know the potential suppliers of the other firms and never anticipate the rewiring behavior of the other firms. Another important remark is that firms know the households' budget B.

2.4. Scale-free initial topology and technological matrix

Instead of using Erdős–Rényi networks (Erdős and Rényi, 1959) as starting configurations, we initialize the topology of the system using a class of scale-free network proposed by Colon and Ghil (2017), which exhibits some empirical features of real production networks. It is based on the empirical study carried out by Fujiwara and Aoyama (2010) on a dataset of over 1 million Japanese firms and their supply relationships, provided by the company Tokyo Shoko Research Ltd. They found that the distributions of in- and out-degrees have a power-law distribution for degrees between 10 and 1000, followed by an exponential decay for higher degrees, and an average connectivity c of 4. The exponents of the cumulative distributions $P_>(k)$ of in- and out-degree are respectively 1.35 ± 0.02 and 1.26 ± 0.02, where the error corresponds to 1.96 times the estimated standard deviation. Note that the numbers of suppliers of each firm, c_i, are generated by this algorithm. The average of the distribution is controlled by parameter c.

We algorithmically generate networks that reproduce these features. As in Colon and Ghil (2017), we use the method first proposed by Goh et al. (2001), in the form used by Chung and Lu (2002). The number of incoming and outgoing links attributed to each node is proportional to a weight or load factor. For a network with n nodes, the factor of the i^{th} node is $(i + i_0 - 1)^{-1/(\gamma-1)}$, where γ is the targeted power-law exponent, and i_0 an integer used to introduce an exponential decay for high degrees. We use γ-values of 1.35 for in-degrees and of 1.26 for out-degrees, while i_0 is set to 12. The connectivity c is defined as the average number of in- and out-degrees per agent, and it is set to 4, i.e., there are 4 times more links than nodes. This value also corresponds to the average number of suppliers per firm, c_i.

This algorithm allows us the generate the initial connectivity matrix, $\mathbb{M}_0$, and the initial set of suppliers per firm $\mathcal{S}_{i,0}$. Next, for each firm i, we draw the number of extra suppliers c'_i from a binomial probability distribution of mean c', a free parameter, and bounded by $n - 1 - c_i$. We then randomly select a number c'_i of potential suppliers with equiprobable weights, thereby generating the set of possible suppliers $\overline{\mathcal{S}_i}$. Since the rewiring dynamics only consist of replacing a supplier with another, the number of suppliers c_i per firm is preserved, and the distribution of in-degrees remains scale-free. Last, we generate the input–output coefficients $\overline{w}_{ji}$ such that $\overline{w}_{ji} = \mathbb{M}_{ji}/c_i$.

2.5. Observables: characterizing dynamical regimes, visited configurations, and rewiring activity

We examine the trajectory of the system $\mathbb{M}_t$ in the space of networks for $t = 0, 1, \ldots, T$, where T is the time length of the simulations. We stop the simulation in one of the three following situations;

- when no firm rewire during a whole round—a fixed point is reached,
- when the same set of configurations repeat itself for five rounds in a row—a signal that a low-dimension attractor may have been reached,
- when a pre-defined number of round, denoted by $\rho_{\max}$, is reached.

We denoted by R the number of rounds reached in a simulation, such that $T = nR$.

We monitor the diversity of configurations visited during a simulation. To that end, we identify unique configurations and define a topological distance. Specifically, the distance between two configurations, say $\mathbb{M}_s$ and $\mathbb{M}_t$, is denoted by $d(\mathbb{M}_s, \mathbb{M}_t)$ and defined by

$$d(\mathbb{M}_s, \mathbb{M}_t) = 1 - \frac{\sum_{ij} \mathbb{M}_{i,j,s}\mathbb{M}_{i,j,t}}{\left(\sum_{ij} \mathbb{M}_{i,j,s}\right)\left(\sum_{ij} \mathbb{M}_{i,j,t}\right)}. \tag{14}$$

In other words, the distance measures the fraction of links that differ between two networks.

Using this distance, we cluster the visited configurations by using dendrograms. Each cluster is represented by the branch of a tree, whose location relative to the other branches is proportional to the distance between clusters. Similar techniques are used, for instance, to visualize the conformation space of large molecules (e.g., Becker and Karplus, 1997).

We characterize the rewiring dynamics by monitoring the time series of rewirings denoted by $\rho_t = 1, \ldots, T$, which is defined as follows: $\rho_t = 1$ if the firm that attempts to rewire at time t does rewire, $\rho_t = 0$ otherwise. Averaging ρ_t over the simulation characterizes the *rewiring rate* of the system; see Tab. 1. We also count the total number of rewirings per firm over the whole simulation, denoted by f_i, and evaluate the average number of rewirings per firm F.

Last, at each time step, we evaluate total utility U_t and how far firms are from their current maximum reachable profit. Specifically, we define, for each firm i and at each time step t, the *profit gap* $\theta_{i,t}$, which is the difference between the current profit and the profit that it could reach by making the best choice of suppliers, evaluated with complete visibility over the network. We study the time series of the *global profit gap* θ_t, which is the sum of the firms' profit gaps, i.e., $\sum_i^n \theta_{i,t}$.

2.6. Numerical simulation: parameter dependence and response to shocks

Table 1 sumarizes the parameters, variables, and observables used in the model. In the baseline, the production-related parameters are set as follows: $a_i = 0.5$, $b_i = 0.8$, $z_i = 1$ for all i in $1, .., n$, as in Bonart et al. (2014). As for baseline network-related parameters, the number of suppliers per firm is $c = 4$, the mean number of extra suppliers per firm is $c' = 4$, and the number of firms is $n = 50$.

In the baseline, we assume, as in standard economic models such as Acemoglu et al. (2012), that firms have complete visibility and information over the whole network. It is as if the tier level of each firm τ is equal to the network diameter. They perfectly anticipate how all firms in the network would adjust their production and prices in response to their rewiring decisions. We assume that firms do not anticipate the rewiring decisions of other firms in response to their own rewiring decisions.

We run simulations in the baseline case and characterize the rewiring dynamics and the set of connectivity matrices $\mathbb{M}_t = 1, \ldots, T$. We analyze the impact of randomly removing firms on the rewiring dynamics, then gradually increasing the network's size. Next, we decrease firms' oversight by decreasing their tier visibility τ in two ways.

First, we assume that all firms homogeneously share the same tier visibility, i.e., $\tau_i = \tau$ for all $i = 1, \ldots, n$, where integer τ is initially as large as the network diameter, then step-wise reduced until it is 0. Second, we analyze the case in which the τ_is are heterogeneous, i.e., each firm can have different visibility over their supply chain, which is likely to be the case in the real world. Specifically, we draw the τ_is according to a lognormal distribution with mean and standard deviation τ. We then observe the dynamic regime of the system according to changing values of τ.

3. Results

Section 3.1 illustrates the basic features of the model in the baseline—i.e., full information on the network and the other firms—using the example of a small network. In small networks, the system settles in stable configurations. In Sec. 3.2, we demonstrate that these stable configurations are generally not unique. The rewiring dynamics is highly path-dependent. Section 3.3 shows how the network reconfigures itself in response to shocks. In Sec. 5, we increase the size of the network and observe extremely long transients in which the system seems to not stabilize. Section 3.5 focuses on the effect of decreasing the oversight of firms. When they have only limited information on the network and the other firms, networks are caught in endless rewiring.

3.1. The rewiring process redistributes profits

In the baseline scenario, in which firms have a perfect knowledge of all firms and the interactions between them, the system reaches a fixed point. Figure 2 presents as an example the simulation of a 10-firm network, in which firms have one supplier and one extra one, i.e., $c_i = 1$ and $c'_i = 1$. All firm-level parameters are equal, i.e., $a_i = 0.5$, $b_i = 0.9$, $z_i = 1$. A fixed point is reached after four rounds and six rewirings. At this point, no isolated rewiring from a firm can further improve its profit. This situation corresponds to a Nash equilibrium. The profit gap, i.e., the difference between the current profit and the maximum profit reachable by a switch of supplier, is null.

Although all firms are similar when taken in isolation and although all supplier-buyer relationships are technologically equivalent—all $\overline{w}_{j,i}$ are equal for $j \in \overline{\mathcal{S}_i}$—their embedding in a heterogeneous web of relationships generates differentiated behavior and outcome. Some firms rewire more than others, some succeed in increasing profits, whereas others incur losses.

Each rewiring is followed by a change in the distribution of profits, which can be significant—see the second and fourth rewiring—or more moderate; see the first and third rewiring. A rewiring directly affects three firms—the buyer, the replaced supplier, and the replacing supplier—but indirectly modifies the profits of many other firms. Newly chosen suppliers are benefiting the most, often more than the rewiring firm itself. The increase in sales experienced by a newly chosen supplier induces positive ripple effects for its own suppliers and clients. On the other hand, dropped suppliers undergo losses that propagate along their supply chains.

Table 1: sumary of the variables, parameters, and observables used in the model. The symbol (∗) indicates free parameters.

Parameters	Definition	Default value
$U_{\max}$ (∗)	Maximum number of rounds simulated	1,000
n (∗)	Number of firms	50
c	Average number of suppliers per firm	4
c_i	Number of suppliers of firm i	see Sec. 2.4
c' (∗)	Average number of extra suppliers per firm	4
c'_i	Number of extra suppliers of firm i	see Sec. 2.4
$\overline{W}$	Technological matrix	see Sec. 2.4
$\overline{w}_{ji}$	Input–output coefficient from supplier j to buyer i	$\frac{\mathbb{M}_{ji}}{c_i}$
$\overline{\mathcal{S}_i}$	Set of possible suppliers of firm i	defined by $\overline{W}$
z (∗)	Firm productivity	1
a (∗)	Share of labor in the input mix	0.5
b (∗)	Overall return to scale	0.9
Variable	**Definition**	
$\mathcal{S}_{i,t}$	Set of suppliers of firm i	
$\mathbb{M}_t$	Connectivity matrix	
$x_{i,t}$	Production of firm i	
$p_{i,t}$	Price of firm i	
$\ell_{i,t}$	Labor hired by firm i	
$g_{j,i,t}$	Quantity of good j purchased by firm i	
Observable	**Definition**	**Formula**
R	Number of rounds reached by a simulation	Capped at $U_{\max}$. Stop conditions apply; see Sec. 2.2
T	Time length of the simulations	$n \times R$
ρ_t	Rewiring occurrence at time t	Directly observed. 0 or 1
γ_q	Rewiring rate at round q	$\langle \rho_t \rangle_{(q-1)n+1 \leq t \leq qn}$
Γ	Average rewiring rate	$\langle \rho_t \rangle_{1 \leq t \leq T}$
f_i	Number of rewirings done by firm i during a simulation	Directly observed
F	Average number of rewirings per firm	$\langle f_i \rangle_{1 \leq i \leq n}$
U_t	Household utility at time t	$-\sum_i \log(p_{i,t})$
$\mathcal{P}_{i,t}$	Profit of firm i at time t	Eq. (2)
$\theta_{i,t}$	Profit gap of firm i at time t	Difference between $\mathcal{P}_{i,t}$ and the maximum profit reachable with rewiring and full information
θ_t	Global profit gap at time t	$\sum_i^n \theta_{i,t}$
$d(\mathbb{M}_u, \mathbb{M}_w)$	Distance between configurations $\mathbb{M}_u$ and $\mathbb{M}_w$	$\frac{\sum_{ij} \mathbb{M}_{i,j,v} \mathbb{M}_{i,j,w}}{\left(\sum_{ij} \mathbb{M}_{i,j,v}\right)\left(\sum_{ij} \mathbb{M}_{i,j,w}\right)}$
ν	Diversity of final configurations	Probability that two systems with the same initial conditions end in two different final configurations

Overall, the total value added generated by this economy remains exactly constant. Profits are simply redistributed among firms. However, the reconfiguration of the network increases household utility by about 0.1%. Such an increase in utility is however not general: utility decreases in about 25% of the runs in similar networks.

3.2. Stable configurations are path dependent

Starting from the same initial network, multiple simulations lead to a wide range of stable configurations; see Fig. 3. Except for very small networks, with five firms or less, two simulations starting with the same initial configurations are extremely likely to terminate on two different stable configurations associated with different utilities for households. These final configurations are stable fixed points in the sense of dynamical systems but are generally not the system optimum in which household utility is maximized.

To characterize the diversity of stable configurations, we generate, for a given number of firms n and average number of alternative suppliers c', an ensemble of 100 initial configurations denoted by $\mathbb{M}_0^{(1)}$, $\mathbb{M}_0^{(2)}$, …, $\mathbb{M}_0^{(100)}$. For each of them, we run 100 simulations and evaluate the proportion of unique final configurations $\nu^{(1)}$, $\nu^{(2)}$, …, $\nu^{(100)}$, which are then averaged to provide a proxy for the diversity of final configurations ν.

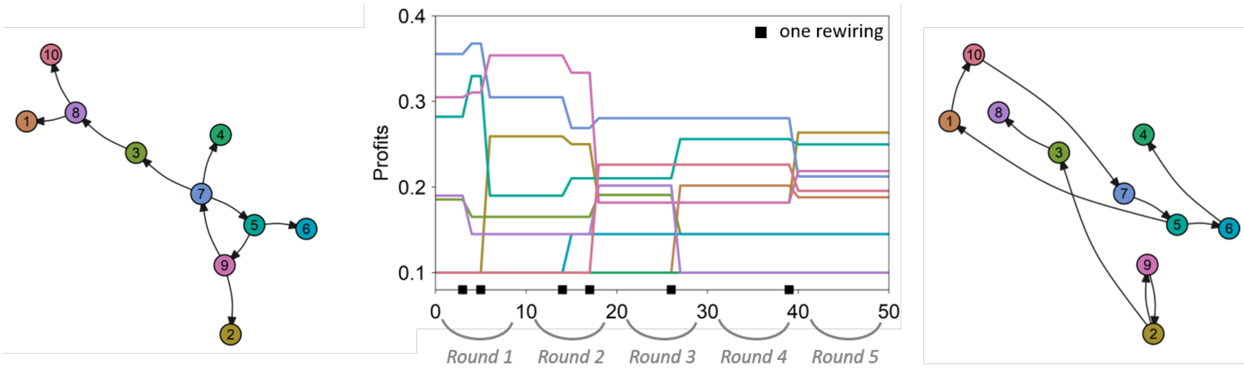

Figure 2: Profits are redistributed through rewiring. The initial 10-firm network displayed on the left panel evolves through the rewiring process presented on the central panel. It reaches a stable configuration shown on the right panel. To ease readability, the position of the nodes does not change between the left and right panels, and one color is assigned to each node. In the central panel, each curve indicates the time evolution of the profit of one firm. The color of the curves corresponds to that of the nodes in the left and right panels. On the x-axis, six black squares pinpoint the rewiring events.

The diversity of stable configurations grows with the number of firms involved, n—moving rightward on the x-axis of Fig. 3—but also with the range of technically feasible input–output relationships, c'—blue bars vs. green bars. The more suppliers firms can switch to, the more combinations of stable links. As noticed above, the total number of network configurations $\mathfrak{C}$ grows exponentially with n, and one can expect that the number of stable ones is also exponential in n, as is the case of generic combinatorial optimization problems. It would be interesting to see whether analytical methods used to enumerate the stable configuration of, e.g. spin-glasses (Mézard et al., 1986), can be applied to the present problem, as it has been done recently in the context of portfolio optimisation (Garnier-Brun et al., 2021), or of complex ecologies (Ros et al., 2022).

The origin of this marked path dependency lies in the random order according to which firms attempt to rewire. A new order is drawn at each round. This process heavily influences the trajectories. Given that many configurations can simultaneously satisfy all firms, the timing at which firms rewire does affect the final configuration.

A few metrics from the ensemble of simulations shown in Fig. 3 help characterize the geometry of the space of stable configurations. The distance, defined in Eq. (14), between the initial and final configuration is 0.29 for $c' = 2$ and 0.4 when $c' = 4$. In other words, on average, initial and final configurations differ by 29% of their edges when $c' = 2$ and by 40% when $c' = 4$. Stabilization results from a series of rewirings, which we call reconfiguration waves, that are larger the more supply options are available to firms. Similarly, the distance between stable configurations grows with the technological options: 12% of the edges when $c' = 2$ and 23% when $c' = 4$.

3.3. Shutting down firms triggers reconfiguration waves.

Removing a firm from a stabilized network triggers a wave of reconfigurations, during which the shock is digested by the network. Right after the removal, clients of the disappeared firm need new suppliers among their set of possible suppliers $\overline{\mathcal{S}_i}$. New supplier–buyer relationships are tied, which in turn modify the profitability of established commercial linkages. Gradually, across the web of input–output links, firms reassess their profits, and many change suppliers until a new stable configuration is reached. If the removed firm then re-enters the network, another wave of reconfigurations occurs. The stable configuration reached after this series of removal–reintroduction generally substantially differs from the original stable configuration.

Such dynamics is illustrated in Fig. 4 using dendrograms. The figure shows three waves of reconfigurations of a baseline network, i.e., 50 firms, $c = 4$ suppliers per firm, and $c' = 4$ extra suppliers per firm. The ensemble of network topologies visited during the rewiring dynamics is clustered according to the distance defined by Eq. 14. The dynamics can be described as follows.

- The initial wave of reconfigurations, shown in blue, lasts for 7 rounds. It transforms the unstable initial

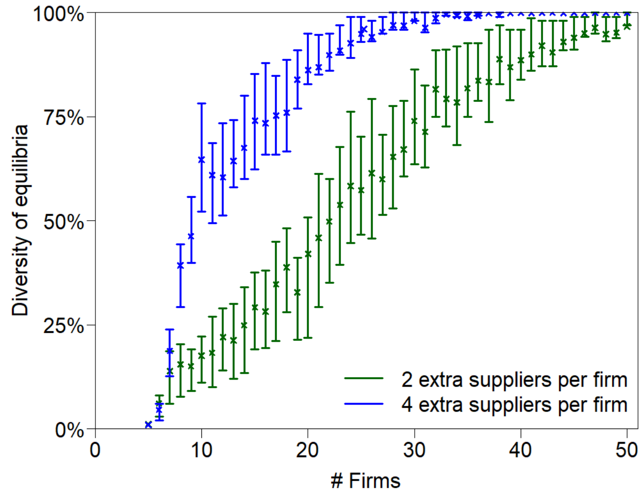

Figure 3: Rewiring generates multiple stable configurations from the same initial network. For each value of n between 5 and 50, the number of firms (x-axis), we generate 100 initial networks, and, for each of them, we simulate 100 times the rewiring dynamics ($100 \times 100 \times 46$ simulations in total). For each n, we evaluate the resulting diversity of final configurations, ν (y-axis). It ranges from 0%—all 100 simulations starting from the same initial network converge to the same configuration—to 100%—all final configurations are different; see text for details on how ν is calculated. The distributions are represented by two features: mean values are indicated by x-signs, and interquartile intervals by vertical bars.

configuration $\mathbb{M}_0$ into a stable one $\mathbb{M}_1$ after 109 rewirings. The resulting configuration differs from $\mathbb{M}_0$ by 79 links—i.e., 40% of its links.

- The removal of a firm then sparks a second wave shown in red. It also lasts for 7 rounds but only involves 51 rewirings. The resulting second stable configuration $\mathbb{M}_2$ differs from $\mathbb{M}_1$ by 36 links, but it is almost as far away from the starting point $\mathbb{M}_0$ as the first stable configuration $\mathbb{M}_1$, by 83 links instead of 79 links respectively.
- Reintroducing the missing firm sparks a third wave, shown in green. After 6 rounds and 28 rewirings, a third stable configuration $\mathbb{M}_3$ is reached, which differs by 19 links from $\mathbb{M}_2$ and by 83 links from $\mathbb{M}_0$.

Shocks do not necessarily push the system away from the initial configuration. Because of the very high dimensionality of the space of stable configurations, many stable configurations are equally distant from one another and also all roughly equally distant from the initial, unstable one. The configurations visited during a reconfiguration wave tend to be clustered together. In Figure 4(a), the major branches of the tree can be associated with high fidelity to specific waves in the statistical sense of classification algorithms. If we repeat such a series of removal–reintroduction, the system keeps drifting to new domains of the topological space. Figure 8 in Appendix C shows a simulation in which a 50-firms network undergoes a series of ten removals–reintroductions, and how each series of visited configurations get clustered.

3.4. Stabilization gets increasingly slow in large networks

The number of rounds needed to stabilize the system grows with network size; see Fig. 5. A 50-firm network with $c = 4$ and $c' = 4$ stabilizes, on average, after 7 rounds and 2.4 switches per firm, whereas a 300-firm network needs 22.5 rounds and 3.4 switches per firm. With more firms, each firm needs to readjust its supplier networks more to accommodate such changes.

This result is linked to the high level of interdependencies found in production networks. Unlike, for instance, road networks, new incoming nodes are always within a small reach from existing ones. Because of this so-called small-world feature, a single rewiring decision in a larger network affects the profitability of a larger number of commercial relationships. The network needs more time to digest a rewiring.

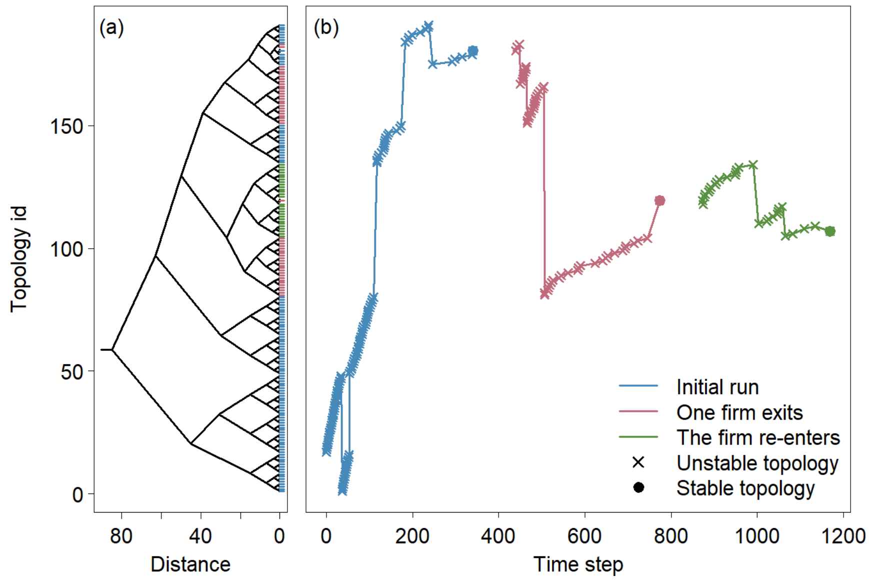

Figure 4: A localized shock on the network triggers a wave of reconfigurations. Panel (a) is a dendrogram. The leaves of the tree represent the configurations visited throughout the rewiring dynamics. They are vertically ordered by a clustering algorithm that groups networks based on their mutual distance defined by Eq. 14. The x-coordinate of the points at which two branches merge corresponds to the distance between the branches' most distant networks. Panel (b) shows the time series of the configurations visited. It shares the same y-axis as the dendrogram. The blue curve corresponds to the initial wave of reconfigurations, which converges to a first stable configuration. Then, the productivity of a randomly-chosen firm is set to 0. This shock sparks a second wave, colored in red, which ends in a second stable configuration. Last, the productivity of that firm is set back to 1, leading to the third wave in green. Baseline parameter values are used: 50 firms, 4 suppliers per firm, 4 extra suppliers per firm, $z = 1$, $a = 0.5$, $b = 0.9$, $\sigma = 0$.

In networks smaller than 200 firms, stabilization is longer when firms have more technological options; see the blue and green bars of Fig. 5. Firms explore more supply choices, leading to longer reconfiguration waves.

For larger networks, we increasingly encounter extreme cases in which stabilization takes a disproportionately long time. Below 200 firms, over 99.5% of the runs have stabilized within 50 rounds. But for larger networks, more than 10% of networks are still getting reconfigured after 50 rounds when $c' = 4$ and 5% after 100 rounds. Such disproportionately slower cases are even more prevalent when lower technological options are available; see the upwards trends for $n \geq 200$ and $c' = 2$ in Fig. 5. More than 29% of the rounds are not stabilized after 100 rounds and 7% are still not after 1,000 rounds.

Whereas limited technological options (low c') accelerate stabilization below 200 firms, the reduced number of possible choices makes it increasingly difficult for larger networks to find a configuration that satisfies all the firms. In other words, limited choices of suppliers quickly guide the network towards stable configurations when the degree of interdependencies is relatively loose. But when firms are added and the number of conflicting interests to be resolved increases, the potential solutions are harder to find, and firms need to search for a longer time. This is reminiscent of what happens in so-called Constraint Satisfaction Problems, see e.g. (Zdeborová, 2009; Sharma et al., 2019), and refs. therein.

3.5. Partial knowledge of the network leads to endless rewiring

The previous results correspond to a situation in which firms have full visibility over their supply chains. They know all network flows, all firm-specific parameters and variables, and can take them into account when formulating expectations on the consequences of their rewiring decisions. The number of tiers on which they have oversight is large enough such that it covers the whole network. Those simulations, located at the extreme right of each Fig. 6's left panels (a-c), always lead to a fixed point network configuration.

Now let firms have only a partial knowledge of their supply chains, i.e., moving leftward in Fig. 6's panels. Under a certain tier level, the network does not stabilize any longer and is caught in endless rewirings.

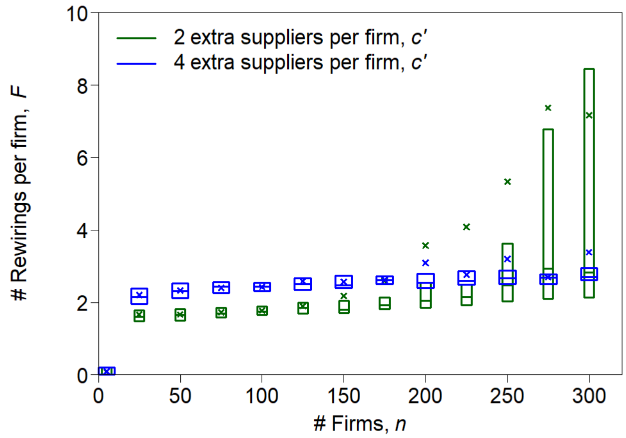

Figure 5: Stabilization becomes increasingly slow for large networks. For each value of n, i.e., the number of firms, we run 100 simulations with different initial configurations and measure the number of rewirings after which the system reaches a fixed point. If a system has not stabilized after 1,000 rounds, we stop the simulation and count the total number of rewirings. The rectangles indicate the interquartile intervals, the solid horizontal lines indicate the medians, and the X-signs indicate the means.

Figure 9 illustrates such a situation with a 60-firm network in which all firms have tier-2 visibility. During 40 rounds, the system keeps rewiring and visits 1,074 configurations, which are all unique. Even for a small network of 20 firms, when all firms have the same tier-1 visibility—which leads in the network topology used to have full information on about 40% of the firms—, the network does not stabilize.

When firm visibility is homogeneous—i.e., $\tau_i = \tau$ for all firm i—we observe a clear bifurcation point along tier visibility τ; see the left panels (a–c) of Fig. 6. Over this point, networks stabilize; under it, it rewires forever. The tier value at which the systems switches from stable to unstable dynamics varies with network size. With 20 firms, the bifurcation occurs at tier 2; with 60 firms, at tier 3; with 100 firms, between tier 3 and 4; see Fig. 6. We checked that the number of potential suppliers, c', does not modify these bifurcation points.

When firm visibility is heterogeneous, we observe a smoother transition. Even when visibility is high on average, we still observe cases of endless rewiring. For instance, in a 60-firm network, about 45% of the run do not stabilize when the average tier visibility is 5, whereas it almost always stabilizes for this tier level in the homogeneous case; see panel (e) vs. panel (b) of Fig. 6. Conversely, even when visibility is low, it may stabilize if some firms have large visibility.

When fixed points are not reached any longer, we observe unstable trajectories in which the system keeps on drifting across the very large network space, but also others in which the systems remain within a smaller set of configurations—a small-dimensional attractor in the theory of dynamical systems. In Fig. 6 we report limit cycles, in which the network is trapped into two configurations and continuously alternates between them. Figure 10 in Appendix C provides an example of a 4-dimensional attractor.

On stable trajectories, the rewiring rate—i.e., the proportion of time steps in which rewirings occur—is much lower than on unstable trajectories. In 60-firm networks, the rewiring rate of stable trajectories is about 25% vs. 68% for unstable trajectories. The rewiring rate does not change with tier level for stable trajectories, whereas, for unstable trajectories, the lower the visibility, the higher the rewiring rate: 46%for tier 2, 75% for tier 1, and 100% for tier 0. All simulations for tier 0 have a 100% rewiring rate. When firms do not take into their suppliers' and clients' reactions to their own rewiring decisions, they make erroneous anticipations about the profitability of these decisions.

Tier visibility largely reduces profit gaps, even for systems in permanent rewiring. Figure 7 shows the global profit gap averaged over the last 5 rounds of the simulations, i.e., $\langle\theta_t\rangle_{T-5n\leq t\leq T}$, for different tier visibility. When firms consider their direct suppliers and clients in their rewiring decisions (tier 1) compared to being fully myopic (tier 0), the gap is on average reduced by a factor of 6 in the homogeneous case (panel a), and 2 in the heterogeneous case (panel b). Extending visibility to tier 2 compared to tier 1 further significantly reduces the profit gap by a factor of 10 in the homogeneous case and a factor 2 in the heterogeneous case. The higher dispersion of the global profit gaps in Fig. 7b results from the large profit gaps of firms with low tier-visibility.

As expected, increased oversight over the network drastically improves the capacity of firms to correctly assess their rewiring options and helps them make better decisions. Having visibility over the supply chain, both upstream toward suppliers and downstream toward clients, decreases the intensity of rewiring, stabilizes the rewiring dynamics, and helps firms choose their suppliers to maximize profit.

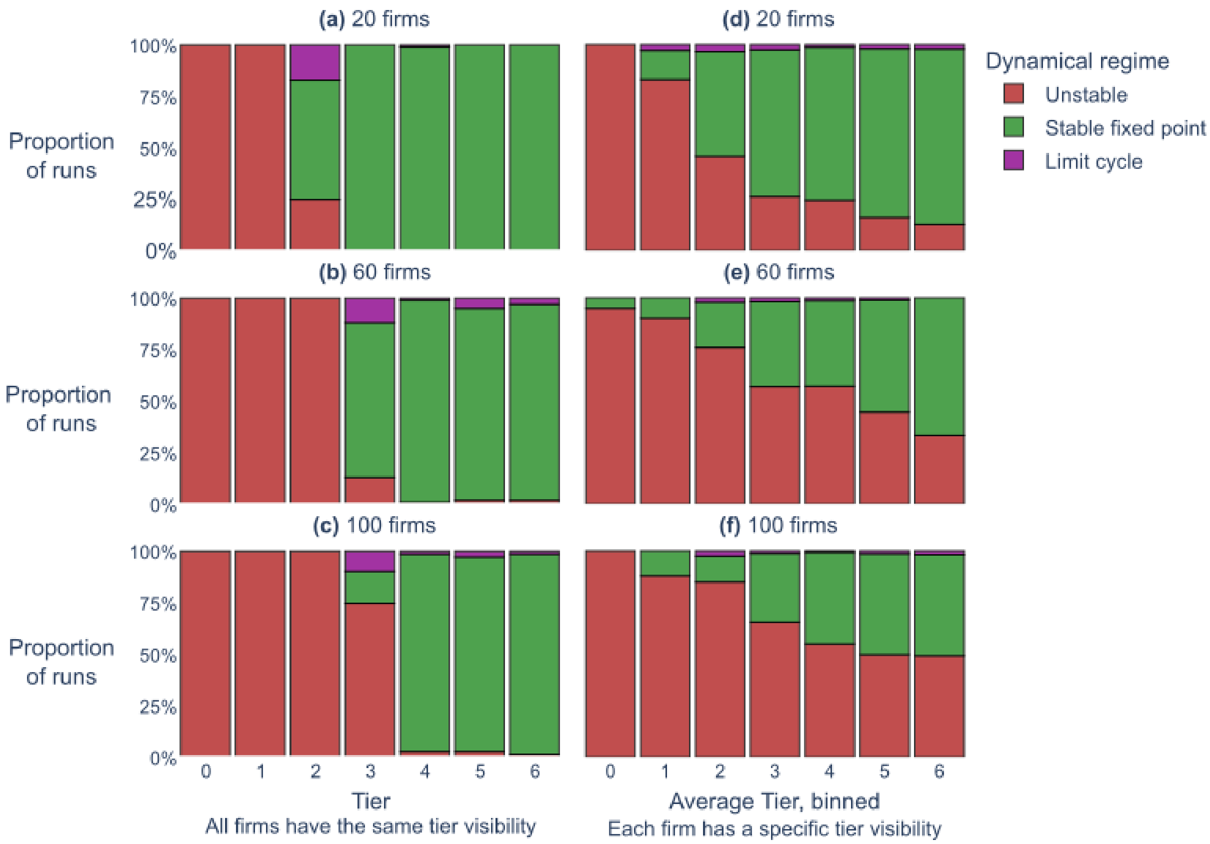

Figure 6: Transition from stability to instability when tier visibility is reduced. In each of the left panels (a-c), for each tier-value τ shown on the x-axis, we run 50 simulations with $c' = 2$ and 50 simulations with $c' = 4$ and $U_{\max} = 50$. In each system, all firms have the same tier visibility, i.e., $\tau_i = \tau$ for all $i = 1, \ldots, n$. We evaluate the proportion of runs falling into three dynamical regimes: stable fixed points, unstable trajectories, and period-2 limit cycles. Note that any run that has not reached fixed points nor limit cycles after 50 rounds is considered unstable. This is motivated by Sec. 3.4's result that, below 200 firms, over 99.5% of the runs have stabilized within 50 rounds. In the right panels (d-f), firm-level tier visibility is heterogeneous. For each tier value shown along the x-axis, we run 100 simulations in which parameters τ_is are drawn according to a log-normal distribution whose mean and standard deviation corresponds to that tier value. Among those 100 simulations, 50 are run with $c' = 2$ and 50 simulations with $c' = 4$.

4. Discussion & Perspective

By adding a simple rewiring dynamics into a standard economic model—with Cobb-Douglas production functions, instantaneous market clearing with frictionless price and production adjustments, and complete information of firms—we uncover the "radical complexity" feature of supplier–buyer networks. As found in many Constrained Satisfaction Problems, the landscape of locally stable network configurations, in which all firms cannot individually increase their profit by a change of supplier, is extremely vast and ruggeed. These

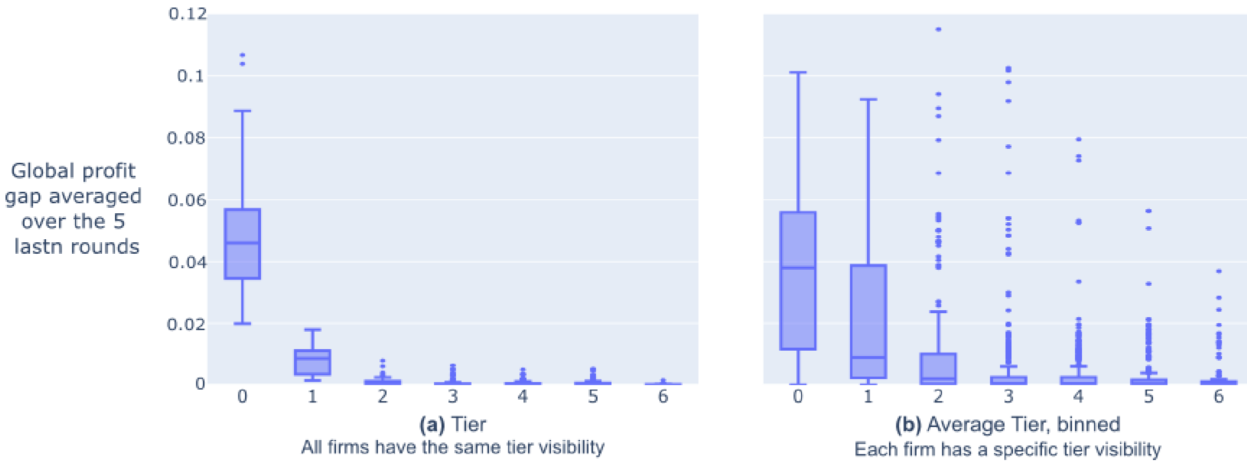

Figure 7: Tier visibility enhances profits. The two panels show how the distribution of the global profit gap, averaged over the last 5 rounds of each simulation, changes with tier visibility. The same simulations as in Fig. 6 are used. The rectangles indicate the interquartile range, the solid horizontal lines indicate the median. The upper fence corresponds to the third quartile plus 1.5 times the interquartile range, The lower fence corresponds to the first quartile mine 1.5 times the interquartile range. The dots are the data points outside the fences.

configurations furthermorey stand far apart in the topological space; see Sec. 3.2. They all correspond to a different level of utility for households, which is not globally maximized.

The only difference with a pure neoclassical framework of rational expectations is that firms do not anticipate the subsequent rewiring decisions of other firms when making their own rewiring decisions. If they were to do so, as in a game theoretic setting of perfect information in an infinitely long game, it would imply that they are able to anticipate chains of hundreds of rewiring, which themselves depend on the random order at which firms rewire. That would require solving, before each single rewiring decision, a very complex combinatorial problem that would exponentially grow in complexity. In such "radically complex" situations, the full rationality assumption is untenable; as argued long ago by H. Simon, agents and firms will opt for "satisficing" solutions.

In fact, our baseline scenario is already very optimistic about the ability of firms to get accurate information and process it. Whether the locally stable fixed points can be reached in practice is highly debatable. Quick relaxations towards stable configurations appear to be only occurring in small networks, which were, due to computational constraints, the focus of our investigation. In networks only as small as than 200 firms—a country like France has more than three million firms!—stable configurations seem harder, if not impossible, to reach; see Sec. 3.4. Because all these firms are only a few links away from each other—the small-world features of production networks are now well established (Bacilieri et al., 2022)—, every rewiring has reverberating effects on all the firms.

Any exit or entry of firms sparks new reconfiguration waves, by which the network "digests" the new conditions; see Sec. 3.3. Entries and exits are very common—Bacilieri et al. (2022) found that 20% of firms enter or exit each year. Other types of shocks on technologies, regulations, or market conditions would also spark such waves, which involve a number of rewirings that increases with network size. Switching supplier takes time, from a few days to a few months. Even if firms had full visibility over the network, reaching a stable configuration is an extremely long process, almost infinite on economic time scales for real networks of millions of firms. As also argued in (Dessertaine et al., 2022), real systems are most likely out of equilibrium, always in the process of digesting new shocks.

Our work also suggest that, because of firms' limited visibility over their supply chain, stable configurations may not even exist. In network of a few hundred firms, having visibility over tier 1 or 2 suppliers or clients is not enough to reach any equilibrium. In other words, even without shocks, the supplier-buyer network is not expected to be stable, but rather to endogenously reconfigure itself indefinitely. It is also not expected to stay close nor to progress towards any "optimal" configuration. Still, from the point of view of firms, increasing oversight drastically improves their capacity to correctly assess rewiring options and make better decisions. But, even with tier-2 visibility, which is in practice already challenging, one cannot hope that the network will reach any stable configuration. Even a few short-sighted firms can destabilize the network. Therefore, we argue that one cannot assume that supplier–buyer networks are in equilibrium.

The unstable dynamics of production networks is, in fact, very diverse. By looking at the return of similar configurations, we were able to detect low-dimension attractors, in which the system oscillates between a limited set of topologies, as in Fig. 10. Other trajectories, instead, seem to relentlessly explore new configurations, without ever falling back to any previously visited ones, as in Fig. 9. Interestingly, the dynamics is sometimes *intermittent*: sudden rewiring avalanches can spontaneously break out; see Fig. 11. This completely endogenous behavior suggests that production networks can undergo sudden reconfiguration waves without external shock. Finding some warning signals that anticipate those avalanches would be of great interest.

What would in reality counteract the rewiring dynamics and slow down further convergence to equilibrium are transaction costs and frictional factors. Transaction costs linked to searching for new suppliers, evaluating them, and contracting a new supply relationship are preventing firms from choosing potentially profit-enhancing suppliers and favoring the status quo. Rewiring may also be impeded by the limited ability to quickly and optimally adjust the quantity of labor and capital to a different input mix.

We should emphasize that the present study is mostly numerical, and lacks at this stage a more analytical approach to the results presented above. Our numerical study only constitutes a cursory exploration of such exotic dynamics, and more in-depth analyses should be carried out, supplemented, when possible, with a rigorous mathematical analysis. The reason why we are confident that these results are indeed robust and generic relies on the strong analogies between our model and the host of similar "rough landscape" problems that have been studied in great depth in the physics literature for the past 40 years. The exponentially large number of locally optimal solutions and hyper-slow dynamics are now well-known to characterize these problems. We are convinced that similar effects exist in many situations of economic interest (see also Galla and Farmer (2013), Garnier-Brun et al. (2021) and Sharma et al. (2022)). Due to the inherent complexity of the optimization problems that economic agents have to solve, one cannot assume that economies are in equilibrium in any meaningful way—a new paradigm we would like to call "radical complexity" (Bouchaud, 2021).

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Appendix A. Solving the market-clearing equations in case of perfect knowledge

This section presents a method to solve the system of $2n+1$ equations formed by Equations (5), (6), and (7). It is similar to that of Acemoglu et al. (2012) and Bonart et al. (2014), expect that $\sum_i w_{ij} = 1 + \epsilon \neq 1$. This system can be solved by introducing n variables $v_1, \dots, v_n$ defined by $v_i = p_i x_i$. Equations (7) become

$$v_i = \frac{1}{n} + \sum_{j=1}^{n} (1 - a_j) b_j w_{ij} v_j, \tag{15}$$

which form a linear system of n equations. They can be written in the following vectorized form: $V = A/n + \tilde{W}V$, where $V = (v_1, \dots, v_n)$ and $\tilde{W}$ is a matrix whose elements are $\tilde{w_{ij}} = (1 - a_j) b_j w_{ij}$. We can solve it for V: $V = (I - \tilde{W})^{-1} 1/n$, where I is the identity matrix.

Knowing all variables $v_1, \dots, v_n$, we can infer the other variables using equations (5) and (6). Specifically, introducing v_i into Eq. (6) yields the wage

$$h = \sum_{j=1}^{n} (1 - a_j) b_j \frac{v_j}{n}. \tag{16}$$

Next, introducing v_i into Eq. (5) yields

$$p_i = {z_i}^{-1} {b_i}^{-b_i} {v_i}^{1-b_i} h^{a_i b_i} \prod_j {p_j}^{(1-a_i) w_{ji} b_i}. \tag{17}$$

By taking the logarithm of the n equations (17), we obtain a linear system of n equations that we can solve for $\log(p_i)$. Knowing $\log(p_i)$ and v_i for $i = 1, \ldots, n$, we can infer all prices p_i and production x_i. Last, the quantity of labor hired by each of the firms and the quantity of goods they purchase, ℓ_i and g_{ji}, can be found using Eqs. (3) and (4).

Appendix B. Solving the market-clearing equations in case of limited knowledge

This section presents a method to solve the system of equations formed by Equations (9), (10), and (8), i.e., when the rewiring firm i know only its tier $\mathcal{T}_{i,r}^K$. There are $2 * n_r + 1$ equations, where n_r is the cardinality of tier $\mathcal{T}_{i,r}^K$. As in App. A, this system can be solved by introducing n_r variables v_j defined by $v_j = p_j x_j$. We use lightened mathematical notations, i.e., $\mathbb{E}(\bullet)$ instead of $\mathbb{E}_r(\bullet_{t+1})$, $\mathcal{T}$ instead of $\mathcal{T}_{i,r}^K$, and we drop the time subscript $_t$ for prices, production levels, and wage, which all relate to the present equilibrium. Equations (18) become

$$\mathbb{E}(v_k) = 1 + \sum_{j \in \mathcal{S}_k \cap \mathcal{T}} (1 - a_j) b_j w_{kj} \mathbb{E}(v_j) + K_1 \tag{18}$$

where

$$K_1 = \sum_{j \in \mathcal{S}_k \setminus \mathcal{T}} (1 - a_j) b_j w_{kj} v_j \tag{19}$$

These n_r equations can be written in the following vectorized form: $\mathbb{E}(V) = (1 + K_1)I + \tilde{W}\mathbb{E}(V)$, where $V = (v_j, j \in \mathcal{T})$, I is the identity matrix, and $\tilde{W}$ is a matrix whose elements are $\tilde{w_{ij}} = (1 - a_j) b_j w_{ij}$. We can solve it for $\mathbb{E}(V)$, such that $\mathbb{E}(V) = (I - \tilde{W})^{-1}(1 + K_1)$.

Introducing v_k into Eq. (10) yields the wage

$$\mathbb{E}(h) = \frac{1}{K_2} \sum_{k \in \mathcal{T}} a_k b_k \mathbb{E}(v_k) \tag{20}$$

where

$$K_2 = \sum_{k \in \mathcal{T}} a_k b_k \frac{v_k}{h}. \tag{21}$$

Since the $\mathbb{E}(v_k)$s are know for $k \in \mathcal{T}$, we can deduct $\mathbb{E}(h)$.

Next, introducing v_i into Eq. (9) yields

$$\mathbb{E}(p_k) = {z_k}^{-1} b_k^{-b_k} \mathbb{E}(v_k)^{1-b_k} \mathbb{E}(h)^{a_k b_k} \prod_{j \in \mathcal{S}_k \cap \mathcal{T}} \mathbb{E}(p_j)^{(1-a_i) w_{ji} b_i} K_3 \tag{22}$$

where

$$K_3 = \prod_{j \in \mathcal{S}_k \setminus \mathcal{T}} {p_{j,t}}^{(1-a_i) w_{ji} b_i}. \tag{23}$$

By taking the logarithm of the n_r equations (22), we obtain a linear system that we can solve for $\log(\mathbb{E}(p_k))$. We can deduct $\mathbb{E}(p_k)$ for $k \in \mathcal{T}$. Knowing $\mathbb{E}(h)$, $\mathbb{E}(p_k)$ and $\mathbb{E}(v_k))$ for $k \in \mathcal{T}$, we can infer the estimated production levels $\mathbb{E}(x_k)$. Last, the quantity of labor hired by each of the firms and the quantity of goods they purchase, ℓ_i and g_{ji}, can be found using Eqs. (3) and (4).

Appendix C. Additional time series

Additional time series are presented to give the reader a wider view of the dynamical behavior of the model.

- Figure 8 presents a longer run similar to that shown in Fig. 4 in Sec. 3.3. Sequential removal-reintroductions of single firms make the system wander across new regions of the topological space.
- Figure 9 presents the simulation of a 60-firm network in which all firms have tier-2 visibility. The system keeps rewiring and vvisits1,074 configurations, which are all unique. The average rewiring rate is 45%, and most firms take part in rewiring.
- Figure 10 presents the simulation of a 20-firm network in which all firms have tier-1 visibility. After about 180 time steps, i.e., 9 rounds, the network alternates between four configurations. Only two firms, firm 12 and firm 17, keep on changing suppliers. Firm 12 alternates between firms 17 and 2 (red dashed arrows), and firm 17 between firms 18 and 3 (green dashed arrows). These two firms are very interdependent. They are sometimes directly linked, and firms 3 and 18, which supply firm 17, are both supplied by firm 12. Because they only anticipate the impact of their rewiring decisions on their direct clients and suppliers, they always miss some part of their intricate interdependencies. Some rewiring decisions lead to increased profit, as expected, such as firm 12 switching from firm 17 to firm 2. Still, the anticipation of the profit generated by the reverse rewiring is erroneous, incentivizing firm 12 to switch back to firm 17 when it is not profitable to do so.
- Figure 11, we first observe the usual pattern: a short transient of about ten rounds (600 time steps) in which most firms rewire, followed by a more moderate background rewiring activity driven by a few firms. But a series of rewiring after 1,100 time steps lead to a surge of rewiring activity. In that period, some firms that had remained quiet since the early transient started rewiring again for a short period, such as firms 39 and 41. A second, milder surge occurs around time step 1,600, which cools down again, and the system falls back to a regime with lower rewiring activity. After that, some firms restart to rewire frequently, such as firms 32 and 33. Others stop rewiring, such as firm 58.

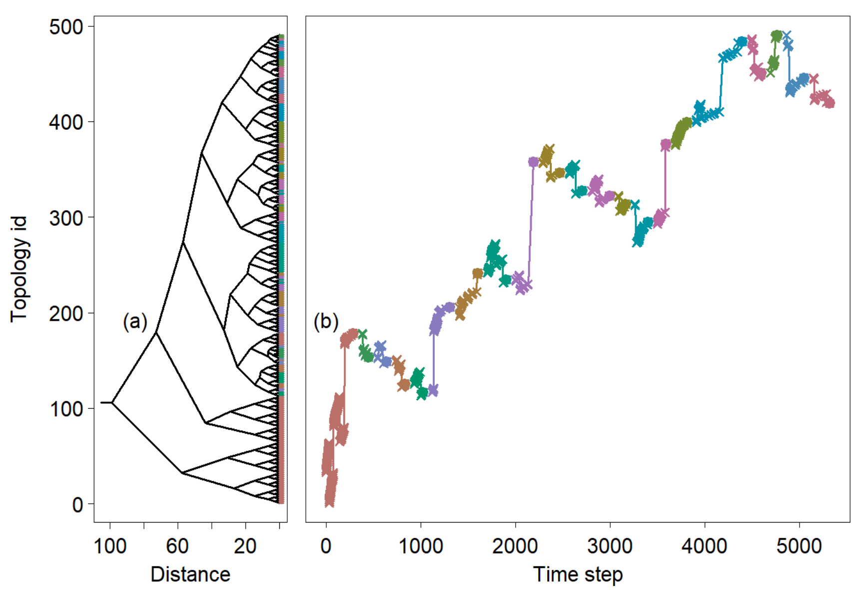

Figure 8: Reconfiguration waves triggered by ten series of removal–reintroduction of a single firm. Details on how to read both panels are given in the caption of Fig. 4. It is a 50-firm network with baseline parameter values: $c = 4$, $c' = 4$, $z = 1$, $a = 0.5$, $b = 0.9$.

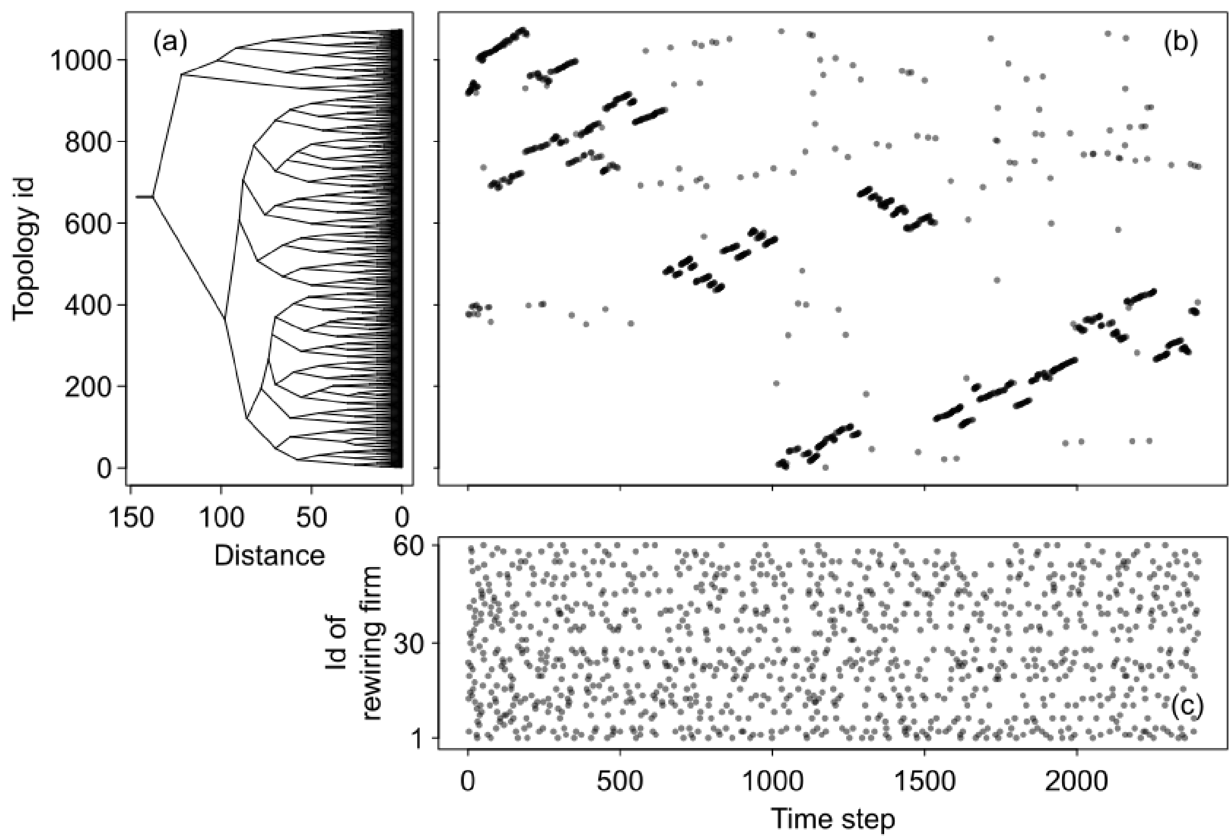

Figure 9: Example of an unstable trajectory. The system is a 60-firm network with 2 suppliers and 2 extra potential suppliers per firm. All firms have tier-2 visibility. Panel (a) presents a dendrogram of the 1,074 configurations visited during the simulation, which are all unique. Panel (b) is the time series of the visited configurations; the y-axis coincides with that of the dendrogram. Panel (c) is the time series showing the id of the rewiring firms.

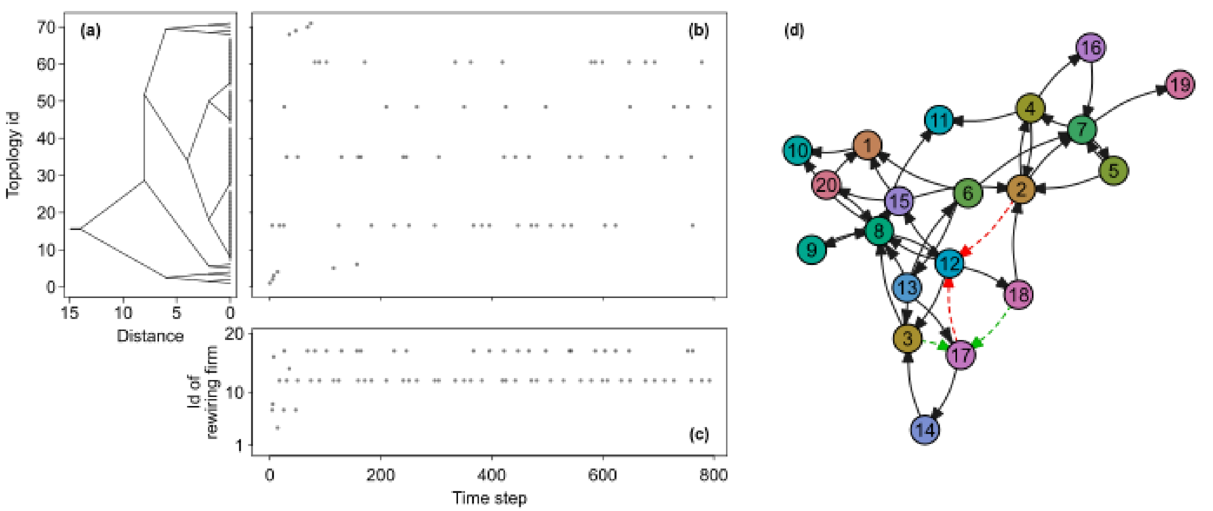

Figure 10: Example of an unstable trajectory reaching an attractor. The system is a 20-firm network with 2 suppliers and 2 extra potential suppliers per firm. All firms have tier-1 visibility. Panel (a) presents a dendrogram of the 71 configurations visited during the simulation, which boils down to 14 unique configurations. Panel (b) is the time series of the visited configurations; the y-axis coincides with that of the dendrogram. Panel (c) is the time series showing the id of the rewiring firms. Panel (d) illustrates the resulting configuration, in which firm 12 switches between firm 2 and firm 12 (red dashed arrows) and firm 17 between firm 3 and firm 18 (green dashed arrows).

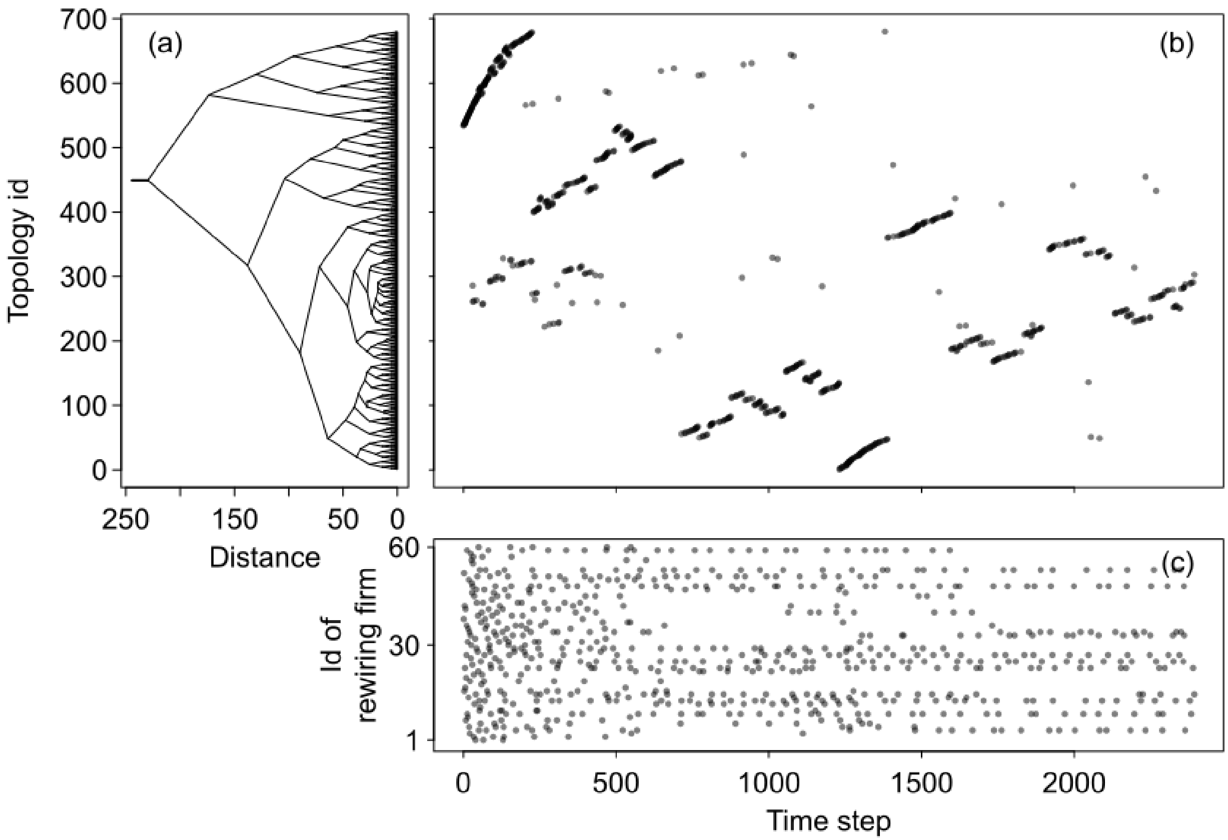

Figure 11: Example of an unstable trajectory with rewiring avalanche. The system is a 60-firm network with 24 suppliers and 4 extra potential suppliers per firm. Firms' visibilities τ_i are tier-2 on average and are lognormally distributed. are tier-1 visibility. Panel (a) presents a dendrogram of the 680 configurations visited during the simulation, which are all unique. Panel (b) is the time series of the visited configurations; the y-axis coincides with that of the dendrogram. Panel (c) is the time series showing the id of the rewiring firms.