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A PRIORI ESTIMATES AND OPTIMAL FINITE ELEMENT APPROXIMATION OF THE MHD FLOW IN SMOOTH DOMAINS

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Abstract. We study a finite element approximation of the initial-boundary value problem of the 3D incompressible magnetohydrodynamic (MHD) system under smooth domains and data. We first establish several important regularities and *a priori* estimates for the velocity, pressure and magnetic field $(\mathbf{u}, p, \mathbf{B})$ of the MHD system under the assumption that $\nabla\mathbf{u} \in L^4(0,T;L^2(\Omega)^{3\times 3})$ and $\nabla\times\mathbf{B} \in L^4(0,T;L^2(\Omega)^3)$. Then we formulate a finite element approximation of the MHD flow. Finally, we derive the optimal error estimates of the discrete velocity and magnetic field in energy-norm and the discrete pressure in L^2-norm, and the optimal error estimates of the discrete velocity and magnetic field in L^2-norm by means of a novel negative-norm technique, without the help of the standard duality argument for the Navier-Stokes equations.

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1. Introduction

This work is concerned with the following 3D incompressible magnetohydrodynamic system that couples the incompressible Navier-Stokes equations with Maxwell equations under the influence of body forces:

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + \nabla p + (\mathbf{u}\cdot\nabla)\mathbf{u} + \tau\,\mathbf{B}\times(\nabla\times\mathbf{B}) = \mathbf{f} \quad \text{in } \ \Omega\times(0,T], \\ \frac{\partial \mathbf{B}}{\partial t} + \mu\nabla\times(\nabla\times\mathbf{B}) - \nabla\times(\mathbf{u}\times\mathbf{B}) = 0 \quad \text{in } \ \Omega\times(0,T], \\ \nabla\cdot\mathbf{u} = 0 \quad \nabla\cdot\mathbf{B} = 0 \quad \text{in } \ \Omega\times(0,T], \end{cases} \tag{1.1}$$

which hold for all $\mathbf{r} = (x, y, z) \in \Omega$ and $t \in (0, T]$. Here Ω is an open bounded domain in R^3 with a smooth boundary, $\mathbf{u}$, p and $\mathbf{B}$ stand for the velocity, pressure and magnetic field, the two parameters ν and μ are the reciprocals of the Reynolds number Re and the magnetic Reynolds Re_m respectively, and the constant $\tau = M^2/(Re\, Re_m)$ is the coupling number, with $M > 0$ being the Hartman number.

For convenience, we shall often write the pressure p as $p(t)$ or $p(\mathbf{r}, t)$, velocity $\mathbf{u}$ as $\mathbf{u}(t)$ or $\mathbf{u}(\mathbf{r}, t)$, and the magnetic field $\mathbf{B}$ as $\mathbf{B}(t)$ or $\mathbf{B}(\mathbf{r}, t)$. Usually the system (1.1) is complemented with the following initial and

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boundary conditions [3, 12, 19, 23, 25, 28]:

$$\begin{aligned}
&\mathbf{u}(0) = \mathbf{u}_0(\mathbf{r}), \quad \mathbf{B}(0) = \mathbf{B}_0(\mathbf{r}) \text{ in } \Omega, \\
&\mathbf{u} = 0, \quad \mathbf{B}\cdot\mathbf{n} = 0, \quad \mathbf{n}\times(\nabla\times\mathbf{B}) = 0 \quad \text{ on } \partial\Omega\times[0,T],
\end{aligned} \tag{1.2}$$

where $\mathbf{u}_0$ and $\mathbf{B}_0$ satisfy that $\nabla\cdot\mathbf{u}_0(\mathbf{r}) = 0$ and $\nabla\cdot\mathbf{B}_0(\mathbf{r}) = 0$, with $\mathbf{n}$ being the unit exterior normal to $\partial\Omega$.

Remark 1.1. Instead of the boundary conditions that $\mathbf{B}\cdot\mathbf{n} = 0$ and $\mathbf{n}\times(\nabla\times\mathbf{B}) = 0$ in (1.2) for the magnetic field $\mathbf{B}$, we can equally consider the boundary condition $B\times\mathbf{n} = 0$, which is also frequently used for the MHD system; see, *e.g.*, [11, 16, 17, 19, 21, 27].

Remark 1.2. In this work we consider only the case that the domain Ω is smooth or convex, so the magnetic field $\mathbf{B}$ has the H^1-regularity and can be approximated by the standard H^1-conforming Lagrangian finite elements. This may make the numerical realization of the resulting discrete system very convenient as the velocity and pressure of the MHD flow are often approximated by the H^1-conforming Lagrangian finite elements in this case. But for more practical applications where the domains are not smooth and non-convex, *e.g.*, non-convex polyhedral domains with reentrant corners, the magnetic field $\mathbf{B}$ is not H^1-regular, then we may need to apply other types of finite elements that are not H^1-conforming, such as edge finite elements as it was done in [25] for stationary MHD system.

For our subsequent analysis, we introduce the following Sobolev spaces

$$M = L^2_0(\Omega), \quad \mathbf{X} = H^1_0(\Omega)^3, \quad \mathbf{W} = \{\mathbf{C}\in\mathbf{H}^1(\Omega); \quad \mathbf{C}\cdot\mathbf{n}|_{\partial\Omega} = 0\},$$

$$\mathbf{H} = \{\xi\in L^2(\Omega)^3, \quad \mathrm{div}\xi = 0, \quad \xi\cdot\mathbf{n}|_{\partial\Omega} = 0\},$$

$$\mathbf{V} = \mathbf{X}\cap\mathbf{H}, \quad \mathbf{W}_0 = \mathbf{W}\cap\mathbf{H}, \quad \mathbf{H}^k(\Omega) = H^k(\Omega)^3 \ \ (k\ge 1)$$

and the following two trilinear forms

$$\begin{aligned}
b(\mathbf{w},\mathbf{u},\mathbf{v}) &= \frac{1}{2}((\mathbf{w}\cdot\nabla)\mathbf{u},\mathbf{v})_\Omega - \frac{1}{2}((\mathbf{w}\cdot\nabla)\mathbf{v},\mathbf{u})_\Omega \\
&= ((\mathbf{w}\cdot\nabla)\mathbf{u} + \frac{1}{2}(\nabla\cdot\mathbf{w})\mathbf{u},\mathbf{v})_\Omega \quad \forall\,\mathbf{w},\mathbf{u},\mathbf{v}\in\mathbf{X}, \\
\mathrm{d}(\mathbf{v},\mathbf{B},\mathbf{C}) &= (\mathbf{v}\times\mathbf{B},\nabla\times\mathbf{C})_\Omega \qquad \forall\mathbf{v}\in\mathbf{X}, \quad \mathbf{B},\ \mathbf{C}\in\mathbf{W}.
\end{aligned}$$

It is straightforward to derive the following variational formulation of the coupled flow system (1.1)–(1.2): Find $(\mathbf{u}(t), p(t), \mathbf{B}(t))\in\mathbf{X}\times M\times\mathbf{W}$ satisfying

$$\begin{aligned}
&(\mathbf{u}_t,\mathbf{v})_\Omega + \nu(\nabla\mathbf{u},\nabla\mathbf{v})_\Omega - (p,\nabla\cdot\mathbf{v})_\Omega + (\nabla\cdot\mathbf{u},q)_\Omega + b(\mathbf{u},\mathbf{u},\mathbf{v}) + \tau\,\mathrm{d}(\mathbf{v},\mathbf{B},\mathbf{B}) \\
&= (\mathbf{f},\mathbf{v})_\Omega \quad \forall\,\mathbf{v}\in\mathbf{X}\,, \quad q\in M,
\end{aligned} \tag{1.3}$$

$$(\mathbf{B}_t,\mathbf{C})_\Omega + \mu(\nabla\times\mathbf{B},\nabla\times\mathbf{C})_\Omega - \mathrm{d}(\mathbf{u},\mathbf{B},\mathbf{C}) = 0 \quad \forall\,\mathbf{C}\in\mathbf{W}\,. \tag{1.4}$$

The global unique solvability of the system (1.3)–(1.4) with slightly different boundary conditions was studied in [19], and the results was analogous to the ones for the Navier-Stokes system [18]. The global unique solvability of a modified system of (1.3)–(1.4) was demonstrated in [12]. In particular, it was shown in [19] that the system (1.3)–(1.4) is globally uniquely solvable for all $t > 0$ in the case when the initial data and source $\mathbf{f}$ are sufficiently small and for $t\in[0,T)$ with some small $T > 0$ in the case with general initial data. The global attractors was investigated in [28] for the 2D magnetohydrodynamic equations.

For the Navier-Stokes equations alone, a semi-discrete finite element scheme was studied in [14, 15], where the approximate velocity $\mathbf{u}_h(t)$ and pressure $p_h(t)$ are determined in a conforming or nonconforming finite element space pair $(\mathbf{X}_h, M_h)$, and the following error estimates were established for all $t \in (0, T]$ that

$$\|\mathbf{u}(t) - \mathbf{u}_h(t)\|_{L^2} + h\|\nabla(\mathbf{u}_h(t) - \mathbf{u}(t))\|_{L^2} \le \kappa h^2,$$
$$\|p(t) - p_h(t)\|_{L^2} \le \kappa\sigma^{-\frac{1}{2}}(t)h$$

if $(\mathbf{X}_h, M_h)$ satisfies the approximation property of the first order to both $\nabla\mathbf{u}$ and p [14], and

$$\|\mathbf{u}(t) - \mathbf{u}_h(t)\|_{L^2} + h\|\nabla(\mathbf{u}_h(t) - \mathbf{u}(t))\|_{L^2} \le \kappa\sigma^{1-\frac{m}{2}}(t)h^m,$$
$$\|p(t) - p_h(t)\|_{L^2} \le \kappa\sigma^{\frac{1}{2}-\frac{m}{2}}(t)h^{m-1}$$

if $(\mathbf{X}_h, M_h)$ satisfies the approximation property of the $(m-1)$th order to both $\nabla\mathbf{u}$ and p [15]. The function $\sigma(t)$ above is given by $\sigma(t) = \min\{1, t\}$, and κ is a generic positive constant depending on the data $T, \mathbf{u}_0, \Omega$ and $\mathbf{f}$.

Several efficient numerical schemes were proposed and analysed recently for the time-dependent MHD problem. A finite element scheme was studied in [24] for both high and low magnetic Reynolds numbers, based on a conservative formulation to ensure the local divergence-free condition of the magnetic field weakly. Long-time dissipative properties and non-linear unconditional stability of a time integration algorithm were investigated in [1], based on a mixed finite element approximation in space. In [22], the behavior of a generalized alternating-direction implicit scheme was analysed for the low magnetic Reynolds number. An Euler semi-implicit scheme was proposed in [8] for a one-fluid or two-fluid MHD system. Some coupling and decoupling fully discrete schemes were explored in [23], while an implicit stabilized finite element scheme was analysed in [3] for the case of variable coefficients ρ, ν, σ. Combined with a finite element discretization in space, the Crank-Nicolson scheme was studied in [31] at small magnetic Reynolds numbers, while a semi-implicit scheme was shown in [13] to converge unconditionally.

In this work, we study the finite element spatial approximation of the MHD system (1.3)–(1.4) under smooth domains and data. The discrete solution $(\mathbf{u}_h(t), p_h(t), \mathbf{B}_h(t))$ is approximated in a conforming finite element space $\mathbf{X}_h \times M_h \times \mathbf{W}_h$, which is assumed to possess the approximation property of the second order to $(\nabla\mathbf{u}, p, \nabla\mathbf{B})$. We will not assume that the initial data, the body force $\mathbf{f}$ and the terminal time T are sufficiently small; instead we require only the regularities $\nabla\mathbf{u} \in L^4(0, T; L^2(\Omega)^{3\times3})$ and $\nabla \times \mathbf{B} \in L^4(0, T; L^2(\Omega)^3)$, as it was done in [8]. Under these two conditions we first establish the H^3-regularity of the exact solution $(\mathbf{u}, \mathbf{B})$. Then we formulate the finite element approximation $(\mathbf{u}_h, p_h, \mathbf{B}_h)$ based on the second order finite element space $\mathbf{X}_h \times M_h \times \mathbf{W}_h$ to the solution $(\mathbf{u}, p, \mathbf{B})$ of the MHD flow and provide the optimal $\mathbf{H}^1$- and L^2-norm error estimates of $(\mathbf{u}_h, \mathbf{B}_h)$ to $(\mathbf{u}, \mathbf{B})$ and p_h to p respectively. Particularly, we emphasize that we are able to achieve the optimal L^2-norm error estimates of $(\mathbf{u}_h, \mathbf{B}_h)$ to $(\mathbf{u}, \mathbf{B})$ by using a special new negative-norm technique without the standard duality argument (that was applied to the single Navier-Stokes equations [14, 15]). Hence our arguments are easier, and more importantly, they get rid of the disadvantages of the duality argument, such as the existence and desired regularities of the solutions to the nonlinear duality problem, as well as the constraint on the time stepsize when time discretization is considered. The optimal $\mathbf{H}^1$- and L^2-norm error estimates obtained in this work are new, and no similar error estimates were established in the existing literature, *e.g.*, [1, 3, 13, 22, 23, 24, 31].

We recall that Heywood and Rannacher did use the negative-norm techniques already in [14, 15] to analyze the finite element solution for the single Navier-Stokes equations, but there are two essential differences as stated below:

(a) The standard duality argument was used in [14, 15] as usual for the L^2-norm error estimates of the velocity in the singer Navier-Stokes equations (see Lems. 5.1–5.2, [14]), instead of the negative-norm techniques. Unfortunately, this standard duality argument for the optimal L^2-norm error estimate does not appear to work for the finite element approximation of the current MHD system due to the great complication of the nonlinear coupling between velocity and magnetic field. In fact, the finite element analysis for the MHD system is much more challenging than the single Navier-Stokes equations. To overcome the difficulty, we shall propose a rather delicate and novel negative-norm technique in this work that enables us to successfully achieve the optimal L^2-norm error estimates of the discrete velocity and magnetic field simultaneously (see Lem. 4.2, Thm. 4.1). The basic idea is to estimate the H^{-1}-norm of the errors of the discrete velocity $\mathbf{u}_h$ and magnetic field $\mathbf{B}_h$ simultaneously by making use of the special testing functions $\mathbf{v}_h = A_h^{-1}(P_h\mathbf{u} - \mathbf{u}_h)$ and $\mathbf{C}_h = A_{2h}^{-1}(R_{0h}\mathbf{B} - \mathbf{B}_h)$ in the finite element error equations for $\mathbf{u}_h$ and $\mathbf{B}_h$, and then use several unique properties of the discrete Stokes operator A_h and Maxwell operator A_{2h} as well as the L^2-projections P_h and R_{0h} onto the discrete divergence spaces. Very importantly, this new strategy will help us achieve the optimal error estimates of the discrete velocity and magnetic field simultaneously in both L^2- and energy-norm. To our best knowledge, this is completely new in literature in terms of finite element analysis for a coupled PDE system like MHD, and has greatly simplified the error estimates of the finite element approximations, and even much simpler than the ones in [14, 15] that handled only the single Navier-Stokes system.

(b) A negative-norm technique was indeed used in [14, 15] for the L^2-norm error estimate of the discrete pressure (see, *e.g.*, Lems. 6.1–6.2, [14]). To do so, one can write the error $p - p_h$ of the discrete pressure in terms of the error $\mathbf{u} - \mathbf{u}_h$ of the discrete velocity directly from the continuous and finite element variational systems. Then the important term $((\mathbf{u} - \mathbf{u}_h)_t, \mathbf{v}_h)$ involved there was simply bounded by the product of the H^{-1}-norm of $\mathbf{u} - \mathbf{u}_h$ and H^1-norm of the testing function $\mathbf{v}_h$, and the H^{-1}-norm of $\mathbf{u} - \mathbf{u}_h$ is further crudely estimated by the L^2-norm of $\mathbf{u} - \mathbf{u}_h$. Clearly, this negative-norm technique is quite natural, and as we shall see, it is essentially different from the ones we propose in this work. Due to the great complication of the nonlinear coupling between velocity and magnetic field, this simple and direct negative-norm technique used in [14, 15] does not work for our MHD system for the optimal error estimate of the numerical pressure. Instead we will combine the energy-norm error estimates in this work for the discrete Stokes projections and discrete Maxwell projections with the optimal simultaneous L^2- and H^1-norm error estimates of the discrete velocity and magnetic field we have developed earlier using our new negative-norm techniques; see the proofs of Lemmas 5.1–5.2 and Theorem 5.3.

The rest of the work is organized as follows. In Section 2, we give some functional settings and regularity results for the solutions to the MHD flow system (1.3)–(1.4). A finite element spatial approximation is then proposed for the MHD system in Section 3, and some basic and important estimates are also presented. We devote our main effort to build up the optimal L^2-norm error estimates for the approximate velocity and magnetic field in Section 4, and the optimal H^1-norm error estimate for the approximate velocity and the optimal L^2-norm error estimate for the approximate pressure in Section 5.

2. Functional setting of the MHD problem

In this section we present the mathematical setting of the system (1.3)–(1.4). For the subsequent analysis on the true solution to the system, we introduce the following Sobolev inequalities and one important identity [7, 10, 12, 28, 29, 30]:

$$
\begin{aligned}
\|\mathbf{v}\|_{L^3} &\le c\|\mathbf{v}\|_{0,\Omega}^{\frac12}\|\nabla\mathbf{v}\|_{0,\Omega}^{\frac12}, \quad \|\mathbf{v}\|_{L^6} \le c\|\nabla\mathbf{v}\|_{0,\Omega}, \quad \forall\, \mathbf{v}\in\mathbf{X},\\
\|\mathbf{v}\|_{L^\infty} + \|\nabla\mathbf{v}\|_{L^3} &\le c\|\nabla\mathbf{v}\|_{0,\Omega}^{1/2}\|\mathbf{v}\|_{2,\Omega}^{1/2}, \quad \forall \mathbf{v}\in\mathbf{H}^2(\Omega)\cap\mathbf{X},\\
\|\mathbf{B}\|_{1,\Omega} &\le c(\|\nabla\times\mathbf{B}\|_{0,\Omega} + \|\nabla\cdot\mathbf{B}\|_{0,\Omega}), \quad \forall \mathbf{B}\in\mathbf{W},\\
\|\mathbf{B}\|_{L^\infty} + \|\nabla\mathbf{B}\|_{L^3} &\le c\|\mathbf{B}\|_{1,\Omega}^{1/2}\|\mathbf{B}\|_{2,\Omega}^{1/2}, \quad \forall \mathbf{B}\in\mathbf{H}^2(\Omega)\cap\mathbf{W},\\
\|\mathbf{B}\|_{L^m} &\le c_1(m)\|\mathbf{B}\|_{1,\Omega}^{\alpha}\|\mathbf{B}\|_{0,\Omega}^{1-\alpha}, \quad \forall \mathbf{B}\in\mathbf{H}^1(\Omega),\\
b(\mathbf{u},\mathbf{v},\mathbf{w}) &= -b(\mathbf{u},\mathbf{w},\mathbf{v}), \quad \forall\, \mathbf{u}, \quad \mathbf{v}, \quad \mathbf{w}\in\mathbf{X},
\end{aligned} \tag{2.1}
$$

$$|b(\mathbf{u},\mathbf{v},\mathbf{w})| \leq N_0\|\nabla\mathbf{u}\|_{0,\Omega}\|\nabla\mathbf{v}\|_{0,\Omega}\|\nabla\mathbf{w}\|_{0,\Omega},\ \forall\,\mathbf{u},\ \mathbf{v},\ \mathbf{w}\in\mathbf{X},$$
$$|\mathrm{d}(\mathbf{u},\mathbf{B},\mathbf{C})| \leq N_1\|\nabla\mathbf{u}\|_{0,\Omega}\|\nabla\mathbf{B}\|_{0,\Omega}\|\nabla\times\mathbf{C}\|_{0,\Omega},\ \forall\,\mathbf{u}\in\mathbf{X},\ \mathbf{B},\ \mathbf{C}\in\mathbf{W},$$

where the two indices α and m are specified by

$$\alpha = 3\left(\frac{1}{2}-\frac{1}{m}\right)\in[0,1] \quad \text{and} \quad m\in[2,6].$$

Here and hereafter, N_0 and N_1 are positive constants depending on Ω, c is used to denote a general positive constant depending on the data (Ω,ν,μ,s), and $\gamma(m)$ and $c_1(m)$ are positive constants depending on Ω and m.

Let P be an L^2-projection operator from $L^2(\Omega)^3$ into $\mathbf{H}$, then we introduce the Stokes operator $A_1=-P\Delta$ with its domain $D(A_1)=\mathbf{H}^2(\Omega)\cap\mathbf{V}$ and the Maxwell's operator $A_2=P(\nabla\times\nabla\times-\nabla\nabla\cdot)$ with its domain $D(A_2)=\mathbf{H}^2(\Omega)\cap\mathbf{W}_0$. Finally, it is easy to show that

$$(A_1\mathbf{u},\mathbf{v})=(\nabla\mathbf{u},\nabla\mathbf{v})=(A_1^{\frac{1}{2}}\mathbf{u},A_1^{\frac{1}{2}}\mathbf{v})\quad\forall\mathbf{u}\in D(A_1),\ \mathbf{v}\in\mathbf{V},$$

and that for any $\mathbf{C}\in\mathbf{W}_0$ and $\mathbf{B}\in D(A_2)$ satisfying $\mathbf{n}\times\nabla\times\mathbf{B}=0$ on $\partial\Omega$,

$$(A_2\mathbf{B},\mathbf{C})=(\nabla\times\mathbf{B},\nabla\times\mathbf{C})+(\nabla\cdot\mathbf{B},\nabla\cdot\mathbf{C})=(A_2^{\frac{1}{2}}\mathbf{B},A_2^{\frac{1}{2}}\mathbf{C}).$$

Moreover, for some positive constants c_0 and c_1, we have the following estimates

$$c_0\|\mathbf{v}\|_{1,\Omega}\leq\|A_1^{\frac{1}{2}}\mathbf{v}\|_{0,\Omega}\leq c_1\|\mathbf{v}\|_{1,\Omega},\quad c_0\|\mathbf{C}\|_{1,\Omega}\leq\|A_2^{\frac{1}{2}}\mathbf{C}\|_{0,\Omega}\leq c_1\|\mathbf{C}\|_{1,\Omega}\quad\forall\mathbf{v}\in\mathbf{V},\ \mathbf{C}\in\mathbf{W}_0. \tag{2.2}$$

Throughout this paper we make the following assumption on the prescribed data for the MHD system (1.3)–(1.4), which specifies the regularity of the data needed for our major results.

Assumption 2.1. The initial data $\mathbf{u}_0\in D(A_1)$, $\mathbf{B}_0\in D(A_2)$ and the force $\mathbf{f}$ meet the following *a priori* bound for some generic constant κ_0:

$$\|\mathbf{f}(t)\|_{1,\Omega}^2+\|\mathbf{f}_t(t)\|_{0,\Omega}^2+\|\mathbf{f}_{tt}(t)\|_{0,\Omega}^2+\|\mathbf{u}_0\|_{2,\Omega}^2+\|\mathbf{B}_0\|_{2,\Omega}^2\leq\kappa_0$$

Here and hereafter, κ and κ_i for $i\geq0$ are generic positive constants depending only on the given data $(\nu,\mu,s,\Omega,T,\mathbf{u}_0,\mathbf{B}_0,\mathbf{f})$.

Assumption 2.1 ensures the existence of a unique strong solution to the problem (1.3)–(1.4) on some small time interval $[0,T]$ such that (*cf.* Thm. 3.2 [26])

$$\mathbf{u}\in C([0,T];\mathbf{V})\cap L^2(0,T;H^2(\Omega)^3),\quad p\in\times L^2(0,T;H^1(\Omega)\cap M),$$

$$\mathbf{B}\in C([0,T];\mathbf{W}_0)\cap L^2(0,T;\mathbf{H}^2(\Omega)),$$

and $\mathbf{u}_t\in L^2(0,T;\mathbf{H})$, $\mathbf{B}_t\in L^2(0,T;\mathbf{H})$, and the equations (1.3)–(1.4) hold for almost all $t\in[0,T)$. But for two dimensions, or three dimensions when the data $\mathbf{u}_0$, $\mathbf{B}_0$, $\mathbf{f}$ are sufficiently small, the solution to the problem (1.3)–(1.4) exists for any $T>0$ and satisfies that [26]

$$\sup_{0\leq t\leq T}(\|\nabla\mathbf{u}(t)\|_{0,\Omega}+\|\nabla\mathbf{B}(t)\|_{0,\Omega})\leq\kappa_0\,.$$

Instead of the condition that the data are sufficiently small in three dimensions, we shall assume in this work the basic existence of the solution to the problem (1.3)–(1.4) on some interval $[0, T)$ and the following *a priori* estimate:

Assumption 2.2. There exists a unique solution $(\mathbf{u}(t), p(t), \mathbf{B}(t)) \in \mathbf{X} \times M \times \mathbf{W}$ to the system (1.3)–(1.4), and it satisfies the regularity

$$\int_0^T (\|\nabla \mathbf{u}(t)\|_{0,\Omega}^4 + \|\nabla \times \mathbf{B}(t)\|_{0,\Omega}^4) \mathrm{d}t \le \kappa.$$

Furthermore, we know the following results [5, 8, 9, 14, 26].

Assumption 2.3. There exists a unique solution to the steady Stokes problem

$$-\Delta \mathbf{v} + \nabla q = \mathbf{g}, \quad \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega,$$

under the conditions $\mathbf{v} = 0$ on $\partial\Omega$ and $\mathbf{g} \in \mathbf{H}^{k-2}(\Omega)$ $(k = 2, 3)$, with the *a priori* estimate:

$$\|\mathbf{v}\|_{k,\Omega} + \|q\|_{k-1,\Omega} \le c\|\mathbf{g}\|_{k-2,\Omega}\,.$$

And there exists a unique solution to the Maxwell's equations

$$\nabla \times \nabla \times \mathbf{C} - \nabla\nabla \cdot \mathbf{C} = \mathbf{h}, \quad \nabla \cdot \mathbf{C} = 0 \quad \text{in} \quad \Omega,$$

under the conditions that $\mathbf{n} \times (\nabla \times \mathbf{C}) = 0$, $\mathbf{C} \cdot \mathbf{n} = 0$ on $\partial\Omega$ and $\mathbf{h} \in \mathbf{H}^{k-2}(\Omega)$ $(k = 2, 3)$ with $\nabla \cdot \mathbf{h} = 0$, and the following *a priori* estimate holds:

$$\|\mathbf{C}\|_{k,\Omega} \le c\|\mathbf{h}\|_{k-2,\Omega}.$$

It follows from Assumption 2.3 that

$$\|\mathbf{v}\|_{2,\Omega} \le c\|A_1\mathbf{v}\|_{0,\Omega} \quad \forall\, \mathbf{v} \in D(A_1)\,; \quad \|\mathbf{C}\|_{2,\Omega} \le c\|A_2\mathbf{C}\|_{0,\Omega} \quad \forall\, \mathbf{C} \in D(A_2). \tag{2.3}$$

The following useful results can be found in [13].

Lemma 2.4. *Under Assumptions 2.1–2.3, it holds for the solution $(\mathbf{u}(t), p(t), \mathbf{B}(t))$ to the problem (1.3)–(1.4) and all $t \in [0, T]$ that*

$$\begin{aligned} &\|\mathbf{u}_t(t)\|_{0,\Omega}^2 + \tau\|\mathbf{B}_t(t)\|_{0,\Omega}^2 + \|\mathbf{u}(t)\|_{2,\Omega}^2 + \|p(t)\|_{1,\Omega}^2 + \|\mathbf{B}(t)\|_{2,\Omega} \le \kappa\,, \\ &\int_0^t (\nu\|A_1^{\frac{1}{2}}\mathbf{u}_t(s)\|_{0,\Omega}^2 + s\mu\|A_2^{\frac{1}{2}}\mathbf{B}_t(s)\|_{0,\Omega}^2)\mathrm{d}s \le \kappa\,. \end{aligned} \tag{2.4}$$

With the help of Lemma 2.4 and Assumptions 2.1–2.3, we next derive several more *a priori* estimates in three lemmas, which will be needed in our subsequent error estimates of finite element solutions.

Lemma 2.5. *Under Assumptions 2.1–2.3, it holds for the solution $(\mathbf{u}(t), p(t), \mathbf{B}(t))$ to the problem (1.3)–(1.4) and all $t \in [0, T]$ that*

$$\int_0^t (\|\mathbf{u}(s)\|_{3,\Omega}^2 + \|p(s)\|_{2,\Omega}^2 + \|\mathbf{B}(s)\|_{3,\Omega}^2)\mathrm{d}s \le \kappa\,. \tag{2.5}$$

Proof. Using Assumption 2.3, (2.1)–(2.2) and the Young inequality, we obtain

$$\begin{aligned}\nu\|\mathbf{u}(t)\|_{3,\Omega}+\|p(t)\|_{2,\Omega}\le{}& c\|A_1^{\frac12}\mathbf{u}_t(t)\|_{0,\Omega}+c\|f(t)\|_{1,\Omega}+c\|(\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t)\|_{0,\Omega}\\&+c\|\mathbf{B}(t)\times\nabla\times\mathbf{B}(t)\|_{0,\Omega}+\|\nabla[(\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t)]\|_{0,\Omega}+c\|\nabla[\mathbf{B}(t)\times\nabla\times\mathbf{B}(t)]\|_{0,\Omega}\\ \le{}& c\|A_1^{\frac12}\mathbf{u}_t(t)\|_{0,\Omega}+c\|f(t)\|_{1,\Omega}+c\|\mathbf{u}(t)\|_{2,\Omega}\|A_1^{\frac12}\mathbf{u}(t)\|_{0,\Omega}\\&+c\|\nabla\times\mathbf{B}(t)\|_{1,\Omega}\|A_1^{\frac12}\mathbf{B}(t)\|_{0,\Omega}+c\|\mathbf{u}(t)\|_{2,\Omega}^2+c\|\nabla[\mathbf{B}(t)\times\nabla\times\mathbf{B}(t)]\|_{0,\Omega},\end{aligned}\tag{2.6}$$

$$\begin{aligned}\mu\|\mathbf{B}(t)\|_{3,\Omega}\le{}& c\|A_2^{\frac12}\mathbf{B}_t(t)\|_{0,\Omega}+c\|\nabla\times(\mathbf{u}(t)\times\mathbf{B}(t))\|_{0,\Omega}+\|\nabla\times\nabla\times(\mathbf{u}(t)\times\mathbf{B}(t))\|_{0,\Omega}\\ \le{}& c\|A_2^{\frac12}\mathbf{B}_t(t)\|_{0,\Omega}+c\|\mathbf{u}(t)\|_{2,\Omega}\|A_2^{\frac12}\mathbf{B}(t))\|_{0,\Omega}+c\|\nabla\times\nabla\times(\mathbf{u}(t)\times\mathbf{B}(t))\|_{0,\Omega},\end{aligned}\tag{2.7}$$

and the following two terms in (2.6) and (2.7) can be further bounded by

$$\begin{aligned}c\|\nabla[\mathbf{B}(t)\times\nabla\times\mathbf{B}(t)]\|_{0,\Omega}&\le c\|\nabla\mathbf{B}\|_{L^3}\|\nabla\times\mathbf{B}\|_{L^6}+c\|\mathbf{B}\|_{L^\infty}\|\nabla(\nabla\times\mathbf{B})\|_{L^2}\\&\le c\|\mathbf{B}(t)\|_{2,\Omega}^2,\\ c\|\nabla\times\nabla\times(\mathbf{u}(t)\times\mathbf{B}(t))\|_{0,\Omega}&=c\|\nabla\times[(\mathbf{B}\cdot\nabla)\mathbf{u}(t)-(\mathbf{u}(t)\cdot\nabla)\mathbf{B}(t))\|_{0,\Omega}\\&\le c\|\mathbf{B}\|_{L^\infty}\|\nabla\nabla\mathbf{u}(t)\|_{L^2}+c\|\nabla\mathbf{B}(t)\|_{L^3}\|\nabla\mathbf{u}(t)\|_{L^6}\\&\quad+c\|(\mathbf{u}(t)\cdot\nabla)(\nabla\times\mathbf{B}(t))\|_{0,\Omega}\le c\|\mathbf{B}(t)\|_{2,\Omega}\|\mathbf{u}(t)\|_{2,\Omega}.\end{aligned}$$

Now the proof of Lemma 2.5 is completed by combining (2.6)–(2.7) with the estimates above and Lemma 2.4. □

Lemma 2.6. *Under Assumptions 2.1–2.3, the solution $(\mathbf{u}(t),p(t),\mathbf{B}(t))$ to the problem (1.3)–(1.4) satisfies the following estimate for all $t\in[0,T]$ that*

$$\begin{aligned}&\sigma(t)[\|A_1^{\frac12}\mathbf{u}_t(t)\|_{0,\Omega}^2+\tau\|A_2^{\frac12}\mathbf{B}_t(t)\|_{0,\Omega}^2]\\&+\sigma(t)[\|\mathbf{u}(t)\|_{3,\Omega}^2+\|p(t)\|_{2,\Omega}^2+\|\mathbf{B}(t)\|_{2,\Omega}^2+\|\nabla\times\mathbf{B}(t)\|_{2,\Omega}^2]\\&+\int_0^t\sigma(s)(\nu\|A_1\mathbf{u}_t(s)\|_{0,\Omega}^2+\tau\mu\|A_2\mathbf{B}_t(s)\|_{0,\Omega}^2)\mathrm{d}s\\&+\int_0^t\sigma(s)(\|\mathbf{u}_{tt}(s)\|_{0,\Omega}^2+\|\mathbf{B}_{tt}(s)\|_{0,\Omega}^2)\mathrm{d}s\le\kappa\,.\end{aligned}\tag{2.8}$$

Proof. We differentiate (1.3) and (1.4) with respect to t respectively to obtain for all $(\mathbf{v},q,\mathbf{C})\in\mathbf{X}\times M\times\mathbf{W}$ that

$$\begin{aligned}&(\mathbf{u}_{tt},\mathbf{v})_\Omega+\nu(\nabla\mathbf{u}_t,\nabla\mathbf{v})_\Omega-(\nabla\cdot\mathbf{v},p_t)_\Omega+(\nabla\mathbf{u}_t,q)_\Omega+b(\mathbf{u}_t,\mathbf{u},\mathbf{v})+b(\mathbf{u},\mathbf{u}_t,\mathbf{v})\\&\quad+\tau\mathrm{d}(\mathbf{v},\mathbf{B}_t,\mathbf{B})+\tau\mathrm{d}(\mathbf{v},\mathbf{B},\mathbf{B}_t)=(\mathbf{f}_t,\mathbf{v})_\Omega,\end{aligned}\tag{2.9}$$

$$(\mathbf{B}_{tt},\mathbf{C})_\Omega+\mu(\nabla\times\mathbf{B}_t,\nabla\times\mathbf{C})_\Omega-\mathrm{d}(\mathbf{u}_t,\mathbf{B},\mathbf{C})-\mathrm{d}(\mathbf{u},\mathbf{B}_t,\mathbf{C})=0\,,\tag{2.10}$$

then it follows by taking the sum of (2.9) with $(\mathbf{v},q)=(A_1\mathbf{u}_t,0)$ and (2.10) with $\mathbf{C}=\tau A_2\mathbf{B}_t$ and using (2.2) and Young's inequality that

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|A_1^{\frac12}\mathbf{u}_t\|_{0,\Omega}^2+\nu\|A_1\mathbf{u}_t\|_{0,\Omega}^2+\frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A_2^{\frac12}\mathbf{B}_t\|_{0,\Omega}^2+\tau\mu\|A_2\mathbf{B}_t\|_{0,\Omega}^2\\&\quad+b(\mathbf{u},\mathbf{u}_t,A_1\mathbf{u}_t)+b(\mathbf{u}_t,\mathbf{u},A_1\mathbf{u}_t)+\tau\mathrm{d}(A_1\mathbf{u}_t,\mathbf{B}_t,\mathbf{B})+\tau\mathrm{d}(A_1\mathbf{u}_t,\mathbf{B},\mathbf{B}_t)\\&\quad-\tau\,\mathrm{d}(\mathbf{u}_t,\mathbf{B},A_2\mathbf{B}_t)-\tau\mathrm{d}(\mathbf{u},\mathbf{B}_t,A_2\mathbf{B}_t)\le\frac{\nu}{8}\|A_1\mathbf{u}_t\|_{0,\Omega}^2+\frac{4}{\nu}\|\mathbf{f}_t\|_{0,\Omega}^2.\end{aligned}\tag{2.11}$$

By means of (2.2)–(2.3) and Young's inequality again, we derive

$$\begin{aligned}
|b(\mathbf{u}_t,\mathbf{u},A_1\mathbf{u}_t)|+|b(\mathbf{u},\mathbf{u}_t,A_1\mathbf{u}_t)| &\le c_0\|\mathbf{u}\|_{2,\Omega}\|A_1^{\frac12}\mathbf{u}_t\|_{0,\Omega}\|A_1\mathbf{u}_t\|_{0,\Omega}\\
&\le \frac{\nu}{16}\|A_1\mathbf{u}_t\|_{0,\Omega}^2+\frac{4}{\nu}c_0^2\|\mathbf{u}\|_{2,\Omega}^2\|A_1^{\frac12}\mathbf{u}_t\|_{0,\Omega}^2,\\
\tau|\mathrm{d}(A_1\mathbf{u}_t,\mathbf{B}_t,\mathbf{B})|+\tau|\mathrm{d}(A_1\mathbf{u}_t,\mathbf{B},\mathbf{B}_t)| &\le \tau c_0\|A_1\mathbf{u}_t\|_{0,\Omega}\|\mathbf{B}\|_{2,\Omega}\|A_2^{\frac12}\mathbf{B}_t\|_{0,\Omega}\\
&\le \frac{\nu}{16}\|A_1\mathbf{u}_t\|_{0,\Omega}^2+\frac{4}{\nu}\tau^2c_0^2\|\mathbf{B}\|_{2,\Omega}^2\|A_2^{\frac12}\mathbf{B}_t\|_{0,\Omega}^2,\\
\tau|\mathrm{d}(\mathbf{u}_t,\mathbf{B},A_2\mathbf{B}_t)|+\tau|\mathrm{d}(\mathbf{u},\mathbf{B}_t,A_2\mathbf{B}_t)| &\le \tau c_0\|A_2\mathbf{B}_t\|_{0,\Omega}(\|\mathbf{u}\|_{2,\Omega}+\|\mathbf{B}\|_{2,\Omega})(\|A_1^{\frac12}\mathbf{u}_t\|_{0,\Omega}+\|A_2^{\frac12}\mathbf{B}_t\|_{0,\Omega})\\
&\le \frac{\tau\mu}{16}\|A_2\mathbf{B}_t\|_{0,\Omega}^2\\
&\quad+\mu^{-1}4^2\tau c_0^2(\|\mathbf{u}\|_{2,\Omega}^2+\|\mathbf{B}\|_{2,\Omega}^2)\,(\|A_1^{\frac12}\mathbf{u}_t\|_{0,\Omega}^2+\|A_2^{\frac12}\mathbf{B}_t\|_{0,\Omega}^2).
\end{aligned}$$

It follows by combining the above three estimates with (2.11) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|A_1^{\frac12}\mathbf{u}_t\|_{0,\Omega}^2+\tau\|A_2^{\frac12}\mathbf{B}_t\|_{0,\Omega}^2)+(\nu\|A_1\mathbf{u}_t\|_{0,\Omega}^2+\tau\mu\|A_2\mathbf{B}_t\|_{0,\Omega}^2)\\
&\le c(1+\|\mathbf{u}\|_{2,\Omega}^2+\|\mathbf{B}\|_{2,\Omega}^2)(\|A_1^{\frac12}\mathbf{u}_t\|_{0,\Omega}^2+\tau\|A_2^{\frac12}\mathbf{B}_t\|_{0,\Omega}^2)+c\|\mathbf{f}_t\|_{0,\Omega}^2.
\end{aligned}\tag{2.12}$$

But multiplying (2.12) by $\sigma(t)$, then integrating with respect to t, applying the Gronwall's lemma and Lemma 2.5, we come to

$$\sigma(t)(\|A_1^{\frac12}\mathbf{u}_t(t)\|_{0,\Omega}^2+\tau\|A_2^{\frac12}\mathbf{B}_t(t)\|_{0,\Omega}^2)+\int_0^t\sigma(t)(\nu\|A_1\mathbf{u}_t\|_{0,\Omega}^2+\tau\mu\|A_2\mathbf{B}_t\|_{0,\Omega}^2)\mathrm{d}s\le\kappa,\tag{2.13}$$

while using (2.9)–(2.10) and (2.1)–(2.2) we directly see

$$\begin{aligned}
\|\mathbf{u}_{tt}\|_{0,\Omega}&\le\|f_t\|_{0,\Omega}+\nu\|A_1\mathbf{u}_t\|_{0,\Omega}+c\|A_1^{\frac12}\mathbf{u}_t\|_{0,\Omega}\|\mathbf{u}\|_{2,\Omega}+\tau c\|A_2^{\frac12}\mathbf{B}_t\|_{0,\Omega}\|\mathbf{B}\|_{2,\Omega},\\
\|\mathbf{B}_{tt}\|_{0,\Omega}&\le\mu\|A_2\mathbf{B}_t\|_{0,\Omega}+c\|A_1^{\frac12}\mathbf{u}_t\|_{0,\Omega}\|\mathbf{B}\|_{2,\Omega}+c\|A_2^{\frac12}\mathbf{B}_t\|_{0,\Omega}\|\mathbf{u}\|_{2,\Omega}.
\end{aligned}$$

Now we can conclude the desired estimate of Lemma 2.6 by combining these two estimates with (2.13), (2.6)–(2.7) and Lemma 2.5. □

Lemma 2.7. *Under Assumptions 2.1–2.3, the solution* $(\mathbf{u}(t),p(t),\mathbf{B}(t))$ *to the problem (1.3)–(1.4) satisfies the following estimate for all* $t\in[0,T]$ *that*

$$\begin{aligned}
&\sigma^2(t)[\|\mathbf{u}_{tt}(t)\|_{0,\Omega}^2+\tau\|\mathbf{B}_{tt}(t)\|_{0,\Omega}^2]\\
&\quad+\int_0^t\sigma^2(t)[\nu\|A_1^{\frac12}\mathbf{u}_{tt}(s)\|_{0,\Omega}^2+\tau\mu\|A_2^{\frac12}\mathbf{B}_{tt}(s)\|_{0,\Omega}^2]\mathrm{d}s\le\kappa,
\end{aligned}\tag{2.14}$$

$$\begin{aligned}
&\int_0^t\sigma^2(s)[\|\mathbf{u}_t(s)\|_{3,\Omega}^2+\|p_t(s)\|_{2,\Omega}^2+\|\mathbf{B}_t(s)\|_{3,\Omega}^2]\mathrm{d}s\\
&\quad+\sigma^2(t)[\|\mathbf{u}_t(t)\|_{2,\Omega}^2+\|p_t(t)\|_{1,\Omega}^2+\|\mathbf{B}_t(t)\|_{2,\Omega}^2]\le\kappa\,.
\end{aligned}\tag{2.15}$$

Proof. We differentiate (2.9) and (2.10) with respect to t respectively to obtain for all $(\mathbf{v}, q, \mathbf{C}) \in \mathbf{X} \times M \times \mathbf{W}$ that

$$\begin{aligned}&(\mathbf{u}_{ttt}, \mathbf{v})_\Omega + \nu(\nabla \mathbf{u}_{tt}, \nabla \mathbf{v})_\Omega - (\nabla \cdot \mathbf{v}, p_{tt})_\Omega + (\nabla \cdot \mathbf{u}_{tt}, q)_\Omega + b(\mathbf{u}_{tt}, \mathbf{u}, \mathbf{v}) + 2b(\mathbf{u}_t, \mathbf{u}_t, \mathbf{v}) \\ &\quad + b(\mathbf{u}, \mathbf{u}_{tt}, \mathbf{v}) + \tau \mathrm{d}(\mathbf{v}, \mathbf{B}_{tt}, \mathbf{B}) + 2\tau \mathrm{d}(\mathbf{v}, \mathbf{B}_t, \mathbf{B}_t) + \tau \mathrm{d}(\mathbf{v}, \mathbf{B}, \mathbf{B}_{tt}) = (\mathbf{f}_{tt}, \mathbf{v})_\Omega, \qquad (2.16)\\ &(\mathbf{B}_{ttt}, \mathbf{C})_\Omega + \mu(\nabla \times \mathbf{B}_{tt}, \nabla \times \mathbf{C})_\Omega - \mathrm{d}(\mathbf{u}_{tt}, \mathbf{B}, \mathbf{C}) - 2d(\mathbf{u}_t, \mathbf{B}_t, \mathbf{C}) - \mathrm{d}(\mathbf{u}, \mathbf{B}_{tt}, \mathbf{C}) = 0, \qquad (2.17)\end{aligned}$$

then take the sum of (2.16) with $(\mathbf{v}, q) = (\mathbf{u}_{tt}, p_{tt})$ and (2.17) with $\mathbf{C} = \tau \mathbf{B}_{tt}$ to further derive by using (2.1) and Young's inequality that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}_{tt}\|_{0,\Omega}^2 + \nu\|A_1^{\frac{1}{2}}\mathbf{u}_{tt}\|_{0,\Omega}^2 + \frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{B}_{tt}\|_{0,\Omega}^2 + \tau\mu\|A_2^{\frac{1}{2}}\mathbf{B}_{tt}\|_{0,\Omega}^2 + b(\mathbf{u}_{tt}, \mathbf{u}, \mathbf{u}_{tt}) \\ &\quad + 2b(\mathbf{u}_t, \mathbf{u}_t, \mathbf{u}_{tt}) + \tau \mathrm{d}(\mathbf{u}_{tt}, \mathbf{B}_{tt}, \mathbf{B}) + 2\tau \mathrm{d}(\mathbf{u}_{tt}, \mathbf{B}_t, \mathbf{B}_t) - 2\tau \mathrm{d}(\mathbf{u}_t, \mathbf{B}_t, \mathbf{B}_{tt}) - \tau \mathrm{d}(\mathbf{u}, \mathbf{B}_{tt}, \mathbf{B}_{tt}) \\ &\le \frac{\nu}{16}\|\nabla \mathbf{u}_{tt}\|_{0,\Omega}^2 + \frac{4}{\nu}\gamma_0^2\|\mathbf{f}_{tt}\|_{0,\Omega}^2. \qquad (2.18)\end{aligned}$$

By means of (2.2)–(2.3) and Young's inequality again, we can estimate

$$\begin{aligned}|b(\mathbf{u}_{tt}, \mathbf{u}, \mathbf{u}_{tt})| &\le c_0\|\mathbf{u}\|_{2,\Omega}\|\mathbf{u}_{tt}\|_{0,\Omega}\|A_1^{\frac{1}{2}}\mathbf{u}_{tt}\|_{0,\Omega} \le \frac{\nu}{16}\|A_1^{\frac{1}{2}}\mathbf{u}_{tt}\|_{0,\Omega}^2 + \frac{4}{\nu}c_0^2\|\mathbf{u}\|_{2,\Omega}^2\|\mathbf{u}_{tt}\|_{0,\Omega}^2, \\ 2|b(\mathbf{u}_t, \mathbf{u}_t, \mathbf{u}_{tt})| &\le c_0\|\nabla \mathbf{u}_t\|_{0,\Omega}^2\|A_1^{\frac{1}{2}}\mathbf{u}_{tt}\|_{0,\Omega} \le \frac{\nu}{16}\|A_1^{\frac{1}{2}}\mathbf{u}_{tt}\|_{0,\Omega}^2 + \frac{4}{\nu}c_0^2\|\nabla \mathbf{u}_t\|_{0,\Omega}^4, \\ \tau|\mathrm{d}(\mathbf{u}_{tt}, \mathbf{B}_{tt}, \mathbf{B})| + \tau|\mathrm{d}(\mathbf{u}, \mathbf{B}_{tt}, \mathbf{B}_{tt})| &\le \tau c_0(\|A_1^{\frac{1}{2}}\mathbf{u}_{tt}\|_{0,\Omega}\|\mathbf{B}\|_{2,\Omega} + \|A_2^{\frac{1}{2}}\mathbf{B}_{tt}\|_{0,\Omega}\|u\|_{2,\Omega})\|\mathbf{B}_{tt}\|_{0,\Omega} \\ &\le \frac{\nu}{16}\|A_1^{\frac{1}{2}}\mathbf{u}_{tt}\|_{0,\Omega}^2 + \frac{\tau\mu}{16}\|A_2^{\frac{1}{2}}\mathbf{B}_{tt}\|_{0,\Omega}^2 \\ &\quad + \left(\frac{4}{\nu}\tau^2 c_0^2\|\mathbf{B}\|_{2,\Omega}^2 + \frac{4}{\mu}\tau c_0^2\|\mathbf{u}\|_{2,\Omega}^2\right)\|\mathbf{B}_{tt}\|_{0,\Omega}^2, \\ \tau|\mathrm{d}(\mathbf{u}_{tt}, \mathbf{B}_t, \mathbf{B}_t)| + \tau|\mathrm{d}(\mathbf{u}_t, \mathbf{B}_t, \mathbf{B}_{tt})| &\le \tau c_0\|A_1^{\frac{1}{2}}\mathbf{u}_{tt}\|_{0,\Omega}\|A_2^{\frac{1}{2}}\mathbf{B}_t\|_{0,\Omega}^2 + \tau c_0\|A_1^{\frac{1}{2}}\mathbf{u}_t\|_{0,\Omega}\|A_2^{\frac{1}{2}}\mathbf{B}_t\|_{0,\Omega}\|A_2^{\frac{1}{2}}\mathbf{B}_{tt}\|_{0,\Omega} \\ &\le \frac{\nu}{16}\|A_1^{\frac{1}{2}}\mathbf{u}_{tt}\|_{0,\Omega}^2 + \frac{\tau\mu}{16}\|A_2^{\frac{1}{2}}\mathbf{B}_{tt}\|_{0,\Omega}^2 \\ &\quad + \left(\frac{4}{\nu}\tau^2 c_0^2\|A_2^{\frac{1}{2}}\mathbf{B}_t\|_{0,\Omega}^2 + \frac{4}{\mu}\tau c_0^2\|A_1^{\frac{1}{2}}\mathbf{u}_t\|_{0,\Omega}^2\right)\|A_2^{\frac{1}{2}}\mathbf{B}_t\|_{0,\Omega}^2.\end{aligned}$$

Combining the above 4 inequalities with (2.18) leads to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}_{tt}\|_{0,\Omega}^2 + \nu\|A_1^{\frac{1}{2}}\mathbf{u}_{tt}\|_{0,\Omega}^2 + \tau\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{B}_{tt}\|_{0,\Omega}^2 + \tau\mu\|A_2^{\frac{1}{2}}\mathbf{B}_{tt}\|_{0,\Omega}^2 \\ &\le c\|\mathbf{f}_{tt}\|_{0,\Omega}^2 + c(\mu + \|\mathbf{B}\|_{2,\Omega}^2 + \|\mathbf{u}\|_{2,\Omega}^2)(\|\mathbf{u}_{tt}\|_{0,\Omega}^2 + \tau\|\mathbf{B}_{tt}\|_{0,\Omega}^2) \\ &\quad + c(\|A_1^{\frac{1}{2}}\mathbf{u}_t\|_{0,\Omega}^2 + \|A_2^{\frac{1}{2}}\mathbf{B}_t\|_{0,\Omega}^2)^2. \qquad (2.19)\end{aligned}$$

Now (2.14) follows readily by multiplying (2.19) by $\sigma^2(t)$ and then integrating with respect to t and using Lemma 2.6.

Next we estimate the higher order spatial derivatives of the time derivatives of $\mathbf{u}$, p and $\mathbf{B}$. We can readily derive from (2.9)–(2.10) and Assumption 2.3 that

$$\begin{aligned}\nu\|\mathbf{u}_t\|_{2,\Omega} + \|p_t\|_{1,\Omega} &\le c\|f_t\|_{0,\Omega} + c\|\mathbf{u}_{tt}\|_{0,\Omega} + c\|(\mathbf{u}_t \cdot \nabla)\mathbf{u}\|_{0,\Omega} + c\|(\mathbf{u} \cdot \nabla)\mathbf{u}_t\|_{0,\Omega} \\ &\quad + c\|\mathbf{B}_t \times (\nabla \times \mathbf{B})\|_{0,\Omega} + c\|\mathbf{B} \times (\nabla \times \mathbf{B}_t)\|_{0,\Omega}, \qquad (2.20)\end{aligned}$$

$$\mu\|\mathbf{B}_t\|_{2,\Omega} \le c\|\mathbf{B}_{tt}\|_{0,\Omega} + c\|\nabla\times(\mathbf{u}_t\times\mathbf{B})\|_{0,\Omega} + c\|\nabla\times(\mathbf{u}\times\mathbf{B}_t)\|_{0,\Omega}, \tag{2.21}$$

$$\begin{aligned}\nu\|\mathbf{u}_t\|_{3,\Omega} + \|p_t\|_{2,\Omega} \le{}& c\|f_t\|_{1,\Omega} + c\|A^{\frac{1}{2}}\mathbf{u}_{tt}\|_{1,\Omega} + c\|(\mathbf{u}_t\cdot\nabla)\mathbf{u}\|_{1,\Omega} + c\|(\mathbf{u}\cdot\nabla)\mathbf{u}_t\|_{1,\Omega}\\ &+c\|\mathbf{B}_t\times(\nabla\times\mathbf{B})\|_{1,\Omega} + c\|\mathbf{B}\times(\nabla\times\mathbf{B}_t)\|_{1,\Omega},\end{aligned} \tag{2.22}$$

$$\begin{aligned}\mu\|\mathbf{B}_t\|_{3,\Omega} \le{}& c\|A_2^{\frac{1}{2}}\mathbf{B}_{tt}\|_{0,\Omega} + c\|\nabla\times(\mathbf{u}_t\times\mathbf{B})\|_{0,\Omega} + c\|\nabla\times(\mathbf{u}\times\mathbf{B}_t)\|_{0,\Omega}\\ &+\|\nabla\times\nabla\times(\mathbf{u}_t\times\mathbf{B})\|_{0,\Omega} + \|\nabla\times\nabla\times(\mathbf{u}\times\mathbf{B}_t)\|_{0,\Omega}.\end{aligned} \tag{2.23}$$

Then we obtain by applying the estimates (2.1)–(2.2) and (2.4) that

$$\begin{aligned}c\|(\mathbf{u}_t\cdot\nabla)\mathbf{u}\|_{0,\Omega} + c\|(\mathbf{u}\cdot\nabla)\mathbf{u}_t\|_{0,\Omega} &\le c\|\mathbf{u}_t\|_{L^6}\|\nabla\mathbf{u}\|_{L^3} + c\|\mathbf{u}\|_{L^\infty}\|\nabla\mathbf{u}_t\|_{L^2} \le c\|\mathbf{u}\|_{2,\Omega}\|A_1^{\frac{1}{2}}\mathbf{u}_t\|_{0,\Omega},\\
c\|\mathbf{B}_t\times(\nabla\times\mathbf{B})\|_{0,\Omega} + c\|\mathbf{B}\times(\nabla\times\mathbf{B}_t)\|_{0,\Omega} &\le c\|\mathbf{B}\|_{2,\Omega}\|A_2^{\frac{1}{2}}\mathbf{B}_t\|_{0,\Omega},\\
c\|\nabla\times(\mathbf{u}_t\times\mathbf{B})\|_{0,\Omega} + c\|\nabla\times(\mathbf{u}\times\mathbf{B}_t)\|_{0,\Omega} &\le c\|\mathbf{B}\|_{2,\Omega}\|A_1^{\frac{1}{2}}\mathbf{u}_t\|_{0,\Omega} + c\|\mathbf{u}\|_{2,\Omega}\|A_2^{\frac{1}{2}}\mathbf{B}_t\|_{0,\Omega},\\
c\|\nabla[(\mathbf{u}_t\cdot\nabla)\mathbf{u}]\|_{0,\Omega} + c\|\nabla[(\mathbf{u}\cdot\nabla)\mathbf{u}_t]\|_{0,\Omega} &\le c\|\nabla\mathbf{u}_t\|_{L^3}\|\nabla\mathbf{u}\|_{L^6}\\
&\quad +c\|\mathbf{u}_t\|_{L^\infty}\|\nabla\cdot\nabla\mathbf{u}\|_{L^2} + c\|\nabla\cdot\nabla\mathbf{u}_t\|_{L^2}\|\mathbf{u}\|_{L^\infty}\\
&\le c\|\mathbf{u}_t\|_{2,\Omega}\|\mathbf{u}\|_{2,\Omega},\\
c\|\nabla[\mathbf{B}_t\times(\nabla\times\mathbf{B})]\|_{0,\Omega} + c\|\nabla[\mathbf{B}\times(\nabla\times\mathbf{B}_t)]\|_{0,\Omega} &\le c\|\mathbf{B}_t\|_{L^\infty}\|\nabla\times\nabla\times\mathbf{B}\|_{L^2} + c\|\nabla\mathbf{B}_t\|_{L^3}\|\nabla\times\mathbf{B}\|_{L^6}\\
&\quad +c\|\nabla\mathbf{B}\|_{L^6}\|\nabla\times\mathbf{B}_t\|_{L^3} + c\|\mathbf{B}\|_{L^\infty}\|\nabla\times\nabla\times\mathbf{B}_t\|_{L^2}\\
&\le c\|\mathbf{B}_t\|_{2,\Omega}\|\mathbf{B}\|_{2,\Omega},\\
c\|\nabla\times[\nabla\times(\mathbf{u}_t\times\mathbf{B})]\|_{0,\Omega} &\le c\|\mathbf{u}_t\|_{L^\infty}\|\mathbf{B}\|_{2,\Omega} + c\|\nabla\mathbf{u}_t\|_{L^3}\|\nabla\mathbf{B}\|_{L^6} + c\|\mathbf{u}_t\|_{2,\Omega}\|\mathbf{B}\|_{L^\infty}\\
&\le c\|\mathbf{u}_t\|_{2,\Omega}\|\mathbf{B}\|_{2,\Omega},\\
c\|\nabla\times[\nabla\times(\mathbf{u}\times\mathbf{B}_t)]\|_{0,\Omega} &= c\|\nabla\times[(\mathbf{B}_t\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{B}_t)]\|_{0,\Omega}\\
&\le c\|\nabla\mathbf{B}_t\|_{L^3}\|\nabla\mathbf{u}\|_{L^6} + c\|\mathbf{B}_t\|_{L^\infty}\|\mathbf{u}\|_{2,\Omega}\\
&\quad +c\|\mathbf{u}\|_{L^\infty}\|\nabla\times\mathbf{B}_t\|_{1,\Omega} \le c\|\mathbf{B}_t\|_{2,\Omega}\|\mathbf{u}\|_{2,\Omega}.\end{aligned} \tag{2.24}$$

Combining (2.24) with (2.20)–(2.21) and (2.24) with (2.22)–(2.23) respectively, we come to

$$\begin{aligned}\|\mathbf{u}_t\|_{2,\Omega}^2 + \|p_t\|_{1,\Omega}^2 + \|\mathbf{B}_t\|_{2,\Omega}^2 \le{}& c\|f_t\|_{0,\Omega}^2 + c\|\mathbf{u}_{tt}\|_{0,\Omega}^2 + \|\mathbf{B}_{tt}\|_{0,\Omega}^2\\ &+c(1+\|\mathbf{u}\|_{2,\Omega}^2 + \|\mathbf{B}\|_{2,\Omega}^2)(\|A_1^{\frac{1}{2}}\mathbf{u}_t\|_{0,\Omega}^2 + \|A_2^{\frac{1}{2}}\mathbf{B}_t\|_{0,\Omega}^2),\end{aligned} \tag{2.25}$$

$$\begin{aligned}\|\mathbf{u}_t\|_{3,\Omega}^2 + \|p_t\|_{2,\Omega}^2 + \|\mathbf{B}_t\|_{3,\Omega}^2 \le{}& c\|f_t\|_{1,\Omega}^2 + c\|\mathbf{B}_{tt}\|_{0,\Omega}^2\\ &+c\|\mathbf{u}_{tt}\|_{1,\Omega}^2 + c(\|\mathbf{u}\|_{2,\Omega}^2 + c\|\mathbf{B}\|_{2,\Omega}^2)(\|\mathbf{u}_t\|_{2,\Omega}^2 + \|A_2\mathbf{B}_t\|_{0,\Omega}^2)\\ &+c(1+\|\mathbf{u}\|_{2,\Omega}^4 + c\|A_2\mathbf{B}\|_{0,\Omega}^4)(\|A_1^{\frac{1}{2}}\mathbf{u}_t\|_{0,\Omega}^2 + \|A_2^{\frac{1}{2}}\mathbf{B}_t\|_{0,\Omega}^2).\end{aligned} \tag{2.26}$$

Now we multiply (2.25) by $\sigma^2(t)$ and then apply (2.14) to obtain

$$\sigma^2(t)[\|\mathbf{u}_t(t)\|_{2,\Omega}^2 + \|p_t(t)\|_{1,\Omega}^2 + \|\mathbf{B}_t(t)\|_{2,\Omega}^2] \le \kappa. \tag{2.27}$$

and multiply (2.26) by $\sigma^2(t)$, integrate with respect to t and apply (2.14) to derive

$$\int_0^t \sigma^2(s)[\|\mathbf{u}_t(s)\|_{3,\Omega}^2 + \|p_t(s)\|_{2,\Omega}^2 + \|\mathbf{B}_t(s)\|_{3,\Omega}^2]\mathrm{d}s \le \kappa,$$

which, along with (2.27), leads to (2.15). □

3. Finite element discretization of the MHD system

In this section we discuss the finite element spatial discretization of the MHD equations (1.3)–(1.4). We first introduce the triangulation of the domain Ω. For the sake of technical treatments, we assume that the boundary of domain Ω is a closed polyhedron; the actual curved smooth boundary case can be treated using some well-developed technicalities for the smooth boundary (*cf.* [20]), in combination with the finite element error estimates established in this work. Let $\mathcal{T}_h$ be a triangulation of the polyhedral domain Ω, and $\mathbf{X}_h \subset \mathbf{X}$, $M_h \subset M$, $\mathbf{V}_h \subset \mathbf{V}$ and $\mathbf{W}_h \subset \mathbf{W}$ be a set of finite element spaces defined on $\mathcal{T}_h$, satisfying the following basic approximation properties [2, 6, 7, 10, 14, 15, 29]:

Assumption 3.1. For each $\mathbf{v} \in \mathbf{H}^i(\Omega) \cap \mathbf{V}$, $q \in H^{i-1}(\Omega) \cap M$ and $\mathbf{C} \in \mathbf{H}^i(\Omega) \cap \mathbf{W}_0$, there exist approximations $\pi_h v \in \mathbf{V}_h$, $\rho_h q \in M_h$ and $J_h C \in \mathbf{W}_h$ such that for $i = 2, 3$,

$$\|\nabla(\mathbf{v} - \pi_h \mathbf{v})\|_{0,\Omega} \le c_1 h^{i-1} \|\mathbf{v}\|_{i,\Omega}, \ \|q - \rho_h q\|_{0,\Omega} \le c_1 h^{i-1} \|q\|_{i-1,\Omega},$$

$$\|\nabla(\mathbf{C} - J_h \mathbf{C})\|_{0,\Omega} \le c_1 h^{i-1} \|\mathbf{C}\|_{i,\Omega} .$$

Moreover, the following inverse inequalities hold for $\mathbf{v}_h \in \mathbf{X}_h$, $\mathbf{C}_h \in \mathbf{W}_h$ and $2 \le p \le q$ that

$$\|\nabla \mathbf{v}_h\|_{0,\Omega} \le c_1 h^{-1} \|\mathbf{v}_h\|_{0,\Omega}, \quad \mathbf{v}_h \in \mathbf{X}_h ,$$

$$\|\mathbf{C}_h\|_{L^q} \le c\, h^{3(\frac{1}{q} - \frac{1}{p})} \|\mathbf{C}_h\|_{L^p}, \quad \|\nabla \mathbf{C}_h\|_{0,\Omega} \le c_1 h^{-1} \|\mathbf{C}_h\|_{0,\Omega}, \quad \mathbf{C}_h \in \mathbf{W}_h ,$$

and the following inf-sup condition holds

$$\sup_{\mathbf{v}_h \in \mathbf{X}_h} \frac{(\nabla \cdot \mathbf{v}_h, q_h)_\Omega}{\|\nabla \mathbf{v}_h\|_{0,\Omega}} \ge \beta_1 \|q_h\|_{0,\Omega} \quad \forall q_h \in M_h$$

where c_1 and β_1 are two positive constants depending only on Ω.

We know the following finite element spaces $\mathbf{X}_h$, M_h, $\mathbf{V}_h$ and $\mathbf{W}_h$ fulfil Assumption 3.1 (see, *e.g.*, [3, 7, 23]):

$$\mathbf{X}_h = \{\mathbf{v}_h \in C(\bar{\Omega}) \cap \mathbf{X}; \mathbf{v}_h|_K \in P_2(K)^3 \quad \forall K \in \mathcal{T}_h\},$$

$$M_h = \{q_h \in C(\bar{\Omega}) \cap M; q_h|_K \in P_1(K) \quad \forall K \in \mathcal{T}_h\},$$

$$\mathbf{V}_h = \{\mathbf{v}_h \in \mathbf{X}_h; (\nabla \cdot \mathbf{v}_h, q_h)_\Omega = 0 \quad \forall q_h \in M_h\},$$

$$\mathbf{W}_h = \{\mathbf{C}_h \in C(\bar{\Omega}) \cap \mathbf{W}; \mathbf{C}_h|_K \in P_2(K)^3 \quad \forall K \in \mathcal{T}_h\}.$$

Letting P_h be the L^2-projection from $L^2(\Omega)^3$ to $\mathbf{V}_h$ and R_{0h} be the L^2-projection from $L^2(\Omega)^3$ to $\mathbf{W}_h$, the following estimates hold for $i = 1, 2, 3$ by using Assumption 3.1 and the argument of [4]:

$$\|\mathbf{v} - P_h \mathbf{v}\|_{0,\Omega} + h\|\nabla(\mathbf{v} - P_h \mathbf{v})\|_{0,\Omega} \le c_2 h^i \|\mathbf{v}\|_{i,\Omega}, \ \forall\, \mathbf{v} \in \mathbf{H}^i(\Omega) \cap \mathbf{V}, \tag{3.1}$$

$$\|\mathbf{C}-R_{0h}\mathbf{C}\|_{0,\Omega}+h\|\nabla(\mathbf{C}-R_{0h}\mathbf{C})\|_{0,\Omega}\le c_2h^i\|\mathbf{C}\|_{i,\Omega},\ \forall\,\mathbf{C}\in\mathbf{H}^i(\Omega)\cap\mathbf{W}_0\,. \tag{3.2}$$

With all the preparations above, we can formulate the finite element approximation of the MHD system (1.3)–(1.4):

Find $(\mathbf{u}_h(t),p_h(t),\mathbf{B}_h(t))\in\mathbf{X}_h\times M_h\times\mathbf{W}_h$ such that $\mathbf{u}_h(0)=P_h\mathbf{u}_0$ and $\mathbf{B}_h(0)=R_{0h}\mathbf{B}_0$, and the following equations hold for all $(\mathbf{v}_h,q_h,\mathbf{C}_h)\in\mathbf{X}_h\times M_h\times\mathbf{W}_h$,

$$\begin{aligned}&(\mathbf{u}_{ht},\mathbf{v}_h)_\Omega+\nu(\nabla\mathbf{u}_h,\nabla\mathbf{v}_h)_\Omega-(\nabla\cdot\mathbf{v}_h,p_h)_\Omega+(\nabla\cdot\mathbf{u}_h,q_h)_\Omega\\&\quad+b(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}_h)+sd(\mathbf{v}_h,\mathbf{B}_h,\mathbf{B}_h)=(\mathbf{f},\mathbf{v}_h)_\Omega, &(3.3)\\&(\mathbf{B}_{ht},\mathbf{C}_h)_\Omega+\mu(\nabla\times\mathbf{B}_h,\nabla\times\mathbf{C}_h)_\Omega+\mu(\nabla\cdot\mathbf{B}_h,\nabla\cdot\mathbf{C}_h)_\Omega\\&\quad-\mathrm{d}(\mathbf{u}_h,\mathbf{B}_h,\mathbf{C}_h)=0\,. &(3.4)\end{aligned}$$

For the subsequent error estimates, we need to introduce several more notation and basic analysis tools. We shall frequently use the discrete Laplacian $-\Delta_h$ defined by

$$(-\Delta_h\mathbf{u}_h,\mathbf{v}_h)=(\nabla\mathbf{u}_h,\nabla\mathbf{v}_h)\ \ \forall\mathbf{u}_h,\ \mathbf{v}_h\in\mathbf{X}_h$$

and the discrete Stokes operator $\mathbf{A}_h:=-P_h\Delta_h$. We will also apply the following important Gadliardo-Nirenberg estimates [14, 15]:

$$\|\nabla\mathbf{v}_h\|_{L^3}+\|\mathbf{v}_h\|_{L^\infty}\le c_3\|\nabla\mathbf{v}_h\|_{0,\Omega}^{\frac12}\|A_h\mathbf{v}_h\|_{0,\Omega}^{\frac12},\quad\|\nabla\mathbf{v}_h\|_{L^6}\le c_3\|A_h\mathbf{v}_h\|_0,\quad\mathbf{v}_h\in\mathbf{V}_h,$$

$$\|A_hP_h\mathbf{v}\|_{0,\Omega}\le c_3\|A_1\mathbf{v}\|_{0,\Omega}\ \ \forall\mathbf{v}\in D(A_1)\,. \tag{3.5}$$

For the space $\mathbf{V}_h$, we will often use the discrete norm $\|\mathbf{v}_h\|_\alpha=\|A_h^{\frac{\alpha}{2}}\mathbf{v}_h\|_0$ for $\alpha\in R$ and any $\mathbf{v}_h\in\mathbf{V}_h$. Then we see

$$\|\mathbf{v}_h\|_1=\|\nabla\mathbf{v}_h\|_0,\quad\|\mathbf{v}_h\|_2=\|A_h\mathbf{v}_h\|_0,\quad\|\mathbf{v}_h\|_{-1}=\|A_h^{-\frac12}\mathbf{v}_h\|_0\ \ \forall\mathbf{v}_h\in\mathbf{V}_h.$$

Furthermore, we define a discrete operator $A_{2h}:\mathbf{W}_h\to\mathbf{W}_h$ by

$$(A_{2h}\mathbf{B}_h,\mathbf{C}_h)_\Omega=(\nabla\times\mathbf{B}_h,\nabla\times\mathbf{C}_h)_\Omega+(\nabla\cdot\mathbf{B}_h,\nabla\cdot\mathbf{C}_h)_\Omega=(A_{2h}^{\frac12}\mathbf{B}_h,A_{2h}^{\frac12}\mathbf{C}_h)_\Omega$$

and the discrete norm $\|\mathbf{B}_h\|_\alpha=\|A_{2h}^{\frac{\alpha}{2}}\mathbf{B}_h\|_{0,\Omega}$ for any $\mathbf{B}_h\in\mathbf{W}_h$ and $\alpha\in R$. We clearly see

$$\|\mathbf{B}_h\|_0^2=\|\mathbf{B}_h\|_{0,\Omega}^2,\quad\|\mathbf{B}_h\|_1^2=\|A_{2h}^{\frac12}\mathbf{B}_h\|_{0,\Omega}^2=\|\nabla\times\mathbf{B}_h\|_{0,\Omega}^2+\|\nabla\cdot\mathbf{B}_h\|_{0,\Omega}^2,$$

$$\|\mathbf{B}_h\|_2^2=\|A_{2h}\mathbf{B}_h\|_0^2,\quad\|\mathbf{B}_h\|_{-1}=\|A_{2h}^{-\frac12}\mathbf{B}_h\|_{0,\Omega}=\sup_{\mathbf{C}_h\in\mathbf{W}_h}\frac{(\mathbf{B}_h,\mathbf{C}_h)_\Omega}{\|A_{2h}^{\frac12}\mathbf{C}_h\|_{0,\Omega}}.$$

For the subsequent convenience, we now introduce a discrete Stokes projection and a discrete Maxwell projection. The discrete Stokes projection (R_h,Q_h) is defined as follows: for any given $(\mathbf{v},q)\in\mathbf{V}\times M$, $R_h(\mathbf{v},q)\in\mathbf{X}_h$ and $Q_h(\mathbf{v},q)\in M_h$ solve for all $(\phi_h,\psi_h)\in\mathbf{X}_h\times M_h$ that

$$(\nabla(R_h(\mathbf{v},q)-\mathbf{v}),\nabla\phi_h)_\Omega-(\nabla\cdot\phi_h,Q_h(\mathbf{v},q)-q)_\Omega+(\nabla\cdot(R_h(\mathbf{v},q)-\mathbf{v}),\psi_h)_\Omega=0\,. \tag{3.6}$$

While the discrete Maxwell projection R_{2h} is defined as follows: for any given $\mathbf{C} \in \mathbf{W}_0$, find $R_{2h}\mathbf{C} \in \mathbf{W}_h$ such that for all $\phi_h \in \mathbf{W}_h$,

$$(\nabla \times (R_{2h}\mathbf{C} - \mathbf{C}), \nabla \times \phi_h)_\Omega + (\nabla \cdot (R_{2h}\mathbf{C} - \mathbf{C}), \nabla \cdot \phi_h)_\Omega = 0\,. \tag{3.7}$$

The next lemma establishes the important approximation properties of the discrete Stokes and Maxwell projections (R_h, Q_h) and R_{2h}.

Lemma 3.2. *Under Assumptions 2.1–2.3 and 3.1, the following error estimates hold for any* $(\mathbf{v}, q) \in (\mathbf{H}^i(\Omega) \cap \mathbf{X}) \times (H^{i-1}(\Omega) \cap M)$ *($i = 2, 3$) that*

$$\begin{aligned}&\|\mathbf{v} - R_h(\mathbf{v}, q)\|_{0,\Omega} + h(\|\nabla(\mathbf{v} - R_h(\mathbf{v}, q))\|_{0,\Omega} + \|q - Q_h(\mathbf{v}, q)\|_{0,\Omega})\\ &\le c\, h^i(\|\mathbf{v}\|_{i,\Omega} + c\|q\|_{i-1,\Omega}),\end{aligned} \tag{3.8}$$

and for any $\mathbf{C} \in \mathbf{H}^{i-1}(\Omega) \cap \mathbf{W}_0$ *($i = 2, 3$) that*

$$\|\mathbf{C} - R_{2h}\mathbf{C}\|_{0,\Omega} + h[\|\nabla \times (\mathbf{C} - R_{2h}\mathbf{C})\|_{0,\Omega} + \|\nabla \cdot (\mathbf{C} - R_{2h}\mathbf{C})\|_{0,\Omega}] \le c\, h^i \|\mathbf{C}\|_{i,\Omega}. \tag{3.9}$$

Proof. For simplicity we write $\tilde{R}_h = R_h(\mathbf{v}, q)$ and $\tilde{Q}_h = Q_h(\mathbf{v}, q)$ below. By taking $\phi_h = P_h\mathbf{v} - \tilde{R}_h$ and $\psi_h = \rho_h q - \tilde{Q}_h$ in (3.6), we obtain

$$\begin{aligned}&\frac{1}{2}(\|\nabla(P_h\mathbf{v} - \tilde{R}_h)\|^2_{0,\Omega} + \|\nabla(\mathbf{v} - \tilde{R}_h)\|^2_{0,\Omega}) - (\nabla \cdot (P_h\mathbf{v} - \tilde{R}_h), q - \rho_h q)_\Omega\\ &= \frac{1}{2}\|\nabla(\mathbf{v} - P_h\mathbf{v})\|^2_{0,\Omega}.\end{aligned} \tag{3.10}$$

Noting that

$$|(\nabla \cdot (P_h\mathbf{v} - \tilde{R}_h), q - \rho_h q)_\Omega| \le \frac{1}{4}\|\nabla(P_h\mathbf{v} - \tilde{R}_h)\|^2_{0,\Omega} + c\|q - \rho_h q\|^2_{0,\Omega},$$

we deduce from (3.10) that

$$\|\nabla(\mathbf{v} - \tilde{R}_h)\|^2_{0,\Omega} \le c\|\nabla(\mathbf{v} - P_h\mathbf{v})\|^2_{0,\Omega} + c\|q - \rho_h q\|^2_{0,\Omega}. \tag{3.11}$$

On the other hand, we derive from (3.6) and Assumption 3.1 that

$$\begin{aligned}\|q - \tilde{Q}_h\|_{0,\Omega} &\le \|q - \rho_h q\|_{0,\Omega} + \|\rho_h q - \tilde{Q}_h\|_{0,\Omega}\\ &\le c\|q - \rho_h q\|_{0,\Omega} + c\|\nabla(\mathbf{v} - \tilde{R}_h)\|_{0,\Omega}.\end{aligned} \tag{3.12}$$

To further estimate $\|\mathbf{v} - \tilde{R}_h\|_{0,\Omega}$, we apply the duality argument. Let $(\mathbf{w}, r) \in \mathbf{X} \times M$ be the unique solution of the auxiliary Stokes equations:

$$-\Delta\mathbf{w} - \nabla r = \mathbf{v} - \tilde{R}_h, \quad \nabla \cdot \mathbf{w} = 0 \quad \text{in} \quad \Omega.$$

By Assumption 2.3, there holds

$$\|\mathbf{w}\|_{2,\Omega} + \|r\|_{1,\Omega} \le c\|\mathbf{v} - \tilde{R}_h\|_{0,\Omega}. \tag{3.13}$$

Now by integrating by parts, we can write

$$\|\mathbf{v}-\tilde{R}_h\|_{0,\Omega}^2 = (\nabla(\mathbf{v}-\tilde{R}_h), \nabla \mathbf{w})_\Omega + (\nabla\cdot(\mathbf{v}-\tilde{R}_h), r)_\Omega - (\nabla\cdot w, q-\tilde{Q}_h)_\Omega. \tag{3.14}$$

Summing the above relation with (3.6) for $(\phi_h, \psi_h) = (\pi_h w, \rho_h r)$, then using (3.13), we derive

$$\begin{aligned}\|\mathbf{v}-\tilde{R}_h\|_{0,\Omega}^2 &= (\nabla(\mathbf{v}-\tilde{R}_h), \nabla(\mathbf{w}-\pi_h\mathbf{w}))_\Omega + (\nabla\cdot(\mathbf{v}-\tilde{R}_h), r-\rho_h r)_\Omega \\ &\quad -(\nabla\cdot(\mathbf{w}-\pi_h\mathbf{w}), q-\tilde{Q}_h)_\Omega \\ &\le c\,(\|\nabla(\mathbf{v}-\tilde{R}_h)\|_{0,\Omega} + \|q-\tilde{Q}_h\|_{0,\Omega})(\|\nabla(\mathbf{w}-\pi_h\mathbf{w})\|_\Omega + \|r-\rho_h r\|_\Omega) \\ &\le c\,h(\|\nabla(\mathbf{v}-\tilde{R}_h)\|_{0,\Omega} + \|q-\tilde{Q}_h\|_{0,\Omega})(\|\mathbf{w}\|_{2,\Omega} + \nu^{-1}\|r\|_{1,\Omega}) \\ &\le c\,h(\|\nabla(\mathbf{v}-\tilde{R}_h)\|_{0,\Omega} + \|q-\tilde{Q}_h\|_{0,\Omega})\|\mathbf{v}-\tilde{R}_h\|_{0,\Omega}.\end{aligned} \tag{3.15}$$

Clearly (3.8) follows by combining (3.15) with (3.11)–(3.12) and applying (3.1) and 3.1.

It remains to show (3.9). We first take $\mathbf{\Phi}_h = R_{2h}\mathbf{C} - R_{0h}\mathbf{C}$ in (3.7) to obtain

$$\begin{aligned}&\frac{1}{2}(\|\nabla\times(R_{2h}\mathbf{C}-R_{0h}\mathbf{C})\|_{0,\Omega}^2 + \|\nabla\times(R_{2h}\mathbf{C}-\mathbf{C})\|_{0,\Omega}^2) \\ &\quad +\frac{1}{2}(\|\nabla\cdot(R_{2h}\mathbf{C}-R_{0h}\mathbf{C})\|_{0,\Omega}^2 + \|\nabla\cdot(R_{2h}\mathbf{C}-\mathbf{C})\|_{0,\Omega}^2) \\ &\le \frac{1}{2}(\|\nabla\times(R_{0h}\mathbf{C}-\mathbf{C})\|_{0,\Omega}^2 + \|\nabla\cdot(R_{0h}\mathbf{C}-\mathbf{C})\|_{0,\Omega}^2).\end{aligned}$$

Combining the above estimate with (2.1) we readily see

$$\|\nabla(\mathbf{C}-R_{2h}\mathbf{C})\|_{0,\Omega} \le c\,\|\nabla(\mathbf{C}-R_{0h}\mathbf{C})\|_{0,\Omega}^2. \tag{3.16}$$

Next we shall use the duality argument again to estimate the L^2-norm $\|\mathbf{C}-R_{2h}\mathbf{C}\|_{0,\Omega}$. Let $\mathbf{w}\in\mathbf{W}_0$ be the unique solution to the auxiliary elliptic system

$$\nabla\times\nabla\times\mathbf{w} = \mathbf{C}-R_{2h}\mathbf{C}, \quad \nabla\cdot\mathbf{w} = 0 \quad \text{in} \quad \Omega,$$

with the boundary conditions $\mathbf{n}\times(\nabla\times\mathbf{w}) = 0$ and $\mathbf{w}\cdot\mathbf{n} = 0$ on $\partial\Omega$. By Assumption 2.3,

$$\|\mathbf{w}\|_{2,\Omega} \le c\|\mathbf{C}-R_{2h}\mathbf{C}\|_{0,\Omega}. \tag{3.17}$$

Now by integrating by parts we can write

$$\|\mathbf{C}-R_{2h}\mathbf{C}\|_{0,\Omega}^2 = (\nabla\times(\mathbf{C}-R_{2h}\mathbf{C}), \nabla\times\mathbf{w})_\Omega + (\nabla\cdot(\mathbf{C}-R_{2h}\mathbf{C}), \nabla\cdot\mathbf{w}). \tag{3.18}$$

Then by summing (3.18) and (3.7) with $\phi_h = J_h\mathbf{w}$, and using (3.17) and 3.1, we deduce

$$\begin{aligned}\|\mathbf{C}-R_{2h}\mathbf{C}\|_{0,\Omega}^2 &= (\nabla\times(\mathbf{C}-R_{2h}\mathbf{C}), \nabla\times(\mathbf{w}-J_h\mathbf{w}))_\Omega \\ &\quad +(\nabla\cdot(\mathbf{C}-R_{2h}\mathbf{C}), \nabla\cdot(\mathbf{w}-J_h\mathbf{w}))_\Omega \\ &\le c\|\nabla(\mathbf{C}-R_{2h}\mathbf{C})\|_{0,\Omega}\|\nabla(\mathbf{w}-J_h\mathbf{w})\|_{0,\Omega} \\ &\le c\,h\|\nabla(\mathbf{C}-R_{2h}\mathbf{C})\|_{0,\Omega}\|\mathbf{w}\|_{2,\Omega} \\ &\le c\,h\|\nabla(\mathbf{C}-R_{2h}\mathbf{C})\|_{0,\Omega}\|\mathbf{C}-R_{2h}\mathbf{C}\|_{0,\Omega}.\end{aligned} \tag{3.19}$$

Clearly (3.9) is now a direct consequence of (3.19), (3.16) and (3.2). □

The following lemma presents some approximation properties that are crucial to our subsequent finite element error estimates.

Lemma 3.3. *Under Assumptions 2.1–2.3 and 3.1, the following error estimates hold for $i = 2, 3$,*

$$\begin{aligned}\|A_h^{-1}P_hA_1\mathbf{v}-\mathbf{v}\|_{0,\Omega}+h\|\nabla(A_h^{-1}P_hA_1\mathbf{v}-\mathbf{v})\|_{0,\Omega} &\le ch^i\|\mathbf{v}\|_{i,\Omega},\\ \|A_1^{-1}PA_h\mathbf{v}_h-\mathbf{v}_h\|_{0,\Omega}+h\|\nabla(A_1^{-1}PA_h\mathbf{v}_h-\mathbf{v}_h)\|_{0,\Omega} &\le ch^i\|\mathbf{v}_h\|_i,\end{aligned} \tag{3.20}$$

$$\begin{aligned}\|A_{2h}^{-1}R_{0h}A_2\mathbf{C}-\mathbf{C}\|_{0,\Omega}+h\|\nabla(A_{2h}^{-1}R_{0h}A_2\mathbf{C}-\mathbf{C})\|_{0,\Omega} &\le ch^i\|\mathbf{C}\|_{i,\Omega},\\ \|A_2^{-1}PA_{2h}\mathbf{C}_h-\mathbf{C}_h\|_{0,\Omega}+h\|\nabla(A_2^{-1}PA_{2h}\mathbf{C}_h-\mathbf{C}_h)\|_{0,\Omega} &\le ch^i\|\mathbf{C}_h\|_i\,.\end{aligned} \tag{3.21}$$

Proof. For a given vector function $\mathbf{g}\in L^2(\Omega)^3$, we consider the variational formulation of the Stokes equations: Find $(\mathbf{v},q)\in\mathbf{X}\times M$ such that

$$(\nabla\mathbf{v},\nabla\mathbf{w})_\Omega-(\nabla\cdot\mathbf{w},q)_\Omega+(\nabla\cdot\mathbf{v},\psi)_\Omega=(\mathbf{g},\mathbf{w})_\Omega\quad\forall\,(\mathbf{w},\psi)\in\mathbf{X}\times M\,, \tag{3.22}$$

and its finite element approximation: Find $(\mathbf{v}_h,q_h)\in\mathbf{X}_h\times M_h$ such that

$$(\nabla\mathbf{v}_h,\nabla\mathbf{w}_h)_\Omega-(\nabla\cdot\mathbf{w}_h,q_h)_\Omega+(\nabla\cdot\mathbf{v}_h,\psi_h)_\Omega=(\mathbf{g},\mathbf{w}_h)_\Omega\quad\forall\,(\phi_h,\psi_h)\in\mathbf{X}_h\times M_h\,. \tag{3.23}$$

It is easy to see that the above two pairs of solutions $(\mathbf{v},q)\in\mathbf{X}\times M$ and $(\mathbf{v}_h,q_h)\in\mathbf{X}_h\times M_h$ satisfy (3.6), namely $(\mathbf{v}_h,q_h)\in\mathbf{X}_h\times M_h$ is the Stokes projection of $(\mathbf{v},q)\in\mathbf{V}\times M$.

Taking $\mathbf{g}=A_1\mathbf{v}$ in (3.23), we see $\mathbf{v}_h=A_h^{-1}P_hA_1\mathbf{v}$. Then the results of Lemma 3.1 imply

$$\begin{aligned}&\|A_h^{-1}P_hA_1\mathbf{v}-\mathbf{v}\|_{0,\Omega}+h\|\nabla(A_h^{-1}P_hA_1\mathbf{v}-\mathbf{v})\|_{0,\Omega}\\ &\quad=\|v_h-\mathbf{v}\|_{0,\Omega}+h\|\nabla(\mathbf{v}_h-\mathbf{v})\|_{0,\Omega}\le ch^i\|\mathbf{v}\|_{i,\Omega}.\end{aligned} \tag{3.24}$$

Similarly, taking $\mathbf{g}=A_h\mathbf{v}_h$ in (3.22), we know $\mathbf{v}=A_1^{-1}PA_h\mathbf{v}_h$. This, with Lemma 3.2 and Assumption 2.3, yields

$$\begin{aligned}&\|A_1^{-1}PA_h\mathbf{v}_h-\mathbf{v}_h\|_{0,\Omega}+h\|\nabla(A_1^{-1}PA_h\mathbf{v}_h-\mathbf{v}_h)\|_{0,\Omega}\\ &=\|\mathbf{v}_h-\mathbf{v}\|_{0,\Omega}+h\|\nabla(\mathbf{v}_h-\mathbf{v})\|_{0,\Omega}\le ch^i\|\mathbf{v}\|_{i,\Omega}\\ &\le ch^i\|\mathbf{g}\|_{i-2,\Omega}=ch^i\|A_h\mathbf{v}_h\|_{i-2}\le ch^i\|\mathbf{v}_h\|_i,\end{aligned}$$

which, along with (3.24), implies (3.20).

Now for a given $\mathbf{h}\in\mathbf{H}$, we consider the Maxwell's problem: Find $\mathbf{C}\in\mathbf{W}$ such that

$$\nabla\times\nabla\times\mathbf{C}=\mathbf{h},\quad\nabla\cdot\mathbf{C}=0\quad\text{in}\quad\Omega\quad\text{and}\quad\mathbf{n}\times\nabla\times\mathbf{C}=0\quad\text{on}\quad\partial\Omega\,, \tag{3.25}$$

and its finite element approximation: Find $\mathbf{C}_h\in\mathbf{W}_h$ such that

$$(\nabla\cdot\mathbf{C}_h,\nabla\cdot\mathbf{\Phi}_h)_\Omega+(\nabla\times\mathbf{C}_h,\nabla\times\mathbf{\Phi}_h)=(\mathbf{h},\mathbf{\Phi}_h)_\Omega\quad\forall\,\mathbf{\Phi}_h\in\mathbf{W}_h\,. \tag{3.26}$$

It is ready to see from the above two equations and (3.7) that $\mathbf{C}_h\in\mathbf{W}_h$ is the Maxwell's projection of $\mathbf{C}\in\mathbf{W}_0$.

Next by setting $\mathbf{h}=A_2\mathbf{C}$, we can immediately see from (3.26) that $\mathbf{C}_h=A_{2h}^{-1}R_{0h}A_2\mathbf{C}$. Then it follows from Lemma 3.2 that

$$\begin{aligned}&\|A_{2h}^{-1}R_{0h}A_2\mathbf{C}-\mathbf{C}\|_{0,\Omega}+h\|\nabla\times(A_{2h}^{-1}R_{0h}A_2\mathbf{C}-\mathbf{C})\|_{0,\Omega}+h\|\nabla\cdot(A_{2h}^{-1}R_{0h}A_2\mathbf{C}-\mathbf{C})\|_{0,\Omega}\\ &=\|\mathbf{C}_h-\mathbf{C}\|_{0,\Omega}+h(\|\nabla\times(\mathbf{C}_h-\mathbf{C})\|_{0,\Omega}+\|\nabla\cdot(\mathbf{C}_h-\mathbf{C})\|_{0,\Omega})\le c\,h^i\|\mathbf{C}\|_{i,\Omega}.\end{aligned} \tag{3.27}$$

Similarly, by setting $\mathbf{h} = A_{2h}\mathbf{C}_h$, we see readily from (3.25) that $\mathbf{C} = A_2^{-1}PA_{2h}\mathbf{C}_h$. Then it follows from Lemma 3.2 and Assumption 2.3 that

$$\begin{aligned}&\|A_2^{-1}PA_{2h}\mathbf{C}_h - \mathbf{C}_h\|_{0,\Omega} + h\|\nabla\times(A_2^{-1}PA_{2h}\mathbf{C}_h - \mathbf{C}_h)\|_{0,\Omega} + h\|\nabla\cdot(A_2^{-1}PA_{2h}\mathbf{C}_h - \mathbf{C}_h)\|_{0,\Omega}\\ &= \|\mathbf{C}_h - \mathbf{C}\|_{0,\Omega} + h\|\nabla\times(\mathbf{C}_h - \mathbf{C})\|_{0,\Omega} + h\|\nabla\cdot(\mathbf{C}_h - \mathbf{C})\|_{0,\Omega}\\ &\le c\,h^i\|\mathbf{C}\|_{i,\Omega} \le c\,h^i\|\mathbf{h}\|_{i-2,\Omega} \le c\,h^i\|A_{2h}\mathbf{C}_h\|_{i-2} \le c\,h^i\|\mathbf{C}_h\|_i\,,\end{aligned}$$

which, along with (3.27) and (2.1), gives (3.21). □

Theorem 3.4. *Under Assumptions 2.1–2.3 and 3.1, the solution* $(\mathbf{u}_h, p_h, \mathbf{B}_h)$ *to the system (3.3)–(3.4) satisfies the following stability estimate*

$$\|\mathbf{u}_h(t)\|_{0,\Omega}^2 + \tau\|\mathbf{B}_h(t)\|_{0,\Omega}^2 + \int_0^t [\nu\|\nabla\mathbf{u}_h(s)\|_{0,\Omega}^2 + \tau\mu\|\nabla\mathbf{B}_h(s)\|_{0,\Omega}^2]\mathrm{d}s \le \kappa. \tag{3.28}$$

Proof. Summing (3.3) with $(\mathbf{v}_h, q_h) = (\mathbf{u}_h, p_h)$ and (3.4) with $\mathbf{C}_h = \tau\mathbf{B}_h$, we obtain the identity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}_h\|_{0,\Omega}^2 + \nu\|\nabla\mathbf{u}_h\|_{0,\Omega}^2 + \frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{B}_h\|_{0,\Omega}^2 + \tau\mu\|\nabla\times\mathbf{B}_h\|_{0,\Omega}^2 + \tau\mu\|\nabla\cdot\mathbf{B}_h\|_{0,\Omega}^2 = (\mathbf{f}, \mathbf{u}_h)_\Omega, \tag{3.29}$$

then applying Young's inequality,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\mathbf{u}_h\|_{0,\Omega}^2 + \tau\|\mathbf{B}_h\|_{0,\Omega}^2) + \nu\|\nabla\mathbf{u}_h\|_{0,\Omega}^2 + \tau\mu\|\nabla\times\mathbf{B}_h\|_{0,\Omega}^2 + \tau\mu\|\nabla\cdot\mathbf{B}_h\|_{0,\Omega}^2 \le c\,\|\mathbf{f}\|_{0,\Omega}^2. \tag{3.30}$$

Integrating both sides of the above inequality from 0 to t, we come to

$$\begin{aligned}&\|\mathbf{u}_h(t)\|_{0,\Omega}^2 + \tau\|\mathbf{B}_h(t)\|_{0,\Omega}^2 + \nu\int_0^t \|\nabla\mathbf{u}_h(s)\|_{0,\Omega}^2\mathrm{d}s\\ &\quad + \tau\mu\int_0^t [\|\nabla\times\mathbf{B}_h(s)\|_{0,\Omega}^2 + \|\nabla\cdot\mathbf{B}_h(s)\|_{0,\Omega}^2]\mathrm{d}s\\ &\le \|\mathbf{u}_0\|_{0,\Omega}^2 + \tau\|B_0\|_{0,\Omega}^2 + c\int_0^T \|\mathbf{f}\|_{0,\Omega}^2\mathrm{d}s,\end{aligned}$$

for all $t \in [0, T]$, which, along with (2.1), implies (3.28). □

4. L^2-NORM ERROR ESTIMATES OF THE FINITE ELEMENT SOLUTION

We are now ready to derive a series of L^2-norm error estimates for the finite element solution $(\mathbf{u}_h, p_h, \mathbf{B}_h)$ to the system (3.3)–(3.4).

Lemma 4.1. *Under Assumptions 2.1–2.3 and 3.1, the finite element solution* $(\mathbf{u}_h, p_h, \mathbf{B}_h)$ *to the system (3.3)–(3.4) satisfies the following error estimate:*

$$\begin{aligned}&\int_0^t [\nu\|\nabla(\mathbf{u}_h(s) - \mathbf{u}(s))\|_{0,\Omega}^2 + \tau\mu\|\nabla(\mathbf{B}(s) - \mathbf{B}_h(s))\|_{0,\Omega}^2]\mathrm{d}s\\ &\quad + \|\mathbf{u}_h(t) - \mathbf{u}(t)\|_{0,\Omega}^2 + \tau\|\mathbf{B}(t) - \mathbf{B}_h(t)\|_{0,\Omega}^2 \le \kappa\,h^4.\end{aligned} \tag{4.1}$$

Proof. Setting $e_h = P_h\mathbf{u} - \mathbf{u}_h,\ \eta_h = \rho_h p - p_h,\ \varepsilon_h = R_{0h}\mathbf{B} - \mathbf{B}_h$, we derive readily from (1.3)–(1.4) and (3.3)–(3.4) that

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}(\mathbf{u}-\mathbf{u}_h),\mathbf{v}_h\right)_\Omega + \nu(\nabla(\mathbf{u}-\mathbf{u}_h),\nabla\mathbf{v}_h)_\Omega - (\nabla\cdot\mathbf{v}_h, p-p_h)_\Omega + (\nabla\cdot e_h, q_h)_\Omega\\
&\quad + b(\mathbf{u}-\mathbf{u}_h,\mathbf{u},\mathbf{v}_h) + b(\mathbf{u},\mathbf{u}-\mathbf{u}_h,\mathbf{v}_h) - b(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\mathbf{u}_h,\mathbf{v}_h)\\
&\quad + \tau\mathrm{d}(\mathbf{v}_h,\mathbf{B}-\mathbf{B}_h,\mathbf{B}) + \tau\mathrm{d}(\mathbf{v}_h,\mathbf{B},\mathbf{B}-\mathbf{B}_h)\\
&\quad - \tau\mathrm{d}(\mathbf{v}_h,\mathbf{B}-\mathbf{B}_h,\mathbf{B}-\mathbf{B}_h) = 0, \qquad (4.2)\\
&\left(\frac{\partial}{\partial t}(\mathbf{B}-\mathbf{B}_h),\mathbf{C}_h\right)_\Omega + \mu(\nabla\times(\mathbf{B}-\mathbf{B}_h),\nabla\times\mathbf{C}_h)_\Omega + \mu(\nabla\cdot(\mathbf{B}-\mathbf{B}_h),\nabla\cdot\mathbf{C}_h)_\Omega\\
&\quad - \mathrm{d}(\mathbf{u}-\mathbf{u}_h,\mathbf{B},\mathbf{C}_h) - \mathrm{d}(\mathbf{u},\mathbf{B}-\mathbf{B}_h,\mathbf{C}_h)\\
&\quad + \mathrm{d}(\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h,\mathbf{C}_h) = 0\,. \qquad (4.3)
\end{aligned}$$

Taking $(\mathbf{v}_h, q_h) = (e_h, \eta_h)$ and $\mathbf{C}_h = \tau\varepsilon_h$ in (4.2) and (4.3), then adding up the resultant equations, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_h\|^2_{0,\Omega} + \frac{\nu}{2}(\|\nabla e_h\|^2_{0,\Omega} + \|\nabla(\mathbf{u}-\mathbf{u}_h)\|^2_{0,\Omega}) + \frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon_h\|^2_{0,\Omega} + \frac{\tau\mu}{2}(\|\nabla\times\varepsilon_h\|^2_{0,\Omega} + \|\nabla\times(\mathbf{B}-\mathbf{B}_h)\|^2_{0,\Omega})\\
&\quad + \frac{\tau\mu}{2}(\|\nabla\cdot\varepsilon_h\|^2_{0,\Omega} + \|\nabla\cdot(\mathbf{B}-\mathbf{B}_h)\|^2_{0,\Omega}) + b(\mathbf{u}-\mathbf{u}_h,\mathbf{u},e_h) + b(\mathbf{u},\mathbf{u}-\mathbf{u}_h,e_h) - b(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-P_h\mathbf{u},e_h)\\
&\quad + \tau\mathrm{d}(e_h,\mathbf{B}-\mathbf{B}_h,\mathbf{B}) + \tau\mathrm{d}(e_h,\mathbf{B},\mathbf{B}-\mathbf{B}_h) + \tau\mathrm{d}(\mathbf{u}-P_h\mathbf{u},\mathbf{B}-\mathbf{B}_h,\mathbf{B}-\mathbf{B}_h)\\
&\quad - \tau\mathrm{d}(\mathbf{u}-\mathbf{u}_h,\mathbf{B},\varepsilon_h) - \tau\mathrm{d}(\mathbf{u},\mathbf{B}-\mathbf{B}_h,\varepsilon_h) - \tau\mathrm{d}(\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h,\mathbf{B}-R_{0h}\mathbf{B})\\
&\le \frac{\nu}{2}\|\nabla(\mathbf{u}-P_h\mathbf{u})\|^2_{0,\Omega} - (\nabla\cdot e_h,\rho_h p - p)_\Omega + \frac{\tau\mu}{2}(\|\nabla\times(\mathbf{B}-R_{0h}\mathbf{B})\|^2_{0,\Omega} + \|\nabla\cdot(\mathbf{B}-R_{0h}\mathbf{B})\|^2_{0,\Omega}). \qquad (4.4)
\end{aligned}$$

But by means of the estimates (2.1)–(2.2), (3.1)–(3.2), (3.5) and Assumption 3.1, we deduce

$$\begin{aligned}
&|b(\mathbf{u}-\mathbf{u}_h,\mathbf{u},e_h) + b(\mathbf{u},\mathbf{u}-\mathbf{u}_h,e_h)| \le c_0\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}\|\mathbf{u}\|_{2,\Omega}\|e_h\|_{0,\Omega}\\
&\le \frac{\nu}{16}\|\nabla(\mathbf{u}-\mathbf{u}_h)\|^2_{0,\Omega} + \frac{4}{\nu}c_0^2\|\mathbf{u}\|^2_{2,\Omega}\|e_h\|^2_{0,\Omega},
\end{aligned}$$

and the following more estimates:

$$\begin{aligned}
&|b(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-P_h\mathbf{u},e_h)| \le N_0\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}\|\nabla(\mathbf{u}-P_h\mathbf{u})\|_{0,\Omega}\|\nabla e_h\|_{0,\Omega}\\
&\le c_0\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}\|\mathbf{u}\|_{2,\Omega}\|e_h\|_{0,\Omega} \le \frac{\nu}{16}\|\nabla(\mathbf{u}-\mathbf{u}_h)\|^2_{0,\Omega} + \frac{4}{\nu}c_0^2\|\mathbf{u}\|^2_{2,\Omega}\|e_h\|^2_{0,\Omega},\\
&\tau|\mathrm{d}(e_h,\mathbf{B}-\mathbf{B}_h,\mathbf{B}) + \mathrm{d}(e_h,\mathbf{B},\mathbf{B}-\mathbf{B}_h)| \le \tau c_0\|\nabla e_h\|_{0,\Omega}\|\mathbf{B}\|_{2,\Omega}\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}\\
&\le \frac{\nu}{16}\|\nabla e_h\|^2_{0,\Omega} + \frac{4}{\nu}\tau^2c_0^2\|\mathbf{B}\|^2_{2,\Omega}\|\mathbf{B}-\mathbf{B}_h\|^2_{0,\Omega},\\
&\tau|\mathrm{d}(\mathbf{u}-P_h\mathbf{u},\mathbf{B}-\mathbf{B}_h,\mathbf{B}-\mathbf{B}_h)| \le \tau N_1\|\nabla(\mathbf{u}-P_h\mathbf{u})\|_{0,\Omega}\|\nabla\times(\mathbf{B}-\mathbf{B}_h)\|_{0,\Omega}\\
&\quad \times(\|\nabla(\mathbf{B}-R_{0h}\mathbf{B})\|_{0,\Omega} + ch^{-1}\|\varepsilon_h\|_{0,\Omega})\\
&\le \frac{\tau\mu}{16}\|\nabla\times(\mathbf{B}-\mathbf{B}_h)\|^2_{0,\Omega} + \frac{4}{\mu}c_0^2\tau\|\mathbf{u}\|^2_{2,\Omega}\|\varepsilon_h\|^2_{0,\Omega} + \frac{4}{\mu}c_0^2\tau\|\nabla(\mathbf{u}-P_h\mathbf{u})\|^2_{2,\Omega}\|\nabla(\mathbf{B}-R_{0h}\mathbf{B})\|^2_{0,\Omega},\\
&\quad \tau|\mathrm{d}(\mathbf{u}-\mathbf{u}_h,\mathbf{B},\varepsilon_h) + \tau\mathrm{d}(\mathbf{u},\mathbf{B}-\mathbf{B}_h,\varepsilon_h)|\\
&\le \tau c_0(\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}\|\mathbf{B}\|_{2,\Omega} + \|\mathbf{u}\|_{2,\Omega}\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega})\|\nabla\times\varepsilon_h\|_{0,\Omega}\\
&\le \frac{\tau\mu}{8}\|\nabla\times\varepsilon_h\|^2_{0,\Omega} + \frac{4}{\mu}\tau c_0^2\|\mathbf{u}\|^2_{2,\Omega}\|\mathbf{B}-\mathbf{B}_h\|^2_{0,\Omega} + \frac{4}{\mu}\quad\tau c_0^2\|\mathbf{B}\|^2_{2,\Omega}\|\mathbf{u}-\mathbf{u}_h\|^2_{0,\Omega},
\end{aligned}$$

$$\tau|\mathrm{d}(\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h,\mathbf{B}-R_{0h}\mathbf{B})| \le \tau N_1\|\nabla\times(\mathbf{B}-R_{0h}\mathbf{B})\|_{0,\Omega}\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}$$
$$\times(\|\nabla(\mathbf{B}-R_{0h}\mathbf{B}\|_{0,\Omega}+ch^{-1}\|\varepsilon_h\|_{0,\Omega})$$
$$\le \frac{\nu}{16}\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}^2+\frac{4\tau^2}{\nu}(c_0^2\|\mathbf{u}\|_{2,\Omega}^2\|\varepsilon_h\|_{0,\Omega}^2+N_1^2\|\nabla\times(\mathbf{B}-R_{0h}\mathbf{B})\|_{0,\Omega}^2\|\nabla(\mathbf{B}-R_{0h}\mathbf{B})\|_{0,\Omega}^2),$$
$$|(\nabla\cdot e_h,\rho_h p-p)_\Omega| \le \frac{\nu}{16}\|\nabla e_h\|_{0,\Omega}^2+\frac{4}{\nu}c_0^2\|p-\rho_h p\|_{0,\Omega}^2.$$

Combining the above estimates and applying (3.2) and 3.1, we derive from (4.4) that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|e_h\|_{0,\Omega}^2+\tau\|\varepsilon_h\|_{0,\Omega}^2)+\nu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}^2+\tau\mu\|\nabla\times(\mathbf{B}-\mathbf{B}_h)\|_{0,\Omega}^2 \le \mathrm{d}(t)(\|e_h\|_{0,\Omega}^2+\tau\|\varepsilon_h\|_{0,\Omega}^2)$$
$$+ch^4(\|\mathbf{u}\|_{3,\Omega}^2+\|p\|_{2,\Omega}^2+\|\mathbf{B}\|_{3,\Omega}^2)+ch^4(\|\mathbf{u}\|_{2,\Omega}^2+\|\mathbf{B}\|_{2,\Omega}^2)\|\mathbf{B}\|_{3,\Omega}^2,$$

with $\mathrm{d}(t)=c[\|\mathbf{u}\|_{2,\Omega}^2+\|\mathbf{B}\|_{2,\Omega}^2]$. Then integrating the above inequality and using Lemma 2.5,

$$\|e_h(t)\|_{0,\Omega}^2+\tau\|\varepsilon_h(t)\|_{0,\Omega}^2+\int_0^t[\nu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}^2+\tau\mu\|\nabla\times(\mathbf{B}-\mathbf{B}_h)\|_{0,\Omega}^2]\mathrm{d}s$$
$$\le \kappa h^4+\int_0^t \mathrm{d}(s)[\|e_h\|_{0,\Omega}^2+\tau\|\varepsilon_h\|_{0,\Omega}^2]\mathrm{d}s. \tag{4.5}$$

Now applying the Gronwall lemma to (4.5) and using Lemma 2.5, we come to

$$\|e_h(t)\|_{0,\Omega}^2+\tau\|\varepsilon_h(t)\|_{0,\Omega}^2+\int_0^t[\nu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}^2+\tau\mu\|\nabla\times(\mathbf{B}-\mathbf{B}_h)\|_{0,\Omega}^2]\mathrm{d}s \le \kappa h^4. \tag{4.6}$$

This, combining with Lemma 3.2 and the following estimates from (3.1)–(3.2),

$$\|P_h\mathbf{u}(t)-\mathbf{u}(t)\|_{0,\Omega}^2 \le ch^4\|\mathbf{u}(t)\|_{2,\Omega}^2, \quad \|R_{0h}\mathbf{B}-\mathbf{B}\|_{0,\Omega}^2 \le ch^4\|\mathbf{B}\|_{2,\Omega}^2,$$

gives the desired estimate (4.1). □

For our desired results, we need to first establish the following important error estimates for two L^2-projections P_h and R_{0h} in H^{-1}-norm.

Lemma 4.2. *Under Assumptions 2.1–2.3 and 3.1, the finite element solution* $(\mathbf{u}_h,p_h,\mathbf{B}_h)$ *to the system (3.3)–(3.4) satisfies the following error estimate:*

$$\|P_h\mathbf{u}(t)-\mathbf{u}_h(t)\|_{-1}^2+\tau\|R_{0h}\mathbf{B}(t)-\mathbf{B}_h(t)\|_{-1}^2$$
$$+\int_0^t[\nu\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}^2+\tau\mu\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}^2]\mathrm{d}s \le \kappa h^6. \tag{4.7}$$

Proof. Taking $(\mathbf{v}_h, q_h) = (A_h^{-1}e_h, 0)$ and $\mathbf{C}_h = \tau A_{2h}^{-1}\varepsilon_h$ in (4.2) and (4.3) respectively, then adding up the resultant equations, we derive

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_h\|_{-1}^2 + \nu\|e_h\|_0^2 + \frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon_h\|_{-1}^2 + \tau\mu\|\varepsilon_h\|_0^2 + (\nabla\cdot(A_h^{-1}e_h), p-\rho_h p)_\Omega \\
&\quad +\nu(A_h^{-1}P_hA_1\mathbf{u}-\mathbf{u}, e_h)_\Omega + \tau\mu(A_{2h}^{-1}R_{0h}A_2\mathbf{B}-\mathbf{B}, \varepsilon_h)_\Omega \\
&\quad +b(\mathbf{u}-\mathbf{u}_h, \mathbf{u}, A_h^{-1}e_h) + b(\mathbf{u}, \mathbf{u}-\mathbf{u}_h, A_h^{-1}e_h) - b(\mathbf{u}-\mathbf{u}_h, \mathbf{u}-\mathbf{u}_h, A_h^{-1}e_h) \\
&\quad +\tau\mathrm{d}(A_h^{-1}e_h, \mathbf{B}-\mathbf{B}_h, \mathbf{B}) + \tau\mathrm{d}(A_h^{-1}e_h, \mathbf{B}, \mathbf{B}-\mathbf{B}_h) \\
&\quad -\tau\mathrm{d}(A_h^{-1}e_h, \mathbf{B}-\mathbf{B}_h, \mathbf{B}-\mathbf{B}_h) - \tau\mathrm{d}(\mathbf{u}-\mathbf{u}_h, \mathbf{B}, A_{2h}^{-1}\varepsilon_h) \\
&\quad -\tau\mathrm{d}(\mathbf{u}, \mathbf{B}-\mathbf{B}_h, A_{2h}^{-1}\varepsilon_h) + \tau\mathrm{d}(\mathbf{u}-\mathbf{u}_h, \mathbf{B}-\mathbf{B}_h, A_{2h}^{-1}\varepsilon_h) = 0. \qquad (4.8)
\end{aligned}$$

But by using the estimates (2.1)–(2.2), (3.1)–(3.2), (3.5) and Lemma 3.3, we can deduce

$$\begin{aligned}
|(\nabla\cdot(A_h^{-1}e_h), p-\rho_h p)_\Omega| &= |(\nabla\cdot(A_h^{-1}e_h - A_1^{-1}PA_hA_h^{-1}e_h), p-\rho_h p)_\Omega| \\
&\le \sqrt{3}\|\nabla(A_h^{-1}e_h - A_1^{-1}PA_hA_h^{-1}e_h)\|_{0,\Omega}\|p-\rho_h p\|_{0,\Omega} \\
&\le c_0h^3\|p\|_{2,\Omega}\|e_h\|_0 \le \frac{\nu}{16}\|e_h\|_0^2 + \frac{4}{\nu}c_0^2h^6\|p\|_{2,\Omega}^2, \\
\nu|(A_h^{-1}P_hA_1\mathbf{u}-\mathbf{u}, e_h)_\Omega| + \tau\mu|(A_{2h}^{-1}R_{0h}A_2\mathbf{B}-\mathbf{B}, \varepsilon_h)_\Omega| &\le \nu c_0h^3\|\mathbf{u}\|_{3,\Omega}\|e_h\|_0 + c_0\tau\mu h^3\|\mathbf{B}\|_{3,\Omega}\|\varepsilon_h\|_0 \\
&\le \frac{\nu}{16}\|e_h\|_0^2 + \frac{s\mu}{16}\|\varepsilon_h\|_0^2 + \frac{4}{\nu}c_0^2h^6\|\mathbf{u}\|_{3,\Omega}^2 \\
&\quad +\frac{8}{\mu}\tau c_0^2h^6\|\mathbf{B}\|_{3,\Omega}^2, \\
|b(\mathbf{u}-\mathbf{u}_h, \mathbf{u}, A_h^{-1}e_h) + b(\mathbf{u}, \mathbf{u}-\mathbf{u}_h, A_h^{-1}e_h)| &\le c_0\|\nabla A_h^{-1}e_h\|_{0,\Omega}\|\mathbf{u}\|_{2,\Omega}\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega} \\
&\le \frac{\nu}{16}(\|e_h\|_0^2 + \|\mathbf{u}-P_h\mathbf{u}\|_{0,\Omega}^2) + \frac{8}{\nu}c_0^2\|\mathbf{u}\|_{2,\Omega}^2\|e_h\|_{-1}^2, \\
|b(\mathbf{u}-\mathbf{u}_h, \mathbf{u}-\mathbf{u}_h, A_h^{-1}e_h)| &\le 2\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}\|A_h^{-1}e_h\|_{L^\infty} \\
&\le c_0\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}\|e_h\|_0 \\
&\le \frac{\nu}{16}\|e_h\|_0^2 + \frac{4}{\nu}c_0^2\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}^2\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}^2, \\
\tau|\mathrm{d}(A_h^{-1}e_h, \mathbf{B}-\mathbf{B}_h, \mathbf{B}) + \mathrm{d}(A_h^{-1}e_h, \mathbf{B}, \mathbf{B}-\mathbf{B}_h)| &\le \tau c_0\|e_h\|_{-1}(\|\mathbf{B}-R_{0h}\mathbf{B}\|_{0,\Omega} + \|\varepsilon_h\|_0)\|\mathbf{B}\|_{2,\Omega} \\
&\le \frac{\tau\mu}{16}(\|\varepsilon_h\|_0^2 + \|\mathbf{B}-R_{0h}\mathbf{B}\|_{0,\Omega}^2) + \frac{8}{\mu}\tau c_0^2\|\mathbf{B}\|_{2,\Omega}^2\|e_h\|_{-1}^2, \\
\tau|\mathrm{d}(A_h^{-1}e_h, \mathbf{B}-\mathbf{B}_h, \mathbf{B}-\mathbf{B}_h) &\le \tau c_0\|e_h\|_0\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}\|\nabla\times(\mathbf{B}-\mathbf{B}_h)\|_{0,\Omega} \\
&\le \frac{\nu}{16}\|e_h\|_0^2 + \frac{8}{\nu}\tau^2c_0^2\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}^2\|\nabla\times(\mathbf{B}-\mathbf{B}_h)\|_{0,\Omega}^2,
\end{aligned}$$

and we continue to obtain

$$\begin{aligned}
\tau|\mathrm{d}(\mathbf{u}-\mathbf{u}_h, \mathbf{B}-\mathbf{B}_h, A_{2h}^{-1}\varepsilon_h)| &\le \tau\sqrt{2}\|\mathbf{u}-\mathbf{u}_h\|_{L^6}\|\mathbf{B}-\mathbf{B}_h\|_{L^2}\|\nabla\times A_{2h}^{-1}\varepsilon_h\|_{L^3} \\
&\le \tau c_0h^{-\frac{1}{2}}\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}\|\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}\|\varepsilon_h\|_{-1} \\
&\le \frac{\tau\mu}{8}(\|\varepsilon_h\|_0^2 + \|R_{0h}\mathbf{B}-\mathbf{B}\|_{0,\Omega}^2) \\
&\quad +\frac{8}{\mu}\tau c_0^2h^{-1}\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}^2\|\varepsilon_h\|_{-1}^2,
\end{aligned}$$

$$\begin{aligned}\tau|\mathrm{d}(\mathbf{u}-\mathbf{u}_h,\mathbf{B},A_{2h}^{-1}\varepsilon_h)+\mathrm{d}(\mathbf{u},\mathbf{B}-\mathbf{B}_h,A_{2h}^{-1}\varepsilon_h)| &\le \tau c_0\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}\|\mathbf{B}\|_{2,\Omega}\|\varepsilon_h\|_{-1}\\ &\quad+\tau c_0\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}\|\mathbf{u}\|_{2,\Omega}\|\varepsilon_h\|_{-1}\\ &\le \frac{\tau\mu}{16}(\|\varepsilon_h\|_0^2+\|\mathbf{B}-R_{0h}\mathbf{B}\|_{0,\Omega}^2)+\frac{\nu}{16}(\|e_h\|_0^2+\|\mathbf{u}-P_h\mathbf{u}\|_{0,\Omega}^2)\\ &\quad+c_0^2\left(\frac{8}{\nu}\tau^2\|\mathbf{B}\|_{2,\Omega}^2+\frac{8}{\mu}\tau\|\mathbf{u}\|_{2,\Omega}^2\right)\|\varepsilon_h\|_{-1}^2,\\ &\quad\|R_{0h}\mathbf{B}-\mathbf{B}\|_{0,\Omega}\le ch^3\|\mathbf{B}\|_{3,\Omega}.\end{aligned}$$

Combining the estimates above with (4.8) and using (3.1)–(3.2) lead to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|e_h\|_{-1}^2+\tau\|\varepsilon_h\|_{-1}^2)+\nu\|e_h\|_0^2+\tau\mu\|\varepsilon_h\|_0^2\\ &\le \mathrm{d}(t)(\|e_h\|_{-1}^2+\tau\|\varepsilon_h\|_{-1}^2)+ch^6(\|\mathbf{u}\|_{3,\Omega}^2+\|p\|_{2,\Omega}+\|\mathbf{B}\|_{3,\Omega}^2)\\ &\quad+c(\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}^2+\|\nabla\times(\mathbf{B}-\mathbf{B}_h)\|_{0,\Omega}^2)(\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}^2+\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}^2),\end{aligned}\tag{4.9}$$

where we have written $\mathrm{d}(t)=c[\|\mathbf{u}\|_{2,\Omega}^2+\|\mathbf{B}\|_{2,\Omega}^2+h^{-1}\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}^2]$. Then integrating (4.9) and applying the Gronwall lemma, Lemmas 2.5 and 4.1, we obtain

$$\begin{aligned}&\|e_h(t)\|_{-1}^2+\tau\|\varepsilon_h(t)\|_{-1}^2+\int_0^t[\nu\|e_h\|_0^2+\tau\mu\|\varepsilon_h\|_0^2]\mathrm{d}s\\ &\le ce^{\int_0^t\mathrm{d}(s)\mathrm{d}s}\left\{h^6\int_0^t(\|\mathbf{u}\|_{3,\Omega}^2+\|p\|_{2,\Omega}^2+\|\mathbf{B}\|_{2,\Omega}^2+\|\nabla\times\mathbf{B}\|_{2,\Omega}^2)\mathrm{d}s\right.\\ &\quad\left.+c\int_0^t[\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}^2+\|\nabla\times(\mathbf{B}-\mathbf{B}_h)\|_{0,\Omega}^2)(\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}^2+\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}^2)]\mathrm{d}s\right\}\\ &\le\kappa h^6,\end{aligned}\tag{4.10}$$

Now the desired estimate (4.7) follows readily from (3.1)–(3.2), and Lemma 2.5. □

With the results in the previous two lemmas, we are now able to establish one of our major optimal error estimates in H^1-norm.

Theorem 4.3. *Under Assumptions 2.1–2.3 and 3.1, the finite element solution* $(\mathbf{u}_h,p_h,\mathbf{B}_h)$ *to the system (3.3)–(3.4) satisfies the following error estimate:*

$$\begin{aligned}&\sigma(t)[\|\mathbf{u}(t)-\mathbf{u}_h(t)\|_{0,\Omega}^2+\|\mathbf{B}(t)-\mathbf{B}_h(t)\|_{0,\Omega}^2]\\ &\quad+\int_0^t\sigma(s)h^2[\nu\|\mathbf{u}(t)-\mathbf{u}_h(t)\|_1^2+\tau\mu\|\mathbf{B}(t)-\mathbf{B}_h(t)\|_1^2]\mathrm{d}s\le\kappa h^6.\end{aligned}\tag{4.11}$$

Proof. Setting $e_h=R_h(\mathbf{u},p)-\mathbf{u}_h,\ \eta_h=Q_h(\mathbf{u},p)-p_h,\ \varepsilon_h=R_{2h}\mathbf{B}-\mathbf{B}_h$, we deduce from (1.3) to (1.4) and (3.3)–(3.4) that

$$\begin{aligned}&(\mathbf{u}_t-\mathbf{u}_{ht},\mathbf{v}_h)_\Omega+\nu(\nabla e_h,\nabla\mathbf{v}_h)_\Omega-(\nabla\cdot\mathbf{v}_h,\eta_h)_\Omega+(\nabla\cdot e_h,q_h)_\Omega\\ &\quad+b(\mathbf{u}-\mathbf{u}_h,\mathbf{u},\mathbf{v}_h)+b(\mathbf{u},\mathbf{u}-\mathbf{u}_h,\mathbf{v}_h)-b(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\mathbf{u}_h,\mathbf{v}_h)\\ &\quad+\mathrm{d}(\mathbf{v}_h,\mathbf{B}-\mathbf{B}_h,\mathbf{B})+\mathrm{d}(\mathbf{v}_h,\mathbf{B},\mathbf{B}-\mathbf{B}_h)\\ &\quad-\mathrm{d}(\mathbf{v}_h,\mathbf{B}-\mathbf{B}_h,\mathbf{B}-\mathbf{B}_h)=0,\end{aligned}\tag{4.12}$$

$$
\begin{aligned}
&(\mathbf{B}_t-\mathbf{B}_{ht},C_h)_\Omega+\mu(A_{2h}^{\frac12}\varepsilon_h,A_{2h}^{\frac12}\mathbf{C}_h)_\Omega-\mu(\mathbf{B}-\mathbf{B}_h,\mathbf{C}_h)_\Omega-\mathrm{d}(\mathbf{u}-\mathbf{u}_h,\mathbf{B},\mathbf{C}_h)\\
&-\mathrm{d}(\mathbf{u},\mathbf{B}-\mathbf{B}_h,C_h)+\mathrm{d}(\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h,\mathbf{C}_h)=0. \qquad (4.13)
\end{aligned}
$$

Then taking $(\mathbf{v}_h,q_h)=(e_h,\eta_h)$ and $\mathbf{C}_h=\tau\varepsilon_h$ in (4.12) and (4.13) respectively, and adding up the resultant equations, we readily see

$$
\begin{aligned}
&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|e_h\|_0^2+\nu\|e_h\|_1^2+\frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon_h\|_0^2+s\mu\|\varepsilon_h\|_1^2\\
&\quad+b(\mathbf{u}-\mathbf{u}_h,\mathbf{u},e_h)+b(\mathbf{u},\mathbf{u}-\mathbf{u}_h,e_h)-b(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\mathbf{u}_h,e_h)\\
&\quad+\tau\mathrm{d}(e_h,\mathbf{B}-\mathbf{B}_h,\mathbf{B})+\tau\mathrm{d}(e_h,\mathbf{B},\mathbf{B}-\mathbf{B}_h)-\tau\mathrm{d}(e_h,\mathbf{B}-\mathbf{B}_h,\mathbf{B}-\mathbf{B}_h)\\
&\quad-\tau\mathrm{d}(\mathbf{u}-\mathbf{u}_h,\mathbf{B},\varepsilon_h)-\tau\mathrm{d}(\mathbf{u},\mathbf{B}-\mathbf{B}_h,\varepsilon_h)+\tau\mathrm{d}(\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h,\varepsilon_h)\\
&\le\|\mathbf{u}_t-R_h(\mathbf{u}_t,p_t)\|_{0,\Omega}\|e_h\|_0+\tau\|\mathbf{B}_t-R_{2h}\mathbf{B}_t\|_{0,\Omega}\|\varepsilon_h\|_0. \qquad (4.14)
\end{aligned}
$$

But using the estimates (2.1)–(2.2), (3.5) and Assumption 3.1, we can derive

$$
\begin{aligned}
|b(\mathbf{u}-\mathbf{u}_h,\mathbf{u},e_h)|+|b(\mathbf{u},\mathbf{u}-\mathbf{u}_h,e_h)|&\le c_0\|e_h\|_1\|\mathbf{u}\|_{2,\Omega}\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}\\
&\le\frac{\nu}{16}\|e_h\|_1^2+\frac{4}{\nu}c_0^2\|\mathbf{u}\|_{2,\Omega}^2\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}^2,\\
|b(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\mathbf{u}_h,e_h)|&\le c_0h^{-1}\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}\|e_h\|_1\\
&\le\frac{\nu}{16}\|e_h\|_1^2+\frac{4}{\nu}c_0^2h^{-2}\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}^2\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}^2,\\
\tau|\mathrm{d}(e_h,\mathbf{B}-\mathbf{B}_h,\mathbf{B})+\mathrm{d}(e_h,\mathbf{B},\mathbf{B}-\mathbf{B}_h)|&\le\tau c_0\|e_h\|_1\|\mathbf{B}\|_{2,\Omega}\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}\\
&\le\frac{\nu}{16}\|e_h\|_1^2+\frac{4}{\nu}\tau^2c_0^2\|\mathbf{B}\|_{2,\Omega}^2\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}^2,\\
\tau|\mathrm{d}(e_h,\mathbf{B}-\mathbf{B}_h,\mathbf{B}-\mathbf{B}_h)|&\le\tau c_0h^{-1}\|e_h\|_1\|\nabla\times(\mathbf{B}-\mathbf{B}_h)\|_{0,\Omega}\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}\\
&\le\frac{\nu}{16}\|e_h\|_1^2+\frac{4}{\nu}\tau^2c_0^2h^{-2}\|\nabla\times(\mathbf{B}-\mathbf{B}_h)\|_{0,\Omega}^2\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}^2,\\
\tau|\mathrm{d}(\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h,\varepsilon_h)|&\le\tau c_0h^{-1}\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}\|\varepsilon_h\|_1\\
&\le\frac{\tau\mu}{8}\|\varepsilon_h\|_1^2+\frac{4}{\mu}\tau c_0^2h^{-2}\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}^2\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
\tau|\mathrm{d}(\mathbf{u}-\mathbf{u}_h,\mathbf{B},\varepsilon_h)+\mathrm{d}(\mathbf{u},\mathbf{B}-\mathbf{B}_h,\varepsilon_h)|&\le\tau\,c_0(\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}\|\mathbf{B}\|_{2,\Omega}+\|\mathbf{u}\|_{2,\Omega}\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega})\|\varepsilon_h\|_1\\
&\le\frac{\tau\mu}{8}\|\varepsilon_h\|_1^2+\frac{4}{\mu}\tau c_0^2(\|\mathbf{B}\|_{2,\Omega}^2\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}^2+\|\mathbf{u}\|_{2,\Omega}^2\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}^2).
\end{aligned}
$$

Applying these estimates to (4.14) yields

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|e_h\|_0^2+\tau\|\varepsilon_h\|_0^2)+\nu\|e_h\|_1^2+\tau\mu\|\varepsilon_h\|_1^2\le c(\|\mathbf{u}\|_{2,\Omega}^2+\|\mathbf{B}\|_{2,\Omega}^2)(\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}^2+\tau\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}^2)\\
&\quad+ch^{-2}(\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}^2+\|\nabla\times(\mathbf{B}-\mathbf{B}_h)\|_{0,\Omega}^2)(\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}^2+\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}^2)\\
&\quad+c(\|\mathbf{u}_t-R_h(\mathbf{u}_t,p_t)\|_{0,\Omega}^2+\|\mathbf{B}_t-R_{2h}(\mathbf{B}_t,\tilde{p}_t)\|_{0,\Omega}^2)^{\frac12}(\|e_h\|_0^2+\tau\|\varepsilon_h\|_0^2)^{\frac12}. \qquad (4.15)
\end{aligned}
$$

Now multiplying both sides of the above inequality by $\sigma(t)$, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}[\sigma(t)(\|e_h\|_0^2+\tau\|\varepsilon_h\|_0^2)]+\nu\sigma(t)\|e_h\|_1^2+\tau\mu\sigma(t)\|\varepsilon_h\|_1^2\\&\le c\sigma(t)(\|\mathbf{u}\|_{2,\Omega}^2+\|\mathbf{B}\|_{2,\Omega}^2)(\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}^2+\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}^2)\\&\quad+c(\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}^2+\tau\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}^2+\|\mathbf{u}-R_h(\mathbf{u},p)\|_{0,\Omega}^2+\|\mathbf{B}-R_{2h}\mathbf{B}\|_{0,\Omega}^2)\\&\quad+ch^{-2}(\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}^2+\|\nabla\times(\mathbf{B}-\mathbf{B}_h)\|_{0,\Omega}^2)(\|\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}^2+\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}^2)\\&\quad+c\sigma^2(t)(\|\mathbf{u}_t-R_h(\mathbf{u}_t,p_t)\|_{0,\Omega}^2+\|\mathbf{B}_t-R_{1h}\mathbf{B}_t\|_{0,\Omega}^2).\end{aligned}\tag{4.16}$$

Then we may see immediately the desired error estimate (4.11) by integrating both sides of the above inequality from 0 to t, using Assumption 3.1 and Lemmas 2.5–2.6, 3.2 and 4.1–4.2. □

5. (H^1-L^2)-NORM ERROR ESTIMATES OF THE APPROXIMATE VELOCITY AND PRESSURE

With the error estimates established in the previous section for the L^2-velocity error $\mathbf{u}-\mathbf{u}_h$ and the L^2-magnetic error $\mathbf{B}-\mathbf{B}_h$, it remains for us to build up the H^1- error of the approximate velocity u_h and the L^2- error of the approximate pressure p_h from the finite element system (3.3)–(3.4). For this purpose, we first establish two auxiliary error estimates in two lemmas.

Lemma 5.1. *Under Assumptions 2.1–2.3 and 3.1, the finite element solution $(\mathbf{u}_h,p_h,\mathbf{B}_h)$ to the system (3.3)–(3.4) satisfies the following error estimate:*

$$\begin{aligned}&\sigma(t)[\nu\|\nabla(R_h(\mathbf{u}(t),p(t))-\mathbf{u}_h(t))\|_{0,\Omega}^2+\tau\mu\|\nabla(R_{2h}\mathbf{B}-\mathbf{B}_h)\|_{0,\Omega}]\\&\quad+\int_0^t\sigma(s)(\|u_t-\mathbf{u}_{ht}\|_{0,\Omega}^2+\tau\|\mathbf{B}_t-\mathbf{B}_{ht}\|_{0,\Omega}^2)\mathrm{d}s\le\kappa h^4.\end{aligned}\tag{5.1}$$

Proof. For convenience, we set $e_h=R_h(\mathbf{u},p)-\mathbf{u}_h,\ \eta_h=Q_h(\mathbf{u},p)-p_h,\ \varepsilon_h=R_{2h}\mathbf{B}-\mathbf{B}_h$. By taking $(\mathbf{v}_h,q_h)=(e_{ht},\eta_{ht})$ in (4.12) and $\mathbf{C}_h=\tau\varepsilon_{ht}$ in (4.13), then adding up the resultant equations, we obtain

$$\begin{aligned}&\|e_{ht}\|_0^2+\tau\|\varepsilon_{ht}\|_0^2+\frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_h\|_1^2+\frac{\tau\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon_h\|_1^2\\&\quad+b(\mathbf{u}-\mathbf{u}_h,\mathbf{u},e_{ht})+b(\mathbf{u},\mathbf{u}-\mathbf{u}_h,e_{ht})-b(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\mathbf{u}_h,e_{ht})\\&\quad+\tau\mathrm{d}(e_{ht},\mathbf{B}-\mathbf{B}_h,\mathbf{B})+\tau\mathrm{d}(e_{ht},\mathbf{B},\mathbf{B}-\mathbf{B}_h)-\tau\mathrm{d}(e_{ht},\mathbf{B}-\mathbf{B}_h,\mathbf{B}-\mathbf{B}_h)\\&\quad-\tau\mathrm{d}(\mathbf{u}-\mathbf{u}_h,\mathbf{B},\varepsilon_{ht})-\tau\mathrm{d}(\mathbf{u},\mathbf{B}-\mathbf{B}_h,\varepsilon_h)+\tau\mathrm{d}(\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h,\varepsilon_{ht})\\&\le\|\mathbf{u}_t-R_h(\mathbf{u}_t,p_t)\|_{0,\Omega}\|e_{ht}\|_0+\tau\|\mathbf{B}_t-R_{2h}\mathbf{B}_t\|_{0,\Omega}\|\varepsilon_{ht}\|_0.\end{aligned}\tag{5.2}$$

But by making use of the estimates (2.1)–(2.2), (3.5) and Assumption 3.1, we can achieve the following bounds

$$\begin{aligned}|b(\mathbf{u}-\mathbf{u}_h,\mathbf{u},e_{ht})|+|b(\mathbf{u},\mathbf{u}-\mathbf{u}_h,e_{ht})|&\le c_0h^{-1}\|e_{ht}\|_0\|\mathbf{u}\|_{2,\Omega}\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}\\&\le\frac{1}{16}\|e_{ht}\|_0^2+4c_0^2h^{-2}\|\mathbf{u}\|_{2,\Omega}^2\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}^2,\\|b(\mathbf{u}-\mathbf{u}_h,\mathbf{u}-\mathbf{u}_h,e_{ht})|&\le 2\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}\|e_{ht}\|_{L^\infty}\\&\le c_0h^{-2}\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}\|e_{ht}\|_0\\&\le\frac{1}{16}\|e_{ht}\|_0^2+4c_0^2h^{-4}\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}^2\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}^2,\end{aligned}$$

$$\begin{aligned}
\tau|\mathrm{d}(e_{ht},\mathbf{B}-\mathbf{B}_h,\mathbf{B})+\mathrm{d}(e_{ht},\mathbf{B},\mathbf{B}-\mathbf{B}_h)| &\le \tau c_0 h^{-1}\|e_{ht}\|_0\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}\|\mathbf{B}\|_{2,\Omega}\\
&\le \frac{1}{16}\|e_{ht}\|_0^2+4\tau^2c_0^2h^{-2}\|\mathbf{B}\|_{2,\Omega}^2\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}^2,\\
\tau|\mathrm{d}(e_{ht},\mathbf{B}-\mathbf{B}_h,\mathbf{B}-\mathbf{B}_h)| &\le c_0h^{-2}\|e_{ht}\|_0\|\nabla\times(\mathbf{B}-\mathbf{B}_h)\|_{0,\Omega}\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}\\
&\le \frac{1}{16}\|e_{ht}\|_0^2+4\tau^2c_0^2h^{-4}\|\nabla\times(\mathbf{B}-\mathbf{B}_h)\|_{0,\Omega}^2\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}^2,\\
\tau|\mathrm{d}(\mathbf{u}-\mathbf{u}_h,\mathbf{B},\varepsilon_{ht})+\mathrm{d}(\mathbf{u},\mathbf{B}-\mathbf{B}_h,\varepsilon_{ht})| &\le \tau c_0h^{-1}(\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}\|\mathbf{B}\|_{2,\Omega}+\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}\|\mathbf{u}\|_{2,\Omega})\|\varepsilon_{ht}\|_0\\
&\le \frac{\tau}{8}\|\varepsilon_{ht}\|_0^2+4\tau c_0^2h^{-2}(\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}^2\|\mathbf{B}\|_{2,\Omega}^2+\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}^2\|\mathbf{u}\|_{2,\Omega}^2),\\
\tau|\mathrm{d}(\mathbf{u}-\mathbf{u}_h,\mathbf{B}-\mathbf{B}_h,\varepsilon_{ht})| &\le \tau c_0h^{-2}\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}\|\varepsilon_{ht}\|_0\\
&\le \frac{s}{8}\|\varepsilon_{ht}\|_0^2+4\tau c_0^2h^{-4}\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}^2\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}^2.
\end{aligned}$$

Applying these bounds and Young's inequality to (5.2), and noting equation (4.13), we derive

$$\begin{aligned}
&\|e_{ht}\|_0^2+\nu\frac{\mathrm{d}}{\mathrm{d}t}\|e_h\|_1^2+\tau\|\varepsilon_{ht}\|_0^2+\tau\mu\frac{\mathrm{d}}{\mathrm{d}t}\|\varepsilon_h\|_1^2\\
&\le 8\|\mathbf{u}_t-R_h(\mathbf{u}_t,p_t)\|_{0,\Omega}^2+8\tau\|\mathbf{B}_t-R_{2h}\mathbf{B}_t\|_{0,\Omega}^2\\
&\quad +ch^{-2}(\|\mathbf{u}\|_{2,\Omega}^2+\|\mathbf{B}\|_{2,\Omega}^2)(\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}^2+\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}^2)\\
&\quad +ch^{-4}(\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}+\|\nabla\times(\mathbf{B}-\mathbf{B}_h)\|_{0,\Omega}^2)(\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}^2+\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}^2),
\end{aligned}$$

then multiplying both sides of the above inequality by $\sigma(t)$ and using Lemma 3.2,

$$\begin{aligned}
&\sigma(t)(\|e_{ht}\|_0^2+\tau\|\varepsilon_{ht}\|_0^2)+\frac{\mathrm{d}}{\mathrm{d}t}[\sigma(t)(\mu\|e_h\|_1^2+\tau\mu\|\varepsilon_h\|_1^2)]\\
&\le (\mu\|e_h\|_1^2+\tau\mu\|\varepsilon_h\|_1^2)\\
&\quad +c\sigma(t)h^4[\|\mathbf{u}_t\|_{2,\Omega}^2+\|p_t\|_{1,\Omega}^2+\|\mathbf{B}_t\|_{1,\Omega}^2+\|\nabla\times\mathbf{B}_t\|_{1,\Omega}^2]\\
&\quad +c\sigma(t)h^{-2}(\|\mathbf{u}\|_{2,\Omega}^2+\|\mathbf{B}\|_{2,\Omega}^2)(\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}^2+\|\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}^2)\\
&\quad +ch^{-4}(\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}+\|\nabla\times(\mathbf{B}-\mathbf{B}_h)\|_{0,\Omega}^2)\\
&\quad \times(\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}^2+\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}^2). \qquad (5.3)
\end{aligned}$$

Now the desired estimate (5.1) follows readily by integrating both sides of the inequality above from 0 to t and using Lemmas 2.6, 3.2, 4.1 and 4.2. □

Lemma 5.2. *Under Assumptions 2.1–2.3 and 3.1, the approximate velocity $\mathbf{u}_h$ in the system (3.3)–(3.4) satisfies the following error estimate:*

$$\sigma^2(t)\|\mathbf{u}_t(t)-u_{ht}(t)\|_{0,\Omega}^2\le\kappa\,h^4. \qquad (5.4)$$

Proof. We set $e_h = P_h\mathbf{u} - \mathbf{u}_h,\ \eta_h = \rho_h p - p_h$, then differentiate (4.2) with respect to t to obtain

$$\begin{aligned}
&(e_{htt}, \mathbf{v}_h)_\Omega + \nu(\nabla(\mathbf{u}_t - \mathbf{u}_{ht}), \nabla \mathbf{v}_h)_\Omega - (\nabla\cdot\mathbf{v}_h, p_t - p_{ht})_\Omega + (\nabla\cdot e_{ht}, q_h)_\Omega \\
&\quad + b(\mathbf{u}_t - \mathbf{u}_{ht}, \mathbf{u}, \mathbf{v}_h) + b(\mathbf{u} - \mathbf{u}_h, \mathbf{u}_t, e_h) + b(\mathbf{u}_t, \mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{u}, \mathbf{u}_t - \mathbf{u}_{ht}, \mathbf{v}_h) \\
&\quad - b(\mathbf{u}_t - \mathbf{u}_{ht}, \mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) - b(\mathbf{u} - \mathbf{u}_h, \mathbf{u}_t - \mathbf{u}_{ht}, \mathbf{v}_h) \\
&\quad + \tau \mathrm{d}(\mathbf{v}_h, \mathbf{B}_t - \mathbf{B}_{ht}, \mathbf{B}) + \tau \mathrm{d}(\mathbf{v}_h, \mathbf{B} - \mathbf{B}_h, \mathbf{B}_t) \\
&\quad + \tau \mathrm{d}(\mathbf{v}_h, \mathbf{B}_t, \mathbf{B} - \mathbf{B}_h) + \tau \mathrm{d}(\mathbf{v}_h, \mathbf{B}, \mathbf{B}_t - \mathbf{B}_{ht}) \\
&\quad - \tau \mathrm{d}(\mathbf{v}_h, \mathbf{B}_t - \mathbf{B}_{ht}, \mathbf{B} - \mathbf{B}_h) - \tau \mathrm{d}(\mathbf{v}_h, \mathbf{B} - \mathbf{B}_h, \mathbf{B}_t - \mathbf{B}_{ht}) = 0
\end{aligned} \tag{5.5}$$

for all $(\mathbf{v}_h, q_h) \in \mathbf{X}_h \times M_h$. Taking $\mathbf{v}_h = e_{ht}$ and $q_h = \eta_{ht}$, and using (2.1), we can write

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_{ht}\|_{0,\Omega}^2 + \frac{\nu}{2}(\|\nabla(u_t - u_{ht})\|_{0,\Omega}^2 + \|e_{ht}\|_1^2) \\
&\quad + b(\mathbf{u}_t - \mathbf{u}_{ht}, \mathbf{u}, e_{ht}) + b(\mathbf{u} - \mathbf{u}_h, \mathbf{u}_t, e_{ht}) \\
&\quad + b(\mathbf{u}_t, \mathbf{u} - \mathbf{u}_h, e_{ht}) + b(\mathbf{u}, \mathbf{u}_t - u_{ht}, e_{ht}) \\
&\quad - b(\mathbf{u}_t - \mathbf{u}_{ht}, \mathbf{u} - \mathbf{u}_h, e_{ht}) - b(\mathbf{u} - \mathbf{u}_h, \mathbf{u}_t - u_{ht}, e_{ht}) \\
&\quad + \tau \mathrm{d}(e_{ht}, \mathbf{B}_t - \mathbf{B}_{ht}, \mathbf{B}) + \tau \mathrm{d}(e_{ht}, \mathbf{B} - \mathbf{B}_h, \mathbf{B}_t) \\
&\quad + \tau \mathrm{d}(e_{ht}, \mathbf{B}_t, \mathbf{B} - \mathbf{B}_h) + \tau \mathrm{d}(e_{ht}, \mathbf{B}, \mathbf{B}_t - \mathbf{B}_{ht}) \\
&\quad - \tau \mathrm{d}(e_{ht}, \mathbf{B}_t - \mathbf{B}_{ht}, \mathbf{B} - \mathbf{B}_h) - \tau \mathrm{d}(e_{ht}, \mathbf{B} - \mathbf{B}_h, \mathbf{B}_t - \mathbf{B}_{ht}) \\
&= \frac{\nu}{2}\|\nabla(\mathbf{u}_t - P_h\mathbf{u}_t)\|_{0,\Omega}^2 + (\nabla\cdot e_{ht}, p_t - \rho_h p_t)_\Omega\,.
\end{aligned} \tag{5.6}$$

Now using the estimates (2.1), (3.1)–(3.2) and Lemma 3.2, we can deduce

$$\begin{aligned}
|(\nabla\cdot e_{ht}, p_t - \rho_h p_t)_\Omega| &\le \frac{\nu}{16}\|e_{ht}\|_1^2 + \frac{16}{\nu}\|p_t - \rho_h p_t\|_{0,\Omega}^2, \\
|b(\mathbf{u} - \mathbf{u}_h, \mathbf{u}_t, e_{ht})| + |b(\mathbf{u}_t, \mathbf{u} - \mathbf{u}_h, e_{ht})| &\le c_0\|e_{ht}\|_1\|\nabla\mathbf{u}_t\|_{0,\Omega}\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{0,\Omega} \\
&\le \frac{\nu}{16}\|e_{ht}\|_1^2 + \frac{4}{\nu}c_0^2\|\nabla\mathbf{u}_t\|_{0,\Omega}^2\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{0,\Omega}^2, \\
|b(\mathbf{u}_t - \mathbf{u}_{ht}, \mathbf{u}, e_{ht})| + |b(\mathbf{u}, \mathbf{u}_t - \mathbf{u}_{ht}, e_{ht})| &\le c_0\|e_{ht}\|_1\|\mathbf{u}\|_{2,\Omega}\|\mathbf{u}_t - \mathbf{u}_{ht}\|_{0,\Omega} \\
&\le \frac{\nu}{16}\|e_{ht}\|_1^2 + \frac{4}{\nu}c_0^2\|\mathbf{u}\|_{2,\Omega}^2\|\mathbf{u}_t - \mathbf{u}_{ht}\|_{0,\Omega}^2, \\
|b(\mathbf{u}_t - \mathbf{u}_{ht}, \mathbf{u} - \mathbf{u}_h, e_{ht})| + |b(\mathbf{u} - \mathbf{u}_h, \mathbf{u}_t - \mathbf{u}_{ht}, e_{ht})| &\le c_0 h^{-1}\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{0,\Omega}\|\mathbf{u}_t - \mathbf{u}_{ht}\|_{0,\Omega}\|e_{ht}\|_1 \\
&\le \frac{\nu}{16}\|e_{ht}\|_1^2 + \frac{4}{\nu}c_0^2 h^{-2}\|\nabla(\mathbf{u} - \mathbf{u}_h)\|_{0,\Omega}^2\|\mathbf{u}_t - \mathbf{u}_{ht}\|_{0,\Omega}^2,
\end{aligned}$$

and we continue to derive

$$\begin{aligned}
\tau|\mathrm{d}(e_{ht}, \mathbf{B} - \mathbf{B}_h, \mathbf{B}_t) + \mathrm{d}(e_{ht}, \mathbf{B}_t, \mathbf{B} - \mathbf{B}_h)| &\le \tau c_0 h^{-1}\|e_{ht}\|_1\|\mathbf{B} - \mathbf{B}_h\|_{0,\Omega}\|\mathbf{B}_t\|_{1,\Omega} \\
&\le \frac{\nu}{16}\|e_{ht}\|_1^2 + \frac{8}{\nu}\tau^2 c_0^2 h^{-2}\|\mathbf{B}_t\|_{1,\Omega}^2\|\mathbf{B} - \mathbf{B}_h\|_{0,\Omega}^2, \\
\tau|\mathrm{d}(e_{ht}, \mathbf{B}_t - \mathbf{B}_{ht}, \mathbf{B}) + \mathrm{d}(e_{ht}, \mathbf{B}, \mathbf{B}_t - \mathbf{B}_{ht})| &\le \tau c_0\|e_{ht}\|_1\|\mathbf{B}_t - \mathbf{B}_{ht}\|_{0,\Omega}\|\mathbf{B}\|_{2,\Omega} \\
&\le \frac{\nu}{16}\|e_{ht}\|_1^2 + \frac{8}{\nu}\tau^2 c_0^2\|\mathbf{B}\|_{2,\Omega}^2\|\mathbf{B}_t - \mathbf{B}_{ht}\|_{0,\Omega}^2, \\
\tau|\mathrm{d}(e_{ht}, \mathbf{B}_t - \mathbf{B}_{ht}, \mathbf{B} - \mathbf{B}_h)| &\le \tau c_0 h^{-1}\|\mathbf{B}_t - \mathbf{B}_{ht}\|_{0,\Omega}\|\nabla\times(\mathbf{B} - \mathbf{B}_h)\|_{0,\Omega}\|e_{ht}\|_1 \\
&\le \frac{\nu}{16}\|e_{ht}\|_1^2 + \frac{4}{\nu}\tau^2 c_0^2 h^{-2}\|\mathbf{B}_t - \mathbf{B}_{ht}\|_{0,\Omega}^2\|\nabla\times(\mathbf{B} - \mathbf{B}_h)\|_{0,\Omega}^2,
\end{aligned}$$

$$
\begin{aligned}
\tau|\mathrm{d}(e_{ht},\mathbf{B}-\mathbf{B}_h,\mathbf{B}_t-\mathbf{B}_{ht})| &\le \tau\sqrt{2}\|e_{ht}\|_{L^\infty}\|\mathbf{B}-\mathbf{B}_h\|_{L^2}\\
&\quad\times(\|\nabla\times(\mathbf{B}_t-R_{2h}\mathbf{B}_t)\|_{0,\Omega}+ch^{-1}\|R_{2h}\mathbf{B}_t-\mathbf{B}_{ht}\|_{0,\Omega})\\
&\le \tau c_0\|e_{ht}\|_1(\|\mathbf{B}_t\|_{1,\Omega}+\|\nabla\times\mathbf{B}_t\|_{1,\Omega})\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}\\
&\quad+\tau c_0 h^{-2}\|e_{ht}\|_1\|\mathbf{B}_t-\mathbf{B}_{ht}\|_{0,\Omega}\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}\\
&\le \frac{\nu}{8}\|e_{ht}\|_1^2+\frac{8}{\nu}\tau^2c_0^2(\|\mathbf{B}_t\|_{1,\Omega}^2+\|\nabla\times\mathbf{B}_t\|_{1,\Omega}^2)\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}^2\\
&\quad+\frac{4}{\nu}\tau^2c_0^2h^{-4}\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}^2\|\mathbf{B}_t-\mathbf{B}_{ht}\|_{0,\Omega}^2.
\end{aligned}
$$

Applying these estimates to (5.6) and using Assumption 3.1 yield

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|e_{ht}\|_0^2 &\le \nu\|\nabla(\mathbf{u}_t-P_h\mathbf{u}_t)\|_{0,\Omega}^2+\frac{16}{\nu}\|p_t-\rho_hp_t\|_{0,\Omega}^2\\
&\quad+c(\|\mathbf{u}\|_{2,\Omega}^2+\|\mathbf{B}\|_{2,\Omega}^2)(\|\mathbf{u}_t-\mathbf{u}_{ht}\|_{0,\Omega}^2+\|\mathbf{B}_t-\mathbf{B}_{ht}\|_{0,\Omega}^2)\\
&\quad+ch^{-2}(\|\mathbf{u}_t\|_{1,\Omega}^2+\|\mathbf{B}_t\|_{1,\Omega}^2)(\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}^2+\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}^2)\\
&\quad+c(\|\mathbf{B}_t\|_{1,\Omega}^2+\|\nabla\times\mathbf{B}_t\|_{1,\Omega}^2)\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}^2\\
&\quad+c(\|\mathbf{u}\|_{2,\Omega}^2+\|\mathbf{B}\|_{2,\Omega}^2+h^{-2}\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}^2\\
&\quad+h^{-2}\|\nabla\times(\mathbf{B}-\mathbf{B}_h)\|_{0,\Omega}^2+h^{-4}\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}^2)\\
&\quad\times(\|\mathbf{u}_t-\mathbf{u}_{ht}\|_{0,\Omega}^2+\|\mathbf{B}_t-\mathbf{B}_{ht}\|_{0,\Omega}^2). \qquad (5.7)
\end{aligned}
$$

Now the estimate (5.4) follows by multiplying both sides of the inequality (5.7) by $\sigma^2(t)$, then integrating from 0 to t and using Assumption 3.1, Lemmas 2.6–2.7, 3.2 and Theorem 4.3. □

We are now ready to demonstrate our last error estimate. By applying Assumption 3.1, (2.1) and (4.2), we find that $\eta_h=\rho_hp-p_h$ satisfies

$$
\begin{aligned}
\|\eta_h\|_{0,\Omega} &\le \beta^{-1}\sup_{\mathbf{v}_h\in\mathbf{X}_h}\frac{(\nabla\cdot\mathbf{v}_h,\eta_h)_\Omega}{\|\nabla\mathbf{v}_h\|_{0,\Omega}}\\
&\le c\|\mathbf{u}_t-\mathbf{u}_{ht}\|_{0,\Omega}+c\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}+c\|\nabla\mathbf{u}\|_{0,\Omega}\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}\\
&\quad+c\|\mathbf{B}\|_{2,\Omega}\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}+c\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}^2\\
&\quad+ch^{-1}\|\mathbf{B}-\mathbf{B}_h\|_{0,\Omega}\|\nabla\times(\mathbf{B}-\mathbf{B}_h)\|_{0,\Omega}\,.
\end{aligned}
$$

Using again Theorem 4.3, Lemmas 4.1, 5.1 and 5.2, we easily see the following estimate:

$$
\begin{aligned}
\sigma(t)\|\eta_h\|_{0,\Omega} &\le \kappa(\sigma(t)\|u_t-\mathbf{u}_{ht}\|_{0,\Omega}+c\sigma^{\frac{1}{2}}(t)\|\nabla(\mathbf{u}(t)-\mathbf{u}_h(t))\|_{0,\Omega}\\
&\quad+c\sigma(t)\|\mathbf{B}(t)\|_{2,\Omega}\|\mathbf{B}(t)-\mathbf{B}_h(t)\|_{0,\Omega}+c\sigma(t)\|\nabla(\mathbf{u}(t)-\mathbf{u}_h(t))\|_{0,\Omega}^2\\
&\quad+c\sigma(t)h^{-1}\|\mathbf{B}(t)-\mathbf{B}_h(t)\|_{0,\Omega}\|\nabla\times(\mathbf{B}(t)-\mathbf{B}_h(t))\|_{0,\Omega}\\
&\le \kappa h^2. \qquad (5.8)
\end{aligned}
$$

Then combining this estimate with Theorem 4.3, we come to the following convergence result.

Theorem 5.3. *Under Assumptions 2.1–2.3 and 3.1, the approximate velocity, pressure and magnetic field* $\mathbf{u}_h$, p_h *and* $\mathbf{B}_h$ *to the system (3.3)–(3.4) satisfies the following error estimate for all* $t\in(0,T]$*:*

$$
\begin{aligned}
&\sigma^{\frac{1}{2}}(t)[\|\mathbf{u}(t)-\mathbf{u}_h(t)\|_{0,\Omega}+\sqrt{\tau}\|\mathbf{B}(t)-\mathbf{B}_h(t)\|_{0,\Omega}]+h\sigma(t)[\|p(t)-p_h(t)\|_{0,\Omega}\\
&\quad+\sigma^{\frac{1}{2}}(t)h[\sqrt{\nu}\|\nabla(\mathbf{u}(t)-\mathbf{u}_h(t))\|_{0,\Omega}+\sqrt{\tau\mu}\|\nabla(\mathbf{B}(t)-\mathbf{B}_h(t))\|_{0,\Omega}]\le\kappa h^3\,.
\end{aligned}
$$

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