ESAIM: M2AN 52 (2018) 29–67
https://doi.org/10.1051/m2an/2017030

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

ON THE HOMOGENIZATION OF THE HELMHOLTZ PROBLEM WITH THIN PERFORATED WALLS OF FINITE LENGTH *

Adrien Semin[1], Bérangère Delourme[2] and Kersten Schmidt[3]

Abstract. In this work, we present a new solution representation for the Helmholtz transmission problem in a bounded domain in $\mathbb{R}^2$ with a thin and periodic layer of finite length. The layer may consists of a periodic pertubation of the material coefficients or it is a wall modelled by boundary conditions with an periodic array of small perforations. We consider the periodicity in the layer as the small variable δ and the thickness of the layer to be at the same order. Moreover we assume the thin layer to terminate at re-entrant corners leading to a singular behaviour in the asymptotic expansion of the solution representation. This singular behaviour becomes visible in the asymptotic expansion in powers of δ where the powers depend on the opening angle. We construct the asymptotic expansion order by order. It consists of a macroscopic representation away from the layer, a boundary layer corrector in the vicinity of the layer, and a near field corrector in the vicinity of the end-points. The boundary layer correctors and the near field correctors are obtained by the solution of canonical problems based, respectively, on the method of periodic surface homogenization and on the method of matched asymptotic expansions. This will lead to transmission conditions for the macroscopic part of the solution on an infinitely thin interface and corner conditions to fix the unbounded singular behaviour at its end-points. Finally, theoretical justifications of the second order expansion are given and illustrated by numerical experiments. The solution representation introduced in this article can be used to compute a highly accurate approximation of the solution with a computational effort independent of the small periodicity δ.

Mathematics Subject Classification. 32S05, 35C20, 35J05, 35J20, 41A60, 65D15.

Received August 17, 2016. Revised March 27, 2017. Accepted June 14, 2017.

1. Introduction

The present work is dedicated to the iterative construction of a second order asymptotic expansion of the solution to an Helmholtz problem posed in a non-convex polygonal domain which excludes a set of similar small obstacles equi-spaced along the line between two re-entrant corners. The distance between two consecutive

Keywords and phrases. Helmholtz equation, thin periodic interface, method of matched asymptotic expansions, method of periodic surface homogenization, corner singularities.

* *Part of this work was carried out where Bérangère Delourme was on research leave at Laboratoire POEMS, INRIA-Saclay, ENSTA, UMR CNRS 2706, France. This work was carried out where Adrien Semin was at Research center Matheon, Institut für Mathematik, Technische Universität Berlin, 10623 Berlin, Germany.*

[1] Brandenburgische Technische Universität Cottbus-Senftenberg, Institut für Mathematik, 03046 Cottbus, Germany.
adrien.semin@math.tu-berlin.de

[2] Université Paris 13, Sorbonne Paris Cité, LAGA, UMR 7539, 93430 Villetaneuse, France.

[3] Research center Matheon, Institut für Mathematik, Technische Universität Berlin, 10623 Berlin, Germany.

obstacles, which appear to be holes in the domain, and the diameter of the obstacles are of the same order of magnitude δ, which is supposed to be small compared to the dimensions of the domain. The presence of this thin periodic layer of holes is responsible for the appearance of two different kinds of singular behaviors. First, a highly oscillatory boundary layer appears in the vicinity of the periodic layer. Strongly localized, it decays exponentially fast as the distance to the periodic layer increases. Additionally, since the thin periodic layer has a finite length and ends in corners of the boundary, corners singularities come up in the neighborhood of its extremities. The objective of this work is to provide a practical asymptotic expansion that takes into account these two types of singular behaviors.

The boundary layer effect occurring in the vicinity of the periodic layer is well-known. It can be described using a two-scale asymptotic expansion (inspired by the periodic homogenization theory) that superposes slowly varying macroscopic terms and periodic correctors that have a two-scale behavior: these functions are the combination of highly oscillatory and decaying functions (periodic of period δ with respect to the tangential direction of the periodic interface and exponentially decaying with respect to d/δ, d denoting the distance to the periodic interface) multiplied by slowly varying functions. This boundary layer effect has been widely investigated since the work of Panasenko [34], Sanchez−Palencia [39,40], Achdou [3,4] and Artola–Cessenat [6,7]. In particular, high order asymptotics have been derived for the Laplace equation [5, 11, 14, 29] and for the Helmholtz equation [36, 37].

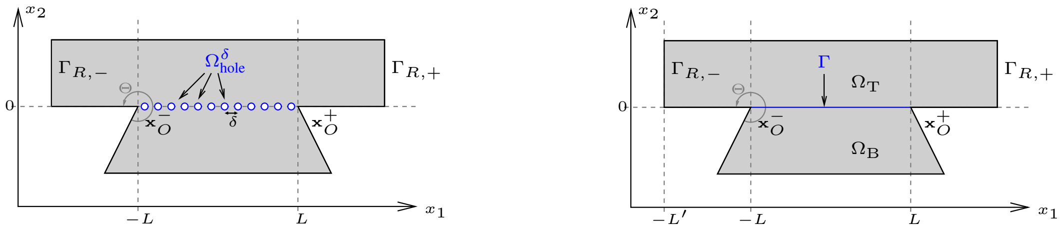
On the other hand, corner singularities appearing when dealing with singularly perturbed boundaries have also been widely investigated. Among the numerous examples of such singularly perturbed problems, we can mention the cases of small inclusions (see Chap. 2 of [31] for the case of one inclusion and [9] for the case of several inclusions), perturbed corners [18], propagation of waves in thin slots [25,26], propagation of waves across a thin interface[16], diffraction by wires [15], diffraction by a muffler containing perforated ducts [10], or the mathematical investigation of patched antennas [8]. Again, this effect can be depicted using two-scale asymptotic expansion methods that are the method of multi-scale expansion (sometimes called compound method) and the method of matched asymptotic expansions [24, 31, 43]. Following these methods, the solution of the perturbed problem may be seen as the superposition of slowly varying macroscopic terms that do not see directly the perturbation and microscopic terms that take into account the local perturbation.

Recently, the authors investigated a Poisson problem in a polygonal domain which excludes a set of similar small obstacles equi-spaced along the line between two re-entrant corners [20, 21]. In their study, they have combined the two different kinds of asymptotic expansions mentioned above in order to deal with both corner singularities and the boundary layer effect. Based on the matched asymptotic expansions, the authors constructed and justified a complete asymptotic expansion. This asymptotic expansion relies on the analysis of the behaviour of the solutions of the Poisson problem in an infinite cone with oscillating boundary with Dirichlet boundary conditions by Nazarov [32]. In the present paper, we are going to extend this work for the Helmholtz equation by constructing explicitly and rigorously the terms of the expansions up to order 2 (with Neumann boundary conditions on the perforations of the layer).

The remainder of the paper is organized as follows. In Section 2 we are going to define the problem, show the main ingredients of the asymptotic expansion following the method of matched asymptotic expansions, and give the main results. The asymptotic expansion of the solution away from the corners is given in Section 3, whereas the problem for the terms of the near field expansion and their behavior towards infinity, is analyzed in Section 4. The terms of this expansion takes into account the boundary layer effect due to the thin layer with small perforations and satisfy transmission conditions. Then, the matching of the far field and near field expansions and the iterative construction of the terms of the asymptotic expansions are conducted in Section 5. Finally, in Section 6 the asymptotic expansion is justified with an error analysis.

2. Description of the problem and main results

In this section, we first define the problem under consideration (Sect. 2.1). Then, we give the Ansatz of the asymptotic expansion (Sect. 2.2). Finally, we give the main result of this paper, which states the existences



(A) The domain ofi nterest Ω^{δ}. (B) The domain $\Omega = \Omega_{\text{T}} \cup \Omega_{\text{B}} \cup \Gamma$.

FIGURE 1. Illustration of the polygonal domain $\mathit{\Omega}$ and the domain of interest $\mathit{\Omega}^{\delta}$.

of the terms of the asymptotic expansion and the convergence of the truncated series toward the exact solution and we show a numerical illustration of the result (Sect. 2.3).

2.1. Description of the problem

2.1.1. Definition of the domain $\mathit{\Omega}^{\delta}$ with a thin perforated wall of finite length

Our domain of interest $\mathit{\Omega}^{\delta}$ consists of a (non-convex) polygon $\mathit{\Omega}$ intersected with the complement of an array of 'small' similar obstacles, see Figure 1a. The polygon $\mathit{\Omega}$, represented on Figure 1b, is the union of the rectangular domain $\mathit{\Omega}_{\text{T}}$ and a symmetric trapezoidal domain $\mathit{\Omega}_{\text{B}}$ (of height $H_{\text{B}} > 0$) that share a common interface $\mathit{\Gamma}$ ($\mathit{\Gamma}$ corresponds to the upper side of $\mathit{\Omega}_{\text{B}}$ and the lower side of $\mathit{\Omega}_{\text{T}}$). More precisely,

$$\mathit{\Omega}_{\text{T}} = \left\{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2, \text{such that} - L' < x_1 < L', \text{ and } 0 < x_2 < H_{\text{T}} \right\}, \quad (L' > L > 0,\ H_{\text{T}} > 0), \tag{2.1}$$

the common interface $\mathit{\Gamma}$ is given by

$$\mathit{\Gamma} := \{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2, -L < x_1 < L \text{and } x_2 = 0 \} \tag{2.2}$$

and

$$\mathit{\Omega} = \mathit{\Omega}_{\text{B}} \cup \mathit{\Omega}_{\text{T}} \cup \mathit{\Gamma}. \tag{2.3}$$

We point out that the polygon $\mathit{\Omega}$ has two re-entrant corners $\mathbf{x}_O^{\pm} = (\pm L, 0)$ of angle of $\mathit{\Theta} > \pi$.

Besides, let $\widehat{\mathit{\Omega}}_{\text{hole}} \in \mathbb{R}^2$ be a *smooth* canonical bounded open set (not necessarily connected) strictly included in the domain $(0,1) \times (-1,1)$. Then, let $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ denote the set of positive integers and let δ be a positive real number (that is supposed to be small) such that

$$\frac{2L}{\delta} = q \in \mathbb{N}^*. \tag{2.4}$$

Now, let $\mathit{\Omega}^{\delta}_{\text{hole}}$ be the thin (periodic) layer consisting of q equi-spaced similar obstacles defined by scaling and shifting the canonical obstacle $\widehat{\mathit{\Omega}}_{\text{hole}}$ (see Fig. 1):

$$\mathit{\Omega}^{\delta}_{\text{hole}} = \bigcup_{\ell=1}^{q} \left\{ -L\mathrm{e}_1 + \delta \{ \widehat{\mathit{\Omega}}_{\text{hole}} + (\ell - 1)\mathrm{e}_1 \} \right\}. \tag{2.5}$$

Here, e_1 and e_2 denote the unit vectors of $\mathbb{R}^2$ and δ is assumed to be smaller than H_{T} and H_{B} such that $\mathit{\Omega}^{\delta}_{\text{hole}}$ does not touch the top or bottom boundaries of $\mathit{\Omega}$. Finally, we define our domain of interest as

$$\mathit{\Omega}^{\delta} = (\mathit{\Omega}_{\text{B}} \cup \mathit{\Omega}_{\text{T}} \cup \mathit{\Gamma}) \setminus \overline{\mathit{\Omega}^{\delta}_{\text{hole}}}.$$

Its boundary $\partial\Omega^\delta$ consists of the union of three sets (see Fig. 1):

- the set of holes $\Gamma^\delta = \partial\Omega^\delta_{\text{hole}}$,
- the lateral boundaries $\Gamma_{R,\pm} = \left\{\mathbf{x} \in \partial\Omega^\delta \ / \ x_1 = \pm L'\right\}$ of Ω_{T}:

$$\Gamma_R = \Gamma_{R,-} \cup \Gamma_{R,+},$$

- the remaining part $\Gamma_N = \partial\Omega^\delta \setminus \left(\Gamma^\delta \cup \Gamma_R\right) = \partial\Omega \setminus \Gamma_R$, namely the boundaries of Ω_{B} except Γ and the upper boundary Ω_{T}.

Note, that in the limit $\delta \to 0$ the repetition of holes degenerates to the interface Γ, the domain Ω^δ to the domain $\Omega^0 := \Omega_{\mathrm{T}} \cup \Omega_{\mathrm{B}} = \Omega \setminus \Gamma$, and its boundary $\partial\Omega^\delta$ to $\partial\Omega \cup \Gamma$.

Remark 2.1. Note that the asymptotic analysis that will be employed in this article can be simply transferred to similar domains with thin periodic layers and different boundary conditions away from the layer. For example, the upper subdomain Ω_{T} can be replaced by a half space where radiation conditions are imposed at infinity.

2.1.2. The Helmholtz problem with a thin perforated wall of finite length.

On the domain Ω^δ we introduce the Helmholtz transmission problem to be considered in this article. Let $k_0 > 0$ be a given positive number, and let $u_{\text{inc}} = \exp(\imath k_0(x_1 - L'))$ be an incident plane wave of wavenumber k_0 coming from the left, we seek u^δ as solution of the total field problem

$$\left\{\begin{aligned} -\Delta u^\delta - (k^\delta)^2(\mathbf{x})u^\delta &= 0, && \text{in} && \Omega^\delta,\\ \nabla u^\delta \cdot \mathbf{n} &= 0, && \text{on} && \Gamma^\delta,\\ \nabla(u^\delta - u_{\text{inc}}) \cdot \mathbf{n} - \imath k_0(u^\delta - u_{\text{inc}}) &= 0, && \text{on} && \Gamma_R^-,\\ \nabla u^\delta \cdot \mathbf{n} - \imath k_0 u^\delta &= 0 && \text{on} && \Gamma_R^+,\\ \nabla u^\delta \cdot \mathbf{n} &= 0, && \text{on} && \Gamma_N. \end{aligned}\right. \tag{2.6}$$

In the previous system of equations, $\mathbf{n}$ stands for the outward unit normal vector of $\partial\Omega^\delta$. In the first equation of (2.6), $k^\delta(\mathbf{x})$ is given by

$$k^\delta(\mathbf{x}) = \begin{cases} k_0 & \text{if} \quad \mathbf{x} \quad \in \Omega^\delta \setminus (-L, L) \times (-\delta, \delta),\\ \widehat{k}\left(\dfrac{x_1}{\delta}, \dfrac{x_2}{\delta}\right) & \text{otherwise,} \end{cases}$$

where the function $\widehat{k}$ (defined on $\mathbb{R}^2$) is a smooth, positive function that is 1-periodic with respect to its first variable s. We also assume that there exists $\eta \in (0,1)$ such that $k(s,t) = k_0$ for $|t| > \eta$ or $|s| > \eta$. In other words, k^δ is a smooth function that is constant equal to k_0 outside the thin layer $(-L, L) \times (-\delta, \delta)$ and periodic of period δ in the vicinity of it. In particular, k^δ is bounded from above and from below independently of δ and k^δ tends almost everywhere to k_0.

The model (2.6) can be seen as a Helmholtz transmission problem in an infinite wave-guide with Neumann boundary conditions on the (rigid) walls, especially, on Γ^δ and Γ_N, which is truncated to a finite domain using first-order absorbing boundary conditions of Robin's type on Γ_R (see *e.g.* Ref. [23]). The following well-posedness result, based on the Fredholm alternative (Thm. 6.6 in [12]), is standard (see for instance Lem. 3.4 in [26] – Prop. 11.3 in [16]).

Proposition 2.2 (Existence, uniqueness and stability)**.** *For any $\delta > 0$ there exists a unique solution u^δ of problem (2.6) in $\mathrm{H}^1(\Omega^\delta)$. Moreover, there exists a constant C (independent of δ) such that*

$$\left\|u^\delta\right\|_{\mathrm{H}^1(\Omega^\delta)} \leqslant C\|\nabla u_{\text{inc}} \cdot \mathbf{n} - \imath k_0 u_{\text{inc}}\|_{(\mathrm{H}^{1/2}(\Gamma_R^-))'}. \tag{2.7}$$

For the sake of completeness, the proof of the previous is written in Appendix 6. We remark that the constant C appearing in the stability estimates (2.7) is independent of δ but depends on k_0, $\hat{k}$, and $\hat{\Omega}_{\text{hole}}$.

The objective of this paper is to describe the behaviour of u^δ as δ tends to 0. Our work relies on a construction of an asymptotic expansion of u^δ as δ tends to 0.

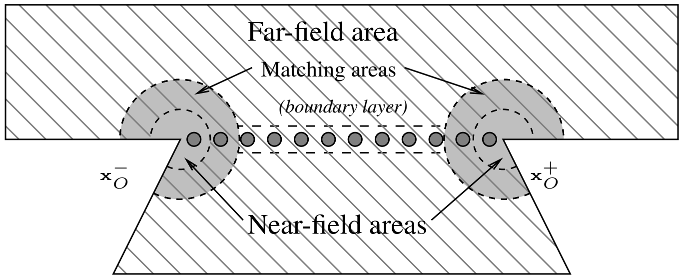


FIGURE 2. Schematic representation of the overlapping subdomains for the asymptotic expansion. The far field area (*hatched*) away from the corners $\mathbf{x}_O^{\pm}$ is overlapping the near field area (*gray*) in the matching zone.

2.2. Ansatz of the asymptotic expansion

As mentioned in the introduction, due to the presence of both the periodic layer and the two re-entrant corners, it seems not possible to write a simple asymptotic expansion valid in the whole domain. We have to take into account both the boundary layer effect in the vicinity of Γ and the additional corner singularities appearing in the neighborhood of the two re-entrant corners $\mathbf{x}_O^{\pm}$. To do so, we shall distinguish differents areas where the expansions are different:

- a *far field area* located 'far' from the corners $\mathbf{x}_O^{\pm}$ (hached area in Fig. 2),
- two *near field zones* located in their vicinities (grey areas in Fig. 2).

The far and near field areas intersect in the (non-empty) matching zone.

2.2.1. Far field expansion

In this section, we write an asymptotic expansion valid away from the two corners $\mathbf{x}_O^{\pm}$ (hatched area in Fig. 2). We shall decompose u^δ as the superposition of a *macroscopic part* (that contains no rapid oscillation) and a *boundary layer contribution* localized in the neighborhood of the thin periodic layer. In the present case the solution u^δ is then expanded in powers of δ, where each power is the sum of an integer and a so-called singular exponent λ_n given by

$$\lambda_n = n\lambda, \quad \lambda = \frac{\pi}{\Theta}\cdot \tag{2.8}$$

More precisely, we choose the ansatz

$$u^\delta(\mathbf{x}) = u^\delta_{\mathrm{FF},0,0}(\mathbf{x}) + \delta^{\lambda_1}\, u^\delta_{\mathrm{FF},1,0}(\mathbf{x}) + \delta\, u^\delta_{\mathrm{FF},0,1}(\mathbf{x}) + \delta^{\lambda_2}\, u^\delta_{\mathrm{FF},2,0}(\mathbf{x}) + \delta^{\lambda_1+1}\, u^\delta_{\mathrm{FF},1,1}(\mathbf{x}) + O(\delta^{\min(\lambda_3,2)}), \tag{2.9}$$

where each term takes the form

$$u^\delta_{\mathrm{FF},n,q}(\mathbf{x}) = \begin{cases} u^\delta_{n,q}(\mathbf{x}) & \text{if} \quad |x_1| > L + 2\delta, \\ \chi\left(\dfrac{x_2}{\delta}\right) u^\delta_{n,q}(\mathbf{x}) + \Pi^\delta_{n,q}\left(x_1, \dfrac{\mathbf{x}}{\delta}\right) & \text{if} \quad |x_1| < L - 2\delta, \end{cases} \tag{2.10}$$

with a smooth transition for $L - 2\delta < |x_1| < L + 2\delta$, which is detailed later in the article (see Sect. 6, Refs. [20] and [21]). Here, $u^\delta_{\mathrm{FF},n,q}$, $(n,q) \in \mathbb{N}^2$, is a combination of *macroscopic terms* $u^\delta_{n,q}$ and *boundary layer correctors* $\Pi^\delta_{n,q}$, and $\chi : \mathbb{R} \mapsto (0,1)$ denotes a smooth cut-off function satisfying

$$\chi(t) = \begin{cases} 1 & \text{if} \quad |t| > 2, \\ 0 & \text{if} \quad |t| < 1. \end{cases} \tag{2.11}$$

The superscript δ in $u^\delta_{n,q}$ and $\varPi^\delta_{n,q}$ indicates that they may depend on δ, however, this dependence is only polynomial in $\ln\delta$. In the next three paragraphs, we shall write the equations satisfied by the macroscopic terms, the boundary layer correctors and the transmissions conditions liking the two kinds of terms. The detailed derivation of these equations is done in Section 3.3.

Macroscopic equations. The macroscopic terms $u^\delta_{n,q}$ are defined in the limit domain $\Omega_\mathrm{T} \cup \Omega_\mathrm{B}$. Based on the usual decay assumption (see *e.g.* Refs. [36] and [37]) on the boundary layer correctors we find that the macroscopic terms satisfy the homogeneous Helmholtz equation

$$-\Delta u^\delta_{n,q} - k_0^2 u^\delta_{n,q} = 0 \qquad \text{in} \qquad \Omega_\mathrm{T} \cup \Omega_\mathrm{B}, \tag{2.12}$$

which is completed with prescribed boundary conditions on Γ_R and Γ_N

$$\begin{aligned}
\nabla(u^\delta_{0,0} - u_\mathrm{inc})\cdot\mathbf{n} - \imath k_0(u^\delta_{0,0} - u_\mathrm{inc}) &= 0, && \text{on} \quad \Gamma_R^-,\\
\nabla u^\delta_{0,0}\cdot\mathbf{n} - \imath k_0 u^\delta_{0,0} &= 0, && \text{on} \quad \Gamma_R^+,\\
\nabla u^\delta_{n,q}\cdot\mathbf{n} - \imath k_0 u^\delta_{n,q} &= 0, \quad (n,q)\neq(0,0), && \text{on} \quad \Gamma_R,\\
\nabla u^\delta_{n,q}\cdot\mathbf{n} &= 0, && \text{on} \quad \Gamma_N.
\end{aligned} \tag{2.13}$$

A priori, they are not continuous across Γ and may become unbounded when approaching the corners $\mathbf{x}^\pm_O$. Hence, the macroscopic terms are not entirely defined:

- we first have to prescribe transmission conditions across the interface Γ (for instance the jump of their trace and the jump of their normal trace across Γ). This information will appear to be a consequence of the boundary layer equations (see the paragraph 'Transmission conditions' below).
- we also have to prescribe the behaviour of the macroscopic terms in the vicinity of the two corner points $\mathbf{x}^\pm_O$. This information will be given through the matching conditions and will be provided through the iterative construction of the first terms (see Sects. 2.2.3, and 5).

Boundary layer corrector equations. The boundary layer correctors $\varPi^\delta_{n,q}(x_1, X_1, X_2)$ (also sometimes denoted as *periodic correctors*) are assumed, as usual in the periodic homogenization theory, to be 1-periodic with respect to the scaled tangential variable X_1. They are defined in the infinite periodicity cell $\mathcal{B} = \{(0,1)\times\mathbb{R}\} \setminus \overline{\widehat{\Omega}_\mathrm{hole}}$ (*cf.* Fig. 3a) and satisfy

$$\begin{cases}
-\Delta_\mathbf{X} \varPi^\delta_{n,q}(x_1,\mathbf{X}) = F^\delta_{n,q}(x_1,\mathbf{X}) & \text{in} \quad \mathcal{B},\\
\partial_\mathbf{n} \varPi^\delta_{n,q}(x_1,\mathbf{X}) = -\partial_{x_1}\varPi^\delta_{n,q-1}(x_1,\mathbf{X})\mathbf{e}_1\cdot\mathbf{n} & \text{on} \quad \partial\widehat{\Omega}_\mathrm{hole},
\end{cases} \tag{2.14}$$

in which $\mathbf{n}$ denotes the normal vector on $\partial\widehat{\Omega}_\mathrm{hole}$. The source terms $F^\delta_{n,q}$, depending on the macroscopic terms $u^\delta_{n,p}$ for $p \le q$ (see 3), are given by

$$\begin{aligned}
F^\delta_{n,0}(x_1,\mathbf{X}) &= \sum_\pm u^\delta_{n,0}(x_1,0^\pm)\chi''_\pm(X_2),\\
F^\delta_{n,1}(x_1,\mathbf{X}) &= \sum_\pm \Big\{\partial_{x_2} u^\delta_{n,0}(x_1,0^\pm)(2\chi'_\pm(X_2) + X_2\chi''_\pm + (X_2))\\
&\qquad + u^\delta_{n,1}(x_1,0^\pm)\chi''_\pm(X_2)\Big\},
\end{aligned}$$

where the cut-off function χ_+ (resp. χ_-) is the restriction of χ for $t \in \mathbb{R}_+$ (resp. $t \in \mathbb{R}_-$), *i.e.*

$$\chi_\pm(t) = \chi(t)\mathbb{1}_{\mathbb{R}_\pm}(t). \tag{2.15}$$

In addition, the periodic correctors are required to be super-algebraically decaying as the scaled variable X_2 tends to $\pm\infty$ (they decay faster than any power of X_2). More precisely, for any $(k,\ell) \in \mathbb{N}^2$, we impose that

$$\lim_{|X_2|\to+\infty} X_2^k \partial_{X_2}^{\ell} \Pi_{n,q}^{\delta} = 0. \tag{2.16}$$

Transmission conditions.

Enforcing the decaying condition (2.16) leads to the missing transmission conditions for the macroscopic terms $u_{n,q}^{\delta}$ on Γ. The complete procedure to obtain these transmission conditions is classical and is fully described in Section 3. In this paragraph, we restrict ourselves to the statement of the results. To do so, we introduce the definition of the jump and mean values of a function u across Γ (for a sufficiently smooth function u defined in a vicinity of Γ):

$$[u]_{\Gamma}(x_1) = \lim_{h\to 0^+} \left(u(x_1,h) - u(x_1,-h)\right), \langle u\rangle_{\Gamma}(x_1) = \frac{1}{2}\lim_{h\to 0^+} \left(u(x_1,h) + u(x_1,-h)\right). \tag{2.17}$$

For $n \in \{0,1,2,3\}$, we obtain that the terms $u_{n,0}^{\delta}$ do not jump across Γ, *i.e.*

$$\left[u_{n,0}^{\delta}\right]_{\Gamma} = \left[\partial_{x_2} u_{n,0}^{\delta}\right]_{\Gamma} = 0 \quad \text{on} \quad \Gamma. \tag{2.18}$$

By contrast, for $n \in \{0,1,2\}$, the terms $u_{n,1}^{\delta}$ satisfy non-homogeneous jump conditions:

$$\begin{cases} \left[u_{n,1}^{\delta}\right]_{\Gamma} = \mathcal{D}_1\, \partial_{x_1} \langle u_{n,0}^{\delta}\rangle_{\Gamma} + \mathcal{D}_2\, \langle \partial_{x_2} u_{n,0}^{\delta}\rangle_{\Gamma} & \text{on} \quad \Gamma, \\ \left[\partial_{x_2} u_{n,1}^{\delta}\right]_{\Gamma} = \mathcal{N}_1\, \langle u_{n,0}^{\delta}\rangle_{\Gamma} + \mathcal{N}_2\, \partial_{x_1}^2 \langle u_{n,0}^{\delta}\rangle_{\Gamma} + \mathcal{N}_3\, \partial_{x_1} \langle \partial_{x_2} u_{n,0}^{\delta}\rangle_{\Gamma} & \text{on} \quad \Gamma. \end{cases} \tag{2.19}$$

Here, the quantities $\mathcal{D}_i$ ($i \in \{1,2\}$) and $\mathcal{N}_i$ ($i \in \{1,2,3\}$), defined by (3.20)−(3.28) are complex-valued constants coming from the periodicity cell problems (2.14). They only depend on $\hat{k}$ and on the geometry of the periodicity cell.

2.2.2. Near field expansions

Let us now describe the asymptotic expansion valid in the two near field zones, namely in the vicinity of the two reentrant corners $\mathbf{x}_O^{\pm}$ (dark gray areas in Fig. 2). In these areas, the solution varies rapidly in all directions. Therefore, we shall see that

$$\begin{aligned} u^{\delta}(\mathbf{x}) = & U_{0,0,\pm}^{\delta}\left(\frac{\mathbf{x}-\mathbf{x}_O^{\pm}}{\delta}\right) + \delta^{\lambda_1} U_{1,0,\pm}^{\delta}\left(\frac{\mathbf{x}-\mathbf{x}_O^{\pm}}{\delta}\right) + \delta\, U_{0,1,\pm}^{\delta}\left(\frac{\mathbf{x}-\mathbf{x}_O^{\pm}}{\delta}\right) \\ & + \delta^{\lambda_2} U_{2,0,\pm}^{\delta}\left(\frac{\mathbf{x}-\mathbf{x}_O^{\pm}}{\delta}\right) + \delta^{\lambda_1+1} U_{1,1,\pm}^{\delta}\left(\frac{\mathbf{x}-\mathbf{x}_O^{\pm}}{\delta}\right) + \delta^{\lambda_3} U_{3,0,\pm}^{\delta}\left(\frac{\mathbf{x}-\mathbf{x}_O^{\pm}}{\delta}\right) + O(\delta^2) \end{aligned} \tag{2.20}$$

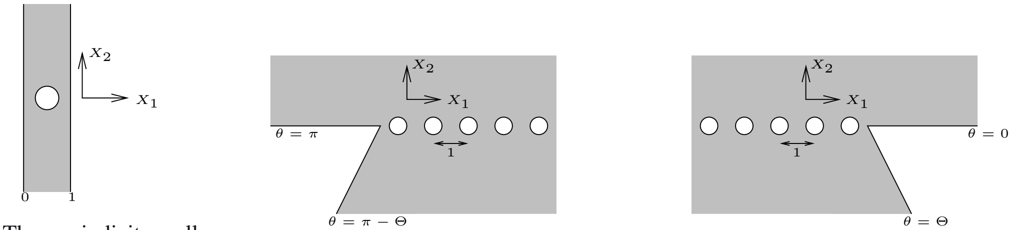
for some near field terms $U_{n,q,\pm}^{\delta}$ defined in the fixed unbounded domains

$$\widehat{\Omega}^{-} = \mathcal{K}^{-} \setminus \bigcup_{\ell\in\mathbb{N}} \left\{\overline{\widehat{\Omega}_{\text{hole}}} + \ell\mathbf{e}_1\right\}, \quad \widehat{\Omega}^{+} = \mathcal{K}^{+} \setminus \bigcup_{\ell\in\mathbb{N}^*} \left\{\overline{\widehat{\Omega}_{\text{hole}}} - \ell\mathbf{e}_1\right\} \tag{2.21}$$

shown in Figure 3b and 3c, where $\mathcal{K}^{\pm}$ are the conical domains

$$\mathcal{K}^{\pm} = \left\{\mathbf{X} = R^{\pm}(\cos\theta^{\pm}, \sin\theta^{\pm}), R^{\pm} \in \mathbb{R}_+^*, \theta^{\pm} \in I^{\pm}\right\} \subset \mathbb{R}^2 \tag{2.22}$$

of angular sectors $I^+ = (0,\Theta)$ and $I^- = (\pi-\Theta,\pi)$. The domains $\widehat{\Omega}^{\pm}$ consist of the angular domains $\mathcal{K}^{\pm}$ minus a infinite half line of equi-spaced similar canonical obtacles. In particular, if the domain $\widehat{\Omega}_{\text{hole}}$ is symmetric with respect to the axis $X_1 = 1/2$, then the domain $\widehat{\Omega}^{-}$ is nothing but the domain $\widehat{\Omega}^{+}$ mirrored with respect to the axis $X_1 = 0$. However, this is not the case in general.



(A) The periodicity cell $\mathcal{B}$.   (B) The domain $\widehat{\Omega}^-$.   (C) The domain $\widehat{\Omega}^+$.

FIGURE 3. The periodicity cell $\mathcal{B}$ and the normalized domains $\widehat{\Omega}^{\pm}$.

Similarly to the far field terms the near field terms $U^{\delta}_{n,q,\pm}$ might also have a polynomial dependence with respect to $\ln \delta$.

Inserting the near field ansatz (2.20) into the Helmholtz equation (2.6) and separating formally the different powers of δ, it is easily seen that the near field term $U^{\delta}_{n,q}$ satisfies

$$\begin{cases} -\Delta_{\mathbf{X}} U^{\delta}_{n,q,\pm} = (\widehat{k}^{\pm})^2(\mathbf{X}) U^{\delta}_{n,q-2,\pm} & \text{in} \quad \widehat{\Omega}^{\pm}, \\ \partial_{\mathbf{n}} U^{\delta}_{n,q,\pm} = 0 & \text{on} \quad \partial\widehat{\Omega}^{\pm}, \end{cases} \tag{2.23}$$

where the perturbed wave number $\widehat{k}^{\pm}(\mathbf{X})$ is given by

$$\widehat{k}^{\pm}(\mathbf{X}) = \begin{cases} \widehat{k}(\mathbf{X}) & \text{if} \quad \pm X_1 < 0, \\ k_0 & \text{otherwise.} \end{cases} \tag{2.24}$$

Again, Equation (2.23) does not define $U^{\delta}_{n,q,\pm}$ entirely because its (possibly increasing) behaviour towards infinity is missing. This behaviour will be given through the matching conditions.

2.2.3. Matching principle

To link the far and near fields expansions (2.9) and (2.20), we assume that they are both valid in two intermediate areas $\Omega^{\delta,\pm}_{\mathcal{M}}$ (dark shaded in Fig. 2) of the following form:

$$\Omega^{\delta,\pm}_{\mathcal{M}} = \left\{ \mathbf{x} = (x_1, x_2) \in \Omega^{\delta}, \sqrt{\delta} \leq d(\mathbf{x}, \mathbf{x}^{\pm}_{O}) \leq 2\sqrt{\delta} \right\}, \tag{2.25}$$

where d denotes the usual Euclidian distance. The reader might just keep in mind that they correspond to a neighborhood of the corners $\mathbf{x}^{\pm}_{O}$ of the re-entrant corners for the far field terms (macroscopic and boundary layer correctors) and to a neighborhood of infinity, *i.e.*, $R^{\pm} \to \infty$, for the near field terms (expressed in the scaled variables).

In practice, for a given order $N_0 \geq 0$, we make a *formal* identification between (2.9) and (2.20):

$$\sum_{\lambda_n + q < N_0} \delta^{\lambda_n + q}\, u^{\delta}_{\mathrm{FF},n,q}(\mathbf{x}) \approx \sum_{\lambda_n + q < N_0} \delta^{\lambda_n + q}\, U^{\delta}_{n,q,\pm}\left(\frac{\mathbf{x} - \mathbf{x}^{\pm}_{O}}{\delta}\right). \tag{2.26}$$

The previous relation can be seen in two different scales (the macroscopic scale and the near field scale) and will relate, on the one hand, the regular part of the far field terms to the increasing behaviour of the near field terms, and, on the other hand, the decreasing behaviour of the near field terms to the singular behaviour of the far field terms. The matching will be conducted for the first terms order by order in Section 5.

Remark 2.3. A crucial point for the matching procedure is that we match only the far and near field expansions away from the layer, *i.e.*, $\theta^- \neq 0$ and $\theta^+ \neq \pi$. Indeed, thanks to the linearity of the canonical cell problem, the periodic correctors appear to be a by-product of the macroscopic terms (see Sect. 3). As a consequence, as soon as the two series match away from the layer, they also match in the vicinity of the layer (see Sect. 5.7).

2.3. Main results

2.3.1. Error estimates

Collectiong the macroscopic problems (2.12)−(2.13)−(2.18)−(2.19), the boundary layer problems (2.14), the near field problems (2.23), and the matching conditions (2.26) permits us to define in step by step the first terms of the asymptotic expansion up to order 2 (see Sect. 5). Then, our main theoretical result deals with the convergence of the truncated macroscopic series in a domain that excludes the two corners and the periodic thin layer:

Theorem 2.4 (Error estimates of the truncated macroscopic expansion)**.** *Let $\Theta \in (\pi, 2\pi)$, and, for a given number $\alpha > 0$, let*

$$\Omega_\alpha = \Omega^\delta \setminus (-L-\alpha, L+\alpha) \times (-\alpha, \alpha).$$

There exists a constant $\delta_0 > 0$, a constant $C > 0$ and a integer $\kappa \in \{0, 1\}$ such that for any $\delta \in (0, \delta_0)$,

$$\left\| u^\delta - u_{0,0} \right\|_{\mathrm{H}^1(\Omega_\alpha)} \leqslant C\delta, \tag{2.27}$$

$$\left\| u^\delta - u_{0,0} - \delta u_{0,1} \right\|_{\mathrm{H}^1(\Omega_\alpha)} \leqslant C\delta^{\lambda_2}. \tag{2.28}$$

and,

$$\left\| u^\delta - u_{0,0} - \delta u_{0,1} - \delta^{\lambda_2} u_{2,0} \right\|_{\mathrm{H}^1(\Omega_\alpha)} \leqslant C\delta^2 (\ln \delta)^\kappa, \qquad \text{if} \quad \Theta \leq \frac{3\pi}{2}, \tag{2.29}$$

$$\left\| u^\delta - u_{0,0} - \delta u_{0,1} - \delta^{\lambda_2} u_{2,0} - \delta^{\lambda_3} u_{3,0} \right\|_{\mathrm{H}^1(\Omega_\alpha)} \leqslant C\delta^2 \qquad \text{if} \quad \Theta \in \left(\frac{3\pi}{2}, 2\pi\right)\cdot \tag{2.30}$$

The proof of the previous theorem, although rather classical (see *e.g.* Chap. 4 in Ref. [31]), is conducted in Section 6: it is based on the construction of an approximation global approximation (defined in (6.5)) of u^δ defined in the whole domain Ω^δ.

2.3.2. Numerical justification

We illustrate numerically the results of Theorem 2.4 using the finite elements method with the numerical C++ library Concepts [17, 22]. For both, the exact and macroscopic problems, we rely on meshes geometrically refined towards the corners and varying polynomial degree [41, 42]. We consider the geometry sketched in the left part of Figure 4 for $\delta = 0.25$, for which the inner angle $\Theta = \frac{3\pi}{2}$ at the two corners $\mathbf{x}_O^\pm$. The upper rectangle representing a wave-guide is $\Omega_\mathrm{T} = (-2.5, 2.5) \times (0, 1)$ and the lower one representing a chamber is $\Omega_\mathrm{B} = (-0.5, 0.5) \times (-1, 0)$. The canonical hole $\widehat{\Omega}^\pm_\mathrm{hole}$ is the disk centered at $(0.5, 0)$ with diameter equal to 0.3. We consider a homogeneous wave number $k^\delta = k_0 = 5\pi$. In Figure 4 we show the difference between the exact solution u^δ and the macroscopic expansion of different order, using that $u_{1,0} = u_{1,1} = 0$, in the $L^2(\Omega_\alpha)$-norm for $\alpha = 0.25$ as a function of δ where $\delta = 1/4, 1/8, \ldots, 1/128$. As might be expected, we exactly recover the convergence rate stated in Theorem 2.4.

3. Analysis of the far field problems: Transmission problem, boundary layer problems and derivation of the transmission conditions

This section is dedicated to the analysis of the far-field problems. In Section 3.1 and Section 3.2, we first recall the functional frameworks that will allow us to define the macroscopic terms and boundary layer terms. Then, Section 3.3 is dedicated to the *formal* derivation of the transmission conditions (2.18)−(2.19) for the macroscopic fields $u^\delta_{n,q}$ across Γ.

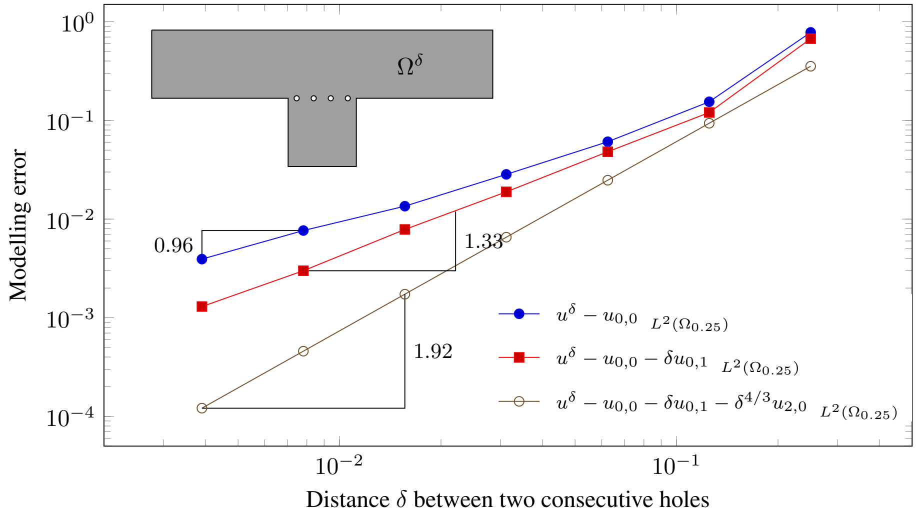


FIGURE 4. The numerically computed errors of macroscopic expansions truncated at different orders in dependence of δ. The computational domain Ω^δ is sketched for $\delta = 0.25$.

3.1. General results of existence for transmission problem

The macroscopic fields satisfy transmission problems of the following form (*cf.* (2.12)–(2.13)–(2.18)–(2.19)):

$$\begin{cases} -\Delta u - k_0^2 u = \mathfrak{f} & \text{in} \quad \Omega_{\mathrm{T}} \cup \Omega_{\mathrm{B}}\ , \\ [u]_\Gamma = \mathfrak{g} & \text{on} \quad \Gamma\ , \\ [\partial_{x_2} u]_\Gamma = \mathfrak{h} & \text{on} \quad \Gamma\ , \\ \nabla u \cdot \mathbf{n} - \imath k_0 u^\delta = \mathfrak{j}, & \text{on} \quad \Gamma_R, \\ \nabla u \cdot \mathbf{n} = 0, & \text{on} \quad \Gamma_N. \end{cases} \tag{3.1}$$

To solve this transmission problem, we consider the space $\mathrm{H}^1(\Omega_{\mathrm{T}} \cup \Omega_{\mathrm{B}})$ defined by

$$\mathrm{H}^1(\Omega_{\mathrm{T}} \cup \Omega_{\mathrm{B}}) = \left\{ v \in \mathrm{L}^2(\Omega) \text{ such that } v_{|\Omega_{\mathrm{T}}} \in \mathrm{H}^1(\Omega_{\mathrm{T}}) \text{ and } v_{|\Omega_{\mathrm{B}}} \in \mathrm{H}^1(\Omega_{\mathrm{B}}) \right\}, \tag{3.2}$$

which incorporates discontinuous functions over Γ (see Fig. 1b). We denote by $\mathrm{H}^{1/2}(\Gamma)$ the restriction of the trace of the functions $\mathrm{H}^1(\Omega_{\mathrm{T}})$ to Γ. Naturally, the space $\mathrm{H}^{1/2}(\Gamma)$ is also the restriction of the trace of the functions of $\mathrm{H}^1(\Omega_{\mathrm{B}})$. We point out that general transmission problems are investigated in [33] using the Kondratev theory. In particular the following well-posedness result is proved (Thms. 3.4 and 3.5 in Ref. [33], Prop. 3.6.1 in Ref. [19]).

Proposition 3.1. *Let* $\mathfrak{f} \in \mathrm{L}^2(\Omega)$, $\mathfrak{g} \in \mathrm{H}^{1/2}(\Gamma)$, $\mathfrak{h} \in \mathrm{L}^2(\Gamma)$, *and* $\mathfrak{j} \in H^{-1/2}(\Gamma_R)$. *Then, problem* (3.1) *has a unique solution* u *belonging to* $\mathrm{H}^1(\Omega_{\mathrm{T}} \cup \Omega_{\mathrm{B}})$.

3.2. Existence and uniqueness result for the boundary layer problem

The boundary layer correctors satisfy problems of the form (see (2.14))

$$\begin{cases} -\Delta_{\mathbf{X}} \Pi = F & \text{in} \quad \mathcal{B}, \\ \partial_{\mathbf{n}} \Pi = G & \text{on} \quad \partial\widehat{\Omega}_{\text{hole}}, \\ \partial_{X_1} \Pi(0, X_2) = \partial_{X_1} \Pi(1, X_2), & X_2 \in \mathbb{R}. \end{cases} \tag{3.3}$$

together with the super-algebric decaying condition 2.16. In this section, we give a standard result of existence and uniqueness associated with this problem. To do so, we introduce the two weighted Sobolev spaces

$$\mathcal{V}^{\pm}(\mathcal{B}) = \left\{ \Pi \in \mathrm{H}^1_{\mathrm{loc}}(\mathcal{B}), \Pi(0, X_2) = \Pi(1, X_2), \text{ and } \left(\Pi\, w_e^{\pm}\right) \in \mathrm{H}^1(\mathcal{B}) \right\}, \tag{3.4}$$

where the weighting functions $w_e^{\pm}(X_1, X_2) = \chi(X_2) \exp(\pm \frac{|X_2|}{2})$. The functions of $\mathcal{V}^-(\mathcal{B})$ correspond to the periodic (w.r.t. X_1) functions of $\mathrm{H}^1_{\mathrm{loc}}(\mathcal{B})$ that grow slower than $\exp(\frac{|X_2|}{2})$ as X_2 tends to $\pm\infty$. By contrast, the functions of $\mathcal{V}^+(\mathcal{B})$ correspond to the periodic functions of $\mathrm{H}^1_{\mathrm{loc}}(\mathcal{B})$ decaying faster than $\exp(-\frac{|X_2|}{2})$ as X_2 tends to $\pm\infty$. Note also that $\mathcal{V}^+(\mathcal{B}) \subset \mathcal{V}^-(\mathcal{B})$.

As soon as $F \in (\mathcal{V}^-(\mathcal{B}))'$ and $G \in L^2(\partial\widehat{\Omega}_{\mathrm{hole}})$, it is known that problem (3.3) has (several) solutions in $\mathcal{V}^-(\mathcal{B})$ (*cf.* Prop. 2.2 of Ref. [32] and Sect. 5 of Ref. [16]). More specifically, problem (3.3) has a finite dimensional kernel of dimension 2, spanned by the functions $\mathcal{N} = \mathbb{1}_{\mathcal{B}}$ and $\mathcal{D}$, where $\mathcal{D}$ is the unique harmonic function of $\mathcal{V}^-(\mathcal{B})$ such that there exists $\mathcal{D}_\infty \in \mathbb{R}$ such that

$$\widetilde{\mathcal{D}}(X_1, X_2) = \mathcal{D}(X_1, X_2) - \chi_+(X_2)(X_2 + \mathcal{D}_\infty) - \chi_-(X_2)(X_2 - \mathcal{D}_\infty)$$

belongs to $\mathcal{V}^+(\mathcal{B})$ ($\chi_\pm$ defined by (2.15)).

The following proposition provides necessary and sufficient conditions for the existence of an exponentially decaying solution (see also Prop. 2.2 of Ref. [32] and Sect. 5 of Ref. [16] for the proof):

Proposition 3.2. *Assume that* $F \in (\mathcal{V}^-(\mathcal{B}))'$ *and* $G \in L^2(\partial\widehat{\Omega}_{\mathrm{hole}})$. *Problem* (3.3) *has a unique solution* $\Pi \in \mathcal{V}^+(\mathcal{B})$ *if and only if* (F, G) *satisfies the following two conditions*

$$\int_{\mathcal{B}} F(\mathbf{X})\mathcal{D}(\mathbf{X}) d\mathbf{X} + \int_{\partial\widehat{\Omega}_{\mathrm{hole}}} G(\mathbf{X})\mathcal{D}(\mathbf{X}) \mathrm{d}\sigma(\mathbf{X}) = 0, \tag{$\mathcal{C}_{\mathcal{D}}$}$$

$$\int_{\mathcal{B}} F(\mathbf{X})\mathcal{N}(\mathbf{X}) d\mathbf{X} + \int_{\partial\widehat{\Omega}_{\mathrm{hole}}} G(\mathbf{X})\mathcal{N}(\mathbf{X}) \mathrm{d}\sigma(\mathbf{X}) = 0. \tag{$\mathcal{C}_{\mathcal{N}}$}$$

3.3. Derivation of the boundary layer correctors problems and the transmission conditions for the macroscopic problems

The previous framework will allows us to derive *formally* the transmission conditions (2.18)−(2.19) for the macroscopic fields $u^\delta_{n,q}$ across Γ. This procedure turns out to be independent of the index n and of the superscript δ (of $u^\delta_{n,q}$) so that we shall omit the index n and the superscript δ in this section. To do so, we completely ignore the corners $\mathbf{x}_O^{\pm}$ and we proceed as if the periodic layer were infinite. For a given $a \in (0, L)$, we restrict the domain Ω^δ to $\Omega^\delta_a = \{\mathbf{x} \in \Omega^\delta \text{ such that } |x_1| < a\}$, and we call Ω_a the limit domain as $\delta \to 0$, *i.e.* $\Omega_a = \{\mathbf{x} \in \Omega_{\mathrm{T}} \cup \Omega_{\mathrm{B}} \text{ such that } |x_1| < a\}$. We start from a (given) term u_0 in Ω_a that is solution of the homogeneous Helmholtz equation

$$-\Delta u_0 - k_0^2 u_0 = 0 \qquad \text{in} \qquad \Omega_{\mathrm{T}} \cap \Omega_a \qquad \text{and} \qquad \Omega_{\mathrm{B}} \cap \Omega_a.$$

Then, using the method of homogenization [35], we extend u_0 to a function v^δ of the form

$$\chi(x_2/\delta)(u_0 + \delta u_1 + \delta^2 u_2) + (1 - \chi(x_2/\delta))(\Pi_0 + \delta \Pi_1 + \delta^2 \Pi_2))$$

that is defined in Ω^δ_a and that satisfies the original Helmholtz problem (2.6) up to a given order (ignoring the lateral boundaries Ω^δ_a):

$$-\Delta v^\delta - (k^\delta)^2 v^\delta \approx 0 \quad \text{in} \quad \Omega^\delta_a \quad \text{and} \quad \partial_{\mathbf{n}} v^\delta \approx 0 \quad \text{on} \quad \Gamma^\delta \cap \partial\Omega^\delta_a.$$

Remark 3.3. The periodic boundary layer being considered as infinite, we point out that the following analysis is entirely classical [1, 4, 6, 40]. Moreover, we emphasize that the upcoming iterative procedure is formal in the sense that we shall provide necessary transmission conditions for the macroscopic terms u_q (without questioning their existence yet).

3.3.1. Step 0: $[u_0]_\Gamma$ and Π_0

We start with the ansatz

$$v^\delta(\mathbf{x}) = u_0(\mathbf{x})\chi(x_2/\delta), \qquad \text{in} \quad \Omega_a^\delta. \tag{3.5}$$

The choice of the cut-off function $\chi(x_2/\delta)$ is intended since $k^\delta(\mathbf{x}) = k_0$ on the support of $\chi(x_2/\delta)$. Reminding that $\big(-\Delta u_0 - k_0^2 u_0\big)\chi(x_2/\delta) = 0$, we see that

$$-\Delta v^\delta - \widehat{k}^2\left(\frac{\mathbf{x}}{\delta}\right)v^\delta = -\frac{1}{\delta^2}u_0(\mathbf{x})\chi''\left(\frac{x_2}{\delta}\right) - \frac{2}{\delta}\frac{\mathrm{d}u_0}{\mathrm{d}x_2}(\mathbf{x})\chi'\left(\frac{x_2}{\delta}\right) \quad \text{and} \quad \partial_{\mathbf{n}} v^\delta = 0 \quad \text{on} \quad \Gamma^\delta \cap \partial\Omega_a^\delta. \tag{3.6}$$

In (3.6), the leading order term is in δ^{-2} and is supported in a vicinity of the limit interface $\Gamma_a = (-a,a)\times\{0\}$. To correct it, it is rational to add to v^δ an exponentially decaying periodic corrector $\Pi_0(x_1, \mathbf{x}/\delta)$:

$$v^\delta(\mathbf{x}) = u_0(\mathbf{x})\chi(x_2/\delta) + \Pi_0(x_1, \mathbf{x}/\delta), \quad \text{in} \quad \Omega_a^\delta \tag{3.7}$$

We note that

$$\begin{aligned}
-\Delta v^\delta - \widehat{k}^2(\mathbf{x}/\delta)v^\delta = {} & \frac{1}{\delta^2}\left(-u_0(\mathbf{x})\chi''\left(\frac{x_2}{\delta}\right) - \Delta_{\mathbf{X}}\Pi_0\left(x_1, \frac{\mathbf{x}}{\delta}\right)\right) \\
& + \frac{1}{\delta}\left(-2\frac{\mathrm{d}u_0}{\mathrm{d}x_2}(\mathbf{x})\chi'\left(\frac{x_2}{\delta}\right) - 2\partial_{x_1}\partial_{X_1}\Pi_0\left(x_1, \frac{\mathbf{x}}{\delta}\right)\right) \\
& - \partial_{x_1}^2\Pi_0\left(x_1, \frac{\mathbf{x}}{\delta}\right) - \widehat{k}^2(\mathbf{x}/\delta)\Pi_0\left(x_1, \frac{\mathbf{x}}{\delta}\right)\cdot
\end{aligned} \tag{3.8}$$

Then, making the change of scale $\mathbf{X} = \mathbf{x}/\delta$ and using a Taylor expansion of $u_0(x_1, \delta X_2)$ for δ small and for $X_2 \neq 0$, the leading term of order δ^{-2} vanishes if Π_0 satisfies

$$\begin{cases} -\Delta_{\mathbf{X}}\Pi_0(x_1,\mathbf{X}) = F_0(x_1,\mathbf{X}) & \text{in} \quad \mathcal{B}, \\ \partial_{\mathbf{n}}\Pi_0 = 0 & \text{on} \quad \partial\widehat{\Omega}_{\mathrm{hole}}, \end{cases} \quad F_0(x_1,\mathbf{X}) = \sum_{\pm} u_0(x_1, 0^\pm)\chi_\pm''(X_2). \tag{3.9}$$

Problem (3.9) is a partial differential equation with respect to the microscopic variables X_1 and X_2, wherein the macroscopic variable x_1 plays the role of a parameter. For a fixed x_1 in $(-a,a)$ (considered as a parameter), $F_0(x_1,\cdot)$ belongs to $(\mathcal{V}^-(\mathcal{B}))'$ since it is compactly supported. Then, in view of Proposition 3.2, there exists an exponentially decaying solution $\Pi_0(x_1,\cdot) \in \mathcal{V}^+(\mathcal{B})$ if and only if the two compatibility conditions ($\mathcal{C}_\mathcal{D}$, $\mathcal{C}_\mathcal{N}$) (Prop. 3.2) are satisfied. The condition ($\mathcal{C}_\mathcal{N}$) is always satisfied while the condition ($\mathcal{C}_\mathcal{D}$) gives $[u_0]_{\Gamma_a}(x_1) = 0$. Taking formally in this relation the limit $a = L$ gives

$$[u_0]_\Gamma(x_1) = 0. \tag{3.10}$$

The previous equality provides a first transmission condition for the limit macroscopic term u_0 (a transmission condition for $[\partial_{\mathbf{n}} u_0]_\Gamma$ is still needed). In addition, under the previous condition, $F_0(x_1,\mathbf{X}) = \chi''(X_2)\langle u_0\rangle_\Gamma(x_1)$, and, using the linearity of problem (3.9), we can obtain a tensorial representation of Π_0, in which macroscopic and microscopic variables are separated:

$$\Pi_0(x_1,\mathbf{X}) = \langle u_0\rangle_\Gamma(x_1)\, V_0(\mathbf{X}). \tag{3.11}$$

Here the profile function $V_0(\mathbf{X})$ is the unique function of $\mathcal{V}^+(\mathcal{B})$ satisfying

$$\begin{cases} -\Delta_{\mathbf{X}} V_0(\mathbf{X}) = F_{V_0}(\mathbf{X}) & \text{in} \quad \mathcal{B}, \\ \partial_{\mathbf{n}} V_0 = 0 & \text{on} \quad \partial\widehat{\Omega}_{\mathrm{hole}}, \\ \partial_{X_1}V_0(0,X_2) = \partial_{X_1}V_0(1,X_2), \;\; X_2 \in \mathbb{R}, \end{cases} \quad F_{V_0}(\mathbf{X}) = \chi''(X_2). \tag{3.12}$$

A direct calculation shows that $V_0(\mathbf{X}) = 1 - \chi(X_2)$.

3.3.2. Step 1: $[\partial_{x_2} u_0]_\Gamma$, $[u_1]_\Gamma$, and Π_1

By definition of Π_0, the leading part in the right hand side of (3.8) is of order δ^{-1}. To cancel these terms, we correct v^δ defined by (3.7), adding a first order corrector, *both* in a vicinity of the layer and away from the layer:

$$v^\delta(\mathbf{x}) = u_0(\mathbf{x})\chi(x_2/\delta) + \Pi_0(x_1, \mathbf{x}/\delta) + \delta u_1(\mathbf{x})\chi(x_2/\delta) + \delta \Pi_1(x_1, \mathbf{x}/\delta), \quad \text{in } \Omega_a^\delta. \tag{3.13}$$

Adding the term Π_1 is natural (indeed, the remaining term in (3.8) is located in the vicinity of the interface Γ. It is of order $1/\delta$, that can be seen as δ (order of the remaining term) times δ^{-2} (order of differentiation after the change of scale)). By contrast, the addition of the term u_1 might be surprising but appears to be mandatory to ensure the exponential decay of Π_1. Then,

$$\begin{aligned}
-\Delta v^\delta - \widehat{k}^2\left(\frac{\mathbf{x}}{\delta}\right) v^\delta = &-\delta(\Delta u_1 + k_0^2 u_1)\chi\left(\frac{x_2}{\delta}\right) + \frac{1}{\delta^2}\left(u_0(x_1,0) - u_0(\mathbf{x})\right)\chi''\left(\frac{x_2}{\delta}\right) \\
&- \frac{2}{\delta}\partial_{x_2} u_0(\mathbf{x})\chi'\left(\frac{x_2}{\delta}\right) - \frac{1}{\delta}u_1(\mathbf{x})\chi''\left(\frac{x_2}{\delta}\right) - \frac{1}{\delta}\Delta_{\mathbf{X}}\Pi_1\left(x_1, \frac{\mathbf{x}}{\delta}\right) \\
&+ \left(\partial_{x_1}^2 + \widehat{k}^2\left(\frac{\mathbf{x}}{\delta}\right)\right) u_0(x_1,0)\left(1 - \chi\left(\frac{x_2}{\delta}\right)\right) - 2\partial_{x_2} u_1(\mathbf{x})\chi'\left(\frac{x_2}{\delta}\right) \\
&- 2\partial_{x_1}\partial_{X_1}\Pi_1\left(x_1, \frac{\mathbf{x}}{\delta}\right) - \delta\left(\partial_{x_1}^2 + \widehat{k}^2(\frac{\mathbf{x}}{\delta})\right)\Pi_1\left(x_1, \frac{\mathbf{x}}{\delta}\right).
\end{aligned} \tag{3.14}$$

and

$$\partial_{\mathbf{n}} v^\delta(\delta\mathbf{X}) = \partial_{\mathbf{n}}\Pi_1(x_1, \mathbf{X}) + \partial_{x_1}\langle u_0(x_1,0)\rangle \mathbf{e}_1 \cdot \mathbf{n}. \tag{3.15}$$

For a given $\mathbf{x}$ such that $x_2 \neq 0$, dividing (3.14) by δ and taking the limit as $\delta \to 0$ in (3.14) leads to

$$-\Delta u_1 - k_0^2 u_1 = 0 \quad \text{in} \quad \Omega_{\mathrm{T}} \cap \Omega_a \quad \text{and} \quad \Omega_{\mathrm{B}} \cap \Omega_a \tag{3.16}$$

Indeed, the terms that contains $1-\chi$, χ' and χ'' are compactly supported and vanish for $|x_2| > 2\delta$, and, by assumption the terms related to Π_1 are exponentially decaying towards $x_2/\delta \to \infty$. To defined Π_1, we make the change of scale $\mathbf{X} = \frac{\mathbf{x}}{\delta}$ in (3.14) (using Taylor expansions of u_0 and u_1 in the vicinity of Γ) and we enforce the term in δ^{-1} in (3.14) to vanish. Together with the Neumann boundary condition (3.15), it is rational to construct Π_1 as a solution to

$$\begin{cases}
-\Delta_{\mathbf{X}}\Pi_1(x_1, \mathbf{X}) = F_1(x_1, \mathbf{X}) & \text{in} \quad \mathcal{B}, \\
\partial_{\mathbf{n}}\Pi_1 = -\partial_{x_1}\langle u_0(x_1,0)\rangle\, \mathbf{e}_1 \cdot \mathbf{n} & \text{on} \quad \partial\widehat{\Omega}_{\text{hole}}, \\
\partial_{X_1}\Pi_1(0, X_2) = \partial_{X_1}\Pi_1(1, X_2), & X_2 \in \mathbb{R},
\end{cases} \tag{3.17}$$

where

$$F_1(x_1, \mathbf{X}) = \sum_{\pm}\left(\partial_{x_2} u_0(x_1, 0^\pm)(2\chi'_\pm(X_2) + X_2\chi''_\pm(X_2)) + u_1(x_1, 0^\pm)\chi''_\pm(X_2)\right). \tag{3.18}$$

As for Π_0, problem (3.17) is a partial differential equation with respect to the microscopic variables X_1 and X_2, where the macroscopic variable x_1 plays the role of a parameter. For a fixed x_1 in $(-L, L)$, $F_1(x_1, \cdot)$ is compactly supported in $\mathcal{B}$, and, consequently, belongs to $(\mathcal{V}^-(\mathcal{B}))'$. Then, thanks to Proposition 3.2, there exists an exponentially decaying solution $\Pi_1(x_1, \cdot) \in \mathcal{V}^+(\mathcal{B})$ if and only if the two compatibility conditions ($\mathcal{C}_{\mathcal{D}}$, $\mathcal{C}_{\mathcal{N}}$) are satisfied. A direct calculation shows that the compatibility condition ($\mathcal{C}_{\mathcal{N}}$) is fulfilled if and only if

$$[\partial_{x_2} u_0]_\Gamma(x_1) = 0, \tag{3.19}$$

and the compatibility condition ($\mathcal{C}_{\mathcal{D}}$) is fulfilled if and only if

$$\begin{aligned}
&[u_1]_\Gamma(x_1) = \mathcal{D}_1\, \partial_{x_1}\langle u_0\rangle_\Gamma(x_1) + \mathcal{D}_2\,\langle \partial_{x_2} u_0\rangle_\Gamma(x_1), \\
&\mathcal{D}_1 = -\int_{\partial\widehat{\Omega}_{\text{hole}}} \mathcal{D}\mathbf{e}_1 \cdot \mathbf{n},\ \mathcal{D}_2 = \int_{\mathcal{B}} (2\chi'(X_2) + X_2\chi''(X_2))\mathcal{D}.
\end{aligned} \tag{3.20}$$

Under the two conditions (3.19)–(3.20), problem (3.17) has a unique solution $\Pi_1 \in \mathcal{V}^+(\mathcal{B})$ that can be written as

$$\Pi_1(x_1,\mathbf{X}) = \langle u_1\rangle_\Gamma(x_1)\, V_0(\mathbf{X}) + \partial_{x_1}\langle u_0\rangle_\Gamma(x_1)\, V_{1,1}(\mathbf{X}) \;+\; \langle \partial_{x_2} u_0\rangle_\Gamma(x_1)\, V_{1,2}(\mathbf{X}). \tag{3.21}$$

Here, $V_{1,1} \in \mathcal{V}^+(\mathcal{B})$ and $V_{1,2} \in \mathcal{V}^+(\mathcal{B})$ are the unique exponentially decaying solutions to the following problems:

$$\begin{cases} -\Delta_{\mathbf{X}} V_{1,1}(\mathbf{X}) = \mathcal{D}_1 \frac{\chi_+''(X_2)-\chi_-''(X_2)}{2} & \text{in} \quad \mathcal{B},\\ \partial_{\mathbf{n}} V_{1,1} = -\mathbf{e}_1\cdot\mathbf{n} & \text{on} \quad \partial\widehat{\Omega}_{\text{hole}},\\ \partial_{X_1} V_{1,1}(0,X_2) = \partial_{X_1} V_{1,1}(1,X_2), & X_2\in\mathbb{R}, \end{cases} \tag{3.22}$$

$$\begin{cases} -\Delta_{\mathbf{X}} V_{1,2}(\mathbf{X}) = F_{V_{1,2}} + \mathcal{D}_2 \frac{\chi_+''(X_2)-\chi_-''(X_2))}{2} & \text{in} \quad \mathcal{B},\\ \partial_{\mathbf{n}} V_{1,2} = 0 & \text{on} \quad \partial\widehat{\Omega}_{\text{hole}}, \quad F_{V_{1,2}} = 2\chi'(X_2) + X_2\chi''(X_2)\\ \partial_{X_1} V_{1,2}(0,X_2) = \partial_{X_1} V_{1,2}(1,X_2), & X_2\in\mathbb{R}, \end{cases} \tag{3.23}$$

3.3.3. Step 2: $[\partial_{x_2} u_1]_\Gamma$ ($[u_2]_\Gamma$ and Π_2)

To define completely u_1, we need to go one order further into the asymptotic expansion. We then correct v^δ defined by (3.13), adding a second order corrector:

$$v^\delta(\mathbf{x}) = u_0(\mathbf{x})\,\chi\left(\frac{x_2}{\delta}\right) + \Pi_0\left(x_1,\frac{\mathbf{x}}{\delta}\right) + \delta\left(u_1(\mathbf{x})\,\chi\left(\frac{x_2}{\delta}\right) + \Pi_1\left(x_1,\frac{\mathbf{x}}{\delta}\right)\right) + +\delta^2\left(u_2(\mathbf{x})\,\chi\left(\frac{x_2}{\delta}\right) + \Pi_2\left(x_1,\frac{\mathbf{x}}{\delta}\right)\right)\cdot \tag{3.24}$$

Again, we apply the Helmholtz operator on v^δ. Then extracting the macroscopic δ^2 order and the δ^0 order close to the layer gives the equations for u_2 and Π_2. The term u_2 is solution of the homogeneous Helmholtz equation

$$-\Delta u_2 - k_0^2 u_2 = 0 \tag{3.25}$$

in $\Omega_{\mathrm{T}}\cap\Omega_a$ and $\Omega_{\mathrm{B}}\cap\Omega_a$. The periodic corrector Π_2 satisfies the following equation

$$\begin{cases} -\Delta_{\mathbf{X}} \Pi_2(x_1,\mathbf{X}) = F_2(x_1,\mathbf{X}) & \text{in} \quad \mathcal{B},\\ \partial_{\mathbf{n}} \Pi_2 = -\partial_{x_1}\Pi_1\,\mathbf{e}_1\cdot\mathbf{n} & \text{on} \quad \partial\widehat{\Omega}_{\text{hole}},\\ \partial_{X_1}\Pi_2(0,X_2) = \partial_{X_1}\Pi_2(1,X_2), & X_2\in\mathbb{R}. \end{cases} \tag{3.26}$$

Here,

$$\begin{aligned} F_2(x_1,\mathbf{X}) = \sum_{\pm} u_2(x_1,0^\pm)\chi_\pm''(X_2) + \frac{\left(\left(\chi_+'(X_2)-\chi_-'(X_2)\right)X_2\right)'}{2}\left[\partial_{x_2} u_1\right]_\Gamma(x_1)\\ +\; F_{V_{1,2}}\,\langle \partial_{x_2} u_1\rangle_\Gamma(x_1) + F_{V_{2,1}}(\mathbf{X})\,\langle u_0\rangle_\Gamma(x_1)\\ +\; F_{V_{2,2}}(\mathbf{X})\,\partial_{x_1}^2\langle u_0\rangle_\Gamma(x_1) \;+\; F_{V_{2,3}}(\mathbf{X})\,\partial_{x_1}\langle\partial_{x_2} u_0\rangle_\Gamma(x_1). \end{aligned} \tag{3.27}$$

F_{V_0} and $F_{V_{1,2}}$ are given by (3.12)–(3.23), and,

$$\begin{aligned} F_{V_{2,1}}(\mathbf{X}) &= k_0^2 g(X_2) + \left(\widehat{k}^2 - k_0^2\right),\\ F_{V_{2,2}}(\mathbf{X}) &= g(X_2) + 2\,\partial_{X_1} V_{1,1}(\mathbf{X}),\\ F_{V_{2,3}}(\mathbf{X}) &= 2\,\partial_{X_1} V_{1,2}(\mathbf{X}),\\ g(X_2) &= \left(\frac{(X_2)^2}{2}(1-\chi(X_2))\right)'' \end{aligned}$$

To obtain formula (3.27), we have replaced Π_0 and Π_1 with their tensorial representations (3.11), (3.21), we have replaced $-\partial_{x_2}^2 u_0(x_1, 0^\pm)$ by $\partial_{x_1}^2 u_0(x_1, 0^\pm) + k_0^2 u_0(x_1, 0^\pm)$.

For a fixed $x_1 \in (-L, L)$, it is easily verified that $F_2(x_1, \cdot)$ belongs to $(\mathcal{V}^-(\mathcal{B}))'$. Then again, the existence of an exponentially decaying corrector $\Pi_2(x_1, \cdot) \in \mathcal{V}^+(\mathcal{B})$ results from the orthogonality condirtions $(\mathcal{C}_\mathcal{N})$–$(\mathcal{C}_\mathcal{D})$. As previously, enforcing the compatibility condition $(\mathcal{C}_\mathcal{N})$ provides the transmission condition for the jump of the normal trace of u_1 across Γ:

$$[\partial_{x_2} u_1]_\Gamma = \mathcal{N}_1 \langle u_0 \rangle_\Gamma + \mathcal{N}_2\, \partial_{x_1}^2 \langle u_0 \rangle_\Gamma + \mathcal{N}_3\, \partial_{x_1} \langle \partial_{x_2} u_0 \rangle_\Gamma, \tag{3.28}$$

where

$$\mathcal{N}_1 = -\int_\mathcal{B} F_{V_{2,1}}(\mathbf{X}), \quad \mathcal{N}_2 = -\int_\mathcal{B} F_{V_{2,2}} + \int_{\partial \widehat{\Omega}_{\rm hole}} V_{1,1} \mathbf{e}_1 \cdot \mathbf{n}, \quad \mathcal{N}_3 = -\int_\mathcal{B} F_{V_{2,3}} + \int_{\partial \widehat{\Omega}_{\rm hole}} V_{1,2} \mathbf{e}_1 \cdot \mathbf{n}. \tag{3.29}$$

Then, enforcing the compatibility condition $(\mathcal{C}_\mathcal{D})$ provides the jump $[u_2]_\Gamma$, and the existence of Π_2 is proved. Naturally, an explicit expression of $[u_2]_\Gamma$ and a tensorial representation of Π_2 can be written, but, for the sake of concision and the relevance of this article, we do not write it here.

Remark 3.4. In the case of a symmetric hole (*i.e.* $(X_1, X_2) \in \mathcal{B} \iff (1 - X_1, X_2) \in \mathcal{B}$), $V_{1,2}$ is symmetric with respect to the axis $X_1 = \frac{1}{2}$, and, consequently, $\mathcal{D}_1 = \mathcal{N}_3 = 0$.

4. Analysis of singular behavior of near field terms

The (first order) near field terms satisfy Laplace problems (see (2.23)) and might grow at infinity. This consideration motivates us to introduce two families of so-called *near field singularities* $S_n^\pm$ ($n \in \mathbb{N}$) that satisfy the following homogeneous near field problems

$$\begin{cases} -\Delta S_n^\pm = 0 & \text{in} \quad \widehat{\Omega}^\pm, \\ \partial_\mathbf{n} U = 0 & \text{on} \quad \partial \widehat{\Omega}^\pm \end{cases} \tag{4.1}$$

and behaves like $(R^\pm)^{\lambda_n}$ for large $R^\pm$.

4.1. Singular asymptotic blocks

In absence of the periodic layer, *i.e.* $\widehat{\Omega^\pm} = \widehat{\mathcal{K}^\pm}$, the function $\ln R^\pm$ and, for $n \in \mathbb{Z} \setminus \{0\}$, the functions $(R^+)^{\lambda_n} \cos(\lambda_n \theta^+)$ (resp. $(R^-)^{\lambda_n} \cos \lambda_n(\theta^- - \pi))$ are particular solutions of the homogeneous Laplace equation with Neumann boundary conditions on $\partial\widehat{\mathcal{K}^\pm}$. However, these functions do not satisfy the homogenous problem (4.1) since they do not fulfill the homogeneous Neumann boundary conditions on the obstacles of the periodic layer. Nevertheless, as done in Section 3 of [32], for any $n \in \mathbb{N}$, starting from the function

$$w_{0,0,\pm}(\ln R^\pm, \theta^\pm) = \ln R^\pm, \quad w_{n,0,+}(\theta^+) = \cos(\lambda_n \theta^+), \quad w_{n,0,-}(\theta^-) = \cos\left(\lambda_n(\theta^- - \pi)\right), \tag{4.2}$$

it is possible to build iteratively a so-called asymptotic block $\mathcal{U}_{n,p,+}$ (for any $p \in \mathbb{N}$) of the form

$$\begin{aligned} \mathcal{U}_{n,p,\pm} = \chi(R^\pm) \sum_{q=0}^{p} \big(& \chi_{\rm macro,\pm}(X_1^\pm, X_2^\pm) (R^\pm)^{\lambda_n - q} w_{n,q,\pm}(\ln R^\pm, \theta^+) \\ & + \chi_\mp(X_1^\pm) |X_1^+|^{\lambda_n - q} p_{n,q,\pm}(\ln|X_1^\pm|, X_1^\pm, X_2^\pm) \big), \end{aligned} \tag{4.3}$$

that 'almost' satisfies problem (4.1) for large $R^\pm$. In (4.3), the cut-off function χ_- has been defined in (2.15) and is represented on the right part of Figure 5. The cut-off function $\chi_{\rm macro,+}$, represented on the left part of Figure 5, is a smooth function that satisfies

$$\chi_{\rm macro,+}(X_1^+, X_2^+) = \chi(X_2^+), \quad X_1^+ < -1. \tag{4.4}$$

and the function $\chi_{\rm macro,-}(X_1^-, X_2^-) = \chi_{\rm macro,+}(-X_1^-, X_2^-)$.

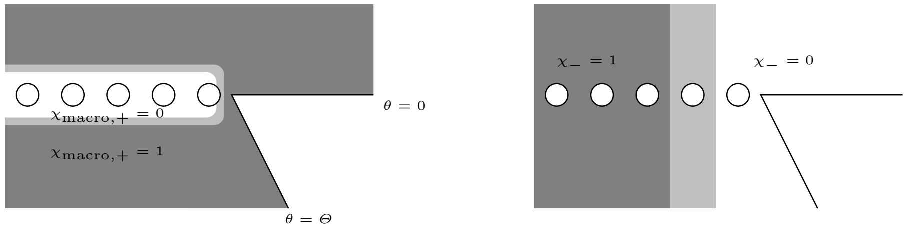


FIGURE 5. Graphic representation of the cut-off functions $\chi_{\text{macro},+}$ (*left part*) and χ_- (*right part*).

The definition of the functions $w_{n,q,\pm}$ and $p_{n,q,\pm}$ is given in Appendix C.2. The functions $w_{n,q,\pm}$ are polynomials in $\ln R^\pm$. The functions $p_{n,q,\pm}$ are polynomials in $\ln|X_1^\pm|$, periodic with respect to $X_1^\pm$ and exponentially decaying as $X_2^\pm$ tends to $\pm\infty$. The construction of these functions is done in such a way that their Laplacian and their Neumann trace become more and more decaying at infinity as $p \to +\infty$: more precisely, we can prove that , for any $\varepsilon > 0$,

$$\Delta \mathcal{U}_{n,p,\pm} = o\left((R^\pm)^{\lambda_n - p - 1 + \varepsilon}\right) \quad \text{and} \quad \partial_n \mathcal{U}_{n,p,\pm} = o\left((R^\pm)^{\lambda_n - p - 1 + \varepsilon}\right) \quad \text{on} \quad \partial\widehat{\Omega}^\pm. \tag{4.5}$$

We point out that the usage of the cut-off functions $\chi_{\text{macro},\pm}$, $\chi_\mp(X_1^\pm)$ and $\chi(R^\pm)$ in (4.3) is only a technical way to construct functions defined on the whole domain $\widehat{\Omega}^\pm$.

The asymptotic blocks $\mathcal{U}_{n,p,\pm}$ turn out to be useful to construct the near field singularities $S_n^\pm$ and to describe their asymptotic for large $R^\pm$.

4.2. The families $S_n^\pm$

We are now in a position to write the main result of this subsection, which proves the existence of the two families $S_n^\pm$ and give their behaviour at infinity.

Proposition 4.1. *Let* $n \in \mathbb{N}^*$, $p(n) = \max(1, 1 + \lceil \lambda_n \rceil)$, *and*

$$C_n^\pm := \begin{cases} -\dfrac{1}{\Theta}\left(\displaystyle\int_{\widehat{\Omega}^\pm} \Delta \mathcal{U}_{n,p(n),\pm} - \int_{\partial\widehat{\Omega}^\pm} \partial_{\mathbf{n}} \mathcal{U}_{n,p(n),\pm}\right) & \textit{if} \quad \lambda_n \in \mathbb{N}, \\ 0 & \textit{otherwise.} \end{cases} \tag{4.6}$$

There exists a unique function $S_n^\pm \in \mathrm{H}^1_{\text{loc}}(\widehat{\Omega}^\pm)$ *satisfying the homogeneous problem* (4.1) *such that the function*

$$\tilde{S}_n^\pm = S_n^\pm - \mathcal{U}_{n,1+\lceil\lambda_n\rceil,\pm} - C_n^\pm \mathcal{U}_{0,1,\pm},$$

tends to 0 *as* $R^\pm$ *goes to infinity. Moreover,* $S_n^\pm$ *admits the following block decomposition for large* $R^\pm$*: for any* $k \in \mathbb{N}^*$,

$$\begin{aligned} S_n^\pm &= \mathcal{U}_{n,1+\lceil\lambda_{n+k}\rceil,\pm} + \sum_{m=1}^{k} \mathscr{L}_{-m}(S_n^\pm)\mathcal{U}_{-m,1+\lceil\lambda_{k-m}\rceil,\pm} + o\left((R^\pm)^{-\lambda_k}\right) \\ &\qquad \textit{if} \quad \lambda_n \notin \mathbb{N}, \\ S_n^\pm &= \mathcal{U}_{n,1+\lceil\lambda_{n+k}\rceil,\pm} + \sum_{m=1}^{k} \mathscr{L}_{-m}(S_n^\pm)\mathcal{U}_{-m,1+\lceil\lambda_{k-m}\rceil,\pm} + o\left((R^\pm)^{-\lambda_k}\right) \\ &\quad + C_n^\pm \mathcal{U}_{0,1+\lceil\lambda_k\rceil,\pm} \qquad \textit{if} \quad \lambda_n \in \mathbb{N}. \end{aligned} \tag{4.7}$$

In the previous proposition $\lceil a \rceil$ denotes the ceiling of a real number a. As demonstrated in 6, for $\lambda_n \notin \mathbb{N}$, the quantity $\int_{\widehat{\Omega}^\pm} \Delta \mathcal{U}_{n,p(n),\pm} - \int_{\partial\widehat{\Omega}^\pm} \partial_{\mathbf{n}} \mathcal{U}_{n,p(n),\pm}$ vanishes (Lem. B.3), which explains why $C_n = 0$ in this case.

The asymptotic formula (4.7) shows that, for large $R^\pm$, $S_n^\pm$ can be decomposed as a sum of 'macroscopic' contributions of the form $(R^\pm)^{\lambda_m - q} s_{m,q}(\theta^\pm, \ln R^\pm)$ modulated by exponentially decaying (in $X_2^\pm$) periodic (in X_1) functions of the form $|X_1^\pm|^{\lambda_m - q} p_{m,q}(\ln |X_1^\pm|, X_1^\pm, X_2^\pm)$ in the vicinity if the periodic layer.

Proof. The existence of the function $\tilde{S}_n^\pm$ results from the application of Proposition B.1 (or Cor. 3.23 of Ref. [13]), noting that the compatibility condition (B.3) (due to the Neumann boundary condition) is satisfied: for $\lambda_n \in \mathbb{N}$, the addition of $C_n^\pm \mathcal{U}_{0,1,\pm}$ is required in order to fulfill this condition (note that, as shown in the proof of Lemma B.2, $\int_{\widehat{\Omega}^\pm} \Delta \mathcal{U}_{0,1,\pm} - \int_{\partial \widehat{\Omega}^\pm} \partial_{\mathbf{n}} \mathcal{U}_{0,1,\pm} = \Theta$). The asymptotic (4.7) then follows from the application of the results of Nazarov [32] (see also Sect. 4 of Ref. [21] for a detailed description of this decomposition). A rigorous estimation of the remainder $o\left((R^\pm)^{-k}\right)$ can be done through the introduction of non-uniform weigthed Sobolev spaces [32]. □

Remark 4.2. We point out that it is not possible to construct a function in $\mathrm{H}^1_{\mathrm{loc}}(\widehat{\Omega}^\pm)$ satisfying the homogeneous problem (4.1) and behaving like $\ln R^\pm$ at infinity (See Lem. B.2 in Appendix B.2).

We complete the family $(S_n^\pm)_{n>0}$ defined in Proposition 4.1 defining the function

$$S_0^\pm = 1, \tag{4.8}$$

which obviously satisfies the homogeneous Laplace equation on $\widehat{\Omega}^\pm$.

5. Iterative construction of the first terms of the expansion

In this section, we propose a step by step iterative procedure to construct the first terms of the expansion up to order δ^2. Since $\theta \in (\pi, 2\pi)$, $0 < \lambda_1 < 1 < \lambda_2 < \lambda_1 + 1 < \lambda_3$. It follows that we shall consider the indexes (n, q) (associated with increasing powers of $\delta^{\lambda_n + q}$) in the following order: $(0,0)$, $(1,0)$, $(0,1)$, $(2,0)$, $(1,1)$ and, in the case of $\Theta > \frac{3\pi}{2}$, the couple $(3,0)$.

5.1. Construction of the limit terms $u_{0,0}^\delta$, $\Pi_{0,0}^\delta$ and $U_{0,0,\pm}^\delta$

The macroscopic term $u_{0,0}^\delta$ and the near field terms $U_{0,0,\pm}^\delta$ satisfy the following problems

$$\left\{\begin{array}{rll} -\Delta u_{0,0}^\delta - k_0^2 u_{0,0}^\delta = 0 & \text{in} & \Omega_{\mathrm{T}} \cup \Omega_{\mathrm{B}}, \\ \left[u_{0,0}^\delta\right]_\Gamma = \left[\partial_{x_2} u_{0,0}^\delta\right]_\Gamma = 0 & \text{on} & \Gamma, \\ \nabla u_{0,0}^\delta \cdot \mathbf{n} = 0 & \text{on} & \Gamma_N, \\ \nabla u_{0,0}^\delta \cdot \mathbf{n} - \imath k_0 u_{0,0}^\delta = 0 & \text{on} & \Gamma_{R,+}, \\ \nabla u_{0,0}^\delta \cdot \mathbf{n} - \imath k_0 u_{0,0}^\delta = -2\imath k_0 & \text{on} & \Gamma_{R,-}, \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{rll} -\Delta U_{0,0,\pm}^\delta = 0 & \text{in} & \widehat{\Omega}^\pm, \\ \partial_{\mathbf{n}} U_{0,0,\pm}^\delta = 0 & \text{on} & \partial\widehat{\Omega}^\pm, \end{array}\right. \tag{5.1}$$

coupled by the matching condition (2.26) (written here by only identifying the term of order 0 in the two series)

$$U_{0,0,\pm}^\delta\left(\frac{x}{\delta}\right) \approx u_{0,0}^\delta(x). \tag{5.2}$$

5.1.1. *Construction of the macroscopic term $u_{0,0}^\delta$*

As well-known, the limit term $u_{0,0}^\delta$ is regular. In fact, $u_{0,0}^\delta = u_{0,0}$ (it does not depend on δ) is defined as the unique solution of (5.1)-(*left*) belonging to $\mathrm{H}^1(\Omega)$. The absence of singular behavior in $u_{0,0}$ can be understood by the following *formal* argument: a singular term in $u_{0,0}$ of the form r^{-s} would necessary counterbalance a term of the form $(R^\pm)^{-s}$, $s > 0$ in $U_{0,0}^\delta$, which, written in terms of the macroscopic variable $r^\pm$, become $\delta^s (r^\pm)^{-s}$, and can therefore not be canceled at order 0. Similarly, due to Remark 4.2 a singular term of the form $\ln r$ is excluded at this stage.

Remark 5.1. More generally, the previous argument shows that for any $(n,q)\in\mathbb{N}^2$, a singular term in $u^\delta_{n,q}$ of the form r^{-s} cannot counterbalance a regular term of the near field term of the same order $U^\delta_{n,q}$.

It is well-known that $u_{0,0}$ admits the following expansion in the matching zones

$$u_{0,0}(r^\pm,\theta^\pm) = \ell_0^\pm(u_{0,0})J_0(k_0 r^\pm) + \sum_{m=1}^{\infty} \ell_m^\pm(u_{0,0})J_{\lambda_m}(k_0 r^\pm)w_{m,0,\pm}(\theta^\pm), \tag{5.3}$$

where the functions $w_{m,0,\pm}$ are defined by (4.2), the functions J_{λ_m} are the Bessel functions of first kind (see *e.g.* Sect. 9.1 of Ref. [2]) and the quantities $\ell_m^\pm(u_{0,0})$ are complex constants. Using the radial decomposition of J_{λ_m}, we see that

$$\begin{aligned} u_{0,0}(r^\pm,\theta^\pm) =& \ell_0^\pm(u_{0,0}) + \frac{\ell_1^\pm(u_{0,0})(k_0/2)^{\lambda_1}}{\Gamma(\lambda_1+1)}\left(r^\pm\right)^{\lambda_1} w_{1,0,\pm}(\theta^\pm) \\ &+ \frac{\ell_2^\pm(u_{0,0})(k_0/2)^{\lambda_2}}{\Gamma(\lambda_2+1)}\left(r^\pm\right)^{\lambda_2} w_{2,0,\pm}(\theta^\pm) + \frac{\ell_3^\pm(u_{0,0})(k_0/2)^{\lambda_3}}{\Gamma(\lambda_3+1)}\left(r^\pm\right)^{\lambda_3} w_{3,0,\pm}(\theta^\pm) + O(r^2). \end{aligned} \tag{5.4}$$

5.1.2. Construction of $U^\delta_{0,0,\pm}$

We now turn to the definition of the near field term $U^\delta_{0,0,\pm}$. In view of (5.4), writting the matching conditions (5.2) in term of the microscopic variable gives $\ell_0^\pm(u_{0,0}) \approx U^\delta_{0,0,\pm}$. As a result, $U^\delta_{0,0,\pm}$ should behave like $\ell_0^\pm(u_{0,0})$ in the matching zones (*i.e.* for $R^\pm$ large). Consequently, it is natural to define $U^\delta_{0,0,\pm}$ as

$$U^\delta_{0,0,\pm} = U_{0,0,\pm} = \ell_0^\pm(u_{0,0}). \tag{5.5}$$

5.1.3. Construction of the periodic corrector $\Pi^\delta_{0,0}$

Finally, using then relations (3.11), the periodic boundary layer corrector is

$$\Pi^\delta_{0,0}(x_1,\mathbf{X}) = \Pi_{0,0}(x_1,\mathbf{X}) = \langle u_0\rangle_\Gamma(x_1)\,(1-\chi(X_2)). \tag{5.6}$$

5.2. Construction of the terms $u^\delta_{1,0}$, $\Pi^\delta_{1,0}$ and $U^\delta_{1,0,\pm}$

Reminding that $u^\delta_{1,0}$ fulfills the jump conditions (2.18) (see also Sect. 3), $u^\delta_{1,0}$ and $U^\delta_{1,0,\pm}$ satisfy

$$\begin{cases} -\Delta u^\delta_{1,0} - k_0^2 u^\delta_{1,0} = 0 & \text{in} \quad \Omega_{\mathrm{T}}\cup\Omega_{\mathrm{B}}, \\ \left[u^\delta_{1,0}\right]_\Gamma = \left[\partial_{x_2} u^\delta_{1,0}\right]_\Gamma = 0 & \text{on} \quad \Gamma, \\ \nabla u^\delta_{1,0}\cdot\mathbf{n} = 0 & \text{on} \quad \Gamma_N, \\ \nabla u^\delta_{1,0}\cdot\mathbf{n} - \imath k_0 u^\delta_{1,0} = 0 & \text{on} \quad \Gamma_R, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta U^\delta_{1,0,\pm} = 0 & \text{in} \quad \widehat{\Omega}^\pm, \\ \partial_{\mathbf{n}} U^\delta_{1,0,\pm} = 0 & \text{on} \quad \partial\widehat{\Omega}^\pm, \end{cases} \tag{5.7}$$

together with the matching condition (2.26) written up to order δ^{λ_1}. Outside the thin periodic layer, and thanks to (5.4) and (5.5), one can verify that this matching condition can be rewritten as

$$\frac{\ell_1^\pm(u_{0,0})(k_0/2)^{\lambda_1}}{\Gamma(\lambda_1+1)}\left(r^\pm\right)^{\lambda_1} w_{1,0,\pm}(\theta^\pm) + \delta^{\lambda_1} u^\delta_{1,0}(r^\pm,\theta^\pm) \approx \delta^{\lambda_1} U^\delta_{1,0,\pm}\left(\frac{r^\pm}{\delta},\theta^\pm\right)\cdot \tag{5.8}$$

Analogously to Section (5.1), we will start with the construction of the macroscopic far field $u^\delta_{1,0}$. Then, we will define the near field term $U^\delta_{1,0}$ and, finally, we will define the associated boundary layer corrector $\Pi^\delta_{1,0}$.

5.2.1. Construction of the macroscopic term $u^\delta_{1,0}$

First, it is reasonable to construct $u^\delta_{1,0}$ as a regular function. Indeed, a singular behaviour in $u^\delta_{1,0,\pm}$ of the form r^{-s} or (resp. $\ln r$) would counterbalance a regular term of the right hand side of (5.8). This singular term would necessary come from a regular term in $U^\delta_{1,0,\pm}$, which, thanks to Remark 5.1 (resp. Rem. 4.2) cannot be cancelled at this stage. It is then reasonable (see Prop. 3.1) to define $u^\delta_{1,0}$ as

$$u^\delta_{1,0} = u_{1,0} := 0. \tag{5.9}$$

5.2.2. Construction of $U^\delta_{1,0,\pm}$

Taking into account (5.9) and writing the matching condition (5.8) in term of the microscopic variables gives

$$\delta^{\lambda_1}\frac{\ell_1^\pm(u_{0,0})(k_0/2)^{\lambda_1}}{\Gamma(\lambda_1+1)}\left(R^\pm\right)^{\lambda_1} w_{1,0,\pm}(\theta^\pm) \approx \delta^{\lambda_1} U^\delta_{1,0,\pm}\left(\frac{r^\pm}{\delta},\theta^\pm\right).$$

Then, $U^\delta_{1,0,\pm}$ has to grow like $\frac{\ell_1^\pm(u_{0,0})(k_0/2)^{\lambda_1}}{\Gamma(\lambda_1+1)}\left(R^\pm\right)^{\lambda_1} w_{1,0,\pm}(\theta^\pm)$ towards infinity. Of course, the term $\left(R^\pm\right)^{\lambda_1} w_{1,0,\pm}(\theta^\pm)$ does not satisfies the homogeneous problem (5.7)-(right). However, Proposition 4.1 ensures the existence of a function $S_1^\pm$, that satisfies (5.7)-(*right*) and behaves like $\left(R^\pm\right)^{\lambda_1} w_{1,0,\pm}(\theta^\pm)$ at infinity. Then, it is natural to define $U^\delta_{1,0,\pm}$ as

$$U^\delta_{1,0,\pm} = U_{1,0,\pm} = \frac{\ell_1^\pm(u_{0,0})(k_0/2)^{\lambda_1}}{\Gamma(\lambda_1+1)} S_1^\pm. \tag{5.10}$$

In view of the asymptotic formula (4.7) for $S_1^\pm$ ($\lambda_1 \notin \mathbb{N}$), outside the periodic layer, the asymptotic of $U_{1,0,\pm}$ is given by

$$\begin{aligned} U_{1,0,\pm} = \frac{\ell_1^\pm(u_{0,0})(k_0/2)^{\lambda_1}}{\Gamma(\lambda_1+1)}\Big\{&(R^\pm)^{\lambda_1} w_{1,0,\pm}(\theta^\pm) + (R^\pm)^{\lambda_1-1} w_{1,1,\pm}(\theta^\pm)\\ &+ (R^\pm)^{-\lambda_1}\mathscr{L}_{-1}(S_1^\pm) w_{-1,0,\pm}(\theta^\pm)\\ &+ (R^\pm)^{-\lambda_2}\mathscr{L}_{-2}(S_1^\pm) w_{-2,0,\pm}(\theta^\pm)\Big\} + O(R^{\lambda_1-2}\ln R). \end{aligned} \tag{5.11}$$

Here we use the fact that $\lambda_1 - 1$ is not a mutliple of λ_1 so that $w_{1,1,\pm}$ is independent of $\ln R^\pm$.

5.2.3. Construction of the periodic corrector $\Pi^\delta_{1,0}$

Thanks to the relation (3.11), and since $u^\delta_{1,0}$ vanishes, its associated boundary corrector also vanishes, and we have

$$\Pi^\delta_{1,0} = \Pi_{1,0} = 0. \tag{5.12}$$

5.3. Construction of the terms $u^\delta_{0,1}$ and $U^\delta_{0,1,\pm}$

Reminding that $u^\delta_{0,1}$ fulfills the jump conditions (2.19) (*cf.* Appendix 3), $u^\delta_{0,1}$ and $U^\delta_{0,1,\pm}$ satisfy the following problems

$$\left\{\begin{aligned} -\Delta u^\delta_{0,1} - k_0^2 u^\delta_{0,1} &= 0 && \text{in} && \Omega_{\mathrm{T}}\cup\Omega_{\mathrm{B}},\\ \left[u^\delta_{0,1}\right]_\Gamma &= g_{0,1} && \text{on} && \Gamma,\\ \left[\partial_{x_2} u^\delta_{0,1}\right]_\Gamma &= h_{0,1} && \text{on} && \Gamma,\\ \nabla u^\delta_{0,1}\cdot\mathbf{n} &= 0 && \text{on} && \Gamma_N,\\ \nabla u^\delta_{0,1}\cdot\mathbf{n} - \imath k_0 u^\delta_{0,1} &= 0 && \text{on} && \Gamma_R, \end{aligned}\right. \quad \text{and} \quad \left\{\begin{aligned} -\Delta U^\delta_{0,1,\pm} &= 0 && \text{in} && \widehat{\Omega}^\pm,\\ \partial_{\mathbf{n}} U^\delta_{0,1,\pm} &= 0 && \text{on} && \partial\widehat{\Omega}^\pm, \end{aligned}\right. \tag{5.13}$$

with $g_{0,1} = \mathcal{D}_1\, \partial_{x_1} \langle u_{0,0}\rangle_\Gamma \,+\, \mathcal{D}_2\, \langle \partial_{x_2} u_{0,0}\rangle_\Gamma$ and $h_{0,1} = \mathcal{N}_1\, \langle u_{0,0}\rangle_\Gamma + \mathcal{N}_2\, \partial^2_{x_1} \langle u_{0,0}\rangle_\Gamma + \mathcal{N}_3\, \partial_{x_1} \langle \partial_{x_2} u_{0,0}\rangle_\Gamma$. Thanks to (5.4)–(5.5)–(5.11), the matching condition (2.26) written up to order δ, can be written as

$$\delta\, u^\delta_{0,1}(r^\pm,\theta^\pm) \approx \delta \frac{\ell_1^\pm(u_{0,0})(k_0/2)^{\lambda_1}}{\Gamma(\lambda_1+1)} (r^\pm)^{\lambda_1-1} w_{1,1,\pm}(\theta^\pm) + \delta\, U^\delta_{0,1,\pm}\left(\frac{r^\pm}{\delta},\theta^\pm\right) \tag{5.14}$$

outside the periodic layer. Analogously to Sections 5.1 and 5.2, we will start with the construction of the macroscopic far field $u^\delta_{0,1}$. Then, we will define the near field term $U^\delta_{0,1}$. As we have already seen in the previous sections, we can rebuild *a posteriori* the boundary layer corrector $\Pi^\delta_{0,1}$, but for the sake of brevity, from now on, we omit this reconstruction.

5.3.1. Construction of the macroscopic term $u^\delta_{0,1}$

First, we remark that $u^\delta_{0,1}$ should contain a singular contribution of order $(r^\pm)^{\lambda_1-1}$ in order to cancel out the first term in the right-hand side of (5.14). In fact, we shall see (and this is a crucial point) that this singular contribution appears to be a consequence of the transmission condition in (5.13)-(*left*). Besides, according to Remark 5.1, $u^\delta_{0,1}$ has no other singular behavior (any other singular behavior would stem from $U^\delta_{0,1,\pm}$ and could not be compensated at this stage).

Let us now investigate problem (5.13)-(*right*). In view of the asymptotic behaviour of $u_{0,0}$ (5.4) in the vicinity of the two corners, the functions $g_{0,1}$ and $h_{0,1}$ blow up at the extremities of Γ. Indeed,

$$\begin{aligned} g_{0,1}(r^\pm) &= \ell_1^\pm(u_{0,0})\, \frac{(k_0/2)^{\lambda_1}}{\Gamma(\lambda_1+1)}\, \left(\mp\, \lambda_1 \mathcal{D}_1 \langle w_{1,0,\pm}\rangle \mp \mathcal{D}_2\, \langle \partial_{\theta^\pm} w_{1,0,\pm}\rangle\right)\, (r^\pm)^{\lambda_1-1} \\ &\quad + O((r^\pm)^{\lambda_2-1}) = \ell_1^\pm(u_{0,0}) \frac{(k_0/2)^{\lambda_1}}{\Gamma(\lambda_1+1)} [(r^\pm)^{\lambda_1-1} w_{1,1,\pm}] + O((r^\pm)^{\lambda_2-1}), \end{aligned}$$

where the functions $w_{1,1,\pm}$ are defined in Appendix (C.2) (note that $V_{1,1} = W_1^{\mathrm{t}}$ and $V_{1,2} = W_1^{\mathrm{n}}$). Similarly,

$$h_{0,1}(r^\pm) = \ell_1^\pm(u_{0,0}) \frac{(k_0/2)^{\lambda_1}}{\Gamma(\lambda_1+1)} \left[\partial_{x_2} \left((r^\pm)^{\lambda_1-1} w_{1,1,\pm}\right)\right] + O((r^\pm)^{\lambda_2-2}).$$

We shall construct $u^\delta_{0,1}$ by lifting explicitly the singular part of $g_{0,1}$ and $h_{0,1}$. To do so, we consider the function

$$\mathbf{J}^\pm_{1,-1}(r^\pm,\theta^\pm) = J_{\lambda_1-1}(k_0 r^\pm)\, w_{1,1,\pm}(\theta^\pm) \tag{5.15}$$

that satisfies the homogeneous Helmholtz equation in $\Omega_{\mathrm{T}} \cup \Omega_{\mathrm{B}}$. According to the asymptotic of the Bessel function of the first kind J_{λ_1-1} (using Eq. (9.1.10) of Ref. [2]), we notice that $g_{0,1} \approx \frac{k_0}{2\lambda_1} \ell_1^\pm(u_{0,0})[\mathbf{J}^\pm_{1,-1}]$ and $h_{0,1} \approx \frac{k_0}{2\lambda_1} \ell_1^\pm(u_{0,0})[\partial_{x_2}\mathbf{J}^\pm_{1,-1}]$ in the neighborhood of the extremities of Γ. It means that $\frac{k_0}{2\lambda_1} \ell_1^\pm(u_{0,0})\mathbf{J}^\pm_{1,-1}$ is potentially a good candidate to lift the singular parts of the $g_{0,1}$ and $h_{0,1}$. It is then natural to define $u^\delta_{0,1}$ as

$$u^\delta_{0,1} = u_{0,1} := \frac{k_0}{2\lambda_1} \left(\ell_1^+(u_{0,0})\chi_L^+ \mathbf{J}^+_{1,-1} + \ell_1^-(u_{0,0})\chi_L^- \mathbf{J}^-_{1,-1}\right) + \hat{u}_{0,1}, \tag{5.16}$$

where $\chi_L^\pm(\mathbf{x}) = 1 - \chi(2r^\pm/L)$ and the function $\hat{u}_{0,1}$ is the unique solution in $\mathrm{H}^1(\Omega_{\mathrm{T}} \cup \Omega_{\mathrm{B}})$ of the following problem:

$$\left\{\begin{aligned} -\Delta \hat{u}_{0,1} - k_0^2 \hat{u}_{0,1} &= \hat{f}_{0,1} && \text{in} \quad \Omega_{\mathrm{T}} \cup \Omega_{\mathrm{B}}, \\ [\hat{u}_{0,1}]_\Gamma &= \hat{g}_{0,1} && \text{on} \quad \Gamma, \\ [\partial_{x_2} \hat{u}_{0,1}]_\Gamma &= \hat{h}_{0,1} && \text{on} \quad \Gamma, \\ \nabla \hat{u}_{0,1} \cdot \mathbf{n} &= 0 && \text{on} \quad \Gamma_N, \\ \nabla \hat{u}_{0,1} \cdot \mathbf{n} - \imath k_0 \hat{u}_{0,1} &= 0 && \text{on} \quad \Gamma_R, \end{aligned}\right. \quad \text{with} \quad \begin{aligned} \hat{f}_{0,1} &= \frac{k_0}{2\lambda_1} \sum_\pm \ell_1^\pm(u_{0,0})[\Delta, \chi_L^\pm]\mathbf{J}^\pm_{1,-1}, \\ \hat{g}_{0,1} &= g_{0,1} - \frac{k_0}{2\lambda_1} \sum_\pm \ell_1^\pm(u_{0,0})\chi_L^\pm[\mathbf{J}^\pm_{1,-1}], \\ \hat{h}_{0,1} &= h_{0,1} - \frac{k_0}{2\lambda_1} \sum_\pm \ell_1^\pm(u_{0,0})\chi_L^\pm[\partial_{x_2}\mathbf{J}^\pm_{1,-1}]. \end{aligned} \tag{5.17}$$

Here $[\Delta, \chi_L^{\pm}]$ denotes the commutator operator given by $[\Delta, \chi_L^{\pm}]v = v\Delta\chi_L^{\pm} + 2\nabla\chi_L^{\pm} \cdot \nabla v$ (for any sufficiently smooth function v). The existence and uniqueness of $\hat{u}_{0,1}$ in $\mathrm{H}^1(\Omega_\mathrm{T} \cup \Omega_\mathrm{B})$ is ensured by Proposition 3.1 since $\hat{f}_{0,1}$ is compactly supported, $\hat{g}_{0,1} \in \mathrm{H}^{1/2}(\Gamma)$ and $\hat{h}_{0,1} \in \mathrm{L}^2(\Gamma)$.

Moreover, the asymptotic expansion of $u_{0,1}$ in the matching zones is given by

$$\begin{aligned} u_{0,1}(r^{\pm}, \theta^{\pm}) = {} & \ell_1^{\pm}(u_{0,0}) \frac{(k_0/2)^{\lambda_1}}{\Gamma(\lambda_1+1)} (r^{\pm})^{\lambda_1 - 1} w_{1,1,\pm}(\theta^{\pm}) + \ell_0^{\pm}(u_{0,1}) \\ & + \ell_2^{\pm}(u_{0,0}) \frac{(k_0/2)^{\lambda_2}}{\Gamma(\lambda_2+1)} (r^{\pm})^{\lambda_2 - 1} w_{2,1,\pm}(\theta^{\pm}) \\ & + \ell_1^{\pm}(u_{0,1}) \frac{(k_0/2)^{\lambda_1}}{\Gamma(\lambda_1+1)} \left(r^{\pm}\right)^{\lambda_1} w_{1,0,\pm}(\theta^{\pm}) \\ & + \ell_3^{\pm}(u_{0,0}) \frac{(k_0/2)^{\lambda_3}}{\Gamma(\lambda_3+1)} (r^{\pm})^{\lambda_3 - 1} w_{3,1,\pm}(\theta^{\pm}) + O\left(r^{\pm}\right), \end{aligned} \tag{5.18}$$

where the quantities $\ell_0^{\pm}(u_{0,1})$ and $\ell_1^{\pm}(u_{0,1})$ are complex constants. Obviously, the first term of (5.18) compensates the first term of the right hand side of the matching condition (5.14). The presence of the terms in factor of $\ell_2^{\pm}(u_{0,0})$ and $\ell_3^{\pm}(u_{0,0})$ results from the transmission condition (see Sect. 3.3 in Ref. [21] for a similar asymptotic). If $\Theta < \frac{3\pi}{2}$ and so $\lambda_3 > 2$, the last listed term of the expansion (5.18) is negligible with respect to $O(r^{\pm})$.

5.3.2. Construction of $U_{0,1,\pm}^{\delta}$

Plugging the asymptotic expansion (5.18) of $u_{0,1}$ into the matching condition (5.14) written in term of the microscopic variable (ignoring the terms in factor of δ^s, $s > 1$, which will be taken into account latter), we obtain

$$\delta U_{0,1,\pm}^{\delta}(R^{\pm}, \theta^{\pm}) \approx \delta \ell_0^{\pm}(u_{0,1}).$$

We then see that $U_{0,1,\pm}^{\delta}$ should behave like $\ell_0^{\pm}(u_{0,1})$ at infinity. Thus, we define $U_{1,0,\pm}^{\delta}$ as

$$U_{0,1,\pm}^{\delta} = U_{0,1,\pm} = \ell_0^{\pm}(u_{0,1}). \tag{5.19}$$

5.4. Construction of the terms $u_{2,0}^{\delta}$ and $U_{2,0,\pm}^{\delta}$

Reminding that $u_{2,0}^{\delta}$ fulfills the jump conditions (2.18), $u_{2,0}^{\delta}$ and $U_{2,0,\pm}^{\delta}$ satisfy

$$\left\{ \begin{aligned} -\Delta u_{2,0}^{\delta} - k_0^2 u_{2,0}^{\delta} &= 0 \quad \text{in} \quad \Omega_\mathrm{T} \cup \Omega_\mathrm{B}, \\ \left[u_{2,0}^{\delta}\right]_{\Gamma} = \left[\partial_{x_2} u_{2,0}^{\delta}\right]_{\Gamma} &= 0 \quad \text{on} \quad \Gamma, \\ \nabla u_{2,0}^{\delta} \cdot \mathbf{n} &= 0 \quad \text{on} \quad \Gamma_N, \\ \nabla u_{2,0}^{\delta} \cdot \mathbf{n} - \imath k_0 u_{2,0}^{\delta} &= 0 \quad \text{on} \quad \Gamma_R, \end{aligned} \right. \quad \text{and} \quad \left\{ \begin{aligned} -\Delta U_{2,0,\pm}^{\delta} &= 0 \quad \text{in} \quad \widehat{\Omega}^{\pm}, \\ \partial_{\mathbf{n}} U_{2,0,\pm}^{\delta} &= 0 \quad \text{on} \quad \partial\widehat{\Omega}^{\pm}, \end{aligned} \right. \tag{5.20}$$

together with the matching condition (2.26) written up to order δ^{λ_2}, which, outside the thin periodic layer gives

$$\begin{aligned} & \frac{\ell_2^{\pm}(u_{0,0})(k_0/2)^{\lambda_2}}{\Gamma(\lambda_2+1)} \Big(\left(r^{\pm}\right)^{\lambda_2} w_{2,0,\pm}(\theta^{\pm}) + \delta (r^{\pm})^{\lambda_2 - 1} w_{2,1,\pm}(\theta^{\pm}) \Big) + \delta^{\lambda_2} u_{2,0}^{\delta} \\ & \qquad \approx \delta^{\lambda_2} \frac{\ell_1^{\pm}(u_{0,0})(k_0/2)^{\lambda_1}}{\Gamma(\lambda_1+1)} \mathscr{L}_{-1}(S_1^{\pm}) \left(r^{\pm}\right)^{-\lambda_1} w_{-1,0,\pm}(\theta^{\pm}) + \delta^{\lambda_2} U_{2,0}^{\delta}. \end{aligned} \tag{5.21}$$

Here, we used the asymptotic expansions (5.3)−(5.18) for the far field terms $u_{0,0}$ and $u_{0,1}$, the definition (5.5)- and (5.19) of the near field terms $U_{0,0}$ and $U_{0,1}$, and the asymptotic expansion (5.11) of $U_{1,0}$. Predicably, the matching process carried out in the previous subsections makes the expression of (5.21) relatively simple.

5.4.1. Construction of the macroscopic term $u^\delta_{2,0}$

In view of the right-hand side of (5.21) (and, here again, Rem. 5.1), we remark that $u^\delta_{2,0,\pm}$ should have a single singular contribution of the form $\frac{\ell_1^\pm(u_{0,0})(k_0/2)^{\lambda_1}}{\Gamma(\lambda_1+1)}\mathscr{L}_{-1}(S_1^\pm)\,(r^\pm)^{-\lambda_1}\,w_{-1,0,\pm}(\theta^\pm)$. As done for $u_{0,1}$ in Section 5.3, we shall construct $u^\delta_{2,0,\pm}$ by lifting explicitly its singular behaviour. We remark that $(r^\pm)^{-\lambda_1}w_{-1,0,\pm}$ does not satisfy the homogeneous Helmholtz equation in $\Omega_{\mathrm{T}}\cup\Omega_{\mathrm{B}}$ (by construction it satisfies the homogeneous Laplace equation). However, we can substitute it with a multiple of the function

$$\mathbf{Y}_1^\pm(r^\pm,\theta^\pm) = Y_{\lambda_1}(k_0 r^\pm)w_{-1,0,\pm}(\theta^\pm), \tag{5.22}$$

which behaves like $-\frac{\Gamma(\lambda_1)}{\pi}\left(\frac{k_0}{2}\right)^{-\lambda_1}(r^\pm)^{-\lambda_1}\,w_{-1,0,\pm}(\theta^\pm)$ in the vicinity of the two corners and satisfies the homogeneous Helmholtz equation in $\Omega_{\mathrm{T}}\cup\Omega_{\mathrm{B}}$. It this then natural to define $u^\delta_{2,0}$ as

$$u^\delta_{2,0} = u_{2,0} := \sum_{\pm}\ell^\pm_{2,0,-1}(u_{0,0})\chi_L^\pm\mathbf{Y}_1^\pm + \hat{u}_{2,0}, \quad \ell^\pm_{2,0,-1}(u_{0,0}) := -\pi\left(\frac{\ell_1^\pm(u_{0,0})\mathscr{L}_{-1}(S_1^\pm)}{\Gamma(\lambda_1)\Gamma(\lambda_1+1)}\right)\left(\frac{k_0}{2}\right)^{\lambda_2}, \tag{5.23}$$

the cut-off functions $\chi_L^\pm$ being defined in (5.16) and the function $\hat{u}_{2,0}$ being the only $\mathrm{H}^1(\Omega)$ solution to the following problem:

$$\begin{cases} -\Delta\hat{u}_{2,0} - k_0^2\hat{u}_{2,0} = \hat{f}_{2,0} & \text{in} \quad \Omega_{\mathrm{T}}\cup\Omega_{\mathrm{B}},\\ [\hat{u}_{2,0}]_\Gamma = [\partial_{x_2}\hat{u}_{2,0}]_\Gamma = 0 & \text{on} \quad \Gamma,\\ \nabla\hat{u}_{2,0}\cdot\mathbf{n} = 0 & \text{on} \quad \Gamma_N,\\ \nabla\hat{u}_{2,0}\cdot\mathbf{n} - \imath k_0\hat{u}_{2,0} = 0 & \text{on} \quad \Gamma_R,\end{cases} \qquad \hat{f}_{2,0} := \sum_{\pm}\ell^\pm_{2,0,-1}(u_{0,0})[\Delta,\chi_L^\pm]\mathbf{Y}_1^\pm. \tag{5.24}$$

The function $\hat{f}_{2,0}$ being in $\mathrm{L}^2(\Omega)$ (it is compactly supported), Proposition 3.1 ensures the well-posedness of (5.24) in $\mathrm{H}^1(\Omega)$. In the vicinity of the two corners, $\hat{f}_{2,0}$ vanishes, so that

$$\hat{u} = \sum_{m=0}^{\infty}\ell_m^\pm(u_{2,0})J_{\lambda_m}(k_0 r^\pm)w_{m,0,\pm}(\theta^\pm), \quad \ell_m^\pm(u_{2,0})\in\mathbb{C}.$$

Using the radial decomposition of the Bessel functions, coupled with the formula

$$Y_{\lambda_1}(k_0 r) = \frac{J_{\lambda_1}(k_0 r)\cos(\lambda_1\pi) - J_{-\lambda_1}(k_0 r)}{\sin(\lambda_1\pi)} \quad \text{(see Eq. (9.1.2) of Ref. [2])},$$

we see that

$$\begin{aligned} u_{2,0}(r^\pm,\theta^\pm) = {}& \frac{\ell_1^\pm(u_{0,0})(k_0/2)^{\lambda_1}}{\Gamma(\lambda_1+1)}\mathscr{L}_{-1}(S_1^\pm)\left(r^\pm\right)^{-\lambda_1}w_{-1,0,\pm}(\theta^\pm) + \ell_0^\pm(u_{2,0})\\ & + \ell_{2,0,1}(u_{2,0})\left(r^\pm\right)^{\lambda_1}w_{1,0,\pm}(\theta^\pm) + O(r^{\max(\lambda_2,-\lambda_1+2)}), \end{aligned} \tag{5.25}$$

where

$$\ell_{2,0,1}(u_{2,0}) = \frac{\ell_1^\pm(u_{2,0})(k_0/2)^{\lambda_1}}{\Gamma(\lambda_1+1)} + \frac{\ell^\pm_{2,0,-1}(u_{0,0})\cos(\lambda_1\pi)(k_0/2)^{\lambda_1}}{\sin(\lambda_1\pi)\Gamma(\lambda_1+1)}. \tag{5.26}$$

By construction, the first term of the right hand side of (5.21) is counterbalanced by the first term of (5.25) multiplied by δ^{λ_2}.

5.4.2. Construction of $U^\delta_{2,0,\pm}$

Writing the matching condition (5.21) with respect to the microscopic variable and taking into account (5.25), we obtain

$$\delta^{\lambda_2}\left(\frac{\ell_2^\pm(u_{0,0})(k_0/2)^{\lambda_2}}{\Gamma(\lambda_2+1)}\left(\left(R^\pm\right)^{\lambda_2}w_{2,0,\pm}(\theta^\pm)+(R^\pm)^{\lambda_2-1}w_{2,1,\pm}(\theta^\pm)\right)+\ell_0^\pm(u_{2,0})\right)\approx\delta^{\lambda_2}U^\delta_{2,0}.$$

We then see that $U^\delta_{2,0,\pm}$ has to grow up like

$$\frac{\ell_2^\pm(u_{0,0})(k_0/2)^{\lambda_2}}{\Gamma(\lambda_2+1)}\left(\left(R^\pm\right)^{\lambda_2}w_{2,0,\pm}(\theta^\pm)+(R^\pm)^{\lambda_2-1}w_{2,1,\pm}(\theta^\pm)\right)+\ell_0^\pm(u_{2,0}).$$

Of course, $\left(R^\pm\right)^{\lambda_2}w_{2,0,\pm}(\theta^\pm)+(R^\pm)^{\lambda_2-1}w_{2,1,\pm}(\theta^\pm)$ does not satisfy the homogeneous problem (5.20)-(*right*). However, Proposition (4.1) ensures the existence of a function $S_2^\pm$, that satisfies (5.20)-(*right*) and such that $S_2^\pm-\left(R^\pm\right)^{\lambda_2}w_{2,0,\pm}(\theta^\pm)+(R^\pm)^{\lambda_2-1}w_{2,1,\pm}(\theta^\pm)$ tends to 0 as $R^\pm$ tends to infinity ($\lambda_2\notin\mathbb{N}$). Consequently, it is natural to define $U^\delta_{2,0,\pm}$ as

$$U^\delta_{2,0,\pm}=U_{2,0,\pm}=\frac{\ell_2^\pm(u_{0,0})(k_0/2)^{\lambda_2}}{\Gamma(\lambda_2+1)}S_2^\pm+\ell_0^\pm(u_{2,0}). \tag{5.27}$$

Outside the periodic layer, $U_{2,0,\pm}$ admits the following asymptotic expansion at infinity

$$\begin{aligned}U_{2,0,\pm}=&\frac{\ell_2^\pm(u_{0,0})(k_0/2)^{\lambda_2}}{\Gamma(\lambda_2+1)}\left(R^\pm\right)^{\lambda_2}w_{2,0,\pm}(\theta^\pm)\\&+\frac{\ell_2^\pm(u_{0,0})(k_0/2)^{\lambda_2}}{\Gamma(\lambda_2+1)}(R^\pm)^{\lambda_2-1}w_{2,1,\pm}(\theta^\pm)+\ell_0^\pm(u_{2,0})\\&+\frac{\ell_2^\pm(u_{0,0})(k_0/2)^{\lambda_2}}{\Gamma(\lambda_2+1)}(R^\pm)^{-\lambda_1}\mathscr{L}_{-1}(S_2^\pm)w_{-1,0,\pm}(\theta^\pm)+O((R^\pm)^{\lambda_2-2}\ln R^\pm).\end{aligned} \tag{5.28}$$

5.5. Construction of the terms $u^\delta_{1,1}$ and $U^\delta_{1,1,\pm}$

Reminding that $u^\delta_{1,1}$ fulfills the jump conditions (2.19) and that $u_{1,0}=0$ (see (5.9)), $u^\delta_{1,1}$ and $U^\delta_{1,1,\pm}$ satisfy the following problems

$$\left\{\begin{aligned}-\Delta u^\delta_{1,1}-k_0^2u^\delta_{1,1}&=0 &&\text{in}\quad \Omega_{\mathrm{T}}\cup\Omega_{\mathrm{B}},\\ \left[u^\delta_{1,1}\right]_\Gamma=\left[\partial_{x_2}u^\delta_{1,1}\right]_\Gamma&=0&&\text{on}\quad\Gamma,\\ \nabla u^\delta_{1,1}\cdot\mathbf{n}&=0&&\text{on}\quad\Gamma_N,\\ \nabla u^\delta_{1,1}\cdot\mathbf{n}-\imath k_0u^\delta_{1,1}&=0&&\text{on}\quad\Gamma_R,\end{aligned}\right.\qquad\text{and}\qquad\left\{\begin{aligned}-\Delta U^\delta_{1,1,\pm}&=0&&\text{in}\quad\widehat{\Omega}^\pm,\\ \partial_{\mathbf{n}}U^\delta_{1,1,\pm}&=0&&\text{on}\quad\partial\widehat{\Omega}^\pm,\end{aligned}\right. \tag{5.29}$$

Outside the thin periodic layer, the matching condition (2.26) written up to order δ^{λ_1+1} gives

$$\delta\frac{\ell_1^\pm(u_{0,1})(k_0/2)^{\lambda_1}}{\Gamma(\lambda_1+1)}\left(r^\pm\right)^{\lambda_1}w_{1,0,\pm}(\theta^\pm)+\delta^{\lambda_1+1}u^\delta_{1,1}(r^\pm,\theta^\pm)\approx\delta^{\lambda_1+1}U^\delta_{1,1,\pm}\left(\frac{r^\pm}{\delta},\theta^\pm\right). \tag{5.30}$$

A analogous analysis than the one made in Section 5.2 yields

$$u^\delta_{1,1}=u_{1,1}=0\quad\text{and}\quad U^\delta_{1,1,\pm}=U_{1,1,\pm}=\frac{\ell_1^\pm(u_{0,1})(k_0/2)^{\lambda_1}}{\Gamma(\lambda_1+1)}S_1^\pm. \tag{5.31}$$

Far from the periodic layer, the asymptotic behaviour of $U_{1,1,\pm}$ is given by

$$U_{1,1,\pm}(\mathbb{R}^\pm,\theta^\pm)=\frac{\ell_1^\pm(u_{0,1})(k_0/2)^{\lambda_1}}{\Gamma(\lambda_1+1)}(R^\pm)^{\lambda_1}w_{1,0,\pm}(\theta^\pm)+O((R^\pm)^{\lambda_1-1}). \tag{5.32}$$

5.6. Construction of the terms $u^\delta_{3,0}$ and $U^\delta_{3,0,\pm}$ for $\Theta > \frac{3\pi}{2}$

Reminding that $u^\delta_{3,0}$ fulfills the jump conditions (2.18), $u^\delta_{3,0}$ and $U^\delta_{3,0,\pm}$ satisfy

$$\begin{cases} -\Delta u^\delta_{3,0} - k_0^2 u^\delta_{3,0} = 0 & \text{in} \quad \Omega_{\mathrm{T}} \cup \Omega_{\mathrm{B}}, \\ \left[u^\delta_{3,0}\right]_\Gamma = \left[\partial_{x_2} u^\delta_{3,0}\right]_\Gamma = 0 & \text{on} \quad \Gamma, \\ \nabla u^\delta_{3,0} \cdot \mathbf{n} = 0 & \text{on} \quad \Gamma_N, \\ \nabla u^\delta_{3,0} \cdot \mathbf{n} - \imath k_0 u^\delta_{3,0} = 0 & \text{on} \quad \Gamma_R, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta U^\delta_{3,0,\pm} = 0 & \text{in} \quad \widehat{\Omega}^\pm, \\ \partial_{\mathbf{n}} U^\delta_{3,0,\pm} = 0 & \text{on} \quad \partial\widehat{\Omega}^\pm, \end{cases} \tag{5.33}$$

Outside the periodic layer, collecting the asymptotic representation (5.4)–(5.4)–(5.18) of the far field terms, the defintions (5.5)–(5.19) of $U_{0,0}$ and $U_{0,1}$ and the asymptotic expansions (5.11)–(5.28)–(5.32) of $U_{1,0}$, $U_{2,0}$, $U_{1,1}$, the matching condition (2.26) written up to order δ^{λ_3} becomes

$$\frac{\ell_3^\pm(u_{0,0})(k_0/2)^{\lambda_3}}{\Gamma(\lambda_3+1)} \left(\left(r^\pm\right)^{\lambda_3} w_{3,0,\pm}(\theta^\pm) + \delta (r^\pm)^{\lambda_3 - 1} w_{3,1,\pm} \right) + \delta^{\lambda_2} \ell_{2,0,1}(u_{2,0}) \left(r^\pm\right)^{\lambda_1} w_{1,0,\pm}$$
$$+ \delta^{\lambda_3} u^\delta_{3,0} \approx \delta^{\lambda_3} \left(\sum_{i=1}^{2} \frac{\ell^\pm_{3-i}(u_{0,0})(k_0/2)^{\lambda_{3-i}}}{\Gamma(\lambda_{3-i}+1)} (r^\pm)^{-\lambda_i} \mathscr{L}_{-i}(S^\pm_{3-i}) w_{-i,0,\pm} \right) + \delta^{\lambda_3} U^\delta_{3,0}. \tag{5.34}$$

5.6.1. *Construction of the macroscopic term* $u^\delta_{3,0}$

In view of the right hand side of (5.34), we remark that $u^\delta_{3,0}$ has two singular contributions of the form $(r^\pm)^{-\lambda_2}$ and $(r^\pm)^{-\lambda_1}$. Defining

$$\mathbf{Y}_2^\pm(r^\pm, \theta^\pm) = Y_{\lambda_2}(k_0 r^\pm) w_{-2,0,\pm}(\theta^\pm), \quad \ell^\pm_{3,0,-i}(u_{0,0}) = -\pi \mathscr{L}_{-i}(S^\pm_{3-i}) \ell^\pm_{3-i}(u_{0,0}) \frac{(k_0/2)^{\lambda_3}}{\Gamma(\lambda_i)\Gamma(\lambda_{3-i}+1)}, \tag{5.35}$$

the function $\sum_{i=1}^2 \delta^{\lambda_3/2} \ell^\pm_{3,0,-i}(u_{0,0}) \mathbf{Y}_i^\pm(r^\pm, \theta^\pm)$ ($\mathbf{Y}_1^\pm$ defined in (5.22)) can counterbalance the first two terms of the right hand side of (5.34). This remark leads us to define $u^\delta_{3,0}$ as

$$u^\delta_{3,0} = u_{3,0} := \hat{u}_{3,0} + \sum_\pm \sum_{i=1}^{2} \ell^\pm_{3,0,-i}(u_{0,0}) \chi_L^\pm \mathbf{Y}_i^\pm, \tag{5.36}$$

where $\chi_L^\pm$ is defined in (5.16) and the function $\hat{u}_{3,0}$ is the unique function of $\mathrm{H}^1(\Omega)$ satisfying

$$\begin{cases} -\Delta \hat{u}_{3,0} - k_0^2 \hat{u}_{3,0} = \hat{f}_{3,0} & \text{in} \quad \Omega_{\mathrm{T}} \cup \Omega_{\mathrm{B}}, \\ \left[\hat{u}_{3,0}\right]_\Gamma = \left[\partial_{x_2} \hat{u}_{3,0}\right]_\Gamma = 0 & \text{on} \quad \Gamma, \\ \nabla \hat{u}_{3,0} \cdot \mathbf{n} = 0 & \text{on} \quad \Gamma_N, \\ \nabla \hat{u}_{3,0} \cdot \mathbf{n} - \imath k_0 \hat{u}_{3,0} = 0 & \text{on} \quad \Gamma_R, \end{cases} \tag{5.37}$$

with

$$\hat{f}_{3,0} := \sum_{i=1}^{2} \sum_\pm \ell^\pm_{3,0,-i}(u_{0,0}) [\Delta, \chi_L^\pm] \mathbf{Y}_i^\pm.$$

The well-posedness of (5.37) directly follows from Proposition 3.1. In the matching zones,

$$u_{3,0}(r^\pm, \theta^\pm) = -\frac{\ell^\pm_{3,0,-2}(u_{0,0})(k_0/2)^{-\lambda_2}}{\sin(\lambda_2 \pi)\Gamma(1-\lambda_2)} \left(r^\pm\right)^{-\lambda_2} w_{-2,0,\pm}(\theta^\pm)$$
$$- \frac{\ell^\pm_{3,0,-1}(u_{0,0})(k_0/2)^{-\lambda_1}}{\sin(\lambda_1 \pi)\Gamma(1-\lambda_1)} \left(r^\pm\right)^{-\lambda_1} w_{-1,0,\pm}(\theta^\pm) + \ell_0^\pm(u_{3,0}) + O(r^{-\lambda_2+2}), \tag{5.38}$$

5.6.2. Construction of $U^{\delta}_{3,0,\pm}$

Writing the matching condition (5.34) in term of the microscopic variables and taking into account (5.38), we obtain

$$\delta^{\lambda_3}\left(\frac{\ell_3^{\pm}(u_{0,0})(k_0/2)^{\lambda_3}}{\Gamma(\lambda_3+1)}\left(\left(r^{\pm}\right)^{\lambda_3} w_{3,0,\pm}(\theta^{\pm}) + (R^{\pm})^{\lambda_3-1} w_{3,1,\pm}(\theta^{\pm})\right)\right.$$
$$\left. + \ell_{2,0,1}(u_{2,0})\left(R^{\pm}\right)^{\lambda_1} w_{1,0,\pm}(\theta^{\pm}) + \ell_0^{\pm}(u_{3,0})\right) \approx \delta^3 U^{\delta}_{3,0}. \quad (5.39)$$

As in Section 5.4, it is natural to define $U^{\delta}_{3,0,\pm}$ as

$$U^{\delta}_{3,0,\pm} = U_{3,0,\pm} := \frac{\ell_3^{\pm}(u_{0,0})(k_0/2)^{\lambda_3}}{\Gamma(\lambda_3+1)} S_3^{\pm} + \frac{\left(\ell_1^{\pm}(u_{2,0})\sin\lambda_1\pi + \ell^{\pm}_{2,0,-1}(u_{2,0})\cos\lambda_1\pi\right)(k_0/2)^{\lambda_2}}{\Gamma(\lambda_1+1)\sin\lambda_1\pi} S_1^{\pm} + \ell_0^{\pm}(u_{3,0}). \quad (5.40)$$

5.7. The 'automatic' matching inside the layer

We end this part by showing, that far and near field expansions automatically match in the matching areas. For the sake of concision, we consider the case $\theta \in (\pi, 3\pi/2)$ and we only investigate the matching area located in the vicinity of the right corner $\mathbf{x}_O^+$.

On the one hand, collecting the results of the present section and Appendix 3, we see that the boundary layer correctors are given by

$$\begin{aligned}
\Pi_{0,0}(x_1,\mathbf{X}) &= \langle u_{0,0}\rangle_{\Gamma}(x_1) V_0(\mathbf{X}), \quad \Pi_{1,0} = 0,\\
\Pi_{0,1}(x_1,\mathbf{X})(x_1,\mathbf{X}) &= \langle u_{0,1}\rangle_{\Gamma}(x_1)\, V_0(\mathbf{X}) + \partial_{x_1}\langle u_{0,0}\rangle_{\Gamma}(x_1)\, V_{1,1}(\mathbf{X})\\
&\quad + \langle \partial_{x_2} u_{0,0}\rangle_{\Gamma}(x_1)\, V_{1,2}(\mathbf{X}),\\
Pi_{2,0}(x_1,\mathbf{X}) &= \langle u_{2,0}\rangle_{\Gamma}(x_1)\, V_0(\mathbf{X}), \qquad \Pi_{1,1} = 0.
\end{aligned} \quad (5.41)$$

Then, the asymptotic expansion for the boundary layer in the matching areas can be directly written introducing the asymptotic formula (5.4)–(5.18)–(5.25) of the macroscopic terms $u_{0,0}$, $u_{0,1}$ and $u_{2,0}$ into (5.41). Writting the obtained asymptotic expansions in term of the microscopic variables, noticing that $V_0 = W_0^{\mathfrak{t}}$, $V_{1,1} = W_1^{\mathfrak{t}}$ and $V_{1,2} = W_1^{\mathfrak{n}}$ (defined in (C.1)–(C.4)), and summing over $(n,q) \in \mathbb{N}_2$, we obtain

$$\begin{aligned}
\sum_{(n,q)\in\mathbb{N}_2} \delta^{\lambda_n+q}\, \Pi_{n,q} &\approx \ell_0^{\pm}(u_{0,0}) W_0^{\mathfrak{t}}\\
&+ \delta^{\lambda_1}\left\{\frac{\ell_1^{\pm}(u_{0,0})(k_0/2)^{\lambda_1}}{\Gamma(\lambda_1+1)}\right\}\left\{|X_1|^{\lambda_1} p_{1,0,+} + |X_1|^{\lambda_1-1} p_{1,1,+}\right.\\
&\left. + |X_1|^{-\lambda_1}\mathscr{L}_{-1}(S_1^{\pm}) p_{-1,0,+} W_0^{\mathfrak{t}}\right\}\\
&+ \delta\, \ell_0^{\pm}(u_{0,1}) W_0^{\mathfrak{t}}\\
&+ \delta^{\lambda_2}\left\{\frac{\ell_2^{\pm}(u_{0,0})(k_0/2)^{\lambda_2}}{\Gamma(\lambda_2+1)}\right\}\left\{|X_1|^{\lambda_2} p_{2,0,+} + |X_1|^{\lambda_2-1} p_{2,1,+}\right\} + \delta^{\lambda_2}\ell_0^{\pm}(u_{2,0})\, W_0^{\mathfrak{t}}\\
&+ \delta^{\lambda_1+1}\left\{\frac{\ell_1^{\pm}(u_{0,1})(k_0/2)^{\lambda_1}}{\Gamma(\lambda_1+1)}\right\}|X_1|^{\lambda_1} p_{1,0,+},
\end{aligned} \quad (5.42)$$

the function $p_{n,r,t,\pm}$ being defined in (C.3).

On the other hand, using the definitions (5.5)−(5.10)−(5.19)−(5.27) of the near field terms, the truncated series of the near field is given by

$$\sum_{(n,q)\in\mathbb{N}_2} \delta^{\lambda_n+q}\, \mathit{\Pi}_{n,q} = \ell_0^+(u_{0,0}) + \delta^{\lambda_1}\frac{\ell_1^+(u_{0,0})(k_0/2)^{\lambda_1}}{\Gamma(\lambda_1+1)} S_1^+ + \delta \ell_0^+(u_{0,1}) + \delta^{\lambda_2}\left(\frac{\ell_2^+(u_{0,0})(k_0/2)^{\lambda_2}}{\Gamma(\lambda_2+1)} S_2^+ + \ell_0^+(u_{2,0})\right) + \delta^{\lambda_1+1}\frac{\ell_1^{\pm}(u_{0,1})(k_0/2)^{\lambda_1}}{\Gamma(\lambda_1+1)} S_1^+. \tag{5.43}$$

Introducing the asymptotic expansions (4.7) of the functions S_1^+ and S_2^+ in the vicinity of the periodic layer into (5.43), we see that the near field expansions (5.43) and (5.42) coÃ¯ncide (up to a given order).

6. Error estimates

To finish this paper, we give the sketch of the proof of Theorem 2.4. As usual for this kind of work (see *e.g.* Sect. 6 of Ref. [20], Sect. 3 of Ref. [26], Sect. 2.5 of Ref. [27]), the proof of the previous result is based on the construction of an approximation $u^\delta_{N_0}$ of u^δ in the whole domain Ω^δ. To do so, we define the following four truncated series (at order N_0), corresponding to the truncated series of the macroscopic terms, the boundary layer terms and the near field terms:

- The truncated series $u^\delta_{\text{macro},N_0}$ of the macroscopic terms: the macroscopic approximation is defined by

$$u^\delta_{\text{macro},N_0}(\mathbf{x}) = \chi^\delta_{\text{macro}}(\mathbf{x}) \sum_{(n,q)\in\mathbb{N}_{N_0}} \delta^{\lambda_n+q} u^\delta_{n,q}(\mathbf{x}), \tag{6.1}$$

  where the set $\mathbb{N}_{N_0}$ is the set of indexes $(n,q)\in\mathbb{N}^2$ for which $\lambda_n+q<N_0$, and the macroscopic cut-off function $\chi^\delta_{\text{macro}}$ is given by

$$\chi^\delta_{\text{macro}}(\mathbf{x}) = \chi_+\left(\frac{x_1-L}{\delta}\right)\chi_-\left(\frac{x_1+L}{\delta}\right)\chi\left(\frac{x_2}{\delta}\right) + \sum_{\pm}\chi_{\text{macro},\pm}\left(\frac{x_1\mp L}{\delta},\frac{x_2}{\delta}\right)\left(1-\chi_\pm\left(\frac{x_1\mp L}{\delta}\right)\right). \tag{6.2}$$

  We notice that the function $\chi^\delta_{\text{macro}}$ is equal to 1 for $|x_1|>L$ and coincides with $\chi\left(\frac{x_2}{\delta}\right)$ in the region $|x_1|<L-\delta$ (The cut-off functions χ and $\chi_\pm$ are defined in (2.11) and (2.15), while the cut-off functions $\chi_{\text{macro},+}$, represented on Fig. 5, satisfies (4.4)).
- The truncated series $\mathit{\Pi}^\delta_{N_0}$ of the periodic correctors is given by

$$\mathit{\Pi}^\delta_{N_0}(\mathbf{x}) = \chi_+\left(\frac{x_1-L}{\delta}\right)\chi_-\left(\frac{x_1+L}{\delta}\right)\chi\left(\frac{2x_2}{\min(H_{\text{B}},H_{\text{T}})}\right)\sum_{(n,q)\in\mathbb{N}_{N_0}}\delta^{\lambda_n+q}\mathit{\Pi}^\delta_{n,q}(\mathbf{x}). \tag{6.3}$$

  The use of the function $\chi_+\left(\frac{x_1-L}{\delta}\right)\chi_-\left(\frac{x_1+L}{\delta}\right)$ permits us to localize the function $\mathit{\Pi}^\delta_{N_0}(\mathbf{x})$ in the domain $|x_1|<L$ while the introduction of the function $\chi\left(\frac{2x_2}{\min(H_{\text{B}},H_{\text{T}})}\right)$ ensures that $\mathit{\Pi}^\delta_{N_0}(\mathbf{x})$ satisfies Neumann boundary condition on Γ_N.
- The truncated near field series $U^\delta_{N_0,\pm}$ are given by

$$U^\delta_{N_0,\pm} = \sum_{(n,q)\in\mathbb{N}_{N_0}}\delta^{\lambda_n+q}U^\delta_{n,q,\pm}\left(\frac{\mathbf{x}-\mathbf{x}_O^\pm}{\delta}\right). \tag{6.4}$$

Based, on these truncated series, the global approximation $u^\delta_{N_0}$ is defined by

$$u^\delta_{N_0} = \chi^\delta_+ \, U^\delta_{N_0,+} \; + \; \chi^\delta_- \, U^\delta_{N_0,-} \; + \; (1 - \chi^\delta_+ - \chi^\delta_-) \left(u^\delta_{\mathrm{macro},N_0} + \Pi^\delta_{N_0} \right), \tag{6.5}$$

where $\chi^\delta_\pm(\mathbf{x}) = \chi(|\mathbf{x} - \mathbf{x}^\pm_O|/\sqrt{\delta})$. We point out that $u^\delta_{N_0}$ coincides with $U^\delta_{N_0,\pm}$ in the vicinity of the two corners, with $\Pi^\delta_{N_0}$ in the vicinity of the layer and with $u^\delta_{\mathrm{macro},N_0}$ away from the corner and the periodic layer.

Remark 6.1. The overall approximation $u^\delta_{N_0}$ can be computed for any real number $\mathbb{N}_0$ as soon as the terms of the the far and near field expansions are defined. In Section 5, we only constructed the first terms of these expansions, but the next order terms can naturally be derived using the same methodology.

The overall approximation being constructed, it remains to evaluate the H^1-norm of the error $e^\delta_{N_0} = u^\delta - u^\delta_{N_0}$ in Ω^δ. It is in fact sufficient to estimate the residue $(\Delta + (k^\delta)^2)e^\delta_{N_0}$ and the Neumann trace $\partial_{\mathbf{n}} e^\delta_{N_0}$. Indeed, the estimation of $\left\| e^\delta_{N_0} \right\|_{\mathrm{H}^1(\Omega^\delta)}$ directly results from a straightforward modification of the uniform stability result (2.7) (Prop. 2.2): there exists a constant $C > 0$ independent of δ (but depending on other parameters such as N_0 and hole shape) such that, for δ small enough,

$$\left\| e^\delta_{N_0} \right\|_{\mathrm{H}^1(\Omega^\delta)} \leqslant C \left(\left\| (\Delta + (k^\delta)^2) e^\delta_{N_0} \right\|_{\mathrm{L}^2(\Omega^\delta)} + \left\| \partial_{\mathbf{n}} e^\delta_{N_0} \right\|_{\mathrm{L}^2(\Gamma^\delta)} \right). \tag{6.6}$$

Similarly to the proof of Proposition 6.3 in reference [20], we decompose the error of the residue into a modeling error (measuring how the truncated far and near field expansions fail to satisfies the Helmholtz equation and the Neumann boundary condition) and a matching error (measuring the difference between the far and near field expansions in the matching areas), and we obtain the following proposition:

Proposition 6.2. *Let $N_0 \in \mathbb{R}$. There exists a constant $C \geq 0$, a constant $\kappa = \kappa(N_0) > 0$ and a constant $\delta_0 > 0$ such that, for any $\delta \in (0, \delta_0)$,*

$$\left\| (\Delta + (k^\delta)^2) e^\delta_{N_0} \right\|_{\mathrm{L}^2(\Omega^\delta)} + \left\| \partial_{\mathbf{n}} e^\delta_{N_0} \right\|_{\mathrm{L}^2(\Gamma^\delta)} \leqslant C (\ln \delta)^\kappa \delta^{\frac{N_0}{2} - \frac{5}{2}}. \tag{6.7}$$

As a consequence, there exists a constant $C > 0$, a constant $\kappa = \kappa(N_0) > 0$ and a constant $\delta_0 > 0$ such that, for any $\delta \in (0, \delta_0)$,

$$\left\| e^\delta_{N_0} \right\|_{\mathrm{H}^1(\Omega^\delta)} \leqslant C (\ln \delta)^\kappa \delta^{\frac{N_0}{2} - \frac{5}{2}}. \tag{6.8}$$

Finally, since $e^\delta_{N_0}$ coincides with $u^\delta - \sum_{(n,q) \in \mathbb{N}_{N_0}} \delta^{\lambda_n + q} u^\delta_{n,q}$ in Ω_α for δ small enough, Theorem 2.4 follows from (6.8) and the triangular inequality.

Appendix A. Proof of Proposition 2.2

The variational formulation associated with (2.6) writes as follows: find $u^\delta \in \mathrm{H}^1(\Omega^\delta)$ such that,

$$\forall v \in \mathrm{H}^1(\Omega^\delta), \qquad a^\delta(u^\delta, v) = \langle \partial_n u_{\mathrm{inc}} - \imath k_0 u_{\mathrm{inc}}, v \rangle_{H^{-1/2}(\Gamma^+_R), H^{1/2}(\Gamma^+_R)} \tag{A.1}$$

where

$$a^\delta(u, v) = \int_{\Omega^\delta} \nabla u \cdot \overline{\nabla v} \, \mathrm{d}x - \int_{\Omega^\delta} (k^\delta)^2 u \, \overline{v} \, \mathrm{d}x \; - \; \imath k_0 \int_{\Gamma_R} u \overline{\nabla v} \mathrm{d}x,$$

and $\langle \cdot, \cdot \rangle_{H^{-1/2}(\Gamma^+_R), H^{1/2}(\Gamma^+_R)}$ stands for the duality pairing between $H^{-1/2}(\Gamma^+_R)$ and $H^{1/2}(\Gamma^+_R)$ extending the $L^2(\Gamma^+_R)$ iner-product. It is easily seen that problem (2.6) is a Fredholm-type problem (Thm. 6.6 in [12]). Let us prove that is has a unique solution. Assume that

$$a^\delta(u, v) = 0 \quad \forall v \in \mathrm{H}^1(\Omega^\delta).$$

Then, taking $v = u^\delta$ leads to $u^\delta = 0$ on $\Gamma_R^\pm$. Since $u^\delta = 0$ satisfies a Robin-type boundary condition on Γ_R, we deduce that $\partial_n u^\delta = 0$ on $\Gamma_R^\pm$. It then follows from the unique continuation theorem that $u^\delta = 0$ in the whole domain Ω^δ.

It remains to prove the uniform stability estimate (2.7). The proof is by contradiction. If (2.7) does not hold, there exists a sequence δ_n going to 0 as n tends to $+\infty$, and a sequence $u_n \in \mathrm{H}^1(\Omega^\delta)$ such that

$$\|u_n\|_{\mathrm{H}^1(\Omega^{\delta_n})} = 1 \quad \text{and} \quad \forall v \in \mathrm{H}^1(\Omega^{\delta_n}), \lim_{n\to+\infty} a^{\delta_n}(u_n, v) = 0. \tag{A.2}$$

First, we construct an extension of $\tilde{u}_n$ of u_n belonging to $\mathrm{H}^1(\Omega)$ (see *e.g.* example 1 in [38]) that satisfies

$$1 \le \|\tilde{u}_n\|_{\mathrm{H}^1(\Omega)} \le C\|u_n\|_{\mathrm{H}^1(\Omega^\delta)} \quad \text{and} \qquad \tilde{u}_n = u_n \text{ on } \Omega^{\delta_n}.$$

Then, for any $v \in \mathrm{H}^1(\Omega)$,

$$\lim_{n\to+\infty} a^{\delta_n}(\tilde{u}_n, v) = \lim_{n\to+\infty} \left(a^{\delta_n}(u_n, v) + \int_{\Omega^{\delta_n}_{\text{hole}}} \nabla\tilde{u}_n \cdot \overline{\nabla v}\, \mathrm{d}x \;-\; (k^\delta)^2 \tilde{u}_n \overline{v}\, \mathrm{d}x \right) = 0.$$

Indeed, since the measure of $\Omega^{\delta_n}_{\text{hole}}$ tends to 0 as δ tends to 0, for any $v \in \mathrm{H}^1(\Omega)$, $\lim\limits_{n\to+\infty} \|v\|_{\mathrm{H}^1(\Omega^{\delta_n}_{\text{hole}})} = 0$. Besides, $\tilde{u}_n$ being bounded in $\mathrm{H}^1(\Omega)$, there exists a function $u_* \in \mathrm{H}^1(\Omega)$ such that, up to a subsequence, $\tilde{u}_n$ weakly tends to u_* in $\mathrm{H}^1(\Omega)$ as n tends to $+\infty$. As a result,

$$\lim_{n\to+\infty} a^{\delta_n}(\tilde{u}_n, v) = \int_\Omega \nabla u_* \cdot \overline{\nabla v}\, dx \;-\; (k_0)^2 \int_\Omega u_* \overline{v} dx - ik_0 \int_\Omega u_* v ds = 0.$$

Naturally, it implies that $u_* = 0$. In particular, it $\lim_{n\to\infty} \|\tilde{u}_n\|_{L^2(\Omega)} = 0$, which in turn implies that $\lim_{n\to\infty} \|\nabla\tilde{u}_n\|_{L^2(\Omega)} = 0$, and contradicts the fact that $\|\tilde{u}_n\|_{\mathrm{H}^1(\Omega)} \ge 1$.

Appendix B. Technical results for the near field singularities

B.1. The variational framework associated with the near field problems

The near field terms $U^\delta_{n,q,\pm}$ satisfy Laplace problems (see (2.23))of the form

$$\begin{cases} -\Delta U = F & \text{in} \quad \widehat{\Omega}^\pm, \\ \partial_{\mathbf{n}} U = G & \text{on} \quad \partial\widehat{\Omega}^\pm. \end{cases} \tag{B.1}$$

As described in Section 3.5 in [13], the standard variational space to solve problem (B.1) is

$$\mathfrak{V}(\widehat{\Omega}^\pm) = \left\{ v \in \mathrm{H}^1_{\text{loc}}(\widehat{\Omega}^\pm),\ \nabla v \in L^2(\widehat{\Omega}^\pm), \quad \frac{v}{(1+R^\pm)\ln(2+R^\pm)} \in L^2(\widehat{\Omega}^\pm) \right\}, \tag{B.2}$$

which, equipped with the norm

$$\|v\|_{\mathfrak{V}(\widehat{\Omega}^\pm)} = \left(\left\| v/(1+R^\pm)\ln(2+R^\pm) \right\|^2_{\mathrm{L}^2(\widehat{\Omega}^\pm)} + \|\nabla v\|^2_{\mathrm{L}^2(\widehat{\Omega}^\pm)} \right)^{1/2}$$

is a Hilbert space. Based on a variational formulation, we can prove the following well-posedness result (see Prop. 3.22 and Cor. 3.23 of Ref. [13] for the proof):

Proposition B.1. *Assume that* $(1+R^\pm)\ln(2+R^\pm)F \in \mathrm{L}^2(\widehat{\Omega}^\pm)$, $(1+R^\pm)^{1/2}\ln(2+R^\pm)G \in \mathrm{L}^2(\partial\widehat{\Omega}^\pm)$, *and the compatibility condition*

$$\int_{\widehat{\Omega}^\pm} F + \int_{\partial\widehat{\Omega}^\pm} G = 0 \tag{B.3}$$

is satisfied. Then, problem (B.1) *has a solution* $u \in \mathfrak{V}(\widehat{\Omega}^\pm)$, *unique up to an additive constant.*

B.2. Absence of logarithmic singularity

As explained in Section 4, we are interesting in building solutions to the homogeneous problem (*i.e.* $F = G = 0$) associated with (B.1) that blow up at infinity. One natural question is to know if such a solution can blow up like $\ln R^{\pm}$ at infinity. The negative answer is given in the following Lemma:

Lemma B.2. *The problem*

$$\begin{cases} -\Delta U = \Delta \mathcal{U}_{0,1,\pm} & \text{in} \quad \widehat{\Omega}^{\pm}, \\ \partial_{\mathbf{n}} U = -\partial_{\mathbf{n}} \mathcal{U}_{0,1,\pm} & \text{on} \quad \partial\widehat{\Omega}^{\pm} \end{cases} \tag{B.4}$$

has no solution in $\mathfrak{V}(\widehat{\Omega}^{+})$. *As a consequence, it is not possible to construct a solution to the homogeneous problem* (B.1) *(*i.e. $F = G = 0$*) that has a logarithmic blow up as* $R^{\pm}$ *tends toward infinity.*

Proof. We first remark that $(1 + R^{\pm})\ln(2 + R^{\pm})\Delta\mathcal{U}_{0,1,\pm} \in \mathrm{L}^2(\widehat{\Omega}^{\pm})$ and $(1 + R^{\pm})^{1/2}\ln(2 + R^{\pm})\partial_{\mathbf{n}}\mathcal{U}_{0,1,\pm} \in \mathrm{L}^2(\partial\widehat{\Omega}^{\pm})$. Then, thanks to Proposition B.1, if problem (B.4) has a solution in $\mathfrak{V}(\widehat{\Omega}^{+})$, the right hand side of (B.4) has to satisfy the compatibility condition (B.3). We shall see that this compatibility condition does not hold.

Following the proof of Theorem 3.25 in reference [13], we shall construct a sequence of domains $\widehat{\Omega}_k^{+}$ that tends to $\widehat{\Omega}^{+}$ as k tends to $+\infty$. To do so, we introduce $M_0 \in (0, 1)$ such that the vertical segment $\{-M_0\} \times (-1, 1)$ does not intersect the obstacle $\widehat{\Omega}_{\text{hole}}$, and we consider the sequence $(M_k)_{k\in\mathbb{N}^*}$ defined by $M_k = M_0 + k$. By construction, the vertical segment $\{-M_k\} \times (-1, 1)$ does not intersect any hole of the domain $\widehat{\Omega}^{+}$. Then, we define,

$$\widehat{\Omega}_k^{+} = \widehat{\Omega}^{+} \cap \mathcal{B}(0, M_k) \qquad \text{and} \qquad \Gamma_k^{+} = (\partial\widehat{\Omega}^{+}) \cap \mathcal{B}(0, M_k). \tag{B.5}$$

We have

$$\int_{\widehat{\Omega}^{+}} \Delta\mathcal{U}_{0,1,+} - \int_{\partial\widehat{\Omega}^{+}} \partial_{\mathbf{n}}\mathcal{U}_{0,1,+} = \lim_{k\to\infty} \int_{\widehat{\Omega}_k^{+}} \Delta\mathcal{U}_{0,1,+} - \int_{\Gamma_k^{+}} \partial_{\mathbf{n}}\mathcal{U}_{0,1,+} \tag{B.6}$$

Applying the Green formula (to the first integral of the right hand side of the previous equality) gives

$$\int_{\widehat{\Omega}^{+}} \Delta\mathcal{U}_{0,1,+} - \int_{\partial\widehat{\Omega}^{+}} \partial_{\mathbf{n}}\mathcal{U}_{0,1,+} = \lim_{k\to\infty} \int_0^{\Theta} \partial_R^{\pm}\mathcal{U}_{0,1,+}(M_k, \theta^{+}) M_k d\theta^{+}. \tag{B.7}$$

But, for large $R^{\pm}$,

$$\mathcal{U}_{0,1,+}(M, \theta^{+}) = \ln R^{\pm} + \chi_{\text{macro},+}(X_1^{+}, X_2^{+}) \frac{1}{R^{\pm}} w_{0,1,+}(\ln R^{\pm}) + \chi_{-}(X_1^{+})|X_1^{+}|^{-1} p_{0,1,+}(\ln|X_1^{+}|, X_1^{+}, X_2^{+})$$

where

$$p_{0,1,+}(\ln|X_1^{+}|, X_1^{+}, X_2^{+}) = g^{\mathfrak{n}}_{0,0,1,+}(\ln|X_1^{+}|)\, W_1^{\mathfrak{n}}\left(X_1^{+}, X_2^{+}\right) + \sum_{p=0}^{1} g^{\mathfrak{t}}_{0,1-p,p,+}(\ln|X_1^{+}|)\, W_p^{\mathfrak{t}}\left(X_1^{+}, X_2^{+}\right),$$

$g^{\mathfrak{t}}_{0,1-p,p,+}$ and $g^{\mathfrak{n}}_{0,0,1,+}$ having a polynomial dependence with respect to $\ln|X_1^{+}|$. Then, a direct computation shows that

$$\partial_{R^{+}}\mathcal{U}_{0,1,+}(R^{+}, \theta^{+}) = \frac{1}{M} + O\left(\frac{\ln R^{+}}{(R^{+})^2}\right), \quad \text{uniformly w.r.t } \theta^{+}.$$

Consequently, taking the limit of the integral in (B.7) gives

$$\int_{\widehat{\Omega}^{+}} \Delta\mathcal{U}_{0,1,+} - \int_{\partial\widehat{\Omega}^{+}} \partial_{\mathbf{n}}\mathcal{U}_{0,1,+} = \Theta \neq 0, \tag{B.8}$$

which means that compatibility condition (B.3) is not satisfied.

Finally, the absence of logarithmic singularity is proved by contradiction: assume that such a function exists. We denote it by $S_{\log}$. Then, in view of Theorem 4.1 of reference [32], $S_{\log}$ can be decomposed as $S_{\log} = \mathcal{U}_{0,1,+} + \hat{S}_{\log}$, $\hat{S}_{\log}$ being in $\mathfrak{V}(\widehat{\Omega}^+)$. Noticing that $\hat{S}_{\log}$ satisfies problem (B.4) that has no solution in $\mathfrak{V}(\widehat{\Omega}^+)$, we obtain a contradiction. □

Lemma B.3. *Let $n \in \mathbb{Z}^*$ and $p(n) = \max(1, 1 + \lceil \lambda_n \rceil)$. If $n < 0$ or $\lambda_n \notin \mathbb{N}$, then*

$$\int_{\widehat{\Omega}^+} \Delta \mathcal{U}_{n,p(n),\pm} - \int_{\partial \widehat{\Omega}^+} \partial_{\mathbf{n}} \mathcal{U}_{n,p(n),\pm} = 0 \tag{B.9}$$

Proof. As in the proof of Lemma B.2, we will define a domain $\widehat{\Omega}_k^\pm$ such that $\lim_{k\to\infty} \widehat{\Omega}_k^\pm = \widehat{\Omega}^\pm$. As previously, we consider $M_0 \in (0,1)$ such that the vertical segment $\{-M_0\} \times (-1,1)$ does not intersect the obstacle $\widehat{\Omega}_{\text{hole}}$, and we consider the sequence M_k, $k \in \mathbb{N}^*$, by $M_k = M_0 + k$. By construction, the vertical segment $\{-M_k\} \times (-1,1)$ does not intersect any hole of the domain $\widehat{\Omega}^+$. We define the boundary I_k,

$$I_k = \{(R_k^+(\theta^+) \cos\theta^+, R_k^+(\theta^+) \sin\theta^+) \in \widehat{\Omega}^+, 0 < \theta^+ < \Theta\},$$

where the function $R_k^+(\theta^+)$ is given by

$$R_k^+(\theta^+) = \begin{cases} -M_k / \cos\theta^+, & |\theta^+ - \pi| \leq \theta_k, \\ \sqrt{M_k^2 + 4}, & \text{otherwise.} \end{cases} \qquad \theta_k = \sin^{-1}(2/\sqrt{M_k^2+4}).$$

For $|\theta^+ - \pi| \geq \theta_k$, I_k coincides with a portion of the circle of radius $\sqrt{M_k^2+4}$ and of center $(0,0)$ while for $|\theta^+ - \pi| \geq \theta)k$, I_k coincides with the segment $\{(-M_k, X_2^+) - 2 \leq X_2^+ \leq 2\}$. Again, analogously to the proof of Lemma B.2, we have,

$$\int_{\widehat{\Omega}^+} \Delta \mathcal{U}_{n,p(n),+} - \int_{\partial \widehat{\Omega}^+} \partial_{\mathbf{n}} \mathcal{U}_{n,p(n),+} = \lim_{k\to\infty} J_k^n \quad J_k^n = \int_{I_k} \partial_{\mathbf{n}} \mathcal{U}_{n,p(n),+} \mathrm{d}\sigma \tag{B.10}$$

where, since $(1+R^+)\ln(2+R^+)\Delta\mathcal{U}_{n,p(n),+} \in \mathrm{L}^2(\widehat{\Omega}^+)$ and $(1+R^+)^{1/2}\ln(2+R^+)\partial_{\mathbf{n}}\mathcal{U}_{n,p(n),+} \in \mathrm{L}^2(\partial\widehat{\Omega}^+)$ the limit of J_k^n is finite. But, applying Lemmas B.4 and B.5 below, we can prove

$$J_k^n = \sum_{m=0}^{\lfloor \lambda_n \rfloor} \sum_{\ell=0}^{L} C_{m\ell} M_k^{\lambda_n - m} (\ln M_k)^\ell + o(1). \tag{B.11}$$

If $n < 0$, we immediately deduce that J_k^n tends to 0 as k tends toward infinity. For $n > 0$, since $\lambda_n \notin \mathbb{N}$, $\lambda_n - m \neq 0$. But, since the limit is finite, the coefficients $C_{m\ell}$ have to vanish and we conclude that $\lim_{k\to\infty} J_k^n = 0$. □

Lemma B.4. *For any $(n,p) \in \mathbb{Z} \times \mathbb{R}$, there exists a sequence $(C_{n,p,t,q})_{t\in\mathbb{N}, q\in\mathbb{N}, q\leq p}$ such that, for any $s \in \mathbb{N}$,*

$$\int_{I_k^+} \partial_{\mathbf{n}} (R^+)^{\lambda_n - p} w_{n,p,+}(\ln R^+, \theta^+) \chi_{\text{macro},+}(X_1^+, X_2^+) \mathrm{d}\sigma(\mathbf{X}) = \sum_{t=0}^{s} \sum_{q=0}^{p} C_{n,p,t,q} (M_k)^{\lambda_n - p - t} (\ln M_k)^q + o\big((M_k)^{\lambda_n - p - s}\big). \tag{B.12}$$

Proof. We decompose I_k^+ into its circular part

$$I_1 = \left\{ \left(R_k \cos\theta^+, R_k \sin\theta^+\right) \in \mathbb{R}^2, \theta^+ \in (0, \pi - \theta_k) \cup (\pi + \theta_k, \Theta) \right\}, \tag{B.13}$$

$R_k = \sqrt{M_k^2+4}$, and its straight part

$$I_2 = \left\{(-M_k, X_2^+) \in \mathbb{R}^2, X_2^+ \in (-2,2)\right\}, \tag{B.14}$$

and we study the integral over these two parts separately.

Integration over I_1: On this part, the normal derivative is $\partial_{\mathbf{n}} = \partial_{R^+}$ and $\chi_{\text{macro},+} = 1$. Using the explicit form (C.7) of the function $w_{n,p,+}(\ln R^+, \theta^+)$, we see that

$$J_1 = \int_{I_{k,c}} \partial_{\mathbf{n}} \left\{(R^+)^{\lambda_n - p} w_{n,p,+}(\ln R^+, \theta)\chi_{\text{macro},+}\right\} \mathrm{d}\sigma = (R_k^+)^{\lambda_n-p} \sum_{q=0}^{p} (\ln R_k^+)^q \int_{|\theta^+ - \pi| \geq \theta_k} v_{n,p,q,+}(\theta^+) d\theta^+.$$

where the functions $v_{n,p,q,+}$ are smooth on the intervals $(0, \pi-\theta_k)$ and $(\pi-\theta_k, \Theta)$. On $(0,\pi-\theta_k)$ (resp. $(\pi-\theta_k,\Theta)$), we denote by $V_{n,p,q,+}$ the primitive of $v_{n,p,q,+}$ that vanishes at 0 (resp. Θ). The function $V_{n,p,q,+}$ is smooth on both $(0,\pi-\theta_k)$ and $(\pi-\theta_k,\Theta)$.

$$J_1 = (R_k^+)^{\lambda_n-p} \sum_{q=0}^{p} (\ln R_k^+)^q (V_{n,p,q,+}(\pi-\theta_k) - V_{n,p,q,+}(\pi+\theta_k)) \tag{B.15}$$

Then, we use Taylor expansion of $V_{n,p,q,+}$ at the point $\theta = \pi^\pm$ ($V_{n,p,q,+}$ is not continuous at π)

$$V_{n,p,q,+}(\pi-\theta_k) = \sum_{r=0}^{N} \frac{V_{n,p,q,+}^{(r)}(\pi^+)}{r!} (\theta_k)^r + o((\theta_k)^N)$$

and the following expansions to conclude:

$$\begin{aligned}
&\forall s \in \mathbb{R},\ \exists\, (\alpha_{i,s})_{i\in\mathbb{N}},\ \forall N \in \mathbb{N}, \quad R_k^s = \sum_{i=0}^{N} M_k^{s-i}\alpha_{i,s} + o((M_k)^{s-N}),\\
&\forall m \in \mathbb{R},\ \exists\, (\beta_{i,m,\ell})_{i\in\mathbb{N},\ell\in\mathbb{N},\ell\leq m},\ \forall N \in \mathbb{N},\\
&\qquad (\ln R_k)^m = \sum_{i=0}^{N}\sum_{\ell=0}^{m} (\ln M_k)^\ell \beta_{i,m,\ell} M_k^{-i} + o((M_k)^{-N}),\\
&\forall m \in \mathbb{R},\ \exists\, (\gamma_{i,m})_{i\in\mathbb{N}},\ \forall N \in \mathbb{N}, \quad (\theta_k)^m = \sum_{i=0}^{N} \gamma_{i,m} M_k^{-i} + o((M_k)^{-N}).
\end{aligned}$$

Integration over I_2: On this part, $\partial_{\mathbf{n}} = -\partial_{X_1}^+$, and $\chi_{\text{macro}}(\mathbf{X}^+) = \chi(X_2)$. Then,

$$J_2 = \int_{(-2,-1)\cup(1,2)} -\partial_{X_1^+} \mathtt{v}_{n,p}(\mathbf{X})\chi(X_2^+) dX_2^+, \quad \text{with } \mathtt{v}_{n,p}(\mathbf{X}) = (R^+)^{\lambda_n-p} \sum_{q=0}^{p} \left(\ln R^+\right)^q w_{n,p,q,+}(\theta^+)$$

We remind that $\mathtt{v}_{n,p}$ is harmonic in both Ω_{T} and Ω_{B}. We shall compute the integral over $(1,2)$, the computation of the integral over $(-2,-1)$ being similar. First, since $\partial_{X_1^+}\mathtt{v}_{n,p}(\mathbf{X}^+) = \cos\theta^+ \partial_{R^+}\mathtt{v}_{n,p}(R^+,\theta^+) - \frac{1}{R^+}\sin(\theta^+)\partial_{\theta^+}\mathtt{v}_{n,p}(R^+,\theta^+)$, there exists smooth functions $v_{n,p,q,+}$ such that

$$\partial_{X_1^+}\mathtt{v}_{n,p}(\mathbf{X}^+) = (R^+)^{\lambda_n-p-1} \sum_{q=0}^{p} \left(\ln R^+\right)^q v_{n,p,q,+}(\theta^+). \tag{B.16}$$

Since $R^+ = M_k\sqrt{1+\frac{X_2^2}{M_k^2}}$ and $\theta^+ = \tan^{-1}\left(\frac{X_2^+}{X_1^+}\right)$ to obtain the following asymptotic formula (reminding that X_2 is bounded):

$$\begin{aligned}
&\forall s \in \mathbb{Z}, \exists (\tilde{\alpha}_{i,s})_{i\in\mathbb{N}}, \forall N \in \mathbb{N}, (R^+)^s = \sum_{i=0}^{N} \tilde{\alpha}_{i,s} (X_2^+)^{2i} (M_k)^{s-2i} + o((M_k)^{s-2N})\\
&\forall m \in \mathbb{R},\ \exists\, (\tilde{\beta}_{i,m,\ell})_{i\in\mathbb{N},\ell\in\mathbb{N},\ell\leq m},\ \forall N \in \mathbb{N},\\
&\quad (\ln R^+)^m = \sum_{i=0}^{N}\sum_{\ell=0}^{m} (\ln M_k)^{\ell} \tilde{\beta}_{i,m,\ell} (X_2^+)^{2i} M_k^{-2i} + o((M_k)^{-2N}),\\
&\exists (\gamma_i)_{i\in\mathbb{N}}, \forall N \in \mathbb{N}, v_{n,p,q,+}(\theta^+) = \sum_{i=0}^{N} \tilde{\gamma}_i\, (X_2)^i\, (M_k)^{-i} + o((M_k)^{-N})
\end{aligned}$$

Introducing the previous formulas into (B.16), integrating exactly with respect to X_2 gives the desired formula. □

Lemma B.5. *For any $(n, q) \in \mathbb{Z}\times\mathbb{R}$ and for any $s \in \mathbb{N}$, there exists a sequence $(C'_{n,q,t,r})_{t\leqslant s, r\leq q}$ such that*

$$\int_{I_k^+} \partial_{\mathbf{n}} (X_1^+)^{\lambda_n - q} p_{n,q,+}(\ln|X_1^+|, X_1^+, X_2^+)\chi_-(X_1^+)\mathrm{d}\sigma(\mathbf{X}) = \sum_{t=0}^{s}\sum_{r=0}^{q} C'_{n,q,t,r} (M_k)^{\lambda_n - q - t} (\ln M_k)^r + o\big((M_k)^{\lambda_n - q - s}\big). \tag{B.17}$$

Proof. As in the proof of Lemma B.4, we decompose I_k^+ into its circular part $I_1$and its straight part I_2 (*cf.* (B.13)−(B.14)), and we study the integral over these two parts separately.

Integration over I_2: on this part, $X_1^+ = -M_k$ $\partial_{\mathbf{n}} = -\partial^+_{X_1}$, and $\chi_-(X_1^+) = \chi(M_k) = 1$ for $k \geqslant 2$. Then,

$$J_2 = \int_{-2}^{2} -\partial_{X_1^+}\left((X_1^+)^{\lambda_n - q} p_{n,q,+}(\ln|X_1^+|, \mathbf{X}^+)\right) dX_2^+.$$

We use expression of $p_{n,q,+}$ given by (C.12) in Appendix C.3, which yields to consider intergrals of the form

$$\int_{-2}^{2} (X_1^+)^{\lambda_n - q - 1} \left(\ln X_1^+\right)^{\kappa} W(X_1^+, X_2^+) dX_2^+ \quad \kappa \in \mathbb{N},$$

where the functions W are one periodic with respect to X_1^+ ($W(X_1^+, X_2^+) = W(-M_0, X_2^+)$). Moreover, since, by assumption the line $X_1^+ = -M_0$ does not intersect any obstacle, the functions $W(-M_0, X_2^+)$ are continuous and bounded for $X_2^+ \in [-2, 2]$. Then, integrating exactly with respect to X_2 gives the desired formula.

Integration over I_1: On this part, the normal derivative is given by

$$\partial_{\mathbf{n}} = \partial_{R^+} = \frac{X_1^+}{R_k}\partial_{X_1^+} + \frac{X_2^+}{R_k}\partial_{X_2^+}, \quad R_k = \sqrt{M_k^2 + 4}. \tag{B.18}$$

Here again, we separate this integral into three arcs:

$$I_1^1 = \left\{\left(R_k\cos\theta^+, R_k\sin\theta^+\right) \in \mathbb{R}^2, \theta^+ \in (0, \pi/2 + \theta'_k) \cup \left(\frac{3\pi}{2} - \theta'_k, \Theta\right)\right\}, \quad \theta'_k = \sin^{-1}(1/R_k)$$

$$I_1^2 = \left\{\left(R_k\cos\theta^+, R_k\sin\theta^+\right) \in \mathbb{R}^2, \theta^+ \in (\pi/2 + \theta'_k, \pi - \theta''_k) \cup \left(\pi + \theta''_k, \frac{3\pi}{2} - \theta'_k\right)\right\}, \quad \theta''_k = \sin^{-1}\left(\frac{\alpha \ln M_k}{R_k}\right)$$

$$I_1^3 = \left\{\left(R_k\cos\theta^+, R_k\sin\theta^+\right) \in \mathbb{R}^2, \theta^+ \in (\pi - \theta''_k, \pi - \theta_k) \cup (\pi + \theta_k, \pi + \theta''_k)\right\}.$$

The parameter $\alpha \in \mathbb{R}_+^*$ (defining θ_k'') will be fixed later. We remark that $R_k\cos(\pi/2+\theta_k') = -1$ and $R_k\sin(\pi \mp \theta_k'') = \pm\alpha \ln M_k$.

Integration over I_1^1: this integration is trivial, because on this integration domain, $X_1^+ \geqslant R_k\cos(\pi/2+\theta_k') \geqslant -1$ and therefore $\chi_-(X_1^+) = 0$.

Integration over I_1^2: because the functions $p_{n,q,+}$ are exponentially decaying with respect to X_2^+, we shall prove that the corresponding integral is $o(M_k)^{\lambda_n - p - s}$. First, we remind that the profile functions $W_i^{\mathbf{t}}$ and $W_i^{\mathbf{n}}$ are in $\mathcal{V}^+(\mathcal{B})$ and are smooth on I_1^2. It follows that $W_i^{\mathbf{t}}$, $W_i^{\mathbf{n}}$ and their derivatives can be bounded by $C_i \exp(-\pi X_2^+)$. Since $|X_1^+| \leq R_k$, it follows that there exists $C > 0$ such that

$$J_1^2 = \int_{I_1^2} \partial_{\mathbf{n}} \left\{ (X_1^+)^{\lambda_n - p} p_{n,p,+}(\ln|X_1^+|, X_1^+, X_2^+)\chi_-(X_1^+) \right\} \mathrm{d}\sigma(\mathbf{X}) \leqslant C \left(R_k\right)^{\lambda_n - p} |\ln R_k|^p \int_{I_1^2} \exp(-\pi X_2^+)\mathrm{d}\sigma(\mathbf{X}) \tag{B.19}$$

We parametrize then the arc I_1^2 by $X_2 \in \pm \Big(\alpha \ln M_k, R_k\cos(\tilde{\theta}_k)\Big)$, which means that $X_1^+ = -\sqrt{R_k^2 - X_2^2}$ and $\mathrm{d}\sigma(\mathbf{X}) = |R_k/X_1^+| dX_2$. In addition, since $|X_1^+| \geqslant 1$ on I_1^2 (by construction), $|R_k^+/X_1^+| \leqslant R_k$. Therefore,

$$J_1^2 \leq \left(R_k\right)^{\lambda_n - p + 1} |C \ln R_k|^p \int_{\alpha \ln M_k}^{\infty} \exp(-\pi X_2^+) dX_2^+ = \frac{C}{\pi}\left(R_k\right)^{\lambda_n - p + 1} |\ln R_k|^p M_k^{-\pi\alpha},$$

which is equivalent to $M_k^{\lambda_n - p + 1 - \pi\alpha} |\ln M_k|^p$ as k tends toward infinity. In the end, choosing $\alpha = (s+2)/\pi$, we see that $J_1^2 = o\big((M_k)^{\lambda_n - p - s}\big)$.

Integration over I_1^3: on this integration domain, $\chi_-(X_1^+) = 1$, and $|X_2^+| \in (2, \alpha \ln M_k)$. It follows that X_2/R_k is uniformly bounded by $\alpha \ln M_k / M_k$, which tends to 0 as M_k tends to infinity. Combining formula (B.18) and the definition (C.12) of the function $p_{n,q,+}$, we see that we have to evaluate the two following kinds of integrals:

$$J_1^3 = \int_{I_1^3} \frac{(X_1^+)^{\lambda_n - q}}{R_k} \ln(|X_1^+|)^\kappa W(X_1^+, X_2^+) \mathrm{d}\sigma, \quad K_1^3 = \int_{I_1^3} \frac{X_2^+ (X_1^+)^{\lambda_n - q}}{R_k} \ln(|X_1^+|)^\kappa W(X_1^+, X_2^+) \mathrm{d}\sigma \tag{B.20}$$

where $\kappa \in \mathbb{N}$ and $W \in \mathcal{V}^+(\mathcal{B})$ (exponentially decaying with respect to X_2^+) is a (generic) 1-periodic function in X_1^+. In fact, W stands for either the profile functions $W_i^{\mathbf{t}}$ and $W_i^{\mathbf{n}}$ (defined in (C.1)−(C.4)) or their partial derivatives with respect to X_1^+ and X_2^+. Consequently, W admits the following Fourier series decomposition for $|X_2^+| > 2$:

$$\exists R \in \mathbb{N}, \exists\, (c_{r,p,\pm})_{r \leqslant R, p \in \mathbb{Z}^*}, \quad W(\mathbf{X}) = \sum_{r=0}^{R} \sum_{p \neq 0} c_{r,p,\pm} \exp(\imath 2\pi p X_1^+), (X_2^+)^r \exp(-2\pi p |X_2^+|), \tag{B.21}$$

the coefficients $c_{r,p,\pm}$ being super-algebraically convergent as $p \to \pm\infty$, *i.e.*

$$\forall r \in \mathbb{N}, r \leq R, \forall \beta \in \mathbb{R}, \quad \sum_{p \neq 0} p^\beta c_{r,p,\pm} \exp(-4\pi p) < \infty. \tag{B.22}$$

Since $X_1^+ = -\sqrt{R_k^2 - (X_2^+)^2} = -M_k\sqrt{1 + \frac{4 - X_2^2}{M_k^2}}$, similarly to the proof of Lemma B.4, the following expansions hold:

$$\forall s \in \mathbb{Z}, \exists (\tilde{\alpha}_{i,j,s})_{(i,j) \in \mathbb{N}^2}, \forall N \in \mathbb{N}, \quad (X_1^+)^s = \sum_{i=0}^{N} \sum_{j=0}^{i} \tilde{\alpha}_{i,j,s} (X_2^+)^{2j} (M_k)^{s - 2i} + M_k^s o((\ln M_k / M_k)^{2N}) \tag{B.23}$$

$\forall m\in\mathbb{R},\ \exists\,(\tilde\beta_{i,j,m,\ell})_{(i,j,\ell)\in\mathbb{N}^3,\ell\le m},\ \forall N\in\mathbb{N},$

$$(\ln|X_1^+|)^m=\sum_{i=0}^{N}\sum_{j=0}^{i}\sum_{\ell=0}^{m}(\ln M_k)^\ell\tilde\beta_{i,j,m,\ell}(X_2^+)^{2j}M_k^{-2i}+o\left(\frac{(\ln M_k)^{m+2N}}{M_k^{2N}}\right)\cdot \quad \text{(B.24)}$$

Then, here again, we parameterize the arc I_1^3 by $X_2^+\in\pm(2,\alpha\ln M_k)$. Expanding $|R_k/X_1^+|$ with respect to X_2^+, we obtain

$$\exists(\tilde\gamma_{i,j})_{(i,j)\in\mathbb{N}^2},\forall N\in\mathbb{N},\quad \mathrm{d}\sigma(\mathbf{X})=\sum_{i=0}^{N}\sum_{j=0}^{i}\tilde\gamma_{i,j}(X_2^+)^{2j}(M_k)^{-2i}dX_2^++o\left(\left(\frac{\alpha\ln M_k}{M_k}\right)^{-2N}\right)dX_2^+, \quad \text{(B.25)}$$

and

$$\exists(\tilde\delta_{1,i,j})_{(i,j)\in\mathbb{N}^2},\forall N\in\mathbb{N},\qquad X_1^+/R_k=\sum_{i=0}^{N}\sum_{j=0}^{i}\tilde\delta_{1,i,j}(X_2^+)^{2j}(M_k)^{-2i}+o\left(\left(\frac{\alpha\ln M_k}{M_k}\right)^{-2N}\right), \quad \text{(B.26)}$$

$$\exists(\tilde\delta_{2,i})_{i\in\mathbb{N}},\forall N\in\mathbb{N},\quad X_2^+/R_k=\sum_{i=0}^{N}\tilde\delta_{2,i}X_2^+(M_k)^{-2i-1}+o\left(\frac{\alpha\ln M_k}{(M_k)^{2N+1}}\right)\cdot \quad \text{(B.27)}$$

It remains to expand $W(X_1^+,X_2^+)$, expressing X_1^+ in terms of X_2^+. More specifically, thanks to (B.21), we have to compute $\exp(2\imath\pi X_1^+)$ for any $p\in\mathbb{Z}^*$. Since $M_k=M_0+k$,

$$\exp(2\imath\pi pX_1^+)=\exp(-2\imath\pi p\sqrt{R_k^2-X_2^2})=\exp(-2\imath\pi pM_0)\exp\left(-2\imath\pi pM_k\left(\sqrt{1+\frac{4-(X_2^+)^2}{M_k^2}}-1\right)\right)\cdot$$

Then, using that $M_k(\sqrt{1+\frac{4-(X_2^+)^2}{M_k^2}}-1)=O(\ln^2M_k/M_k)$ which tends to 0 as M_k tends to 0, for p fixed, we can make a Taylor expansion of this exponential term with respect to X_2:

$$\begin{aligned}\exists(\tilde\zeta_i)_{i\in\mathbb{N}},\quad \forall N\in\mathbb{N},\exp&\left(-2\imath\pi pM_k\left(\sqrt{1+\frac{4-(X_2^+)^2}{M_k^2}}-1\right)\right)\\ &=\sum_{n=0}^{N}\frac{(-2\imath\pi p)^n}{n!}\left[M_k\left(\sqrt{1+\frac{4-(X_2^+)^2}{M_k^2}}-1\right)\right]^n+\mathcal{R}_N(p)\phi((\ln^2M_k/M_k)^N)\end{aligned} \quad \text{(B.28)}$$

where the remainder $\mathcal{R}_N(p)$ is polynomial with respect to p and behaves like $(2\pi p)^N/(N!)$ for N fixed as $p\to\infty$, and the function $\phi(x)$ is $o(x)$ as $x\to0$. In (B.28), expanding the polynomial sum with respect to X_2^+ and neglecting the terms in $o(M_k)^{-N}$ gives

$$\begin{aligned}\exists(\tilde\zeta_{i,j})_{(i,j)\in\mathbb{N}^2},\quad \forall N\in\mathbb{N},\exp&\left(2\imath\pi pM_k\left(\sqrt{1+\frac{4-(X_2^+)^2}{M_k^2}}-1\right)\right)\\ &1+\sum_{n=1}^{N}\frac{(2\imath\pi p)^n}{n!}M_k^n\sum_{i=1}^{\lfloor(N+n)/2\rfloor}\sum_{j=0}^{i}\tilde\zeta_{i,j}(X_2^+)^{2j}(M_k)^{-2i}+\frac{(2\pi p)^N}{N!}\tilde\phi((\ln^2M_k/M_k)^N),\end{aligned} \quad \text{(B.29)}$$

where $\tilde{\phi}(x)$ is also $o(x)$ as $x \to 0$. Finally, we insert (B.29) in (B.21) and we obtain

$$W(\mathbf{X}) = \sum_{r=0}^{R}\sum_{p\neq 0} c_{r,p,\pm}\left(1+\sum_{n=1}^{N}\frac{(2\imath\pi p)^n}{n!}M_k^n\sum_{\ell=1}^{\lfloor (N+n)/2\rfloor}\sum_{j=0}^{\ell}\tilde{\zeta}_{\ell,j}(X_2^+)^{2j}(M_k)^{-2\ell}\right)(X_2^+)^r\exp(-2\pi p|X_2^+|)$$
$$+\sum_{r=0}^{R}\sum_{p\neq 0} c_{r,p,\pm}(X_2^+)^r\exp(-2\pi p|X_2^+|)\mathcal{R}_N(p)\tilde{\phi}((\ln^2 M_k/M_k)^N) \tag{B.30}$$

To estimate the remainder in (B.30), we use that

$$(X_2^+)^r\exp(-2\pi p|X_2^+|)\mathcal{R}_N(p)\tilde{\phi}((\ln^2 M_k/M_k)^N) \leqslant (\alpha\ln M_2^+)^R\exp(-4\pi p)\mathcal{R}_N(p)\tilde{\phi}((\ln^2 M_k/M_k)^N),$$

which, together with (B.22) taking $\beta = -2-N$ gives

$$\sum_{r=0}^{R}\sum_{p\neq 0} c_{r,p,\pm}(X_2^+)^r\exp(-2\pi p|X_2^+|)\mathcal{R}_N(p)\tilde{\phi}((\ln^2 M_k/M_k)^N) = o\left(\frac{(\ln M_k)^{2N+R}}{(M_K)^N}\right).$$

Finally, we insert the expansions (B.23)–(B.27)–(B.30) written up to order $N = \lambda_n - q - s$ into (B.20). We obtain

$$J_1^3 = \sum_{t=0}^{s}\sum_{r=0}^{\kappa}(M_k)^{\lambda_n-q-t}(\ln M_k)^r\sum_{i=0}^{Q}\sum_{p\neq 0} c_{i,p,t,r,\pm}\int_{\pm 2}^{\pm\alpha\ln M_k}(X_2^+)^i\exp(-2\pi p|X_2^+|)dX_2^+$$
$$+\,o\big((\ln M_k)^{\tilde{Q}}(M_k)^{\lambda_n-q-s}\big), \tag{B.31}$$

where Q and $\tilde{Q}$ are positive integers depending on s, R and κ. Note also that the sum over p converges using again (B.22) with $\beta = -2-Q$. To conclude, it remains to estimate each integral that appears on (B.31). A direct integration by parts gives, for any numbers $0 < a < b$,

$$\int_a^b (X_2^+)^i\exp(-2\pi p X_2^+)dX_2^+ = i!\sum_{k=0}^{i}\frac{(2\pi p)^{k-1-i}}{k!}\big(a^i\exp(-2p\pi a) - b^i\exp(-2p\pi b)\big). \tag{B.32}$$

We use then (B.32) for $a = 2$ and $b = \alpha\ln M_k$, such that $b^i\exp(-2\pi p b) = (\alpha\ln M_k)^i(M_k)^{-2\pi p\alpha}$. Using that $\alpha = (s+2)/\pi$, the sum of $b^i\exp(-2\pi b)$ over i is negligible with respect to $(M_k)^{t-s}$ $(-t-s-4<0)$. Then (B.32) becomes

$$\int_2^{\alpha\ln M_k}(X_2^+)^i\exp(-2\pi p|X_2^+|)dX_2^+ = i!\exp(-4\pi p)\sum_{k=0}^{i}\frac{(2\pi p)^{k-1-i}2^i}{k!} + o((M_k)^{t-s}). \tag{B.33}$$

Inserting (B.33) in (B.31) gives the desired result for J_1^3, the analysis of K_1^3 being similar. □

Appendix C. Complete definition of the asymptotic blocks

The definition of the asymptotic block $\mathcal{U}_{n,p,\pm}$ (4.3) requires the definition of the functions $w_{n,q,\pm}$ and $p_{n,q,\pm}$. To do that, we first need to introduce two families of boundary layer functions $W_i^{\mathfrak{t}}$ and $W_i^{\mathfrak{n}}$.

C.1. Two families of boundary layer profile functions $W_i^{\mathfrak{t}}$ and $W_i^{\mathfrak{n}}$

Let $W_i^{\mathfrak{t}} = 0$ for any negative integer i, and, for $i \geq 0$, we define $W_i^{\mathfrak{t}} \in \mathcal{V}^+(\mathcal{B})$ as the unique decaying solution to

$$\left\{\begin{array}{rcl} -\varDelta_{\mathbf{X}}W_i^{\mathfrak{t}}(\mathbf{X}) &=& F_i^{\mathfrak{t}}(\mathbf{X}) + \dfrac{\mathcal{D}_i^{\mathfrak{t}}}{2}[g_0(\mathbf{X})] + \dfrac{\mathcal{N}_i^{\mathfrak{t}}}{2}[g_1(\mathbf{X})] \quad \text{in} \quad \mathcal{B},\\ \partial_{\mathbf{n}}W_i^{\mathfrak{t}} &=& G_i^{\mathfrak{t}}(\mathbf{X}) \quad \text{on} \quad \partial\widehat{\Omega}_{\text{hole}},\\ \partial_{X_1}W_i^{\mathfrak{t}}(0,X_2) &=& \partial_{X_1}W_i^{\mathfrak{t}}(1,X_2), \quad X_2\in\mathbb{R}, \end{array}\right. \tag{C.1}$$

where $G_i^{\mathfrak{t}}(\mathbf{X}) = -W_{i-1}^{\mathfrak{t}}\mathbf{e}_1 \cdot \mathbf{n}$ and

$$\begin{aligned} F_i^{\mathfrak{t}}(\mathbf{X}) = {} & 2\partial_{X_1} W_{i-1}^{\mathfrak{t}}(\mathbf{X}) \,+\, W_{i-2}^{\mathfrak{t}}(\mathbf{X}) \,+\, (-1)^{\lfloor i/2 \rfloor} \left(2 \langle g_i(\mathbf{X}) \rangle\, \delta_i^{\text{even}}\right) \\ & + \sum_{k=2}^{i-1} (-1)^{\lfloor k/2 \rfloor} \frac{[g_k(\mathbf{X})]}{2} \delta_k^{\text{even}} \mathcal{D}_{i-k}^{\mathfrak{t}} + \sum_{k=2}^{i-1} (-1)^{\lfloor k/2 \rfloor} \frac{[g_k(\mathbf{X})]}{2}\, \delta_k^{\text{odd}}\, \mathcal{N}_{i-k+1}^{\mathfrak{t}}. \end{aligned} \tag{C.2}$$

In (C.2), the constants $\mathcal{D}_i^{\mathfrak{t}}$ and $\mathcal{N}_i^{\mathfrak{t}}$ are given by

$$\mathcal{D}_i^{\mathfrak{t}} = \int_{\mathcal{B}} F_i^{\mathfrak{t}}\, \mathcal{D} + \int_{\partial \widehat{\Omega}_{\text{hole}}} G_i^{\mathfrak{t}}\, \mathcal{D}, \qquad \mathcal{N}_i^{\mathfrak{t}} = -\int_{\mathcal{B}} F_i^{\mathfrak{t}} \mathcal{N} - \int_{\partial \widehat{\Omega}_{\text{hole}}} G_i^{\mathfrak{t}} \mathcal{N}. \tag{C.3}$$

and, for $k \in \mathbb{N}$, $\langle g_k(\mathbf{X}) \rangle := \frac{1}{2}[\varDelta, \chi_+ + \chi_-]\left(\frac{X_2^k}{k!}\right)$, $[g_k(\mathbf{X})] := [\varDelta, \chi_+ - \chi_-]\left(\frac{X_2^k}{k!}\right)$. Moreover, δ_k^{odd} is equal to the remainder of the euclidian division of k by 2 (*i.e.* δ_k^{odd} is equal to 1 if k is odd and equal to 0 if k is even), $\delta_k^{\text{even}} = 1 - \delta_k^{\text{odd}}$ and, $\lfloor r \rfloor$ denotes the floor of a real number r.

Similarly, let $W_i^{\mathfrak{n}} = 0$, for $i \le 0$. Then, for $i \ge 1$, we define $W_i^{\mathfrak{n}} \in \mathcal{V}^+(\mathcal{B})$ as the unique decaying solution to

$$\left\{ \begin{aligned} -\varDelta_{\mathbf{X}} W_i^{\mathfrak{n}}(\mathbf{X}) &= F_i^{\mathfrak{n}}(\mathbf{X}) + \frac{\mathcal{D}_i^{\mathfrak{n}}}{2}[g_0(\mathbf{X})] + \frac{\mathcal{N}_i^{\mathfrak{n}}}{2}[g_1(\mathbf{X})] \quad \text{in } \mathcal{B}, \\ \partial_{\mathbf{n}} W_i^{\mathfrak{n}} &= G_i^{\mathfrak{n}}(\mathbf{X}) \quad \text{on } \partial \widehat{\Omega}_{\text{hole}}, \\ \partial_{X_1} W_i^{\mathfrak{n}}(0, X_2) &= \partial_{X_1} W_i^{\mathfrak{n}}(1, X_2), \quad X_2 \in \mathbb{R}, \end{aligned} \right. \tag{C.4}$$

where $G_i^{\mathfrak{n}}(\mathbf{X}) = -W_{i-1}^{\mathfrak{n}}\mathbf{e}_1 \cdot \mathbf{n}$ and

$$\begin{aligned} F_i^{\mathfrak{n}}(\mathbf{X}) = {} & 2\partial_{X_1} W_{i-1}^{\mathfrak{n}}(\mathbf{X}) \,+\, W_{i-2}^{\mathfrak{n}}(\mathbf{X}) \,+\, (-1)^{\lfloor i/2 \rfloor} \left(2 \langle g_i(\mathbf{X}) \rangle\, \delta_i^{\text{odd}}\right) \\ & + \sum_{k=2}^{i-1} (-1)^{\lfloor k/2 \rfloor} \frac{[g_k(\mathbf{X})]}{2} \delta_k^{\text{even}} \mathcal{D}_{i-k}^{\mathfrak{n}} + \sum_{k=2}^{i-1} (-1)^{\lfloor k/2 \rfloor} \frac{[g_k(\mathbf{X})]}{2} \delta_k^{\text{odd}}\, \mathcal{N}_{i-k+1}^{\mathfrak{n}}, \end{aligned} \tag{C.5}$$

the constants $\mathcal{D}_i^{\mathfrak{n}}$ and $\mathcal{N}_i^{\mathfrak{n}}$ being given by

$$\mathcal{D}_i^{\mathfrak{n}} = \int_{\mathcal{B}} F_i^{\mathfrak{n}}\, \mathcal{D} + \int_{\partial \widehat{\Omega}_{\text{hole}}} G_i^{\mathfrak{n}}\, \mathcal{D}, \qquad \mathcal{N}_i^{\mathfrak{n}} = -\int_{\mathcal{B}} F_i^{\mathfrak{n}} \mathcal{N} - \int_{\partial \widehat{\Omega}_{\text{hole}}} G_i^{\mathfrak{n}} \mathcal{N}. \tag{C.6}$$

Remark C.1. The well posedness of problem (C.4) and problem (C.1) results from the application of Proposition 3.2 noticing that the right-hand sides of problem (C.4) and problem (C.1) satisfy the conditions ($\mathcal{C}_{\mathcal{D}}$)–($\mathcal{C}_{\mathcal{N}}$).

C.2. Definition of the profile functions $w_{n,q,\pm}$

We shall construct the functions $w_{n,q,\pm}$ as

$$w_{n,q,\pm}(\ln R^{\pm}, \theta^{\pm}) = \sum_{s=0}^{q} w_{n,q,s,\pm}(\theta^{\pm})(\ln R^{\pm})^s, \quad q \in \mathbb{N}, \quad w_{n,q,s,\pm} \in \mathcal{C}^{\infty}(\overline{I_1^{\pm}}) \cap \mathcal{C}^{\infty}(\overline{I_2^{\pm}}), \tag{C.7}$$

where $I_1^{\pm} = (a^{\pm}, \gamma^{\pm})$, $I_2^{\pm} = (\gamma^{\pm}, b^{\pm})$ with $a^+ = 0$, $\gamma^+ = \pi$, $b^+ = \varTheta$, and, $a^- = \pi - \varTheta$, $\gamma^- = 0$, $b^- = \pi$. The construction is done by induction on q. The functions $w_{n,0,\pm}$ have already been defined in (4.2):

$$w_{0,0,\pm}(\ln R^{\pm}, \theta^{\pm}) = \ln R^{\pm}, \quad w_{n,0,+}(\theta^+) = \cos(\lambda_n \theta^+), \quad w_{n,0,-}(\theta^-) = \cos\left(\lambda_n(\theta^- - \pi)\right),$$

For $q \ge 1$, we construct $w_{n,q,\pm}$ of the form (C.7) such that the function

$$\mathtt{v}_{n,q,\pm}(R^{\pm}, \theta^{\pm}) = (R^{\pm})^{\lambda_n - q} w_{n,q,\pm}(\ln R^{\pm}, \theta^{\pm})$$

satisfies

$$\left\{\begin{array}{rcl} \Delta \mathtt{v}_{n,q,\pm} &=& 0 \text{ in } \mathcal{K}_1^{\pm} \cap \mathcal{K}_2^{\pm}, \\ \partial_\theta \mathtt{v}_{n,q,\pm}(a^{\pm}) = \partial_\theta \mathtt{v}_{n,q,\pm}(b^{\pm}) &=& 0, \\ [\mathtt{v}_{n,q,\pm}(R^{\pm},\gamma^{\pm})]_{\partial\mathcal{K}_1^{\pm}\cap\partial\mathcal{K}_2^{\pm}} &=& (R^{\pm})^{\lambda_n - q}\, \mathtt{a}_{n,q,\pm}(\ln R^{\pm}), \\ [\partial_{\theta^{\pm}} \mathtt{v}_{n,q,\pm}(R^{\pm},\gamma^{\pm})]_{\partial\mathcal{K}_1^{\pm}\cap\partial\mathcal{K}_2^{\pm}} &=& (R^{\pm})^{\lambda_n - q}\, \mathtt{b}_{n,q,\pm}(\ln R^{\pm}), \end{array}\right. \quad \forall q \in \mathbb{N}^*, \tag{C.8}$$

where, for $j = \{1,2\}$, $\mathcal{K}_j^{\pm} = \left\{(R^{\pm}\cos\theta^{\pm}, R^{\pm}\sin\theta^{\pm}) \in \mathcal{K}^{\pm},\ R^{\pm} \in \mathbb{R}^*, \theta^{\pm} \in I_j^{\pm}\right\}$, and,

$$\mathtt{a}_{n,q,\pm}(\ln R^{\pm}) = \sum_{r=0}^{q-1} \left(\mathcal{D}^{\mathtt{t}}_{q-r} g^{\mathtt{t}}_{n,r,q-r,\pm}(\ln R^{\pm}) + \mathcal{D}^{\mathtt{n}}_{q-r} g^{\mathtt{n}}_{n,r,q-r,\pm}(\ln R^{\pm})\right), \tag{C.9}$$

$$\mathtt{b}_{n,q,\pm}(\ln R^{\pm}) = \sum_{r=0}^{q-1} \left(\mathcal{N}^{\mathtt{t}}_{q+1-r} h^{\mathtt{t}}_{n,r,q-r,\pm}(\ln R^{\pm}) + \mathcal{N}^{\mathtt{n}}_{q+1-r} h^{\mathtt{n}}_{n,r,q-r,\pm}(\ln R^{\pm})\right). \tag{C.10}$$

The reals coefficients $\mathcal{D}^{\mathtt{t}}_i$, $\mathcal{D}^{\mathtt{n}}_i$, $\mathcal{N}^{\mathtt{t}}_i$ and $\mathcal{N}^{\mathtt{t}}_i$ are defined in (C.3)−(C.6). The functions $g^{\mathtt{t}}_{n,r,t,\pm}$, $g^{\mathtt{n}}_{n,r,t,\pm}$ are defined by the following relations: for $r \in \mathbb{N}$, $t \in \mathbb{N}$,

$$\begin{aligned} &(R^{\pm})^{\lambda_n - r - t}\ g^{\mathtt{t}}_{n,r,t,\pm}(\ln R^{+}) \\ &\quad = (\mp 1)^t \frac{\partial^t}{(\partial R^{\pm})^t} \left[(R^{\pm})^{\lambda_n - r} \langle w_{n,r,\pm}(\gamma^{\pm}, \ln R^{\pm})\rangle_{\partial\mathcal{K}_1^{\pm}\cap\partial\mathcal{K}_2^{\pm}}\right], \\ &(R^{\pm})^{\lambda_n - r - t}\ g^{\mathtt{n}}_{n,r,t,\pm}(\ln R^{+}) \\ &\quad = (\mp 1)^t \frac{\partial^{t-1}}{(\partial R^{\pm})^{t-1}} \left[(R^{\pm})^{\lambda_n - r - 1} \langle \partial_{\theta^{\pm}} w_{n,r,\pm}(\gamma^{\pm}, \ln R^{\pm})\rangle_{\partial\mathcal{K}_1^{\pm}\cap\partial\mathcal{K}_2^{\pm}}\right], (t \geq 1) \end{aligned}$$

$g^{\mathtt{n}}_{n,r,0,\pm} = 0$, $h^{\mathtt{t}}_{n,r,t,\pm} = \mp g^{\mathtt{t}}_{n,r,t+1,\pm}$ and $h^{\mathtt{n}}_{n,r,t,\pm} = \mp g^{\mathtt{n}}_{n,r,t+1,\pm}$.

The existence $w_{n,q,\pm}$ of the form (C.7) results from the following Lemma (see also Chap. 3 in Ref. [30] and the Sect. 6.4.2 in Ref. [28] for the proof).

Lemma C.2. *Let $j \in \mathbb{N}$, $\lambda \in \mathbb{R}$, $(\mathtt{a},\mathtt{b}) \in \mathbb{R}^2$ and*

$$N = \begin{cases} j & \text{if} \quad \lambda \notin \frac{\pi}{\Theta}\mathbb{Z}, \\ j+1 & \text{if} \quad \lambda \in \frac{\pi}{\Theta}\mathbb{Z}^*, \\ j+2 & \text{if} \quad \lambda = 0. \end{cases}$$

There exist $N+1$ functions $g_k \in \mathcal{C}^\infty(\overline{I_1^{\pm}}) \cap \mathcal{C}^\infty(\overline{I_2^{\pm}})$, $(0 \leq k \leq N)$, such that the function $\mathtt{v}(R^{\pm},\theta^{\pm}) = (R^{\pm})^{\lambda}\left(\sum_{k=0}^{N} (\ln R^{\pm})^k g_k(\theta^{\pm})\right)$ satisfies

$$\left\{\begin{array}{rcl} \Delta \mathtt{v} &=& 0 \text{ in } \mathcal{K}_1^{\pm} \cap \mathcal{K}_2^{\pm}, \\ \partial_\theta \mathtt{v}(R^{\pm},a^{\pm}) = \partial_\theta \mathtt{v}(R^{\pm},b^{\pm}) &=& 0, \\ [\mathtt{v}(R^{\pm},\gamma^{\pm})]_{\partial\mathcal{K}_1^{\pm}\cap\partial\mathcal{K}_2^{\pm}} &=& \mathtt{a}\,(R^{\pm})^{\lambda}\,\ln(R^{\pm})^j, \\ [\partial_\theta \mathtt{v}(R^{\pm},\gamma^{\pm})]_{\partial\mathcal{K}_1^{\pm}\cap\partial\mathcal{K}_2^{\pm}} &=& \mathtt{b}\,(R^{\pm})^{\lambda}\,\ln(R^{\pm})^j. \end{array}\right. \tag{C.11}$$

Remark C.3. If $\lambda_n - q \in \frac{\pi}{\Theta}\mathbb{Z}^*$, the function $w_{n,q,\pm}$ is not uniquely defined by (C.7) because we can add any multiple of the function $\theta^{\pm} \mapsto w_{\frac{\Theta}{\pi}(\lambda_n - q),0,\pm}(\theta^{\pm})$. In that case, we restore the uniqueness taking the orthogonal projection of $w_{n,q,0,\pm}$ with respect to $w_{\frac{\Theta}{\pi}(\lambda_n-q),0,\pm}$, *i.e.*

$$\int_{a^\pm}^{b^\pm} w_{n,q,0,\pm}(\ln R^\pm,\theta^\pm) w_{\frac{\Theta}{\pi}(\lambda_n-q),0,\pm}(\theta^\pm)d\theta^\pm = 0, \quad \left(\lambda_n - q \in \frac{\pi}{\Theta}\mathbb{Z}^*, q \geq 1\right).$$

Similarly, for $n > 0$, if $\lambda_n - q = 0$, the function $w_{n,q,\pm}$ is not uniquely defined by (C.7), because we can add any multiple of the functions 1 and $\ln R^\pm$. Here again, the uniqueness is restored by imposing $\int_{a^\pm}^{b^\pm} w_{n,q,0,\pm} d\theta^\pm = \int_{a^\pm}^{b^\pm} w_{n,q,1,\pm} d\theta^\pm = 0$.

C.3. Definition of the profile functions $p_{n,q,\pm}$

Finally, the functions $p_{n,q,\pm}$ are given by

$$p_{n,q,\pm}(\ln|X_1^\pm|,\mathbf{X}^\pm) = \sum_{i=0}^{q} g^{\mathfrak{t}}_{n,q-i,i,+}(\ln|X_1^\pm|)\, W_i^{\mathfrak{t}}(\mathbf{X}^\pm) + \sum_{i=1}^{q} g^{\mathfrak{n}}_{n,q-i,i,+}(\ln|\ln X_1^\pm|)\, W_i^{\mathfrak{n}}(\mathbf{X}^\pm). \tag{C.12}$$

Acknowledgements. The authors would like to thank Robert Gruhlke (TU Berlin) for helpful discussions and the support of the implementation in Concepts related to the project. They gratefully acknowledge the financial support by the Einstein Foundation Berlin (grant number IPF-2011-98).

References

[1] T. Abboud and H. Ammari, Diffraction at a curved grating: TM and TE cases, homogenization. *J. Math. Anal. Appl.* **202** (1996) 995–1026.

[2] M. Abramowitz and I.A. Stegun, Handbook of mathematical functions with formulas, graphs, and mathematical tables, vol. 55 of *National Bureau of Standards Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. (1964).

[3] Y. Achdou, Etude de la réflexion d'une onde électromagnétique par un métal recouvert d'un revêtement métallisé. Technical report, INRIA (1989).

[4] Y. Achdou, Effect of a thin metallized coating on the reflection of an electromagnetic wave. *C. R. Acad. Sci. Paris, Ser. I* **314** (1992) 217–222.

[5] Y. Achdou, O. Pironneau and F. Valentin, Effective boundary conditions for laminar flows over periodic rough boundaries. *J. Comput. Phys.* **147** (1998) 187–218.

[6] M. Artola and M. Cessenat, Diffraction d'une onde électromagnétique par une couche composite mince accolée à un corps conducteur épais. I. Cas des inclusions fortement conductrices. *C. R. Acad. Sci. Paris, Ser. I* **313** (1991) 231–236.

[7] M. Artola and M. Cessenat, Scattering of an electromagnetic wave by a slender composite slab in contact with a thick perfect conductor. II. Inclusions (or coated material) with high conductivity and high permeability. *C. R. Acad. Sci. Paris, Ser. I* **313** (1991) 381–385.

[8] A. Bendali, A. Makhlouf and S. Tordeux, Field behavior near the edge of a microstrip antenna by the method of matched asymptotic expansions. *Quart. Appl. Math.* **69** (2011) 691–721.

[9] V. Bonnaillie−Noël, M. Dambrine, S. Tordeux and G. Vial, Interactions between moderately close inclusions for the Laplace equation. *Math. Models Meth. Appl. Sci.* **19** (2009) 1853–1882.

[10] A.-S. Bonnet-Ben Dhia, D. Drissi and N. Gmati, Mathematical analysis of the acoustic diffraction by a muffler containing perforated ducts. *Math. Models Meth. Appl. Sci.* **15** (2005) 1059–1090.

[11] D. Bresch and V. Milisic, High order multi-scale wall-laws, Part I: the periodic case. *Quart. Appl. Math.* **68** (2010) 229–253.

[12] H. Brezis, Functional analysis, Sobolev spaces and partial differential equations. Universitext. Springer, New York (2011).

[13] G. Caloz, M. Costabel, M. Dauge and G. Vial, Asymptotic expansion of the solution of an interface problem in a polygonal domain with thin layer. *Asymptot. Anal.* **50** (2006) 121–173.

[14] I. S. Ciuperca, M. Jai and C. Poignard, Approximate transmission conditions through a rough thin layer: the case of periodic roughness. *European J. Appl. Math.* **21** (2010) 51–75.

[15] X. Claeys, On the theoretical justification of Pocklington's equation. *Math. Models Meth. Appl. Sci.* **19** (2009) 1325–1355.

[16] X. Claeys and B. Delourme, High order asymptotics for wave propagation across thin periodic interfaces. *Asymptot. Anal.* **83** (2013) 35–82.

[17] Concepts Development Team. Webpage of Numerical C++ Library Concepts 2. http://www.concepts.math.ethz.ch (2016).

[18] M. Dauge, S. Tordeux and G. Vial, Selfsimilar perturbation near a corner: matching versus multiscale expansions for a model problem. In Around the research of Vladimir Maz'ya. II, vol. 12 of *Int. Math. Ser. (N. Y.)*. Springer-Verlag, New York (2010) 95–134.

[19] B. Delourme, *Modèles et asymptotiques des interfaces fines et périodiques en électromagnétisme* Ph.D. thesis, Université Pierre et Marie Curie (2010).

[20] B. Delourme, K. Schmidt and A. Semin, When a thin periodic layer meets corners: asymptotic analysis of a singular poisson problem. Technical report (2015).

[21] B. Delourme, K. Schmidt and A. Semin, On the homogenization of thin perforated walls of finite length. *Asymptotic Anal.* **97** (2016) 211–264.

[22] Ph. Frauenfelder and Ch. Lage, Concepts – an object-oriented software package for partial differential equations. *ESAIM: M2AN* **36** (2002) 937–951.

[23] C.I. Goldstein, A finite element method for solving Helmholtz type equations in waveguides and other unbounded domains. *Math. Comput.* **39** (1982) 309–324.

[24] A.M. Il′in, Matching of asymptotic expansions of solutions of boundary value problems, vol. 102 of Translations of Mathematical Monographs. Translated from the Russian by V. Minachin (V.V. Minakhin). *American Mathematical Society*, Providence, RI (1992).

[25] P. Joly and S. Tordeux, Matching of asymptotic expansions for wave propagation in media with thin slots. I. The asymptotic expansion. *Multiscale Model. Simul.* **5** (2006) 304–336.

[26] P. Joly and S. Tordeux, Matching of asymptotic expansions for waves propagation in media with thin slots. II. The error estimates. *ESAIM: M2AN* **42** (2008) 193–221.

[27] P. Joly and A. Semin. Construction and analysis of improved kirchoff conditions for acoustic wave propagation in a junction of thin slots. *ESAIM Proc.* **25** (2008) 44–67.

[28] V.A. Kozlov, V.G. Mazya and J. Rossmann, Elliptic boundary value problems in domains with point singularities, vol. 52 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI (1997).

[29] A.L. Madureira and F. Valentin. Asymptotics of the Poisson problem in domains with curved rough boundaries. *SIAM J. Math. Anal.* **38** (2006/07) 1450–1473.

[30] A. Makhlouf, *Justification et amélioration de modèles d'antenne patch par la méthode des développements asymptotiques raccordés.* Ph.D. thesis, Institut National des Sciences Appliquées de Toulouse (2008).

[31] V. Maz'ya, S. Nazarov and B. Plamenevskij, Asymptotic theory of elliptic boundary value problems in singularly perturbed domains. Vol. 111 of *Operator Theory: Advances and Applications.* Translated from the German by Georg Heinig and Christian Posthoff. Birkhäuser Verlag, Basel (2000).

[32] S.A. Nazarov. The Neumann problem in angular domains with periodic and parabolic perturbations of the boundary. *Tr. Mosk. Mat. Obs.* **69** (2008) 182–241.

[33] S. Nicaise and A.-M. Sändig, General interface problems. I, II. *Math. Methods Appl. Sci.* **17** (1994) 395–429, 431–450.

[34] G. Panasenko, High order asymptotics of solutions of problems on the contact of periodic structures. *Sbornik: Math.* **38** (1981) 465–494.

[35] G.A. Pavliotis and A.M. Stuart, Multiscale Methods: Averaging and Homogenization. Springer (2008).

[36] J.-R. Poirier, A. Bendali, and P. Borderies, Impedance boundary conditions for the scattering of time-harmonic waves by rapidly varying surfaces. *IEEE Trans. Antennas and Propagation* **54** (2006) 995–1005.

[37] J.-R. Poirier, A. Bendali, P. Borderies and S. Tournier, High order asymptotic expansion for the scattering of fast oscillating periodic surfaces. In *Proc. 9th Int. Conf. on Mathematical and Numerical Aspects of Waves Propagation (Waves 2009), Pau, France* (2009).

[38] J. Rauch and M. Taylor, Potential and scattering theory on wildly perturbed domains *J. Functional Anal.* **18** (1975) 27–59.

[39] E. Sánchez−Palencia, Nonhomogeneous media and vibration theory. Vol. 127 of *Lecture Notes in Physics.* Springer Verlag, Berlin (1980).

[40] E. Sánchez−Palencia, Un problème d'écoulement lent d'un fluide incompressible au travers d'une paroi finement perforée. In Homogenization methods: theory and applications in physics (Bréau-sans-Nappe 1983), vol. 57 of Collect. Dir. Études Rech. Élec. France. Eyrolles, Paris (1985) 371–400.

[41] K. Schmidt and P. Kauf, Computation of the band structure of two-dimensional photonic crystals with *hp* finite elements. *Comput. Methods Appl. Mech. Engrg.* **198** (2009) 1249–1259.

[42] C. Schwab, *p*- and *hp*-finite element methods: Theory and applications in solid and fluid mechanics. Oxford University Press, Oxford, UK (1998).

[43] M. Van Dyke, Perturbation methods in fluid mechanics. *Applied Mathematics and Mechanics.* Vol. 8. Academic Press, New York (1964).