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LINEAR QUADRATIC MEAN FIELD GAME WITH CONTROL INPUT CONSTRAINT*

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Abstract. In this paper, we study a class of linear-quadratic (LQ) mean-field games in which the individual control process is constrained in a closed convex subset Γ of full space $\mathbb{R}^m$. The decentralized strategies and consistency condition are represented by a class of mean-field forward-backward stochastic differential equation (MF-FBSDE) with projection operators on Γ. The wellposedness of consistency condition system is obtained using the monotonicity condition method. The related ϵ-Nash equilibrium property is also verified.

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1. Introduction

Our starting point comes from the recently well-studied mean-field games (MFGs) for large-population system, which arises naturally in various fields such as economics, engineering, social science and operational research, etc. The most salient feature of large-population system is the existence of a large number of individually negligible agents (or players) which are interrelated in their dynamics and (or) cost functionals via the state-average (in linear case) or more generally, the generated empirical measure over the whole population (in nonlinear case). Because of this highly complicated coupling feature, it is intractable for a given agent to employ the centralized optimization strategies based on the information of all its peers in large-population system. Actually, this will bring considerably high computational complexity in a large-scale manner. Alternatively, one reasonable yet practical direction is to investigate the related decentralized strategies based on local information only. By local information, we mean that the related strategies should be designed upon the individual state (or, random noise) of the given agent, together with some mass-effect quantities, which can be computed in off-line manner.

Keywords and phrases. ϵ-Nash equilibrium, mean-field forward-backward stochastic differential equation (MF-FBSDE), linear-quadratic constrained control, projection, monotonic condition..

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Along this research direction, one efficient and tractable methodology leading to decentralized strategies is the MFGs, which generally result in a coupled system of HJB equation and Fokker-Planck (FP) equation in nonlinear case. In principle, the procedure of MFGs consists of the following four main steps (see [2,5,12,13,15,19], etc). In Step 1, it is necessary to analyze the asymptotic behavior of state-average when the agent number N tends to infinity and introduce the related state-average limiting term. Of course, this limiting term is undetermined at this moment, thus it should be treated as some exogenous "frozen" term. Step 2 turns to study the related limiting optimization problem (which is also called auxiliary or tracking problem) by adopting the frozen limit term instead of its state-average. The initial high-coupled optimization problems of all agents are therefore decoupled and only parameterized by this generic frozen limit. The related decentralized optimal strategy can be analyzed using standard control techniques such as dynamic programming principle (DPP) or stochastic maximum principle (SMP) (see *e.g.*, [21]). As a result, some HJB equation (due to DPP) or Hamiltonian system (due to SMP) will be obtained to characterize this decentralized optimality. Step 3 aims to determine the frozen state-average limit by some consistency condition: While applying the optimal decentralized strategies derived in Step 2, the state-average limit here should be reproduced as the agents number tends to infinity. Accordingly, some fixed-point analysis should be applied and some FP equation will be introduced by coupling with the HJB equation in Step 2. As the necessary verification, Step 4 will show that the derived decentralized strategies should possess the ϵ-Nash equilibrium properties. A comprehensive survey of MFG can be found in [4].

For further analysis of MFGs, the interested readers may refer to [8] for a survey of mean-field games focusing on the partial differential equation aspect and related real applications; [2] for more recent MFG studies and the related mean-field type control; [5] for the probabilistic analysis of a large class of stochastic differential games for which the interaction between the players is of mean-field type; [6] for the mean-field game where considerable interrelated banks share the system risk and common noise; [18] for a class of risk-sensitive mean-field stochastic differential games; [14] for MFGs with nonlinear diffusion dynamics and their relations to McKean-Vlasov particle system. It is remarkable that there exists a substantial literature body to the study of MFGs in the linear-quadratic (LQ) framework. Here, we mention a few of them which are more relevant to our current work: [11] the mean-field LQ games with a major player and a large number of minor players, [13] the mean-field LQ games with non-uniform agents through the state-aggregation by empirical distribution, [16] the mean-field LQ mixed games with continuum-parameterized minor players.

In this paper, we discuss the linear-quadratic (LQ) mean-field game where the individual control is constrained in a closed convex set Γ of full space: $\Gamma \subset \mathbb{R}^m$. The LQ problems with control constraint arise naturally from various practical applications. For instance, the no-shorting constraint in portfolio selection leads to the LQ control with positive control ($\Gamma = \mathbb{R}^m_+$, the positive orthant). Moreover, due to general market accessibility constraint, it is also interesting to study the LQ control with more general closed convex cone constraint (see [9]). As a response, this paper investigates the LQ dynamic game of large-population system with general closed convex control constraint. The control constraint brings some new features to our study here: (1) the related consistency condition (CC) system is no longer linear, and it becomes a class of nonlinear FBSDE with projection operator. (2) Due to the nonlinearity of (1), the standard Riccati equation with feedback control is no longer valid to represent the consistency condition of limit state-average process. Instead, the consistency condition is embedded into a class of mean-field FBSDE with a generic driven Brownian motion. Our investigation is mainly sketched as follows. First, applying the maximum principle, the optimal decentralized response is characterized through some Hamiltonian system with projection operator upon the constrained set Γ. Second, the consistency condition system is connected to the well-posedness of some mean-field forward-backward stochastic differential equation (MF-FBSDE). Next, we present some monotonicity condition of this MF-FBSDE to obtain its uniqueness and existence. Last, the related approximate Nash equilibrium property is also verified. We derive the MFG strategy in its open-loop manner. Consequently, the approximate Nash equilibrium property is verified under the open-loop strategies perturbation and some estimates of forward-backward SDE are involved. In addition, all agents are set to be statistically identical thus the limiting control problem and fixed-point arguments are given for a *representative* agent. In case the agents are heterogeneous with different parameters, the similar procedure to MFG strategies can be proceeded via the introduction of index indicator and empirical state-average statistics.

The reminder of this paper is structured as follows: Section 2 formulates the LQ MFGs with control constraint. The decentralized strategies are derived with the help of a forward-backward SDE with projection operators. The consistency condition is also established. Section 3 verifies the $\epsilon-$Nash equilibrium of the decentralized strategies. Section 4 is appendix.

2. Mean-Field LQG games with control constraint

Throughout this paper, we denote the k-dimensional Euclidean space by $\mathbb{R}^k$ with standard Euclidean norm $|\cdot|$ and standard Euclidean inner product $\langle\cdot,\cdot\rangle$. The transpose of a vector (or matrix) x is denoted by $x^\top$. $\mathrm{Tr}(A)$ denotes the trace of a square matrix A. Let $\mathbb{R}^{m\times n}$ be the Hilbert space consisting of all $(m\times n)$-matrices with the inner product $\langle A,B\rangle:=\mathrm{Tr}(AB^\top)$ and the norm $|A|:=\langle A,A\rangle^{\frac{1}{2}}$. Denote the set of symmetric $k\times k$ matrices with real elements by S^k. If $M\in S^k$ is positive (semi)definite, we write $M>\ (\geq)\ 0$. $L^\infty(0,T;\mathbb{R}^k)$ is the space of uniformly bounded $\mathbb{R}^k-$valued functions. If $M(\cdot)\in L^\infty(0,T;S^k)$ and $M(t)>\ (\geq)\ 0$ for all $t\in[0,T]$, we say that $M(\cdot)$ is positive (semi) definite, which is denoted by $M(\cdot)>\ (\geq)\ 0$. $L^2(0,T;\mathbb{R}^k)$ is the space of all $\mathbb{R}^k-$valued functions satisfying $\int_0^T|x(t)|^2\mathrm{d}t<\infty$.

Consider a finite time horizon $[0,T]$ for fixed $T>0$. We assume $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{0\leq t\leq T},P)$ is a complete, filtered probability space on which a standard N-dimensional Brownian motion $\{W_i(t),\ 1\leq i\leq N\}_{0\leq t\leq T}$ is defined. For given filtration $\mathbb{F}=\{\mathcal{F}_t\}_{0\leq t\leq T}$, let $L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)$ denote the space of all $\mathcal{F}_t$-progressively measurable $\mathbb{R}^k$-valued processes satisfying $\mathbb{E}\int_0^T|x(t)|^2\mathrm{d}t<\infty$. Let $L^{2,\mathcal{E}_0}_{\mathbb{F}}(0,T;\mathbb{R}^k)\subset L^2_{\mathbb{F}}(0,T;\mathbb{R}^k)$ be the subspace satisfying $\mathbb{E}x_t\equiv 0$ for $x.\in L^{2,\mathcal{E}_0}_{\mathbb{F}}(0,T;\mathbb{R}^k)$.

Now let us consider a large-population system with N weakly-coupled negligible agents $\{\mathcal{A}_i\}_{1\leq i\leq N}$. The state x^i for each $\mathcal{A}_i$ satisfies the following controlled linear stochastic system:

$$
\begin{cases}
\mathrm{d}x^i(t)=[A(t)x^i(t)+B(t)u_i(t)+F(t)x^{(N)}(t)+b(t)]\mathrm{d}t\\
\qquad\qquad +[D(t)u_i(t)+\sigma(t)]\mathrm{d}W_i(t),\\
x^i(0)=x\in\mathbb{R}^n,
\end{cases}
\tag{2.1}
$$

where $x^{(N)}(\cdot)=\dfrac{1}{N}\displaystyle\sum_{i=1}^N x^i(\cdot)$ is the state-average, $(A(\cdot),B(\cdot),F(\cdot),b(\cdot);D(\cdot),\sigma(\cdot))$ are matrix-valued functions with appropriate dimensions to be identified. For sake of presentation, we set all agents are homogeneous or statistically symmetric with same coefficients $(A,B,F,b;D,\sigma)$ and deterministic initial states x.

Example 2.1. In case $A(t)=-F(t)=a,b(t)=0$, our current model can be reduced to the system risk model in [6] where u_i (α_t therein) denotes the borrowing-lending rate. In case the rate is constraint by market (*e.g.*, see [7] the convex constraint with market segments and investment restriction), we have the input constraint model given by (2.1).

Now we identify the information structure of large population system: $\mathbb{F}^i=\{\mathcal{F}^i_t\}_{0\leq t\leq T}$ is the natural filtration generated by $\{W_i(t),0\leq t\leq T\}$ and augmented by all $P-$null sets in $\mathcal{F}$. $\mathbb{F}=\{\mathcal{F}_t\}_{0\leq t\leq T}$ is the natural filtration generated by $\{W_i(t),1\leq i\leq N,0\leq t\leq T\}$ and augmented by all $P-$null sets in $\mathcal{F}$. Thus, $\mathbb{F}^i$ is the individual decentralized information of ith Brownian motion while $\mathbb{F}$ is the centralized information driven by all Brownian motion components. Note that the heterogeneous noise W_i is specific for individual agent $\mathcal{A}_i$ but $x^i(t)$ is adapted to $\mathcal{F}_t$ instead of $\mathcal{F}^i_t$ due to the coupling state-average $x^{(N)}$.

The (centralized) admissible control $u_i\in\mathcal{U}^c_{\mathrm{ad}}$ where the (centralized) admissible control set $\mathcal{U}^c_{\mathrm{ad}}$ is defined as

$$\mathcal{U}^c_{\mathrm{ad}}:=\{u_i(\cdot)|u_i(\cdot)\in L^2_{\mathbb{F}}(0,T;\Gamma),\quad 1\leq i\leq N\}$$

where $\Gamma \subset \mathbb{R}^m$ is a closed convex set. By "centralized," we mean $\mathbb{F}$ is the centralized information generated by all Brownian motion components. Typical examples of such set is $\Gamma = \mathbb{R}^m_+$ which represents the positive control. Moreover, we also define decentralized control as $u_i \in \mathcal{U}^{d,i}_{\mathrm{ad}}$, where the decentralized admissible control set $\mathcal{U}^{d,i}_{\mathrm{ad}}$ is defined as

$$\mathcal{U}^{d,i}_{\mathrm{ad}} := \{u_i(\cdot) | u_i(\cdot) \in L^2_{\mathbb{F}^i}(0,T;\Gamma), \quad 1 \le i \le N\}.$$

Note that both $\mathcal{U}^{d,i}_{\mathrm{ad}}$ and $\mathcal{U}^{c}_{\mathrm{ad}}$ are defined in open-loop sense, and $\mathcal{U}^{d,i}_{\mathrm{ad}} \subset \mathcal{U}^{c}_{\mathrm{ad}}$. Let $u = (u_1, \dots, u_i, \dots, u_N)$ denote the set of control strategies of all N agents and $u_{-i} = (u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_N)$ denote the control strategies set except the ith agent $\mathcal{A}_i$. Introduce the cost functional of $\mathcal{A}_i$ as

$$\begin{aligned}\mathcal{J}_i(u_i, u_{-i}) = \frac{1}{2}\mathbb{E}\bigg[\int_0^T \Big\langle Q(t)\big(x^i(t) - x^{(N)}(t)\big), x^i(t) - x^{(N)}(t) \Big\rangle \, \mathrm{d}t + \int_0^T \langle R(t)u_i(t), u_i(t)\rangle \, \mathrm{d}t \\ + \Big\langle G\big(x^i(T) - x^{(N)}(T)\big), x^i(T) - x^{(N)}(T) \Big\rangle \bigg]. \end{aligned} \tag{2.2}$$

We impose the following assumptions:

(H1) $A(\cdot), F(\cdot) \in L^\infty(0,T;S^n), B(\cdot), D(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times m}), b(\cdot), \sigma(\cdot) \in L^\infty(0,T;\mathbb{R}^n)$;
(H2) $Q(\cdot) \in L^\infty(0,T;S^n), Q(\cdot) \ge 0, R(\cdot) \in L^\infty(0,T;S^m), R(\cdot) > 0$ and $R^{-1}(\cdot) \in L^\infty(0,T;S^m)$, $G \in S^n$, $G \ge 0$.

It follows that (2.1) admits a unique solution $x^i(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$ under admissible control $u_i \in \mathcal{U}^c_{\mathrm{ad}}$ with (**H1**) and (**H2**). Now, we formulate the large population LQG games with control constraint (**CC**).

Problem (CC). Find an open-loop Nash equilibrium strategies set $\bar{u} = (\bar{u}_1, \bar{u}_2, \dots, \bar{u}_N)$ satisfying

$$\mathcal{J}_i(\bar{u}_i(\cdot), \bar{u}_{-i}(\cdot)) = \inf_{u_i(\cdot)\in\mathcal{U}^c_{\mathrm{ad}}} \mathcal{J}_i(u_i(\cdot), \bar{u}_{-i}(\cdot)),$$

where $\bar{u}_{-i}$ represents $(\bar{u}_1, \dots, \bar{u}_{i-1}, \bar{u}_{i+1}, \dots, \bar{u}_N)$, the strategies of all agents except $\mathcal{A}_i$.

The study of (**CC**) is of heavy computational burden due to the highly-complicated coupling structure among these agents. Alternatively, one efficient method to search the approximate Nash equilibrium is the mean-field game theory, which bridges the "centralized" LQG games to the limiting LQG control problems, as the number of agents tends to infinity. To this end, we need to construct some auxiliary control problem using the frozen state-average limit. Based on it, we can find the decentralized strategies by consistency condition. More details are given below. Introduce the following auxiliary problem for $\mathcal{A}_i$:

$$\begin{cases} \mathrm{d}x^{i,\dagger}(t) = [A(t)x^{i,\dagger}(t) + B(t)u_i(t) + F(t)z(t) + b(t)]\mathrm{d}t \\ \qquad\qquad\quad + [D(t)u_i(t) + \sigma(t)]\mathrm{d}W_i(t), \\ x^{i,\dagger}(0) = x \in \mathbb{R}^n, \end{cases} \tag{2.3}$$

and limiting cost functional is given by

$$\begin{aligned} J_i(u_i) = \frac{1}{2}\mathbb{E}\bigg[\int_0^T \big\langle Q(t)\big(x^{i,\dagger}(t) - z(t)\big), x^{i,\dagger}(t) - z(t)\big\rangle + \langle R(t)u_i(t), u_i(t)\rangle \, \mathrm{d}t \\ + \big\langle G\big(x^{i,\dagger}(T) - z(T)\big), x^{i,\dagger}(T) - z(T)\big\rangle \bigg], \end{aligned} \tag{2.4}$$

where z is the average limit of realized states which should be determined by the consistency-condition (CC) in our later analysis (see (2.9)). Note that the auxiliary state $x^{i,\dagger}$ is different to the true state x^i. Also, the admissible control u_i in (2.3), (2.4) $\in \mathcal{U}_{\rm ad}^{d,i}$ whereas in (2.1), (2.2), the admissible control $\in \mathcal{U}_{\rm ad}^{c}$ (for sake of simplicity, we still denote them with the same notation). Now we formulate the following limiting stochastic optimal control (SOC) problem with control constraint (**LCC**).

Problem (LCC). For the ith agent, $i = 1, 2, \ldots, N$, find $u_i^*(\cdot) \in \mathcal{U}_{\rm ad}^{d,i}$ satisfying

$$J_i(u_i^*(\cdot)) = \inf_{u_i(\cdot)\in\mathcal{U}_{\rm ad}^{d,i}} J_i(u_i(\cdot)).$$

Then $u_i^*(\cdot)$ is called a decentralized optimal control for Problem (**LCC**). Note that the cost functional is strictly convex and coercive thus it admits a unique optimal control u_i^* (*e.g.*, see [1], Thm. 2.6.1.) Now we apply the maximum principle method to characterize u_i^* with the optimal state $x^{i,*}$. First, introduce the following adjoint process

$$\begin{cases} dp^i = -\big[A^\top p^i - Q(x^{i,*} - z)\big]\mathrm{d}t + q^i \mathrm{d}W_i(t), \\ p^i(T) = -G\big(x^{i,*}(T) - z(T)\big). \end{cases}$$

Applying the maximum principle, the Hamiltonian function can be expressed by

$$\begin{aligned} H^i = H^i(t, p^i, q^i, x^i, u_i) &= \big\langle p^i, Ax^i + Bu_i + Fz + b\big\rangle \\ &+ \big\langle q^i, Du_i + \sigma\big\rangle - \frac{1}{2}\big\langle Q(x^i - z), x^i - z\big\rangle - \frac{1}{2}\left\langle Ru_i, u_i\right\rangle. \end{aligned} \tag{2.5}$$

Since $\mathit{\Gamma}$ is a closed convex set, then maximum principle reads as the following local form

$$\left\langle \frac{\partial H^i}{\partial u_i}(t, p^{i,*}, q^{i,*}, x^{i,*}, u^{i,*}), u - u^{i,*} \right\rangle \le 0, \quad \text{for all } u \in \mathit{\Gamma},\ \text{a.e. } t \in [0, T],\ \mathbb{P} - a.s. \tag{2.6}$$

Hereafter, time argument is suppressed in case when no confusion occurs. Noticing (2.5), then (2.6) yields that

$$\big\langle B^\top p^{i,*} + D^\top q^{i,*} - Ru^{i,*}, u - u^{i,*}\big\rangle \le 0,\ \text{for all } u \in \mathit{\Gamma},\ \text{a.e. } t \in [0, T],\ \mathbb{P} - a.s.$$

or equivalently (noticing $R > 0$),

$$\left\langle R^{\frac{1}{2}}[R^{-1}(B^\top p^{i,*} + D^\top q^{i,*}) - u^{i,*}], R^{\frac{1}{2}}(u - u^{i,*})\right\rangle \le 0,\ \text{for all } u \in \mathit{\Gamma},\ \text{a.e. } t \in [0, T],\ \mathbb{P} - a.s. \tag{2.7}$$

As $R(\cdot) > 0$, we take the following norm on $\mathit{\Gamma} \subset \mathbb{R}^m$ (which is equivalent to its Euclidean norm)

$$\|x\|_R^2 = \langle\langle x, x\rangle\rangle := \left\langle R^{\frac{1}{2}}x, R^{\frac{1}{2}}x\right\rangle,$$

and by the well-known results of convex analysis, we obtain that (2.7) is equivalent to

$$u^{i,*}(t) = \mathbf{P}_{\mathit{\Gamma}}[R^{-1}(t)(B^\top(t)p^{i,*}(t) + D^\top(t)q^{i,*}(t))], \quad \text{a.e. } t \in [0, T],\ \mathbb{P} - a.s.,$$

where $\mathbf{P}_{\mathit{\Gamma}}(\cdot)$ is the projection mapping from $\mathbb{R}^m$ to its closed convex subset $\mathit{\Gamma}$ under the norm $\|\cdot\|_R$. For more details, see Appendix. From now on, we denote

$$\varphi(p, q) := \mathbf{P}_{\mathit{\Gamma}}\left[R^{-1}(t)\left(B^\top(t)p + D^\top(t)q\right)\right].$$

Here, for simplicity, the dependence of φ on time variable t is suppressed. The related Hamiltonian system becomes

$$\begin{cases} \mathrm{d}x^{i,*} = \Big[Ax^{i,*} + B\varphi(p^{i,*}, q^{i,*}) + Fz + b\Big]\mathrm{d}t + \Big[D\varphi(p^{i,*}, q^{i,*}) + \sigma\Big]\mathrm{d}W_i(t), \\ dp^{i,*} = -\big[A^\top p^{i,*} - Q(x^{i,*} - z)\big]\mathrm{d}t + q^{i,*}\mathrm{d}W_i(t), \\ x^{i,*}(0) = x, \qquad p^{i,*}(T) = -G\big(x^{i,*}(T) - z(T)\big). \end{cases}$$

Based on above analysis, it follows that

$$z(\cdot) = \lim_{N\to+\infty} \frac{1}{N}\sum_{i=1}^{N} x^{i,*}(\cdot) = \mathbb{E}x^{i,*}(\cdot). \tag{2.8}$$

Here, the first equality of (2.8) is due to the consistency condition: the frozen term $z(\cdot)$ should equal to the average limit of all realized states $x^{i,*}(\cdot)$; the second equality is due to the law of large numbers. Thus, by replacing z by $\mathbb{E}x^{i,*}$ in above Hamiltonian system, we get the following system

$$\begin{cases} \mathrm{d}x^{i,*} = \Big[Ax^{i,*} + B\varphi(p^{i,*}, q^{i,*}) + F\mathbb{E}x^{i,*} + b\Big]\mathrm{d}t + \Big[D\varphi(p^{i,*}, q^{i,*}) + \sigma\Big]\mathrm{d}W_i(t), \\ dp^{i,*} = -\big[A^\top p^{i,*} - Q(x^{i,*} - \mathbb{E}x^{i,*})\big]\mathrm{d}t + q^{i,*}\mathrm{d}W_i(t), \\ x^{i,*}(0) = x, \qquad p^{i,*}(T) = -G\big(x^{i,*}(T) - \mathbb{E}x^{i,*}(T)\big). \end{cases}$$

As all agents are statistically identical, thus we can suppress subscript "i" and the following consistency condition system arises for generic agent:

$$\begin{cases} \mathrm{d}x^* = \Big[Ax^* + B\varphi(p^*, q^*) + F\mathbb{E}x^* + b\Big]\mathrm{d}t + \Big[D\varphi(p^*, q^*) + \sigma\Big]\mathrm{d}W_t, \\ -dp^* = \big[A^\top p^* - Q(x^* - \mathbb{E}x^*)\big]\mathrm{d}t - q^*\mathrm{d}W_t, \\ x_0^* = x, \qquad p_T^* = -G\big(x_T^* - \mathbb{E}x_T^*\big). \end{cases} \tag{2.9}$$

Here, W stands for a generic Brownian motion on $(\Omega, \mathcal{F}, P)$, and denote $\mathbb{F}^W$ the natural filtration generated by it and augmented by all null-sets. $L^2_{\mathbb{F}^W}, L^{2,\mathcal{E}_0}_{\mathbb{F}^W}$ are defined in the similar way with $L^2_{\mathbb{F}}, L^{2,\mathcal{E}_0}_{\mathbb{F}}$ before. The system (2.9) is a nonlinear mean-field forward-backward SDE (MF-FBSDE) with projection operator. It characterizes the state-average limit $z = \mathbb{E}x$ and MFG strategies $\bar{u}_i = \varphi(p, q)$ for a generic agent in the combined manner. As an important issue, we need to prove the above consistency condition system admits a unique solution. We first present the following uniqueness and existence result.

Theorem 2.2. *Under* (**H1**), (**H2**), *there exists a unique adapted solution* $(x, p, q) \in L^2_{\mathbb{F}^W}(0, T; \mathbb{R}^n) \times L^{2,\mathcal{E}_0}_{\mathbb{F}^W}(0, T; \mathbb{R}^n) \times L^2_{\mathbb{F}^W}(0, T; \mathbb{R}^n)$ *to system* (2.9).

Proof. (**Uniqueness**) Suppose that there exists two solutions: (x^1, p^1, q^1), (x^2, p^2, q^2) and denote

$$\hat{x} = x^1 - x^2, \quad \hat{p} = p^1 - p^2, \quad \hat{q} = q^1 - q^2.$$

Then, we have

$$\begin{cases} d\hat{x} = \Big[A\hat{x} + B\widehat{\varphi}(\hat{p}, \hat{q}) + F\mathbb{E}\hat{x}\Big]\mathrm{d}t + D\widehat{\varphi}(\hat{p}, \hat{q})\mathrm{d}W_t, \\ -d\hat{p} = \big[A^\top \hat{p} - Q(\hat{x} - \mathbb{E}\hat{x})\big]\mathrm{d}t - \hat{q}\mathrm{d}W_t, \\ \hat{x}_0 = 0, \qquad \hat{p}_T = -G\big(\hat{x}_T - \mathbb{E}\hat{x}_T\big) \end{cases} \tag{2.10}$$

with

$$\widehat{\varphi}(\hat{p}, \hat{q}) := \varphi(p^1, q^1) - \varphi(p^2, q^2) := \mathbf{P}_\Gamma[R^{-1}(B^\top p^1 + D^\top q^1)] - \mathbf{P}_\Gamma[R^{-1}(B^\top p^2 + D^\top q^2)].$$

First, taking the expectation in the second equation of (2.10) yields $\mathbb{E}\hat{p}=0$. Applying Itô's formula to $\langle \hat{p},\hat{x}\rangle$ and taking expectations on both sides (also, noting $\mathbb{E}\hat{p}=0$, and the monotonicity property of $\widehat{\varphi}$), we have:

$$\begin{aligned}
0 &= \mathbb{E}\left\langle G\big(\hat{x}_T-\mathbb{E}\hat{x}_T\big),\hat{x}_T\right\rangle \\
&\quad + \mathbb{E}\int_0^T \big\langle (B^T\hat{p}_s+D^T\hat{q}_s),\widehat{\varphi}_s(\hat{p},\hat{q})\big\rangle + \langle \hat{x}_s, Q(\hat{x}_s-\mathbb{E}\hat{x}_s)\rangle \,\mathrm{d}s + \mathbb{E}\int_0^T \langle \hat{p}_s, F\mathbb{E}\hat{x}_s\rangle\,\mathrm{d}s \\
&\geq \mathbb{E}\left\langle G^{\frac12}\big(\hat{x}_T-\mathbb{E}\hat{x}_T\big),G^{\frac12}\big(\hat{x}_T-\mathbb{E}\hat{x}_T\big)\right\rangle + \mathbb{E}\int_0^T \left\langle Q^{\frac12}(\hat{x}_s-\mathbb{E}\hat{x}_s),Q^{\frac12}(\hat{x}_s-\mathbb{E}\hat{x}_s)\right\rangle \mathrm{d}s.
\end{aligned}$$

Thus, $G\big(\hat{x}_T-\mathbb{E}\hat{x}_T\big)=0$ and $Q\big(\hat{x}-\mathbb{E}\hat{x}\big)=0$ which implies $\hat{p}_s\equiv 0, \hat{q}_s\equiv 0$. Next, we have $\widehat{\varphi}(\hat{p},\hat{q})\equiv 0$ which further implies $\mathbb{E}\hat{x}_s\equiv 0$, hence $\hat{x}_s\equiv 0$. Hence the uniqueness follows.

(**Existence**) Consider a family of parameterized FBSDE as follows:

$$\begin{cases}
\mathrm{d}x^\alpha = \Big[\alpha\mathbf{B}(x^\alpha,p^\alpha,q^\alpha,\mathbb{E}x^\alpha)+\phi\Big]\mathrm{d}t + \Big[\alpha\Xi(p^\alpha,q^\alpha)+\psi\Big]\mathrm{d}W_t,\\
-dp^\alpha = \big[\alpha\mathbf{F}(x^\alpha,p^\alpha,\mathbb{E}x^\alpha)+\gamma-\mathbb{E}\gamma\big]\mathrm{d}t - q^\alpha\mathrm{d}W_t,\\
x_0^\alpha = x, \qquad p_T^\alpha = -\alpha G\big(x_T^\alpha-\mathbb{E}x_T^\alpha\big)+\xi-\mathbb{E}\xi,
\end{cases}$$

with

$$\begin{cases}
\mathbf{B} := Ax + B\varphi(p,q) + F\mathbb{E}x + b,\\
\Xi := D\varphi(p,q)+\sigma,\\
\mathbf{F} := A^Tp - Q(x-\mathbb{E}x).
\end{cases}$$

Here, (ϕ,ψ,γ) are given processes in $L^2_{\mathbb{F}^W}(0,T;\mathbb{R}^n)\times L^2_{\mathbb{F}^W}(0,T;\mathbb{R}^n)\times L^2_{\mathbb{F}^W}(0,T;\mathbb{R}^n)$, and ξ is a $\mathbb{R}^n$-valued square integrable random variable which is $\mathbb{F}^W_T$-measurable. When $\alpha=0$, we have a decoupled FBSDE whose solvability is trivial:

$$\begin{cases}
\mathrm{d}x = \phi\mathrm{d}t + \psi\mathrm{d}W_t,\\
-dp = (\gamma-\mathbb{E}\gamma)\mathrm{d}t - q\mathrm{d}W_t,\\
x_0 = x, \qquad p_T = \xi-\mathbb{E}\xi.
\end{cases}$$

Denote $\mathcal{M}(0,T)=L^2_{\mathbb{F}^W}(0,T;\mathbb{R}^n)\times L^{2,\mathcal{E}_0}_{\mathbb{F}^W}(0,T;\mathbb{R}^n)\times L^2_{\mathbb{F}^W}(0,T;\mathbb{R}^n)$. Now we introduce a mapping $I_{\alpha_0}:(x,p,q)\in\mathcal{M}(0,T)\longrightarrow (X,P,Q)\in\mathcal{M}(0,T)$ via the following FBSDE:

$$\begin{cases}
dX_t = \Big[\alpha_0\mathbf{B}(X_t,P_t,Q_t,\mathbb{E}X_t)+\delta\mathbf{B}(x_t,p_t,q_t,\mathbb{E}x_t)+\phi_t\Big]\mathrm{d}t\\
\qquad\quad + \Big[\alpha_0\Xi(P_t,Q_t)+\delta\Xi(p_t,q_t)+\psi_t\Big]\mathrm{d}W_t,\\
-dP_t = \Big[\alpha_0\mathbf{F}(X_t,P_t,\mathbb{E}X_t)+\gamma_t-\mathbb{E}\gamma+\delta\mathbf{F}(x_t,p_t,\mathbb{E}x_t)\Big]\mathrm{d}t - Q_t\mathrm{d}W_t,\\
X_0 = x, \qquad P_T = -\alpha_0G\big(X_T-\mathbb{E}X_T\big)-\delta G(x_T-\mathbb{E}x_T)+\xi-\mathbb{E}\xi.
\end{cases}$$

Considering $I_{\alpha_0}:(x,p,q)\longrightarrow(X,P,Q)$ and $I_{\alpha_0}:(x',p',q')\longrightarrow(X',P',Q')$ and

$$(\widehat{X},\widehat{P},\widehat{Q}) = (X-X',P-P',Q-Q')$$

$$\begin{cases}
d\widehat{X}_t = \Big[\alpha_0\widehat{\mathbf{B}}(\widehat{X}_t,\widehat{P}_t,\widehat{Q}_t,\mathbb{E}\widehat{X}_t)+\delta\widehat{\mathbf{B}}(\hat{x}_t,\hat{p}_t,\hat{q}_t,\mathbb{E}\hat{x}_t)\Big]\mathrm{d}t\\
\qquad\quad + \Big[\alpha_0\widehat{\Xi}(\widehat{P}_t,\widehat{Q}_t)+\delta\widehat{\Xi}(\hat{p}_t,\hat{q}_t)\Big]\mathrm{d}W_t,\\
-d\widehat{P}_t = \Big[\alpha_0\widehat{\mathbf{F}}(\widehat{X}_t,\widehat{P}_t,\mathbb{E}\widehat{X}_t)+\delta(\widehat{\mathbf{F}}(\hat{x}_t,\hat{p}_t,\mathbb{E}\hat{x}_t)\Big]\mathrm{d}t - \widehat{Q}_t\mathrm{d}W_t,\\
\widehat{X}_0 = 0, \qquad \widehat{P}_T = -\alpha_0G\big(\widehat{X}_T-\mathbb{E}\widehat{X}_T\big)-\delta G(\hat{x}_T-\mathbb{E}\hat{x}_T),
\end{cases}$$

with

$$\begin{cases} \widehat{\mathbf{B}} := \mathbf{B}(X_t, P_t, Q_t, \mathbb{E}X_t) - \mathbf{B}(X'_t, P'_t, Q'_t, \mathbb{E}X'_t), \\ \widehat{\Xi} := \Xi(P_t, Q_t) - \Xi(P'_t, Q'_t), \\ \widehat{\mathbf{F}} := \mathbf{F}(X_t, P_t, \mathbb{E}X_t) - \mathbf{F}(X'_t, P'_t, \mathbb{E}X'_t). \end{cases}$$

Note that $\mathbb{E}\widehat{P}_t \equiv 0$ because

$$\widehat{\mathbf{F}}(X_t, P_t, \mathbb{E}X_t) = A^\top \widehat{P} - Q(\widehat{X} - \mathbb{E}\widehat{X}), \qquad \text{and} \quad \mathbb{E}p_t \equiv 0.$$

Applying Itô formula to $\left\langle \widehat{P}, \widehat{X} \right\rangle$ and taking expectations on both sides:

$$\begin{aligned} \mathbb{E}\left\langle \widehat{X}_T, -\alpha_0 G\big(\widehat{X}_T - \mathbb{E}\widehat{X}_T\big) - \delta G(\hat{x}_T - \mathbb{E}\hat{x}_T) \right\rangle = \mathbb{E}\int_0^T & \left\langle \widehat{X}_s, -\alpha_0 \widehat{\mathbf{F}}(\widehat{X}_s, \widehat{P}_s, \mathbb{E}\widehat{X}_s) \right\rangle + \left\langle \widehat{X}_s, -\delta \widehat{\mathbf{F}}(\hat{x}_s, \hat{p}_s, \mathbb{E}\hat{x}_s) \right\rangle \\ & + \left\langle \widehat{P}_s, \alpha_0 \widehat{\mathbf{B}}(\widehat{X}_s, \widehat{P}_s, \widehat{Q}_s, \mathbb{E}\widehat{X}_s) \right\rangle + \left\langle \widehat{P}_s, \delta \widehat{\mathbf{B}}(\hat{x}_s, \hat{p}_s, \hat{q}_s, \mathbb{E}\hat{x}_s) \right\rangle \\ & + \left\langle \widehat{Q}_s, \alpha_0 \widehat{\Xi}(\widehat{P}_s, \widehat{Q}_s) \right\rangle + \left\langle \widehat{Q}_s, \delta \widehat{\Xi}(\hat{p}_s, \hat{q}_s) \right\rangle \mathrm{d}s. \end{aligned}$$

Rearranging the above terms, we have

$$\begin{aligned} & \alpha_0 \mathbb{E}\left\langle \widehat{X}_T, G\big(\widehat{X}_T - \mathbb{E}\widehat{X}_T\big) \right\rangle + \mathbb{E}\int_0^T \alpha_0 \left[\left\langle \widehat{X}_s, -\widehat{\mathbf{F}}(\widehat{X}_s, \widehat{P}_s, \mathbb{E}\widehat{X}_s) \right\rangle \right. \\ & \quad \left. + \left\langle \widehat{P}_s, \widehat{\mathbf{B}}(\widehat{X}_s, \widehat{P}_s, \widehat{Q}_s, \mathbb{E}\widehat{X}_s) \right\rangle + \left\langle \widehat{Q}_s, \widehat{\Xi}(\widehat{P}_s, \widehat{Q}_s) \right\rangle \right] \mathrm{d}s \\ & = \mathbb{E}\int_0^T \delta \left[\left\langle \widehat{X}_s, \widehat{\mathbf{F}}(\hat{x}_s, \hat{p}_s, \mathbb{E}\hat{x}_s) \right\rangle + \left\langle \widehat{P}_s, -\widehat{\mathbf{B}}(\hat{x}_s, \hat{p}_s, \hat{q}_s, \mathbb{E}\hat{x}_s) \right\rangle \right. \\ & \quad \left. + \left\langle \widehat{Q}_s, -\widehat{\Xi}(\hat{p}_s, \hat{q}_s) \right\rangle \right] \mathrm{d}s - \delta \mathbb{E}\left\langle \widehat{X}_T, G(\hat{x}_T - \mathbb{E}\hat{x}_T) \right\rangle. \end{aligned}$$

Hence,

$$\begin{aligned} & \alpha_0 \mathbb{E}|G^{\frac{1}{2}}(\widehat{X}_T - \mathbb{E}\widehat{X}_T)|^2 + \mathbb{E}\int_0^T \alpha_0 |Q^{\frac{1}{2}}(\widehat{X}_s - \mathbb{E}\widehat{X}_s)|^2 \mathrm{d}s \\ & \le \alpha_0 \mathbb{E}\left\langle \widehat{X}_T, G\big(\widehat{X}_T - \mathbb{E}\widehat{X}_T\big) \right\rangle + \mathbb{E}\int_0^T \alpha_0 \left[\left\langle \widehat{X}_s, -\widehat{\mathbf{F}}(\widehat{X}_s, \widehat{P}_s, \mathbb{E}\widehat{X}_s) \right\rangle \right. \\ & \quad \left. + \left\langle \widehat{P}_s, \widehat{\mathbf{B}}(\widehat{X}_s, \widehat{P}_s, \widehat{Q}_s, \mathbb{E}\widehat{X}_s) \right\rangle + \left\langle \widehat{Q}_s, \widehat{\Xi}(\widehat{P}_s, \widehat{Q}_s) \right\rangle \right] \mathrm{d}s \\ & = \mathbb{E}\int_0^T \delta \left[\left\langle \widehat{X}_s, \widehat{\mathbf{F}}(\hat{x}_s, \hat{p}_s, \mathbb{E}\hat{x}_s) \right\rangle + \left\langle \widehat{P}_s, -\widehat{\mathbf{B}}(\hat{x}_s, \hat{p}_s, \hat{q}_s, \mathbb{E}\hat{x}_s) \right\rangle \right. \\ & \quad \left. + \left\langle \widehat{Q}_s, -\widehat{\Xi}(\hat{p}_s, \hat{q}_s) \right\rangle \right] \mathrm{d}s - \delta \mathbb{E}\left\langle \widehat{X}_T, G(\hat{x}_T - \mathbb{E}\hat{x}_T) \right\rangle \\ & \le \delta C_1 \mathbb{E}\int_0^T (|\hat{x}_s|^2 + |\hat{p}_s|^2 + |\hat{q}_s|^2) \mathrm{d}s + \delta C_1 \mathbb{E}\hat{x}_T^2 + \delta C_1 \mathbb{E}\int_0^T (|\widehat{X}_s|^2 + |\widehat{P}_s|^2 + |\widehat{Q}_s|^2) \mathrm{d}s + \delta C_1 \mathbb{E}\widehat{X}_T^2. \end{aligned}$$

Here, the first inequality uses the monotonicity property of $\varphi(p, q)$ (Prop. 4.3). The second inequality is due to the basic geometric inequality and Lipschitz property of projection operator (Prop. 4.2).

Then, by standard estimates of BSDE:

$$\begin{aligned}\mathbb{E}\int_0^T\left(|\widehat{P}_s|^2+|\widehat{Q}_s|^2\right)\mathrm{d}s&\le\delta C_2\mathbb{E}\int_0^T(|\hat{x}_s|^2+|\hat{p}_s|^2+|\hat{q}_s|^2)\mathrm{d}s+\delta C_2\mathbb{E}|\hat{x}_T|^2\\&\quad+C_2\left(\alpha_0\mathbb{E}|G^{\frac{1}{2}}(\widehat{X}_T-\mathbb{E}\widehat{X}_T)|^2+\mathbb{E}\int_0^T\alpha_0|Q^{\frac{1}{2}}(\widehat{X}_s-\mathbb{E}\widehat{X}_s)|^2\mathrm{d}s\right)\\&\le\delta C_3\mathbb{E}\int_0^T(|\hat{x}_s|^2+|\hat{p}_s|^2+|\hat{q}_s|^2)\mathrm{d}s+\delta C_3\mathbb{E}|\hat{x}_T|^2.\end{aligned}$$

Next, by the standard estimate of forward SDEs:

$$\begin{aligned}\mathbb{E}\int_0^T|\widehat{X}_s|^2\mathrm{d}s+\mathbb{E}|\widehat{X}_T|^2&\le\delta C_4\mathbb{E}\int_0^T(|\hat{x}_s|^2+|\hat{p}_s|^2+|\hat{q}_s|^2)\mathrm{d}s+C_4\mathbb{E}\int_0^T\left(|\widehat{P}_s|^2+|\widehat{Q}_s|^2\right)\mathrm{d}s\\&\le\delta C_5\delta(\mathbb{E}\int_0^T(|\hat{x}_s|^2+|\hat{p}_s|^2+|\hat{q}_s|^2)\mathrm{d}s+\delta C_5\mathbb{E}|\hat{x}_T|^2.\end{aligned}$$

Based on the above estimates, we know the mapping I satisfying

$$\mathbb{E}\int_0^T\left(|\widehat{X}_s|^2+|\widehat{P}_s|^2+|\widehat{Q}_s|^2\right)\mathrm{d}s+\mathbb{E}|\widehat{X}_T|^2\le K\delta\left(\mathbb{E}\int_0^T\left(|\widehat{x}_s|^2+|\widehat{p}_s|^2+|\widehat{q}_s|^2\right)\mathrm{d}s+\mathbb{E}|\widehat{x}_T|^2\right).$$

It follows the mapping is a contraction and the existence follows immediately using the arguments presented in [10,17]. $\square$

3. ϵ-Nash equilibrium for problem (CC)

In above sections, we can characterize the decentralized strategies $\{\bar{u}_t^i, 1\le i\le N\}$ of Problem (**CC**) through the auxiliary (**LCC**) and consistency condition system. For sake of presentation, we alter the notation of consistency condition system to be $(\alpha^i,\beta^i,\gamma^i)$:

$$\begin{cases}\mathrm{d}\alpha^i=\Big[A\alpha^i+B\varphi(\beta^i,\gamma^i)+F\mathbb{E}\alpha^i+b\Big]\mathrm{d}t+\Big[D\varphi(\beta^i,\gamma^i)+\sigma\Big]\mathrm{d}W_i(t),\\ \mathrm{d}\beta^i=-\big(A^\top\beta^i-Q(\alpha^i-\mathbb{E}\alpha^i)\big)\mathrm{d}t+\gamma^i\mathrm{d}W_i(t),\\ \alpha^i(0)=x,\qquad \beta^i(T)=-G\big(\alpha^i(T)-\mathbb{E}\alpha^i(T)\big).\end{cases}\tag{3.1}$$

Now, we turn to verify the ϵ-Nash equilibrium of them. To start, we first present the definition of ϵ-Nash equilibrium.

Definition 3.1. A set of strategies, $\bar{u}_t^i\in\mathcal{U}_{\mathrm{ad}}^c$, $1\le i\le N$, for N agents, is called to satisfy an ϵ-Nash equilibrium with respect to costs $\mathcal{J}^i,\ 1\le i\le N$, if there exists $\epsilon=\epsilon(N)\ge0$, $\lim_{N\to+\infty}\epsilon(N)=0$, such that for any $1\le i\le N$, we have

$$\mathcal{J}^i(\bar{u}_t^i,\bar{u}_t^{-i})\le\mathcal{J}^i(u_t^i,\bar{u}_t^{-i})+\epsilon,\tag{3.2}$$

when any alternative strategy $u^i\in\mathcal{U}_{\mathrm{ad}}^c$ is applied by $\mathcal{A}_i$.

Remark 3.2. If $\epsilon=0$, then Definition 3.1 is reduced to the usual exact Nash equilibrium.

Now, we state the main result of this paper and its proof will be given later.

Theorem 3.3. *Under* **(H1)–(H2)**, $(\bar{u}_1,\bar{u}_2,\dots,\bar{u}_N)$ *is an ϵ-Nash equilibrium of* **Problem (CC)**.

The proof of Theorem 3.3 needs several lemmas which are presented later. For agent $\mathcal{A}_i$, recall that its decentralized open-loop optimal strategy is $\bar{u}_i = \varphi(\beta^i, \gamma^i)$. The decentralized state $\breve{x}_t^i$ is

$$\begin{cases} \mathrm{d}\breve{x}^i = \Big[A\breve{x}^i + B\varphi(\beta^i,\gamma^i) + F\breve{x}^{(N)} + b\Big]\,\mathrm{d}t + \big[D\varphi(\beta^i,\gamma^i) + \sigma\big]\,\mathrm{d}W_i(t), \\ \mathrm{d}\alpha^i = \big[A\alpha^i + B\varphi(\beta^i,\gamma^i) + F\mathbb{E}\alpha^i + b\big]\,\mathrm{d}t + \big[D\varphi(\beta^i,\gamma^i) + \sigma\big]\,\mathrm{d}W_i(t), \\ \mathrm{d}\beta^i = -\big[A^\top \beta^i - Q(\alpha^i - \mathbb{E}\alpha^i)\big]\mathrm{d}t + \gamma^i \mathrm{d}W_i(t), \\ \breve{x}^i(0) = \alpha^i(0) = x, \qquad \beta^i(T) = -G\big(\alpha^i(T) - \mathbb{E}\alpha^i(T)\big), \end{cases} \tag{3.3}$$

where $\breve{x}^{(N)} = \frac{1}{N}\sum_{i=1}^N \breve{x}^i$. We recall that $(\alpha^i, \beta^i, \gamma^i)$ satisfies (3.1).

For each $1 \le i \le N$, the monotonic fully coupled FBSDE (3.1) has a unique solution $(\alpha^i, \beta^i, \gamma^i) \in L^2_{\mathbb{F}^i}(0,T;\mathbb{R}^n) \times L^2_{\mathbb{F}^i}(0,T;\mathbb{R}^n) \times L^2_{\mathbb{F}^i}(0,T;\mathbb{R}^n)$. Thus, the system of all first equation of (3.3), $1 \le i \le N$, has also a unique solution $(\breve{x}^i)_i \in (L^2_{\mathbb{F}}(0,T;\mathbb{R}^n))^{\otimes N}$. Here, $\otimes N$ denotes the n-tuple Cartesian product. Moreover, since $\{W_i\}_{i=1}^N$ is N-dimensional Brownian motions whose components are independent and identically distributed, we have $(\alpha^i, \beta^i, \gamma^i), 1 \le i \le N$ are independent and identically distributed.

Now, let us present the following lemmas.

Lemma 3.4.

$$\mathbb{E} \sup_{0\le t\le T} \left|\breve{x}^{(N)}(t) - \mathbb{E}\alpha^i(t)\right|^2 = O\left(\frac{1}{N}\right)\cdot \tag{3.4}$$

Proof. On one hand, let us add up both sides of the first equation of (3.3) with respect to all $1 \le i \le N$ and multiply $\frac{1}{N}$, we obtain (recall that $\breve{x}^{(N)} = \frac{1}{N}\sum_{i=1}^N \breve{x}^i$)

$$\begin{cases} \mathrm{d}\breve{x}^{(N)} = \left[A\breve{x}^{(N)} + \dfrac{1}{N}\displaystyle\sum_{i=1}^N B\varphi(\beta^i,\gamma^i) + F\breve{x}^{(N)} + b\right]\mathrm{d}t \\ \qquad\qquad + \dfrac{1}{N}\displaystyle\sum_{i=1}^N \big[D\varphi(\beta^i,\gamma^i) + \sigma\big]\,\mathrm{d}W_i(t), \\ \breve{x}^{(N)}(0) = x. \end{cases} \tag{3.5}$$

On the other hand, by taking the expectation on both sides of the second equation of (3.3), it follows from Fubini's theorem that $\mathbb{E}\alpha^i$ satisfies the following equation:

$$\begin{cases} d(\mathbb{E}\alpha^i) = \big[A\mathbb{E}\alpha^i + \mathbb{E}\left(B\varphi(\beta^i,\gamma^i)\right) + F\mathbb{E}\alpha^i + b\big]\,\mathrm{d}t, \\ \mathbb{E}\alpha^i(0) = x. \end{cases} \tag{3.6}$$

From (3.5) and (3.6), by denoting $\varDelta(t) := \breve{x}^{(N)}(t) - \mathbb{E}\alpha^i(t)$, we have

$$\begin{cases} d\varDelta = \left[A\varDelta + \dfrac{1}{N}\displaystyle\sum_{i=1}^N B\varphi(\beta^i,\gamma^i) - B\mathbb{E}\varphi(\beta^i,\gamma^i) + F\varDelta\right]\mathrm{d}t \\ \qquad\quad + \dfrac{1}{N}\displaystyle\sum_{i=1}^N \Big[D\varphi(\beta^i,\gamma^i) + \sigma\Big]\mathrm{d}W_i(t), \\ \varDelta(0) = 0, \end{cases}$$

and the inequality $(x+y)^2 \leq 2x^2+2y^2$ yields that, for any $t\in[0,T]$,

$$\mathbb{E}\sup_{0\leq s\leq t}|\Delta(s)|^2 \leq 2\mathbb{E}\sup_{0\leq s\leq t}\left|\int_0^s \Big[(A+F)\Delta(r)+\frac{1}{N}\sum_{i=1}^N B\varphi(\beta^i(r),\gamma^i(r))-B\mathbb{E}\varphi(\beta^i(r),\gamma^i(r))\Big]dr\right|^2 + 2\mathbb{E}\sup_{0\leq s\leq t}\left|\frac{1}{N}\sum_{i=1}^N\int_0^s \Big[D\varphi(\beta^i(r),\gamma^i(r))+\sigma(r)\Big]\mathrm{d}W_i(r)\right|^2.$$

From the Cauchy–Schwartz inequality and the BDG inequality, we obtain that there exists a constant C_0 independent of N (which may vary line by line) such that

$$\begin{aligned}\mathbb{E}\sup_{0\leq s\leq t}|\Delta(t)|^2 \leq & C_0\mathbb{E}\int_0^t \left[|\Delta(s)|^2+\left|\frac{1}{N}\sum_{i=1}^N \varphi(\beta^i(s),\gamma^i(s))-\mathbb{E}\varphi(\beta^i(s),\gamma^i(s))\right|^2\right]\mathrm{d}s \\ & + \frac{C_0}{N^2}\mathbb{E}\left(\sum_{i=1}^N\int_0^t \left|D\varphi(\beta^i(s),\gamma^i(s))+\sigma(s)\right|^2\mathrm{d}s\right).\end{aligned} \tag{3.7}$$

Since $(\beta^i,\gamma^i), 1\leq i\leq N$ are independent identically distributed, for each fixed $s\in[0,T]$, let us denote that $\mu(s)=\mathbb{E}\varphi(\beta^i(s),\gamma^i(s))$ (note that μ does not depend on i), we have

$$\begin{aligned}\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^N\varphi(\beta^i(s),\gamma^i(s))-\mu(s)\right|^2 &= \frac{1}{N^2}\mathbb{E}\left|\sum_{i=1}^N\Big[\varphi(\beta^i(s),\gamma^i(s))-\mu(s)\Big]\right|^2 \\ &= \frac{1}{N^2}\mathbb{E}\sum_{i=1}^N\left|\varphi(\beta^i(s),\gamma^i(s))-\mu(s)\right|^2 \\ &\quad + \frac{1}{N^2}\mathbb{E}\sum_{i=1,j=1,j\neq i,}^N \Big\langle\varphi(\beta^i(s),\gamma^i(s))-\mu(s),\varphi(\beta^j(s),\gamma^j(s))-\mu(s)\Big\rangle.\end{aligned}$$

Since $(\beta^i,\gamma^i), 1\leq i\leq N$ are independent, we have

$$\begin{aligned}&\frac{1}{N^2}\mathbb{E}\sum_{i=1,j=1,j\neq i,}^N \Big\langle\varphi(\beta^i(s),\gamma^i(s))-\mu(s),\varphi(\beta^j(s),\gamma^j(s))-\mu(s)\Big\rangle \\ =&\frac{1}{N^2}\sum_{i=1,j=1,j\neq i,}^N \Big\langle\mathbb{E}\varphi(\beta^i(s),\gamma^i(s))-\mu(s),\mathbb{E}\varphi(\beta^j(s),\gamma^j(s))-\mu(s)\Big\rangle = 0.\end{aligned}$$

Then, due to the fact that $(\beta^i,\gamma^i), 1\leq i\leq N$ are identically distributed, we can obtain that there exists a constant C_0 independent of N such that

$$\begin{aligned}&\int_0^t\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^N B\varphi(\beta^i(s),\gamma^i(s))-B\mathbb{E}\varphi(\beta^i(s),\gamma^i(s))\right|^2\mathrm{d}s \\ \leq& C_0\int_0^t\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^N\varphi(\beta^i(s),\gamma^i(s))-\mu(s)\right|^2\mathrm{d}s \\ =&\frac{C_0}{N^2}\int_0^t\mathbb{E}\sum_{i=1}^N\left|\varphi(\beta^i(s),\gamma^i(s))-\mu(s)\right|^2\mathrm{d}s \\ =&\frac{C_0}{N}\int_0^t\mathbb{E}\left|\varphi(\beta^i(s),\gamma^i(s))-\mu(s)\right|^2\mathrm{d}s = O\left(\frac{1}{N}\right),\end{aligned}$$

where the last equality comes from the fact that $\varphi(\beta^i,\gamma^i)\in L^2_{\mathcal{F}^i}(0,T;\Gamma)$.

Let us now estimate the second term of (3.7), using the fact that (β^i,γ^i) are identically distributed, we have

$$\frac{C_0}{N^2}\mathbb{E}\left(\sum_{i=1}^N\int_0^t\left|D\varphi(\beta^i(s),\gamma^i(s))+\sigma(s)\right|^2\mathrm{d}s\right)=O\left(\frac{1}{N}\right)\cdot$$

Therefore, from the above analysis, we get from (3.7) that

$$\mathbb{E}\sup_{0\le s\le t}|\Delta(s)|^2\le C_0\mathbb{E}\int_0^t|\Delta(s)|^2+O\left(\frac{1}{N}\right),\ \text{for any } t\in[0,T].$$

Finally, by using Gronwall's inequality, we complete the proof. ☐

Lemma 3.5.

$$\sup_{1\le i\le N}\mathbb{E}\sup_{0\le t\le T}\left|\breve{x}^i(t)-\alpha^i(t)\right|^2=O\left(\frac{1}{N}\right).\tag{3.8}$$

Proof. From (3.3) and (3.1), we have that

$$\begin{cases}\ \mathrm{d}\breve{x}^i=\left[A\breve{x}^i+B\varphi(\beta^i,\gamma^i)+F\breve{x}^{(N)}+b\right]\mathrm{d}t+\left[D\varphi(\beta^i,\gamma^i)+\sigma\right]\mathrm{d}W_i(t),\\ \ \mathrm{d}\alpha^i=\left[A\alpha^i+B\varphi(\beta^i,\gamma^i)+F\mathbb{E}\alpha^i+b\right]\mathrm{d}t+\left[D\varphi(\beta^i,\gamma^i)+\sigma\right]\mathrm{d}W_i(t),\\ \breve{x}^i(0)=\alpha^i(0)=x,\end{cases}\tag{3.9}$$

where $(\alpha^i,\beta^i,\gamma^i)$ is the unique solution to the following FBSDE:

$$\begin{cases}\ \mathrm{d}\alpha^i=\left[A\alpha^i+B\varphi(\beta^i,\gamma^i)+F\mathbb{E}\alpha^i+b\right]\mathrm{d}t+\left[D\varphi(\beta^i,\gamma^i)+\sigma\right]\mathrm{d}W_i(t),\\ \ \mathrm{d}\beta^i=-\left[A^\top\beta^i-Q(\alpha^i-\mathbb{E}\alpha^i)\right]\mathrm{d}t+\gamma^i\mathrm{d}W_i(t),\\ \alpha^i(0)=x,\qquad \beta^i(T)=-G\left(\alpha^i_T-\mathbb{E}\alpha^i_T\right).\end{cases}$$

From (3.9), we have

$$\begin{cases}\ d(\breve{x}^i-\alpha^i)=\left[A(\breve{x}^i-\alpha^i)+F(\breve{x}^{(N)}-\mathbb{E}\alpha^i)\right]\mathrm{d}t,\\ \breve{x}^i(0)-\bar{x}^i(0)=0.\end{cases}$$

The classical estimate for the SDE yields that

$$\mathbb{E}\sup_{0\le t\le T}\left|\breve{x}^i(t)-\alpha^i(t)\right|^2\le C_0\mathbb{E}\int_0^T\left|\breve{x}^{(N)}(s)-\mathbb{E}\alpha^i(s)\right|^2\mathrm{d}s,$$

where C_0 is a constant independent of N. Noticing (3.4) of Lemma 3.4, we obtain (3.8). The proof is completed. ☐

Lemma 3.6. *For all* $1\le i\le N$*, we have*

$$\left|\mathcal{J}_i(\bar{u}_i,\bar{u}_{-i})-J_i(\bar{u}_i)\right|=O\left(\frac{1}{\sqrt{N}}\right)\cdot$$

Proof. Recall (2.2), (2.4) and (2.8), we have

$$\begin{aligned}\mathcal{J}_i(\bar{u}_i,\bar{u}_{-i})=\frac{1}{2}\mathbb{E}\bigg[\int_0^T\Big(\Big\langle Q(t)\big(\breve{x}^i(t)-\breve{x}^{(N)}(t)\big),\breve{x}^i(t)-\breve{x}^{(N)}(t)\Big\rangle+\langle R(t)\bar{u}_i(t),\bar{u}_i(t)\rangle\Big)\mathrm{d}t\\ +\Big\langle G\big(\breve{x}^i(T)-\breve{x}^{(N)}(T)\big),\breve{x}^i(T)-\breve{x}^{(N)}(T)\Big\rangle\bigg]\end{aligned}$$

and

$$J_i(\bar u_i) = \frac{1}{2}\mathbb{E}\bigg[\int_0^T \Big(\big\langle Q(t)\big(\alpha^i(t)-\mathbb{E}\alpha^i(t)\big), \alpha^i(t)-\mathbb{E}\alpha^i(t)\big\rangle + \langle R(t)\bar u_i(t), \bar u_i(t)\rangle\Big)\mathrm{d}t + \big\langle G\big(\alpha^i(T)-\mathbb{E}\alpha^i(T)\big), \alpha^i(T)-\mathbb{E}\alpha^i(T)\big\rangle\bigg],$$

then

$$\begin{aligned}\mathcal{J}_i(\bar u_i,\bar u_{-i})-J_i(\bar u_i) = \frac{1}{2}\mathbb{E}\bigg[&\int_0^T \Big(\Big\langle Q(t)\big(\breve x^i(t)-\breve x^{(N)}(t)\big), \breve x^i(t)-\breve x^{(N)}(t)\Big\rangle - \big\langle Q(t)\big(\alpha^i(t)-\mathbb{E}\alpha^i(t)\big), \alpha^i(t)-\mathbb{E}\alpha^i(t)\big\rangle\Big)\,\mathrm{d}t \\ &+\Big\langle G\big(\breve x^i(T)-\breve x^{(N)}(T)\big), \breve x^i(T)-\breve x^{(N)}(T)\Big\rangle - \big\langle G\big(\alpha^i(T)-\mathbb{E}\alpha^i(T)\big), \alpha^i(T)-\mathbb{E}\alpha^i(T)\big\rangle\bigg].\end{aligned} \tag{3.10}$$

From

$$\langle Q(a-b), a-b\rangle - \langle Q(c-d), c-d\rangle = \langle Q(a-b-(c-d)), a-b-(c-d)\rangle + 2\langle Q(a-b-(c-d)), c-d\rangle,$$

and Lemma 3.4, Lemma 3.5 as well as $\mathbb{E}\sup_{0\le t\le T}\left|\alpha^i(t)\right|^2 \le C_0$, for some constant C_0 independent of N which may vary line by line in the following, we have

$$\begin{aligned}&\left|\mathbb{E}\bigg[\int_0^T \Big(\Big\langle Q(t)\big(\breve x^i(t)-\breve x^{(N)}(t)\big), \breve x^i(t)-\breve x^{(N)}(t)\Big\rangle - \big\langle Q(t)\big(\alpha^i(t)-\mathbb{E}\alpha^i(t)\big), \alpha^i(t)-\mathbb{E}\alpha^i(t)\big\rangle\Big)\,\mathrm{d}t\right| \\ &\le C_0\int_0^T \mathbb{E}\left|\breve x^i(t)-\breve x^{(N)}(t)-(\alpha^i(t)-\mathbb{E}\alpha^i(t))\right|^2\mathrm{d}t \\ &\quad + C_0\int_0^T \mathbb{E}\left[\left|\breve x^i(t)-\breve x^{(N)}(t)-(\alpha^i(t)-\mathbb{E}\alpha^i(t))\right|\cdot\left|\alpha^i(t)-\mathbb{E}\alpha^i(t)\right|\right]\mathrm{d}t \\ &\le C_0\int_0^T \mathbb{E}\left|\breve x^i(t)-\alpha^i(t)\right|^2\mathrm{d}t + C_0\int_0^T \mathbb{E}\left|\breve x^{(N)}(t)-\mathbb{E}\alpha^i(t)\right|^2\mathrm{d}t \\ &\quad + C_0\int_0^T \left(\mathbb{E}\left|\breve x^i(t)-\breve x^{(N)}(t)-(\alpha^i(t)-\mathbb{E}\alpha^i(t))\right|^2\right)^{\frac{1}{2}}\left(\mathbb{E}\left|\alpha^i(t)-\mathbb{E}\alpha^i(t)\right|^2\right)^{\frac{1}{2}}\mathrm{d}t \\ &\le C_0\int_0^T \mathbb{E}\left|\breve x^i(t)-\alpha^i(t)\right|^2\mathrm{d}t + C_0\int_0^T \mathbb{E}\left|\breve x^{(N)}(t)-\mathbb{E}\alpha^i(t)\right|^2\mathrm{d}t \\ &\quad + C_0\int_0^T \left(\mathbb{E}\left|\breve x^i(t)-\alpha^i(t)\right|^2 + \mathbb{E}\left|\breve x^{(N)}(t)-\mathbb{E}\alpha^i(t)\right|^2\right)^{\frac{1}{2}}\mathrm{d}t \\ &= O\left(\frac{1}{\sqrt{N}}\right).\end{aligned}$$

With similar argument, we can show that

$$\left|\mathbb{E}\bigg[\Big\langle G\big(\breve x^i(T)-\breve x^{(N)}(T)\big), \breve x^i(T)-\breve x^{(N)}(T)\Big\rangle - \big\langle G\big(\alpha^i(T)-\mathbb{E}\alpha^i(T)\big), \alpha^i(T)-\mathbb{E}\alpha^i(T)\big\rangle\bigg]\right| = O\left(\frac{1}{\sqrt{N}}\right).$$

The proof is completed by noticing (3.10). ☐

Our remaining analysis is to prove the control strategies set $(\bar{u}_1, \bar{u}_2, \dots, \bar{u}_N)$ is an ϵ-Nash equilibrium for **Problem (CC)**. For any fixed i, $1 \le i \le N$, we consider the perturbation control $u_i \in \mathcal{U}_{\text{ad}}^{d,i}$ and we have the following state dynamics ($j \neq i$):

$$\begin{cases} \mathrm{d}y^i = \Big[Ay^i + Bu_i + Fy^{(N)} + b\Big]\,\mathrm{d}t + [Du_i + \sigma]\,\mathrm{d}W_i(t), \\ \mathrm{d}y^j = \Big[Ay^j + B\varphi(\beta^j, \gamma^j) + Fy^{(N)} + b\Big]\,\mathrm{d}t + \big[D\varphi(\beta^j, \gamma^j) + \sigma\big]\,\mathrm{d}W_j(t), \\ \mathrm{d}\alpha^j = \big[A\alpha^j + B\varphi(\beta^j, \gamma^j) + F\mathbb{E}\alpha^j + b\big]\,\mathrm{d}t + \big[D\varphi(\beta^j, \gamma^j) + \sigma\big]\,\mathrm{d}W_j(t), \\ \mathrm{d}\beta^j = -\big[A^\top \beta^j - Q(\alpha^j - \mathbb{E}\alpha^j)\big]\mathrm{d}t + \gamma^j \mathrm{d}W_j(t), \\ y^i(0) = y^j(0) = \alpha^j(0) = x, \qquad \beta^j(T) = -G\big(\alpha^j(T) - \mathbb{E}\alpha^j(T)\big), \end{cases} \tag{3.11}$$

where $y^{(N)} = \dfrac{1}{N}\displaystyle\sum_{i=1}^N y^i$. The wellposedness of above system is easily to obtain. To prove $(\bar{u}_1, \bar{u}_2, \dots, \bar{u}_N)$ is an ϵ-Nash equilibrium, we need to show that for $1 \le i \le N$,

$$\inf_{u_i \in \mathcal{U}_{\text{ad}}^i} \mathcal{J}_i(u_i, \bar{u}_{-i}) \ge \mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) - \epsilon.$$

Then we only need to consider the perturbation $u_i \in \mathcal{U}_{\text{ad}}^{d,i}$ such that $\mathcal{J}_i(u_i, \bar{u}_{-i}) \le \mathcal{J}_i(\bar{u}_i, \bar{u}_{-i})$. Thus we have

$$\mathbb{E}\int_0^T \langle Ru_i(t), u_i(t)\rangle \mathrm{d}t \le \mathcal{J}_i(u_i, \bar{u}_{-i}) \le \mathcal{J}_i(\bar{u}_i, \bar{u}_{-i}) \le J_i(\bar{u}_i) + O\left(\frac{1}{\sqrt{N}}\right),$$

which implies that

$$\mathbb{E}\int_0^T |u_i(t)|^2 \mathrm{d}t \le C_0, \tag{3.12}$$

where C_0 is a constant independent of N.

Now, for the ith agent, we consider the perturbation in the **Problem (LCC)**. We introduce the following system of the decentralized limiting state with perturbation control ($j \neq i$):

$$\begin{cases} \mathrm{d}\bar{y}^i = \big[A\bar{y}^i + Bu_i + F\mathbb{E}\alpha^i + b\big]\,\mathrm{d}t + [Du_i + \sigma]\,\mathrm{d}W_i(t), \\ \mathrm{d}\alpha^j = \big[A\alpha^j + B\varphi(\beta^j, \gamma^j) + F\mathbb{E}\alpha^j + b\big]\,\mathrm{d}t + \big[D\varphi(\beta^j, \gamma^j) + \sigma\big]\,\mathrm{d}W_j(t), \\ \mathrm{d}\beta^j = -\big[A^\top \beta^j - Q(\alpha^j - \mathbb{E}\alpha^j)\big]\mathrm{d}t + \gamma^j \mathrm{d}W_j(t), \\ \bar{y}^i(0) = \alpha^j(0) = x, \qquad \beta^j(T) = -G\big(\alpha^j(T) - \mathbb{E}\alpha^j(T)\big). \end{cases} \tag{3.13}$$

We have the following results:

Lemma 3.7.

$$\mathbb{E}\sup_{0\le t\le T}\Big|y^{(N)}(t) - \mathbb{E}\alpha^i(t)\Big|^2 = O\left(\frac{1}{N}\right)\cdot \tag{3.14}$$

Proof. By (3.11), we get

$$\begin{cases} \mathrm{d}y^{(N)} = \left[(A+F)y^{(N)} + \dfrac{1}{N}Bu_i + \dfrac{1}{N}\displaystyle\sum_{j=1, j\neq i}^N B\varphi(\beta^j, \gamma^j) + b\right]\mathrm{d}t \\ \qquad\qquad + \dfrac{1}{N}\displaystyle\sum_{j=1}^N \sigma \mathrm{d}W_j(t) + \dfrac{1}{N}Du_i \mathrm{d}W_i(t) + \dfrac{1}{N}\displaystyle\sum_{j=1, j\neq i}^N D\varphi(\beta^j, \gamma^j)\mathrm{d}W_j(t), \\ y^{(N)}(0) = x. \end{cases} \tag{3.15}$$

Let us denote $\mathit{\Pi} := y^{(N)} - \mathbb{E}\alpha^i$, and recall (3.6) which is

$$\begin{cases} d(\mathbb{E}\alpha^i) = \left[A\mathbb{E}\alpha^i + \mathbb{E}B\varphi(\beta^i,\gamma^i) + F\mathbb{E}\alpha^i + b\right]\mathrm{d}t, \\ \mathbb{E}\alpha^i(0) = x, \end{cases}$$

we have

$$\begin{cases} d\mathit{\Pi} = \left[(A+F)\mathit{\Pi} + \dfrac{1}{N}Bu_i + \left(\dfrac{1}{N}\displaystyle\sum_{j=1,j\neq i}^{N} B\varphi(\beta^j,\gamma^j) - \mathbb{E}B\varphi(\beta^j,\gamma^j)\right)\right]\mathrm{d}t \\ \qquad\quad + \dfrac{1}{N}\displaystyle\sum_{j=1}^{N}\sigma \mathrm{d}W_j(t) + \dfrac{1}{N}Du_i\mathrm{d}W_i(t) + \dfrac{1}{N}\displaystyle\sum_{j=1,j\neq i}^{N} D\varphi(\beta^j,\gamma^j)\mathrm{d}W_j(t), \\ \mathit{\Pi}(0) = 0. \end{cases}$$

By the Cauchy–Schwartz inequality as well as the BDG inequality, we obtain that there exists a constant C_0 independent of N which may vary line by line such that, for any $t \in [0,T]$,

$$\begin{aligned} \mathbb{E}\sup_{0\le s\le t}|\mathit{\Pi}(s)|^2 \le{}& C_0\mathbb{E}\int_0^t \left(|\mathit{\Pi}(s)|^2 + \frac{1}{N^2}|u_i(s)|^2\right)\mathrm{d}s \\ &+ C_0\mathbb{E}\int_0^t \left|\frac{1}{N}\sum_{j=1,j\neq i}^{N}\varphi(\beta^j(s),\gamma^j(s)) - \mathbb{E}\varphi(\beta^j(s),\gamma^j(s))\right|^2 \mathrm{d}s \\ &+ \frac{C_0}{N^2}\mathbb{E}\sum_{j=1}^{N}\int_0^t |\sigma(s)|^2\,\mathrm{d}s \\ &+ \frac{C_0}{N^2}\mathbb{E}\int_0^t |u_i(s)|^2\mathrm{d}s + \frac{C_0}{N^2}\mathbb{E}\sum_{j=1,j\neq i}^{N}\int_0^t |\varphi(\beta^j(s),\gamma^j(s)|^2\mathrm{d}s. \end{aligned} \tag{3.16}$$

On the one hand, by denoting $\mu(s) := \mathbb{E}\varphi(\beta^j(s),\gamma^j(s))$ (note that since $(\alpha^j,\beta^j,\gamma^j)$, $1\le j\le N$, $j\neq i$, are independent identically distributed, thus μ is independent of j), we have

$$\begin{aligned} \mathbb{E}\left|\frac{1}{N}\sum_{j=1,j\neq i}^{N}\varphi(\beta^j(s),\gamma^j(s)) - \mu(s)\right|^2 &\le 2\mathbb{E}\left|\frac{1}{N}\sum_{j=1,j\neq i}^{N}\varphi(\beta^j(s),\gamma^j(s)) - \frac{N-1}{N}\mu(s)\right|^2 + 2\mathbb{E}\left|\frac{1}{N}\mu(s)\right|^2 \\ &= 2\frac{(N-1)^2}{N^2}\mathbb{E}\left|\frac{1}{N-1}\sum_{j=1,j\neq i}^{N}\varphi(\beta^j(s),\gamma^j(s)) - \mu(s)\right|^2 + \frac{2}{N^2}\mathbb{E}|\mu(s)|^2. \end{aligned}$$

Then, due to the fact that $(\beta^i, \gamma^i), 1 \le i \le N$ are identically distributed and $\varphi(\beta^i, \gamma^i) \in L^2_{\mathbb{F}^i}(0, T; \Gamma)$, similarly to Lemma 3.4 we can obtain that there exists a constant C_0 independent of N such that

$$\begin{aligned}
&\int_0^t \mathbb{E}\left|\frac{1}{N}\sum_{j=1, j\neq i}^{N} \varphi(\beta^j(s), \gamma^j(s)) - \mathbb{E}\varphi(\beta^j(s), \gamma^j(s))\right|^2 \\
&\le \frac{C_0(N-1)^2}{N^2}\int_0^t \mathbb{E}\left|\frac{1}{N-1}\sum_{j=1, j\neq i}^{N} \varphi(\beta^j(s), \gamma^j(s)) - \mu(s)\right|^2 \mathrm{d}s + \frac{C_0}{N^2}\int_0^t \mathbb{E}|\mu(s)|^2 \mathrm{d}s \\
&= \frac{C_0(N-1)}{N^2}\int_0^t \mathbb{E}\left|\varphi(\beta^j(s), \gamma^j(s)) - \mu(s)\right|^2 \mathrm{d}s + \frac{C_0}{N^2}\int_0^t \mathbb{E}|\mu(s)|^2 \mathrm{d}s \\
&= O\left(\frac{1}{N}\right)\cdot
\end{aligned}$$

In addition, due to (3.12), we get

$$\frac{C_0}{N^2}\mathbb{E}\int_0^t |u_i(s)|^2 \mathrm{d}s + \frac{C_0}{N^2}\mathbb{E}\sum_{j=1}^{N}\int_0^t |\sigma(s)|^2 \mathrm{d}s = O\left(\frac{1}{N}\right),$$

and similarly, since (β^j, γ^j), $1 \le j \le N$, $j \neq i$, are identically distributed, we have

$$\frac{C_0}{N^2}\mathbb{E}\sum_{j=1, j\neq i}^{N}\int_0^t |\varphi(\beta^j(s), \gamma^j(s)|^2 \mathrm{d}s = O\left(\frac{1}{N}\right)\cdot$$

Therefore, from above estimates, we get from (3.16) that, for any $t \in [0, T]$,

$$\mathbb{E}\sup_{0\le s\le t}|\Pi(s)|^2 \le C_0\mathbb{E}\int_0^t |\Pi(s)|^2 \mathrm{d}s + O\left(\frac{1}{N}\right)\cdot$$

Finally, by using Gronwall's inequality, we complete the proof. □

Lemma 3.8.

$$\mathbb{E}\sup_{0\le t\le T}\left|y_t^i - \bar{y}_t^i\right|^2 = O\left(\frac{1}{N}\right)\cdot \tag{3.17}$$

Proof. From respectively the first equation of (3.11) and (3.13), we obtain

$$\begin{cases} d(y^i - \bar{y}^i) = \left[A(y^i - \bar{y}^i) + F(y^{(N)} - \mathbb{E}\alpha^i)\right]\mathrm{d}t, \\ y^i(0) - \bar{y}^i(0) = 0. \end{cases}$$

With the help of classical estimates of SDE, Gronwall's inequality and (3.14) of Lemma 3.7, it is easily to obtain (3.17). The proof is completed. □

Lemma 3.9. *For all* $1 \le i \le N$, *for the perturbation control* u_i, *we have*

$$\left|\mathcal{J}_i(u_i, \bar{u}_{-i}) - J_i(u_i)\right| = O\left(\frac{1}{\sqrt{N}}\right)\cdot$$

Proof. Recall (2.2), (2.4) and (2.8), we have

$$
\begin{aligned}
&\mathcal{J}_i(u_i,\bar u_{-i})-J_i(u_i)\\
=&\frac{1}{2}\mathbb{E}\bigg[\int_0^T\Big(\Big\langle Q(t)\big(y^i(t)-y^{(N)}(t)\big),y^i(t)-y^{(N)}(t)\Big\rangle-\big\langle Q(t)\big(\bar y^i(t)-\mathbb{E}\alpha^i(t)\big),\bar y^i(t)-\mathbb{E}\alpha^i(t)\big\rangle\Big)\,\mathrm{d}t\\
&+\Big\langle G\big(y^i(T)-y^{(N)}(T)\big),y^i(T)-y^{(N)}(T)\Big\rangle-\big\langle G\big(\bar y^i(T)-\mathbb{E}\alpha^i(T)\big),\bar y^i(T)-\mathbb{E}\alpha^i(T)\big\rangle\bigg].
\end{aligned}
$$

Using Lemmas 3.7 and 3.8 as well as $\mathbb{E}\sup_{0\le t\le T}\big(|\bar y^i(t)|^2+|\alpha^i(t)|^2\big)\le C_0$, for some constant C_0 independent of N which may vary line by line in the following, we have

$$
\begin{aligned}
&\left|\mathbb{E}\bigg[\int_0^T\Big(\Big\langle Q(t)\big(y^i(t)-y^{(N)}(t)\big),y^i(t)-y^{(N)}(t)\Big\rangle-\big\langle Q(t)\big(\bar y^i(t)-\mathbb{E}\alpha^i(t)\big),\bar y^i(t)-\mathbb{E}\alpha^i(t)\big\rangle\Big)\,\mathrm{d}t\right|\\
&\le C_0\int_0^T\mathbb{E}\left|y^i(t)-y^{(N)}(t)-(\bar y^i(t)-\mathbb{E}\alpha^i(t))\right|^2\mathrm{d}t\\
&\quad+C_0\int_0^T\mathbb{E}\Big[\left|y^i(t)-y^{(N)}(t)-(\bar y^i(t)-\mathbb{E}\alpha^i(t))\right|\cdot\left|\bar y^i(t)-\mathbb{E}\alpha^i(t)\right|\Big]\mathrm{d}t\\
&\le C_0\int_0^T\mathbb{E}\left|y^i(t)-\bar y^i(t)\right|^2\mathrm{d}t+C_0\int_0^T\mathbb{E}\left|y^{(N)}(t)-\mathbb{E}\alpha^i(t)\right|^2\mathrm{d}t\\
&\quad+C_0\int_0^T\left(\mathbb{E}\left|y^i(t)-y^{(N)}(t)-(\bar y^i(t)-\mathbb{E}\alpha^i(t))\right|^2\right)^{\frac12}\left(\mathbb{E}\left|\bar y^i(t)-\mathbb{E}\alpha^i(t)\right|^2\right)^{\frac12}\mathrm{d}t\\
&\le C_0\int_0^T\mathbb{E}\left|y^i(t)-\bar y^i(t)\right|^2\mathrm{d}t+C_0\int_0^T\mathbb{E}\left|y^{(N)}(t)-\mathbb{E}\alpha^i(t)\right|^2\mathrm{d}t\\
&\quad+C_0\int_0^T\left(\mathbb{E}\left|y^i(t)-\bar y^i(t)\right|^2+\mathbb{E}\left|y^{(N)}(t)-\mathbb{E}\alpha^i(t)\right|^2\right)^{\frac12}\mathrm{d}t\\
&=O\left(\frac{1}{\sqrt N}\right)\cdot
\end{aligned}
$$

With similar argument, we can show that

$$
\left|\mathbb{E}\bigg[\Big\langle G\big(y^i(T)-y^{(N)}(T)\big),y^i(T)-y^{(N)}(T)\Big\rangle-\big\langle G\big(\bar y^i(T)-\mathbb{E}\alpha^i(T)\big),\bar y^i(T)-\mathbb{E}\alpha^i(T)\big\rangle\bigg]\right|=O\left(\frac{1}{\sqrt N}\right)\cdot
$$

Hence, we get the desired result. □

Proof of Theorem 3.3. Now, we consider the ϵ-Nash equilibrium for $\mathcal{A}_i$ for **Problem (CC)**. Combining Lemmas 3.6 and 3.9, we have

$$
\mathcal{J}_i(\bar u_i,\bar u_{-i})=J_i(\bar u_i)+O\left(\frac{1}{\sqrt N}\right)\le J_i(u_i)+O\left(\frac{1}{\sqrt N}\right)=\mathcal{J}_i(u_i,\bar u_{-i})+O\left(\frac{1}{\sqrt N}\right).
$$

Consequently, Theorem 3.3 holds with $\epsilon=O\left(\frac{1}{\sqrt N}\right)\cdot$ □

Appendix A.

For the readers' convenience, let us recall the following properties of projection $\mathbf{P}_\Gamma$ onto a closed convex set, see [3], Chapter 5.

Theorem A.1. *For a nonempty closed convex set $\Gamma \subset \mathbb{R}^m$, for every $x \in \mathbb{R}^m$, there exists a unique $x^* \in \Gamma$, such that*

$$|x - x^*| = \min_{y\in\Gamma} |x - y| =: dist(x, \Gamma).$$

Moreover, x^* is characterized by the property

$$x^* \in \Gamma, \quad \langle x^* - x, x^* - y\rangle \le 0 \qquad \forall y \in \Gamma. \tag{A.1}$$

The above element x^* is called the projection of x onto Γ and is denoted by $\mathbf{P}_\Gamma[x]$.

From above theorem, it is easy to show that

Proposition A.2. *Let $\Gamma \subset \mathbb{R}^m$ be a nonempty closed convex set, then we have*

$$\left|\mathbf{P}_\Gamma[x] - \mathbf{P}_\Gamma[y]\right|^2 \le \langle \mathbf{P}_\Gamma[x] - \mathbf{P}_\Gamma[y], x - y\rangle . \tag{A.2}$$

Proof. From (A.1), we have

$$\langle \mathbf{P}_\Gamma[x] - x, \mathbf{P}_\Gamma[x] - z\rangle \le 0 \qquad \forall z \in \Gamma. \tag{A.3}$$

and

$$\langle \mathbf{P}_\Gamma[y] - y, \mathbf{P}_\Gamma[y] - z\rangle \le 0 \qquad \forall z \in \Gamma. \tag{A.4}$$

Choosing $z = \mathbf{P}_\Gamma[y]$ in (A.3) and $z = \mathbf{P}_\Gamma[x]$ in (A.4), then adding the corresponding inequalities, we obtain

$$\langle \mathbf{P}_\Gamma[x] - x, \mathbf{P}_\Gamma[x] - \mathbf{P}_\Gamma[y]\rangle + \langle \mathbf{P}_\Gamma[y] - y, \mathbf{P}_\Gamma[y] - \mathbf{P}_\Gamma[x]\rangle \le 0,$$

which yields obviously

$$\left|\mathbf{P}_\Gamma[x] - \mathbf{P}_\Gamma[y]\right|^2 \le \langle \mathbf{P}_\Gamma[x] - \mathbf{P}_\Gamma[y], x - y\rangle . \qquad \square$$

Proposition A.3. *Let $\Gamma \subset \mathbb{R}^m$ be a nonempty closed convex set, then the projection $\mathbf{P}_\Gamma$ does not increase the distance, i.e.*

$$\left|\mathbf{P}_\Gamma[x] - \mathbf{P}_\Gamma[y]\right| \le \left|x - y\right|. \tag{A.5}$$

Proof. From (A.2), we have

$$\left|\mathbf{P}_\Gamma[x] - \mathbf{P}_\Gamma[y]\right|^2 \le \langle \mathbf{P}_\Gamma[x] - \mathbf{P}_\Gamma[y], x - y\rangle \le |\mathbf{P}_\Gamma[x] - \mathbf{P}_\Gamma[y]| \cdot |x - y|,$$

which gives directly (A.5). $\square$

Now let us consider $\mathbb{R}^m$ and the projection $\mathbf{P}_\Gamma$ both with the norm $\|\cdot\|_{R_0} := \langle R_0^{\frac{1}{2}} \cdot, R_0^{\frac{1}{2}} \cdot\rangle$, from (A.2), we have

Proposition A.4. *Let $\Gamma \subset \mathbb{R}^m$ be a nonempty closed convex set, then*

$$\langle\langle \mathbf{P}_\Gamma[x] - \mathbf{P}_\Gamma[y], x - y\rangle\rangle = \left\langle R^{\frac{1}{2}}\Big(\mathbf{P}_\Gamma[x] - \mathbf{P}_\Gamma[y]\Big), R^{\frac{1}{2}}(x - y)\right\rangle \ge 0.$$

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