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ANALYSIS OF A QUEUEING SYSTEM IN RANDOM ENVIRONMENT WITH AN UNRELIABLE SERVER AND GEOMETRIC ABANDONMENTS

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Abstract. This paper studies a single server queueing model in a multi-phase random environment with server breakdowns and geometric abandonments, where server breakdowns only occur while the server is in operation. At a server breakdown instant (*i.e.*, an abandonment opportunity epoch), all present customers adopt the so-called geometric abandonments, that is, the customers decide sequentially whether they will leave the system or not. In the meantime, the server abandons the service and a repair process starts immediately. After the server is repaired, the server resumes its service, and the system enters into the operative phase i with probability $q_i, i = 1, 2, \ldots, d$. Using probability generating functions and matrix geometric approach, we obtain the steady state distribution and various performance measures. In addition, some numerical examples are presented to show the impact of parameters on the performance measures.

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1. Introduction

Queues with service interruptions or queues with unreliable servers have been widely and successfully used as mathematical models of several computer systems and manufacturing systems. In these queueing systems, the servers may well be subject to unpredictable breakdowns while serving a customer. One example is that in computer systems, the machine may be subject to scheduled backups and unpredictable failures; another example is that in manufacturing systems, the machine may breakdown because of machine or job related problems. Since the introduction of breakdowns and their characteristics, there has been considerable attention to this topic. Among some classical papers on repairable servers, we refer the readers to see the papers by Cao and Cheng [2], Li *et al.* [17], Gray *et al.* [10] Kalidass and Kasturi [13], and Ke [14, 15]. Several authors also have considered queueing models with two phases of service and server breakdowns. For instance, Wang [25] considered an M/G/1 queue with a second optional service and server breakdowns. Choudhury and Deka [3] investigated an M/G/1 retrial queueing system with two phases of service subject to the server breakdown and repair, and carried out an extensive analysis of this model. Furthermore, Choudhury and Deka [4] studied a

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single server queueing system with two phases of service subject to server breakdown and Bernoulli vacation. In [4], the authors derived the joint distribution of the state of the server and queue size, Laplace-Stieltjes transform (LST) of the busy period distribution and the waiting time distribution. Recently, Cordeiro and Kharoufeh [5] studied an unreliable M/M/1 retrial queue in a random environment and presented the orbit size distribution by matrix analytic approach. Liu *et al.* [20] studied an M/G/1 retrial G-queue with preemptive resume and feedback under N-policy subject to the server breakdowns and repairs, in which the server breakdowns caused by negative customers. By using the supplementary variables method, the authors derived the steady-state solutions for both queueing measures and reliability quantities. Economou and Kanta [9] considered the Markovian single-server queue with breakdowns and repairs. In [9], the authors studied the queueing system from an economic viewpoint with the assumption that the customers can observe the queue length and will decide whether to join or balk the system according to the information about the server's state based on a linear reward-cost structure. Later, Li *et al.* [18] investigated the corresponding unobservable cases (the customers have no information on the queue length when they make their decision to join or balk) of the same model studied in [9]. Moreover, customers abandonment due to a slowdown in the service rate also has been studied, see *e.g.*, Perel and Yechiali [23]. In [23], the authors examined the case where customers become impatient and abandon the queue while the system resides in the slow phase.

There are also plenty of references on queueing systems with disasters (catastrophes). For these queueing models, while the server is breakdown, all present customers in the system may be forced to leave the system simultaneously. Since the introduction of catastrophes, there has been considerable attention to this theme. The interested readers are referred to Yechiali [26], Kim and Lee [16], Jiang *et al.* [12], Jiang and Liu [11], etc.

The study of the number of customers in system reduced according to a geometric distribution is a recent endeavor. For more detailed and excellent studies on this policy, the readers may refer to Artalejo *et al.* [1], Economou and Gomez-Corral [8], Dimou *et al.* [6, 7]. For instance, in [8], the authors dealt with a population of individuals that grows stochastically according to a batch Markovian arrival process and is subject to renewal generated geometric catastrophes. In [7], the authors considered a single server vacation queueing model, where the customers become impatient during the absence of the server, and abandon the system according to a geometric distribution. Then, in [6], the authors studied the single server queue with catastrophes and geometric reneging, where the customers become impatient and leave the system according to a geometric distribution while the server is in repair. In fact, the mechanism for the geometric abandonment is well-motivated by applications in various fields, especially in manufacturing systems and perishable inventory systems. For example, in a production system, we can think of a secondary facility (auxiliary facility) that inspects the system, where the inspector has a capacity of exponential time units in the auxiliary facility. When the system suffers from some external factors (external attacks or shocks), the inspector starts to detect the system by considering the present customers sequentially, looking at their service times and removes the customers from the system to the auxiliary facility as many as his capacity. Then, the customers begin sequentially to be transferred for processing to the auxiliary facility, and the reduction of the number of customers ceases at the first individual who determines to stay in the system, or when all present customers are transferred to the auxiliary facility.

Inspired by applications of this queueing model, in this paper, we aim to complement the studies of [22] for the abandoning issue in the unreliable server with multi-phase random environment. The difference between [22] and the present paper is that we assume the customers leave the system according to a geometric distribution while the server is breakdown. Further, we also assume that server breakdowns only occur while the server is in operation, *i.e.*, server breakdowns do not occur while the server is idle or in repair. Not only do we obtain the stationary queue length distribution, but also we derive various performance measures including the LST of the sojourn time of an arbitrary customer and the length time of a cycle.

The paper is organized as follows: In Section 2 we give the model description. In Section 3 we obtain the sufficient and necessary stability condition and derive the steady state distribution by probability generating functions (PGFs). In Section 4 we use the matrix geometric approach to derive the steady state distribution. Sections 5 and 6 are devoted to giving various performance measures analysis including the sojourn time of an arbitrary customer and the length of a cycle. Numerical results are presented in Section 7. Section 8 concludes the paper.

2. Model description

In this paper, we consider a single server queueing model where the service may be interrupted while the server is in operation, *i.e.*, the server experiences failure when it is in operation, but can not fail if it is idle or under repair. Under environment $i, i = 1, 2, \ldots, d$, the Poisson arrival rate is λ_i. Customers are served under the first-come-first-served (FCFS) discipline, and the service times follow an exponential distribution with parameter μ_i. Interruptions occur in accordance with a Poisson process at rate η_i. When the system resides in operative phase $i, i = 1, 2, \ldots, d$, it occasionally suffers a breakdown and moves to phase 0 (repair phase), immediately. Meanwhile, customers in the system adopt the so-called geometric abandonments, that is, the customers decide sequentially whether they will leave the system or not. Each customer abandons the system with probability p or remains in the system with probability $\bar{p} = 1 - p$. Then, all present customers start sequentially to leave the system and the reduction of the number of customers ceases at the first individual who determines to stay in the system, or when all present customers abandon it. For more details, we may refer interested readers to [6, 7]. This abandonment policy can be interpreted as a strategy that at a server breakdown epoch the number of customers in the system is decreased according to a geometric distribution. We further assume that the repair times are exponentially distributed with rate ξ. In phase 0, the Poisson arrival rate is λ_0, and the server stops working completely. After the system is repaired, the server resumes its service, and the system enters into the operative phase i with probability $q_i, i = 1, 2, \ldots, d$, where $\sum_{i=1}^{d} q_i = 1$.

3. Stability condition and stationary queue length distribution

In this section, we give a necessary and sufficient condition for the stability of the system and derive the stationary queue length distribution by PGFs. At time t, the system can be described by $X(t) = \{L(t), J(t), t \geq 0\}$, where $L(t)$ denotes the number of customers in the system at time t, and $J(t)$ denotes the phase that the system operates at time t. Then, $X(t) = \{L(t), J(t), t \geq 0\}$ is a continuous-time Markov chain with state space $\Omega = \{(n, i), n \geq 0, i = 0, 1, 2 \cdots, d\}$. Next, we consider the continuous-time Markov chain, whose state-transition-rate matrix is given by

$$Q = \begin{pmatrix} B_0 & A_0 & & & \\ B_1 & A_1 & A_0 & & \\ B_2 & A_2 & A_1 & A_0 & \\ B_3 & A_3 & A_2 & A_1 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

where all partitioned matrices are square ones with $(d+1) \times (d+1)$ orders and

$$B_0 = \begin{pmatrix} -(\lambda_0 + \xi) & \xi q_1 & \cdots & \xi q_d \\ 0 & -\lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\lambda_d \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \eta_1 p & \mu_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \eta_d p & 0 & \cdots & \mu_d \end{pmatrix},$$

$$A_0 = \mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_d),$$

$$A_1 = \begin{pmatrix} -(\lambda_0 + \xi) & \xi q_1 & \cdots & \xi q_d \\ \eta_1 \bar{p} & -(\lambda_1 + \mu_1 + \eta_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \eta_d \bar{p} & 0 & \cdots & -(\lambda_d + \mu_d + \eta_d) \end{pmatrix},$$

$$A_2=\begin{pmatrix}0 & 0 & \cdots & 0\\ \eta_1 p\bar{p} & \mu_1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \eta_d p\bar{p} & 0 & \cdots & \mu_d\end{pmatrix},\quad A_k=\begin{pmatrix}0 & 0 & \cdots & 0\\ \eta_1 p^{k-1}\bar{p} & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \eta_d p^{k-1}\bar{p} & 0 & \cdots & 0\end{pmatrix},\quad k\geq 3,$$

$$B_k=\begin{pmatrix}0 & 0 & \cdots & 0\\ \eta_1 p^{k} & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \eta_d p^{k} & 0 & \cdots & 0\end{pmatrix},\quad k\geq 2.$$

At first, we investigate the sufficient and necessary stability condition of our model. We will finish its proof based on Theorem 1.7.1 in [21].

Theorem 3.1. *The system under consideration is stable if and only if*

$$\lambda_0+\sum_{k=1}^{d}\frac{\xi q_k\lambda_k}{\eta_k}<\sum_{k=1}^{d}\frac{\xi q_k}{\eta_k}\frac{\mu_k\bar{p}+\eta_k p}{\bar{p}}.$$

Proof. Based on the mean drift result in [21], the system would be stable and the stationary distribution exists if and only if

$$\boldsymbol{x}A_0\boldsymbol{e}<\boldsymbol{x}\sum_{k=2}^{\infty}(k-1)A_k\boldsymbol{e},$$

where $\boldsymbol{e}$ is a column vector with $(d+1)$ dimensions and all its elements equal to one,

$$\sum_{k=2}^{\infty}(k-1)A_k=\begin{pmatrix}0 & 0 & \cdots & 0\\ p\eta_1/\bar{p} & \mu_1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ p\eta_d/\bar{p} & 0 & 0 & \mu_d\end{pmatrix},$$

$\boldsymbol{x}=(x_0,x_1,...,x_d)$ is the invariant probability vector of

$$A=\sum_{k=0}^{\infty}A_k=\begin{pmatrix}-\xi & \xi q_1 & \cdots & \xi q_d\\ \eta_1 & -\eta_1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \eta_d & 0 & \cdots & -\eta_d\end{pmatrix},$$

which satisfies $\boldsymbol{x}A=\boldsymbol{0},\boldsymbol{x}\boldsymbol{e}=1$. Then an immediate result is that

$$x_0=\frac{1}{1+\sum\limits_{i=1}^{d}\frac{\xi q_i}{\eta_i}},\quad x_k=\frac{\xi q_k}{\eta_k}\frac{1}{1+\sum\limits_{i=1}^{d}\frac{\xi q_i}{\eta_i}},\quad k=1,2,\ldots,d,$$

and the stability condition $\boldsymbol{x}A_0\boldsymbol{e} < \sum_{k=2}^{\infty}(k-1)\boldsymbol{x}A_k\boldsymbol{e}$ translates into

$$\sum_{k=0}^{d} x_k\lambda_k < \sum_{k=1}^{d} x_k\left(\mu_k + \eta_k\frac{p}{\bar{p}}\right).$$

Simplifying the above inequality, we have

$$\lambda_0 + \sum_{k=1}^{d}\frac{\xi q_k\lambda_k}{\eta_k} < \sum_{k=1}^{d}\frac{\xi q_k}{\eta_k}\frac{\mu_k\bar{p}+\eta_k p}{\bar{p}},$$

which is the sufficient and necessary condition for the system to be stable.

The proof is finished. □

Remark 3.2. According to [12, 22], the probability that the system resides in phase k can be obtained by

$$\pi_{\cdot,k} = \frac{\frac{q_k}{\eta_k}}{\frac{1}{\xi}+\sum\limits_{i=1}^{d}\frac{q_i}{\eta_i}} = \frac{q_k}{\eta_k\alpha}, \quad 1\le k\le d, \quad \pi_{\cdot,0} = \frac{\frac{1}{\xi}}{\frac{1}{\xi}+\sum\limits_{i=1}^{d}\frac{q_i}{\eta_i}} = \frac{1}{\xi\alpha},$$

where

$$\alpha = \frac{1}{\xi} + \sum_{i=1}^{d}\frac{q_i}{\eta_i}.$$

For the stability condition $\lambda_0 + \sum_{k=1}^{d}\frac{\xi q_k\lambda_k}{\eta_k} < \sum_{k=1}^{d}\frac{\xi q_k}{\eta_k}\frac{\mu_k\bar{p}+\eta_k p}{\bar{p}}$, multiplying the inequality by $\frac{1}{\xi\alpha}$, then we have

$$\lambda_0\pi_{\cdot,0} + \sum_{k=1}^{d}\lambda_k\pi_{\cdot,k} < \sum_{k=1}^{d}(\mu_k + \frac{\eta_k p}{\bar{p}})\pi_{\cdot,k}.$$

So the intuitive interpretation of the theorem is straightforward: $\bar{\lambda} = \lambda_0\pi_{\cdot,0} + \sum_{k=1}^{d}\lambda_k\pi_{\cdot,k}$ is the average arrival rate, $\bar{\mu} = \sum_{k=1}^{d}(\mu_k + \frac{\eta_k p}{\bar{p}})\pi_{\cdot,k}$ is the exit rate from the system, either by service completion or abandonment. For stationary condition, the exit rate from the system, either by service completion or abandonment must exceed the average arrival rate.

Next, we focus on the computation of the stationary distribution for $X(t) = \{L(t), J(t), t\ge 0\}$. Assume that

$$\pi_{n,i} = \lim_{t\to\infty} P\{L(t)=n, J(t)=i\}, n\ge 0, \quad i = 0,1,2,\ldots,d.$$

To this end we consider the following partial PGFs

$$\Pi_0(z) = \sum_{n=0}^{\infty}\pi_{n,0}z^n, \quad \Pi_i(z) = \sum_{n=1}^{\infty}\pi_{n,i}z^n, \quad i = 1,2,\ldots,d,$$

then, the set of balance equations can be obtained as follows:

$$(\lambda_0+\xi)\pi_{0,0}=\sum_{i=1}^{d}\eta_i\sum_{k=1}^{\infty}\pi_{k,i}p^k, \tag{3.1}$$

$$(\lambda_0+\xi)\pi_{k,0}=\lambda_0\pi_{k-1,0}+\sum_{i=1}^{d}\eta_i\sum_{n=k}^{\infty}\pi_{n,i}p^{n-k}\bar{p}, \quad k\geq 1, \tag{3.2}$$

$$\lambda_i\pi_{0,i}=\mu_i\pi_{1,i}+\xi q_i\pi_{0,0}, \quad i=1,2,\ldots,d, \tag{3.3}$$

$$(\lambda_i+\mu_i+\eta_i)\pi_{k,i}=\lambda_i\pi_{k-1,i}+\mu_i\pi_{k+1,i}+\xi q_i\pi_{k,0}, \quad k\geq 1, \quad i=1,2,\ldots,d. \tag{3.4}$$

Multiplying both sides of equation (3.2) by z^k and summing for all $k\geq 1$, we obtain

$$(\lambda_0+\xi)(\Pi_0(z)-\pi_{0,0})=\lambda_0 z\Pi_0(z)+\sum_{i=1}^{d}\eta_i\bar{p}\sum_{n=1}^{\infty}p^n\pi_{n,i}\sum_{k=1}^{n}\left(\frac{z}{p}\right)^k. \tag{3.5}$$

We now need to distinguish two cases. If $z\neq p$, then we have

$$(\lambda_0+\xi-\lambda_0 z)\Pi_0(z)=(\lambda_0+\xi)\pi_{0,0}+\sum_{i=1}^{d}\eta_i\bar{p}\frac{z}{p-z}(\Pi_i(p)-\Pi_i(z)). \tag{3.6}$$

From (3.1), we have

$$(\lambda_0+\xi)\pi_{0,0}=\sum_{i=1}^{d}\eta_i\Pi_i(p).$$

Substituting the above equation into (3.6), and after some manipulations, we obtain the following relation between $\Pi_0(z)$ and $\Pi_i(z)$:

$$(\lambda_0+\xi-\lambda_0 z)(z-p)\Pi_0(z)=\bar{p}z\sum_{i=1}^{d}\eta_i\Pi_i(z)+(\lambda_0+\xi)p(z-1)\pi_{0,0}. \tag{3.7}$$

If $z=p$, equation (3.5) has the form

$$(\lambda_0+\xi-\lambda_0 p)\Pi_0(p)=(\lambda_0+\xi)\pi_{0,0}+\bar{p}p\sum_{i=1}^{d}\eta_i\Pi_i^{'}(p). \tag{3.8}$$

It is easy to see that (3.8) can be alternatively derived from (3.6), by differentiating and taking $z\to p$. Multiplying (3.3) and (3.4) by appropriate powers of z and summing over $k\geq 0$, we obtain

$$[\lambda_i z(1-z)+(\mu_i+\eta_i)z-\mu_i]\Pi_i(z)=\xi q_i z\Pi_0(z)+\lambda_i z\pi_{0,i}(z-1). \tag{3.9}$$

Define

$$f_0(z)=(\lambda_0+\xi-\lambda_0 z)(z-p), \quad f_i(z)=\lambda_i z(1-z)+(\mu_i+\eta_i)z-\mu_i, \quad i=1,2\cdots,d.$$

The set of equations (3.7) and (3.9) can be written in a matrix form as

$$M(z)\Pi(z) = (z-1)b(z), \tag{3.10}$$

where

$$M(z) = \begin{pmatrix} f_0(z) & -\bar{p}\eta_1 z & -\bar{p}\eta_2 z & \cdots & -\bar{p}\eta_d z \\ -\xi q_1 z & f_1(z) & 0 & \cdots & 0 \\ -\xi q_2 z & 0 & f_2(z) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\xi q_d z & 0 & 0 & \cdots & f_d(z) \end{pmatrix},$$

$$\Pi(z) = \begin{pmatrix} \Pi_0(z) \\ \Pi_1(z) \\ \Pi_2(z) \\ \vdots \\ \Pi_d(z) \end{pmatrix}, b(z) = \begin{pmatrix} b_0(z) \\ b_1(z) \\ b_2(z) \\ \vdots \\ b_d(z) \end{pmatrix} = \begin{pmatrix} (\lambda_0+\xi)p\pi_{0,0} \\ \lambda_1 z\pi_{0,1} \\ \lambda_2 z\pi_{0,2} \\ \vdots \\ \lambda_d z\pi_{0,d} \end{pmatrix}.$$

Since $|M(z)|$ is a polynomial of degree $2d+2$ and $|M(1)| = 0$, we define $N(z)$ of degree $2d+1$ by $|M(z)| = (z-1)N(z)$. For all values of z at which $M(z)$ is nonsingular, we use Cramer's rule to obtain $\Pi_i(z)$, and write

$$\Pi_i(z) = \frac{|M_{i+1}(z)|}{N(z)}, \quad i = 0,1,2,\ldots,d, \tag{3.11}$$

where $|M|$ is the determinant of matrix M and $M_i(z)$ is the matrix obtained from $M(z)$ by replacing its ith column by $b(z)$. The functions $\Pi_i(z)$ are expressed in terms of $d+1$ unknown boundary probabilities, $\pi_{0,0}, \pi_{0,1}, \cdots, \pi_{0,d}$, appearing in $b(z)$. In order to derive these boundary probabilities, we utilize the roots of $N(z)$ lie in $(0,1)$. Similarly to the proof of Theorem 2 in [19] and Theorem 2.1 in [24], we have the following theorem:

Theorem 3.3. *If the stability condition* $\lambda_0 + \sum_{k=1}^{d} \frac{\xi q_k \lambda_k}{\eta_k} < \sum_{k=1}^{d} \frac{\xi q_k}{\eta_k} \frac{\mu_k \bar{p} + \eta_k p}{\bar{p}}$ *holds, then the polynomial* $N(z)$ *possesses exactly* d *distinct roots in the open interval* $(0,1)$*. Else,* $N(z)$ *has an additional root in* $(0,1)$*.*

Proof. First, we introduce a series of polynomials as follows:

$$N_0(z) = 1, N_1(z) = |M_1(z)| = f_0(z), N_2(z) = |M_2(z)| = \begin{vmatrix} f_0(z) & -\eta_1\bar{p}z \\ -\xi q_1 z & f_1(z) \end{vmatrix},$$

$$N_k(z) = |M_k(z)| = \begin{vmatrix} f_0(z) & -\eta_1\bar{p}z & \cdots & -\eta_{k-1}\bar{p}z \\ -\xi q_1 z & f_1(z) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\xi q_{k-1} z & 0 & \cdots & f_{k-1}(z) \end{vmatrix}, \quad 1 \le k \le d,$$

$$N(z) = \frac{|M(z)|}{z-1}.$$

That is, $N_k(z), k = 1,2,\ldots,d$ are the determinants of the main-diagonal minors of $M(z)$ starting from the upper left corner of the matrix.

Next, we provide several recursive equations, which are crucial for our subsequent analysis. We first introduce Schur complement. If A and D are square matrices, A is invertible, then

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A| \left| D - CA^{-1}B \right|.$$

In the case that A is singular, the inverse of A in the equation can be replaced by a generalized inverse. According to Schur complement, when $1 \le k \le d-1$, we have

$$\begin{aligned} N_{k+1}(z) &= (\left|f_k(z) - C_{k+1}M_k^{-1}(z)B_{k+1}\right| + 1)N_k(z) - N_k(z) \\ &= (\left|f_k(z) - C_{k+1}M_k^{-1}(z)B_{k+1}\right| + 1)N_k(z) \\ &\quad - (\left|f_{k-1}(z) - C_kM_{k-1}^{-1}(z)B_k\right|)N_{k-1}(z) \\ &= a_k(z)N_k(z) - b_k(z)N_{k-1}(z), 1 \le k \le d-1, \end{aligned} \tag{3.12}$$

$$\begin{aligned} (z-1)N(z) &= (\left|f_d(z) - C_{d+1}M_d^{-1}(z)B_{d+1}\right| + 1)N_d(z) - N_d(z) \\ &= (\left|f_d(z) - C_{d+1}M_d^{-1}(z)B_{d+1}\right| + 1)N_d(z) \\ &\quad - (\left|f_{d-1}(z) - C_dM_{d-1}^{-1}(z)B_d\right|)N_{d-1}(z) \\ &= a_d(z)N_d(z) - b_d(z)N_{d-1}(z), \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} C_{k+1} &= (\xi q_k z, \overbrace{0, \ldots, 0}^{k-1}), B_{k+1} = (-\eta_k \bar{p} z, \overbrace{0, \ldots, 0}^{k-1})^T, 1 \le k \le d-1, \\ C_{d+1} &= (\xi q_d z, \overbrace{0, \ldots, 0}^{d-1}), B_{d+1} = (-\eta_d \bar{p} z, \overbrace{0, \ldots, 0}^{d-1})^T. \end{aligned}$$

For these d+2 polynomials, we have the following properties:

- **(i)** By definition, $N_0(z) = 1$ and therefore has no roots in $(0, +\infty)$;
- **(ii)** For $1 \le k \le d-1$, $N_k(z)$ and $N_{k+1}(z)$ have no common roots in $(0, +\infty)$. Because, if they do have such a common root z_0, then we have $N_{k-1}(z_0) = 0$ from equation (3.12). Iterating, we eventually arrive at $N_0(z_0) = 0$, but $N_0(z)$ possesses no roots. In an analogous fashion, $N(z)$ and $N_d(z)$ do not possess any common roots in $(0, +\infty)$.
- **(iii)** Define $\text{sign}(x) = 1$ if $x > 0$ and $\text{sign}(x) = -1$ if $x < 0$. For $1 \le k \le d-1$, from equation (3.12), if z_0 is a root of $N_k(z)$, then $\text{sign}(N_{k-1}(z_0) \cdot N_{k+1}(z_0)) = -1$. From equation (3.13), if $N_d(z_0) = 0$ and $0 \le z_0 \le 1$, then $N(z)$ and $N_{d-1}(z)$ are the same in sign. If $N_d(z_0) = 0$ and $z_0 \ge 1$, then $N(z)$ is opposite in sign to $N_{d-1}(z)$.
- **(iv)** $N_k(1) > 0, k = 0, 1, 2, \ldots, d,$ and $N(1) > 0$. Substituting $z = 1$ into the expressions of $N_k(z), k = 1, 2, \ldots, d,$ we directly obtain $N_k(1) > 0, k = 0, 1, 2, \ldots, d$. In order to obtain $N(1) > 0$, we make use of the properties of determinant. Using the L'Hospital's rule, we have

$$N(1) = \lim_{z \to 1} \frac{|M(z)|}{z-1} = \lim_{z \to 1} \frac{d}{dz} |M(z)| = (\eta_1 \eta_2 \cdots \eta_d) \left(\xi p - \lambda_0 \bar{p} + \sum_{i=1}^{d} \frac{\bar{p} \xi q_i (\mu_i - \lambda_i)}{\eta_d} \right).$$

From the stability condition, we have

$$\xi \frac{p}{\bar{p}} + \sum_{i=1}^{d} \frac{\xi q_i (\mu_i - \lambda_i)}{\eta_i} - \lambda_0 > 0,$$

which leads to

$$\xi p - \lambda_0 \bar{p} + \sum_{i=1}^{d} \frac{\bar{p}\xi q_i(\mu_i - \lambda_i)}{\eta_i} > 0,$$

i.e., $N(1) > 0$.

(v) $\text{sign}(N_k(0)) = (-1)^k, k = 1, 2, \ldots, d$. Note that $f_0(0) = -p(\lambda + \xi), \quad f_i(0) = -\mu_i, i = 1, 2, \ldots, d$. When $0 \le k \le d$, $N_k(0) = -p(\lambda + \xi)(-\mu_i)^{k-1} = (-1)^k \mu_i(\lambda + \xi)$. Since $N(z) = \frac{|M(z)|}{z-1}$ and $\text{sign}(|M(0)|) = (-1)^{d+1}$, we have $\text{sign}(N(0)) = (-1)^d$.

(vi) $\text{sign}(N_k(+\infty)) = (-1)^k, k = 1, 2, \ldots, d$. It is because that the highest-power coefficient of the polynomial $N_k(z)$ is $(-1)^k \lambda_0 \lambda_1 \cdots \lambda_{k-1}$.

To give properties **(ii)** and **(iii)**, equations (3.12) and (3.13) are needed. It must be pointed that when $N_k(z_0) = N_{k+1}(z_0) = 0, 1 \le k \le d-1$, equation (3.12) can be used. Although the inverses of $M_k(z_0)$ and $M_{k+1}(z_0)$ do not exist, when $M_k(z_0)$ is singular, the inverse of matrix $M_k(z_0)$ can be replaced by a generalized inverse, for example, the Moore-Penrose pseudoinverse.

Based on **(iv)**, **(v)** and **(vi)**, the quadratic polynomial $N_1(z)$ has a root $z_{1,1}$ in $(0,1)$ and a root $z_{1,2}$ in $(1, +\infty)$. According to the property **(iii)**, we have $N_2(z_{1,1}) < 0$ and $N_2(z_{1,2}) < 0$. Since $N_2(z)$ is of degree four, it has four roots in $(0, +\infty)$. From **(iv)**, **(v)** and **(vi)**, we conclude that the four roots $z_{2,1}$, $z_{2,2}$, $z_{2,3}$ and $z_{2,4}$ lie in open interval $(0, z_{1,1})$, $(z_{1,1}, 1)$, $(1, z_{1,2})$ and $(z_{1,2}, +\infty)$. Proceeding further, $N_3(z)$ includes distinct roots in each of the intervals $(0, z_{2,1})$, $(z_{2,1}, z_{2,2})$, $(z_{2,2}, 1)$, $(1, z_{2,3})$, $(z_{2,3}, z_{2,4})$ and $(z_{2,4}, \infty)$, etc.

After repeating this procedure, we conclude that $N_d(z)$ possesses $2d$ real roots in which d real roots lie in $(0,1)$ and d roots lie in $(1, +\infty)$. The $2d$ roots are denoted, in an ascending order, by $z_{d,i}, i = 1, 2, \ldots, 2d$. From the above properties, we have

$$\begin{aligned} &\text{sign}(N_{d-1}(z_{d,i})) = (-1)^{d+i}, \quad i = 1, 2, \ldots, d, \\ &\text{sign}(N_{d-1}(z_{d,i})) = (-1)^{d+i+1}, \quad i = d+1, d+2, \ldots, 2d. \end{aligned}$$

According to the property **(iii)**, which gives the relation of the sign between $N(z)$ and $N_{d-1}(z)$, we have

$$\text{sign}(N(z_{d,i})) = (-1)^{d+i}, \quad i = 1, 2, \ldots, 2d.$$

Obviously, there is at least one real root of $N(z)$ between any consecutive roots of $N_d(z)$. Note that $\text{sign}(N(0)) = (-1)^d$ and $\text{sign}(N(z_{d,1})) = (-1)^{d+1}$, $N(z)$ has a root in $(0, z_{d,1})$. Moreover, $\text{sign}(N(z_{d,d})) = (-1)^{2d} > 0$ and $N(1) > 0$, so $N(z)$ has no root in $(z_{d,d}, 1)$. Analogously, $\text{sign}(N(z_{d,d+1})) = (-1)^{2d+1} < 0$, $N(z)$ has a root in $(1, z_{d,d+1})$. Since $N(z)$ is a polynomial of degree $2d+1$, we have that $N(z)$ has exactly d real roots in the open interval $(0,1)$.

If the stability condition does not hold, *i.e.*, if

$$\lambda_0 + \sum_{k=1}^{d} \frac{\xi q_k \lambda_k}{\eta_k} > \sum_{k=1}^{d} \frac{\xi q_k}{\eta_k} \frac{\mu_k \bar{p} + \eta_k p}{\bar{p}},$$

then $N(1) < 0$. Since $\text{sign}(N(z_{d,d})) = (-1)^{2d} > 0$ and $\text{sign}(N(z_{d,d+1})) = (-1)^{2d+1} < 0$, so $N(z)$ has an additional root in $(z_{d,d}, 1)$ and has no root in $(1, z_{d,d+1})$. Therefore, $N(z)$ has $d+1$ real roots in the open interval $(0,1)$.

The proof is completed. □

According to the result of Theorem 3.3, we respectively denote the d distinct roots of $N(z)$ in the open interval $(0,1)$ by $z_1, z_2, \ldots, z_d$. Because $\Pi_i(z) \geq 0$, from the expressions of $\Pi_i(z), i = 0, 1, 2, \ldots, d$, we have

$$|M_i(z_k)| = 0, \quad k = 1, 2, \ldots, d, \quad i = 1, 2, \ldots, d+1.$$

In fact, the above equations provide d linear equations for the $d+1$ unknown boundary probabilities $\pi_{0,i}, i = 0, 1, \ldots, d$. From the d equations, we can express $\pi_{0,i}, i = 1, \ldots, d$ in term of $\pi_{0,0}$, and then, $\Pi_i(z), i = 0, 1, 2, \ldots, d$ can be expressed by $\pi_{0,0}$. Using the normalization condition

$$\sum_{i=0}^{d} \Pi_i(1) + \sum_{i=1}^{d} \pi_{0,i} = 1,$$

we can obtain the unique unknown probability $\pi_{0,0}$ and the expressions $\Pi_i(z), i = 0, 1, 2, \ldots, d$. Further, the PGF of the number of customers in the system can be obtained by

$$\Pi(z) = \sum_{i=0}^{d} \Pi_i(z) + \sum_{i=1}^{d} \pi_{0,i}, \tag{3.14}$$

and the expected number of customers in the system is

$$E[L] = \frac{\mathrm{d}}{\mathrm{d}z} \Pi(z)|_{z=1}.$$

4. Matrix geometric approach

In this section, we apply an alternative method, the matrix geometric approach, in order to analyze the current queueing model. It is easy to verify that the state-transition-rate matrix Q is a GI/M/1-type. In order to analyze the system effectively via the matrix geometric approach, we utilize the rate matrix R, which is the minimal non-negative solution of $\sum_{k=0}^{\infty} R^k A_k = 0$.

Due to the structure of A_1, it is not trivial to get a closed-form expression for the rate matrix R. However, R can be obtained numerically using well-known algorithms [21]. Next, we solve the matrix equation numerically with the following iteration procedure to obtain R. Consider a sequence of matrices $\{R(n), n \geq 0\}$, obtained by successive substitutions, starting with $R(0) = 0$, and then $R(n) = -(A_0 + \sum_{k=2}^{\infty} R(n-1)^k A_k) A_1^{-1}$ for $n \geq 1$. If for some $\varepsilon > 0$, such that $\|R(n+1) - R(n)\|_\infty < \varepsilon$, we stop the iterative procedure. Once R is obtained, by using the matrix geometric approach, the steady-state probability vector $\boldsymbol{\Pi} = (\boldsymbol{\Pi_0}, \boldsymbol{\Pi_1}, \cdots)$ of Q with $\boldsymbol{\Pi_k} = (\pi_{k,0}, \pi_{k,1}, \ldots, \pi_{k,d})$ has a matrix geometric form:

$$\boldsymbol{\Pi_k} = \boldsymbol{\Pi_0} R^k, \quad k \geq 1.$$

The boundary vector $\boldsymbol{\Pi_0}$ can be obtained by

$$\boldsymbol{\Pi_0} B_0 + \boldsymbol{\Pi_1} B_1 + \boldsymbol{\Pi_2} B_2 + \cdots = \boldsymbol{\Pi_0} \sum_{k=0}^{\infty} R^k B_k = \mathbf{0},$$
$$(\boldsymbol{\Pi_0} + \boldsymbol{\Pi_1} + \boldsymbol{\Pi_2} + \cdots)\boldsymbol{e} = \boldsymbol{\Pi_0}(I - R)^{-1}\boldsymbol{e} = 1.$$

The expected number of customers in the system is

$$E[L] = \sum_{k=0}^{\infty} k\boldsymbol{\Pi_k e} = \boldsymbol{\Pi_0} R(I-R)^{-2}\boldsymbol{e}.$$

5. Stationary sojourn time analysis

In this section, considering a tagged customer, we derive the LST of the stationary sojourn time distribution of an arbitrary customer, where the sojourn time is defined to be the overall time from the arrival till the departure from the system, due to either a service completion or the occurrence of a service interruption. Let W and $W^*(s)$ respectively denote the stationary sojourn time of an arbitrary customer and its corresponding LST.

In order to calculate $W^*(s)$, we consider the following possible two cases:

Case 1: the tagged customer arrives in the state $(n,i), n \geq 0, i = 1,2,\dots,d$;
Case 2: the tagged customer arrives in the state $(n,0), n \geq 0$.

In Case 1, let $W_{n,i}$ and $W^*_{n,i}(s)$ denote the sojourn time and its LST in this case. Define D_i as the interarrival time of a service interruption, $S_{k,i}$ as the total service times of k customers in phase i. Then, we have

$$\begin{aligned} W^*_{n,i}(s) = &\sum_{k=0}^{n} P(S_{k,i}<D_i<S_{k+1,i})E[e^{-sD_i}|S_{k,i}<D<S_{k+1,i}] \cdot \left(\sum_{m=0}^{n-k} p^m \bar{p} R^*(s) \sum_{j=1}^{d} q_j W^*_{n-k-m,j}(s) + p^{n-k+1}\right) \\ &+ P(S_{n+1,i}<D_i)E[e^{-sS_{n+1,i}}|S_{n+1,i}<D_i], \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} &P(S_{k,i}<D_i<S_{k+1,i})E[e^{-sD_i}|S_{k,i}<D_i<S_{k+1,i}] \\ &= \frac{\eta_i}{\eta_i+s}[S^*_{k,i}(\eta_i+s) - S^*_{k+1,i}(\eta_i+s)] = \frac{\eta_i}{\eta_i+s}\left(1-\frac{\mu_i}{\mu_i+\eta_i+s}\right)\left(\frac{\mu_i}{\mu_i+\eta_i+s}\right)^k \\ &= \frac{\eta_i}{\mu_i+\eta_i+s}\left(\frac{\mu_i}{\mu_i+\eta_i+s}\right)^k, \quad P(S_{n+1,i}<D_i)E[e^{-sS_{n+1,i}}|S_{n+1,i}<D_i] = \left(\frac{\mu_i}{\mu_i+\eta_i+s}\right)^{n+1} \end{aligned}$$

and $R^*(s) = \frac{\xi}{\xi+s}$. Simplifying equation (5.1), we have

$$\begin{aligned} W^*_{n,i}(s) &= \sum_{k=0}^{n} \frac{\eta_i}{\mu_i+\eta_i+s}\left(\frac{\mu_i}{\mu_i+\eta_i+s}\right)^k \cdot \left(\sum_{m=0}^{n-k} p^m \bar{p}\frac{\xi}{s+\xi}\sum_{j=1}^{d} q_j W^*_{n-k-m,j}(s) + p^{n-k+1}\right) + \left(\frac{\mu_i}{\mu_i+\eta_i+s}\right)^{n+1} \\ &= \sum_{k=0}^{n} \frac{\eta_i}{\mu_i+\eta_i+s}\left(\frac{\mu_i}{\mu_i+\eta_i+s}\right)^k \left(\sum_{m=0}^{n-k} p^m \bar{p}\frac{\xi}{s+\xi}\sum_{j=1}^{d} q_j W^*_{n-k-m,j}(s)\right) \\ &\quad + \sum_{k=0}^{n} \frac{\eta_i p^{n+1}}{\mu_i+\eta_i+s}\left(\frac{\mu_i}{p(\mu_i+\eta_i+s)}\right)^k + \left(\frac{\mu_i}{\mu_i+\eta_i+s}\right)^{n+1}. \end{aligned} \tag{5.2}$$

In order to obtain $W^*_{n,i}(s)$, we introduce the mixed transforms $Q_i(s,z) = \sum\limits_{n=0}^{\infty} W^*_{n,i}(s)z^n, |z| < 1, s \geq 0, i = 1, 2, \ldots, d$. Multiplying (5.2) by z^n and summing over n, we have

$$Q_i(s,z) = \frac{\eta_i \xi \bar{p} \sum\limits_{j=1}^{d} q_j Q_j(s,z)}{(s+\xi)(\mu_i+\eta_i+s-\mu_i z)(1-pz)} + \frac{p\eta_i}{(\mu_i+\eta_i+s-\mu_i z)(1-pz)} + \frac{\mu_i}{\mu_i+\eta_i+s-\mu_i z}. \tag{5.3}$$

Define

$$\begin{aligned} X_i(s,z) &= \frac{\eta_i \xi \bar{p}}{(s+\xi)(\mu_i+\eta_i+s-\mu_i z)(1-pz)}, \\ Y_i(s,z) &= \frac{p\eta_i}{(\mu_i+\eta_i+s-\mu_i z)(1-pz)} + \frac{\mu_i}{\mu_i+\eta_i+s-\mu_i z}, \end{aligned}$$

then, we have

$$Q_i(s,z) = X_i(s,z)\sum_{j=1}^{d} q_j Q_j(s,z) + Y_i(s,z),$$

which can be written in matrix form as

$$E(s,z)Q(s,z) = Y(s,z), \tag{5.4}$$

where

$$E(s,z) = \begin{pmatrix} 1-q_1X_1(s,z) & -q_2X_1(s,z) & \cdots & -q_dX_1(s,z) \\ -q_1X_2(s,z) & 1-q_2X_2(s,z) & \cdots & -q_dX_2(s,z) \\ \vdots & \vdots & \ddots & \vdots \\ -q_1X_d(s,z) & -q_2X_d(s,z) & \cdots & 1-q_dX_d(s,z) \end{pmatrix},$$

$$Q(s,z) = \begin{pmatrix} Q_1(s,z) \\ Q_2(s,z) \\ \vdots \\ Q_d(s,z) \end{pmatrix}, Y(s,z) = \begin{pmatrix} Y_1(s,z) \\ Y_2(s,z) \\ \vdots \\ Y_d(s,z) \end{pmatrix}.$$

Hence, we have

$$Q_i(s,z) = \frac{|E_i(s,z)|}{|E(s,z)|}, \quad i = 1, 2, \ldots, d,$$

where $E_i(s,z)$ is derived through replacing the ith column of $E(s,z)$ with $Y(s,z)$. Once $Q_i(s,z)$ is obtained, $W^*_{n,i}(s)$ can be uniquely determined. In Case 2, let $W_{n,0}$ and $W^*_{n,0}(s)$ denote the sojourn time and its LST in this case. Then, we can easily have

$$W^*_{n,0}(s) = R^*(s)\sum_{j=1}^{d} q_j W^*_{n,j}(s) = \frac{\xi}{\xi+s}\sum_{j=1}^{d} q_j W^*_{n,j}(s).$$

Combining the two cases, the LST of the stationary sojourn time distribution of an arbitrary customer can be obtained as

$$W^*(s) = \sum_{n=0}^{\infty} \pi_{n,0} W^*_{n,0}(s) + \sum_{n=0}^{\infty} \sum_{j=1}^{d} \pi_{n,j} W^*_{n,j}(s), \tag{5.5}$$

and the mean stationary sojourn time of an arbitrary customer is obtained by

$$E[W] = -\frac{\mathrm{d}W^*(s)}{\mathrm{d}s}|_{s=0}.$$

In fact, for a general value of $d \geq 2$, it is difficult to obtain the explicit expressions of $W^*(s)$. To conclude this section, we give the following analysis on how the explicit solution of $W^*(s)$ and $E[W]$ can be obtained for $d = 1$.

Special case for $d = 1$. In this case, equation (5.4) can be written as

$$Q_1(s,z) = \frac{(s+\xi)p\eta_1 + (s+\xi)\mu_1(1-pz)}{(s+\xi)(\mu_1+\eta_1+s-\mu_1 z)(1-pz) - \eta_1\xi\bar{p}}.$$

Let $r_1(z) = (s+\xi)(\mu_1+\eta_1+s-\mu_1 z)(1-pz) - \eta_1\xi\bar{p}$, we will prove the polynomial has two roots in $(1,+\infty)$ in following part. Since

$$\begin{aligned}
&r_1(0) = (s+\xi)(\mu_1+\eta_1+s) - \eta_1\xi\bar{p} > 0,\\
&r_1(1) = (s+\xi)(\eta_1+s)(1-p) - \eta_1\xi\bar{p} > 0,\\
&r(1/p) = -\eta_1\xi\bar{p} < 0, r_1((\mu_1+\eta_1+s)/\mu_1) = -\eta_1\xi\bar{p} < 0, r_1(+\infty) > 0,
\end{aligned}$$

then, the polynomial has two roots in $(1,+\infty)$, denoted by $y_1 \leq y_2$, where y_1 lies in $\left(1, \min\left(\frac{1}{p}, \frac{\mu_1+\eta_1+s}{\mu_1}\right)\right)$, and y_2 lies in $\left(\max\left(\frac{1}{p}, \frac{\mu_1+\eta_1+s}{\mu_1}\right), +\infty\right)$. $r_1(z)$ can be rewritten as

$$r_1(z) = \mu_1 p(s+\xi)(z-y_1)(z-y_2).$$

The partial fraction expansion of $Q_1(s,z)$ with respect to z implies that

$$Q_1(s,z) = \frac{p\eta_1 + \mu_1(1-pz)}{\mu_1 p(z-y_1)(z-y_2)} = \frac{1}{\mu_1 p}\left(\frac{M_1(s)}{(z-y_1)} + \frac{M_2(s)}{(z-y_2)}\right),$$

where the coefficients $M_1(s)$ and $M_2(s)$ can be obtained by

$$\begin{aligned}
M_1(s) &= \lim_{z\to y_1} \frac{p\eta_1 + \mu_1(1-pz)}{(z-y_2)} = \frac{p\eta_1 + \mu_1(1-py_1)}{y_1 - y_2},\\
M_2(s) &= \lim_{z\to y_2} \frac{p\eta_1 + \mu_1(1-pz)}{(z-y_1)} = \frac{p\eta_1 + \mu_1(1-py_2)}{y_2 - y_1}.
\end{aligned}$$

Using partial fraction decomposition for $Q_1(s,z)$ and expanding in powers of z, we have

$$Q_1(s,z) = \frac{1}{\mu_1 p}\left(\frac{M_1(s)}{(z-y_1)} + \frac{M_2(s)}{(z-y_2)}\right) = -\frac{M_1(s)}{\mu_1 p y_1}\sum_{n=0}^{\infty}\left(\frac{z}{y_1}\right)^n - \frac{M_2(s)}{\mu_1 p y_2}\sum_{n=0}^{\infty}\left(\frac{z}{y_2}\right)^n, \tag{5.6}$$

and the closed-form expression for $W_{n,1}^*(s)$

$$W_{n,1}^*(s) = -\frac{M_1(s)}{\mu_1 p y_1}\left(\frac{1}{y_1}\right)^n - \frac{M_2(s)}{\mu_1 p y_2}\left(\frac{1}{y_2}\right)^n.$$

For $W_{n,0}^*(s)$, we can easily have

$$W_{n,0}^*(s) = R^*(s)W_{n,1}^*(s) = \frac{\xi}{\xi + s}W_{n,1}^*(s).$$

Then, equation (5.5) translates into

$$W^*(s) = \sum_{n=0}^{\infty}\pi_{n,0}W_{n,0}^*(s) + \sum_{n=0}^{\infty}\pi_{n,1}W_{n,1}^*(s),$$

and the mean stationary sojourn time of an arbitrary customer is obtained by

$$E[W] = -\frac{\mathrm{d}W^*(s)}{\mathrm{d}s}|_{s=0}.$$

6. Cycle analysis

In this section, we mainly focus on the cycle analysis. To avoid confusion, a cycle under consideration is defined as the period between two consecutive instants at which a repair process commences. Let C represent the length of a cycle. For $i = 1, 2, \cdots, d$, define A_i as the interarrival times of two consecutive customers in operative phase i, D_i as the interarrival time of a service interruption in phase i, and $H_{k,i}$ as the busy period caused by k customers in operative phase i. Then, we consider the following two cases.

Case 1: No customer arrives in repair period;
Case 2: $k, k \geq 1$ customers arrive in repair period.

Let U_i and $V_{k,i}$ denote the time duration of the system is in operative i under the condition of Case 1 and Case 2, respectively. Then, in Case 1, we have

$$U_i = \begin{cases} U_{1,i}, D_i < H_{1,i}, \\ U_{2,i} + U_i, D > H_{1,i}, \end{cases} \quad i = 1, 2, \ldots, d,$$

where $U_{1,i} = A_i + (D_i|D_i < H_{1,i})$ and $U_{2,i} = A_i + (H_{1,i}|D_i > H_{1,i})$. The LST of $U_{1,i}$ and $U_{2,i}$ are given by

$$U_{1,i}^*(s) = A_i{}^*(s)E[e^{-sD_i}|D_i < H_{1,i}] = \frac{\lambda_i}{s+\lambda_i}\frac{\eta_i}{s+\eta_i}\frac{1 - H_{1,i}^*(s+\eta_i)}{P(D_i < H_{1,i})},$$
$$U_{2,i}^*(s) = A_i{}^*(s)E[e^{-sH_{1,i}}|D_i > H_{1,i}] = \frac{\lambda_i}{s+\lambda_i}\frac{H_{1,i}^*(s+\eta_i)}{P(D_i > H_{1,i})}.$$

From the above equations, we have

$$U_i^*(s) = P(D_i < H_{1,i})U_{1,i}^*(s) + P(D_i > H_{1,i})U_{2,i}^*(s)U_i^*(s).$$

Simplifying the expression of $U_i^*(s)$ yields

$$U_i^*(s) = \frac{P(D_i < H_{1,i})U_{1,i}^*(s)}{1 - P(D_i > H_{1,i})U_{2,i}^*(s)} = \frac{\lambda_i \eta_i (1 - H_{1,i}^*(s+\eta_i))}{(s+\eta_i)(s+\lambda_i - \lambda_i H_{1,i}^*(s+\eta_i))},$$

where $H_{1,i}^*(s+\eta_i)$ satisfies $\lambda_i z^2 - (s+\eta_i+\lambda_i+\mu_i)z + \mu_i = 0$. Similarly to the approach of Case 1, for Case 2, we have

$$V_{k,i} = \begin{cases} V_{k,i,1}, D_i < H_{k,i}, \\ V_{k,i,2} + U_i, D_i > H_{k,i}, \end{cases} \quad k \geq 1, \quad i = 1, 2, \ldots, d,$$

where $V_{k,i,1} = (D_i | D_i < H_{k,i})$ and $V_{k,i,2} = (H_{k,i} | D_i > H_{k,i})$. The expressions of $V_{k,i,1}^*(s)$ and $V_{k,i,2}^*(s)$ are derived by

$$\begin{aligned} V_{k,i,1}^*(s) &= E[e^{-sD_i} | D_i < H_{k,i}] = \frac{\eta_i}{s+\eta_i} \frac{1 - H_{k,i}^*(s+\eta_i)}{P(D_i < H_{k,i})}, \\ V_{k,i,2}^*(s) &= E[e^{-sH_{k,i}} | D_i > H_{k,i}] = \frac{H_{k,i}^*(s+\eta_i)}{P(D_i < H_{k,i})}. \end{aligned}$$

Then, substituting $V_{k,i,1}^*(s)$ and $V_{k,i,2}^*(s)$ into

$$V_{k,i}^*(s) = P(D_i < H_{k,i})V_{k,i,1}^*(s) + P(D_i > H_{k,i})V_{k,i,2}^*(s)U_i^*(s),$$

we have

$$V_{k,i}^*(s) = \frac{\eta_i(1 - H_{k,i}^*(s+\eta_i))}{s+\eta_i} + H_{k,i}^*(s+\eta_i)U_i^*(s),$$

where $H_{k,i}^*(s+\eta_i)$ satisfies $H_{k,i}^*(s+\eta_i) = (H_{1,i}^*(s+\eta_i))^k$. Define a_k as the probability that $k, k \geq 0$ customers arrive in repair period. It is easy to find that

$$a_k = P\left(\sum_{j=0}^{k} A_{j,0} < R < \sum_{j=0}^{k+1} A_{j,0}\right) = \int_0^\infty \xi \frac{(\lambda_0 t)^k}{k!} e^{-(\lambda_0+\xi)t} \mathrm{d}t,$$

where $\{A_{k,0}, k \geq 0\}$ is an independent and identically distributed sequence of the interarrival times of customers in repair period with $A_{0,0} = 0$. Combining the results of the two cases, the expression of $C^*(s)$ can be written as

$$\begin{aligned} C^*(s) = \pi_{0,0} &\left(a_0 E[e^{-sR} | R < A_{1,0}] \sum_{i=1}^{d} q_i U_i^*(s) + \sum_{k=1}^{\infty} a_k E[e^{-sR} | \sum_{j=0}^{k} A_{j,0} < R < \sum_{j=0}^{k+1} A_{j,0}] \sum_{i=1}^{d} q_i V_{k,i}^*(s) \right) \\ &+ \sum_{n=1}^{\infty} \pi_{n,0} \sum_{k=0}^{\infty} a_k E[e^{-sR} | \sum_{j=0}^{k} A_{j,0} < R < \sum_{j=0}^{k+1} A_{j,0}] \sum_{i=1}^{d} q_i V_{n+k,i}^*(s), \end{aligned}$$

where

$$a_0 E[e^{-sR} | R < A_{1,0}] = \frac{\xi}{s+\lambda_0+\xi},$$

$$a_k E[e^{-sR}|\sum_{j=0}^{k} A_{j,0} < R < \sum_{j=0}^{k+1} A_{j,0}] = \frac{\xi}{\xi+s}\left(1-\frac{\lambda_0}{\lambda_0+\xi+s}\right)\left(\frac{\lambda_0}{\lambda_0+\xi+s}\right)^k = \frac{\xi}{\lambda_0+\xi+s}\left(\frac{\lambda_0}{\lambda_0+\xi+s}\right)^k.$$

7. Numerical results

In this section, we first present some special cases by setting special parameter values to validate above results with existing models. Without loss of generality, we assume $d = 2$, *i.e.*, the system has two operative phases

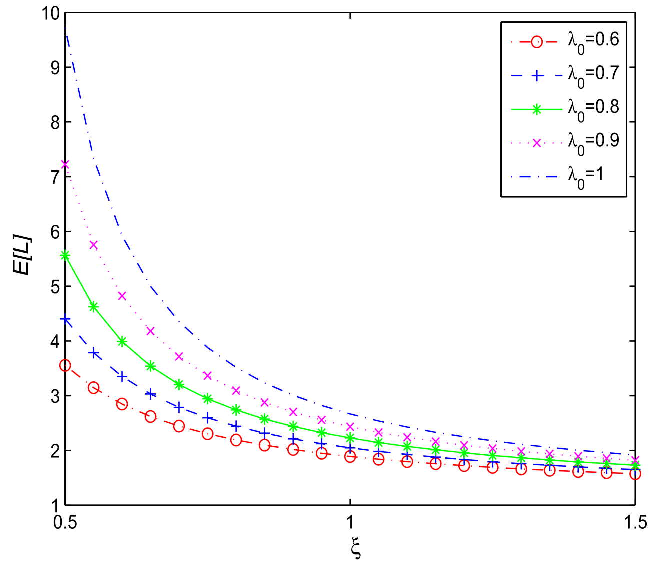

Figure 1. $E[L]$ *versus* ξ,($p = 0.4$).

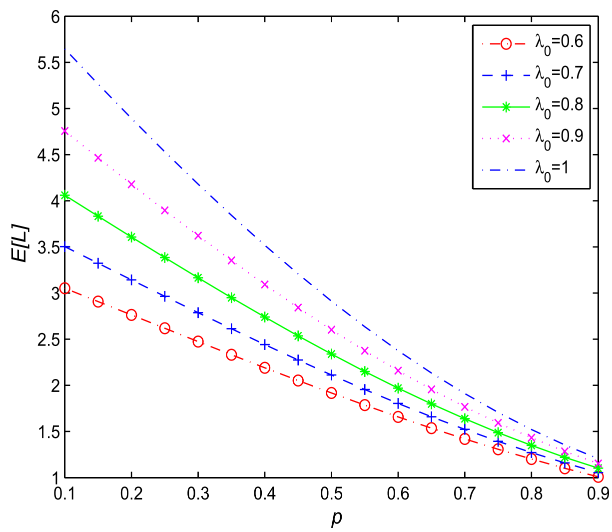

Figure 2. $E[L]$ *versus* p,($\xi = 0.8$).

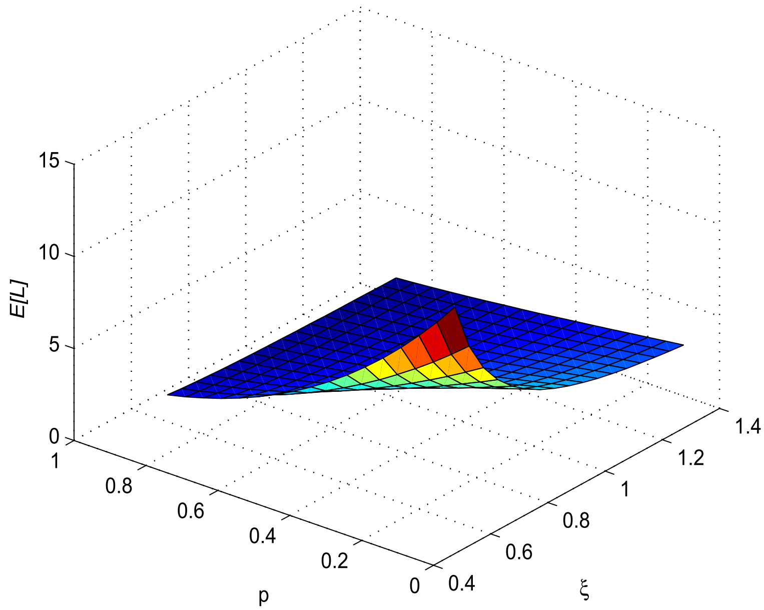

FIGURE 3. $E[L]$ *versus* p and ξ, ($\lambda_0 = 1$).

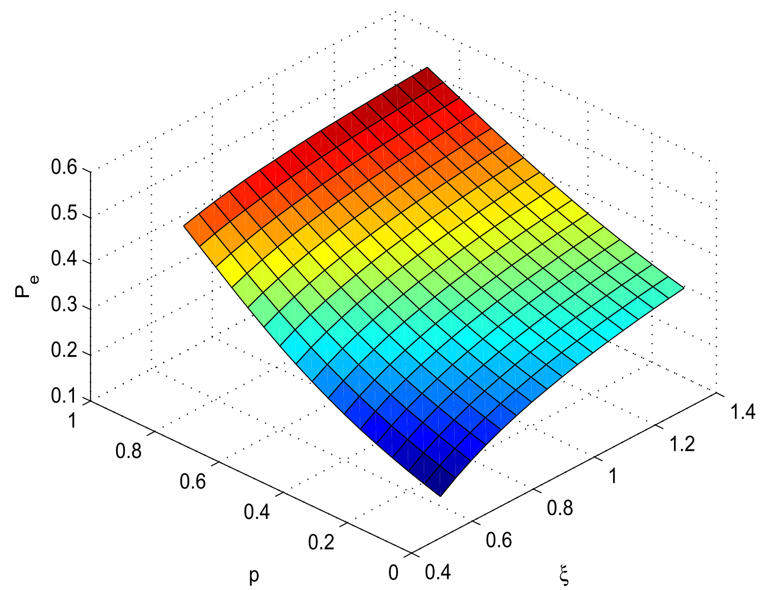

FIGURE 4. P_e *versus* p and ξ, ($\lambda_0 = 1$).

and a repair phase. From the result obtained by (3.11), we have

$$\begin{aligned}N(z) = {}& [\xi - \lambda_0(z-1)](z-1)(z-p)(\lambda_1 z - \mu_1)(\lambda_2 z - \mu_2) - [\xi - \lambda_0(z-1)](z-p)(\lambda_2 z - \mu_2)\eta_1 z \\ & - [\xi - \lambda_0(z-1)](z-p)(\lambda_1 z - \mu_1)\eta_2 z + \xi q_1 \bar{p}\eta_1 z^2(\lambda_2 z - \mu_2) \\ & + \xi q_2 \bar{p}\eta_2 z^2(\lambda_1 z - \mu_1) - [\lambda_0(z-p) - \xi p]\eta_1\eta_2 z^2,\end{aligned}$$

TABLE 1. The roots of $N(z)$, $p = 0.4, q_1 = 0.3$.

z	μ_2					
	$\mu_2 = 1.5$	$\mu_2 = 1.8$	$\mu_2 = 2.1$	$\mu_2 = 2.4$	$\mu_2 = 2.7$	$\mu_2 = 3$
z_1	0.3587	0.3658	0.3704	0.3736	0.3760	0.3778
z_2	0.6688	0.6890	0.7020	0.7110	0.7174	0.7223

TABLE 2. The rate matrix R, $p = 0.4, q_1 = 0.3$.

	$\mu_2 = 1.5$	$\mu_2 = 1.8$
R	$\begin{pmatrix} 0.7170 & 0.1550 & 0.3306 \\ 0.1490 & 0.5814 & 0.1054 \\ 0.1445 & 0.0453 & 0.4906 \end{pmatrix}$	$\begin{pmatrix} 0.6979 & 0.1388 & 0.3063 \\ 0.1423 & 0.5745 & 0.1001 \\ 0.1220 & 0.0336 & 0.4368 \end{pmatrix}$
	$\mu_2 = 2.1$	$\mu_2 = 2.4$
R	$\begin{pmatrix} 0.6824 & 0.1280 & 0.2818 \\ 0.1367 & 0.5697 & 0.0936 \\ 0.1049 & 0.0263 & 0.3912 \end{pmatrix}$	$\begin{pmatrix} 0.6699 & 0.1204 & 0.2591 \\ 0.1321 & 0.5663 & 0.0870 \\ 0.0917 & 0.0214 & 0.3530 \end{pmatrix}$
	$\mu_2 = 2.7$	$\mu_2 = 3$
R	$\begin{pmatrix} 0.6598 & 0.1149 & 0.2389 \\ 0.1283 & 0.5638 & 0.0809 \\ 0.0813 & 0.0179 & 0.3209 \end{pmatrix}$	$\begin{pmatrix} 0.6515 & 0.1108 & 0.2210 \\ 0.1251 & 0.5620 & 0.0753 \\ 0.0730 & 0.0154 & 0.2939 \end{pmatrix}$

TABLE 3. A comparison of two method in computing the stationary distribution $\boldsymbol{\Pi_0}$ and $E[L]$, $p = 0.4$, $q_1 = 0.3$.

		$\pi_{0,0}$	$\pi_{0,1}$	$\pi_{0,2}$	$E[L]$
PGF	$\mu_2 = 1.5$	0.0091	0.0293	0.0538	12.6618
	$\mu_2 = 1.8$	0.0145	0.0427	0.1120	6.5468
	$\mu_2 = 2.1$	0.0178	0.0512	0.1671	4.4772
	$\mu_2 = 2.4$	0.0198	0.0521	0.2239	3.4220
	$\mu_2 = 2.7$	0.0208	0.0525	0.2739	2.7905
	$\mu_2 = 3$	0.0212	0.0522	0.3185	2.3685
MGM	$\mu_2 = 1.5$	0.0091	0.0293	0.0539	12.6329
	$\mu_2 = 1.8$	0.0146	0.0429	0.1118	6.5428
	$\mu_2 = 2.1$	0.0179	0.0494	0.1697	4.4647
	$\mu_2 = 2.4$	0.0198	0.0520	0.2241	3.4207
	$\mu_2 = 2.7$	0.0208	0.0526	0.2738	2.7908
	$\mu_2 = 3$	0.0212	0.0521	0.3186	2.3674

which possesses exactly two distinct roots z_1 and z_2 in open interval (0,1). Using the results

$$|A_1(z_k)| = 0, \quad k = 1, 2,$$

and combining the normalization condition

$$\Pi_0(1) + \Pi_1(1) + \Pi_2(1) + \pi_{0,1} + \pi_{0,2} = 0,$$

we construct three linear equations for the unknown boundary probabilities $\pi_{0,0}$, $\pi_{0,1}$ and $\pi_{0,2}$. Solving the set of linear equations, the expressions of $\pi_{0,0}$, $\pi_{0,1}$ and $\pi_{0,2}$ and the expected number of customers in the system $E[L]$ can be obtained, respectively.

Next, based on the theoretical framework given in Sections 3 and 4, we will give some tables to show the roots of $N(z)$ in the open interval $(0,1)$ and the rate matrix R. Meanwhile, we show that the stationary distribution and the mean queue length $E[L]$ are evaluated through matrix geometric method (MGM) exactly match with the one obtained from probability generating function (PGF). We assume $\lambda_0 = 1$, $\lambda_1 = 1.5$, $\lambda_2 = 1$, $\mu_1 = 2$, $\xi = 0.8$, $\eta_1 = 0.4$ and $\eta_2 = 0.6$ (Tabs. 1–3).

Next, we will provide some figures to illustrate the impact of some parameters on the mean queue length $E[L]$ in the system and the probability that the system is empty $P_e = \boldsymbol{\Pi_0 e}$. Without loss of generality, we assume $d = 5$ and

$$\begin{aligned}
&(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (1.5, 1, 0.8, 2, 2.5),\\
&(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5) = (0.4, 0.6, 0.2, 0.5, 0.8)\\
&(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5) = (2.5, 2, 1.5, 3, 3.5),\\
&(q_1, q_2, q_3, q_4, q_5) = (0.1, 0.25, 0.2, 0.4, 0.05).
\end{aligned}$$

In Figures 1 and 2, we respectively pay attention to the curves of $E[L]$ with the change of ξ from 0.5 to 1.5 and p from 0.1 to 0.9 for different values of $\lambda_0 = 0.6, 0.7, 0.8, 0.9, 1$. Obviously, from Figure 1, with the increase of ξ, $E[L]$ becomes smaller. From Figure 2, we find that $E[L]$ decreases with the increase of p. It is reasonable that, as ξ increases, the times that the system resides in repair period become smaller, and the system has more times in operative phase, which leads to more customers leaving the system by either the service completion or the occurrence of breakdowns. For the curves in Figure 2, it may be because, as p increases, at the instant of breakdown, customers have a higher probability to leave the system, which contributes to the decrease of the expected number of customers in the system. It is noteworthy that, for Figures 1 and 2, if ξ and p are fixed, the smaller λ_0 is, the smaller $E[L]$ becomes, which is identical to the intuitive expectations. In fact, smaller λ_0 means less arriving customers while the system is in repair phase, which leads to the smaller value of $E[L]$.

In Figures 3 and 4, we assume $\lambda_0 = 1$ and investigate the values $E[L]$ and P_e regarding the combinations of the values p and ξ. As expected, from Figure 3, $E[L]$ decreases as p and ξ increase from 0.1 to 0.9 and 0.6 to 1.4. From Figure 4, P_e increases as p and ξ increase, which has an opposite variation trend. It is reasonable that, with the increase of p and ξ, the system is more likely to be empty, which causes a increase of P_e.

8. Conclusion

In this paper, we studied a queue with unreliable server in a multi-phase random environment with geometric abandonments and aimed to establish the theoretical foundations for applications and obtain the explicit computation expressions for the performance measures. By using the mean drift result in [21], we first gave the sufficient and necessary stability condition for the system. Based on Theorem 3.3, we then derived the stationary queue length distribution. Further, we provided the elaborate analysis of various performance measures including the stationary sojourn time of an arbitrary customer and the length of a cycle. Finally, we gave some numerical examples to show the impact of parameters on the performance measures. We expect that the results can be applied to more practical queueing systems.

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