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COMBINATION OF TWO UNDERESTIMATORS FOR UNIVARIATE GLOBAL OPTIMIZATION

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Abstract. In this work, we propose a new underestimator in branch and bound algorithm for solving univariate global optimization problems. The new underestimator is a combination of two underestimators, the classical one used in αBB method (see Androulakis *et al.* [*J. Glob. Optim.* **7** (1995) 337–3637]) and the quadratic underestimator developed in Hoai An and Ouanes [*RAIRO: OR* **40** (2006) 285–302]. We show that the new underestimator is tighter than the two underestimators. A convex/concave test is used to accelerate the convergence of the proposed algorithm. The convergence of our algorithm is shown and a set of test problems given in Casado *et al.* [*J. Glob. Optim.* **25** (2003) 345–362] are solved efficiently.

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1. Introduction

We consider the following problem

$$(P)\begin{cases} \min b(s) \\ s \in [s^0, s^1] \subset R \end{cases}$$

where b is a nonconvex and C^2-continuous function on the real closed interval $[s^0, s^1]$. Several methods have been studied in the literature for univariate global optimization problems (see [6, 13] and references therein).

Univariate global optimization problems attract attention of researchers not only because they arise in many real-life applications but also the methods for these problems are useful for the extension for the multivariate case (see [8, 12]) or by reducing the multidimensional case to the univariate case (see [10]). The lower bounding method is one of the most widely used methods to find global minimum for sure. Among them we can cite the classical αBB method developed in [1, 2, 3], and the method using a quadratic underestimator developed in [9]. The efficiency of a method is in the construction of tight underestimator which allows us to discard big regions which do not contain the global minimum. We propose a new underestimator that combines the two underestimators given in [3, 9]. The main contributions of our paper are: (i) a construction of a new

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underestimator which is tighter than the two underestimators used in [3, 9], (ii) a convex/concave test is used in order to accelerate the convergence of our branch and bound algorithm, (iii) a set of test problems found in [4] are solved efficiently.

The structure of the paper is as follows. The two underestimators developed in [3, 9] with their properties are presented in Section 2. In Section 3, a new underestimator is stated with its properties and the convex/concave test is explained. In Section 4, the algorithm is described and its convergence is shown. Computational results are presented in Section 5.

2. Background

2.1. Underestimator in αBB method [3]

Let K_α be a nonnegative real number such that $K_\alpha \geq \max\{0, -b''(s)\}$, for all s in the real interval $[s^0, s^1]$. The underestimator in αBB method on the interval $[s^0, s^1]$ is given by

$$LB_\alpha(s) = b(s) - \frac{K_\alpha}{2}(s - s^0)(s^1 - s).$$

This underestimator has the following properties :

1. It is convex ($i.e.$ $LB''_\alpha(s) = b''(s) + K_\alpha \geq 0$ because $K_\alpha \geq -b''(s), \forall s \in [s^0, s^1]$).

2. It coincides with the function b at the endpoints of the interval $[s^0, s^1]$ ($i.e.$ by construction of $LB_\alpha(s)$).

3. It is an underestimator of $b(s)$ ($i.e.$ $b(s) - LB_\alpha(s) = \frac{K_\alpha}{2}(s - s^0)(s^1 - s) \geq 0, \forall s \in [s^0, s^1]$).

For more details see [3].

2.2. Quadratic underestimator [9]

Let K be a nonnegative real number such that $K \geq |b"(s)|$, for all s in the interval $[s^0, s^1]$. The quadratic underestimator developed in [9] on the interval $[s^0, s^1]$ is given by

$$LB_q(s) = b(s^0)\frac{s^1 - s}{s^1 - s^0} + b(s^1)\frac{s - s^0}{s^1 - s^0} - \frac{K}{2}(s - s^0)(s^1 - s).$$

This quadratic underestimator has the following properties :

1. It is convex and quadratic ($i.e.$ $LB''_q(s) = K \geq 0$).
2. It coincides with the function b at the endpoints of the interval $[s^0, s^1]$ ($i.e.$ by construction of $LB_q(s)$).
3. It is an underestimator of $b(s)$ ($i.e.$ $(b(s) - LB_q(s))'' = (b''(s) - K) \leq 0, \forall s \in [s^0, s^1]$) which implies that $(b(s) - LB_q(s))$ is concave on $[s^0, s^1]$, it vanishes at the endpoints of $[s^0, s^1]$ then $(b(s) - LB_q(s)) \geq 0, \forall s \in [s^0, s^1]$.

For more details see [9].

3. New underestimator

Let K_q be a real nonnegative number such that $K_q \geq \max\{0, b''(s)\}$, for all s on the interval $[s^0, s^1]$ and let $L_h b(s)$ the linear interpolant of b on this interval given by $L_h b(s) = b(s^0)\frac{s^1 - s}{s^1 - s^0} + b(s^1)\frac{s - s^0}{s^1 - s^0}$ (see [5]). The new underestimator on the interval $[s^0, s^1]$ is given by

$$LB(s) = \frac{K_q b(s) + K_\alpha L_h b(s)}{K_\alpha + K_q} - \frac{K_\alpha K_q}{2(K_\alpha + K_q)}(s - s^0)(s^1 - s).$$

Proposition 3.1.

i) *$LB(s)$ coincides with $b(s)$ at the endpoints of* $[s^0, s^1]$.
ii) *$LB(s)$ is convex on the interval* $[s^0, s^1]$.

Proof.

i) It is obvious by construction of $LB(s)$.
ii) For all s in the interval $[s^0, s^1]$, we have

$$LB''(s) = \frac{K_q b''(s)}{K_\alpha + K_q} + \frac{K_\alpha K_q}{K_\alpha + K_q} = \frac{K_q(K_\alpha + b''(s))}{K_\alpha + K_q} \geq 0.$$

Indeed K_α, K_q are nonnegative numbers and for all s in $[s^0, s^1]$, we have $(K_\alpha + b''(s)) \geq 0$. Then LB is convex.

□

In the following theorem we show that $LB(s)$ is an underestimator of $b(s)$.

Theorem 3.2. $LB(s) \leq b(s), \forall s \in [s^0, s^1]$.

Proof. By computing the second derivative of $(LB(s) - b(s))$, we have

$$(LB(s) - b(s))'' = \frac{K_q(K_\alpha + b''(s))}{K_\alpha + K_q} - b''(s) = \frac{K_\alpha(K_q - b''(s))}{K_\alpha + K_q} \geq 0.$$

As K_α and K_q are nonnegative numbers and $K_q - b''(s) \geq 0$ on $[s^0, s^1]$, then $(LB(s) - b(s))$ is convex. It also vanishes at the endpoints of this interval. Hence $(LB(s) - b(s))$ is nonpositive which implies that $LB(s) \leq b(s), \forall s \in [s^0, s^1]$. □

We will show in the two next theorems that the new underestimator is tighter than the classical underestimator developed in αBB method [3] and the quadratic underestimator developed in [9], respectively.

Theorem 3.3. $LB(s) \geq LB_\alpha(s), \forall s \in [s^0, s^1]$.

Proof. In the same way as above, for all s in $[s^0, s^1]$, we have

$$(LB(s) - LB_\alpha(s))'' = \frac{K_\alpha(K_q - b''(s))}{K_\alpha + K_q} - K_\alpha = -\frac{K_\alpha(K_\alpha + b''(s))}{K_\alpha + K_q} \leq 0,$$

since K_α and K_q are nonnegative numbers and $(K_\alpha + b''(s))$ is nonnegative on the interval $[s^0, s^1]$, then $(LB(s) - LB_\alpha(s))$ is concave. It also vanishes at the endpoints of this interval, hence $LB(s) - LB_\alpha(s) \geq 0$ which implies that $LB(s)$ is tighter than $LB_\alpha(s)$ on the interval $[s^0, s^1]$. □

Theorem 3.4. $LB(s) \geq LB_q(s), \forall s \in [s^0, s^1]$.

Proof. By computing the second derivative of $(LB(s) - LB_q(s))$, we have

$$(LB(s) - LB_q(s))'' = \frac{K_q b''(s)}{K_\alpha + K_q} + \frac{K_\alpha K_q}{K_\alpha + K_q} - K.$$

We have the relation between K, K_α, and K_q which is $K = \max\{K_\alpha, K_q\}$.

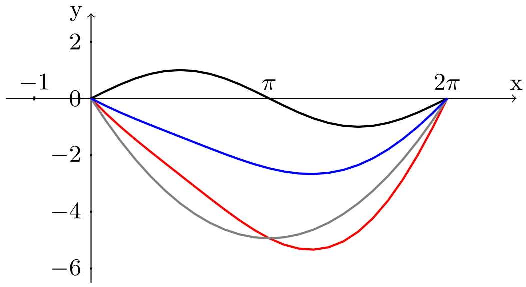

FIGURE 1. Comparison of the new underestimator $LB(x)$ (*blue*) with the two underestimators $LB_\alpha(x)$ (*red*) and $LB_q(x)$ (*gray*) for the function $\sin x$ (*black*).

First case: $K = K_\alpha \geq K_q$

$$\begin{aligned}(LB(s) - LB_q(s))'' &= \frac{K_q b''(s)}{K_\alpha + K_q} + \frac{K_\alpha K_q}{K_\alpha + K_q} - K_\alpha = \frac{K_q b''(s) - K_\alpha^2}{K_\alpha + K_q} \\ &\leq \frac{K_q b''(s) - K_q^2}{K_\alpha + K_q} = (b''(s) - K_q)\frac{K_q}{K_\alpha + K_q}.\end{aligned}$$

As K_q, K_α are nonnegative numbers and $(b''(s) - K_q) \leq 0, \forall s \in [s^0, s^1]$, then $(LB(s) - LB_q(s))$ is concave on $[s^0, s^1]$. It also vanishes at the endpoints of the interval, hence $LB(s) - LB_q(s) \geq 0, \forall s \in [s^0, s^1]$ which implies that $LB(s)$ is tighter than $LB_q(s)$ on the interval $[s^0, s^1]$.

Second case: $K = K_q \geq K_\alpha$

$$\begin{aligned}(LB(s) - LB_q(s))'' &= \frac{K_q b''(s)}{K_\alpha + K_q} + \frac{K_\alpha K_q}{K_\alpha + K_q} - K_q \\ &= \frac{K_q b''(s) + K_\alpha K_q - K_q K_\alpha - K_q K_q}{K_\alpha + K_q} \\ &= (b''(s) - K_q)\frac{K_q}{K_\alpha + K_q}.\end{aligned}$$

In the same way as in above, we prove that $LB(s)$ is tighter than $LB_q(s)$.

□

Example 3.5. We consider a simple example, $b(s) = \sin s, s \in [0, 2\pi]$ as shown in Figure 1. For this example, we compare $LB(s)$ with $LB_\alpha(s)$ and $LB_q(s)$. We have

$$LB_\alpha(s) = \sin s - \frac{1}{2}s(2\pi - s), LB_q(s) = -\frac{1}{2}s(2\pi - s) \text{ and } LB(s) = \frac{1}{2}LB_\alpha(s).$$

For all s in $[0, 2\pi]$, $LB_\alpha''(s) = -\sin s + 1$ is nonnegative, then it is convex on this interval, as LB_α vanishes at 0 and 2π, hence $LB_\alpha(s) \leq 0, \forall s \in [0, 2\pi]$. $LB_q''(s) = 1$, then $LB_q(s)$ is convex, as it vanishes at 0 and 2π, hence $LB_q(s) \leq 0, \forall s \in [0, 2\pi]$.

- We have $LB(s) = \frac{1}{2}LB_\alpha(s) \geq LB_\alpha(s)$. As $LB_\alpha(s) \leq 0$ then $LB(s)$ is tighter than $LB_\alpha(s)$ on $[0, 2\pi]$.
- $(LB(s) - LB_q(s))'' = \frac{1}{2}(-\sin s + 1) - 1 = \frac{1}{2}(-\sin s - 1) \leq 0, \forall s \in [0, 2\pi]$ then $(LB(s) - LB_q(s))$ is concave on $[0, 2\pi]$, as it vanishes at 0 and 2π then $(LB(s) - LB_q(s)) \geq 0, \forall s \in [0, 2\pi]$, hence $LB(s) \geq LB_q(s), \forall s \in [0, 2\pi]$ *i.e.* $LB(s)$ is tighter than $LB_q(s)$.

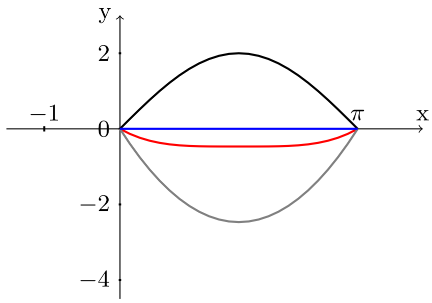

FIGURE 2. The new underestimator $LB(x)$ (*blue*), the αBB underestimators $LB_\alpha(x)$ (*red*), the quadratic underestimator $LB_q(x)$ (*gray*) and the function $\sin x$ (*black*).

3.1. Convex/concave test

By using the definitions of K_α and K_q, we develop a convex/concave test in order to accelerate the convergence of our branch and bound algorithm. We will give here a description of the convex/concave test. At iteration k we compute K_α^k and K_q^k on the interval $[s^k, s^{k+1}]$ by using the following inequalities $K_\alpha^k \geq \max\{0, \max_{s\in[s^k,s^{k+1}]}(-b''(s))\}$, and $K_q^k \geq \max\{0, \max_{s\in[s^k,s^{k+1}]} b''(s)\}$ respectively(*i.e.* by using interval analysis method).

i) If $K_\alpha^k = 0$, then $-b''(s) \leq 0, \forall s \in [s^k, s^{k+1}]$), then b is convex on the interval $[s^k, s^{k+1}]$, any local search gives a global minimum on this interval.

ii) If $K_q^k = 0$, then $b''(s) \leq 0, \forall s \in [s^k, s^{k+1}]$) then b is concave on the interval $[s^k, s^{k+1}]$ and its global minimum is reached at the endpoints of this interval.

iii) If $K_\alpha^k = K_q^k = 0$, then $b(s)$ is affine on $[s^k, s^{k+1}]$ and its global minimum is reached at the endpoints of this interval.

Remark 3.6. If the convex/concave test is satisfied for all considered subintervals during the algorithm, then the algorithm stops, because for each subinterval, we have an exact minimum, then we find a global minimum, or global minima if the objective function has several minima.

Example 3.7. We consider again the same example, $b(s) = \sin s, s \in [0, \pi]$.

One has $K_\alpha = 1, K = 1$, and $K_q = 0$.

$K_q = 0$ then b is concave on $[0, \pi]$, hence its global minimum is reached at the endpoints of this interval (*i.e.* at 0 and π).

For this example the convex/concave test allows us to stop the algorithm at the first iteration and to find the global minimum while the two methods in [3, 9] can't do that at the first iteration (see Fig. 2).

4. ALGORITHM AND ITS CONVERGENCE

Definition 4.1. [11] Let s^* be an arbitrary global optimal solution of problem (P) and $\varepsilon > 0$, $\overline{s}^k$ is an ε-global optimal solution of (P) if it satisfies the inequality $b(\overline{s}^k) \leq b(s^*) + \varepsilon$.

We now present our branch and bound algorithm.

4.1. Algorithm

1. **Initialization step :**
   i) Let ε be a given sufficiently small number, let $[s^0, s^1]$ the initial interval, compute K_α^0 and K_q^0 such that
   $K_\alpha^0 \geq \max\{0, -b''(s)\}, \forall s \in [s^0, s^1]$, and $K_q^0 \geq \max\{0, b''(s)\}, \forall s \in [s^0, s^1]$.

ii) Convex/concave test

If $K_\alpha^0 = 0$ stop b is convex, any local search gives an optimal solution.

If $K_q^0 = 0$ stop b is concave, the optimal solution is reached at the endpoints of $[s^0, s^1]$.

iii) Set $k := 0; T^0 = [s^0, s^1]; M := \{T^0\}$

iv) Solve the convex problem

$\min\left\{LB_0(s) : s \in T^0\right\}$

with

$$LB_0(s) = \frac{K_q^0 b(s) + K_\alpha^0 \left(b(s^0)\frac{s^1 - s}{s^1 - s^0} + b(s^1)\frac{s - s^0}{s^1 - s^0}\right) - \frac{K_\alpha^0 K_q^0}{2}(s - s^0)(s^1 - s)}{K_\alpha^0 + K_q^0}$$

to obtain an optimal solution s_0^*.

v) Set $UB_0 := \min\left\{b(s^0), b(s^1), b(s_0^*)\right\}$

Set $\overline{s}^0$ the best current solution *i.e.* $UB_0 = b(\overline{s}^0)$.

Set $LB_0 := LB(T^0) = LB_0(s_0^*)$.

2. **Iteration step** While $UB_k - LB_k > \varepsilon$, do

2.1 Let $T^k = [s^k, s^{k+1}] \in M$ be the interval such that $LB_k = LB(T^k)$ and let s_k^* be the solution of the convex problem on T^k.

2.2 Bisect T^k into two intervals $T_1^k = [s^k, s_k^*]$ and $T_2^k = [s_k^*, s^{k+1}]$

Set $T_1^k := [s_1^k, s_1^{k+1}]$ and $T_2^k := [s_2^k, s_2^{k+1}]$.

2.3 For $i = 1, 2$ do

a. Convex/concave test

– Compute K_α^{ki} and K_q^{ki} on T_i^k.

– If $K_\alpha^{ki} = 0$, b is convex on T_i^k, any local search gives an optimal solution s_{ki}^* on T_i^k, then update $LB(T_i^k) = UB(T_i^k) = b(s_{ki}^*)$ and goto c.

– If $K_q^{ki} = 0$, b is concave on T_i^k, then update

$LB(T_i^k) = UB(T_i^k)$=min$\{b(s_i^k), b(s_i^{k+1})\}$ and goto c.

b. Set s_{ki}^* the solution of the convex problem

$\min\left\{LB^{ki}(s) : s \in T_i^k\right\}$

with

$$LB^{ki}(s) = \frac{K_q^{ki} b(s) + K_\alpha^{ki}\left(b(s_i^k)\frac{s_i^{k+1} - s}{s_i^{k+1} - s_i^k} + b(s_i^{k+1})\frac{s - s_i^k}{s_i^{k+1} - s_i^k}\right) - \frac{K_\alpha^{ki} K_q^{ki}}{2}(s - s_i^k)(s_i^{k+1} - s)}{K_\alpha^{ki} + K_q^{ki}}$$

and set $LB(T_i^k) = LB^{ki}(s_{ki}^*)$.

c. To fit into M the intervals $T_i^k : M \leftarrow M \bigcup \{T_i^k : UB_k - LB(T_i^k) \geq \varepsilon\} \setminus \{T^k\}$.

d. Update $UB_{k+1} := \min\{UB_k, b(s_i^k), b(s_i^{k+1}), b(s_{ki}^*)\}$.

Set $\overline{s}^{k+1}$ the best current solution *i.e.* $UB_{k+1} = b(\overline{s}^{k+1})$.

2.4 Update $LB_{k+1} := \min\{LB(T) : T \in M\}$.

2.5 Delete from M all intervals T such that $LB(T) > UB_{k+1} - \varepsilon$.

2.6 Set $k := k + 1$.

2.7 End while.

3. **Output :** $\overline{s}^k$ is an ε- global optimal solution of (P).

4.2. Convergence

The following theorem shows the convergence of our branch and bound algorithm.

Theorem 4.2. *The sequence $\{\overline{s}^k\}$ generated by the algorithm converges to an optimal solution of problem (P).*

Proof. If the algorithm stops at iteration k which may be obtained by the convex/concave test or the stopping rule $UB_k - LB_k \leq \varepsilon$ then one obtains an exact or an ε-global optimal solution.

If the algorithm is infinite then it generates an infinite sequence $\{T^k\}$ of intervals whose lengths h_k decrease to zero, then the whole sequence $\{T^k\}$ shrinks to a singleton, we must show that $\lim_{k\to\infty}(UB_k - LB_k) = 0$.

One has

$$
\begin{aligned}
0 &\le UB_k - LB_k \\
&= b(\overline{s}^k) - LB(s_k^*) \\
&= b(\overline{s}^k) - \frac{K_q b(s_k^*) + K_\alpha L_h b(s_k^*)}{K_\alpha + K_q} + \frac{K_\alpha K_q}{2(K_\alpha + K_q)}(s_k^* - s_i^k)(s_i^{k+1} - s_k^*) \\
&= \frac{K_q(b(\overline{s}^k) - b(s_k^*)) + K_\alpha(b(\overline{s}^k) - L_h b(s_k^*)) + \dfrac{K_\alpha K_q}{2}(s_k^* - s^k)(s^{k+1} - s_k^*)}{K_\alpha + K_q}.
\end{aligned}
$$

We have the following inequalities:

i) One has

$$
\begin{aligned}
K_q(b(\overline{s}^k) - b(s_k^*)) &= K_q b'(\xi_1^k)(\overline{s}^k - s_k^*) \\
&\le K_q C_1(s^{k+1} - s^k),
\end{aligned}
$$

with $C_1 \ge |b'(\xi_1^k)| \ge 0, \xi_1^k$ between $\overline{s}^k$ and s_k^*(*i.e.* by the mean value theorem).

ii) by definition of $L_h b$, one has

$$
L_h b(s_k^*) \ge \min\{b(s^k), b(s^{k+1})\},
$$

suppose that this minimum is $b(s^k)$ then

$$
\begin{aligned}
K_\alpha(b(\overline{s}^k) - L_h b(s_k^*)) &\le K_\alpha(b(\overline{s}^k) - b(s^k)) \\
&= K_\alpha b'(\xi_2^k)(\overline{s}^k - s^k) \\
&\le K_\alpha C_2(s^{k+1} - s^k),
\end{aligned}
$$

with $C_2 \ge |b'(\xi_2^k)| \ge 0, \xi_2^k$ between $\overline{s}^k$ and s^k(*i.e.* by the mean value theorem).
We have the same reasoning if we suppose that $\min\{b(s^k), b(s^{k+1})\} = b(s^{k+1})$.

iii) One has

$$
\begin{aligned}
\frac{K_\alpha K_q}{2}(s_k^* - s^k)(s^{k+1} - s_k^*) &\le \max_{s \in [s^k, s^{k+1}]} \frac{K_\alpha K_q}{2}(s - s^k)(s^{k+1} - s) \\
&= \frac{K_\alpha K_q}{8}(s^{k+1} - s^k)^2
\end{aligned}
$$

(*i.e.* maximum of concave function on $[s^k, s^{k+1}]$).

By using the results shown in **i)**, **ii)** and **iii)**, we have

$$
\begin{aligned}
0 &\le UB_k - LB_k \\
&\le (s^{k+1} - s^k)\frac{K_q C_1 + K_\alpha C_2 + \dfrac{K_\alpha K_q}{8}(s^{k+1} - s^k)}{K_\alpha + K_q}
\end{aligned}
$$

and

$$\begin{aligned}0 &\le \lim_{k\to\infty}(UB_k - LB_k)\\ &\le \lim_{k\to\infty}\left((s^{k+1}-s^k)\frac{K_qC_1+K_\alpha C_2+\dfrac{K_\alpha K_q}{8}(s^{k+1}-s^k)}{K_\alpha+K_q}\right)\\ &= 0\end{aligned}$$

($i.e.$ $T^k = [s^k, s^{k+1}]$ shrinks to a singleton). Hence the sequence $\{\overline{s}^k\}$ converges to a global optimal solution of problem (P).

We have shown that

$$\begin{aligned}0 &\le UB_k - LB_k\\ &\le (s^{k+1}-s^k)\frac{K_qC_1+K_\alpha C_2+\dfrac{K_\alpha K_q}{8}(s^{k+1}-s^k)}{K_\alpha+K_q}\\ &\le C(s^{k+1}-s^k)\\ &\quad \text{with}\, \mathrm{C} = \frac{\mathrm{K_qC_1+K_\alpha C_2}+\dfrac{\mathrm{K_\alpha K_q}}{8}(\mathrm{s}^1-\mathrm{s}^0)}{\mathrm{K_\alpha+K_q}} > 0.\end{aligned}$$

then for any $\varepsilon > 0$ there exists $n \in N$ such that $\forall k \ge n$,
we have

$$0 \le b(\overline{s}^k) - b(s^*) \le UB_k - LB_k \le C(s^{k+1}-s^k) \le \varepsilon$$

($i.e.$ $\lim_{k\to\infty}(s^{k+1}-s^k) = 0$) where s^* is an arbitrary global optimal solution, then $\overline{s}^k$ is an ε-global optimal solution of (P). □

5. Computational results

We solve a set of test problems found in [4] and compare the number of partitions(subintervals) used in [7] to find global minimum and the number of subintervals used in our branch and bound algorithm to find global minimum. The experimental environment is implemented in MATLAB programs and executed on a DELL Computer with the configuration of Intel Core I3 CPU M370 at 2.40 GHz and 4GB RAM. Let us notice that Intlab was used for interval analysis to compute K_α, K and K_q.

In Table 1, we report the performance comparison results of the proposed BB algorithm and the algorithm given in [7].

- **NbF** is the number of partitions (subintervals) used in [7]
- **GoF** is the global optimum found in [7]
- **NbO** is the number of subintervals used in our branch and bound algorithm
- **GoO** is the global optimum found by our branch and bound algorithm
- **GsO** is global solutions found by our branch and bound algorithm

A proposed branch and bound algorithm allows us to find all the optimal solutions for all problems and the number of subintervals used in our algorithm is less than the number of partitions (subintervals) used in [7] to construct a convex envelope in order to find global minima for each function.

TABLE 1. Comparison in term of **subintervals** used in our Branch and Bound algorithm with the number of partitions(**subintervals**) used in the algorithm given in [7] to find global minimum of test functions found in [4].

Fct	NbF	GoF	NbO	GoO	GsO
1	16	−1	7	−1	7.853981 14.137166 20.420352
2	8	−1	3	−1	6.283185
3	1	0	1	0	0.999991
4	64	−17.58287	27	−17.582871	6.894525
5	1024	−0.020903	1	−0.020903	0.067812
6	8	−0.952897	13	−0.952896	2.839347
7	8	−6.262872	1	−6.262872	6.920063
8	16	0.077590	3	0.077589	0.902209
9	16	0.211315	1	0.211314	0.224882
10	16	−0.478362	5	−0.478361	0.724896
11	4	−5.815675	3	−5.815674	5.872866
12	8	−7.047444	5	−7.047444	5.134338
13	8	−4.60138	5	−4.601307	5.199786
14	4	−0.14110	3	−0.141100	0.408237
15	16	−0.870885	11	−0.870885	4.858047
16	16	−9.031249	13	−9.031249	5.791796
17	1	0.475689	1	0.475688	−0.787891
18	8	0	5	0	3.141584 6.283180
19	64	−1	3	−1	−0.000044
20	1	1	1	1	0
21	16	1	3	1	0.000055
22	4	−0.918397	3	−0.918397	3.251079
23	64	−0.824239	15	−0.824239	−0.679575
24	4	−0.027864	5	−0.027864	3.926986
25	8	3.5	5	3.5	2.094394 4.188791
26	8	0.367879	7	0.367879	5.759584 3.665190 1.570799
27	8	−0.451388	7	−0.451387	5.006390 1.864797
28	8	−1	7	−1	4.712397 10.995573 17.278759
29	2	−0.410135	3	−0.410315	3.862077
30	16	−0.718282	9	−0.718281	2.617937
31	128	−14.59265	1	−14.592642	0.685999

(*continued*)

TABLE 1. continued.

Fct	NbF	GoF	NbO	GoO	GsO
32	1024	−1	15	−1	0.212206;0.090945 0.057874;0.042441 0.033506;0.027679 0.023578;0.020536
33	32	−12.03125	37	−12.031249	−6.774573
34	4	−0.535534	5	−0.535533	2.414211
35	512	−13.92245	1	−13.9	0.937823
36	1	−0.35	1	−0.35	2.000012
37	4	−32.78126	3	−32.781261	0.713679
38	8	7	7	7	2.999991
39	16	−1	9	−1	1.381976 3.618035
40	4	−89	5	−89	2.000058

6. Conclusion

We proposed in this paper a new underestimator in branch and bound algorithm for univariate global optimization problems. This new underestimator is tighter than the classical underestimator developed in αBB method and the quadratic underestimator developed in [9]. The computational results show the efficiency of our proposed branch and bound algorithm. The extension of our algorithm for the multivariate case is currently in progress.

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