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A VIRTUAL ELEMENT METHOD FOR THE VIBRATION PROBLEM OF KIRCHHOFF PLATES

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Abstract. The aim of this paper is to develop a virtual element method (VEM) for the vibration problem of thin plates on polygonal meshes. We consider a variational formulation relying only on the transverse displacement of the plate and propose an $H^2(\Omega)$ conforming discretization by means of the VEM which is simple in terms of degrees of freedom and coding aspects. Under standard assumptions on the computational domain, we establish that the resulting scheme provides a correct approximation of the spectrum and prove optimal order error estimates for the eigenfunctions and a double order for the eigenvalues. Finally, we report several numerical experiments illustrating the behaviour of the proposed scheme and confirming our theoretical results on different families of meshes. Additional examples of cases not covered by our theory are also presented.

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1. Introduction

The *Virtual Element Method* (VEM), introduced in [5, 6], is a recent generalization of the Finite Element Method which is characterized by the capability of dealing with very general polygonal/polyhedral meshes. The interest in numerical methods that can make use of general polytopal meshes has recently undergone a significant growth in the mathematical and engineering literature; among the large number of papers on this subject, we cite as a minimal sample [5, 18, 25, 35, 36].

Indeed, polytopal meshes can be very useful for a wide range of reasons, including meshing of the domain (such as cracks) and data (such as inclusions) features, automatic use of hanging nodes, use of moving meshes, adaptivity. Moreover, the VEM presents the advantage to easily implement highly regular discrete spaces. Indeed, by avoiding the explicit construction of the local basis functions, the VEM can easily handle general polygons/polyhedrons without complex integrations on the element (see [6] for details on the coding aspects of the method). The VEM has recently been applied successfully to a wide range of problems, see for instance [1, 8, 16, 17, 19, 21, 26, 27, 31, 33, 37].

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The numerical approximation of eigenvalue problems for partial differential equations encountered in engineering applications is the object of great interest, from both the practical and theoretical points of view. We refer to [12, 13] and the references therein for the state of the art in this subject area. In particular, this paper focuses on the so called thin plate vibration problem, which involves the biharmonic operator. Among the existing techniques to solve this problem, various finite element methods have been introduced and analyzed. In particular, we mention nonconforming methods and different mixed formulations for the Kirchhoff model, see for instance [4, 20, 30, 32, 34]. More recently, in [14] a discontinuous Galerkin method has been proposed and analyzed for the vibration and buckling problems of thin plates. On the other hand, the construction of conforming finite elements for $H^2(\Omega)$ is difficult in general, since they usually involve a large number of degrees of freedom (see [22]).

Recently, thanks to the flexibility of VEM, it has been showed in [9, 16] that virtual elements can be used to build global discrete spaces of arbitrary regularity that are simple in terms of degrees of freedom and coding aspects (see also [3, 10]). Thus, in the present contribution, we follow a similar approach in order to solve an eigenvalue problem modelling the two-dimensional plate vibration problem considering a conforming C^1 discrete formulation.

The aim of this paper is to introduce and analyze a C^1-VEM which applies to general polygonal meshes, made by possibly non-convex elements, for the two-dimensional plate vibration problem. We begin with a variational formulation of the spectral problem relying only on the transverse displacement of the plate. Then, we exploit the capability of VEM to build highly regular discrete spaces and propose a conforming $H^2(\Omega)$ discrete formulation. In particular, we consider the discrete virtual space introduced in [3] to solve the Cahn-Hilliard equation, which is a modification of the C^1 virtual elements in [9, 16]. More precisely, the functions of the discrete space will have continuous values and continuous gradients across edges. Therefore, it will be contained in $H^2(\Omega)$ and yields a conforming solution. The resulting discrete bilinear form is continuous and elliptic. This method makes use of a very simple set of degrees of freedom, namely 3 degrees of freedom per vertex of the mesh. By using the abstract spectral approximation theory (see [23, 24]), under rather mild assumptions on the polygonal meshes, we establish that the resulting scheme provides a correct approximation of the spectrum and prove optimal order error estimates for the eigenfunctions and a double order for the eigenvalues. We remark that the present method is new on triangular meshes for the discretization of fourth order eigenvalue problems and in this case the computational cost is almost $3N_v$, where N_v denotes the number of vertices, thus it provides a very competitive alternative in comparison to other classical techniques based on finite elements.

The outline of this article is as follows: In Section 2, we introduce the variational formulation of the vibration eigenvalue problem, define a solution operator and establish its spectral characterization. In Section 3, we introduce the virtual element discrete formulation, describe the spectrum of a discrete solution operator and prove some auxiliary results. In Section 4, we prove that the numerical scheme provides a correct spectral approximation and establish optimal order error estimates for the eigenvalues and eigenfunctions. Several numerical tests that allow us to assess the convergence properties of the method, to confirm that it is not polluted with spurious modes and to check whether the experimental rates of convergence agree with the theoretical ones are reported in Section 5. Finally, we summarize some conclusions in Section 6.

Throughout the article we will use standard notations for Sobolev spaces, norms and seminorms. Moreover, we will denote by C a generic constant independent of the mesh parameter h, which may take different values in different occurrences.

Finally, given a linear bounded operator $T : X \to X$, defined on a Hilbert space X, we denote its spectrum by $\mathrm{sp}(T) := \{z \in \mathbb{C} : (zI - T) \text{ is not invertible}\}$ and by $\rho(T) := \mathbb{C} \setminus \mathrm{sp}(T)$ the resolvent set of T. Moreover, for any $z \in \rho(T)$, $R_z(T) := (zI - T)^{-1} : X \to X$ denotes the resolvent operator of T corresponding to z.

2. The spectral problem

Let $\Omega \subset \mathbb{R}^2$ be a polygonal bounded domain corresponding to the mean surface of a plate in its reference configuration, clamped on its whole boundary Γ. The plate is assumed to be homogeneous, isotropic, linearly elastic, and sufficiently thin as to be modeled by Kirchhoff-Love equations. We denote by w the transverse displacement of the mean surface of the plate.

The plate vibration problem reads as follows:

Find $(\lambda, w) \in \mathbb{R} \times H^2(\Omega)$, $w \neq 0$, such that

$$\begin{cases} \Delta^2 w = \lambda w & \text{in } \Omega, \\ w = \partial_n w = 0 & \text{on } \Gamma, \end{cases} \tag{2.1}$$

where $\lambda = \omega^2$, with $\omega > 0$ being the vibration frequency, and ∂_n denotes the normal derivative. To simplify the notation we have taken the Young modulus and the density of the plate, both equal to 1.

To obtain a weak formulation of the spectral problem (2.1), we multiply the corresponding equation by $v \in H_0^2(\Omega)$ and integrate twice by parts in Ω. Thus, we obtain:

Find $(\lambda, w) \in \mathbb{R} \times H_0^2(\Omega)$, $w \neq 0$, such that

$$a(w, v) = \lambda b(w, v) \qquad \forall v \in H_0^2(\Omega), \tag{2.2}$$

in (2.2) the bilinear forms are defined for any $w, v \in H_0^2(\Omega)$ by

$$a(w, v) := \int_\Omega D^2 w : D^2 v,$$
$$b(w, v) := \int_\Omega wv,$$

with ":" denotes the usual scalar product of 2×2-matrices, $D^2 v := (\partial_{ij} v)_{1 \leq i,j \leq 2}$ denotes the Hessian matrix of v. We note that those are bounded bilinear symmetric forms. Moreover, it is immediate to prove that the eigenvalues of the problem above are real and positive.

Next, we define the solution operator associated with the variational eigenvalue problem (2.2):

$$T : \ H_0^2(\Omega) \longrightarrow H_0^2(\Omega),$$
$$f \longmapsto Tf := u,$$

where $u \in H_0^2(\Omega)$ is the solution of the corresponding source problem:

$$a(u, v) = b(f, v) \qquad \forall v \in H_0^2(\Omega). \tag{2.3}$$

The following lemma allows us to establish the well-posedness of this source problem.

Lemma 2.1. *There exists a constant $\alpha_0 > 0$, depending on Ω, such that*

$$a(v, v) \geq \alpha_0 \, \|v\|_{2,\Omega}^2 \qquad \forall v \in H_0^2(\Omega).$$

Proof. The result follows immediately from the fact that $\|D^2 v\|_{0,\Omega}$ is a norm on $H_0^2(\Omega)$, equivalent with the usual norm. □

We deduce from Lemma 2.1 that the linear operator T is well defined and bounded. Notice that $(\lambda, w) \in \mathbb{R} \times H_0^2(\Omega)$ solves problem (2.2) (and hence problem (2.1)) if and only if $Tw = \mu w$ with $\mu \neq 0$ and $w \neq 0$, in which case $\mu := \frac{1}{\lambda}$. Moreover, it is easy to check that T is self-adjoint with respect to the $a(\cdot, \cdot)$ inner product. Indeed, given $f, g \in H_0^2(\Omega)$,

$$a(Tf, g) = b(f, g) = b(g, f) = a(Tg, f) = a(f, Tg).$$

The following is an additional regularity result for the solution of problem (2.3) and consequently, for the eigenfunctions of T.

Lemma 2.2. *There exist $s \in (\frac{1}{2}, 1]$ and $C > 0$ such that, for all $f \in L^2(\Omega)$, the solution u of problem (2.3) satisfies $u \in H^{2+s}(\Omega)$ and*

$$\|u\|_{2+s,\Omega} \leq C\|f\|_{0,\Omega}.$$

Proof. The proof follows from the classical regularity result for the biharmonic problem with its right-hand side in $L^2(\Omega)$ (*cf.* [29]). $\square$

The constant s, in the lemma given above, is the Sobolev regularity for the biharmonic equation with homogeneous Dirichlet boundary conditions. This constant only depends on the domain Ω. If Ω is convex, then $s = 1$. Otherwise, the lemma holds for all $s < s_0$, where $s_0 \in (\frac{1}{2}, 1)$ depends on the largest reentrant angle of Ω (see [29] for the precise equation determining s_0). Hence, because of the compact inclusion $H^{2+s}(\Omega) \hookrightarrow H_0^2(\Omega)$, T is a compact operator. Therefore, we have the following spectral characterization result.

Lemma 2.3. *The spectrum of T satisfies $\mathrm{sp}(T) = \{0\} \cup \{\mu_k\}_{k\in\mathbb{N}}$, where $\{\mu_k\}_{k\in\mathbb{N}}$ is a sequence of real positive eigenvalues which converges to 0. The multiplicity of each eigenvalue is finite.*

3. Spectral approximation

In this section, first we recall the mesh construction and the assumptions considered to introduce the discrete virtual element spaces. Then, we will introduce a virtual element discretization of the eigenvalue problem (2.2) and provide a spectral characterization of the resulting discrete eigenvalue problem.

Let $\{\mathcal{T}_h\}_h$ be a sequence of decompositions of Ω into polygons K. Let h_K denote the diameter of the element K and h the maximum of the diameters of all the elements of the mesh, i.e., $h := \max_{K\in\mathcal{T}_h} h_K$. In what follows, we denote by N_K the number of vertices of K, by e a generic edge of $\mathcal{T}_h$ and for all $e \in \partial K$, we define a unit normal vector $\boldsymbol{n}_K^e$ that points outside of K.

For the analysis, we will make the following assumptions as in [5,10]: there exists a positive real number $C_{\mathcal{T}}$ such that, for every h and every $K \in \mathcal{T}_h$,

A1: the ratio between the shortest edge and the diameter h_K of K is larger than $C_{\mathcal{T}}$;
A2: $K \in \mathcal{T}_h$ is star-shaped with respect to every point of a ball of radius $C_{\mathcal{T}} h_K$.

For any subset $S \subseteq \mathbb{R}^2$ and nonnegative integer k, we indicate by $\mathbb{P}_k(S)$ the space of polynomials of degree up to k defined on S.

Now, we consider a simple polygon K (meaning open simply connected set whose boundary is a non-intersecting line made of a finite number of straight line segments) and we define the following finite-dimensional space

$$\begin{aligned} V_h^K := \big\{ v_h \in H^2(K) : \Delta^2 v_h \in \mathbb{P}_2(K), v_h|_{\partial K} \in C^0(\partial K), v_h|_e \in \mathbb{P}_3(e)\ \forall e \in \partial K, \\ \nabla v_h|_{\partial K} \in C^0(\partial K)^2, \partial_n v_h|_e \in \mathbb{P}_1(e)\ \forall e \in \partial K \big\}. \end{aligned}$$

We observe that any $v_h \in V_h^K$ satisfies the following conditions:

- the trace (and the trace of the gradient) on the boundary of K is continuous;
- $\mathbb{P}_2(K) \subseteq V_h^K$.

We now introduce two sets $\mathbf{D_1}$ and $\mathbf{D_2}$ of linear operators from V_h^K into $\mathbb{R}$. For all $v_h \in V_h^K$, they are defined as follows:

- $\mathbf{D_1}$ contains linear operators evaluating v_h at the N_K vertices of K;
- $\mathbf{D_2}$ contains linear operators evaluating ∇v_h at the N_K vertices of K.

Note that, as a consequence of definition of V_h^K, the output values of the two sets of operators $\mathbf{D_1}$ and $\mathbf{D_2}$ are sufficient to uniquely determine v_h and ∇v_h on the boundary of K.

In order to construct the discrete scheme, we need some preliminary definitions. First, we split the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, introduced in the previous section, as follows:

$$a(u,v) = \sum_{K\in\mathcal{T}_h} a_K(u,v), \qquad u,v \in H_0^2(\Omega),$$

$$b(u,v) = \sum_{K\in\mathcal{T}_h} b_K(u,v), \qquad u,v \in H_0^2(\Omega),$$

with

$$a_K(u,v) := \int_K D^2u : D^2v, \qquad u,v \in H^2(K),$$

and

$$b_K(u,v) := \int_K uv, \qquad u,v \in H^2(K).$$

Now, we define the projector $\Pi_K^{\Delta} :\ V_h^K \longrightarrow \mathbb{P}_2(K) \subseteq V_h^K$ for each $v \in V_h^K$ as the solution of

$$a_K\left(\Pi_K^{\Delta}v, q\right) = a_K(v,q) \qquad \forall q \in \mathbb{P}_2(K), \tag{3.1a}$$

$$((\Pi_K^{\Delta}v, q))_K = ((v,q))_K \qquad \forall q \in \mathbb{P}_1(K), \tag{3.1b}$$

where $((\cdot,\cdot))_K$ is defined as follows:

$$((u,v))_K = \sum_{i=1}^{N_K} u(P_i)v(P_i) \qquad \forall u,v \in C^0(\partial K),$$

where $P_i, 1 \le i \le N_K$, are the vertices of K. We note that the bilinear form $a_K(\cdot,\cdot)$ has a non-trivial kernel, given by $\mathbb{P}_1(K)$. Hence, the role of condition (3.1b) is to select an element of the kernel of the operator. In order to show that the projector Π_K^{Δ} is computable, we integrate twice by parts on the right hand side of (3.1a) to obtain:

$$a_K\left(\Pi_K^{\Delta}v, q\right) = \int_{\partial K} (D^2 q \boldsymbol{n}_K^e) \cdot \nabla v_h \qquad \forall q \in \mathbb{P}_2(K). \tag{3.2}$$

Thus, from the definition of $((\cdot,\cdot))_K$, we observe that the right hand sides of (3.2) and (3.1b) are computable only on the basis of the output values of the operators in $\mathbf{D_2}$ and $\mathbf{D_1}$, respectively.

Now, we introduce our local virtual space:

$$W_h^K := \left\{ v_h \in V_h^K : \int_K (\Pi_K^{\Delta} v_h) q = \int_K v_h q \qquad \forall q \in \mathbb{P}_2(K) \right\}.$$

It is easy to check that $W_h^K \subseteq V_h^K$. Therefore, the operator Π_K^{Δ} is well defined on W_h^K and computable only on the basis of the output values of the operators in $\mathbf{D_1}$ and $\mathbf{D_2}$.

In Lemma. 2.1 of [3] has been established that the set of operators $\mathbf{D_1}$ and $\mathbf{D_2}$ constitutes a set of degrees of freedom for the space W_h^K. Moreover, it is easy to check that $\mathbb{P}_2(K) \subseteq W_h^K$. This will guarantee the good approximation properties for the space.

Additionaly, we have that the $L^2(\Omega)$ projection operator $\Pi_K^0 : W_h^K \to \mathbb{P}_2(K)$ is computable from the set of degrees freedom. In fact, for all $v_h \in W_h^K$, the function $\Pi_K^0 v_h \in \mathbb{P}_2(K)$ is defined by:

$$\int_K (\Pi_K^0 v_h) q = \int_K v_h q \qquad \forall q \in \mathbb{P}_2(K). \tag{3.3}$$

Now, due to the particular property appearing in definition of the space W_h^K, the right hand side in (3.3) is computable using $\Pi_K^{\Delta} v_h$, and thus $\Pi_K^0 v_h$ depends only on the values of the degrees of freedom for v_h and ∇v_h. Actually, it is easy to check that on the space W_h^K the projectors $\Pi_K^0 v_h$ and $\Pi_K^{\Delta} v_h$ are the same operator. In fact:

$$\int_K (\Pi_K^0 v_h) q = \int_K v_h q = \int_K (\Pi_K^{\Delta} v_h) q \qquad \forall q \in \mathbb{P}_2(K). \tag{3.4}$$

In what follows, we keep the notation Π_K^{Δ} for both operators.

We can now present the global virtual space: for every decomposition $\mathcal{T}_h$ of Ω into simple polygons K, we define

$$W_h := \left\{ v_h \in H_0^2(\Omega) : \ v_h|_K \in W_h^K \right\}.$$

A set of degrees of freedom for W_h is given by all pointwise values of v_h on all vertices of $\mathcal{T}_h$ together with all pointwise values of ∇v_h on all vertices of $\mathcal{T}_h$, excluding the vertices on Γ (where the values vanishes). Thus, the dimension of W_h is three times the number of interior vertices.

On the other hand, let $s_K(\cdot,\cdot)$ and $s_K^0(\cdot,\cdot)$ be any symmetric positive definite bilinear forms to be chosen as to satisfy:

$$c_0 a_K(v_h, v_h) \leq s_K(v_h, v_h) \leq c_1 a_K(v_h, v_h) \qquad \forall v_h \in W_h^K \quad \text{with} \quad \Pi_K^{\Delta} v_h = 0, \tag{3.5}$$

$$c_2 b_K(v_h, v_h) \leq s_K^0(v_h, v_h) \leq c_3 b_K(v_h, v_h) \qquad \forall v_h \in W_h^K. \tag{3.6}$$

We will introduce bilinear forms $s_K(\cdot,\cdot)$ and $s_K^0(\cdot,\cdot)$ satisfying (3.5)–(3.6) in Section 5.

Then, we set

$$a_h(u_h, v_h) := \sum_{K \in \mathcal{T}_h} a_{h,K}(u_h, v_h), \qquad u_h, v_h \in W_h,$$

$$b_h(u_h, v_h) := \sum_{K \in \mathcal{T}_h} b_{h,K}(u_h, v_h), \qquad u_h, v_h \in W_h,$$

where $a_{h,K}(\cdot,\cdot)$ and $b_{h,K}(\cdot,\cdot)$ are the local bilinear forms on $W_h^K \times W_h^K$ defined by

$$a_{h,K}(u_h, v_h) := a_K\big(\Pi_K^{\Delta} u_h, \Pi_K^{\Delta} v_h\big) + s_K\big(u_h - \Pi_K^{\Delta} u_h, v_h - \Pi_K^{\Delta} v_h\big), \qquad u_h, v_h \in W_h^K, \tag{3.7}$$

$$b_{h,K}(u_h, v_h) := b_K\big(\Pi_K^{\Delta} u_h, \Pi_K^{\Delta} v_h\big) + s_K^0\big(u_h - \Pi_K^{\Delta} u_h, v_h - \Pi_K^{\Delta} v_h\big), \qquad u_h, v_h \in W_h^K. \tag{3.8}$$

The construction of the bilinear forms $a_{h,K}(\cdot,\cdot)$ and $b_{h,K}(\cdot,\cdot)$ guarantees the usual consistency and stability properties of VEM, as noted in the proposition below. Since the proof follows standard arguments in the Virtual Element literature (see [3,5,7]), it is omitted.

Proposition 3.1. *The local bilinear forms $a_{h,K}(\cdot,\cdot)$ and $b_{h,K}(\cdot,\cdot)$ on each element K satisfy*

- *Consistency: for all $h > 0$ and for all $K \in \mathcal{T}_h$, we have that*

$$a_{h,K}(p, v_h) = a_K(p, v_h) \qquad \forall p \in \mathbb{P}_2(K), \quad \forall v_h \in W_h^K, \tag{3.9}$$

$$b_{h,K}(p, v_h) = b_K(p, v_h) \qquad \forall p \in \mathbb{P}_2(K), \quad \forall v_h \in W_h^K. \tag{3.10}$$

- *Stability and boundedness: There exist positive constants $\alpha_i, i = 1,2,3,4$, independent of K, such that:*

$$\alpha_1 a_K(v_h, v_h) \leq a_{h,K}(v_h, v_h) \leq \alpha_2 a_K(v_h, v_h) \qquad \forall v_h \in W_h^K, \tag{3.11}$$

$$\alpha_3 b_K(v_h, v_h) \leq b_{h,K}(v_h, v_h) \leq \alpha_4 b_K(v_h, v_h) \qquad \forall v_h \in W_h^K. \tag{3.12}$$

Now, we are in a position to write the virtual element discretization of problem (2.2).

Find $(\lambda_h, w_h) \in \mathbb{R} \times W_h$, $w_h \neq 0$, such that

$$a_h(w_h, v_h) = \lambda_h b_h(w_h, v_h) \qquad \forall v_h \in W_h. \tag{3.13}$$

We observe that by virtue of (3.11), the bilinear form $a_h(\cdot,\cdot)$ is bounded. Moreover, as shown in the following lemma, it is also uniformly elliptic.

Lemma 3.2. *There exists a constant $\beta > 0$, independent of h, such that*

$$a_h(v_h, v_h) \geq \beta \|v_h\|^2_{2,\Omega} \qquad \forall v_h \in W_h.$$

Proof. Thanks to (3.11) and Lemma 2.1, it is easy to check that the above inequality holds with $\beta := \alpha_0 \min\{\alpha_1, 1\}$. $\square$

The discrete version of the operator T is given by

$$\begin{aligned} T_h: \ W_h &\longrightarrow W_h, \\ f_h &\longmapsto T_h f_h := u_h, \end{aligned}$$

where $u_h \in W_h$ is the solution of the corresponding discrete source problem

$$a_h(u_h, v_h) = b_h(f_h, v_h) \qquad \forall v_h \in W_h.$$

Because of Lemma 3.2, the linear operator T_h is well defined and bounded uniformly with respect to h. Once more, as in the continuous case, $(\lambda_h, w_h) \in \mathbb{R} \times W_h$ solves problem (3.13) if and only if $T_h w_h = \mu_h w_h$ with $\mu_h \neq 0$ and $w_h \neq 0$, in which case $\mu_h := \frac{1}{\lambda_h}$. Moreover, T_h is self-adjoint with respect to $a_h(\cdot,\cdot)$. Because of this, it is easy to prove the following spectral characterization.

Theorem 3.3. *The spectrum of T_h consists of $M_h := \dim(W_h)$ eigenvalues, repeated according to their respective multiplicities. All of them are real and positive.*

In order to prove that the solutions of the discrete problem (3.13) converge to those of the continuous problem (2.2), the standard procedure would be to show that T_h converges in norm to T as h goes to zero. However, such a proof does not seem straightforward in our case. In fact, the operator T_h is not well defined for any $f \in H^2_0(\Omega)$, since the definition of bilinear form $b_{h,K}(\cdot,\cdot)$ in (3.8) needs the degrees of freedom and in particular the pointwise values of ∇f, but it is for any $f \in W_h$.

To circumvent this drawback, we will resort to the spectral theory from [23, 24]. In spite of the fact that the main use of this theory is when T is a noncompact operator, it can also be applied to compact operators, and we will show that in our case it works.

With this aim, we first recall the following approximation result which is derived by interpolation between Sobolev spaces (see for instance Thm. I.1.4 of [28]) from the analogous result for integer values of s. In its turn, the result for integer values is stated in Proposition 4.2 of [5] and follows from the classical Scott–Dupont theory (see [15] and Prop. 3.1 of [3])

Proposition 3.4. *If the Assumption* **A2** *is satisfied, then there exists a constant $C > 0$, such that for every $v \in H^{2+s}(K)$ with $s \in (1/2, 1]$, there exists $v_\pi \in \mathbb{P}_2(K)$ such that*

$$|v - v_\pi|_{\ell,K} \leq C h_K^{2+s-\ell} |v|_{2+s,K}, \quad \ell = 0, 1, 2.$$

For the analysis, we will introduce the broken H^2-seminorm:

$$|v|^2_{2,h} := \sum_{K \in \mathcal{T}_h} |v|^2_{2,K},$$

which is well defined for every $v \in L^2(\Omega)$ such that $v|_K \in H^2(K)$ for all polygon $K \in \mathcal{T}_h$.

Now, for $v \in W_h$, let Π_h be defined in $L^2(\Omega)$ by $(\Pi_h v)|_K := \Pi_K^{\Delta} v$ for all $K \in \mathcal{T}_h$, where Π_K^{Δ} has been defined in (3.1a)–(3.1b).

Lemma 3.5. *Let $v \in W_h$. Then, there exists $C > 0$ such that*

$$\|v - \Pi_h v\|_{0,\Omega} \le Ch^2 \|v\|_{2,\Omega}.$$

Proof. Let $v \in W_h$. Now, let $\Pi_K^{\Delta} v \in \mathbb{P}_2(K)$ as defined in (3.1a)–(3.1b). We have for all $r \in \mathbb{P}_2(K)$ that

$$\|v - \Pi_K^{\Delta} v\|_{0,K}^2 = \int_K \left(v - \Pi_K^{\Delta} v\right)\left(v - \Pi_K^{\Delta} v\right) = \int_K \left(v - \Pi_K^{\Delta} v\right)(v - r).$$

Thus,

$$\|v - \Pi_K^{\Delta} v\|_{0,K} \le \inf_{r \in \mathbb{P}_2(K)} \|v - r\|_{0,K} \le Ch_K^2 \|v\|_{2,K},$$

where we have used (3.4) and ([31], Prop. 4.1), and the result follows. □

Now, the remainder of this section is devoted to prove the following properties which will be used in the sequel:

Lemma 3.6. *There exists $C > 0$ such that, for all $f_h \in W_h$, if $u = Tf_h$ and $u_h = T_h f_h$, then*

$$\|(T - T_h) f_h\|_{2,\Omega} = \|u - u_h\|_{2,\Omega} \le C \left(\|\Pi_h f_h - f_h\|_{0,\Omega} + \|u - u_I\|_{2,\Omega} + |u - u_\pi|_{2,h} \right),$$

for all $u_I \in W_h$ and for all $u_\pi \in L^2(\Omega)$ such that $u_\pi|_K \in \mathbb{P}_2(K) \quad \forall K \in \mathcal{T}_h$.

Proof. Let $f_h \in W_h$. For $u_I \in W_h$, we set $v_h := u_h - u_I$. Thus

$$||(T - T_h) f_h||_{2,\Omega} \le ||u - u_I||_{2,\Omega} + ||v_h||_{2,\Omega}. \tag{3.14}$$

Now, thanks to Lemma 3.2, the definition of $a_{h,K}(\cdot,\cdot)$ and those of T and T_h, we have

$$\begin{aligned}
\beta ||v_h||_{2,\Omega}^2 \le a_h(v_h, v_h) &= a_h(u_h, v_h) - a_h(u_I, v_h) = b_h(f_h, v_h) - \sum_{K \in \mathcal{T}_h} a_{h,K}(u_I, v_h) \\
&= b_h(f_h, v_h) - \sum_{K \in \mathcal{T}_h} \left\{ a_{h,K}(u_I - u_\pi, v_h) + a_{h,K}(u_\pi, v_h) \right\} \\
&= b_h(f_h, v_h) - \sum_{K \in \mathcal{T}_h} \left\{ a_{h,K}(u_I - u_\pi, v_h) + a_K(u_\pi - u, v_h) + a_K(u, v_h) \right\} \\
&= b_h(f_h, v_h) - b(f_h, v_h) - \sum_{K \in \mathcal{T}_h} \left\{ a_{h,K}(u_I - u_\pi, v_h) + a_K(u_\pi - u, v_h) \right\}.
\end{aligned} \tag{3.15}$$

Then, we bound the first term on the right hand side of the previous inequality as follows

$$\begin{aligned}
b_h(f_h, v_h) - b(f_h, v_h) &= \sum_{K \in \mathcal{T}_h} \left\{ b_{h,K}(f_h, v_h) - b_K(f_h, v_h) \right\} \\
&= \sum_{K \in \mathcal{T}_h} \left\{ b_{h,K}(f_h - \Pi_K^{\Delta} f_h, v_h) - b_K(f_h - \Pi_K^{\Delta} f_h, v_h) \right\} \\
&\le \sum_{K \in \mathcal{T}_h} \left\{ b_{h,K}(f_h - \Pi_K^{\Delta} f_h, f_h - \Pi_K^{\Delta} f_h)^{1/2} b_{h,K}(v_h, v_h)^{1/2} + ||f_h - \Pi_K^{\Delta} f_h||_{0,K} ||v_h||_{0,K} \right\} \\
&\le C \sum_{K \in \mathcal{T}_h} ||f_h - \Pi_K^{\Delta} f_h||_{0,K} ||v_h||_{0,K},
\end{aligned} \tag{3.16}$$

where we have used the consistency, Cauchy–Schwarz inequality and stability of $b_{h,K}(\cdot,\cdot)$.

Thus, from (3.15), using the above bound together with the Cauchy–Schwarz and triangular inequalities, we obtain

$$\begin{aligned}\beta \|v_h\|_{2,\Omega}^2 &\le C \sum_{K\in\mathcal{T}_h} \left\|\Pi_K^{\Delta} f_h - f_h\right\|_{0,K} \|v_h\|_{0,K} + \sum_{K\in\mathcal{T}_h} \left(\alpha_2 |u_I - u_\pi|_{2,K} + |u_\pi - u|_{2,K}\right) |v_h|_{2,K} \\ &\le C \left(\sum_{K\in\mathcal{T}_h} \left\|\Pi_K^{\Delta} f_h - f_h\right\|_{0,K}^2 + |u_I - u|_{2,K}^2 + |u_\pi - u|_{2,K}^2 \right)^{1/2} \|v_h\|_{2,\Omega} \\ &\le C \left(\|\Pi_h f_h - f_h\|_{0,\Omega} + \|u_I - u\|_{2,\Omega} + |u_\pi - u|_{2,h} \right) \|v_h\|_{2,\Omega}.\end{aligned}$$

Therefore, the proof follows from (3.14) and the above inequality. □

The next step is to find an appropriate term u_I that can be used in the above lemma. Thus, we have the following result.

Proposition 3.7. *Assume* **A1–A2** *are satisfied, let $v \in H^{2+s}(\Omega)$ with $s \in (1/2, 1]$. Then, there exist $v_I \in W_h$ and $C > 0$ such that*

$$\|v - v_I\|_{2,\Omega} \le C h^s |v|_{2+s,\Omega}.$$

Proof. The proof follows repeating the arguments from Proposition 4.4 of [10], (see also [3], Prop. 3.1). □

As we have mentioned before, to prove that T_h provides a correct spectral approximation of T, we will resort to the theory developed in [23] for noncompact operators. To this end, we first introduce some notations. For any linear operator $S : H_0^2(\Omega) \longrightarrow H_0^2(\Omega)$, we define the norm

$$\|S\|_h := \sup_{0\neq v_h\in W_h} \frac{\|S v_h\|_{2,\Omega}}{\|v_h\|_{2,\Omega}}.$$

Moreover, we recall the definition of the gap $\widehat{\delta}$ between two closed subspaces $\mathcal{X}$ and $\mathcal{Y}$ of $H_0^2(\Omega)$:

$$\widehat{\delta}(\mathcal{X}, \mathcal{Y}) := \max\{\delta(\mathcal{X}, \mathcal{Y}), \delta(\mathcal{Y}, \mathcal{X})\},$$

where

$$\delta(\mathcal{X}, \mathcal{Y}) := \sup_{x\in\mathcal{X}:\ \|x\|_{2,\Omega}=1} \delta(x, \mathcal{Y}) \quad \text{with} \quad \delta(x, \mathcal{Y}) := \inf_{y\in\mathcal{Y}} \|x - y\|_{2,\Omega}.$$

The theory from [23] guarantees approximation of the spectrum of T, provided the following two properties are satisfied:

- (P1): $\|T - T_h\|_h \to 0, \quad \text{as } h \to 0,$
- (P2): $\forall \phi \in H_0^2(\Omega), \quad \lim_{h\to 0} \delta(\phi, W_h) = 0.$

Property (P2) follows immediately from the approximation property of the virtual element space (see Prop. 3.7) and the density of smooth functions in $H_0^2(\Omega)$. Property (P1) is a consequence of the following lemma.

Lemma 3.8. *There exist $C > 0$ and $s \in (1/2, 1]$, independent of h, such that*

$$\|T - T_h\|_h \le C h^s.$$

Proof. Given $f_h \in W_h$, we have that (see Lem. 3.6)

$$\|(T - T_h) f_h\|_{2,\Omega} = \|u - u_h\|_{2,\Omega} \le C \left(\|\Pi_h f_h - f_h\|_{0,\Omega} + \|u - u_I\|_{2,\Omega} + |u - u_\pi|_{2,h} \right),$$

now, using Lemma 3.5, Propositions 3.4 and 3.7, and Lemma 2.2, we have

$$\|(T - T_h) f_h\|_{2,\Omega} \le C \left(h^2 \|f_h\|_{2,\Omega} + h^s \|f_h\|_{0,\Omega} \right) \le C h^s \|f_h\|_{2,\Omega}.$$

The proof is complete. □

4. Convergence and error estimates

In this section, we will adapt the arguments from [23,24] to prove convergence of our spectral approximation as well as to obtain error estimates for the approximate eigenvalues and eigenfunctions.

The following results are consequence of property (P1) (see [23]):

Lemma 4.1. *Suppose that* (P1) *holds true and let $F \subset \rho(T)$ be closed. Then, there exist positive constants C and h_0 independent of h, such that for $h < h_0$*

$$\sup_{v_h \in W_h} ||R_z(T_h)v_h||_{2,\Omega} \leq C||v_h||_{2,\Omega} \qquad \forall z \in F.$$

Theorem 4.2. *Let $U \subset \mathbb{C}$ be an open set containing* $\mathrm{sp}(T)$. *Then, there exists $h_0 > 0$ such that* $\mathrm{sp}(T_h) \subset U$ *for all $h < h_0$.*

An immediate consequence of this theorem is that the proposed virtual element method does not introduce spurious modes with eigenvalues interspersed among those with a physical meaning.

By applying the results from [23] to our problem, we conclude the spectral convergence of T_h to T as $h \to 0$. More precisely, let $\mu \neq 0$ be an isolated eigenvalue of T with multiplicity m and let $\mathcal{C}$ be an open circle in the complex plane centered at μ, such that μ is the only eigenvalue of T lying in $\mathcal{C}$ and $\partial\mathcal{C} \cap \mathrm{sp}(T) = \emptyset$. Then, according to Section 2 in [23] for h small enough there exist m eigenvalues $\mu_h^{(1)}, \ldots, \mu_h^{(m)}$ of T_h (repeated according to their respective multiplicities) which lie in $\mathcal{C}$. Therefore, these eigenvalues $\mu_h^{(1)}, \ldots, \mu_h^{(m)}$ converge to μ as h goes to zero.

The next step is to obtain error estimates for the spectral approximation. With this aim, we will use the theory from [24]. However, we cannot apply the results from this reference directly to our problem, because of the variational crimes in the bilinear forms used to define the operator T_h. Therefore, we need to extend the results from this reference to our case. With this purpose, we follow an approach recently presented in [11].

Consider the eigenspace $\mathcal{E}$ of T corresponding to μ and the T_h-invariant subspace $\mathcal{E}_h$ spanned by the eigenspaces of T_h corresponding to $\mu_h^{(1)}, \ldots, \mu_h^{(m)}$. As a consequence of Lemma 4.1, we have that

$$||(zI - T_h)v_h||_{2,\Omega} \geq C||v_h||_{2,\Omega} \qquad \forall v_h \in W_h, \quad \forall z \in \partial\mathcal{C},$$

for h small enough.

Let $\mathcal{P}_h : H_0^2(\Omega) \to W_h \subseteq H_0^2(\Omega)$ be the projector with range W_h defined by the relation

$$a(\mathcal{P}_h u - u, v_h) = 0 \qquad \forall v_h \in W_h.$$

Notice that $\mathcal{P}_h$ is bounded uniformly on h (namely $||\mathcal{P}_h u||_{2,\Omega} \leq ||u||_{2,\Omega}$) and

$$|u - \mathcal{P}_h u|_{2,\Omega} = \inf_{v_h \in W_h} |u - v_h|_{2,\Omega}.$$

Let us define

$$\widehat{T}_h := T_h \mathcal{P}_h : H_0^2(\Omega) \to W_h.$$

Notice that $\mathrm{sp}(\widehat{T}_h) = \mathrm{sp}(T_h) \cup \{0\}$.

Next, we introduce the following spectral projectors (the second one, is well defined at least for h small enough):

- The spectral projector of T relative to μ: $F := \frac{1}{2\pi i} \int_{\partial\mathcal{C}} R_z(T) \mathrm{d}z$;
- The spectral projector of $\widehat{T}_h$ relative to $\mu_h^{(1)}, \ldots, \mu_h^{(m)}$: $\widehat{F}_h := \frac{1}{2\pi i} \int_{\partial\mathcal{C}} R_z(\widehat{T}_h) \mathrm{d}z$.

We have the following result (see Lem. 1 in [24]).

Lemma 4.3. *There exist $h_0 > 0$ and $C > 0$ such that*

$$||R_z(\widehat{T}_h)|| \leq C \qquad \forall z \in \partial\mathcal{C}, \quad \forall h \leq h_0.$$

Proof. It is identical to the proof of Lemma 11 in [11]. □

Consequently, for h small enough, the spectral projector $\widehat{F}_h$ is bounded uniformly in h.

Now, we define

$$\gamma_h := \delta(\mathcal{E}, W_h) \quad \text{and} \quad \eta_h := \sup_{w \in \mathcal{E}} \frac{||w - \Pi_h w||_{0,\Omega}}{||w||_{2,\Omega}}.$$

From Lemmas 2.2 and 3.5, we have that

$$\gamma_h \leq C h^{\tilde{s}} \qquad \text{and} \qquad \eta_h \leq C h^2, \tag{4.1}$$

where $\tilde{s} \in (1/2, 1]$ is such that $\mathcal{E} \subset H^{2+\tilde{s}}(\Omega)$.

The following result establishes an error estimate for the eigenfunctions.

Theorem 4.4. *If $\mathcal{E} \subset H^{2+\tilde{s}}(\Omega)$ with $\tilde{s} \in (1/2, 1]$, there exist positive constants h_0 and C such that, for all $h < h_0$,*

$$\widehat{\delta}(\mathcal{E}, \mathcal{E}_h) \leq C h^{\tilde{s}}.$$

Proof. It follows by arguing exactly as in the proof of Theorem 1 from [24] and using (4.1). □

Finally, we have the following result that provides an error estimate for the eigenvalues.

Theorem 4.5. *There exist positive constants C and h_0 independent of h, such that, for all $h < h_0$,*

$$\left|\lambda - \lambda_h^{(i)}\right| \leq C h^{2\tilde{s}}, \qquad i = 1, \ldots, m,$$

where $\tilde{s} \in (1/2, 1]$ is such that $\mathcal{E} \subset H^{2+\tilde{s}}(\Omega)$.

Proof. Let w_h be such that $(\lambda_h^{(i)}, w_h)$ is a solution of (3.13) with $\|w_h\|_{2,\Omega} = 1$. According to Theorem 4.4, $\delta(w_h, \mathcal{E}) \leq C h^{\tilde{s}}$. It follows that there exists (λ, w) eigenpair solution of (2.2) such that

$$\|w - w_h\|_{2,\Omega} \leq C h^{\tilde{s}}. \tag{4.2}$$

From the symmetry of the bilinear forms and the facts that $a(w, v) = \lambda b(w, v)$ for all $v \in H_0^2(\Omega)$ (*cf.* (2.2)) and $a_h(w_h, v_h) = \lambda_h^{(i)} b_h(w_h, v_h)$ for all $v_h \in W_h$ (*cf.* (3.13)), we have

$$\begin{aligned} a(w - w_h, w - w_h) - \lambda b(w - w_h, w - w_h) &= a(w_h, w_h) - \lambda b(w_h, w_h) \\ &= a(w_h, w_h) - a_h(w_h, w_h) + \lambda_h^{(i)} b_h(w_h, w_h) - \lambda b(w_h, w_h) \\ &= a(w_h, w_h) - a_h(w_h, w_h) + \lambda_h^{(i)} \left[b_h(w_h, w_h) - b(w_h, w_h)\right] + (\lambda_h^{(i)} - \lambda) b(w_h, w_h), \end{aligned}$$

from which we obtain the following identity:

$$\begin{aligned} (\lambda_h^{(i)} - \lambda) b(w_h, w_h) &= a(w - w_h, w - w_h) - \lambda b(w - w_h, w - w_h) \\ &\quad + (a_h(w_h, w_h) - a(w_h, w_h)) - \lambda_h^{(i)} \left[b_h(w_h, w_h) - b(w_h, w_h)\right]. \end{aligned} \tag{4.3}$$

The next step is to bound each term on the right hand side above. The first and the second ones are easily bounded using the Cauchy–Schwarz inequality and (4.2):

$$|a(w - w_h, w - w_h) - \lambda b(w - w_h, w - w_h)| \leq C \left(|w - w_h|_{2,\Omega}^2 + \|w - w_h\|_{0,\Omega}^2\right) \leq C h^{2\tilde{s}}. \tag{4.4}$$

For the third term, for $w \in \mathcal{E}$, we consider $w_\pi \in L^2(\Omega)$ defined on each $K \in \mathcal{T}_h$ so that $w_\pi|_K \in \mathbb{P}_2(K)$ and the estimate of Proposition 3.4 holds true. Then, we use (3.9) and (3.11) to write

$$\begin{aligned}|a_h(w_h, w_h) - a(w_h, w_h)| &= \Big| \sum_{K \in \mathcal{T}_h} \Big\{ a_{h,K}(w_h - w_\pi, w_h) - a_K(w_h - w_\pi, w_h) \Big\} \Big| \\ &\leq \sum_{K \in \mathcal{T}_h} (1 + \alpha_2) a_K(w_h - w_\pi, w_h - w_\pi) \\ &\leq C \sum_{K \in \mathcal{T}_h} |w_h - w_\pi|_{2,K}^2.\end{aligned}$$

Then, adding and subtracting w, using triangular inequality, Proposition 3.4 and (4.2), we obtain

$$|a_h(w_h, w_h) - a(w_h, w_h)| \leq C h^{2\tilde{s}}. \tag{4.5}$$

For the last term in (4.3), using that Π_K^{Δ} is also the L^2-projector (see (3.4)), we obtain

$$\begin{aligned}|b_h(w_h, w_h) - b(w_h, w_h)| &\leq C \sum_{K \in \mathcal{T}_h} \|w_h - \Pi_K^{\Delta} w_h\|_{0,K}^2 \\ &\leq C \sum_{K \in \mathcal{T}_h} \|w_h - w_\pi\|_{0,K}^2 \\ &\leq C \sum_{K \in \mathcal{T}_h} (\|w - w_\pi\|_{0,K}^2 + \|w - w_h\|_{0,K}^2) \leq C h^{2\tilde{s}},\end{aligned}$$

where we have used again Proposition 3.4 and (4.2).

On the other hand, using Lemma 3.2, we have

$$1 = \|w_h\|_{2,\Omega}^2 \leq \frac{1}{\beta} \lambda_h^{(i)} b_h(w_h, w_h) \leq \lambda_h^{(i)} C \|w_h\|_{0,\Omega}^2,$$

thus, the theorem follows from (4.3)–(4.5) and the inequalities above. □

5. Numerical results

We report in this section a couple of tests which have allowed us to assess the theoretical results proved above. With this aim, we have implemented in a MATLAB code the proposed VEM on arbitrary polygonal meshes, by following the ideas presented in [6].

Now, to complete the choice of the VEM, we had to fix the bilinear forms $s_K(\cdot,\cdot)$ and $s_K^0(\cdot,\cdot)$ satisfying (3.5) and (3.6), respectively. Proceeding as in [5], a natural choice for $s_K(\cdot,\cdot)$ is given by

$$s_K(u_h, v_h) := \sigma_K \sum_{i=1}^{N_K} [u_h(P_i) v_h(P_i) + h_{P_i}^2 \nabla u_h(P_i) \cdot \nabla v_h(P_i)] \quad \forall u_h, v_h \in W_h^K, \tag{5.1}$$

where $P_1, \ldots, P_{N_K}$ are the vertices of K, h_{P_i} corresponds to the maximum diameter of the elements with P_i as a vertex and $\sigma_K > 0$ is a multiplicative factor to take into account the magnitude of the material parameter and the h-scaling, for instance, in the numerical tests we have picked $\sigma_K > 0$ as the mean value of the eigenvalues of the local matrix $a_K\big(\Pi_K^{\Delta} u_h, \Pi_K^{\Delta} v_h\big)$. This ensures that the stabilizing term scales as $a_K(v_h, v_h)$. Now, a choice for $s_K^0(\cdot,\cdot)$ is given by

$$s_K^0(u_h, v_h) := \sigma_K^0 \sum_{i=1}^{N_K} [u_h(P_i) v_h(P_i) + h_{P_i}^2 \nabla u_h(P_i) \cdot \nabla v_h(P_i)] \quad \forall u_h, v_h \in W_h^K. \tag{5.2}$$

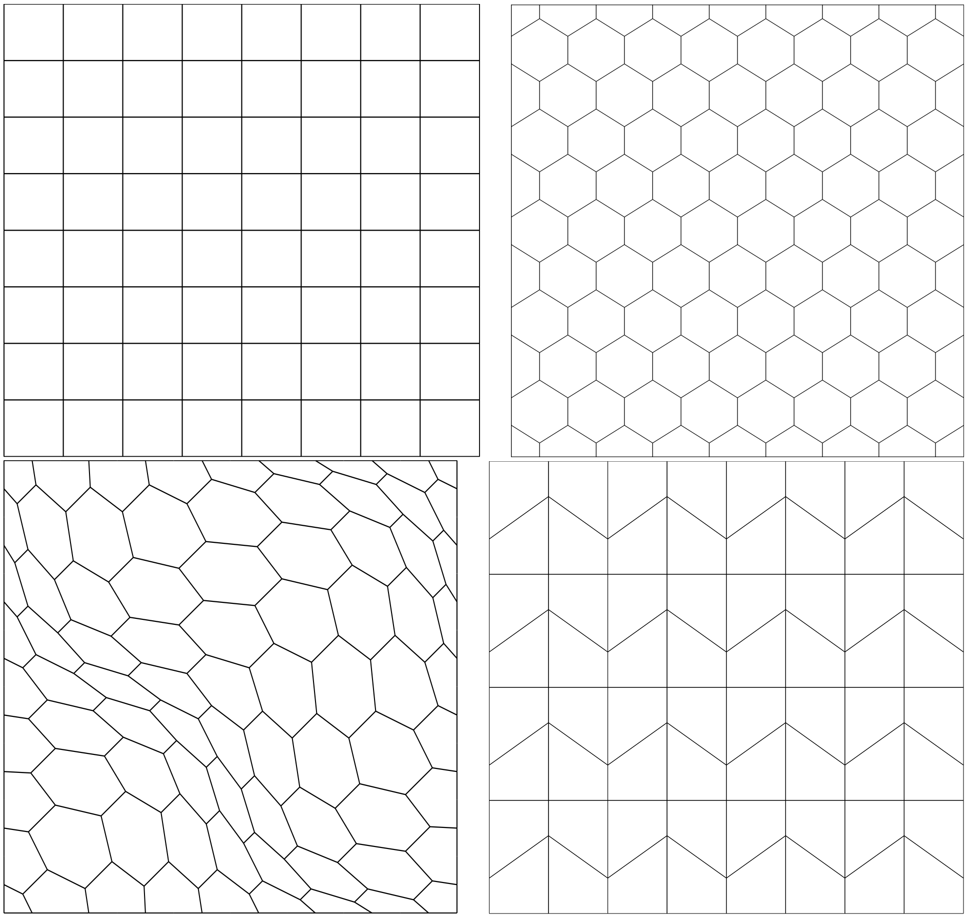

FIGURE 1. Sample meshes: $\mathcal{T}_h^1$ (*top left*), $\mathcal{T}_h^2$ (*top right*), $\mathcal{T}_h^3$ (*bottom left*) and $\mathcal{T}_h^4$ (*bottom right*), for $N = 8$.

In this case, we have taken the parameter $\sigma_K^0 > 0$ as the mean value of the eigenvalues of the local matrix $b_K\big(\Pi_K^{\Delta} u_h, \Pi_K^{\Delta} v_h\big)$ to ensure (3.6). A proof of (3.5) and (3.6) for the above (standard) choices could be derived following the arguments in [7] (see also [3]). Finally, we mention that the above definitions of the bilinear forms $s_K(\cdot,\cdot)$ and $s_K^0(\cdot,\cdot)$ are according to the analysis presented in [31] in order to avoid spectral pollution.

We have tested the method by using different families of meshes (see Fig. 1):

- $\mathcal{T}_h^1$: rectangular meshes;
- $\mathcal{T}_h^2$: hexagonal meshes;
- $\mathcal{T}_h^3$: non-structured hexagonal meshes made of convex hexagons;
- $\mathcal{T}_h^4$: trapezoidal meshes which consist of partitions of the domain into $N \times N$ congruent trapezoids, all similar to the trapezoid with vertices $(0,0)$, $\big(\frac{1}{2},0\big)$, $\big(\frac{1}{2},\frac{2}{3}\big)$ and $\big(0,\frac{1}{3}\big)$.

The refinement parameter N used to label each mesh is the number of elements on each edge of the plate.

5.1. Simply supported plate

First, we have considered a simply supported plate, because analytical solutions are available in this case (see [2,4]). Even though our theoretical analysis has been developed only for clamped plates, we think that the

TABLE 1. Lowest eigenvalues of a simply supported square plate computed on different meshes with the method analyzed in this paper.

	Mesh	$N=32$	$N=64$	$N=128$	Order	Extrapolated	Exact
λ_1		390.0184	389.7307	389.6599	2.02	389.6366	389.6364
λ_2	$\mathcal{T}_h^1$	2430.2171	2433.9024	2434.8914	1.90	2435.2523	2435.2273
λ_3		2430.2171	2433.9024	2434.8914	1.90	2435.2523	2435.2273
λ_4		6259.8318	6240.2949	6235.6906	2.09	6234.2872	6234.1818
λ_1		389.0957	389.4908	389.5987	1.87	389.6395	389.63634
λ_2	$\mathcal{T}_h^2$	2412.1885	2429.0389	2433.6393	1.87	2435.3783	2435.2273
λ_3		2433.8095	2434.8277	2435.1197	1.80	2435.2376	2435.2273
λ_4		6199.2905	6224.8431	6231.7684	1.88	6234.3634	6234.1818

TABLE 2. Lowest eigenvalues of a clamped square plate computed on different meshes with the VEM method analyzed in this paper and the one in [32].

	Mesh	$N=32$	$N=64$	$N=128$	Order	Extrapolated	[32]
λ_1		1283.2286	1291.5607	1294.0225	1.76	1295.0526	1294.9369
λ_2	$\mathcal{T}_h^3$	5268.2854	5353.1383	5377.7322	1.79	5387.6973	5386.6675
λ_3		5326.6504	5368.5269	5381.6191	1.68	5387.5416	5386.6675
λ_4		11406.3068	11622.9583	11686.8981	1.76	11713.7035	11710.9076
λ_1		1289.7221	1293.6088	1294.6010	1.97	1294.9410	1294.9369
λ_2	$\mathcal{T}_h^4$	5318.4039	5368.9773	5382.1939	1.94	5386.8279	5386.6675
λ_3		5351.6510	5377.6743	5384.3950	1.95	5386.7517	5386.6675
λ_4		11664.9586	11698.2652	11707.5973	1.84	11711.1942	11710.9076

results of the previous sections should hold true for more general boundary conditions as well. The results that follow give some numerical evidence of this. For the computations we took $\Omega := (0,1)^2$.

In Table 1, we report the four lowest eigenvalues (λ_i, $i = 1,2,3,4$) computed by our method with two different families of meshes and $N = 32, 64, 128$ for a simply supported plate. The table includes computed orders of convergence, as well as more accurate values extrapolated by means of a least-squares fitting. The last column shows the exact eigenvalues. It can be seen from Table 1 that the method converges to the exact values with an optimal quadratic order. Notice that, for the $\mathcal{T}_h^1$ meshes, the second computed eigenvalue is double, because the meshes preserve the symmetry of the domain leading to an eigenvalue of multiplicity 2 in the continuous problem.

5.2. Clamped plate

In this numerical test, we took $\Omega := (0,1)^2$ and considered clamped boundary condition on the whole of $\partial\Omega$. We present numerical experiments which confirm the theoretical results proved above.

Table 2 shows the four lowest eigenvalues computed with successively refined meshes of each type for a clamped plate. The table includes orders of convergence, as well as accurate values extrapolated by means of a least-squares fitting. Moreover, we compare the performance of the proposed method with the one presented in [32] with a mixed formulation for solving the plate vibration problem and a Galerkin method based on piecewise linear and continuous finite elements. With this aim, we include in the last column of Table 2 the values obtained by extrapolating those computed with method in [32] on uniform triangular meshes as those shown in Figure 2, for the same problem.

It is clear that the eigenvalue approximation order of our method is quadratic and that the results obtained by the two methods agree perfectly well.

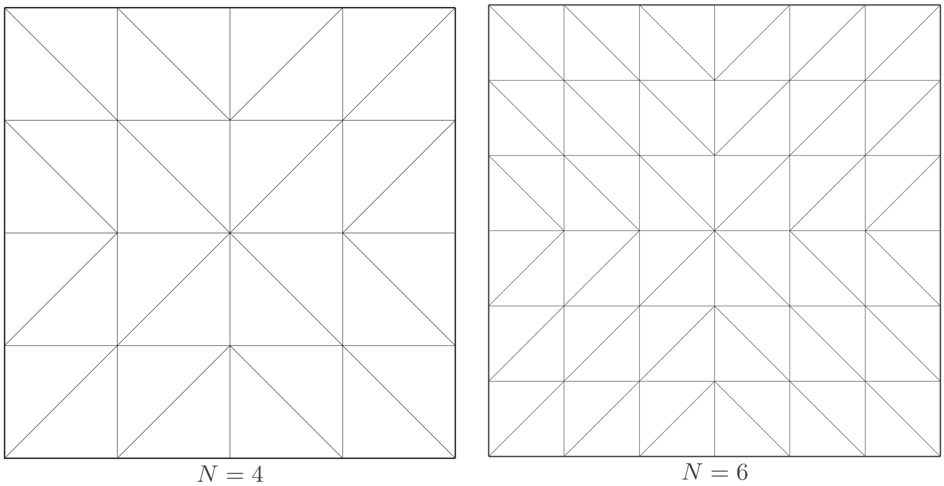

FIGURE 2. Uniform meshes.

TABLE 3. Lowest eigenvalues of an L-shaped clamped plate computed on uniform triangular meshes with the VEM method analyzed in this paper and the one in [32].

	$N = 32$	$N = 64$	$N = 128$	Order	Extrapolated	[32]
λ_1	6827.5421	6753.6207	6725.1315	1.28	6707.4264	6704.2982
λ_2	11 128.5787	11 073.4576	11 059.3867	1.97	11 054.5647	11 055.5189
λ_3	14 989.9367	14 926.5156	14 910.6489	2.00	14 905.3676	14 907.0816
λ_4	26 325.7078	26 195.9206	26 163.4597	2.00	26 152.6488	26 157.9673

5.3. L-shaped plate

Now, we present two numerical experiments which confirm the theoretical results proved above. We have computed the eigenvalues of an L-shaped plate: $\Omega := (0, 1) \times (0, 1) \setminus [0.5, 1) \times [0.5, 1)$.

In the first test, we considered clamped boundary condition on the whole of $\partial\Omega$ and we have used uniform triangular meshes as those shown in Figure 3. Once again, we compare the performance of the proposed method with the one presented in [32].

We report in Table 3 the four lowest eigenvalues computed with the method analyzed in this paper. The table includes orders of convergence, as well as accurate values extrapolated by means of a least-squares fitting. The last column shows the values obtained by extrapolating those computed with method in [32] on the same uniform triangular meshes.

In this case, for the first eigenvalue, the method converges with order close to 1.28, which is the expected one because of the singularity of the solution (see [29]). Instead, the method converges with larger orders for the second, third and fourth eigenvalues.

In this case, we mention the following advantages of the proposed VEM method: the computational cost of our method is smaller than the method studied in [32]. In fact, the number of unknowns for our VEM method is, $3N_v$, where N_v denotes the number of vertices, whereas in [32] is $4N_v$. Moreover, in this case, the eigenvalue problem to be solved is much simpler than the one arising from the formulation studied in [32]. In fact, the latter leads to a degenerate generalized matrix eigenvalue problem, which is shown to be well posed in [32], (Appendix) but that cannot be solved with standard eigensolvers.

We show in Figure 3 the eigenfunctions corresponding to the four lowest eigenvalues for an L-shaped clamped plate.

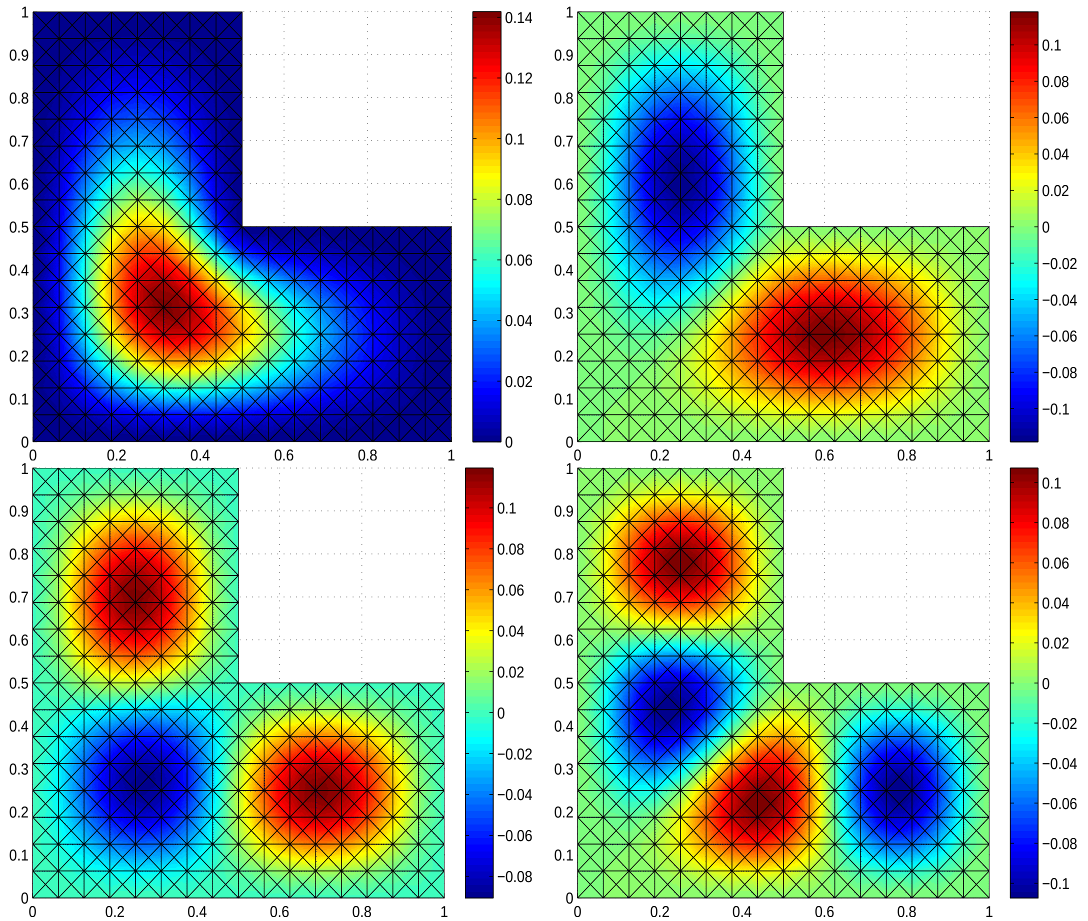

FIGURE 3. Eigenfunctions of the plate problem with clamped boundary condition associated with eigenvalues λ_1 (*top left*) λ_2 (*top right*), λ_3 (*bottom left*) and λ_4 (*bottom right*).

TABLE 4. Lowest eigenvalues of an L-shaped clamped-free plate computed on triangular meshes with the VEM method analyzed in this paper.

	$N = 32$	$N = 64$	$N = 128$	Order	Extrapolated
λ_1	1198.2579	1195.3003	1194.4606	1.82	1194.1302
λ_2	4576.4950	4556.9217	4551.0233	1.73	4548.4764
λ_3	6807.0921	6785.8226	6780.4745	2.00	6778.6710
λ_4	15 094.2896	15 019.3352	14 998.6457	1.86	14 990.8077

Finally, Table 4 shows the four lowest eigenvalues computed with successively refined triangular meshes for an L-shaped clamped-free plate. The table includes orders of convergence, as well as accurate values extrapolated by means of a least-squares fitting. We observe from the results reported in Table 4 that the order of convergence is again quadratic in this case.

We show in Figure 4 the eigenfunctions corresponding to the four lowest eigenvalues.

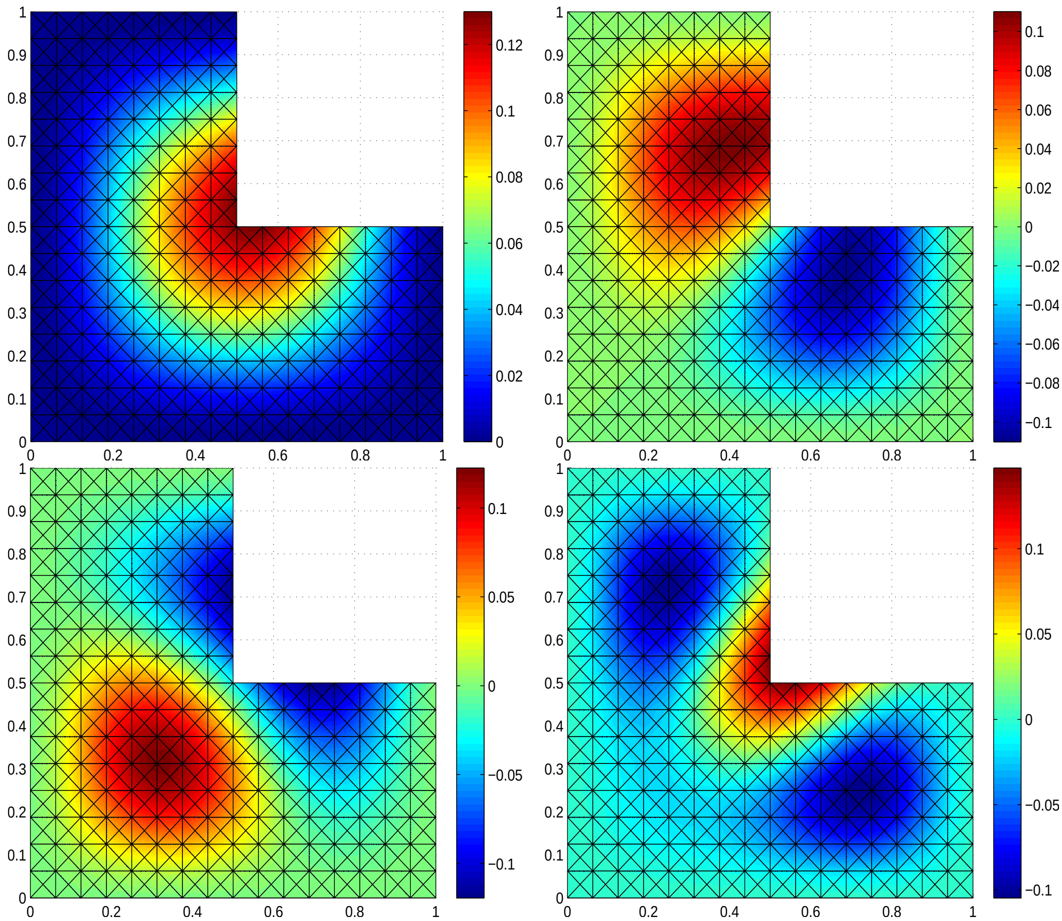

FIGURE 4. Eigenfunctions of the plate problem with mixed boundary condition associated with eigenvalues λ_1 (*top left*) λ_2 (*top right*), λ_3 (*bottom left*) and λ_4 (*bottom right*).

5.4. Effect of the stability constants

The aim of this test is to analyze the influence of the stabilizing bilinear forms $s_K(\cdot,\cdot)$ and $s_K^0(\cdot,\cdot)$ introduced in (5.1) and (5.2), respectively, on the computed spectrum, to know whether the quality of the computations can be affected.

With this aim, for any $\alpha > 0$, we consider the following scaled stabilizing bilinear forms $\alpha s_K(\cdot,\cdot)$ and $\alpha s_K^0(\cdot,\cdot)$. In this test, we have taken the same configuration as in Section 5.2. Therefore, the results for $\alpha = 1$ on different meshes are reported in Table 2.

In Table 5, we report the lowest eigenvalues computed by the method with varying values of α on a fixed mesh $\mathcal{T}_h^4$ with refinement level $N = 16$ (see Fig. 1). We have observed the eigenfunctions associated to each eigenvalue and no spurious eigenvalues were detected for any choice of the parameter α. Moreover, it can be seen in Table 5, that the computed spectrum is well approximated for a wide range of values of α. On the other hand, for small and large values of α, the computed eigenvalues are sensible to this parameter.

However, the eigenvalues for $\alpha \geq 1/4$ converge to the same values with an optimal quadratic order as the mesh is refined, this can be seen in Table 6, where we report the lowest eigenvalues computed with varying values of α on the family of meshes $\mathcal{T}_h^4$. The table also includes orders of convergence, as well as accurate values

TABLE 5. Computed lowest eigenvalues for $\alpha = 4^k$ with $-3 \leq k \leq 3$.

	$\alpha = 1/64$	$\alpha = 1/16$	$\alpha = 1/4$	$\alpha = 1$	$\alpha = 4$	$\alpha = 16$	$\alpha = 64$
λ_1	678.2631	884.2427	1116.2400	1275.3202	1427.0062	1818.0463	3282.2829
λ_2	1573.9238	2458.2514	3892.7704	5146.3829	6083.8491	8275.8511	16 596.5175
λ_3	2981.4564	3623.2656	4477.9219	5259.3630	6266.1534	8638.6246	17 136.2878
λ_4	3159.0171	5555.2757	9188.7834	11 581.8220	15 108.1552	25 967.6692	54 081.8017

TABLE 6. Lowest eigenvalues for different values of α of a clamped square plate computed on the family of meshes $\mathcal{T}_h^4$ with the VEM method analyzed in this paper and the one in [32].

	α	$N = 32$	$N = 64$	$N = 128$	Order	Extrapolated	[32]
λ_1		945.2274	1154.7930	1253.1204	1.09	1340.3447	1294.9369
λ_2	$\alpha = 1/64$	2528.0163	3884.5960	4867.1361	0.47	7412.8204	5386.6675
λ_3		4303.2208	4996.5611	5270.7107	1.34	5449.4967	5386.6675
λ_4		7700.0166	9702.2703	11 006.5891	0.62	13 433.8908	11 710.9076
λ_1		1133.7070	1247.0043	1282.3102	1.68	1298.3651	1294.9369
λ_2	$\alpha = 1/16$	3851.0044	4852.8168	5238.5617	1.38	5478.5748	5386.6675
λ_3		4732.4857	5194.7780	5336.1008	1.71	5398.3001	5386.6675
λ_4		9357.1501	10 894.0565	11 482.6150	1.38	11 851.4313	11 710.9076
λ_1		1241.0596	1280.6680	1291.3094	1.90	1295.1848	1294.9369
λ_2	$\alpha = 1/4$	4858.3819	5238.8944	5348.5413	1.80	5392.4526	5386.6675
λ_3		5106.2979	5311.7571	5367.5790	1.88	5388.4009	5386.6675
λ_4		10 796.1790	11 450.7700	11 643.3385	1.77	11 722.7741	11 710.9076
λ_1		1328.6944	1303.4367	1297.0628	1.99	1294.9288	1294.9369
λ_2	$\alpha = 4$	5565.8430	5431.8202	5397.9674	1.99	5386.6545	5386.6675
λ_3		5610.9332	5443.2945	5400.8574	1.98	5386.4086	5386.6675
λ_4		12 601.1773	11 936.9889	11 767.6059	1.97	11 709.4491	11 710.9076
λ_1		1785.9613	1420.0748	1326.7370	1.97	1294.7091	1294.9369
λ_2	$\alpha = 64$	8146.0204	6081.0251	5561.9715	1.99	5386.8201	5386.6675
λ_3		8194.4611	6098.0923	5569.3074	1.99	5392.1072	5386.6675
λ_4		25 490.6782	15 186.1020	12 597.4474	1.99	11 722.9278	11 710.9076

extrapolated by means of a least-squares fitting. The last column shows the values obtained by extrapolating those computed with the finite element method introduced in [32] on triangular meshes. On the other hand, from the same table, we see that for very small values of α the lowest eigenvalues converge to wrong results and finer meshes are needed for the computed eigenvalues to lie close to the reference value.

This analysis suggests, that the user of VEM for this kind of spectral problems, has to be aware of the risk of degeneration of the eigenvalues for certain values of the parameter α. The way of minimizing this risk in this case is to take values of $\alpha \in [1/4, 4]$, where the method is robust with respect to the parameter.

6. Conclusions

The mathematical and numerical analysis for the vibration problem of Kirchhoff-Love plates approximation by virtual elements was addressed in this paper. The variational formulation is written in terms of the transverse displacement of the plate and a conforming $H^2(\Omega)$ discrete formulation was proposed to numerically approximate the eigenvalue problem. It is established that the resulting scheme provides a correct spectral approximation and that the error estimates are of the optimal order for the eigenfunctions and eigenvalues. The proposed method is new on triangular meshes for the discretization of fourth order eigenvalue problems, and in this case the computational cost is almost $3N_v$, where N_v denotes the number of vertices, thus providing a very competitive alternative in comparison to other classical techniques based on finite elements. The theoretical results obtained were validated numerically. Even though our theoretical analysis has been developed only for clamped plates, additional examples have been considered and we evidenced that the results of the previous sections hold true for more general boundary conditions as well. Moreover, we included a numerical test to check the influence of the stabilizing bilinear forms. No spurious eigenvalues were found for any chosen stability parameter and the computed spectrum is well approximated for a wide range of values of the parameter.

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