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NUMERICAL SOLUTION OF

MATRIX INTERPOLATION PROBLEMS

by

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To my family

Mei, Koko, Yanyan

PREFACE

This thesis is submitted in accordance with the regulations for the degree of Doctor of Philosophy in the University of Glasgow. No part of it has been previously submitted by the author for a degree at any other University.

The results contained in this thesis are claimed as original except where indicated in the text.

I would take this opportunity of expressing my deep gratitude to my supervisor, Dr. N.J. Young for suggesting the problems which are contained in this thesis, and also for his guidance, constant interest and encouragement, without which this thesis would not have been presented.

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CONTENTS

SUMMARY AND INTRODUCTION

Quite a number of engineering design techniques for circuit and system theory depend heavily on the construction of an analytic function of a minimal norm in the open unit disc D satisfying some interpolation conditions.

The above problem was solved constructively by Pick [32] and Nevanlinna [29] at the beginning of this century. This is the well known Nevanlinna-Pick interpolation problem.

In 1967 Saito and Youla [39] first introduced Nevanlinna-Pick theory into engineering literature. They showed how interpolation theoretic constructions corresponded to circuit theoretic ones.

In 1968 two very important articles were published by D. Sarason [40] and V.M.Adamjan, D. Arov and M. Krein(A-A-K) [1][2] giving the solution of the classic interpolation problems in operator terms. These two approaches are quite different.

The approach of A-A-K related significant implications of interpolation theory for some important engineering topics, such as broadband matching [18][19], digital filter design [16] (which is widely used for stationary stochastic processes), model reduction [22], cascade synthesis of networks [10] and linear estimation theory [21]. Their results have led to the use of approximation of functions in the "Hankel norm" for such applications. This Hankel norm approximation has three important properties from an engineering viewpoint.

Firstly, the Hankel norm lies between the most popular least squares norm (L_2-norm) and the most stringent Chebyshev norm (L_∞-norm).

Secondly, this norm is the largest singular value of a Hankel matrix and this value is known to be insensitive with respect to perturbations.

Thirdly, the best approximation in the Hankel norm can be explicitly computed, thanks to A-A-K, it reduces to finding singular values and vectors of an infinite Hankel matrix.

There are two methods of doing this, proposed by S.Y. Kung [22][23] and Delsarte, Genin and Kamp [9] in 1979, 1980. The most important of these algorithms is that of Kung. His point of departure is to consider an impulse response function for an unknown stable linear system. For a pre-assigned error tolerance, it is required to obtain a best approximation to the function of degree as small as possible from input/output data. Such a problem has a long history and has been approached in various ways, but each method leads only to a sub-optimal solution.

It appears that using the A-A-K approach is fertile and rich in both pure mathematics and applied engineering. But none of these methods can claim to be stable, accurate and efficient for numerical computation. However there is a significant dis-covery in the scalar case by N.J. Young [4] in which a completely new algorithm for such interpolation problems was proposed and proved to be very successful in pratical tests. Young's algorithm is based on Sarason's theory [40].

In realistic engineering problems one is more concerned with systems with several inputs and outputs than 1-input and 1-output. In mathematical terms this corresponds to studying interpolation by functions whose values are matrices rather than scalars.

The aim of this thesis is to extend Sarason's theory and Young's algorithm to solve such interpolation problems numerically. As far as we can see, the generalization is not easy to achieve. The difficulties come in many ways.

Firstly, one must find a suitable setting for the generalized Sarason operator.

Secondly, the reduction from an operator to a matrix problem involves decomposing a certain space of rational matrix functions in such a way that the advantages of the scalar method are retained. This can be done through some kind of factorization.

Thirdly, and most significantly the interpolating function of minimal norm is not unique: thus in order to compute a minimizing function we must impose further conditions to ensure uniqueness, or make some arbitary choice and this entails quite new considerations.

Let M_{mxn} denote the space of mxn matrices over the complex field. For $f \in H^{\infty}_{mxn}$, the space of bounded analytic functions on the op-en unit disc D with values in M_{mxn}, we write

$$\| f \|_{\infty} = \sup_{z \in D} \| f(z) \| ,$$

where $\| \ \|$ is the operator norm in M_{mxn}. By Fatou's theorem, any function $f \in H^{\infty}_{mxn}$ has a radial limit almost everywhere on D, and hence defines a function also denoted by f in L^{∞}_{mxn}, the space of bounded measurable functions on the unit circle ∂D with values in M_{mxn}(modulo equality a. e. w. r. t. Lebesgue measure). The maximum principle shows that the L^{∞}_{mxn} norm agrees with the natural norm of H^{∞}_{mxn} (Chebyshev norm)

$$\| f \|_{L^{\infty}} = \| f \|_{H^{\infty}} = \text{ess} \sup_{z \in \partial D} \| f(z) \| .$$

In particulat when m=n=1, H^{∞} is the space of bounded analytic functions which is the case treated by Nevanlinna and Pick:

[N-P] Given distinct points $\alpha_1, \alpha_2, \ldots, \alpha_n$ in D, complex numbers $\omega_1, \omega_2, \ldots, \omega_n$, find a function $f \in H^{\infty}$ such that $f(\alpha_i)=\omega_i$, i=1,2,..., n and $\| f \|_{\infty}$ is minimized.

We can reformulate the [N-P] problem in terms of distances. Suppose φ is any bounded analytic function satisfying $\varphi(\alpha_i)=\omega_i$, i=1,2,...,.n, and let $b(z)=\prod_{j=1}^{n} \frac{z-\alpha_j}{1-\bar{\alpha}_j z}$ be a Blaschke product of α_i(i=1,2,...,n) of degree n. Then a function $f \in H^{\infty}$, such that $f(\alpha_i)=\omega_i$, if and only if $(f-\varphi)(\alpha_i)=0$, in other words $f-\varphi=bk$ for some k in H^{∞}, so that

$$\inf \{ f: f(\alpha_i) = \omega_i,\ i = 1,2,\ldots,n \}$$

$$= \| \varphi + bH^{\infty} \|_{H^{\infty}/bH^{\infty}} \overset{\text{def}}{=} \inf_{g \in H^{\infty}} \| \varphi + bg \|_{\infty} \overset{\text{def}}{=} \text{dis}(\varphi, bH^{\infty}).$$

Observe that bH^{∞} consists of all functions in H^{∞} which vanish at all the zeros of b and have zeros at least the same order as b at these points. The [N-P] problem is mathematically equivalent to

[N-P]' Given a Blaschke product b of degree n, and a function $\varphi \in H^{\infty}$, find a function $f \in \varphi + bH^{\infty}$ such that $\| f \|_{\infty}$ is minimized.

D. Sarason showed that the infimum is attained and equals the norm of the Sarason operator $\varphi(S_b^*)$, where S_b^* is the forward shift operator S^* compressed to $H^2 \ominus bH^2$. Since the zeros of b are in D, $H^2 \ominus bH^2$ is an n-dimensional Hilbert space, and we can choose a suitable basis for $H^2 \ominus bH^2$ and express S_b^* in matrix form. If the zeros of b(z)=0 are known, then it is easy to write down a basis in terms of the zeros of b. However it is well known that a numerical instability can occur in solving the equation b(z)=0. Young [4] found a convenient and natural basis of $H^2 \ominus bH^2$ in terms of the coefficients of the numerator of b. This plays a key role in Young's algorithm; the matrix of S_b with respect to this basis is a companion matrix, the computation of a matrix of $\varphi(S_b^*)$ can therefore be reduced to finding $g(C_b^T)$, where g is a polynomial with degree less than n and C_b^T is the transpose of the companion matrix of b. This can be done with an operation count of $O(n^3)$ rather than the $O(n^4)$ one might expect. Moreover, although this basis is not orthonormal, the Gram matrix of the basis can be obtained by a very simple, recursive formula. This formula is also one of the key techniques that have been developed in Young's algorithm.

Now let us state the [N-P] problem in the matrix valued case:

[M-N-P] Given an inner matrix $B \in H^\infty_{m\times m}$, and a function F in $H^\infty_{m\times n}$ find

(1) $\| F + BH^\infty_{m\times n} \|_{H^\infty_{m\times n}/BH^\infty_{m\times n}}$,

(2) a function $G \in F + BH^\infty_{m\times n}$ such that

$$\| G \|_\infty = \| F + BH^\infty_{m\times n} \|_{H^\infty_{m\times n}/BH^\infty_{m\times n}} .$$

An inner matrix is defined to be an element of H^{∞}_{mxm} which is unitary on ∂D, almost everywhere.

Firstly, we need to characterize the norm in $H^{\infty}_{mxn}/BH^{\infty}_{mxn}$. This can be done by establishing an isometric bounded linear mapping between $H^{\infty}_{mxn}/BH^{\infty}_{mxn}$ and a class of operators T from $H^2_{nx1} \ominus (\det B)H^2_{nx1}$ to $H^2_{mx1} \ominus BH^2_{mx1}$. We study this characterization in Chapter two by using the dual extremal approach. Such operators T are generalizations of the Sarason operator from the scalar case to the matrix valued case. An example is constructed to show that two subspaces, $H^2_{nx1} \ominus (\det B)H^2_{nx1}$ and $H^2_{mx1} \ominus BH^2_{mx1}$, are necessary in the matrix valued case: these coincide in the scalar case.

Some techniques of operator theory and function theory are necessary for our theory and algorithms, and these are presented in Chapter one.

In Chapter three we give a description of a direct decomposition of $H^2_{mx1} \ominus BH^2_{mx1}$, in terms of the coefficients of the numerator of B. We make our choice in such a way that the Gram matrix of the decomposition can be calculated economically. Theorems 3.2.2, 3.3.1 and 3.3.2 are the main theorems in this chapter.

In Chapter four a computational form of the orthogonal projection from $H^2_{mx1} \ominus (\det B)H^2_{mx1}$ to $H^2_{mx1} \ominus BH^2_{mx1}$ is obtained; and this plays an important role in our algorithm. Therefore the advantages in Young's algorithm are retained.

The whole of Chapter five deals with some aspects of implementation, comments on our algorithm and compares the other algorithms.

Finally two numerical examples are presented in Chapter six.

CHAPTER ONE

PRELIMINARIES

1.1 General concepts

The general concepts contained in this thesis can be found in the following list of books.

(1) Operator theory, I. Gohberg and S. Goldberg[14],

(2) Hardy space, K. Hoffman [20], H. Helson [17], Nagy-Foias [27] and R.G. Douglas [12]

(3) Interpolation theory, J.L. Walsh [47]

(4) Numerical analysis, G.W. Stewart [42] and A. Gourlay [15]

(5) Hankel operators, D. Sarason [41], S. Power [34].

1.2 Singular value decompositions

Let H_1, H_2 be separable Hilbert spaces. Let $L(H_1,H_2)$ denote the Banach space of all bounded linear operators from H_1 to H_2. Let $\{f_i^{(j)}\}_{i=1}^{\infty}$, $j=1,2$ be an orthonormal base for H_j. The matrix M corresponding to A and $\{f_i^{(j)}\}$ is (a_{ij});

$$a_{ij} = (Af_j^{(1)}, f_i^{(2)})_2 .$$

Since

$$(A^*f_j^{(2)}, f_i^{(1)})_1 = (f_j^{(2)}, Af_i^{(1)})_2 = \overline{(Af_i^{(1)}, f_j^{(2)})_2},$$

where $(\; , \;)_1$ and $(\; , \;)_2$ are the inner products for H_1 and H_2. The matrix corresponding to A^* and $\{f_i^{(j)}\}$ is the complex conjugate $M^* = (\bar{a}_{ji})$ of the matrix $M = (a_{ij})$ of A. The range of $A \in L(H_1, H_2)$, written Range A, is the subspace

$AH_1 = \{Ax;\ x \in H_1\}$. If Range A is finite dimensional, A is called an operator of finite rank and dim Range A is called the rank of A. If a bounded linear operator $A \in L(H_1, H_2)$ has finite rank r then there exist positive numbers $s_0 \geq s_1 \geq s_2 \geq \ldots\ldots$, and orthonormal sequences e_0, e_1, ..., e_{r-1} in H_1 and f_0, $f_1, \ldots$, f_{r-1} in H_2 such that

$$A = \sum_{j=0}^{r-1} s_j(\quad, e_j)f_j \tag{1}$$

in the sense that, for all $x \in H_1$,

$$Ax = \sum_{j=0}^{r-1} s_j(x, e_j)f_j \ .$$

We make the convention that $s_j = 0$ for $j \geq r$.
The s_j are called the <u>Singular values of A</u> and are unique, being the eigenvalues of $(A^*A)^{\frac{1}{2}}$, together with 0. e_j and f_j are called <u>singular vectors of A.</u> An ordered pair $\langle e, f \rangle \in H_1 x H_2$ is called a Schmidt pair corresponding to s if

$$Ae = sf \quad \text{and} \quad A^*f = se \ . \tag{2}$$

Thus $\langle e_j, f_j \rangle$ is the Schmidt pair corresponding to s_j.
The relation (1) is called a <u>Singular value decomposition of A(SVD)</u>. This is the most reliable characterization for computing the rank of a matrix. The rank of a matrix is equal to the number of nonzero singular values of the matrix. The SVD in matrix form is defined as follows. Let A be an mxn matrix having rank r. Then there exist mxm and nxn unitary matrices U and V such that

$$A = U^* \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} V \tag{3}$$

where $\sum_r$ is a diagonal matrix with the nonzero singular values of A along its diagonal.

1.3. Gram matrices

Let $\{u_i\}_{i=1}^{n}$ be a basis for H. The Gram matrix of the basis is defined to be the nxn matrix $G=\{(u_j,u_i)\}$. Let H_i be an n_i-dimension Hilbert space, i = 1,2, and let A, Q_1, Q_2 be linear operators in $L(H_1,H_2)$, $L(H_1)$, $L(H_2)$ having matrices $[A]$, $[Q_1]$, $[Q_2]$ respectively with respect to the bases $u_1, u_2, \dots, u_{n_1}$ of H_1 and $v_1, v_2, \dots, v_{n_2}$ of H_2. Then

(1) If Q_1 and Q_2 are invertible then the matrix of A, $[A]_0$ with respect to $Q_1u_1, Q_1u_2, \dots\dots, Q_1u_{n_1}$ and $Q_2v_1, Q_2v_2, \dots\dots, Q_2v_{n_2}$ is

$$[A]_0 = [Q_2]^{-1}[A][Q_1] , \qquad (4)$$

(2) $Q_1u_1, Q_1u_2, \dots, Q_1u_{n_1}$ is an orthonormal bases of H_1 if and only if $[Q_1][Q_1]^* = G_1^{-1}$, where G_1 is the Gram matrix of $u_1, u_2, \dots, u_{n_1}$.

These facts are elementary can be founded in many texts on linear algebra [24].

CHAPTER TWO

THE GENERALIZED SARASON OPERATOR

Introduction:

Given an inner matrix $B \in H^{\infty}_{mxm}$, we show how to relate $H^{\infty}_{mxn}/BH^{\infty}_{mxn}$ to operators T acting on certain subspaces of $H^2_{nx1} \ominus (\det B)H^2_{nx1}$ and $H^2_{mx1} \ominus BH^2_{mx1}$. Such operators T are generalizations of the Sarason operator from the scalar case to the matrix valued case. Now a generalization of the Nehari theorem [28] can be used to show that the quotient norm of $H^{\infty}_{mxn}/BH^{\infty}_{mxn}$ can be expressed in terms of a certain Hankel operator (the A-A-K operator) Γ acting from H^2_{nx1} to H^2_{mx1}: one might therefore expect that the generalized Sarason operator T would be closely related to Γ, and in fact T is a unitary multiplied by the nonzero part of Γ. Functions of minimum norm in any coset of BH^{∞}_{mxn} in H^{∞}_{mxn} can be obtained using the A-A-K one step extension [3]. The analogue of this one step extension is here examined in the Sarason-type formulation.

Contents:

2.1 Analytic vector functions

Let $\partial D = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle and let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc. Let $M_{m\times n}$ denote the linear space of all mxn matrices over the complex field $\mathbb{C}$. For $A \in M_{m\times n}$, $1 \le p \le \infty$, let $\|A\|_p$ be the Schatten-Von Neumann norm:

$$\|A\|_p = (\sum_{n\ge 0} s_n{}^p)^{1/p} , \qquad (1)$$

$$\|A\|_\infty = \sup_n \{ s_n \} ,$$

where $\{s_n\}_{n\ge 0}$ is the sequence of all singular values of A. $\| \ \|_1$ and $\| \ \|_2$ are the familiar trace and Hilbert-Schmidt norms and $\| \ \|_\infty$ is the operator norm. Let $L^p_{m\times n}$, $1 \le p \le \infty$, denote the Banach space of all measurable functions on ∂D with values in $M_{m\times n}$, modulo the subspace of functions equal to zero almost everywhere, such that

$$\|f\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} \|f(e^{i\theta})\|_p^p \, d\theta\right)^{1/p} < \infty , \qquad (2)$$

and let $L^\infty_{m\times n}$ be the space of essentially bounded $M_{m\times n}$-valued functions on ∂D with the essential supremum norm:

$$\|f\|_\infty = \operatorname*{ess\,sup}_{z\in\partial D} \|f(z)\|_\infty < \infty \qquad (3)$$

The spaces we are concerned with are $L^1_{m\times n}$, $L^2_{m\times n}$ and $L^\infty_{m\times n}$ and their subspaces. Let $H^1_{m\times n}$ be the subspace of functions g in $L^1_{m\times n}$ with the property that for every pair of vectors $x \in \mathbb{C}^n$, $y \in \mathbb{C}^m$, the scalar function $(g(z)x,y)$ is in H^1. In particular, $L^2_{m\times n}$ is a Hilbert space under the inner product

$$(f,g)_2 = \frac{1}{2\pi}\int_0^{2\pi} \operatorname{trace}(f(e^{i\theta})g(e^{i\theta})^*)\,d\theta \qquad (4)$$

where * denote the complex conjugate transpose in M_{mxn}. A function G in L^2_{mxn} is called analytic if the scalar function $(G(z)x,y)$ belongs to H^2 for each vector $x \in \mathbb{C}^n$, $y \in \mathbb{C}^m$. The analytic functions in L^2_{mxn} form a subspace, which we denote by H^2_{mxn}. In fact any function $f \in H^2_{mxn}$ can be extended to an analytic function on D with values in M_{mxn} having the following expansion

$$f(z) = f_0+f_1z+f_2z^2+\dots$$
$$= \sum_{i=0}^{\infty} f_i z^i, \quad f_i \in M_{mxn}$$

with

$$\|f\|_2^2 = \sup_{0 \le r < 1} \frac{1}{2\pi}\int_0^{2\pi} \|f(re^{i\theta})\|_2^2 \, d\theta < \infty .$$

The key operator for generalizing Sarason approach is the backward shift operator $\underset{\sim}{S}$ acting on H^2_{mx1}: this is defined by

$$\underset{\sim}{S}(f_0+f_1z+f_2z^2+\dots) = f_1+f_2z+\dots$$

or equivalently by

$$\underset{\sim}{S}f(z) = \begin{cases} \frac{1}{z}(f(z)-f(0)) & \text{if } z \neq 0 , \\ f'(0) & \text{if } z=0 . \end{cases} \qquad (5)$$

When m=1, we write S in place of $\underset{\sim}{S}$. It is easy to show that the adjoint operator $\underset{\sim}{S}^*$ of $\underset{\sim}{S}$ is the forward shift operator:

$$\underset{\sim}{S}^* f(z) = zf(z).$$

and $\underset{\sim}{S}\,\underset{\sim}{S}^* = I$, $I-\underset{\sim}{S}^*\underset{\sim}{S}$ is rank m, where I is the identity operator on H^2_{mx1}. We shall assume a familarity with the basic properties of such spaces and operators [17] [20] [27]. Let H^∞_{mxn} be the space of bounded analytic functions on D with values in M_{mxn}. By Fatou's theorem, any function φ in H^∞_{mxn} has a radial limit almost everywhere on D, and hence defines a function in L^∞_{mxn}. The maximum principle shows that the $L^{\infty}_{m\times n}$ norm agrees with the natural norm of $H^{\infty}_{m\times n}$ (Chebyshev norm):

$$\|f\|_{L^{\infty}} = \|f\|_{H^{\infty}} \overset{\text{def}}{=} \sup_{z\in\partial D} \|f(z)\|_{\infty} . \qquad (6)$$

Hence we can identify $H^{\infty}_{m\times n}$ with a subspace of $L^{\infty}_{m\times n}$. Let $RH^{\infty}_{m\times n}$ be the space of rational matrices with no poles on ∂D; in this thesis we restrict attention to $RH^{\infty}_{m\times n}$ for practical reasons. The adjoint of $\varphi\in H^{\infty}_{m\times n}$ is defined by:

$$\varphi^{*}(z) = [\varphi(\bar{z})]^{*}, \qquad z \in \partial D$$

which is also analytic, so that $\varphi^{*} \in H^{\infty}_{n\times m}$.

<u>Definition 2.1.1</u> Let φ be in $H^{\infty}_{m\times n}$, φ is a <u>rigid matrix</u> if $\varphi(z)^{*}\varphi(z)=I_n$ for a.e. $z\in\partial D$. When m=n φ is called an <u>inner matrix</u> if $\varphi(z)$ is a unitary matrix for a.e. $z\in\partial D$.

A Potapov Blaschke product [33] is the standard example for an inner matrix: the general form is

$$B(z) = \prod_{j=1}^{\infty} V_j \begin{bmatrix} \frac{|\alpha_j|}{\alpha_j}\frac{z-\alpha_j}{1-\bar{\alpha}_j z} I_p & 0 \\ 0 & I_q \end{bmatrix} W_j , \qquad (7)$$

where p+q=m, V_j, W_j are constant unitary matrices, $\alpha_j \in D, \forall j$.

2.2 <u>The isomorphism between $H^{\infty}_{m\times n}/BH^{\infty}_{m\times n}$ and $H^{\infty}(\beta, B)$</u>

To any φ in $H^{\infty}_{m\times n}$ there corresponds a multiplication operator M_{φ} from $H^{2}_{n\times r}$ to $H^{2}_{m\times r}$, defined by

$$(M_{\varphi}f)(z) = \varphi(z)f(z), \quad f \in H^{2}_{n\times r} , \qquad z \in D.$$

Let B be an inner matrix in $H^{\infty}_{m\times m}$. Let $\beta(z) = (\det B(z))I_m = B(z)\cdot \operatorname{adj} B(z) = \operatorname{adj} B(z)\cdot B(z)$. We form two spaces

$H^2_{n\times 1} \ominus \beta H^2_{n\times 1}$ and $H^2_{m\times 1} \ominus BH^2_{m\times 1}$. The orthogonal projections in $H^2_{n\times 1}$, $H^2_{m\times 1}$ with range $H^2_{n\times 1} \ominus \beta H^2_{n\times 1}$, $H^2_{m\times 1} \ominus BH^2_{m\times 1}$ will be denoted by P_β, P_B, respectively. In a later chapter we will show that $H^2_{n\times 1} \ominus \beta H^2_{n\times 1}$ and $H^2_{m\times 1} \ominus BH^2_{m\times 1}$ are invariant under the backward shift operator $\underset{\sim}{S}$. We therefore introduce $\underset{\sim}{S}_\beta$, $\underset{\sim}{S}_B$ the restriction of $\underset{\sim}{S}$ to $H^2_{n\times 1} \ominus \beta H^2_{n\times 1}$, $H^2_{m\times 1} \ominus BH^2_{m\times 1}$, respectively. i.e.

$$P^*_\beta \underset{\sim}{S}^*_\beta = \underset{\sim}{S}^* P^*_\beta , \quad P_B' \underset{\sim}{S}^* = \underset{\sim}{S}^*_B P_B .$$

For $\varphi \in H^\infty_{m\times n}$ let $P_B M_P P^*_\beta$ denote the projection onto $H^2_{m\times 1} \ominus BH^2_{m\times 1}$ of the multiplication operators M_φ acting on $H^2_{n\times 1} \ominus \beta H^2_{n\times 1}$: we call $P_B M_\varphi P^*_\beta$ <u>a generalized Sarason operator</u>. Then the generalized Sarason operators $P_B M_\varphi P^*_\beta$ are precisely the operators that intertwine $\underset{\sim}{S}^*_\beta$ and $\underset{\sim}{S}^*_B$, since

$$\begin{aligned}(P_B M_\varphi P^*_\beta)\underset{\sim}{S}^*_\beta &= P_B M_\varphi P^*_\beta \underset{\sim}{S}^*_\beta \\ &= P_B M_\varphi \underset{\sim}{S}^* P^*_\beta \\ &= P_B \underset{\sim}{S}^* M_\varphi P^*_\beta \\ &= \underset{\sim}{S}^*_B P_B M_\varphi P^*_\beta = \underset{\sim}{S}^*_B (P_B M_\varphi P^*_\beta).\end{aligned}$$

It is more important that the converse is also true.

<u>Theorem 2.2.1</u> If T is an operator from $H^2_{n\times 1} \ominus \beta H^2_{n\times 1}$ to $H^2_{m\times 1} \ominus BH^2_{m\times 1}$ that intertwines $\underset{\sim}{S}^*_\beta$, $\underset{\sim}{S}^*_B$, then there is a function $\varphi \in H^\infty_{m\times n}$ such that

$$\| \varphi \| = \| T \| \quad \text{and} \quad T = P_B M_\varphi P^*_\beta \tag{8}$$

This theorem is a special case of the well known Nagy-Foias lifting theorem [26] [11]. This theorem can be proved by using the duality approach followed by Sarason in the scalar case [40]. The key point of this approach is the isometric isomorphism between Sarason operators and the quotient space $H^{\infty}_{m\times n}/BH^{\infty}_{m\times n}$. This holds in the matrix valued case.

Let $H^{\infty}(\beta, B)$ denote the space of Sarason operators

$$H^{\infty}(\beta, B) = \{ P_B M_{\varphi} P_{\beta}^{*} : \varphi \in H^{\infty}_{m\times n} \} .$$

There is a natural map of $H^{\infty}_{m\times n}$ onto $H^{\infty}(\beta, B)$ defined by

$$\varphi \longrightarrow P_B M_{\varphi} P_{\beta}^{*} \quad . \tag{9}$$

This is a bounded linear mapping and the Kernel of this bounded linear mapping is $BH^{\infty}_{m\times n}$. We therefore get a natural algebraic isomorphism from $H^{\infty}_{m\times n}/BH^{\infty}_{m\times n}$ onto $H^{\infty}(\beta, B)$. Moreover this isomorphism is norm preserving. To prove this, it is necessary to identify the dual space of $H^{\infty}_{m\times n}/BH^{\infty}_{m\times n}$.

<u>Lemma 2.2.2</u> Let j be the map from $H^{\infty}_{m\times n}$ to $H^{\infty}(\beta, B)$ defined by

$$j\varphi = P_B M_{\varphi} P_{\beta}^{*} \quad , \quad \varphi \in H^{\infty}_{m\times n} ,$$

then

$$\text{Ker } j = BH^{\infty}_{m\times n} \quad .$$

<u>Proof:</u> We first show Ker $j \subseteq BH^{\infty}_{m\times n}$. Suppose $\varphi \in$ Ker j then

$$\varphi \, (H^2_{n\times 1} \ominus \beta H^2_{n\times 1}) \subseteq BH^2_{m\times 1}$$

and

$$\varphi H^2_{n\times 1} = \varphi (H^2_{n\times 1} \ominus \beta H^2_{n\times 1}) + \varphi \beta H^2_{n\times 1}.$$

It follows from

$$\beta I = B . \text{ adj } B = \text{adj } B . B \quad .$$

That

$$\begin{aligned} \varphi H^2_{n\times 1} &\subseteq BH^2_{m\times 1} + \beta I . \varphi H^2_{n\times 1} \\ &= BH^2_{m\times 1} + B(\text{adj } B)\varphi H^2_{n\times 1} \\ &\subseteq BH^2_{m\times 1} \quad . \end{aligned}$$

but

$$\varphi \in H^{\infty}_{mxn}, \text{ this implies } \varphi \in BH^{\infty}_{mxn}.$$

On the other hand, if $\varphi \in BH^{\infty}_{mxn}$.

then

$$\varphi H^2_{nx1} \subseteq BH^2_{mx1}.$$

so

$$\varphi (H^2_{nx1} \ominus \beta H^2_{nx1}) \subseteq BH^2_{mx1}.$$

This means $\varphi \in$ Ker j.

The proof is complete.

<u>Lemma 2.2.3</u> (1) The space L^{∞}_{mxr} is the dual of L^1_{mxr} under the duality

$$< \Phi , f >= \frac{1}{2\pi} \int_0^{2\pi} \mathrm{trace}(\Phi(e^{i\theta})f(e^{i\theta})^T)d\theta, \qquad (10)$$

$\Phi \in L^{\infty}_{mxr}$, $f \in L^1_{mxr}$, where $f(e^{i\theta})^T$ denote the transpose of the matrix.

(2) The space $H^{\infty}_{mxn}/BH^{\infty}_{mxn}$ is the dual space of $\bar{B}zH^1_{mxn}/zH^1_{mxn}$.

(3) Each function $f \in H^1_{mxr}$ has a factorization

$$f = f_1 f_2 , \qquad (11)$$

where $f_1 \in H^2_{mxk}$, $f_2 \in H^2_{kxr}$ for some positive integer k, and

$$f_2^* f_2 = (f^* f)^{\frac{1}{2}}, \quad f_1^* f_1 = f_2 f_2^* .$$

<u>Proof:</u> A complete proof is given in [40, §9], [30, §2] which is analogous to the well known theorem for the scalar case.

Let $\mathbb{C}^r$ denote the Hilbert space of r-dimensional complex column vectors, and let $\{e_i\}_{i=1}^r$ be the standard orthonormal basis for $\mathbb{C}^r$. Let $H^2_{mx1} \otimes C^r$ denote the space

$$\{ g_1e_1+g_2e_2+\dots+g_re_r : g_i \in H^2_{mx1} \}$$

with inner product

$$(f,g) = \sum_{j=1}^{r} (f_j,g_j)_2 = \sum_{j=1}^{r} \frac{1}{2\pi}\int_0^{2\pi} \text{trace}(f_j(e^{i\theta})g_j(e^{i\theta})^*)d\theta.$$

The space $H^2_{mx1}\otimes C^r$ may be regarded as the orthogonal sum of r copies of H^2_{mx1} which is H^2_{mxr}. For T an operator on H^2_{mx1}, $T\otimes I^r$ acts on $H^2_{mx1}\otimes C^r$ by

$$(T\otimes I_r)(g_1e_1+\dots+g_re_r) = Tg_1e_1+\dots+Tg_re_r .$$

The operator $T\otimes I_r$ may also be represented as rxr diagonal matrix operator with entry T.

$$T\otimes I_r = \begin{bmatrix} T & & 0 \\ & T & \\ 0 & & T \end{bmatrix} \text{rxr} .$$

<u>Theorem 2.2.4</u> If $f \in zH^1_{mxn}$ then there is a positive integer r and functions

$$g_2 \in H^2_{nxr} \ominus \beta H^2_{nxr}, \quad g_1 \in H^2_{mxr} \ominus BH^2_{mxr}$$

with

$$\| g_1\|_2^2 \leq \| f\|_1, \quad \| g_2\|_2^2 \leq \| f\|_1$$

such that

$$< \Phi , \bar{B}f> = ((P_BM_FP_\beta^*\otimes I_r)g_2,g_1)_2 \qquad (12)$$

for all $\Phi \in H^\infty_{mxn}$. Conversely, if $g_2 \in H^2_{nxr} \ominus\beta H^2_{nxr}$, $g_1 \in H^2_{mxr} \ominus BH^2_{mxr}$ then there is a function $f \in zH^1_{mxn}$ such that (12) holds for all Φ in H^∞_{mxn}.

<u>Proof:</u> Let $f \in zH^1_{mxn}$. By the factorization of $z^{-1}f$, there exist $f_1 \in zH^2_{mxr}$, $f_2^T \in H^2_{rxn}$

such that

$$f = f_1 f_2^T, \quad f_1^* f_1 = f_2^T f_2^{T^*}$$

and

$$f_2^{T^*} f_2^T = (f^* f)^{\frac{1}{2}} ,$$

then

$$\langle \Phi, \bar{B}f \rangle = \frac{1}{2\pi} \int_0^{2\pi} \text{trace}(\Phi(e^{i\theta})[\bar{B}(e^{i\theta})f(e^{i\theta})]^T)d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \text{trace}(\Phi(e^{i\theta})f(e^{i\theta})^T B (e^{i\theta})^*)d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \text{trace}(\Phi(e^{i\theta})f_2(e^{i\theta})f_1(e^{i\theta})^T \bar{B}(e^{i\theta})^T)d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \text{trace}(\Phi(e^{i\theta})f_2(e^{i\theta})(B(e^{i\theta})\bar{f}_1(e^{i\theta}))^*)d\theta$$

$$= (\Phi f_2, B\bar{f}_1)_2 .$$

For $f_1 \in zH^2_{mxr}$, the function $\bar{f}_1 \in (H^2_{mxr})^{\perp}$, $B\bar{f}_1 \in (BH^2_{mxr})^{\perp}$ and

$$B\bar{f}_1 \in (H^2_{mxr} \ominus BH^2_{mxr}) \oplus H^{2\perp}_{mxr} = [(H^2_{mx1} \ominus BH^2_{mx1}) \oplus H^{2\perp}_{mx1}] \otimes \mathbb{C}^r.$$

$P_B \otimes I_r$ is the orthogonal projection from H^2_{mxr} onto $H^2_{mxr} \ominus BH^2_{mxr}$. Hence

$$B\bar{f}_1 - (P_B \otimes I_r)B\bar{f}_1 \in (H^2_{mxr})^{\perp} = H^{2\perp}_{mx1} \otimes \mathbb{C}^r$$

and setting

$$(P_B \otimes I_r)B\bar{f}_1 = g_1,$$

we have

$$(\Phi f_2, B\bar{f}_1)_2 = (\Phi f_2, (P_B \otimes I_r)B\bar{f}_1)_2$$

$$= (\Phi f_2, g_1)_2 .$$

Moreover, $P_\beta \otimes I_r$ is the orthogonal projection from H^2_{nxr} onto $H^2_{nxr} \ominus \beta H^2_{nxr}$. The function

$$f_2 - (P_\beta \otimes I_r)f_2 \in \beta H^2_{nxr}$$

and therefore so is

$$\Phi(f_2-(P_\beta \otimes I_r)f_2) \in \Phi\beta H^2_{nxr} = \beta\Phi H^2_{nxr} \subseteq \beta H^2_{mxr} \subseteq BH^2_{mxr},$$

setting

$$g_2 = (P_\beta \otimes I_r)f_2,$$

we have

$$\begin{aligned}(\Phi f_2, g_1)_2 &= (\Phi(P_\beta \otimes I_r)f_2, g_1)_2 \\ &= (\Phi g_2, g_1)_2 \\ &= ((P_B \otimes I_r)\Phi g_2, g_1)_2 \\ &= ((P_B \otimes I_r)\Phi(P_\beta \otimes I_r)^* g_2, g_1)_2 \\ &= ([(P_B M_\Phi P_\beta^*)\otimes I_r]g_2, g_1)_2 ,\end{aligned}$$

and

$$\begin{aligned}\| g_1\|_2^2 &= \| (P_B \otimes I_r)B\bar{f}_1\|_2^2 \\ &= \| B\bar{f}_1\|_2^2 \leq \| \bar{f}_1\|_2^2 = \| f_1\|_2^2 \\ &= \| f_1^* f_1\|_2 = \| (ff^*)^{\frac{1}{2}}\|_2 \\ &= \| f \|_1 .\end{aligned}$$

Similarly $\| g_2\|_2^2 \leq \| f\|_1$.

Conversely, suppose $g_2 \in H^2_{nxr} \ominus \beta H^2_{nxr}$, $g_1 \in H^2_{mxr} \ominus BH^2_{mxr}$

then

$$\bar{B}g_1 \in H^{2\perp}_{mxr} , \quad B\bar{g}_1 \in zH^2_{mxr} ,$$

and we may take

$$f = B\bar{g}_1 g_2^T .$$

Combining Lemmas 2.2.2, 2.2.3 and Theorem 2.2.4 gives us the following theorem.

<u>Theorem 2.2.5</u> The natural isomorphism of $H^{\infty}_{mxn}/BH^{\infty}_{mxn}$ onto $H^{\infty}(\beta, B)$ is norm preserving.

<u>Proof:</u> Let φ be a function in H^{∞}_{mxn} such that

$$1 = \| \varphi + BH^{\infty}_{mxn} \| = \inf_{g \in H^{\infty}_{mxn}} \| \varphi + Bg \| .$$

Let $\varepsilon > 0$.

As $H^{\infty}_{mxn}/BH^{\infty}_{mxn}$ is the dual of $\bar{B}zH^{1}_{mxn}/zH^{1}_{mxn}$ there is a function $f \in zH^{1}_{mxn}$ such that

$$\| f \| = 1 \quad \text{and} \quad |\langle \varphi, \bar{B}f \rangle| > 1-\varepsilon .$$

By Theorem 2.2.4 there are functions

$$g_2 \in H^{2}_{nxr} \ominus \beta H^{2}_{nxr}, \quad g_1 \in H^{2}_{mxr} \ominus BH^{2}_{mxr},$$

with

$$\| g_1 \|_2 \leq 1 \quad \text{and} \quad \| g_2 \|_2 \leq 1$$

such that

$$|\langle \varphi, Bf \rangle| = |((P_B M_{\varphi} P_{\beta}^{*} \otimes I_r) g_2, g_1)_2|$$

$$\leq \| P_B M_{\varphi} P_{\beta}^{*} \otimes I_r \| = \| P_B M_{\varphi} P_{\beta}^{*} \|,$$

i.e. $\| P_B M_{\varphi} P_{\beta}^{*} \| > 1-\varepsilon$.

As ε is arbitary we have

$$\| P_B M_{\varphi} P_{\beta}^{*} \| = 1 = \| \varphi + BH^{\infty}_{mxn} \|_{H^{\infty}_{mxn}/BH^{\infty}_{mxn}} .$$

The proof is complete.

An example can easily be constructed to show that two spaces $H^{2}_{nx1} \ominus \beta H^{2}_{nx1}$, $H^{2}_{mx1} \ominus BH^{2}_{mx1}$ are necessary. Theorem 2.2.5 may fail if $H^{2}_{nx1} \ominus \beta H^{2}_{nx1}$ is replaced by $H^{2}_{nx1} \ominus BH^{2}_{nx1}$, even when m=n. This is not so in the scalar case.

Example 2.2.6 Let $\varphi(z) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, for all $z \in D$ and let $B(z) = \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}$.

Then

$$H^2_{2x1} \ominus BH^2_{2x1} = \left\{ f \in H^2_{2x1} : f(0)=f(z)=\begin{pmatrix} c \\ 0 \end{pmatrix},\ c \in \mathbb{C} \right\}$$

$$= \left\{ \begin{pmatrix} c \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} z + \begin{pmatrix} 0 \\ 0 \end{pmatrix} z^2 + \ldots,\ c \in \mathbb{C} \right\},$$

$$H^2_{2x1} \ominus (\det B)H^2_{2x1} = \left\{ g \in H^2_{2x1} : g(0)=g(z)=\begin{pmatrix} c_1 \\ c_2 \end{pmatrix},\ c_i \in \mathbb{C}, i=1,2 \right\}.$$

For $M_\varphi P_B^* = \left\{ h \in H^2_{2x1} : h(z)=h(0)=\begin{pmatrix} 0 \\ d \end{pmatrix},\ d \in \mathbb{C} \right\}$, so $P_B M_\varphi P_B^* = \{0\}$.

However $\| \varphi + BH^\infty_{2x2} \| = 0$. For if $\| \varphi + BH^2_{2x2} \| = 0$, then $\varphi \in BH^\infty_{2x2}$; i.e.

$$\varphi(z) = \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_{11}(z) & u_{12}(z) \\ u_{21}(z) & u_{22}(z) \end{bmatrix}.$$

This implies

$$u_{12}(z) = \frac{1}{z}, \text{ but } u_{12} \in H^\infty.$$

Therefore

$$\| P_B M_\varphi P_\beta^* \| \neq \| P_B M\, P_B^* \|.$$

By a maximising vector for an operator T we mean a vector $x \neq 0$ and such that

$$\| Tx \| = \| T \| \| x \|.$$

Theorem 2.2.7 Let T be the Sarason operator $P_B M_F P_\beta^*$ acting from $H^2_{nx1} \ominus \beta H^2_{nx1}$ to $H^2_{mx1} \ominus BH^2_{mx1}$. Assume T has maximising vector u_0. Then there is a function G in $F + BH^\infty_{mxn}$ such that

$$\|G\|_\infty = \|T\|$$

and

$$Gu_0 = Tu_0. \qquad (13)$$

Proof: It follows from Theorem 2.2.1 that there exists $G \in F+BH^\infty_{m\times n}$ such that $\|G\|_\infty = \|T\|$. If T has a maximising unit vector u_0,

then

$$\|T\| = \|Tu_0\| = \|P_B M_G P_\beta^* u_0\| = \|P_B M_G u_0\|$$

$$\leq \|Gu_0\| \leq \|T\|.$$

It follows that

$$P_B M_G P_\beta^* u_0 = P_B G u_0 = Tu_0$$

and

$$Gu_0 = Tu_0 .$$

Remark: In the case m=1, G is uniquely determined by (13) when such a u_0 exists : $G = Tu_0/u_0$. But this is not true in the general matrix valued case. However, G would be determined if we had sufficiently many linearly independent maximising vectors. Suppose $\|T\|$ is a singular value of T with multiplicity n: then they ensure that we have exactly n independent maximising vectors $u_1, \ldots u_n$ corresponding to $\|T\|$. By (13) we have that

$$Gu_i = Tu_i ,$$

thus

$$G[u_1, u_2, \ldots u_n] = [Tu_1, Tu_2, \ldots Tu_n] ,$$

and G can be determined by

$$G = [Tu_1, \ldots, Tu_n][u_1, \ldots u_n]^{-1} ,$$

as long as $[u_1(z), u_2(z),\dots, u_n(z)]$ is nonsingular. It is very unlikely that $\|T\|$ is a singular value of T with multiplicity n, and $[u_1(z),\dots, u_n(z)]$ is nonsingular. Motivated by the A-A-K results [3] we extend the two subspaces $H^2_{nx1} \ominus \beta H^2_{nx1}$, $H^2_{mx1} \ominus BH^2_{mx1}$ to $H^2_{nx1} \ominus \underset{\sim}{\beta} H^2_{nx1}$, $H^2_{mx1} \ominus \underset{\sim}{B} H^2_{mx1}$ in such a way that $H^2_{mx1} \ominus BH^2_{mx1} \subseteq H^2_{mx1} \ominus \underset{\sim}{B} H^2_{mx1}$, $H^2_{nx1} \ominus \beta H^2_{nx1} \subseteq H^2_{nx1} \ominus \underset{\sim}{\beta} H^2_{nx1}$ and, for a suitable choice of $\underset{\sim}{F}$, $\|T\|$ is a singular value of the larger generalized Sarason operator $\underset{\sim}{T} = P_{\underset{\sim}{B}} M_{\underset{\sim}{F}} P^*_{\underset{\sim}{\beta}}$ with multiplicity n, $\|T\| = \|\underset{\sim}{T}\|$ and the corresponding maximising vectors $\underset{\sim}{u}_1, \underset{\sim}{u}_2,\dots, \underset{\sim}{u}_n$ with $[\underset{\sim}{u}_1(z),\dots, \underset{\sim}{u}_n(z)]$ is nonsingular. Such so called <u>one step extension operator</u> $\underset{\smile}{T}$ exists and we will examine them in the following section.

2.3 The A-A-K operator and the Sarason operator

There is another way to characterize $H^\infty_{mxn}/BH^\infty_{mxn}$, due to Nehari [28] in the scalar case and A-A-K [3] in the general case, in terms of Hankel operators and Hankel matrices. It is not surprising that they are closely related. Young [46] pointed out that the Sarason operator is a unitary multiple of the nonzero part of the A-A-K operator. In fact this is still true in the matrix valued case. The one step extension idea in the A-A-K approach is one of the methods we are using to form an extremal function. The existence of such a one step extension follows from the fundamental study by A-A-K in [3]. In particular, if the so called symbol function is rational then the existence of such a one step extension is equivalent to the existence of a solution to a matrix quadratic equation

-Riccati equation. This follows from Kung [22] [23]. In view of the relation between the Sarason operator and the A-A-K operators the one step extension method can also be used in our approach. This is different from Arsene, Ceausescu and Foias' 1-PCID method [5] in this particular situation.

Let $\ell^2_{m\times 1}$ denote the set of all square summable infinite sequences, i.e. $\{\xi_j\}_{j=-\infty}^{\infty}$, $\xi_j \in M_{m\times 1}$ such that

$$\|\xi\|_2^2 = \sum_{j=-\infty}^{\infty} \text{trace } (\xi_j \xi_j^*) < \infty$$

and by $\ell^{2+}_{m\times 1}$, $\ell^{2-}_{m\times 1}$, respectively, the set of all square summable sequences such that $\xi_j=0$, $j=-1,-2,\ldots\ldots\ldots$ and $\xi_j=0$, $j=0,1,2,\ldots.$ The generating function ξ in $L^2_{m\times 1}$, corresponding to $\{\xi_j\}_{j=-\infty}^{+\infty}$ is defined by

$$\xi(z) = \sum_{-\infty}^{\infty} \xi_j z^j \;, \quad z \in \partial D.$$

The L^2-norm is defined as

$$\|\xi\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} \text{trace } (\xi(e^{i\theta})\xi(e^{i\theta})^*)d\theta$$

and ξ_j is the j-th Fourier coefficient of $\xi(z)$;

$$\xi_j = \hat{\xi}(j) = \frac{1}{2\pi}\int_0^{2\pi} \xi(e^{i\theta})e^{-ij\theta}d\theta.$$

We shall not distinguish between $\ell^2_{m\times 1}$ and $L^2_{m\times 1}$, $\ell^{2+}_{m\times 1}$ and $H^2_{m\times 1}$, $\ell^{2-}_{m\times 1}$ and $H^{2-}_{m\times 1} = L^2_{m\times 1} \ominus H^2_{m\times 1}$. For any $f \in L^2_{m\times 1}$ there exists a unique partition

$$f(z) = \sum_{i=-\infty}^{\infty} f_i z^i = f_+(z) + f_-(z),$$

where

$$f_+(z) = \sum_{i=0}^{\infty} f_i z^i \;, \quad f_+ \in H^2_{m\times 1} \qquad \text{and}$$

$$f_-(z) = \sum_{i=1}^{\infty} f_{-i} z^{-i}, \quad f_- \in H^{2-}_{m\times 1} \;.$$

Let π_2^+, π_2^- be the orthogonal projections from $L^2_{m\times 1}$ onto $H^2_{m\times 1}$ and $H^{2-}_{m\times 1}$. Let P_j be the orthogonal projection maps $(\xi_0, \xi_1, \ldots\ldots) \in \ell^+_{m\times 1}$ onto its j-th component ξ_j. Let $\{e_j\}_{j=1}^m$ be the standard orthonormal basis for $\mathbb{C}^m$; then $\{z^j \otimes e_i \mid -\infty < j < \infty,\ i=1,2,\ldots m\}$ is a basis for $L^2_{m\times 1}$, where $z^j \otimes e_i$ denotes the mx1 column vector with value z^j at i th coordinate and 0 elsewhere. We consider this basis ordered as follows $\ldots\ldots, z^j\otimes e_1,\ z^j\otimes e_2, \ldots\ldots, z^j\otimes e_m,$ $z^{j+1}\otimes e_1, \ldots\ldots,\ z^{j+1}\otimes e_m, \ldots\ldots$.

Let R be the operator on $L^2_{m\times 1}$ defined by

$$Rf(z) = \frac{1}{z} f\left(\frac{1}{z}\right), \quad z \in \partial D.$$

i.e.

$$R\left(\sum_{-\infty}^{\infty} f_i z^i\right) = \sum_{-\infty}^{\infty} f_i z^{-i-1} .$$

Obviously, $f(z) = \frac{1}{z} Rf\left(\frac{1}{z}\right)$ and if $f \in H^2_{m\times 1}$, then $Rf \in H^{2-}_{m\times 1}$, and vice versa.

<u>Lemma 2.3.1</u> (1) $R\pi_2^+ R = \pi_2^-$.

(2) $RR^* = R^*R = I$.

<u>Proof:</u> The proof is straight-forward.

(1) Let $f \in L^2_{m\times 1}$ and $f(z) = \sum_{-\infty}^{\infty} f_i z^i$, then

$$Rf(z) = \sum_{-\infty}^{\infty} f_i z^{-i-1} ,$$

thus

$$\pi_2^+ Rf(z) = f_{-1} + f_{-2} z + \ldots ,$$

and

$$R\pi_2^+ Rf(z) = \frac{1}{z}\left(f_{-1} + \frac{f_{-2}}{z} + \ldots.\right)$$

$$= f_-(z) = \pi_2^- f(z).$$

(2) it follows that the adjoint of R is given by

$$R^*f(z) = \frac{1}{z} f(\frac{1}{z}).$$

The proof is complete.

Given a function $\varphi \in L^\infty_{mxn}$, the Hankel operator H_φ acting from H^2_{nx1} to H^2_{mx1} is defined by

$$H_\varphi f = \pi_2^+ R \varphi f, \quad f \in H^2_{nx1},$$

where φ is called the symbol (transfer function) of H_φ. We can write down the Hankel matrix Γ_φ for H_φ in terms of the Fourier coefficients of φ, namely if $\varphi \in L^\infty_{mxn}$ and $\varphi(z) = \sum_{i=1}^\infty \varphi_i z^{-i}$, $\varphi_i \in M_{mxn}$, then the matrix Γ_φ has the entry φ_{j+k-1} in the j-th row, k-th column position; it is constant on the cross diagonals.

$$\Gamma_\varphi \equiv \begin{bmatrix} \varphi_1, & \varphi_2, & \varphi_3, \ldots\ldots\ldots \\ \varphi_2, & \varphi_3, & \varphi_4, \ldots\ldots\ldots \\ & & \\ & & \end{bmatrix}.$$

In other words, Γ_φ is the matrix of H_φ with respect to the standard orthonormal basis $\{z^j \otimes e_i : j=0,1,2,\ldots, i=1,2,\ldots m\}$ and φ_{j+k-1} is the j+k-1 Fourier coefficient of φ.
Suppose $\{\xi_j\}_{j=1}^\infty \in \ell^{2+}_{nx1}$, $\{\eta_j\} \in \ell^{2+}_{mx1}$
and

$$\Gamma_\varphi \{\xi_j\} = \{\eta_j\},$$

then

$$\eta_j = \sum_{k=1}^\infty \varphi_{j+k-1} \xi_k .$$

Before we go any further, let us observe the following example which shows the key point of the relation between the Sarason operator and the A-A-K operator.

<u>Example 2.3.3</u> Let $B(z) = \begin{bmatrix} z^2 & 0 \\ 0 & z^2 \end{bmatrix}$; then $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}z\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\z\end{pmatrix}$

is a basis of $H^2_{2x1} \ominus BH^2_{2x1}$.

for any $g \in H^2_{2x1}$,

$$P_Bg(z) = P_B\begin{pmatrix} g_1(z) \\ g_2(z) \end{pmatrix} = \begin{pmatrix} g_1(0)+g'(0)z \\ g_2(0)+g_2'(0)z \end{pmatrix} \qquad (15)$$

$$= g(0)+g'(0)z.$$

Let $\tilde{B}(z) = [B(z)]^*$ and consider

$$H_{\tilde{B}}g(z) = \Pi_2^+ R\tilde{B}g(z) = \Pi_2^+(zg(\tfrac{1}{z})) = g'(0)+g(0)z. \qquad (16)$$

Compare (15) (16): we see that

$$z^2(\tfrac{1}{z}(g'(0)+\tfrac{g(0)}{z})) = z^2R(g'(0)+g(0)z)$$

$$= BR(g'(0)+g(0)z). \qquad (17)$$

The formula (17) in fact is the relation we are looking for.

<u>Theorem 2.3.4</u> Let B be an inner matrix in H^{∞}_{mxm}.

Then for all $g \in H^2_{mx1}$,

$$P_Bg = BRH_{\tilde{B}}g = BR\Pi_2^+R\tilde{B}g \ , \qquad (18)$$

where

$$\tilde{B}(z) = [B(z)]^* \ .$$

<u>Proof:</u> For any $g \in H^2_{m\times 1}$

$$g - B\pi_2^+\tilde{B}g = B\tilde{B}g - B\pi_2^+\tilde{B}g$$

$$= B(\tilde{B}g - \pi_2^+\tilde{B}g)$$

$$= B((1-\pi_2^+)\tilde{B}g) = BR\pi_2^+R\tilde{B}g.$$

Pick $f \in H^2_{m\times 1} \ominus BH^2_{m\times 1}$; then $P_B g = g$ and $\tilde{B}f \in H^{2\;\perp}_{m\times 1}$, so

$$BRH_{\tilde{B}}f = f - B\pi_2^+\tilde{B}f = f.$$

On the other hand, for $f \in BH^2_{m\times 1}$, so $P_B f = 0$.

Let $f = Bh$, $h \in H^2_{m\times 1}$;

then

$$BRH_{\tilde{B}}f = f - B\pi_2^+\tilde{B}Bh$$

$$= f - B\pi_2^+h = f - Bh = 0.$$

The proof is complete.

<u>Remark:</u> If $f \in \beta H^2_{n\times 1}$, then $f = \beta g$ for $g \in H^2_{n\times 1}$ and

$$BRH_{\tilde{B}}f = f - B\pi_2^+\tilde{B}f$$

$$= \beta g - B\pi_2^+\tilde{B}B\cdot \mathrm{adj}Bg$$

$$= \beta g - B\ \mathrm{adj}Bg = \beta g - \beta g = 0.$$

Therefore, if $f \in H^2_{n\times 1}$, then $BRH_{\tilde{B}}f = BRH_{\tilde{B}}P_\beta^* f$.

The fundamental fact about Hankel operators is due to Nehari and A-A-K.

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Theorem 2.3.5 Let $\varphi \in L^\infty_{m\times n}$ and let H_φ be the Hankel operator with symbol φ. Then

$$\| H_\varphi \| = \operatorname{dis}(\varphi, H^\infty_{m\times n}) \overset{def}{=} \inf_{g \in H^\infty_{m\times n}} \| \varphi - g \|_\infty .$$

Proof: Reference also Gohberg [7] [30].

Theorem 2.3.6 Let B be an inner matrix in $H^\infty_{m\times m}$ and let $F \in H^\infty_{m\times n}$. Then

(1) $P_B M_F = BRH_{\tilde{B}F}$,

(2) $$\| F+BH^\infty_{m\times n} \|_{H^\infty_{m\times n}/BH^\infty_{m\times n}} = \operatorname{dis}(\tilde{B}F, H^\infty_{m\times n}) \qquad (19)$$

$$= \| H_{\tilde{B}F} \| = \| P_B M_F P_\beta^* \| .$$

Proof: (2) follows from Theorem 2.2.5 and 2.3.5.

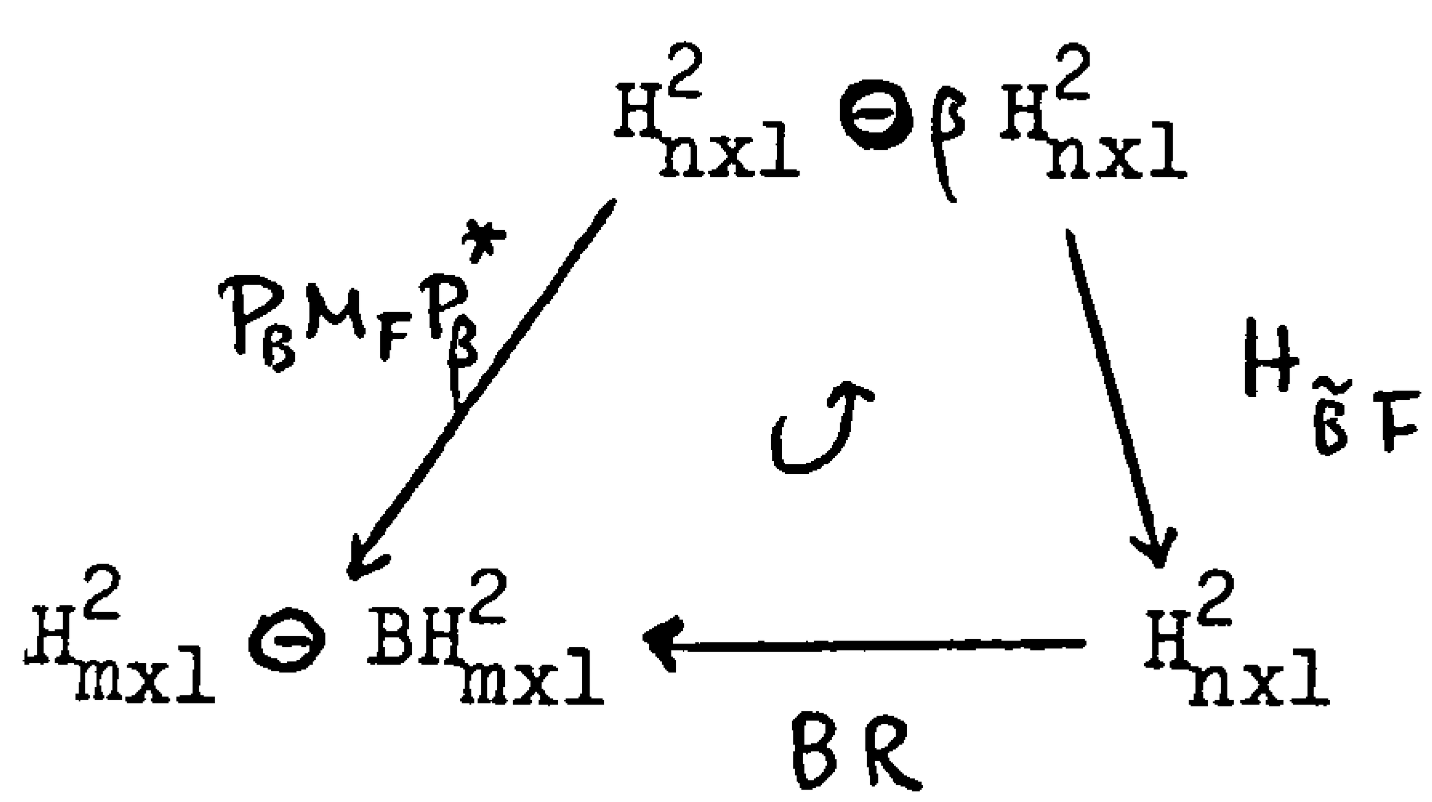
(1) For $g \in H^2_{m\times 1}$,

$$P_B M_F g = P_B F g = BR\,\pi_2^+ R\tilde{B}Fg$$
$$= BRH_{\tilde{B}F}\, g .$$

As we have proved in Theorem 2.3.6 that

$$P_B M_F P_\beta^* = BRH_{\tilde{B}F} \Big|_{H^2_{n\times 1} \ominus \beta H^2_{n\times 1}} ,$$

and BR is a unitary operator. The Sarason operator is therefore a unitary multiple of the nonzero part of the A-A-K operator.



(20)

The most remarkable result of the A-A-K theory concerns the one step extension. For completeness we restate the one step extension of the A-A-K operator and the corresponding extension in the Sarason operator framework. This extension technique is one of the methods whereby extremal functions can be calculated.

Definition 2.3.7 Let $\varphi(z) = \sum_{i=1}^{\infty} \varphi_i z^{-i}$, $\varphi \in L^{\infty}_{mxn}$.
A one step extension of the Hankel matrix Γ_φ, denoted by $\Gamma_{\tilde{\varphi}}$ is a Hankel matrix with symbol $\tilde{\varphi}$, where $\tilde{\varphi}(z) = \frac{\varphi_0}{z} + \frac{\varphi_1}{z^2} + \ldots$
i.e.

$$\Gamma_{\tilde{\varphi}} = \begin{bmatrix} \varphi_0, \varphi_1, \varphi_2, \ldots\ldots \\ \varphi_1, \varphi_2, \quad \ldots\ldots\ldots \\ \vdots \quad \vdots \end{bmatrix} \qquad (21)$$

where φ_i are mxn matrices.

The one step extension problem is: Given a Hankel matrix Γ_φ, $\|\Gamma_\varphi\| \leq \rho$, does there exist a one step extension Hankel matrix $\Gamma_{\tilde{\varphi}}$ such that ρ is a singular value of $\Gamma_{\tilde{\varphi}}$ with multiplicity n .

The existence of such a φ_0 and the description of all of them we will summarize in the following theorem. For details see [3] [23] or Dym and Gohberg [13].

Theorem 2.3.8 Given a rational function φ of finite order such that $\|\Gamma_\varphi\| \leq \rho$ the one step extensions $\Gamma_{\tilde{\varphi}}$ as in (21) such that ρ is a singular value of $\Gamma_{\tilde{\varphi}}$ with multiplicity n are defined by those and only those $\varphi_0 \in M_{mxn}$ which are of the form

$$\varphi_0 = \rho A U B + C \ , \qquad (22)$$

Where U is an arbitary mxn isometric matrix.

$$A^2 = 1 - P_1\Gamma_\varphi(\rho^2-(\underset{\sim}{S}\Gamma_\varphi)^*(\underset{\sim}{S}\Gamma_\varphi))^{-1}\Gamma_\varphi{}^*P_1{}^*,$$

$$B^2 = 1 - P_1\Gamma_\varphi{}^*(\rho^2-\underset{\sim}{S}\Gamma_\varphi(\underset{\sim}{S}\Gamma_\varphi)^*)^{-1}\Gamma_\varphi P_1{}^*,$$

$$C = P_1\Gamma_\varphi(\rho^2-(\underset{\sim}{S}\Gamma_\varphi)^*(\underset{\sim}{S}\Gamma_\varphi))^{-1}(\underset{\sim}{S}\Gamma_\varphi)^*\Gamma_\varphi{}^*P_1{}^*.$$

<u>Remark:</u> Let $\{(\xi^{(i)},\eta^{(i)}),\ i = 1,2,\ldots\ldots,n\}$ be a set of linearly independent Schmidt pairs of Γ_φ corresponding to ρ ; i.e.

$$\Gamma_\varphi\xi^{(i)} = \rho\eta^{(i)}, \quad \Gamma_\varphi^*\eta^{(i)} = \rho\xi^{(i)}.$$

Let $x = (x_0, x_1,\ldots\ldots)$ and $y = (y_0, y_1,\ldots\ldots)$ be the corresponding sequences of nxn matrices in H^2_{nxn} with

$$x_j = (\xi_j^{(1)},\xi_j^{(2)},\ldots\ldots,\xi_j^{(n)}),$$

and

$$y_j = (\eta_j^{(i)},\eta_j^{(2)},\ldots\ldots,\eta_j^{(n)}),$$

$j = 0,1,2,3,\ldots\ldots$,

and let

$$x(z) = \textstyle\sum_{i=0}^{\infty} x_i z^i, \quad y(z) = \sum_{i=0}^{\infty} y_i z^i .$$

Then x(z) and y(z) are nonsingular for each $z \in D$. For a complete proof of this property see H. Dym and I. Gohberg [13] or Kung [23] for the details.

From Theorem 2.3.6 if B is an inner matrix in H^∞_{mxm}, $F \in H^\infty_{mxn}$ then

$$P_B M_F P_\beta^* = BRH_{\tilde{B}F}P_\beta^* .$$

Let $T = P_B M_F P_\beta^*$, $U_B = BR$ and let $\varphi = BF$. If (ξ,η) is the Schmidt pair for ρ of H_φ , i.e.

$$H_\varphi\xi = \rho\eta, \quad H_\varphi^*\eta = \rho\xi .$$

It follows that

$$T\xi = U_B H_\varphi \xi = \rho U_B \eta \quad \text{and} \quad T^* U_B \eta = \rho \xi .$$

This implies that $(\xi, U_B\eta)$ is the Schmidt pair corresponding to ρ of T. Conversely if $(\xi, U_B\eta)$ is the Schmidt pair for ρ of T, then (ξ, η) is the Schmidt pair corresponding to ρ of H_φ. Suppose $H_{\tilde{\varphi}}$ is the one step extension of H_φ: then the corresponding Sarason operator $\tilde{T}$ of $H_{\tilde{\varphi}}$ is

$$\tilde{T} = U_{\tilde{B}} \Gamma_{\tilde{\varphi}} .$$

Since

$$\tilde{\varphi}(z) = \frac{\varphi_0}{z} + \frac{1}{z}\varphi(z) = \tilde{\tilde{B}}\hat{\tilde{F}} ,$$

for some φ_0 is the form of (22),

so

$$\tilde{B}\tilde{\varphi}(z) = \tilde{B}(z)\left(\frac{\varphi_0}{z} + \frac{1}{z}\varphi(z)\right) = \tilde{F}(z).$$

Letting

$$\tilde{B}(z) = zB(z),$$

we have

$$B(z)\varphi_0 + B(z)\varphi(z) = \tilde{F}(z),$$

and

$$B\varphi = F .$$

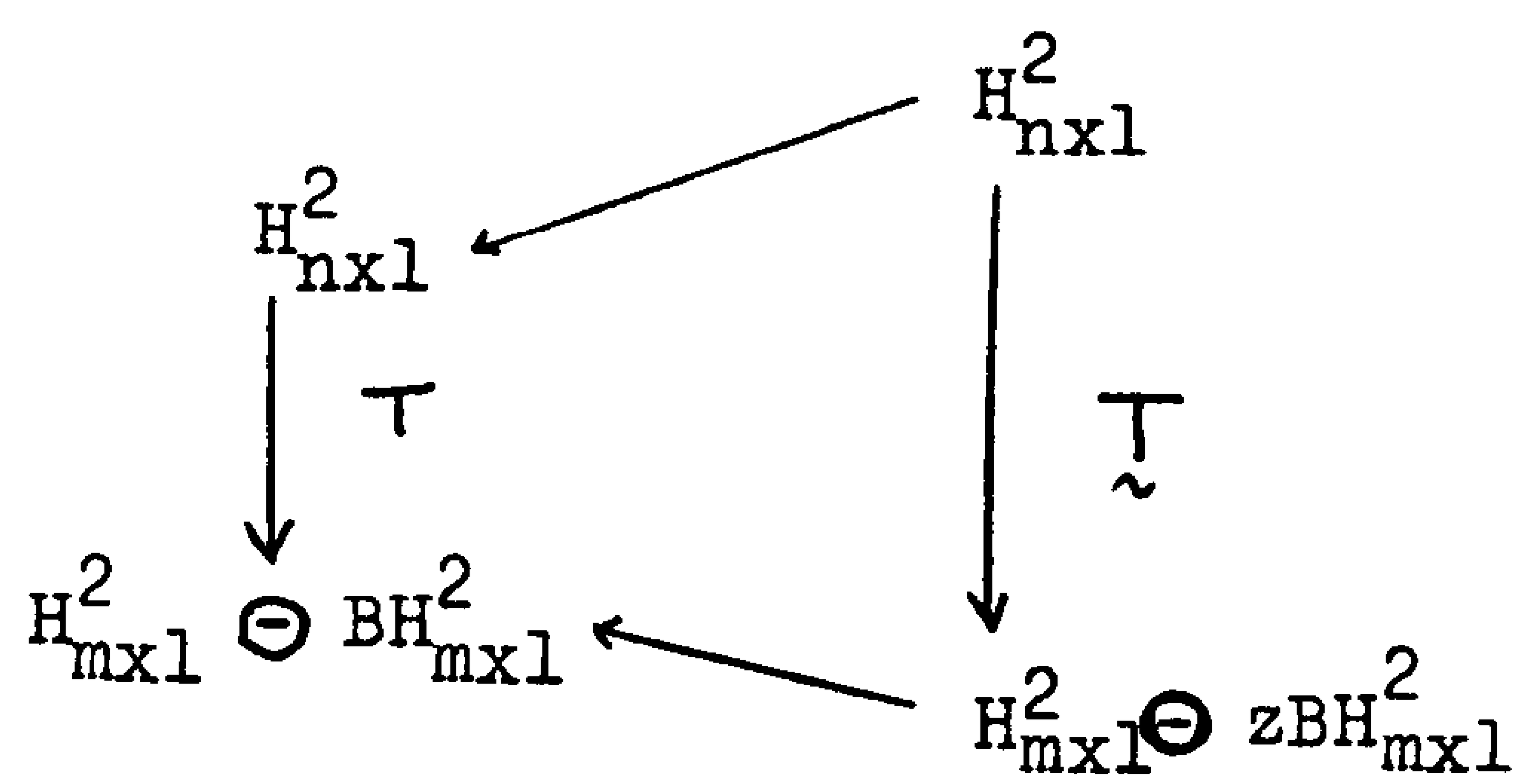
Therefore

$$\tilde{F}(z) = F(z) + B(z)\varphi_0 ,$$

that is

$$\tilde{T} = P_{\tilde{B}} M_{\tilde{F}} ,$$

with $\tilde{B} = zB$ and $\tilde{F} = F + B\varphi_0$ for φ_0 is the form of (22).



<u>Theorem 2.3.9</u> Given B an inner matrix in RH^{∞}_{mxm} and $F \in RH^{\infty}_{mxn}$, and let the generalized Sarason operator $T = P_B M_F P^*_\beta$ satisfy $\| T \| \leq \rho$. Then there exists $F_0 \in M_{mxn}$ such that if $\underset{\sim}{F} = F + BF_0$ then ρ is a singular value of $\underset{\sim}{T} = P_{zB} M_{\underset{\sim}{F}} P^*_{z \beta I}$ with multiplicity n.

<u>Proof:</u> Let H_φ be the Hankel operator with symbol $\varphi = \tilde{B}F$ and by the Theorem 2.3.6 $T = U_B H_\varphi P^*_\beta$, where $U_B = BR$ and

$$\| U_B H_\varphi P^*_\beta \| = \| T \| \leq \rho.$$

Choosing F_0 to be the form of (22)

and let

$$\underset{\sim}{\varphi} = \frac{F_0}{z} + \frac{\varphi}{z},$$

then from Theorem 2.3.8 ρ is a singular value of $H_{\underset{\sim}{\varphi}}$ of multiplicity n. In view of the relation of the Sarason operator and the A-A-K operator

$$\underset{\sim}{\varphi} = \tilde{\underset{\sim}{B}} \underset{\sim}{F}$$

for some $\underset{\sim}{F} \in H^{\infty}_{mxn}$, and some inner matrix $\underset{\sim}{B} \in H^{\infty}_{mxm}$.

Letting

$$\underset{\sim}{F} = F + BF_0,$$

then

$$F + BF_0 = \underset{\sim}{B}\underset{\sim}{\varphi} = \underset{\sim}{B}\left(\frac{F_0}{z} + \frac{\varphi}{z}\right)$$
$$= \underset{\sim}{B}\left(\frac{F_0}{z} + \frac{\tilde{B}F}{z}\right).$$

Thus setting $\underset{\sim}{B}$ to be zB which is inner

and

$$\underset{\sim}{B} R H_{\tilde{\underset{\sim}{B}}\underset{\sim}{F}} = P_{zB} M_F P^*_{z \beta I} = \underset{\sim}{T},$$

$$\| \underset{\sim}{T} \| = \| \Gamma_{\underset{\sim}{\varphi}} \| = \rho .$$

<u>Remark:</u> This one step extension is different from Arsene's 1-PC1D [5] and is much earier to handle computationally.

2.4 One step extension matrices

Let $F \in RH^{\infty}_{mxn}$ and let $B \in RH^{\infty}_{mxm}$ be an inner matrix. The generalized Sarason operator is

$$T = P_B M_F P^{*}_{\beta} \; : H^2_{nx1} \ominus \beta H^2_{nx1} \longrightarrow H^2_{mx1} \ominus BH^2_{mx1}.$$

Let $\|T\| = \rho$, then by Theorem 2.3.9, there always exists $F_0 \in M_{mxn}$ such that if $\tilde{F} = F + BF_0$, $\tilde{B} = zB$, $\tilde{\beta} = z\beta$, then ρ is a singular value of $\tilde{T} = P_{\tilde{B}} M_{\tilde{F}} P^{*}_{\tilde{\beta}}$ with multiplicity n. In view of the relation between a Hankel operator and the Sarason operator such an F_0 can be obtained in terms of a Sarason-type formula.

Let P_z, P_{zB} be the orthogonal projections from H^2_{mx1} onto $H^2_{mx1} \ominus zH^2_{mx1}$, $H^2_{mx1} \ominus zBH^2_{mx1}$, respectively. Then we have the following simple lemma.

<u>Lemma 2.4.1</u> $\quad P_{zB} = P_B + BP_z M_{\tilde{B}}$.

<u>Proof:</u> Pick $f \in H^2_{mx1} \ominus zBH^2_{mx1}$; then $P_{zB} f = 0$. It is not hard to varify that $H^2_{mx1} \ominus zBH^2_{mx1}$ can be decomposed into $H^2_{mx1} \ominus BH^2_{mx1}$ and $B(H^2_{mx1} \ominus zH^2_{mx1})$ (see Lemma 4.1.2), i.e.

$$H^2_{mx1} \ominus zBH^2_{mx1} = (H^2_{mx1} \ominus BH^2_{mx1}) \oplus B(H^2_{mx1} \ominus zH^2_{mx1})$$

and

$$f = f_1 + f_2 ,$$

where

$$f_1 \in H^2_{mx1} \ominus BH^2_{mx1} \quad \text{and} \quad f_2 \in B(H^2_{mx1} \ominus zH^2_{mx1}).$$

Then

$$(P_B + BP_z M_{\tilde{B}})(f) = P_B f_1 + BP_z M_{\tilde{B}} f_2 .$$

Let

$$f_2 = Bg \quad \text{for some} \quad g \in H^2_{mx1} \ominus zH^2_{mx1},$$

then

$$BP_zM_{\tilde{B}}Bg = BP_zg = Bg = f_2 \ .$$

Hence

$$P_{zB}f = (P_B + BP_zM_{\tilde{B}})(f) \quad \text{for } f \in H^2_{mx1} \ominus zBH^2_{mx1}.$$

On the other hand, for $f \in zBH^2_{mx1}$, $f \in BH^2_{mx1}$ and so $P_Bf = 0$. Let $f = zBg$, $g \in H^2_{mx1}$;

then

$$BP_zM_{\tilde{B}}zBg = BP_zzg = 0.$$

Therefore

$$P_{zB}f = (P_B + BP_zM_{\tilde{B}})(f), \quad \text{for } f \in zBH^2_{mx1} \ .$$

The proof is complete.

By a simple calculation, the one step extension Sarason operator $\underset{\sim}{T}$ and T have the following relation

$$\underset{\sim}{T} = P_{zB}M_{F+BF_0}P^*_{z\beta}$$

$$= (P_B + BP_zM_{\tilde{B}})(F + BF_0)P^*_{z\beta}$$

$$= T + BP_zM_{\tilde{B}F}P^*_{z\beta} + BP_zF_0P^*_{z\beta}. \tag{23}$$

Let

$$h = P_zM_{\tilde{B}F} \quad \text{and} \quad \xi = P_z \ ,$$

and let

$$T_1 = BP_zM_{\tilde{B}F}P^*_{z\beta} + BP_zF_0P^*_{z\beta}.$$

Lemma 2.4.2 Let $u_0 \in H^2_{n\times 1} \ominus \beta H^2_{n\times 1}$ be a maximising vector for T. Then

$$\| \tilde{T} \| = \| T \| \text{ if and only if } h(u_0)+F_0u_0(0) = 0$$

and

$$\| \tilde{T} \big|_{u_0^\perp} \| \leq \| T \| .$$

Proof: If $\| \tilde{T} \| = \| T \|$, then $\| \tilde{T} \big|_{u_0^\perp} \| \leq \| T \|$, so we have to show $h(u_0) + F_0u_0(0) = 0$. Suppose $h(u_0) + F_0u_0(0) \neq 0$, then $T_1u_0 \neq 0$. Since

$$\| \tilde{T} \|^2 = \| T \|^2 + \| T_1 \|^2 ,$$

so

$$\| \tilde{T} \| > \| T \| .$$ This is contradiction.

Conversely, if $h(u_0) + F_0u_0(0) = 0$, then $T_1u_0 = 0$, and

$$\| \tilde{T}u_0 \|^2 = \| Tu_0 \|^2 = \| T \|^2 \| u_0 \|^2 ,$$

thus

$$\| \tilde{T}u_0 \| = \| T \| \| u_0 \| .$$

By hypothesis $\| \tilde{T} \big|_{u_0^\perp} \| \leq \| T \|$, this implies $\| \tilde{T} \| \leq \| T \|$. But $\| T \| \leq \| \tilde{T} \|$. Therefore $\| T \| = \| \tilde{T} \|$. The proof is complete.

In fact, we also require ρ to be a singular value of $\tilde{T}$ with multiplicity n. This means that $\| \tilde{T} \| = \rho$, $\| \tilde{T} \big|_{u_0^\perp} \| = \rho$ and the nullity of $\rho^2 - \tilde{T}_0^*\tilde{T}_0$ is not less than n-1, where $\tilde{T}_0 = \tilde{T}\big|_{u_0^\perp}$. Let us denote the nullity of $\rho^2-\tilde{T}_0^*\tilde{T}_0$ by $\nu(\rho^2-\tilde{T}_0^*\tilde{T}_0)$. In order to characterize the required F_0 a well known result on the factorization is used.

Lemma 2.4.3 Let K, G, H be Hilbert spaces, and $A \in L(K, G)$, $B \in L(K, H)$. Then $A^*A \leq B^*B$ if and only if there exists a contraction $X \in L(H, G)$ ($\|X\| \leq 1$) such that $A = XB$.

Proof: see Lemma 2.1 of [11].

Lemma 2.4.4 Let $Q \in L(u_0^{\perp})$, $X \in L(u_0^{\perp}, \mathbb{C}^m)$. Then

$$\nu(Q^*(1-X^*X)Q) \geq \nu(1-X^*X).$$

Proof:

$$\nu(Q^*(1-X^*X)Q) \geq \nu(Q) + \dim(\mathrm{Ker}(1-X^*X) \cap \text{Range } Q)$$

$$= \dim u_0^{\perp} - \dim \text{Range } Q + \nu(1-X^*X)$$

$$+ \dim \text{Range } Q - \dim(\ker(1-X^*X) \vee \text{Range } Q)$$

$$\geq \nu(1-X^*X).$$

Theorem 2.4.5 Let u_0 be a maximising vector for T. Then ρ is a singular value of $\underset{\sim}{T}$ with multiplicity n, and $\|\underset{\sim}{T}\| = \|T\| = \rho$, if and only if $F_0 \in M_{m\times n}$ satisfies

(1) $h(u_0) + F_0 u_0(0) = 0,$ (24)

(2) there exists $X \in L(u_0^{\perp}, \mathbb{C}^m)$ such that $\|X\| = 1$,

$\nu(Q^*(1-X^*X)Q) \geq n-1$ and $h_0 + F_0\mathcal{E}_0 = XQ$ (25)

where

$$h_0 = h\,\Big|_{u_0^{\perp}}, \quad \mathcal{E}_0 = \mathcal{E}\,\Big|_{u_0^{\perp}} \quad \text{and} \quad T_0 = T\,\Big|_{u_0^{\perp}}$$

$$Q^*Q = \rho^2 - T_0^*T_0, \quad \text{where} \quad Q \in L(u_0^{\perp}).$$

Proof: ρ is a singular value of $\underset{\sim}{T}$ with multiplicity n and $\| \underset{\sim}{T} \| = \rho$ if and only if $\| \underset{\sim}{T} \| = \rho$, $\| \underset{\sim}{T}|_{u_0^\perp} \| = \rho$ and $\nu(\rho^2 - \underset{\sim}{T}_0^* \underset{\sim}{T}_0) = n-1$. $\| \underset{\sim}{T}|_{u_0^\perp} \| = \rho$ and $\nu(\rho^2 - \underset{\sim}{T}_0^* \underset{\sim}{T}_0) \geq n-1$ if and only if $\rho^2 - \underset{\sim}{T}_0^* \underset{\sim}{T}_0 \geq 0$ and $\nu(\rho^2 - \underset{\sim}{T}_0^* \underset{\sim}{T}_0) \geq n-1$. $\rho^2 - \underset{\sim}{T}_0^* \underset{\sim}{T}_0 \geq 0$ is equivalent to

$$(h_0 + F_0 \mathcal{E}_0)^* (h_0 + F_0 \mathcal{E}_0) \leq \rho^2 - T_0^* T_0. \qquad (26)$$

Let $Q \in L(u_0^\perp)$ be such that $Q^* Q = \rho^2 - T_0^* T_0$. Then by Lemma 2.4.3, (26) holds if and only if there exists $X \in L(u_0^\perp, \mathbb{C}^m)$, $\| X \| = 1$ such that $h_0 + F_0 \mathcal{E}_0 = XQ$. Moreover,

$$\begin{aligned} \nu(\rho^2 - \underset{\sim}{T}_0^* \underset{\sim}{T}_0) &= \nu(\rho^2 - T_0^* T_0 - (h_0 + F_0 \mathcal{E}_0)^* (h_0 + F_0 \mathcal{E}_0)) \\ &= \nu(Q^* Q - Q^* X^* X Q) \\ &= \nu(Q^* (1 - X^* X) Q) \\ &\geq \nu(1 - X^* X). \end{aligned}$$

Therefore, combine Lemma 2.4.2 and the above to complete the proof.

Observe from (24) and (25) that X satisfies a finite number of relations of the type $X x_i = y_i$; if $g \in u_0^\perp$ and $g(0) = 0$, then

$$\begin{aligned} (XQ)(g) &= (h_0 + F_0 \mathcal{E})(g) \qquad (27) \\ &= h_0(g) + F_0 g(0) \\ &= h_0(g), \end{aligned}$$

and if $g \in u_0^\perp$ and $g(0) = u_0(0)$, then

$$\begin{aligned} (XQ)(g) &= (h_0 + F_0 \mathcal{E})(g) \qquad (28) \\ &= h_0(g) + F_0 u_0(0) = h(g - u_0). \end{aligned}$$

Indeed, if $X \in L(u_0^\perp, \mathbb{C}^m)$ such that $\|X\| = 1$, $\nu(1-X^*X) \geq n-1$ and satisfies (26)&(27), then we can construct F_0 such that (24) and (25) are satisfied. In other words, ρ is a singular value of $\underset{\sim}{T}$ with multiplicity n and $\|\underset{\sim}{T}\| = \rho$. Before we prove our main theorem, we need two simple results.

<u>Lemma 2.4.6</u> $\{\varepsilon(g) : g \in u_0^\perp\} \subsetneq \mathbb{C}^n$ if and only if u_0 is a constant, where $u_0 \in H^2_{n\times 1} - \{0\}$.

<u>Proof:</u> If $\{\varepsilon(g) : g \in u_0^\perp\}$ is properly contained in $\mathbb{C}^n$, then there exists $a \in \mathbb{C}^n$ such that $a \neq 0$ and

$$a \perp \varepsilon(g), \quad \text{for all } g \in u_0^\perp .$$

Let $\underset{\sim}{a}(z) = a$; then $a \perp \varepsilon(g)$, so $(a, g(0))_{\mathbb{C}^n} = 0$, i.e.

$$\frac{1}{2\pi i}\oint_C \underset{\sim}{a}(z)^* g(z)\, \frac{dz}{z} = 0,$$

this implies $\underset{\sim}{a} \in u_0^{\perp\perp}$, thus $\underset{\sim}{a} = ku_0$, i.e. $\underset{\sim}{a}(z) = ku_0(z)$, therefore, $u_0(z) = \frac{a}{k}$ is a constant function.
Conversely, if u_0 is a constant function, then $u_0(z) = u_0(0)$.
Let $g \in u_0^\perp$, then $g \perp u_0$, so

$$\frac{1}{2\pi i}\oint_C u_0(z)^* \frac{g(z)}{z}\, dz = 0$$

i.e.

$$u_0(0)^* g(0) = 0 .$$

This means $u_0(0) \perp \{\varepsilon(g) : g \in u_0^\perp\}$ so that

$$\{\varepsilon(g) : g \in u_0^\perp\} \subsetneq \mathbb{C}^n .$$

<u>Lemma 2.4.7</u> Let $X \in L(u_0^\perp, \mathbb{C}^m)$. Then

$$\nu(1 - X^*X) = \nu(1 - XX^*).$$

Proof: Define a mapping $j: Ker(1-X^*X) \longrightarrow Ker(1-XX^*)$ by

$$j(f) = Xf.$$

Then the Lemma is proved if j is bijective. First, let us show the injectivety; if $f \in Ker(1-X^*X)$ and $Xf = 0$, then $X^*Xf = f = 0$. Surjectivity, let $g \in Ker(1-XX^*)$, then there exists $X^*g \in Ker(1-X^*X)$ such that $j(X^*g)=g$, for $(1-XX^*)g = 0$, then

$$j(X^*g) = X X^*g = g ,$$

and

$$(1-X^*X)X^*g = X^*g - X^*XX^*g$$

$$= X^*g - X^*g = 0 .$$

Theorem 2.4.8 Let $X \in L(u_0^\perp, \mathbb{C}^m)$, $\|X\| = 1$ and $\nu(1-X^*X) \geq n-1$ and

$$XQ(g) = h(g) \quad \text{if } g \in u_0^\perp,\ g(0) = 0 , \qquad (27)$$

$$XQ(g) = h(g-u_0) \quad \text{if } g \in u_0^\perp,\ g(0) = u_0(0). \qquad (28)$$

Then there exists $F_0 \in M_{mxn}$ such that

$$h(u_0) + F_0u_0(0) = 0 ,$$

and

$$h_0 + F_0\mathcal{E}_0 = XQ .$$

Furthermore, if $\underset{\sim}{F} = F + BF_0$, $\underset{\sim}{B} = zB$, $\underset{\sim}{\beta} = z\beta$, then ρ is a singular value of $\underset{\sim}{T} = P_{zB}M_{\underset{\sim}{F}}P_{\underset{\sim}{\beta}}^*$ with multiplicity at least n.

Proof: We consider two cases.

Case 1. When u_0 is not a constant function, then by

Lemma 2.4.6 $\{\mathcal{E}(g) : g \in u_0^{\perp}\} = \mathbb{C}^n$.

Define

$$F_0(\mathcal{E}(g)) = (XQ - h)(g), \quad g \in u_0^{\perp}. \qquad (29)$$

F_0 is well defined, for if $\mathcal{E}(g) = 0 = g(0)$, then $(XQ-h)(g)=0$. Therefore $h_0+F_0\mathcal{E}_0 = XQ$. Since $\{\mathcal{E}(g) : g \in u_0^{\perp}\} = \mathbb{C}^n$, we can pick up $g \in u_0^{\perp}$ such that $g(0) = u_0(0)$, then by (28) and the definition of F_0, we have

$$\begin{aligned} F_0(u_0(0)) = F_0(g(0)) &= F_0(\mathcal{E}(g)) \\ &= (XQ-h)(g) \\ &= XQ(g) - h(g) \\ &= -h(u_0), \end{aligned}$$

i.e.

$$h(u_0) + F_0(u_0(0)) = 0.$$

Therefore F_0 satisfies (24) and (25).

<u>Case 2</u>. When u_0 is a constant function, then

$$u_0(0) \perp \{\mathcal{E}(g) : g \in u_0^{\perp}\}.$$

Define

$$\begin{aligned} &F_0(\mathcal{E}(g)) = (XQ - h)(g) \quad g \in u_0^{\perp} \\ &\text{and } F_0(u_0(0)) = -h(u_0). \end{aligned} \qquad (30)$$

This is well defined on $\mathbb{C}^n$, and satisfies (24) and (25). From Theorem 2.4.5 and the above it follows that ρ is a singular value of one step extension Sarason operator $\underset{\sim}{T} = P_{zB}M_{F+BF_0}P^*_{z\beta}$ with multiplicity at least n.

The proof is complete.

CHAPTER THREE

AN ORTHOGONAL DIRECT DECOMPOSITION OF $H^2_{m\times 1} \ominus BH^2_{m\times 1}$

Introduction:

In Chapter two we have already set up the theoretical part of the generalized Sarason operator $T = P_B M_F P_\beta^*$ acting from $H^2_{n\times 1} \ominus \beta H^2_{n\times 1}$ to $H^2_{m\times 1} \ominus BH^2_{m\times 1}$. The matrix computation of the operator T is rather technical. There are several problems that need to be solved:

(1). Finding a suitable direct decomposition of $H^2_{m\times 1} \ominus BH^2_{m\times 1}$ in such a way that the Gram matrix of this decomposition can be calculated easily. In this chapter, we will give a full description of a decomposition which generalizes Young's algorithm [4]. Theorems 3.2.2, 3.3.1 and 3.3.2 are the main results.

(2). Forming the projection P_B. This problem can be solved but the technical details are laborious. The idea comes from the fact that $H^2_{m\times 1} \ominus BH^2_{m\times 1}$ is contained in $H^2_{m\times 1} \ominus \beta H^2_{m\times 1}$, hence T can be written as $T = \underset{\sim}{\pi} P_\beta M_F P_\beta^*$, where $P_\beta M_F P_\beta^*$ is the generalized Sarason operator acting on $H^2_{m\times 1} \ominus \beta H^2_{m\times 1}$ which is the direct sums of m copies of $H^2 \ominus \beta H^2$. Hence the calculation of $P_\beta M_F P_\beta^*$ can be effected applying Young's algorithm to every entry F, and this does yield an efficient method. Therefore finding a computational formula for $\underset{\sim}{\pi}$ is extremely important to us, and it is fortunate that there is a formula which is neat and simple, depending on a recursive relation. This subject comprises the main part of the next chapter.

Contents:

3.1 An operational calculus

Let $\mathcal{P}H^\infty$ be the set of all polynomial functions in D

and let

$\mathcal{P}H^\infty_{mxn}$ be the space of all mxn matrices with elements in $\mathcal{P}H^\infty$. Let N and E be in $\mathcal{P}H^\infty_{mxn}$; a matrix M in said to be a common left divisor of N and E iff there exist N_1 and E_1 in $\mathcal{P}H^\infty_{mxn}$ such that $N = MN_1$ and $E = ME_1$; both N and E are said to be right multiples of M; a matrix $L \in \mathcal{P}H^\infty_{mxn}$ is said to be a greatest common left divisor (gcld) of N and E iff

(1) it is a common left divisor of N and E, and

(2) it is a right multiple of every common left divisor of N and E.

When gcld L is unimodular (i.e. detL = 1), then N and E are said to be left coprime. We define similarly a greatest common right divisor (gcrd) and right coprime. Consider $\varphi(z)$ in RH^∞_{mxm}. If we write $\varphi(z)$ as a matrix polynomial fraction

$$\varphi(z) = N_r(z)D_r(z)^{-1}$$
$$= D_\ell(z)^{-1}N_\ell(z),$$

then there can be many right and left matrix fraction descriptions(MFDs) of $\varphi(z)$; an MFD $\varphi(z) = N(z)D(z)^{-1}$ will be said to be irreducible if N(z) and D(z) are coprime. Irreducible MFD of $\varphi(z)$ are not unique, because if $N(z)D(z)^{-1}$ is irreducible so is $N(z)W(z)(D(z)W(z))^{-1}$ for any unimodular W(z). Suppose that we have an irreducible right MFD:

$$\varphi(z) = N(z)D(z)^{-1},$$

then the poles of $\varphi(z)$ are, by definition, the roots of detD(z) = 0, and the zeros of $\varphi(z)$ are the roots of detN(z)=0. It is well known [37] that for any mxm rational matrix $\varphi(z)$ having all its poles outside the unit disc there exists a left factorization:

$$\varphi(z) = \varphi_{\ell}^{o}(z)\varphi_{\ell}^{\infty}(z) ,$$

and also a right factorization

$$\varphi(z) = \varphi_{r}^{o}(z)\varphi_{r}^{\infty}(z) ,$$

where $\varphi_{\ell}^{o}(z)$, $\varphi_{\ell}^{\infty}(z)$, $\varphi_{r}^{o}(z)$ and $\varphi_{r}^{\infty}(z)$ are in RH_{mxm}^{∞}, and they have the following properties:

(1) $\varphi_{\ell}^{o}, \varphi_{r}^{o}$ are inner matrices and their zeros are the zeros of $\varphi(z)$ located inside ∂D;

(2) $\varphi_{\ell}^{\infty}, \varphi_{r}^{\infty}$ are maximal phase, i.e. none of their zeros and poles are inside ∂D.

We will show in the next section that maximal phase factors play no role in our problem. A detailed algorithm for obtaining an irredicible MFD one can find in [43] [31] [21].

Let $\mathbb{C}^m \otimes H^2$ denote the space of mxl column vector functions on D with entries belonging to H^2. $\mathbb{C}^m \otimes H^2$ is a Banach space with respect to the norm

$$\| f \|_{\mathbb{C}^m \otimes H^2} = \left(\sum_{j=1}^{m} \| f_j \|_{H^2}^2 \right)^{\frac{1}{2}} \qquad (1)$$

where f_j denotes the $(j,1)$ entry of f.

Suppose $f \in \mathbb{C}^m \otimes H^2$, $f = (f_1, f_2, \ldots , f_m)^T$, where $f_j \in H^2$ and $f_j(z) = \sum_{i=0}^{\infty} a_i^{(j)} z^i$. Then

$$\| f \|_{\mathbb{C}^m \otimes H^2}^2 = \sum_{j=1}^{m} \| f_j \|_{H^2}^2$$

$$= \sum_{j=1}^{m} \frac{1}{2\pi} \int_0^{2\pi} f_j(e^{i\theta}) \bar{f}_j(e^{i\theta}) d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sum_{j=1}^{m} f_j(e^{i\theta})\ \bar{f}_j(e^{i\theta})d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \text{trace}(f(e^{i\theta})f(e^{i\theta})^* d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \| f(e^{i\theta}) \|_2^2 = \| f \|_2^2 \quad .$$

Thus if $g(z) = \sum_{i=0}^{\infty} a_i z^i$, where $a_i = (a_i^{(1)}, a_i^{(2)}, \ldots\ldots, a_i^{(m)})^T$, then $g \in H^2_{mx1}$ and $\| f \|_{\mathbb{C}^m \otimes H^2} = \| g \|_2$. In fact $\mathbb{C}^m \otimes H^2$ and H^2_{mx1} are isometrically isomorphic. The isomorphism can be characterized as following

$$\iota \begin{pmatrix} \sum_{i=0}^{\infty} a_i^{(1)}\ z^i \\ \vdots \\ \sum_{i=0}^{\infty} a_i^{(m)}\ z^i \end{pmatrix} = \sum_{i=0}^{\infty} a_i z^i \ . \qquad (2)$$

We will use this isomorphism ι throughout the thesis without mentioning afterward. $\{ e_i \otimes z^j \mid i=1,2,..,m\ j=0,1,\ldots, \}$ is the standard basis for $\mathbb{C}^m \otimes H^2$, where $e_i \otimes z^j$ denotes mx1 column vector with values z^j at i-th coordinate and 0 elesewhere, we consider this basis ordered as following $e_1 \otimes z^0$, $e_1 \otimes z^1, \ldots,$ $e_1 \otimes z^m, \ldots\ldots, e_2 \otimes z^0, e_2 \otimes z^1, \ldots\ldots\ldots$ Therefore the isomorphism ι can also be characterized as follws

$$\iota\ (e_i \otimes z^j) = z^j \otimes e_i \quad . \qquad (3)$$

Roughly speaking ι is a permutation of the basis.

When $T \in L(H^2)$, then RH^{∞} contains functions which are analytic on some neighborhood of the spectrum of T, $\sigma(T)$. Let U be an open subset of $\mathbb{C}$ whose boundary C consists of a finite number of rectifiable Jordan curves. Suppose $U \supset \sigma(T)$, $\varphi \in RH^{\infty}$, then $U \cup C$ is contained in the domain of analyticity of φ. The operator $\varphi(T)$ is defined by the equation

$$\varphi(T) = \frac{1}{2\pi i} \oint_C \varphi(z)(zI-T)^{-1}dz . \qquad (4)$$

The integral exists as a limit of Riemann sums in the norm of $L(H^2)$. This operational calculus can be generalized directly to any $F \in RH^{\infty}_{mxn}$, which is analytic on a neighborhood U of the spectrum of T, $F(z) = [F_{ij}(z)]_{mxn}$ as follows:

$$F(T) = [F_{ij}(T)]_{mxn}$$

where

$$F_{ij}(T) = \frac{1}{2\pi i} \oint_C F_{ij}(z)(zI-T)^{-1}dz$$

or equivalently,

$$F(T) = \frac{1}{2\pi i} \oint_C F(z) \otimes (zI-T)^{-1}dz . \qquad (5)$$

By the definition of (5) we have some immediate results which will be useful in the sequel.

<u>Lemma 3.1.1</u> Let $F \in RH^{\infty}_{mxn}$, $T \in L(H^2)$ and let F be analytic on some neighborhood U of the spectrum of T. Then

$$F^*(T^*) = [F(T)]^* .$$

<u>Proof:</u> Let C be the boundary of U consisting of a finite number of rectifiable Jordan curves.

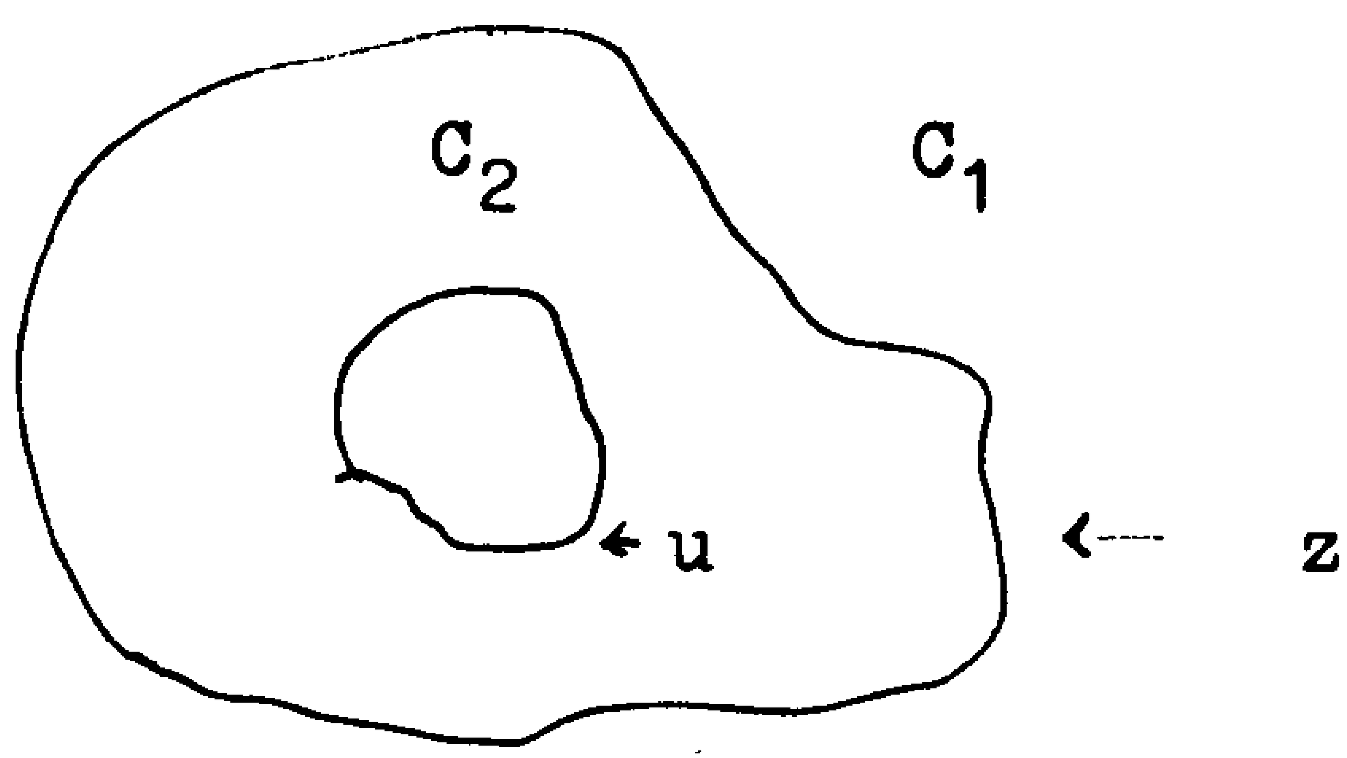
$$[F(T)]^* = \left[\frac{1}{2\pi i} \oint_C F(z)\otimes(zI-T)^{-1}dz\right]^*$$
$$= -\frac{1}{2\pi i} \oint_{C'} [F(z)]^*\otimes(\bar{z}I-T^*)^{-1}d\bar{z}$$
$$= \frac{1}{2\pi i} \oint_C F^*(\bar{z})\otimes(\bar{z}I-T^*)^{-1}d\bar{z}$$
$$= F^*(T^*),$$

where C' is counterclockwise.

Lemma 3.1.2. Let $F, G \in RH^{\infty}_{m\times n}$, $T \in L(H^2)$ and let F, G be analytic on some neighborhood U of the spectrum of T. Then

$$(F\cdot G)(T) = F(T)\cdot G(T).$$

Proof: Let F(T) be evaluated by an integral around a suitable curve C_1. As for G(T), let it be evaluated by an integral around a curve C_2 which lies entirely in the interior of C_1.



$$F(T)G(T) = \left[\frac{1}{2\pi i}\oint_{C_1} F(z)\otimes(zI-T)^{-1}dz\right]\left[\frac{1}{2\pi i}\oint_{C_2} G(u)\otimes(uI-T)^{-1}du\right]$$
$$= (2\pi i)^{-2} \oint_{C_1}\oint_{C_2} F(z)G(u)\otimes(zI-T)^{-1}(uI-T)^{-1}dzdu$$
$$= (2\pi i)^{-2} \oint_{C_1}\oint_{C_2} F(z)G(u)\otimes\frac{1}{u-z}\left[(zI-T)^{-1}-(uI-T)^{-1}\right]dzdu$$
$$= (2\pi i)^{-2}\oint_{C_1} F(z)\otimes(zI-T)^{-1}\left(\oint_{C_2} G(u)\otimes\left(\frac{1}{u-z}\right)du\right)dz \; -$$
$$(2\pi i)^{-2}\oint_{C_2}\left(\oint_{C_1} F(z)\otimes\left(\frac{1}{u-z}\right)dz\right) G(u)\otimes(uI-T)^{-1}du$$

$$= (2\pi i)^{-2} \oint_{C_2} \left(\oint_{C_1} F(z) \otimes \frac{1}{z-u}\, dz \right) G(u) \otimes (uI-T)^{-1} du$$

$$= (2\pi i)^{-2} (2\pi i) \oint_{C_2} F(u) G(u) \otimes (uI-T)^{-1} du$$

$$= \frac{1}{2\pi i} \oint_{C_2} F(u) G(u) \otimes (uI-T)^{-1} du$$

$$= (F \cdot G)(T).$$

In the above calculation, we have used the functional equations of the resolvent and the classical fact that an integral of the form $\oint_C (\alpha - \beta)^{-1} d\alpha$ is $2\pi i$ if β lies within C and 0 if β lies outside C. The proof is complete.

For $B \in RH^{\infty}_{m\times m}$ there exists $\mathrm{adj}B \in RH^{\infty}_{m\times m}$ such that

$$B(z) \cdot \mathrm{adj}B(z) = \mathrm{adj}B(z) \cdot B(z) = (\det B)(z)\, I_m = \beta(z) I_m,$$

where β is a scalar analytic function. It is easy to show that if B is inner, then βI, adjB are also inner [27]. therefore, it follows from Lemma 3.1.2, for any $T \in L(H^2)$ and B is analytic on some neighborhood U of the spectrum of T, that

$$B(T) \cdot \mathrm{adj}B(T) = (\beta I)(T) = I_{\mathbb{C}^m} \otimes \beta(T) = \beta(\underset{\sim}{T}),$$

where $I_{\mathbb{C}^m} \otimes \beta(T)$ denotes the diagonal operator on $\mathbb{C}^m \otimes H^2$ with entries $\beta(T)$, and $\underset{\sim}{T} = I_{\mathbb{C}^m} \otimes T$. In particular, when B is unimodular, then B(T) is invertible.

<u>Lemma3.1.3.</u> Let $W \in RH^{\infty}_{m\times m}$ be unimodular, $T \in L(H^2)$ and let W be analytic on some neighborhood U of the spectrum of T. Then W(T) is invertible.

<u>Proof:</u> Let W be unimodular, then $\det W(z) = 1$, and $W(z)^{-1} = \mathrm{adj}W(z)/\det W(z)$, by Lemma 3.1.2, $W(T)^{-1} = \mathrm{adj}W(T)$.

The following ι is the isomorphism from $\mathbb{C}^m \otimes H^2$ to $H^2_{m\times 1}$ defined in (2) above.

Theorem 3.1.4 Let $\varphi \in RH^{\infty}_{m\times m}$, let S be the backward shift operator on H^2 and M_φ be a multiplication operator Then

(1). $\iota \varphi(S^*)\iota^* = M_\varphi$.

(2). If φ is inner, then $\iota \varphi^*(S)\varphi(S^*)\iota^* = I_{H^2_{m\times 1}}$.

(3). If φ is inner, then $\text{Range}(I_{H^2_{m\times 1}} - \iota\varphi(S^*)\varphi^*(S)\iota^*)$

$$= \text{Ker}\, \iota\varphi^*(S)\iota^* .$$

(4). If φ is inner, then $\text{Ker}\, \iota\varphi^*(S)\iota^* = H^2_{m\times 1} \ominus \varphi H^2_{m\times 1}$.

Proof: (1) Apply both sides to the orthonormal base $\{ z^j \otimes e_i \; : \; j=0,1,\ldots,i=1,2,\ldots,m \}$

$$\begin{aligned}
(\iota\varphi(S^*)\iota^*)\,(z^j\otimes e_i) &= \iota\,\varphi(S^*)(e_i\otimes z^j) \\
&= \iota\left(\frac{1}{2\pi i}\oint_C \varphi(u)\otimes(uI-S^*)^{-1}du\right)(e_i\otimes z^j) \\
&= \iota\left(\frac{1}{2\pi i}\oint_C \varphi(u)e_i\otimes(uI-S^*)^{-1}\, z^j du\right) \\
&= \iota\left(\frac{1}{2\pi i}\oint_C \varphi(u)e_i\otimes \frac{z^j}{u-z}\, du\right) \\
&= \iota\,(\varphi(z)e_i\otimes z^j) = \varphi(z)\,\iota\,(e_i\otimes z^j) \\
&= \varphi(z)(z^j\otimes e_i) = M_\varphi\,(z^j\otimes e_i)\,,
\end{aligned}$$

where $(uI-S^*)^{-1}z^j = \frac{z^j}{u-z}$ and the integral is taken counter-clockwise round the curve $C \supsetneq \partial D$.

(2). If φ is inner, then

$$\begin{aligned}
\iota\,\varphi^*(S)\varphi(S^*)\iota^* &= \iota\,\varphi^*(S)\,\iota^*\iota\,\varphi(S^*)\iota^* \\
&= [\iota\varphi(S^*)\iota^*]^*[\iota\varphi(S^*)\iota^*] \\
&= M^*_\varphi M_\varphi = M_{\varphi^*\varphi} = I_{H^2_{m\times 1}} .
\end{aligned}$$

(3). Let $h \in \mathrm{Range}(I_{H^2_{m\times 1}} - \iota\varphi(S^*)\varphi^*(S)\iota^*)$;

then

$$h = (I_{H^2_{m\times 1}} - \iota\varphi(S^*)\varphi^*(S)\iota^*)f$$

for some $f \in H^2_{m\times 1}$, and

$$\begin{aligned}\iota\varphi^*(S)\iota^*h &= (\iota\varphi^*(S)\iota^* - \iota\varphi^*(S)\iota^*\iota\varphi(S^*)\varphi^*(S)\iota^*)f \\ &= (\iota\varphi^*(S)\iota^* - \iota\varphi^*(S)\iota^*)f = 0.\end{aligned}$$

This implies $h \in \mathrm{Ker}\,\iota\varphi^*(S)\iota^*$.

Conversely, if $h \in \mathrm{Ker}\,\iota\varphi^*(S)\iota^*$, then $\iota\varphi^*(S)\iota^*h = 0$,

thus

$$\iota\varphi(S^*)\varphi^*(S)\iota^*h = 0,$$

therefore

$$\begin{aligned}h &= h - \iota\varphi(S^*)\varphi^*(S)\iota^*h \\ &= (I_{H^2_{m\times 1}} - \iota\varphi(S^*)\varphi^*(S)\iota^*)h \in \mathrm{Range}(I - \iota\varphi(S^*)\varphi^*(S)\iota^*)\end{aligned}$$

The proof of (3) is complete.

(4) follows that

$$\begin{aligned}\mathrm{Ker}\,\iota\varphi^*(S)\iota^* &= \mathrm{Ker}(\iota\varphi(S^*)\iota^*)^* \\ &= \mathrm{Ker}\,M_\varphi^* = (\varphi H^2_{m\times 1})^\perp \\ &= H^2_{m\times 1} \ominus \varphi H^2_{m\times 1}\,.\end{aligned}$$

3.2 The Standard decomposition of $H^2_{m\times 1} \ominus BH^2_{m\times 1}$

Let $B \in RH^\infty_{m\times m}$. The Smith-McMillan form [6] of B(z) is given by

$$B(z) = A(z)M(z)C(z), \tag{6}$$

where

(1). A(z), C(z) are unimodular polynomial matrices.

(2). M(z) has the form

$$M(z) = \mathrm{diag}\left(\frac{e_1(z)}{\psi_1(z)}, \frac{e_2(z)}{\psi_2(z)}, \ldots \frac{e_t(z)}{\psi_t(z)}, 0, \ldots, 0\right) \tag{7}$$

for some positive integer t, and

(a) $e_i(z)$ and $\psi_i(z)$, i=1,2,...,t are monic coprime polynomials;

(b) $e_i(z) \mid e_{i+1}(z)$, i=1,2,...,t-1;

(c) $\psi_i(z) \mid \psi_{i-1}(z)$, i=2,3,...,t , and $\psi_1(z)$ is the least common multiple of all the denominators of the entries of B(z).

Clearly the polynomials $e_i(z)$ and $\psi_i(z)$ are uniquely determined by B(z). We can factor M(z) as follows:

$$\begin{aligned} M(z) &= \psi_\ell(z)^{-1}E_\ell(z) \\ &= E_r(z)\psi_r(z)^{-1} . \end{aligned}$$

where

$$\psi_\ell(z) = \mathrm{diag}(\psi_1(z), \psi_2(z), \ldots, \psi_t(z), \ 1,1,\ldots,1),$$

$$E_\ell(z) = \mathrm{diag}(e_1(z), e_2(z), \ldots, e_t(z), \ 0,0,\ldots,0).$$

There are similar expressions for $\psi_r(z)$ and $E_r(z)$.
Let us define

$$D_\ell(z) = \psi_\ell(z)A(z)^{-1}, \quad N_\ell(z) = E_\ell(z)C(z).$$

Since, for i=1, 2,....,t, $e_i(z)$ and $\psi_i(z)$ are coprime, it follows that $\psi_\ell(z)$ and $E_\ell(z)$ are coprime matrices. The same holds for $D_\ell(z)$ and $N_\ell(z)$ because A(z) and C(z) are unimodular. For the right coprime polynomial factorization, we consider

$$D_r(z) = C(z)^{-1}\psi_r(z) , \quad N_r(z) = A(z)E_r(z).$$

From (6)(7) it follows that the finite poles of B(z) are the zeros of the polynomial $\psi_j(z)$ in its McMillan form (6). Similarly, the zeros of $e_i(z)$ are the zeros of B(z). Therefore B(z) has an irreducible right MFD

$$B(z) = N_r(z)D_r(z)^{-1} . \tag{8}$$

Moreover, if $N(z)D(z)^{-1}$ is another irreducible right MFD, then there exists a unimodular U such that $D(z) = D_r(z)U(z)$, $N(z) = N_r(z)U(z)$. This property can be easily proved by the fact [21] that N(z) and D(z) are right coprime if and only if there are polynomial matrices X(z) and Y(z) such that $X(z)N(z) + Y(z)D(z) = I_m$. In other words, the irreducible MFDs are unique up to multiple by unimodular matrices.

Theorem 3.2.1 Let $B \in RH^{\infty}_{mxm}$ be an inner matrix. Suppose we have an irreducible right MFD

$$B(z) = N(z)\, D(z)^{-1}.$$

Then

$$H^2_{mx1} \ominus BH^2_{mx1} = \mathrm{Ker}\, \iota\, B^*(S)\, \iota^* = \mathrm{Ker}\, \iota\, N^*(S)\, \iota^* .$$

Proof: The theorem is proved if $D^*(S)$ is invertible. Let the Smith-McMillan form of B(z) be

$$B(z) = N_r(z)D_r(z)^{-1} = A(z)E_r(z)[C(z)^{-1}\psi_r(z)]^{-1}$$

as in the form (6)(8) above .

By the uniqeness of the irreducible MFD,

$$D(z) = D_r(z)\, U(z)$$

for some unimodular matrix U(z) and $\det D_r(z) = \det D(z)$. This shows that the zeros of D(z) are the zeros of $D_r(z)$. Since B is an inner matrix, all the poles of B(z) are outside the unit circle. From Lemma3.1.3 $C(S)^{-1}\psi_r(S) = D_r(S)$ is invertible, and

$$B^*(z) = [N(\bar{z})D(\bar{z})^{-1}]^* = D^*(z)^{-1}N^*(z) ,$$

so

$$B^*(S) = D^*(S)^{-1}N^*(S) .$$

It follows that

$$\mathrm{Ker}\, \iota\, B^*(S)\, \iota^* = \mathrm{Ker}\, \iota\, N^*(S)\, \iota^* .$$

Remark: $D(z)$ is called maximal phase, and this function plays no role in our problem. $N(z)$ is called the numerator of $B(z)$.

Let $B \in RH^{\infty}_{m \times m}$ be an inner matrix with irreducible MFD

$$B(z) = N(z)\, D(z)^{-1}$$

where $N(z)$ is a regular polynomial matrix of degree k, i.e.

$$N(z) = B_0 + B_1 z + \dots\dots + B_k z^k \;, \quad \text{with} \quad \det B_k \neq 0 \;. \qquad (9)$$

Such an inner matrix we will call r-inner matrix . It is not hard to show that $\mathrm{Ker} \lfloor B^*(S) \rfloor^* = \mathrm{Ker} \lfloor N^*(S) \rfloor^*$ is the space of all vector valued sequences $x = (x_0, x_1, \dots\dots) \in \ell^{2\,+}_{m \times 1}$ satisfying the recurrence relation

$$B_0^* x_r + B_1^* x_{r+1} + \dots\dots\dots + B_{k-1}^* x_{r+k-1} + B_k^* x_{r+k} = 0 \qquad (10)$$

for $r = 0, 1, 2, \dots\dots$.

Clearly $\mathrm{Ker} \lfloor B^*(S) \rfloor^*$ is invariant under the backward shift operator $\underset{\sim}{S}$, that is

$$\underset{\sim}{S}(\mathrm{Ker} \lfloor B^*(S) \rfloor^*) \subseteq \mathrm{Ker} \lfloor B^*(S) \rfloor^* .$$

It is helpful to have a more explicit decription of $\mathrm{Ker} \lfloor B^*(S) \rfloor^* = H^2_{m \times 1} \ominus BH^2_{m \times 1}$ in terms of the coefficients of the numerator of $B(z)$, and this is quite important to the problem of decomposition of $H^2_{m \times 1} \ominus BH^2_{m \times 1}$.

Theorem 3.2.2 Let $B \in RH^{\infty}_{m \times m}$ be an r-inner matrix. Then $H^2_{m \times 1} \ominus BH^2_{m \times 1}$ is the subspace of $H^2_{m \times 1}$ consisting of all rational functions of the form

$$\hat{N}(z)^{-1} \Big(\sum_{j=0}^{k-1} T_j(z)\, x_j \Big) \;, \quad x_j \in C^m \;, \qquad (11)$$

where

$$\hat{N}(z) = B_k^* + B_{k-1}^* z + \dots\dots + B_0^* z^k = z^k N(1/\bar{z})^*$$

$$T_j(z) = B_k^* z^j + B_{k-1}^* z^{j+1} + \dots\dots + B_{j+1}^* z^{k-1} \;.$$

<u>Proof:</u> Let $X \in \operatorname{Ker} \iota B^*(S)\iota^*$ and $X(z) = \sum_{i=0}^{\infty} x_i z^i$.

By (10) $\{x_i\}_{i=0}^{\infty}$ satisfies the recurrence relation

$$B_0^* x_r + B_1^* x_{r+1} + \dots\dots + B_k^* x_{r+k} = 0$$

for $r = 0,1,2,\dots$.

Then by a matrix calculation it is easy to verify

$$B_{k-j}^* z^j X(z) = B_{k-j}^* x_0 z^j + B_{k-j}^* x_1 z^{j+1} + \dots + B_{k-j}^* x_{k-j} z^k + \dots$$

and

$$\left(\sum_{j=0}^{k} B_{k-j}^* z^j\right) X(z) = \sum_{j=0}^{k} \left(B_{k-j}^* z^j x_0 + B_{k-j}^* z^{j+1} x_1 + \dots + B_{k-j}^* z^k x_{k-j}\right)$$

Now let

$$\hat{N}(z) = \sum_{j=0}^{k} B_{k-j}^* z^j ,$$

$$T_j(z) = B_k^* z^j + B_{k-1}^* z^{j+1} + \dots\dots + B_{j+1}^* z^{k-1},$$

$j=0,1,2,\dots\dots$ i.e.

$$\hat{N}(z) X(z) = \sum_{j=0}^{k-1} T_j(z) x_j .$$

By the hypothesis that B is r-inner matrix, $\hat{N}(z)^{-1}$ is in $H^{\infty}_{m \times m}$. This can be proved as follows : Let $\alpha_1, \alpha_2, \dots, \alpha_s$ be the zeros of $\det N(z) = \det N_r(z) = \det(A(z)E_r(z)) = \det E_r(z)$. As in (7), it follows that each $e_j(z)$ can be written as a product of linear factors

$$e_j(z) = \prod_{\ell=1}^{s} (z-\alpha_\ell)^{k_{j\ell}} .$$

By assumption, $|\alpha_\ell| < 1$ and $N(z)$ is regular matrix of degree k with $\det B_k \neq 0$. Thus

$$\sum_j \sum_\ell k_{j\ell} = mk ,$$

and

$$\det\left(z^k \operatorname{diag}\left(e_1\left(\tfrac{1}{\bar z}\right)^*, e_2\left(\tfrac{1}{\bar z}\right)^*, \dots\dots, e_m\left(\tfrac{1}{\bar z}\right)^*\right)\right)$$
$$= z^{mk} e_1\left(\tfrac{1}{\bar z}\right)^* \dots\dots e_m\left(\tfrac{1}{\bar z}\right)^* = \prod_{j=1}^{m} \prod_{\ell=1}^{s} (1-\bar{\alpha}_j z)^{k_{j\ell}} .$$

$A(z)$ is unimodular, so is $A(\frac{1}{\bar{z}})$, and

$$\begin{aligned}\det\hat{N}(z) &= \det(z^k N(\tfrac{1}{\bar{z}})^*) \\ &= \det(z^k E_1(\tfrac{1}{\bar{z}})^* A(\tfrac{1}{\bar{z}})^*) \\ &= \det(z^k E_1(\tfrac{1}{\bar{z}})^*)\det A(\tfrac{1}{\bar{z}})^* \\ &= \prod_{j=1}^{m}\prod_{\ell=1}^{s}(1-\bar{\alpha}_j z)^{k_{j\ell}}.\end{aligned}$$

This means that all the zeros in $\det\hat{N}(z)$ are outside the unit disc, and $\hat{N}(z)^{-1}$ will be in $H^{\infty}_{m\times m}$. Therefore $X(z)$ can be written in the following form:

$$\begin{aligned}X(z) &= \hat{N}(z)^{-1}\Big(\sum_{j=0}^{k} T_j(z)x_j\Big) \\ &= \frac{\operatorname{adj}\hat{N}(z)}{\det\hat{N}(z)}\Big(\sum_{j=0}^{k-1} T_j(z)x_j\Big) \\ &= \sum_{j=0}^{k-1}\Big(\frac{\operatorname{adj}\hat{N}(z)}{\det\hat{N}(z)} T_j(z)\Big)x_j.\end{aligned}$$

This completes the proof.

<u>Remark:</u> When $m=1$ one can see [4][35][36][46].

From the observation (11) we have a direct decomposition of $H^2_{m\times 1}\ominus BH^2_{m\times 1}$, and each summand can be identified with $\mathbb{C}^m$. Indeed, let G_j be the subspace of $H^2_{m\times 1}\ominus BH^2_{m\times 1}$ defined by

$$G_j = \Big\{x \in H^2_{m\times 1}\ominus BH^2_{m\times 1} \;\Big|\; x(z) = \sum_{n=0}^{\infty} x_n z^n,\ x_0=\dots=x_{j-1}=x_{j+1}=\dots=x_{k-1}=0\Big\} \qquad (12)$$

$j = 0,1,2,\dots,k-1$.

and let

$$\tau_j : \mathbb{C}^m \longrightarrow G_j$$

be defined by

$$(\tau_j x)(z) = xz^j + O(z^k)x \tag{13}$$

$x \in \mathbb{C}^m$, $z \in D$.

It follows from Theorem 3.2.2 that

$$(\tau_j x)(z) = \hat{N}(z)^{-1}T_j(z)x . \tag{14}$$

This map is well defined, linear and 1-1 onto. Let us show injectivity; the remainder can be seen easily. If $\tau_j x = 0$, i.e. $(\tau_j x)(z) = 0$ for all $z \in D$, by definition

$$(\hat{N}(z)^{-1}T_j(z))x = 0 ,$$

or equivalently

$$(B_k^* z^j + B_{k-1}^* z^{j+1} + \ldots + B_{j+1}^* z^{k-1})x = 0$$

and det B_k^* = det $B_k \neq 0$ imples $x = 0$.

By means of the invertible mapping τ_j we can therefore identify $\mathbb{C}^m$ with G_j , and this decomposes $H^2_{mx1} \ominus BH^2_{mx1}$ into direct decomposition $G_0 \oplus G_1 \oplus \ldots \oplus G_{k-1}$.

$\{\tau_j(e_i) : j = 0,1,\ldots,k-1,\ i = 1,2,\ldots,m\}$ is a basis for $H^2_{mx1} \ominus BH^2_{mx1}$, where $\tau_j(e_i)$ denotes the i th column of the matrix function $\hat{N}(z)^{-1}T_j(z)$, i.e.

$$\tau_j(e_i)(z) = [\hat{N}(z)^{-1}T_j(z)]\, e_i \ .$$

We consider this basis ordered as $\tau_0(e_1)$, $\tau_0(e_2), \ldots, \tau_0(e_m)$, $\tau_1(e_1)$, $\tau_1(e_2), \ldots, \tau_1(e_m), \ldots\ldots$, and will call this <u>the standard basis for</u> $H^2_{mx1} \ominus BH^2_{mx1}$. These symbols will be used throughout the thesis without further mention.

The matrix of the restriction of the backward shift $\underset{\sim}{S}_B$ of $\underset{\sim}{S}$ to $H^2_{mx1} \ominus BH^2_{mx1}$ with respect to the standard basis $\{\tau_j(e_i)\}$, is the companion matrix C_{N^*} of $N^*(z)$:

For $j = 1,2,\ldots,k-1$, $z \in D$, $x \in C^m$

$$\underset{\sim}{S}_B(\tau_j(x)(z)) = \underset{\sim}{S}_B(\hat{N}(z)^{-1}T_j(z)\,x)$$

$$= \frac{1}{z}(\hat{N}(z)^{-1}T_j(z) - \hat{N}(0)^{-1}T_j(0))x\ .$$

Since for $j \geqslant 1$, $T_j(0) = 0$, so for $j \geqslant 1$

$$\underset{\sim}{S}_B(\tau_j(x)(z)) = \frac{1}{z}(\hat{N}(z)^{-1}T_j(z)x)$$

$$= \hat{N}(z)^{-1}\left(\frac{B_k^* z^j + B_{k-1}^* z^{j+1} + \ldots + B_{j+1}^* z^{k-1}}{z}\right)x$$

$$= \hat{N}(z)^{-1}(B_k^* z^{j-1} + B_{k-1}^* z^j + \ldots + B_j^* z^{k-1} - B_j^* z^{k-1})x$$

$$= (\hat{N}(z)^{-1}T_{j-1}(z) - \hat{N}(z)^{-1}B_j^* z^{k-1})\,x$$

$$= \tau_{j-1}(x)(z) - \hat{N}(z)^{-1}B_k^* z^{k-1}(B_k^{*\,-1}B_j^* x)$$

$$= \tau_{j-1}(x)(z) - \tau_{k-1}(B_k^{*\,-1}B_j^* x)(z), \qquad (15)$$

while

$$\underset{\sim}{S}_B(\tau_0(x)(z)) = S_B(\hat{N}(z)^{-1}T_0(z)x)$$

$$= \frac{1}{z}(\hat{N}(z)^{-1}T_0(z) - \hat{N}(0)^{-1}T_0(0))x$$

$$= \frac{1}{z}(\hat{N}(z)^{-1}T_0(z) - B_k^{*\,-1}B_k^*)\,x$$

$$= \frac{1}{z}\hat{N}(z)^{-1}(T_0(z) - \hat{N}(z))x$$

$$= \hat{N}(z)^{-1}(-B_0^* z^{k-1})x$$

$$= \hat{N}(z)^{-1}B_k^* z^{k-1}(-B_k^{*\,-1}B_0^*\,x)$$

$$= \tau_{k-1}(-B_k^{*\,-1}B_0^*\,x)(z)\ . \qquad (16)$$

Putting (15)(16) together we have the following result.

<u>Theorem 3.2.3</u> Let $B \in RH^{\infty}_{m\times m}$ be an r-inner matrix, $\underset{\sim}{S}_B$ be the restriction of the backward shift operator $\underset{\sim}{S}$ on $H^2_{m\times 1} \ominus BH^2_{m\times 1}$. The block matrix of $\underset{\sim}{S}_B$ with respect to the standard basis $\{\tau_j(e_i) : j = 0,1,2,..,k-1, i= 1,2,3,..,m\}$ is the companion matrix C_{N^*} of N^*, where

$$N^*(z) = B_0^* + B_1^* z + \dots\dots + B_k^* z^k$$

$$C_{N^*} = \begin{bmatrix} 0, & I_m, & 0 & & 0 \\ 0, & 0 & I_m & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0, & - & - & - & I_m \\ -B_k^{*-1}B_0^*, & -B_k^{*-1}B_1^*, & \dots\dots\dots, & & -B_k^{*-1}B_{k-1}^* \end{bmatrix}. \qquad (17)$$

<u>Remark:</u> When m=n=1, there are three different bases for $H^2_{m\times 1} \ominus BH^2_{m\times 1}$ which were given by Young and Pták in [4][35] and the matrices of $\underset{\sim}{S}_B$ with respect to these bases were also discussed there.

<u>3.3 The Gram matrix of the decomposition</u>

$P_B : H^2_{m\times 1} \longrightarrow H^2_{m\times 1} \ominus BH^2_{m\times 1}$ is the orthogonal projection operator; then $P_B^* : H^2_{m\times 1} \ominus BH^2_{m\times 1} \longrightarrow H^2_{m\times 1}$ is the natural injection and $P_B P_B^*$ is the identity operator on $H^2_{m\times 1} \ominus BH^2_{m\times 1}$, while $P_B^* P_B$ is the Hermitian projection operator on $H^2_{m\times 1}$ which maps each function onto its projection on $H^2_{m\times 1} \ominus BH^2_{m\times 1}$. Now by Theorem 3.1.4 $\llcorner B(S^*)\llcorner^*$ is the operator on $H^2_{m\times 1}$ of multiplication by the inner matrix B, and is an isometry, so that $\iota B^*(S)B(S^*)\iota^*$ is the identity operator. Hence $\iota B(S^*)B^*(S)\iota^*$ is the orthogonal projection operator on the range $\iota B(S^*)\iota^*$, and so $I_{H^2_{m\times 1}} - \iota B(S^*)B^*(S)\iota^*$ is the orthogonal projection on

$$[\text{Range } \iota B(S^*)\iota^*]^{\perp} = \text{Ker } \iota B^*(S)\iota^* = H^2_{m\times 1} \ominus BH^2_{m\times 1}$$

Thus

$$P_B^* P_B = I_{H^2_{m\times 1}} - \iota B^*(S)^* B^*(S)\iota^*. \qquad (18)$$

Let $\mathbb{C}^m \otimes \mathbb{C}^k$ denote the direct sum of k copies of $\mathbb{C}^m$. Define a mapping $\underset{\sim}{K} : H^2_{m\times 1} \longrightarrow \mathbb{C}^m \otimes \mathbb{C}^k$ by

$$\underset{\sim}{K}\left(\sum_{j=0}^{\infty} x_j z^j \right) = (x_0, x_1, \ldots\ldots, x_{k-1}). \qquad (19)$$

Then $\underset{\sim}{K}^*$ is the natural injection of $\mathbb{C}^m \otimes \mathbb{C}^k$ into $H^2_{m\times 1}$ and $\underset{\sim}{K}\underset{\sim}{K}^*$ is the identity operator on $\mathbb{C}^m \otimes \mathbb{C}^k$, and $\underset{\sim}{K}^*\underset{\sim}{K}$ is the Hermitian orthogonal projection operator on $H^2_{m\times 1}$ with range Ker $\underset{\sim}{S}^k$. When m=1 we write K instead of $\underset{\sim}{K}$. Let

$$\nu : \text{Ker } \underset{\sim}{S}^k \longrightarrow H^2_{m\times 1} \ominus BH^2_{m\times 1}$$

be the linear mapping defined by

$$\nu(x_0 + x_1 z + \ldots + x_{k-1} z^{k-1}) = \zeta_0(x_0)(z) + \ldots\ldots + {}_{k}\zeta_1(x_{k-1})(z). \quad (20)$$

The property (11)(14) of the standard basis can then be written

$$\nu^{-1} = \underset{\sim}{K} P_B^* . \qquad (21)$$

Hence we have

$$\begin{aligned}
(\nu^*\nu)^{-1} &= \nu^{-1}\,\nu^{*-1} \\
&= \underset{\sim}{K} P_B^* P_B \underset{\sim}{K}^* \\
&= \underset{\sim}{K}\,(I - \iota B(S^*)B^*(S)\iota^*)\,\underset{\sim}{K}^* \\
&= \underset{\sim}{K}\underset{\sim}{K}^* - \underset{\sim}{K}\,\iota B(S^*)B^*(S)\iota^*\underset{\sim}{K}^* \\
&= I_{\mathbb{C}^m \otimes \mathbb{C}^k} - \underset{\sim}{K}\,\iota B(S^*)B^*(S)\iota^*\,\underset{\sim}{K}^*
\end{aligned}$$

$$= I_{\mathbb{C}^m \otimes \mathbb{C}^k} - [\underset{\sim}{K} \iota B^*(S) \iota^* \underset{\sim}{K}^*]^* [\underset{\sim}{K} \iota B^*(S) \iota^* \underset{\sim}{K}^*].$$

Now use the integral formula for $B^*(S)$ and the fact that $\underset{\sim}{K} \iota$ is the mxm diagonal operator with values $K : H^2 \longrightarrow \mathbb{C}^k$, i.e.

$$\underset{\sim}{K} \iota = I_{\mathbb{C}^m} \otimes K .$$

We find

$$\begin{aligned} \underset{\sim}{K} \iota B^*(S) \iota^* \underset{\sim}{K}^* &= (I_{\mathbb{C}^m} \otimes K)\left(\frac{1}{2\pi i} \oint_C B^*(z) \otimes (zI-S)^{-1} dz\right)(I_{\mathbb{C}^m} \otimes K^*) \\ &= \frac{1}{2\pi i} \oint_C B^*(z) \otimes K(zI-S)^{-1} K^* \, dz \\ &= \frac{1}{2\pi i} \oint_C B^*(z) \otimes (zI-KSK^*)^{-1} \, dz . \qquad (22) \end{aligned}$$

Write $S_k = KSK^*$. Making suitable choices of curve C , we can infer (22) that

$$\begin{aligned} (\nu^* \nu)^{-1} &= I_{\mathbb{C}^m \otimes \mathbb{C}^k} - B^*(S_k)^* B^*(S_k) \\ &= I_{\mathbb{C}^m \otimes \mathbb{C}^k} - B(S_k^*) B^*(S_k) . \qquad (23) \end{aligned}$$

It is easy to calculate that, with respect to the natural basis in H^2, $\{ z^j : j=0,1,2,\ldots.k-1 \}$ of Ker S^k, S_k has matrix

$$\begin{bmatrix} 0, & 1, & 0, & \ldots\ldots & ,0 \\ 0, & 0, & 1, & \ldots\ldots & ,0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0, & 0, & \ldots\ldots & & ,1 \\ 0, & 0, & \ldots\ldots & & ,0 \end{bmatrix}_{k \times k} .$$

<u>Theorem 3.3.1</u> Let $B \in RH^\infty_{m\times m}$ be an r-inner matrix and let ν be the operator from Ker $\underset{\sim}{S}^k$ onto $H^2_{m\times 1} \ominus BH^2_{m\times 1}$ defined by (20). Then

$$(\nu^* \nu)^{-1} = I_{\mathbb{C}^m \otimes \mathbb{C}^k} - B(S_k^*) B^*(S_k) \qquad (24) ,$$

where

$$S_k = KSK^* .$$

As a matter of fact $\nu^*\nu$ is the Gram matrix of the standard basis $\{\tau_j(e_i) \mid j=0,1,2,\ldots,k-1,\ i=1,2,\ldots,m\}$ The Gram matrix of $\{\tau_j(e_i)\}$ is the operator

$$\underset{\sim}{G}(\tau_0, \tau_1, \ldots\ldots, \tau_{k-1}) : C^m \otimes C^k \longrightarrow C^m \otimes C^k ,$$

with block matrix whose (i,j) entry, say $\underset{\sim}{G}_{ij}$, is the operator on $\mathbb{C}^m$ defined by

$$\underset{\sim}{G}_{ij} \ : \ \mathbb{C}^m \longrightarrow \mathbb{C}^m$$

$$\underset{\sim}{G}_{ij}(x) = \tau_{i-1}^* \tau_{j-1}(x), \tag{25}$$

or write the operator in matrix form:

$$\underset{\sim}{G}(\tau_0, \tau_1, \ldots, \tau_{k-1}) = \begin{bmatrix} \tau_0^* \tau_0, & \cdots\cdots & , \tau_0^* \tau_{k-1} \\ \tau_1^* \tau_0, & \cdots\cdots & , \tau_1^* \tau_{k-1} \\ \vdots & & \vdots \\ \tau_{k-1}^* \tau_0, & \cdots\cdots & , \tau_{k-1}^* \tau_{k-1} \end{bmatrix} .$$

This definition indeed generalizes the Gram matrix. In particular, according to the decomposition of $H^2_{mx1} \ominus BH^2_{mx1}$ the Gram matrix can be expressed by B.

<u>Theorem 3.3.2</u>. Let $B \in RH^{\infty}_{mxm}$ be an r-inner matrix, and let ν be the operator from $C^m \otimes C^k$ onto $H^2_{mx1} \ominus BH^2_{mx1}$ defined by (20). Let $\underset{\sim}{G}$ be the Gram matrix of $\{\tau_j(e_i)\}$. Then

$$\underset{\sim}{G} = \nu^* \nu .$$

<u>Proof:</u> Let P_j be the j-th coordinate projection from $\mathbb{C}^m \otimes \mathbb{C}^k$ onto $\mathbb{C}^m$; then

$$\nu = \sum_{j=0}^{k-1} \tau_j P_j ,$$

and

$$\nu^* \nu = \Big(\sum_{j=0}^{k-1} \tau_j P_j \Big)^* \Big(\sum_{\ell=0}^{k-1} \tau_\ell P_\ell \Big)$$

$$= \Big(\sum_{j=0}^{k-1} P_j^* \tau_j^* \Big) \Big(\sum_{\ell=0}^{k-1} \tau_\ell P_\ell \Big)$$

$$= \sum_j \sum_\ell P_j^* \tau_j^* \tau_\ell P_\ell .$$

It is easy to see that $P_i^* \tau_{i-1}^* \tau_{j-1} P_j$ is exactly $\underset{\sim}{G}_{ij}$. Therefore $\underset{\sim}{G} = \nu^* \nu$.

<u>Remark:</u> When m=n=1, one can see [44][45].

CHAPTER FOUR

COMPUTATION OF PROJECTIONS

Introduction:

All the notation of Chapter two and Chapter three is retained. In this chapter we study the relation between two spaces $H^2_{mx1} \ominus BH^2_{mx1}$ and $H^2_{mx1} \ominus \beta H^2_{mx1}$. Since β is a scalar inner function in H^2, the space $H^2_{mx1} \ominus \beta H^2_{mx1}$ can be indentified with the direct sum of m copies of $H^2 \ominus \beta H^2$, i.e. $H^2_{mx1} \ominus \beta H^2_{mx1} = (H^2 \ominus_\beta H^2) \otimes C^m$. In view of this important property the generalized Sarason operator $T = P_B M_F P_\beta^*$ can be formulated as $\underset{\sim}{\Pi} P_\beta M_F P_\beta^*$, where $\underset{\sim}{\Pi}$ is the orthogonal projection from $H^2_{mx1} \ominus \beta H^2_{mx1}$ to $H^2_{mx1} \ominus BH^2_{mx1}$. This is the key idea in our matrix computation of the operator T. In this way T can be calculated very efficiently. With respect to our decompositions of $H^2_{mx1} \ominus \beta H^2_{mx1}$ and $H^2_{mx1} \ominus BH^2_{mx1}$, $\underset{\sim}{\Pi}$ is expressible by a block matrix in terms of the coefficients of the numerator of B(z). Theorem 4.2.1 is the main theorem.

Contents:

4.1 The direct decomposition of $H^2_{m\times 1} \ominus \beta H^2_{m\times 1}$

Let B be an inner matrix in $RH^{\infty}_{m\times m}$. The determinant of B, β, is a scalar inner function in H^{∞}. An inner matrix and its determinant are related in the following way

$$\beta\, H^2_{m\times 1} \subseteq BH^2_{m\times 1}\,.$$

It follows that

$$(BH^2_{m\times 1})^{\perp} \subseteq (\beta H^2_{m\times 1})^{\perp}\,.$$

In view of Theorem 3.2.2, this means

$$H^2_{m\times 1} \ominus BH^2_{m\times 1} \subseteq H^2_{m\times 1} \ominus \beta H^2_{m\times 1}\,, \qquad (1)$$

or

$$\mathrm{Ker}\, \iota B^*(S)\iota^* \subseteq \mathrm{Ker}\, \iota\, \beta^*(S) I\, \iota^*$$

where ι is the isomorphism from $\mathbb{C}^m \otimes H^2$ onto $H^2_{m\times 1}$. Let K_B, K_β denote $\mathrm{Ker}\,\iota B^*(S)\iota^*$ and $\mathrm{Ker}\, \iota\, \beta^*(S) I \iota^*$.

Then

$$K_\beta = K_B \oplus K_B^{\perp} \cap K_\beta$$

$$= K_B \oplus (\iota B(S^*)\iota^*)\, \mathrm{Ker}\, \iota\, \mathrm{adj}\, B^*(S)\iota^*\,. \qquad (2)$$

In order to show the identity (2) we need the following simple lemma.

<u>Lemma 4.1.1</u> Let $B \in RH^{\infty}_{m\times m}$ be an inner matrix and let adj B(z) denote the adjugate of B(z). Then

$$\iota\, (\mathrm{Range}\, B(S^*) \cap \mathrm{Ker}(\mathrm{adj}\, B^*(S).B^*(S)))\iota^*$$
$$= (\iota B(S^*)\iota^*)\, \mathrm{Ker}\, \iota\, \mathrm{adj}\, B^*(S)\iota^*\,.$$

<u>Proof:</u> Let $h \in (\iota B(S^*)\iota^*)\, \mathrm{Ker}\, \iota\, \mathrm{adj}\, B^*(S)\iota^*$,

then

$$h = (\iota B(S^*)\iota^*)(f)\,,$$

where

$$f \in \mathrm{Ker}\, \iota\, \mathrm{adj}\, B^*(S)\iota^*, \text{ i.e } \iota\, \mathrm{adj}\, B^*(S)\iota^* f = 0\,,$$

and

$$(\iota \operatorname{adj} B^*(S)\iota^*)(\iota B^*(S)\iota^* h)$$

$$= \iota \operatorname{adj} B^*(S)\iota^* \iota B^*(S)\iota^* \iota B(S^*)\iota^* f$$

$$= \iota \operatorname{adj} B^*(S)B^*(S)B(S^*)\iota^* f$$

$$= \iota \operatorname{adj} B^*(S)\iota^* f$$

$$= 0 .$$

This implies $h \in \operatorname{Ker} \iota \operatorname{adj} B^*(S)\iota^*$ and $h \in \operatorname{Range} \iota B(S^*)\iota^*$.

The other hand is trivial.

Since

$$K_B^{\perp} \cap K_{\beta} = \operatorname{Range}(\iota B^*(S)\iota^*)^* \cap \operatorname{Ker} \iota \beta^*(S) I \iota^*$$

$$= (\operatorname{Range} \iota B(S^*)\iota^*) \cap \operatorname{Ker} \iota \operatorname{adj} B^*(S).B^*(S)\iota^*,$$

We have by lemma 4.1.1

$$K_B^{\perp} \cap K_{\beta} = \iota B(S^*)\iota^*(\operatorname{Ker} \iota \operatorname{adj} B^*(S)\iota^*).$$

Thus

$$K_{\beta} = K_B \oplus B(H^2_{m\times 1} \ominus \operatorname{adj} BH^2_{m\times 1}) .$$

In fact we have the following result. The proof is similar to Lemma 4.1.1.

<u>Lemma 4.1.2</u> Let $B \in RH^{\infty}_{m\times m}$ be an inner matrix, and let B_1 and B_2 be inner matrices such that

$$B = B_1 B_2 .$$

Then

$$H^2_{m\times 1} \ominus BH^2_{m\times 1} = (H^2_{m\times 1} \ominus B_1 H^2_{m\times 1}) \oplus B_1(H^2_{m\times 1} \ominus B_2 H^2_{m\times 1}).$$

Let $B \in H^{\infty}_{m\times n}$ be an inner matrix with an irreducible right MFD

$$B(z) = N(z)D(z)^{-1} \qquad (3)$$

with N(z) a regular polynomial matrix of degree k, say

$$N(z) = B_0 + B_1 z + \dots + B_k z^k \text{ , with } \det B_k \neq 0 . \quad (4)$$

Then the determinant of B(z) is

$$\beta(z) = \det B(z) = \frac{\det N(z)}{\det D(z)} , \quad (5)$$

under the hypothesis that N(z) is regular with degree k, β is a rational function in H^∞ with the numerator b_N of degree km;

$$b_N(z) = b_0 + b_1 z + \dots + b_{km} z^{km} , \quad (6)$$

and

$$H^2_{m\times 1} \ominus \beta H^2_{m\times 1} = \operatorname{Ker} \iota (I_{\mathbb{C}^m} \otimes \overset{*}{\beta}(S))\iota^*$$

$$= \operatorname{Ker} \iota (I_{\mathbb{C}^m} \otimes b_N^*(S))\iota^*$$

$$= H^2_{m\times 1} \ominus b_N H^2_{m\times 1} = (H^2 \ominus b_N H^2)\otimes \mathbb{C}^m \quad (7)$$

when m=1, $\operatorname{Ker} \iota (I_{\mathbb{C}^m} \otimes b_N^*(S))\iota^* = \operatorname{Ker} b_N^*(S)$.

From Theorem 3.2.2 or from [4] [35] [36] it follows that $H^2 \ominus b_N H^2$ is the space of all $\hat{b}_N(z)^{-1} w(z)$ where w(z) is a polynomial of degree less than km, and

$$\hat{b}_N(z) = z^{km} b_N\left(\frac{1}{\bar{z}}\right)^- = \bar{b}_{km} + \bar{b}_{km-1} z + \dots \bar{b}_0 z^{km} . \quad (8)$$

Now let

$$f_j(z) = \hat{b}_N(z)^{-1}(\bar{b}_{km} z^j + \bar{b}_{km-1} z^{j+1} + \dots + \bar{b}_{j+1} z^{km-1}) \quad (9)$$

j = 0,1,2,...km-1.

It looks more natural if we write (9) in power series form:

$$f_j(z) = z^j + O(z^{km}) . \quad (10)$$

Since $\bar{b}_{km} z^j + \bar{b}_{km-1} z^{j+1} + \dots + \bar{b}_{j+1} z^{km-1}$ can be written

$$z^j \hat{b}_N(z) + z^{km} \hat{b}_N(z) k(z)$$

for some k in H^2. Thus if we define F_j to be the subspace of $H^2_{mx1} \ominus \beta H^2_{mx1}$ by

$$F_j = \langle f_j \otimes e_i \mid i=1,2,\ldots\ldots m \rangle. \qquad (11)$$

where $\langle\ \rangle$ denotes the linear span. Then let

$$\sigma_j : \mathbb{C}^m \longrightarrow F_j$$

be defined by

$$\sigma_j(x) = f_j(z)x = (f_j \otimes x)(z), \qquad (12)$$

$0 \leq j \leq km-1$.

Therefore, there are km summands in the decomposition of $H^2_{mx1} \ominus \beta H^2_{mx1}$; $F_0 \oplus F_1 \oplus \ldots\ldots \oplus F_{km-1}$, and each summand can be identified with $\mathbb{C}^m$, $\{f_j \otimes e_i : j=0,1,2,\ldots,km-1,\ i=1,2,\ldots,m\}$ is a basis for $H^2_{mx1} \ominus \beta H^2_{mx1}$, where $f_j \otimes e_i = \sigma_j(e_i)$. We consider this basis ordered as $f_0 \otimes e_1, f_1 \otimes e_1, \ldots, f_{km-1} \otimes e_1, f_0 \otimes e_2, f_1 \otimes e_2, \ldots, f_{km-1} \otimes e_2, \ldots\ldots$, and will call this the standard basis for $H^2_{mx1} \ominus \beta H^2_{mx1}$. These symbols will be used throughout the thesis without further mention. From the observation of the basis of $H^2_{mx1} \ominus \beta H^2_{mx1}$, $H^2_{mx1} \ominus \beta H^2_{mx1}$ is the direct sum of m copies of $H^2 \ominus \beta H^2$, i.e. $H^2_{mx1} \ominus \beta H^2_{mx1} = (H^2 \ominus \beta H^2) \otimes \mathbb{C}^m$, and if we let P_β be the orthogonal projection of H^2_{mx1} onto $H^2_{mx1} \ominus \beta H^2_{mx1}$, then $P_\beta = P \otimes I_m$, mxm diagonal matrix operator with entry P, P is the orthogonal projection of H^2 to $H^2 \ominus \beta H^2$. Let $I_m \otimes P = \iota^*(P \otimes I_m)\iota$. Certainly, $H^2_{mx1} \ominus \beta H^2_{mx1}$ is invariant under $\underset{\sim}{S}$. The restriction of $\underset{\sim}{S}$ to $H^2_{mx1} \ominus \beta H^2_{mx1}$ is denoted by $\underset{\sim}{S}_\beta$.

$$\underset{\sim}{S}_\beta = \underset{\sim}{S} \Big|_{H^2_{mx1} \ominus \beta H^2_{mx1}}.$$

As we have seen in the previous chapter, Theorem 3.2.2 the matrix of $\underset{\sim}{S}_\beta$ with respect to $\{f_j \otimes e_i\}$ is

$$I_m \otimes C_{b_N^*} \,, \tag{13}$$

where $C_{b_N^*}$ is the companion matrix of the polynomial b_N^*, and I_m is the mxm identity matrix.

$$b_N^*(z) = \bar{b}_0 + \bar{b}_1 z + \ldots + \bar{b}_{km} z^{km} \;;$$

in other words,

$$C_{b_N^*} = \begin{bmatrix} 0 & 1 & 0\ldots\ldots & 0 \\ 0 & 0 & 1\ldots\ldots & 0 \\ & & & \vdots \\ 0\ldots\ldots & & & 1 \\ \bar{b}_{km}^{-1}\bar{b}_0,\ldots\ldots & & & \bar{b}_{km}^{-1}\bar{b}_{km-1} \end{bmatrix}_{km \times km} ,$$

$I_m \otimes C_{b_N^*}$ is the tensor product of two matrices.

Let G be the Gram matrix of $\{f_j : j=0,1,2,\ldots,km-1\}$: then the Gram matrix G_β of $\{f_j \otimes e_i : j=0,1,2,\ldots,km-1, \; i=1,2,\ldots m\}$ is

$$G_\beta = I_m \otimes G \,. \tag{14}$$

There is a remarkable, simple result in Young [4]; it is a formula for the inverse G^{-1} of the Gram matrix of $\{f_j : j=0,1,2,\ldots km-1\}$, to wit

$$G^{-1} = 1 - Q^* Q$$

where

$$Q = \begin{bmatrix} Q_1, & Q_2,\ldots\ldots\ldots\ldots, & Q_{km} \\ 0\;, & Q_1,\ldots\ldots & Q_{km-1} \\ & & \\ 0\;, & 0\;,\ldots\ldots & Q_1 \end{bmatrix} \tag{15}$$

and

$$Q_1 = \frac{\bar{b}_0}{b_{km}}$$

$$Q_i = (\bar{b}_{i-1} - b_{km-i+1} Q_1 - b_{km-i+2} Q_2 - \ldots - b_{km-1} Q_{i-1}) / b_{km}$$

$2 \le i \le km$.

4.2 Computation of projections

Let $\underset{\sim}{\Pi}$ be the orthogonal projection from $H^2_{mx1} \ominus \beta H^2_{mx1}$ into $H^2_{mx1} \ominus BH^2_{mx1}$. Then

$$P_B = \underset{\sim}{\Pi} P_\beta \, . \tag{16}$$

The generalized Sarason operator $T = P_B M_F P_\beta^*$ acting from $H^2_{nx1} \ominus \beta H^2_{nx1}$ to $H^2_{mx1} \ominus BH^2_{mx1}$ can be written as

$$T = \underset{\sim}{\Pi} P_\beta M_F P_\beta^* \tag{17}$$

i.e.

$$\begin{array}{ccc} H^2_{nx1} \ominus \beta H^2_{nx1} & \xrightarrow{P_\beta M_F P_\beta^*} & H^2_{mx1} \ominus \beta H^2_{mx1} \\ & \searrow^{T} & \downarrow \underset{\sim}{\Pi} \\ & & H^2_{mx1} \ominus BH^2_{mx1} \, . \end{array}$$

The operator $P_\beta M_F P_\beta^*$ acts from $H^2_{nx1} \ominus \beta H^2_{nx1}$ to $H^2_{mx1} \ominus \beta H^2_{mx1}$ and $P_\beta M_F P_\beta^* = P_\beta \iota F(S^*) \iota^* P_\beta^* = (P_\beta \iota F^*(S) \iota^* P_\beta^*)^*$,

$$\begin{aligned} P_\beta \iota F^*(S) \iota^* P_\beta^* &= \iota (I_m \otimes P) F^*(S) (I_m \otimes P^*) \iota^* \\ &= \iota (I_m \otimes P) \frac{1}{2\pi i} \oint_C F^*(z) \otimes (zI - S)^{-1} dz (I_m \otimes P^*) \iota^* \\ &= \iota \frac{1}{2\pi i} \oint_C F^*(z) \otimes P(zI - S)^{-1} P^* \iota^* \\ &= \iota \frac{1}{2\pi i} \oint_C F^*(z) \otimes (zI - PSP^*)^{-1} \iota^* \\ &= \iota F^*(S|_{H^2 \ominus \beta H^2}) \iota^* \, . \end{aligned}$$

The adjoint of $\iota F^*(S) \iota^* \Big|_{H^2_{mx1} \ominus \beta H^2_{mx1}}$ is the compression $\iota F(S^*) \iota^*$ to $H^2_{mx1} \ominus \beta H^2_{mx1}$.

Therefore

$$P_\beta M_F P_\beta^* = P_\beta \iota F(S^*) \iota^* P_\beta^* = \iota F(S_\beta^*) \iota^* \tag{18}$$

With respect to the basis $\{f_j \otimes e_i \mid j = 0,1,2,\ldots,km-1, i = 1,2,\ldots,m\}$, the matrix of $P_\beta M_F P_\beta^*$ can be calculated by applying Young's algorithm to every entry of F. Therefore forming the matrix of the orthogonal projection $\underset{\sim}{\pi}$ with respect to $\{\tau_j(e_i) : j= 0,1.2,\ldots,k-1, i = 1,2,\ldots m\}$ and $\{f_j \otimes e_i : j = 0,1,2,\ldots km-1, i = 1,2,\ldots m\}$ is the vital step to us.

The main idea comes from the fact that $H^2_{mx1} \ominus BH^2_{mx1}$ is contained in $H^2_{mx1} \ominus \beta H^2_{mx1}$. Firstly, we are going to show that given any $x \in C^m$, $\tau_j(x)$ is in $H^2_{mx1} \ominus BH^2_{mx1}$, and therefore in $H^2_{mx1} \ominus \beta H^2_{mx1}$, which can be expressed in terms of $\{\sigma_j(x) : j = 0,1,2,\ldots, km-1\}$. It needs a laborious calculation.

Since

$$b_N(z)I_m = \det N(z) = N(z)\cdot \operatorname{adj} N(z) = \operatorname{adj} N(z)\cdot N(z)$$

degree $b_N(z) = km$, thus the degree of adj $N(z)$ is km-k.

Let

$$\operatorname{adj} N(z) = C_0+C_1 z+\ldots\ldots+C_{km-k} z^{km-k} ,$$

thus

$$\operatorname{adj} \hat{N}(z) = C^*_{km-k}+C^*_{km-k-1}z+\ldots\ldots+C^*_1 z^{km-k-1}+C^*_0 z^{km-K}.$$

Moreover,

$$\tau_j(x)(z) = \hat{N}(z)^{-1}T_j(z)x = \hat{N}(z)^{-1}\operatorname{adj} \hat{N}(z)^{-1}\operatorname{adj} \hat{N}(z)T_j(z)x$$

$$= [\operatorname{adj} \hat{N}(z)\hat{N}(z)]^{-1}(\operatorname{adj} \hat{N}(z)T_j(z)x)$$

$$= \hat{b}_N(z)^{-1}(\operatorname{adj} \hat{N}(z)T_j(z)x) \qquad (19)$$

and

$$(b_0+b_1z+\dots+b_{km}z^{km})I_m = N(z)\cdot\text{adj } N(z) = \text{adj } N(z)\cdot N(z)$$

$$= (B_0+B_1z+\dots+B_kz^k)(C_0+C_1z+\dots+C_{km-k}z^{km-k})$$

$$= (C_0+C_1z+\dots+C_{km-k}z^{km-k})(B_0+B_1z+\dots+B_kz^k). \qquad (20)$$

By comparing the coefficients of both sides of (20) we get the following relations:

$$\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{k-1} \end{bmatrix} \otimes I_m = \begin{bmatrix} B_0 & 0 \dots & & 0 \\ B_1 & B_0 & & \\ \vdots & \vdots & \ddots & \\ B_{k-1} & B_{k-2} & \dots & B_0 \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ \\ C_{k-1} \end{bmatrix},$$

$$\begin{bmatrix} b_k \\ \vdots \\ \\ b_{km-k} \end{bmatrix} \otimes I_m = \begin{bmatrix} C_0 & C_1 & & C_k \\ C_1 & C_2 & & C_{k+1} \\ \vdots & \vdots & & \\ C_{km-2k}, & \dots & & C_{km-k} \end{bmatrix} \begin{bmatrix} B_k \\ B_{k-1} \\ \\ B_0 \end{bmatrix},$$

$$\begin{bmatrix} b_{km-k+1} \\ \vdots \\ \\ b_{km-1} \\ b_{km} \end{bmatrix} \otimes I_m = \begin{bmatrix} B_1 & B_2 & & & B_k \\ B_2 & B_3 & & B_k & 0 \\ \vdots & \vdots & & & \\ B_{k-1} & B_k & 0 & & 0 \\ B_k & 0 & 0 & & 0 \end{bmatrix} \begin{bmatrix} C_{km-k} \\ C_{km-k-1} \\ \\ \\ C_{km-2k+1} \end{bmatrix}.$$

$\tau_j(x)(z)$ can be written as follows

$$\tau_j(x)(z)=\hat{N}(z)^{-1}T_j(z)x$$

$$=\hat{b}_N(z)^{-1}(\text{adj } \hat{N}(z)T_j(z)x)$$

$$=\hat{b}_N(z)^{-1}((C^*_{km-k}+C^*_{km-k-1}z+\ldots+C^*_0 z^{km-k})(B^*_k z^j+\ldots+B^*_{j+1}z^{k-1}))x$$

$$=\hat{b}_N(z)^{-1}(C^*_{km-k}B^*_k z^j+(C^*_{km-k}B^*_{k-1}+C^*_{km-k-1}B^*_k)z^{j+1}+\ldots\ldots$$

$$+\ldots\ldots+C^*_0 B^*_{j+1}z^{km-1})x$$

$$=\hat{b}_N(z)^{-1}\Big\{\bar{b}_{km}z^j+\bar{b}_{km-1}z^{j+1}+\ldots\ldots+\bar{b}_{km-k+j+1}z^{k-1}$$

$$+\Big[\bar{b}_{km-k+j}I_m-(C^*_{km-k}B^*_j)\ z^k$$

$$+\Big[\bar{b}_{km-k+j-1}I_m-(C^*_{km-k-1}B^*_j+C^*_{km-k}B^*_{j-1})\Big]z^{k+1}$$

$$+\ldots\ldots\ldots\ldots$$

$$+\Big[\bar{b}_{km-k}I_m-(C^*_{km-k-j}B^*_j+C^*_{km-k-j+1}B^*_{j-1}+\ldots+C^*_{km-k}B^*_0\Big]z^{k+j}$$

$$+\ldots\ldots\ldots\ldots$$

$$+\Big[\bar{b}_{j+1}I_m-(C^*_1B^*_j+C^*_2B^*_{j-1}+\ldots+C^*_{j+1}B^*_0)\Big]z^{km-1}\Big\}\ x$$

$$=\hat{b}_N(z)^{-1}(\bar{b}_{km}z^j+\bar{b}_{km-1}z^{j+1}+\ldots+\bar{b}_{j+1}z^{km-1})x\ -$$

$$\hat{b}_N(z)^{-1}\Big[(C^*_{km-k}B^*_j z^k)+(C^*_{km-k-1}B^*_j+C^*_{km-k}B^*_{j-1})z^{k+1}+\ldots$$

$$+\ldots\ldots\ldots+(C^*_1B^*_j+C^*_2B^*_{j-1}+\ldots+C^*_{j+1}B^*_0)z^{km-1}\Big]x$$

$$=\sigma_j(x)(z)+\sigma_k(y^{(j)}_k)(z)+\ldots+\sigma_{km-1}(y^{(j)}_{km-1})\ (z) \qquad (21)$$

for $j=0,1,2,\ldots k-1$, where

$$y^{(j)}_r = A^{(j)}_r x$$

for some $A^{(j)}_r \in M_{mxm}$, $r=k,k+1,\ldots km-1$.

Then

$$\sigma_k(y_k^{(j)})+\sigma_{k+1}(y_{k+1}^{(j)})+\ldots+\sigma_{km-1}(y_{km-1}^{(j)})\ (z)$$

$$= \hat{b}_N(z)^{-1}(\bar{b}_{km}A_k^{(j)}z^k+\bar{b}_{km-1}A_k^{(j)}z^{k+1}+\ldots+\bar{b}_{k+1}A_k^{(j)}z^{km-1}$$

$$+\bar{b}_{km}A_{k+1}^{(j)}z^{k+1}\ +\ldots+\bar{b}_{k+2}A_{k+1}^{(j)}z^{km-1}$$

$$+\ldots\ldots\ldots\ldots$$

$$+\bar{b}_{km}A_{km-1}^{(j)}z^{km-1})\ x$$

$$= \hat{b}_N(z)^{-1}\ [\bar{b}_{km}A_k^{(j)}z^k+(\bar{b}_{km-1}A_k^{(j)}+\bar{b}_{km}A_{k+1}^{(j)})z^{k+1}$$

$$+(\bar{b}_{km-2}A_k^{(j)}+\bar{b}_{km-1}A_{k+1}^{(j)}+\bar{b}_{km}A_{k+2}^{(j)})z^{k+2}$$

$$+\ldots\ldots\ldots\ldots$$

$$+(\bar{b}_{km-j}A_k^{(j)}+\bar{b}_{km-j+1}A_{k+1}^{(j)}+\ldots+\bar{b}_{km}A_{k+j}^{(j)})z^{k+j}$$

$$+\ldots\ldots\ldots\ldots$$

$$+(\bar{b}_{k+1}A_k^{(j)}\ +\bar{b}_{k+2}A_{k+1}^{(j)}+\ldots\ldots+\bar{b}_{km}A_{km-1}^{(j)})z^{km-1}]\ x. \qquad (22)$$

From (21) and (22) we have the following relations.

Let

$$\delta_0^{(j)} = C_{km-k}^{*}B_j^{*}$$

$$\delta_1^{(j)} = C_{km-k-1}^{*}B_j^{*}+C_{km-k}^{*}B_{j-1}^{*}$$

$$\vdots \qquad (23)$$

$$\delta_r^{(j)} = C_{km-k-r}^{*}B_j^{*}+C_{km-k-r+1}^{*}B_{j-1}^{*}+\ldots+C_{km-k}^{*}B_{j-r}^{*}$$

$0 \leq r \leq km-k-1.$

Then

$$\delta_0^{(j)} = -(\bar{b}_{km}A_k^{(j)})$$

$$\delta_1^{(j)} = -(\bar{b}_{km-1}A_k^{(j)} + \bar{b}_{km}A_{k+1}^{(j)})$$

$$\delta_2^{(j)} = -(\bar{b}_{km-2}A_k^{(j)} + \bar{b}_{km-1}A_{k+1}^{(j)} + \bar{b}_{km}A_{k+2}^{(j)}) \qquad (24)$$

$$\delta_r^{(j)} = -(\bar{b}_{km-r}A_k^{(j)} + \bar{b}_{km-r+1}A_{k+1}^{(j)} + \cdots\cdots + \bar{b}_{km}A_{k+r}^{(j)})$$

where $0 \leq r \leq km-k-1$.

Let us define the sequence $\{\alpha_i \mid i = 0,1,2,\ldots\ldots,km-k-1\}$ by the following relations.

$$\alpha_0 = -\bar{b}_{km}^{-1}$$

$$\alpha_1 = \alpha_0(\alpha_0\bar{b}_{km-1})$$

$$\alpha_2 = \alpha_0(\alpha_0\bar{b}_{km-2}+\alpha_1\bar{b}_{km-1}) \qquad (25)$$

$$\alpha_r = \alpha_0(\alpha_0\bar{b}_{km-r}+\alpha_1\bar{b}_{km-r+1}+\cdots\cdots\cdots+\alpha_{r-1}\bar{b}_{km-1})$$

where $r = 0,1,\ldots.km-k-1$.

Now combining the formulas (24) and (25), this gives us

$$A_k^{(j)} = (-\bar{b}_{km}^{-1})\,\delta_0^{(j)} = \alpha_0\,\delta_0^{(j)}$$

$$A_{k+1}^{(j)} = (-\bar{b}_{km}^{-1})\,(\bar{b}_{km-1}A_k^{(j)} + \delta_1^{(j)})$$

$$= (-\bar{b}_{km}^{-1})\,(\bar{b}_{km-1}\,\alpha_0\,\delta_0^{(j)} + \delta_1^{(j)})$$

$$= \alpha_0(\alpha_0\bar{b}_{km-1})\,\delta_0^{(j)} + \alpha_0\,\delta_1^{(j)}$$

$$= \alpha_1\,\delta_0^{(j)} + \alpha_0\,\delta_1^{(j)}$$

and

$$A^{(j)}_{k+r} = (-\bar{b}_{km}{}^{-1})\ (\bar{b}_{km-r}A^{(j)}_{k}+\bar{b}_{km-r+1}A^{(j)}_{k+1}+\ldots+\bar{b}_{km-1}A^{(j)}_{k+r})$$

$$= \alpha_0(\alpha_0\bar{b}_{km-r}+\alpha_1\bar{b}_{km-r+1}+\ldots+\alpha_{r-1}\bar{b}_{km-1})\delta^{(j)}_0$$

$$+\ldots\ldots\ldots\ldots$$

$$+\alpha_0(\alpha_0\bar{b}_{km-1})\delta^{(j)}_{r-1}$$

$$+\alpha_0\,\delta^{(j)}_r$$

so

$$A^{(j)}_{k+r} = \alpha_r\delta^{(j)}_0+\alpha_{r-1}\delta^{(j)}_1+\ldots+\alpha_1\delta^{(j)}_{r-1}+\alpha_0\delta^{(j)}_r$$

where $r=0,1,\ldots km-k-1$. $j=0,1,\ldots k-1$. By (23)

$$A^{(j)}_{k+r} = \alpha_r(C^*_{km-k}B^*_j)$$

$$+\alpha_{r-1}(C^*_{km-k-1}B^*_j+C^*_{km-k}B^*_{j-1})$$

$$+$$

$$+\alpha_0(C^*_{km-k-r}B^*_j+C^*_{km-k-r+1}B^*_{j-1}+\ldots+C^*_{km-k}B^*_{j-r})$$

$$= (\alpha_r C^*_{km-k}+\alpha_{r-1}C^*_{km-k-1}+\ldots+\alpha_0 C^*_{km-k-r})B^*_j$$

$$+(\qquad \alpha_{r-1}C^*_{km-k}\quad +\ldots+\alpha_0 C^*_{km-k-r+1})B^*_{j-1}$$

$$+\alpha_0 C^*_{km-k}\ )B^*_{j-r}.$$

From these observations we have

Let

$$\underset{\sim}{\alpha} \equiv \begin{vmatrix} \alpha_0 & 0 & 0\ldots\ldots\ldots & & 0 \\ \alpha_1 & \alpha_0 & 0\ldots\ldots\ldots & & 0 \\ & & \ddots & & \vdots \\ \alpha_{r-1} & \alpha_{r-2}\cdots & & \alpha_0 & 0 \\ & & & & \ddots \\ \alpha_{km-k-1} & \cdots & \cdots & & \alpha_0 \end{vmatrix}_{(km-k)x(km-k)} \otimes I_m$$

$$\underset{\sim}{C} = \begin{bmatrix} C^*_{km-k} & 0 & \cdots & 0 \\ C^*_{km-k-1} & C^*_{km-k} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C^*_{km-2k+1} & \cdots & \cdots & C^*_{km-k} \\ C^*_{km-2k} & C^*_{km-2k+1} & \cdots & C^*_{km-k-1} \\ \vdots & \vdots & & \vdots \\ C^*_1 & C^*_2 & \cdots & C^*_k \end{bmatrix}_{(km-k)xk} .$$

Then

$$\underset{\sim}{W} = \underset{\sim}{\alpha}\,\underset{\sim}{C} = [W_{rs}] = \begin{bmatrix} w_0 & 0 & 0 & \cdots & 0 \\ w_1 & w_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{k-1} & w_{k-2} & \cdots & \cdots & w_0 \\ w_k & w_{k-1} & \cdots & \cdots & w_1 \\ \vdots & \vdots & & & \vdots \\ w_{km-k-1} & \cdots & \cdots & \cdots & w_{km-2k} \end{bmatrix}_{(km-k)xk} ,$$

where

$$w_0 = \alpha_0 C^*_{km-k}$$

$$w_1 = \alpha_0 C^*_{km-k-1} + \alpha_1 C^*_{km-k}$$

$$\vdots$$

$$w_r = \alpha_0 C^*_{km-k-r} + \alpha_1 C^*_{km-k-r-1} + \cdots\cdots + \alpha_r C^*_{km-k}$$

$r = 0,1,\ldots km-k-1$. Define $\underset{\sim}{E}$ by

$$\underset{\sim}{E} = \begin{bmatrix} B^*_0 & B^*_1 & \cdots & \cdots & B^*_{k-1} \\ 0 & B^*_0 & B^*_1 & \cdots & B^*_{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & B^*_0 \end{bmatrix}_{kxk}$$

and it is easy to see that (r,j+1) entry in $\underset{\sim}{W}\cdot\underset{\sim}{E}$ is

$$
\begin{aligned}
& w_r B_j^* + w_{r-1} B_{j-1}^* + \cdots + w_{r-1} B_0^* \\
= & (\alpha_0 C_{km-k-r}^* + \alpha_1 C_{km-k-r+1}^* + \cdots + \alpha_r C_{km-k}^*) B_j^* + \cdots \\
& + \cdots\cdots + (\alpha_0 C_{km-k}^*) B_{j-r}^* \\
= & A_{k+r}^{(j)} ,
\end{aligned}
$$

where r=0,1,...km-k-1.

In summary, the calculation of the injection is contained in the following theorem.

<u>Theorem 4.2.1</u> Let J be the natural injection of $H^2_{mx1} \ominus BH^2_{mx1}$ into $H^2_{mx1} \ominus \beta H^2_{mx1}$. The matrix [J] of J with respect to the standard bases $\{\sigma_j(e_i) : j=0,1,2,\ldots,k-1, i=1,2,\ldots,m\}$, $\{f_j \otimes e_i : j=0,1,2,\ldots,km-1, i=1,2,\ldots,m\}$ is a block matrix :

$$
[J] = \left[\begin{array}{c} \begin{matrix} I_m & & \bigcirc \\ & I_m & \\ \bigcirc & & \ddots \; I_m \end{matrix} \\ \hline \underset{\sim}{\alpha} \; \underset{\sim}{C} \; \underset{\sim}{E} \end{array} \right] \begin{array}{l} k\text{x}k \\ \\ (nmk-k)\text{x}(km-k) \end{array} . \tag{26}
$$

$\underset{\sim}{\alpha}$, $\underset{\sim}{C}$ and $\underset{\sim}{E}$ are block Toeplitz matrices given by

$$
\underset{\sim}{\alpha} = \begin{bmatrix} \alpha_0 & 0 & 0 & \cdots & 0 \\ \alpha_1 & \alpha_0 & 0 & & 0 \\ \vdots & & \ddots & & \vdots \\ \alpha_{km-k-1} & \cdots & \cdots & \cdots & \alpha_0 \end{bmatrix} \otimes I_m \quad (km-k)\text{x}(km-k) \tag{27}
$$

and

$$\alpha_0 = -\bar{b}_{km}^{-1}$$

$$\alpha_r = \alpha_0(\alpha_0\bar{b}_{km-r} + \alpha_1\bar{b}_{km-r+1} + \cdots\cdots + \alpha_{r-1}\bar{b}_{km-1}$$

r = 1,2,....km-k-1,

$$\underset{\sim}{C} = \begin{bmatrix} C^*_{km-k} & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ C^*_{km-2k+1} & \cdots & & C^*_{km-k} \\ C^*_{km-2k} & \cdots & & C^*_{km-k-1} \\ \vdots & & & \vdots \\ C^*_1 & \cdots & & C^*_k \end{bmatrix}_{(km-k)\times k} \qquad (28)$$

$$\underset{\sim}{E} = \begin{bmatrix} B^*_0 & B^*_1 & \cdots & \cdots & B^*_{k-1} \\ 0 & B^*_0 & B^*_1 & \cdots & B^*_{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B^*_0 \end{bmatrix}_{k\times k} . \qquad (29)$$

Corollary 4.2.2 Let $\underset{\sim}{\Pi}$ be the orthogonal projection from $H^2_{mx1} \ominus \beta H^2_{mx1}$ into $H^2_{mx1} \ominus BH^2_{mx1}$. The matrix $[\underset{\sim}{\Pi}]$ of $\underset{\sim}{\Pi}$ with respect to the standard bases $\{ f_j \otimes e_i : j = 0,1,2,\ldots km-1, i = 1,2,\ldots m\}$, $\{\sigma_j(e_i) : j = 0,1,2,\ldots k-1, \ i = 1,2,\ldots m\}$ is a block matrix

$$[\underset{\sim}{\Pi}] = G_B^{-1} [J]^* G_\beta , \qquad (30)$$

G_B and G_β are the Gram matrices of $\{\sigma_j(e_i)\}$ and $\{f_j \otimes e_i\}$, res-pectively.

Proof: Let J be the natural injection of $H^2_{m\times 1} \ominus BH^2_{m\times 1}$ into $H^2_{m\times 1} \ominus \beta H^2_{m\times 1}$, then J^* is the orthogonal projection of $H^2_{m\times 1} \ominus \beta H^2_{m\times 1}$ onto $H^2_{m\times 1} \ominus BH^2_{m\times 1}$. From the above constructive proof, the matrix of J with respect to $\{\sigma_j(e_i)\}$ and $\{f_j \otimes e_i\}$ is a block matrix

$$[J] = \left[\begin{array}{ccc} & I & \\ \hline \underset{\sim}{\alpha} & \underset{\sim}{C} & \underset{\sim}{E} \end{array}\right].$$

Choose matrices $[Q_B]$, $[Q_\beta]$ such that $[Q_B][Q_B]^* = G_B^{-1}$, $[Q_\beta][Q_\beta]^* = G_\beta^{-1}$, and let Q_B, Q_β be the operators on $H^2_{m\times 1} \ominus BH^2_{m\times 1}$ and $H^2_{m\times 1} \ominus \beta H^2_{m\times 1}$, respectively, such that the matrices of Q_B, Q_β with respect to $\{\sigma_j(e_i)\}$, $\{f_j \otimes e_i\}$ are $[Q_B]$ and $[Q_\beta]$. By 1.3 (2) $\{Q_B(\sigma_j(e_i))\}$ and $\{Q_\beta(f_j \otimes e_i)\}$ are orthonormal bases of $H^2_{m\times 1} \ominus BH^2_{m\times 1}$ and $H^2_{m\times 1} \ominus \beta H^2_{m\times 1}$, the matrix of $[J]$ with respect to $\{Q_B(\sigma_j(e_i))\}$ and $\{Q_\beta(f_j \otimes e_i)\}$ is

$$[Q_\beta]^{-1}[J][Q_B].$$

Thus the matrix of J^* with respect to these bases is $[Q_B]^*[J]^*[Q_\beta]^{*^{-1}}$, and therefore the matrix of $\underset{\sim}{\Pi}$ with respect to $\{\sigma_j(e_i)\}$ and $\{f_j \otimes e_i\}$ is

$$[\underset{\sim}{\Pi}] = [Q_B][Q_B]^*[J]^*[Q_\beta]^{*^{-1}}[Q_\beta]^{-1}$$
$$= G_B^{-1}[J]^* G_\beta .$$

The proof is complete.

CHAPTER FIVE

IMPLEMENTATION AND COMMENTS

Introduction:

In this chapter we shall present our algorithm for the matrix valued Nevanlinna-Pick problem and compare it with other algorithms. The final computational version of the generalized Sarason operator is given in Theorem 5.5.1. We will comment only on a few of the significant procedures in the algorithm. Since the computation of the corresponding matrix of the Sarason operator is rather complicated, a higher level programming language is required. ALGOL 68 [25] was chosen not only because of its elegant, mathematical style but also because of its powerful operators and flexible structure. The standard ALGOL 68 NAG or FORTRAN NAG library provides routines for some of our procedures. Two simple numerical examples accompany this theoretical work and are given in the final chapter.

Contents:

5.1 The matrix form of the generalized Sarason operator

For the sake of concrete computation, it is helpful to have a restatement of the solution of the [M-N-P] problem in terms of matrices rather than operators. The main results of our theory are theorems 2.2.5, 2.2.7, 2.3.9, 3.3.2 and 4.2.1.

Theorem 5.1.1 Let $F \in RH^{\infty}_{m\times n}$ and let $B \in RH^{\infty}_{m\times m}$ be an inner matrix with an irreducible MFD, $B(z)=N(z)D(z)^{-1}$ with numerator $N(z)$ of degree k and $N(z)=B_0+B_1z+\dots+B_kz^k$ with $\det B_k \neq 0$, and let $\rho(z)=\det N(z)$. Let $\{\tau_j(e_i) \mid j=0,1,2,\dots \dots k-1,\ i=1,2,\dots m\}$, $\{f_j \otimes e_i \mid j=0,1,2,\dots km-1,\ i=1,2,\dots n\}$ from (3-14) (4-12) be the standard bases of $H^2_{m\times 1} \ominus BH^2_{m\times 1}$ and $H^2_{n\times 1} \ominus \rho H^2_{n\times 1}$, and let $[J]$ be the matrix of the injection from $H^2_{m\times 1} \ominus BH^2_{m\times 1}$ into $H^2_{m\times 1} \ominus \rho H^2_{m\times 1}$ with respect to the standard bases. Let S_ρ have matrix $C_{\rho *}$ with respect to $\{f_j\}_{j=0}^{km-1}$. Let G_B, G_ρ be the Gram matrices of $\{\tau_j(e_i)\}$ and $\{f_j \otimes e_i\}$, and let $[U_B], [U_\rho]$ be mkxmk, nmkxnmk matrices such that

$$[U_B][U_B]^* = G_B^{-1}, \quad [U_\rho][U_\rho]^* = G_\rho^{-1}.$$

Then

(1) The infimum M of $\|G\|_\infty$, over all $G \in RH^{\infty}_{m\times n}$ such that $G \in F+BH^{\infty}_{m\times n}$ is given by

$$M = s_0$$

where $s_0 \geq s_1 \geq s_2 \geq \dots$ are the singular values of the matrix

$$A = [U_B]^*[J]^*F(C^*_{\rho *})[U_\rho]^{*-1} \quad . \qquad (1)$$

There exists $F_0 \in M_{m\times n}$ from (2-22) such that if $\tilde{\rho}=z\rho$, $\tilde{B}=zB$ and $\tilde{F}=F+BF_0$, then M is a singular value of the one step extension matrix

$$\underset{\sim}{A} = [U_{\underset{\sim}{B}}]^*[\underset{\sim}{J}]^* F(C^*_{\beta *})[U_{\beta}]^{*^{-1}} \tag{2}$$

with multiplicity n, where $[U_{\underset{\sim}{B}}], [U_{\beta}], [\underset{\sim}{J}]$ and $C_{\beta *}$ are defined the same way as the above corresponding $\underset{\sim}{B}$, $\underset{\sim}{F}$ and β.

(2) One extremal function G for which this infimum is attained is given by

$$G = [v_0, v_1, \ldots, v_{n-1}][u_0, u_1, \ldots, u_{n-1}]^{-1} \tag{3}$$

where

$$v_r = \sum \eta^{(r)}_{j-1+mi}(\tau_i(e_j)), \qquad u_r = \sum \xi^{(r)}_{i+(j-1)(k+1)m} f_i \otimes e_j$$

$\{\underset{\sim}{\tau}_i(e_j)\}$ and $\{\underset{\sim}{f}_i \otimes e_j\}$ are the standard bases of $H^2_{mx1} \ominus zBH^2_{mx1}$ and $H^2_{nx1} \ominus z\beta H^2_{nx1}$.

and

$$[U_{\underset{\sim}{\beta}}]^* x_r = (\xi^{(r)}_0, \xi^{(r)}_1, \ldots, \xi^{(r)}_{nmk+n-1})$$

$$[U_{\underset{\sim}{B}}]^* \underset{\sim}{A} x_r = (\eta^{(r)}_0, \eta^{(r)}_1, \ldots, \eta^{(r)}_{mk+m-1})$$

for r=0,1,2,...n-1, and x_r is a right eigenvector of $\underset{\sim}{A}^*\underset{\sim}{A}$ corresponding to the eigenvalue $\underset{\sim}{s}^2_r (= M^2)$

<u>Proof:</u> Let $Q_{\beta'}$ be the operator on $H^2 \ominus \beta H^2$ whose matrix with respect to $\{f_j\}$ is $[U_{\beta'}]$. By 1.3 (1)(2), $\{Q_{\beta'} f_j\}$ is an orthonormal basis of $H^2 \ominus \beta H^2$. The matrix of S_β with respect to this basis is

$$[U_{\beta'}]^{-1} C_{\beta *}[U_{\beta'}].$$

Since $\{Q_{\beta'} f_j \otimes e_i\}$ is an orthonormal basis of $H^2_{mx1} \ominus \beta H^2_{mx1}$, the matrix of $F(S^*_\beta)$ with respect to $\{Q_{\beta'} f_j \otimes e_i\}$ is

$$F(([U_{\beta'}]^{-1}C_{\beta *}[U_{\beta'}])^{*})$$

and

$$C^{*}_{\beta *} = C^{T}_{\beta}\ ,\quad [U_{\beta}] = I_m \otimes [U_{\beta'}],$$

thus

$$F([U_{\beta'}]^{*}C^{T}_{\beta}[U_{\beta'}]^{*^{-1}}) = [U_{\beta}]^{*}F(C^{T}_{\beta})[U_{\beta}]^{*^{-1}}.$$

Let Q_B be the operator on $H^2_{mx1} \ominus BH^2_{mx1}$ whose matrix with respect to $\{\sigma_j(e_i)\}$ is $[U_B]$, $\{Q_B(\sigma_j(e_i))\}$ is an orthonormal basis of $H^2_{mx1} \ominus BH^2_{mx1}$, and the matrix $[J]$ of the orthogonal injection J with respect to $\{Q_B(\sigma_j(e_i))\}$ and $\{Q_{\beta'}f_j \otimes e_i\}$ is

$$[U_{\beta}]^{-1}[J]\,[U_B].$$

Therefore the matrix A of the generalized Sarason operator $T = P_B M_F P^{*}_{\beta}$ with respect to $\{Q_{\beta'}f_j \otimes e_i\}$ and $\{Q_B(\sigma_j(e_i))\}$ is

$$A = [U_B]^{*}[J]^{*}[U_{\beta}]^{*^{-1}}[U_{\beta}]^{*}F(C^{T}_{\beta})[U_{\beta}]^{*^{-1}}$$
$$= [U_B]^{*}[J]^{*}F(C^{T}_{\beta})[U_{\beta}]^{*^{-1}},$$

and the matrix of $A^{*}A$ is

$$[U_{\beta}]^{-1}F(C^{T}_{\beta})^{*}[J][U_B][U_B]^{*}[J]^{*}F(C^{T}_{\beta})[U_{\beta}]^{*^{-1}}.$$

Let

$$D = [J][U_B][U_B]^{*}[J]^{*},$$

then

$$A^{*}A = [U_{\beta}]^{-1}F(C^{T}_{\beta})^{*}DF(C^{T}_{\beta})[U_{\beta}]^{*^{-1}}.$$

Thus M^2, which by theorem 2.2.5 is

$$\| J^{*}F(S^{*}_{\beta})\|^2$$

is the largest eigenvalue of the latter matrix:

$$M^2 = \sup\{\lambda \in \mathbb{R} : \det(\lambda I - A^{*}A) = 0\}$$
$$= \sup\{\lambda \in \mathbb{R} : \det(\lambda [U_{\beta}][U_{\beta}]^{*} - F(C^{T}_{\beta})^{*}DF(C^{T}_{\beta})) = 0\}.$$

M^2 is equal to the largest generalized eigenvalue λ_0 of the problem. Moreover, if $x_0 \in \mathbb{C}^{kmn} - \{0\}$

and

$$(\lambda_0 G_\beta^{-1} - F(C_\beta^T)^* DF(C_\beta^T))x_0 = 0 ,$$

then

$$(\lambda_0 - [U_\beta]^{-1} F(C_\beta^T)^* DF(C_\beta^T)[U_\beta]^{*-1})[U_\beta]^* x_0 = 0 ,$$

and hence $[U_\beta]^* x_0$ is an eigenvector of A^*A which implies that $[U_\beta]^* x_0$ is a maximising vector for A. Hence if

$$[U_\beta]^* x_0 = (\xi_0, \xi_1, \ldots, \xi_{kmn-1})$$

and

$$f_j \otimes e_i = g_{j+(i-1)km},$$

$$Q_\beta f_j \otimes e_i = g'_{j+(i-1)km},$$

then $\sum_{j=0}^{kmn-1} \xi_j g'_j$ is a maximising vector for the generalized Sarason operator $T = P_B M_F P_\beta^*$ (Theorem 2.2.7). If we write

$$[U_\beta] = [U_{ij}]_{km \times km} ,$$

then

$$\begin{aligned}
\sum_{j=0}^{kmn-1} \xi_j g'_j &= \sum_j \xi_j (\sum_r U_{rj} g_r) \\
&= \sum_{j,r} \xi_j U_{rj} g_r \\
&= \sum_{j,r} (U_{rj} \xi)_r g_r \\
&= \sum_r ([U_\beta][U_\beta]^* x_0)_r g_r \\
&= \sum_r (G_\beta^{-1} x_0)_r g_r .
\end{aligned}$$

Let

$$G_\beta^{-1}x_0 = (\delta_0, \delta_1, \delta_2, .., \delta_{kmn-1}) ;$$

then

$$u_0 = \delta_0(f_0 \otimes e_1) + \delta_1(f_1 \otimes e_1) + \ldots + \delta_{km-1}(f_{km-1} \otimes e_1) + \ldots\ldots\ldots + \delta_{kmn-1}(f_{km-1} \otimes e_m)$$

is a maximising vector for T.

Let $\sigma_j(e_i) = b_{i+mj}$, then Tu_0 is given by

$$[U_B]^*[J]^*F(C_\beta^T)[U_\beta]^{*^{-1}}[U_\beta]^*x_0$$
$$= [U_B]^*[J]^*F(C_\beta^T)x_0 ,$$

hence

$$Tu_0 = \sum_j ([U_B]^*[J]^*F(C_\beta^T)x_0)_j Q_B b_j$$
$$= \sum_j ([U_B][U_B]^*[J]^*F(C_\beta^T)x_0)_j b_j$$
$$= \sum_j (G_B^{-1}[J]^*F(C_\beta^T)x_0)_j b_j$$
$$= \sum \eta_j b_j = v_0 ,$$

where

$$G_B^{-1}[J]^*F(C_\beta^T)x_0 = (\eta_0, \eta_1, \ldots, \eta_{km-1}).$$

By the above method we can calculate the maximising vector u_0 and Tu_0. It follows from Theorem 2.2.7 that there exists $G \in F + BH^\infty_{mxn}$ such that $\|G\|_\infty = \|T\|$ and $Gu_0 = Tu_0$. But this equation is not enough to solve for the rational function G. However, by Theorem 2.3.9, there exists $F_0 \in M_{mxn}$ such that the one step extension Sarason operator $\tilde{T} = P_{\tilde{B}} M_{\tilde{F}} P_{\tilde{\beta}}^*$ has singular value M with multiplicity n, where $\tilde{B} = zB$, $\tilde{F} = F + BF_0$. By the same procedure as above, the matrix $\tilde{A}$ of $\tilde{T}$ can be formed. Corresponding to the first n singular values of $\tilde{A}$, there are n linearly independent maximising vectors u_0, u_1, $u_2 \ldots u_{n-1}$ such that $[u_0(z), \ldots, u_{n-1}(z)]$ is nonsingular for every $z \in D$. By using Theorem 2.2.7 there exist $\underset{\sim}{G} \in \underset{\sim}{F} + \underset{\sim}{B}H^{\infty}_{mxn} \subset F + BH^{\infty}_{mxn}$ such that

$$\| \underset{\sim}{G} \| = \| \underset{\sim}{T} \| = \| T \|$$

and

$$\underset{\sim}{G}[u_0, \ldots, u_{n-1}] = [\underset{\sim}{T}u_0, \underset{\sim}{T}u_1, \ldots, \underset{\sim}{T}u_n].$$

Therefore

$$G(z) = [\underset{\sim}{T}u_0(z), \ldots, \underset{\sim}{T}u_n(z)]\,[u_0(z), \ldots, u_{n-1}(z)]^{-1}$$

$z \in D$.

5.2 Details of algorithms

Here we give the main details of the algorithms developed in Chapter two, three and four. We shall explain how the generalized Sarason operator and the interpolating function of minimal norm can be computed numerically with the aid of the following five main steps.

Step 1: Find bases for $H^2_{mx1} \ominus BH^2_{mx1}$ and $H^2_{nx1} \ominus \beta H^2_{nx1}$:

Perform the irreducible MFD on $B(z)=N(z)D(z)^{-1}$;
write down the determinant of $N(z)$; say $\beta(z)$;
choose standard bases for $H^2_{nx1} \ominus BH^2_{nx1}$ and $H^2 \ominus \beta H^2$, in terms of the coefficients of $N(z)$ and $\beta(z)$ from (3-14) (4-12);
form the inverse of the Gram matrices G_B and G_β for these bases;
perform Choleski decomposition [42] to yield two upper triangular matrices $[U_B]^*$ and $[U_\beta]^*$ such that $[U_B][U_B]^*=G_B^{-1}$, $[U_\beta][U_\beta]^*=G_\beta^{-1}$.

Comments: To obtain the irreducible MFD, one can use Gaussian elimination. We follow the algorithm in [31, P192]. However, we feel such irreducible MFD can be avoided. Details of this are studied in the next section. Procedures " gbinverse", "g2inverse " and "f03ahb" in the ALGOL68 NAG library are provided for these calculations. Since $H^2_{mx1} \ominus \beta H^2_{mx1} \cong C^m \otimes (H^2 \ominus \beta H^2)$, we only need to calculate the Gram matrix for the basis in $H^2 \ominus \beta H^2$. One can take advantage of this to reduce storage requirements.

Step 2: Form $F(C_\beta^T)$:

Write down the matrix C_{β^*} of S_β with respect to the basis in $H^2 \ominus \beta H^2$ from (4-9); evaluate the matrix $F(C^*_{\beta^*})=F(C_\beta^T)$, where $F \in RH^\infty_{mxn}$.

Comments: We observed that C_β is the companion matrix of β, and $F(C_\beta^T)=[F_{ij}(C_\beta^T)]_{mxn}$. There is a remarkable way to calculate $F_{ij}(C_\beta^T)$ which can be reduced to finding $g_{ij}(C_\beta^T)$, where g_{ij} is a polynomial with degree less than m; the degree of β. This is an important feature in Young's algorithm. The idea is to find a polynomial g_{ij} such that $\beta \mid F_{ij}-g_{ij}$. This can be done by using the Euclidean algorithm to find a polynomial S_{ij} such that

$$D_{ij}S_{ij} = I(\mathrm{mod}\ \beta)$$

so that

$$N_{ij}S_{ij} = g_{ij}(\mathrm{mod}\ \beta)$$

where

$$F_{ij} = \frac{N_{ij}}{D_{ij}}$$

For detains see [4, § 4]. Procedures "invertmodp", "multpolymodp", "shiftmodp" and "fuofcompanion" are designed for the above calculations. In order to remove the highest term of a polynomial with negligibly small coefficients, we introduce a procedure "adjustdegree" in the Euclidean algorithm.

Step 3: Form the matrix of the generalized Sarason operator:

Form the matrix of the orthogonal injection [J] from (4-26) with respect to the bases in step 1; form $A=[U_B]^*[J]^*F(C_\beta^T)[U_\beta]^{*-1}$ and calculate the largest singular value of A.

Comments: We use a procedure "proj" for calculating the matrix of the injection operator. One simple but important operator contained in "proj" is the isomorphism ι between $H^2_{m\times 1}$ and $C^m \otimes H^2$. In other words, we need to pay attention to the order of the bases. Procedures "permutationrow" and "permutationcolumn" are designed for this. Since the matrices $[U_B]^*$ and $[U_\beta]^*$ are upper triangular, we can use procedures "ua" and "auinverse" using back substitution to form the matrix A. The most straightforward method of calculating the largest singular value of A is to compute the eigenvalues of A^*A(or AA^*) using "fo2axf" from NAG library. However, a more stable way to do this is by a singular value decomposition of A using a standard routine "fo2waf" which is only available for a real matrix. But by means of Householder transformations and the QR decomposition one can reduce the singular value problem for a complex matrix to the corresponding problem for a real bidiagonal matrix. This technique has also been used in the scalar method. In the case m=1, then to step 5 directly.

Step 4: Find the one step extension matrices

Calculate a one step extension matrix F_0 from (2-22) or Theorem 2.4.8;

form the new inner matrix $\underset{\sim}{B} = zB$, $\underset{\sim}{\beta} = z\beta$ and $\underset{\sim}{F} = F+BF_0$;

find the full singular value decomposition of

$A = [U_{\underset{\sim}{B}}]^*[\underset{\sim}{J}]^*\underset{\sim}{F}(C_{\underset{\sim}{\beta}}^T)[U_{\underset{\sim}{\beta}}]^{*-1}$.

Comments: From Theorem 2.4.6, the one step extension F_0 can be determined by finding X. Let X_0 be the matrix of X with respect to some bases of $u_0^{\perp}$ and C^m. It follows that (2-27) and (2-28) give us a finite number of relations of the type $X_0 m_i = n_i$ with m_i linearly independent, and so X_0 satisfies

$$X_0 M = N$$

By the QR decomposition M can be decomposed into the product of a unitary matrix U_m and an upper triangular matrix R_m; since $m_1, m_2, \ldots$ are linearly independent, R_m has nonzero diagonal. By back substitution $X_0 U_m$ can be written in the following form:

$$X_0 U_m = [\, A \mid B \,]$$

where A is an mx(nk-n) known matrix and B is an mx(n-1) unknown matrix. Therefore the remaining problem is to determine B in such a way that $\| X \| = 1$ and $\nu(1-X^*X) \geq n-1$ Since $\| X \| \leq 1$, $1-AA^*-BB^* \geq 0$ or $BB^* \leq 1 - AA^*$. Let us apply a cholesky decomposition to $1-AA^*$; we have

$$1 - AA^* = LL^*$$

and let

$$B = L \left[\begin{array}{c} \overbrace{\begin{matrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \\ & & & 1 \end{matrix}}^{n-1} \\ \hline 0 \end{array}\right] n-1 = LU_B$$

then

$$\nu(1-AA^*-BB^*) = \nu(L(1-U_BU_B^*)L^*) \geq n-1.$$

F_0 can be constructed from (2-29) or (2-30).

From the construction of F_0 it follows that $\| A \|$ is the largest singular value of $\underset{\sim}{A}$ and has multiplicity n. The n Schmidt pairs of $\underset{\sim}{A}$ corresponding to $\| A \|$ will be calculted. Therefore, n linearly independent maximising vectors $u_0(z)$, $u_1(z)$,... $u_{n-1}(z)$ in $H^2_{nx1} \ominus z\beta H^2_{nx1}$ can be constructed with respect to the basis of $H^2_{nx1} \ominus z\beta H^2_{nx1}$ and $[u_0(z),..., u_{n-1}(z)]$ is nonsingular for every $z \in D$. Consequently, the corresponding $Tu_0(z), Tu_1(z),.....,$ $Tu_{n-1}(z)$ can also be constructed. Procedures "onestepext", "combinkerb", "combine2", "ubstartmat", "ubetastarmat" and "matrixvector" are provided for the step 4 calculations.

Step 5: Calculate and check an extremal function

Form the extremal function

$$G(z) = [Tu_0(z),... Tu_{n-1}(z)][u_0(z),... u_{n-1}(z)]^{-1} ;$$

check G(z).

Comments: There are three main properties that can be used to check this result. Firstly, $[u_0(z),...u_{n-1}(z)]$ is nonsingular for $z \in D$, and $u_i \in H^\infty$, so the denominators of u_i must have no zeros inside the unit circle. We use Cohn's algorithm [8] to find the numbers of zeros of a given polynomial. Secondly, we have that $G \in \underset{\sim}{F} + \underset{\sim}{B}H^{\infty}_{mxn} \subseteq F + BH^{\infty}_{mxn}$ i.e. for every $z \in D$, $G(z)=F(z)+B(z)g(z)$ for some $g \in H^{\infty}_{mxn}$. For example, if $B(\alpha)= 0$ then $G(\alpha)=F(\alpha)$, $(\alpha \in D)$. We write a procedure "evaluemnratn" to check G. Thirdly, by Theorem 2.2.7 $\sup_{z \in \partial D} \| G(z)\|_{\infty} = \| \underset{\sim}{A} \| = \| A \|$, $G(z)$ has constant modulus on the unit circle, we also write a procedure "checksmnratn" that calculates $\| G(z) \|$ at six points on the unit circle; we expect the result to be constant and equal to $\| A \|$.

5.3 Other algorithms

To derive the Nevanlinna-Type algorithm of Ph. Delsarte, Y. Genin and Y. Kamp [9] from our result is rather easy. Let us take the inner matrix B to be a scalar matrix with entry a Blaschke product having simple distinct zeros at $\alpha_1, \alpha_2, \ldots, \alpha_n$; in other words

$$B(z) = b_1 b_2 \ldots\ldots b_n(z) I_n ,$$

where $b_j(z) = \frac{z-\alpha_j}{1-\bar{\alpha}_j z}$. Then $\{a_j \otimes e_i \ : \ j=1,2,\ldots n, \ i=1,2,\ldots n\}$ is a basis of $H^2_{nx1} \ominus BH^2_{nx1} = \mathbb{C}^n \otimes (H^2 \ominus b_1 b_2 \ldots b_n H^2)$, where $a_j(z) = \frac{1}{1-\bar{\alpha}_j z}$. Since the inner matrix B is scalar, the generalized Sarason operator can therefore act from $\mathbb{C}^n \otimes (H^2 \ominus b_1 b_2 \ldots b_n H^2)$ to $\mathbb{C}^m \otimes (H^2 \ominus b_1 \ldots b_n H^2)$. i.e.

$$H^2_{nx1} \ominus \beta H^2_{nx1} = \mathbb{C}^n \otimes (H^2 \ominus b_1 b_2 \ldots b_n H^2)$$

and

$$H^2_{mx1} \ominus BH^2_{mx1} = \mathbb{C}^m \otimes (H^2 \ominus b_1 \ldots\ldots b_n H^2).$$

The orthogonal projection from $H^2_{mx1} \ominus \beta H^2_{mx1}$ to $H^2_{mx1} \ominus BH^2_{mx1}$ is an identity operator. Let us consider the basis ordered as follows $a_1 \otimes e_1$, $a_1 \otimes e_2, \ldots, a_1 \otimes e_n, \ldots, a_2 \otimes e_1, \ldots, a_2 \otimes e_n$, ... The compressed shift operator S_β with respect to $\{a_j \otimes e_i\}$ is a diagonal block matrix $D = \mathrm{diag}\{\bar{\alpha}_1 I_n, \bar{\alpha}_2 I_n, \ldots \alpha_n I_n\}$ The Gram matrices G_β, G_B for these bases are

$$[r_{ij}] \otimes I_n \ , \quad [r_{ij}] \otimes I_m$$

respectively, and $r_{ij} = (a_j, a_i) = \dfrac{1}{1-\bar{\alpha}_j \alpha_i}$.

Let $\Phi \in RH^\infty_{mxn}$, by step 1 and step 3 in 5.2, there exists matrices $[U_B]$, $[U_\beta]$ such that $[U_B][U_B]^* = G_B^{-1}$, $[U_\beta][U_\beta]^* = G_\beta^{-1}$ and $A = [U_B]^* \Phi(D^*)[U_\beta]^{*-1}$. The corresponding eigenvalue problem from (5-4) is

$$(\lambda I - AA^*)x = 0$$

or

$$(\lambda I - [U_B]^* \Phi(D^*)[U_\beta]^{*-1}[U_\beta]^{-1}\Phi(D^*)^*[U_B])x = 0.$$

This implies

$$(\lambda I - [U_B]^* \Phi(D^*) G_\beta \Phi(D^*)^* [U_B])x = 0$$

Thus

$$\left[\lambda [U_B]^{*-1}[U_B]^{-1} - \Phi(D^*) G_\beta \Phi(D^*)^*\right][U_B]x = 0\ ;$$

i.e.

$$(\lambda G_B - \Phi(D^*) G_\beta \Phi(D^*)^*)[U_B]x = 0.$$

Therefore the problem reduces to the following

$$\left[\frac{\lambda - \Phi(\alpha_i)\bar{\Phi}(\alpha_j)^*}{1-\alpha_i\bar{\alpha}_j}\right]_{nxn} [U_B]\, x = 0\ .$$

$(\quad)_{nxn}$ is the matrix that the Nevanlinna-Type algorithm is concerned about.

There are two disadvantages of this algorithm: we need to solve the equation of $B(z)=0$, and we also require to know the distinct simple zeros of $B(z)=0$. A numerical instability can occur in solving the equation $B(z)=0$ and also in the case of α's being very close to each other.

The method of Kung is based on the A-A-K one step extension theory. He solves the corresponding interpolation problems in terms of "the minimal basis" and "algebraic Riccati equation". In view of the relation of the A-A-K operator and the generalized Sarason operator, which we studied in Chapter two, one can see that the one step extension matrix from (2-22) is exactly the same in both theories. In terms of our notation, Kung shows that the one step extension matrix can be obtained as follows:

Let F, $B \in RH^{\infty}_{m\times m}$ and B be inner. Let $\varphi(z)=[B(z)]^{*}F(z) = \frac{N(z)}{a(z)}$, where $a(z)$ is the least common multiple of the denominators of $\varphi(z)$. And $N(z)$ is a matrix polynomial. Let $\mathrm{dis}(F, BH^{\infty}_{m\times m}) = s_0 = \|\Gamma_{\varphi}\|$, and let $\check{a}(z)=z^{n}a(\frac{1}{z})$, $\check{N}(z)=z^{n-1}N(\frac{1}{z})$, where n is the degree of $a(z)$. Then the singular vectors of the one step extension Hankel operator $\Gamma_{\tilde{\varphi}}$ are the solutions $x(z)$ of the following equations

$$\begin{bmatrix} a(z)I_m, & -N(z), & s_0^2\check{a}^{*}(z)I_m, & 0 \\ 0, & s_0^2 a(z)I_m, & -\check{N}^{*}(z), & \check{a}^{*}(z)I_m \end{bmatrix} x(z) = 0, \tag{10}$$

Where $\tilde{\varphi} = \frac{F_0}{z} + \frac{1}{z}\varphi$, $F_0 \in M_{m\times m}$.

If we write

$$x(z) = \begin{bmatrix} x_{11}(z) & , & x_{12}(z) \\ x_{21}(z) & , & x_{22}(z) \\ x_{31}(z) & , & x_{32}(z) \\ x_{41}(z) & , & x_{42}(z) \end{bmatrix},$$

where $x_{ij}(z)$ is a mxm matrix polynomial, then one step extension matrix F_0 is a solution of the algebraic Riccati equation

$$[I_m, F_0] \begin{bmatrix} x_{11}(0), - x_{12}(0) \\ x_{21}(0), - x_{22}(0) \end{bmatrix} \begin{bmatrix} I_m \\ F_0^* \end{bmatrix} = 0. \qquad (11)$$

The extremal function such that the norm attained is given by

$$G(z) = \pi_2^- \left\{ [x_{11}(z)-x_{12}(z)F_0^*][x_{21}(z)-x_{22}(z)F_0^*]^{-1} \right\} \qquad (12)$$

There are two difficulties in Kung's method:
Firstly, to obtain the accurate Hankel norm $\|\Gamma_\varphi\|$, we require a lengthy computation, and this certainly creates a rounding error in the solution space of (10). Secondly, to calculate the co-analytic part of $[x_{11}(z)-x_{12}(z)F_0^*][x_{21}(z)-x_{22}(z)F_0^*]^{-1}$ in (12), one has to compute the poles of the above rational function. This can cause numerical instability. Numerical tests of the two algorithms will be carried out in subsequent work.

5.4 Conclusion

On close inspection of our algorithm, we can make the following two conjectures.

Firstly, is the irreducible MFD on the inner matrix B necessary?

We factorize B(z) into an irreducible MFD, $N(z)D(z)^{-1}$, then $\mathrm{Ker}\, \iota B^*(S)\iota^* = \mathrm{Ker}\, \iota N^*(S)\iota^*$, and we can decompose $\mathrm{Ker}\, \iota N^*(S)\iota^*$ into $G_0 \oplus G_1 \oplus \ldots \oplus G_{k-1}$, for some k, such that G_j can be identified with $\mathbb{C}^m$, i.e.

$$\tau_j(x)(z) = \hat{N}(z)^{-1} T_j(z)x \in G_j \subset H^2_{m\times 1}$$

for every $x \in \mathbb{C}^m$.

However, it may be possible to avoid using irreducible MFD on B(z). We are lead to this conjecture by the following observations.

Consider B(z) to be the Potapov Blaschke product of degree n;

$$B(z) = \prod_{j=1}^{n} V_j \begin{bmatrix} b_j(z)I_p & 0 \\ 0 & I_q \end{bmatrix} W_j ,$$

where $b_j(z) = \dfrac{z-\alpha_j}{1-\bar{\alpha}_j z}$, $|\alpha_j| < 1$.

Let

$$h(z) = \prod_{j=1}^{n} (1-\bar{\alpha}_j z) = h_0 + h_1 z + \ldots + h_n z^n ,$$

and let

$$B(z) = \frac{N'(z)}{h(z)} = \frac{1}{h(z)} (B_0' + B_1' z + \ldots + B_n' z^n)$$

Assume $\det B_n' \neq 0$, and let

$$\beta(z) = \det B(z) = \lambda b_0 \ldots b_n = \lambda \frac{h(z)}{\hat{h}(z)} , \quad |\lambda| = 1,$$

then

$\mathrm{Ker}\, \iota N'^*(S)\iota^*$ is the subspace of $L^2_{m\times 1}$ consisting of all functions of the form

$$\hat{N}'(z)^{-1} \left(\sum_{j=0}^{n-1} T_j'(z) x_j \right), \quad \text{for } x_j \in \mathbb{C}^m,$$

where

$$T_j'(z) = B_n^* z^j + \dots + B_{j+1}^* z^{n-1} .$$

Let G_j' be the subspace of $L^2_{m\times 1}$ defined by

$$G_j' = \left\{ \sum_{n=-\infty}^{\infty} x_n z^n \in L^2_{m\times 1} \;\middle|\; x_0 = x_1 = \dots = x_{j-1} = x_j = \dots = x_n = 0 \right\}$$

$j=0,1,2,\dots n-1$.

Then

$$\operatorname{Ker} \iota B^*(S)\iota^* = (G_0' \oplus G_1' \oplus \dots \oplus G_{n-1}') \cap H^2_{m\times 1}$$

i.e.

$$H^2_{m\times 1} \ominus BH^2_{m\times 1} = (\operatorname{Ker} \iota N'^*(S)\iota^*) \cap H^2_{m\times 1}.$$

This suggest us that there exist some $x \in \mathbb{C}^m$ such that

$$\tau_j(x)(z) = \hat{N}'^{-1}(z) T_j'(z) x \in G_j' \subset H^2_{m\times 1}.$$

Let $b(z)=\det N'(z)$, and let $K_\beta = \operatorname{Ker} \iota N'^*(S)\iota^*$, then $H^2_{n\times 1} \ominus \beta H^2_{n\times 1} \subseteq K_\beta$. Therefore, the generalized Sarason operator $T = P_B M_F P_\beta^*$ can be modified as

$$T = \Pi_B j_\beta P_\beta M_F P_\beta^*$$

$$\begin{array}{ccccc}
H^2_{n\times 1} \ominus \beta H^2_{n\times 1} & \xrightarrow{P_\beta M_F P_\beta^*} & H^2_{n\times 1} \ominus \beta H^2_{n\times 1} & & \\
 & & \downarrow j_\beta & & \\
 & & K_\beta & \xrightarrow{\Pi_B} & K_B,
\end{array}$$

where j_β is the orthogonal projection from $H^2_{n\times 1} \ominus \beta H^2_{n\times 1}$ into $\operatorname{Ker} \iota b^*(S)\iota^*$, and Π_B is the orthogonal projection from K_β to K_B. The matrix form of Π_B with respect to the decompositions of $\operatorname{Ker} \iota b^*(S)\iota^*$ and $\operatorname{Ker} \iota N'^*(S)\iota^*$ can be obtained in the same way as in Chapter four. The matrix of j_B with respect to the decompositions of $H^2_{nx1} \ominus \beta H^2_{nx1}$ and K_β is

$$\left[\frac{I_n}{C^n_{h^*}}\right] \otimes I_n ,$$

where

$$C_{h^*} = \begin{bmatrix} 0 & 1 & 0,\dots\dots\dots 0 \\ 0 & 0 & 1,\dots\dots\dots 0 \\ -\frac{\bar{h}_0}{\bar{h}_n},\dots\dots\dots\dots\dots & & & -\frac{\bar{h}_{n-1}}{\bar{h}_n} \end{bmatrix}.$$

<u>Secondly, is the one step extension operator T absolutely necessary</u> ?

When we form the extremal function with minimal norm, we use the one step extension method to get enough maximising vectors. But this increases the storage requirement. However, Young proposed a very promising method to deal with this problem, the mathematical proof of which is presently being examined.

Let

$$T = P_B M_F P_\beta^* : H^2_{nx1} \ominus \beta H^2_{nx1} \longrightarrow H^2_{mx1} \ominus BH^2_{mx1},$$

let

$$x_0 = H^2_{nx1} \ominus \beta H^2_{nx1},$$

$$x_i = \langle u_0, u_1, \dots\dots, u_{n-1} \rangle^{\perp} \cap H^2_{nx1} \ominus \beta H^2_{nx1}$$

where u_i is the maximising vector of

$$T_i = T \Big|_{x_i}$$

i=0,1,2,...,n-1.

Let $\| T_i \| = t_i$, and

$$Tu_i = t_i w_i .$$

Then there is a function $G \in F + BH^{\infty}_{mxn}$ such that $\| G \|_{\infty} = \| T \|$ is given by

$$G = [u_0, u_1, \ldots\ldots, u_{n-1}]^* [t_0 w_0, t_1 w_1, \ldots\ldots, t_{n-1} w_{n-1}] .$$

CHAPTER SIX

NUMERICAL EXAMPLES

To illustrate the foregoing results let us see how the algorithm performs in calculating the matrix of a generalized Sarason operator and in finding a minimising function.

Example 1. Consider n=1, m=2, $F(z) = \begin{bmatrix} \frac{2+(1+i)z}{1-0.2z} \\ \frac{(1+2i)+3z}{1-0.3z} \end{bmatrix} = \begin{bmatrix} f_{11}(z) \\ f_{21}(z) \end{bmatrix}$

and the Potapov Blaschke product B(z) from (2-7) as

$V_1=W_1=V_2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $W_2=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $V_3=W_3=V_4=W_4=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and

$b_i(z) = \frac{z-\alpha_i}{1-\bar{\alpha}_i z}$, $|\alpha_i| < 1$. i.e.

$$B(z) = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{(z-\alpha_1)(z-\alpha_2)}{(1-\alpha_1 z)(1-\bar{\alpha}_2 z)}, & \frac{(z-\alpha_1)(z-\alpha_2)}{(1-\bar{\alpha}_1 z)(1-\bar{\alpha}_2 z)} \\ -\frac{(z-\alpha_3)(z-\alpha_4)}{(1-\bar{\alpha}_3 z)(1-\bar{\alpha}_4 z)}, & \frac{(z-\alpha_3)(z-\alpha_4)}{(1-\bar{\alpha}_3 z)(1-\bar{\alpha}_4 z)} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} b_1b_2, b_1b_2 \\ -b_3b_4, b_3b_4 \end{bmatrix}.$$

Step 1 take $\alpha_1=0.3$, $\alpha_2=0.5$, $\alpha_3=0.4$, $\alpha_4=0.6$. An irreducible MFD of B(z) is $N(z)D(z)^{-1}$, where

$$N(z) = \frac{1}{\sqrt{2}}\begin{bmatrix} 0.15 & 0.15 \\ -0.24 & 0.24 \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} -0.8 & -0.8 \\ 1 & -1 \end{bmatrix} z + \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} z^2$$

$$= B_0 + B_1 z + B_2 z^2,$$

$$\beta(z) = 0.036 - 0.342z + 1.19z^2 - 1.8z^3 + z^4 .$$

$$\hat{N}(z)^{-1} = \begin{bmatrix} 1-0.8z+0.15z^2 & 0 \\ 0 & 1-z+0.24z^2 \end{bmatrix}^{-1}$$

$$\hat{\beta}(z) = 1-1.8z+1.19z^2-0.342z^2+0.036z^2 .$$

The standard basis for $H^2_{2x1} \ominus BH^2_{2x1}$ is

$$\tau_0(e_1)(z) = \frac{1}{\sqrt{2}} \begin{bmatrix} \dfrac{1-0.8z}{1-0.8z+0.15z^2} \\ \dfrac{1-0.8z}{1-z+0.24z^2} \end{bmatrix}, \quad \tau_0(e_2)(z) = \frac{1}{\sqrt{2}} \begin{bmatrix} \dfrac{-1+z}{1-0.8z+0.15z^2} \\ \dfrac{1-z}{1-z+0.24z^2} \end{bmatrix},$$

$$\tau_1(e_1)(z) = \frac{1}{\sqrt{2}} \begin{bmatrix} \dfrac{z}{1-0.8z+0.15z^2} \\ \dfrac{z}{1-z+0.24z^2} \end{bmatrix}, \quad \tau_1(e_2)(z) = \frac{1}{\sqrt{2}} \begin{bmatrix} \dfrac{-z}{1-0.8z+0.15z^2} \\ \dfrac{z}{1-z+0.24z^2} \end{bmatrix}.$$

and the standard basis for $H^2 \ominus \beta H^2$ is

$$f_1(z) = \frac{1-1.8z+1.19z^2-0.342z^2}{\hat{\beta}(z)}, \quad f_2(z) = \frac{z-1.8z^2+1.19z^3}{\hat{\beta}(z)},$$

$$f_3(z) = \frac{z^2-1.8z^3}{\hat{\beta}(z)}, \quad f_4(z) = \frac{z^3}{\hat{\beta}(z)} .$$

On applying Choleski decomposition to G_B^{-1} and G_β^{-1}, we have

$$[U_B][U_B]^* = G_B^{-1}, \quad I_2 \otimes [U_\beta][U_\beta]^* = G_\beta^{-1},$$

where

$$[U_\beta]^* = \begin{bmatrix} 0.9994 & 0.0100 & -0.0234 & 0.0105 \\ 0 & 0.9601 & 0.1978 & -0.1085 \\ 0 & 0 & 0.6797 & 0.5885 \\ 0 & 0 & 0 & 0.2425 \end{bmatrix},$$

$$[U_B]^* = \begin{bmatrix} 0.9887 & 0 & 0.1032 & 0 \\ 0 & 0.9708 & 0 & 0.1879 \\ 0 & 0 & 0.7103 & 0 \\ 0 & 0 & 0 & 0.5740 \end{bmatrix}.$$

<u>Step 2</u> The matrix $F(C^T)$ equals $P(C^T)$,

where

$$P(z) = \begin{bmatrix} (1.9994-0.0004i)+(1.4055+1.0039i)z \\ \qquad +(0.2616+0.1869i)z^2+(0.0818+0.0584i)z^3 \\ (0.9943+1.9990i)+(3.3529+0.6096i)z \\ \qquad +(0.8159+0.1483i)z^2+(0.5321+0.0967i)z^3 \end{bmatrix}.$$

<u>Step 3</u> The matrix of the generalized Sarason operator A is a 4x4 matrix.

$$A=\begin{bmatrix} 2.1004+0.0869i, & 0.0744-0.0857i, & -0.3040-0.0871i, & -0.1719-0.0382i \\ 1.1511+2.0410i, & -0.0993+0.3194i, & -0.5704-0.2618i, & -0.4715-0.3036i \\ 1.1761+0.8401i, & 2.4099+0.6646i, & 1.8168+0.4110i, & 0.7355+0.1542i \\ 2.6267+0.4776i, & 3.1079+1.6521i, & 2.6873+1.7077i, & 1.5905+1.0879i \end{bmatrix}$$

and $\| A \| = 6.8303$ $(=\rho)$. The left singular vector of A is

$$[\,0.1304-0.0357i, \quad 0.0870+0.1576i, \quad 0.4879-0.0590i, \quad 0.8413\,].$$

<u>Step 5</u> A maximising vector for the adjoint of the generalized Sarason operator T^* is

$$v_0(z) = \begin{bmatrix} \dfrac{(0.0315-0.1331i)+(-0.1117+0.0749i)z}{1-0.8z+0.15z^2} \\ \\ \dfrac{(0.1508+0.0832i)+(0.4749-0.0995i)z}{1-z+0.24z^2} \end{bmatrix}.$$

and

$$u_0(z) = \frac{1}{\rho}T^*v_0(z) = (3.4558-0.9060i)+(-2.5321-0.1942i)z +(0.1816+0.6375i)z^2+(0.1015-0.1542i)z^3.$$

The extremal function(unique, since we are dealing with F of type 2x1) G(z) is

$$G(z) = \begin{bmatrix} G_{11}(z) \\ G_{21}(z) \end{bmatrix} = \begin{bmatrix} \dfrac{(6.0155-1.6446i)+(5.9659+0.8326i)z +(-10.5376+0.4172i)z^2+(2.8755-0.1949i)z^3}{(3.4558-0.9060i)+(-2.5321-0.1942i)z +(0.1816+0.6375i)z^2+(0.1015-0.1542i)z^3} \\ \\ \dfrac{(3.9378+7.1382i)+(16.2045-11.4672i)z +(-14.8931+5.6760i)z^2+(2.9032-0.8635i)z^3}{(3.4558-0.9060i)+(-2.5321-0.1942i)z +(0.1816+0.6375i)z^2+(0.1015-0.1542i)z^3} \end{bmatrix}$$

$$= v_0(z)u_0(z)^{-1}.$$

<u>Step 6.</u> Checks of the result.

Since $G(z) = F(z)+B(z)g(z)$, for some $g(z) = \begin{bmatrix} g_{11}(z) \\ g_{21}(z) \end{bmatrix}$,

$$G(z) = \begin{bmatrix} F_{11}(z)+b_1(z)b_2(z)g_{11}(z)+b_1(z)b_2(z)g_{21}(z) \\ F_{21}(z)-b_3(z)b_4(z)g_{11}(z)+b_3(z)b_4(z)g_{21}(z) \end{bmatrix}.$$

Therefore $G_{11}(\alpha_2) = F_{11}(\alpha_2)$ and $G_{21}(\alpha_4) = F_{21}(\alpha_4)$,

$$\begin{bmatrix} F_{11}(\alpha_2) \\ F_{21}(\alpha_4) \end{bmatrix} = \begin{bmatrix} 2.777+0.555i \\ 3.415+2.439i \end{bmatrix} = \begin{bmatrix} G_{11}(\alpha_2) \\ G_{21}(\alpha_4) \end{bmatrix}.$$

Let $\theta = 0, \frac{\pi}{6}, \frac{2\pi}{6}, \frac{3\pi}{6}, \frac{4\pi}{6}, \frac{5\pi}{6}$, and $z_\theta = \cos\theta + i\sin\theta$, which is on the unit circle. Then $\| G(z_0) \| = 6.8303 = \rho$. Constancy is observed up to 13 decimal places.

The Cohn algorithm shows that the denominator of $G(z)$ has no zeros in the unit circle. Therefore G is in H^∞_{2x1}.

Example 2. Consider m=n=2. The simplest candidate for this case is $F(z) = \begin{bmatrix} z & 3z \\ z & 2+z \end{bmatrix}$ and $B(z) = \frac{1}{\sqrt{2}}\begin{bmatrix} z^2 & z^2 \\ -z^2 & z^2 \end{bmatrix}$.

The purpose of this simple example is to show how to use the one step extension method to form an extremal function with minimal norm.

The norm of the Sarason operator $T = P_B M_F P_\beta^*$ is 3.9681 $(=\rho)$. From step 4', one extension matrix F_0 is

$$F_0 = \begin{bmatrix} -2.0216 & -0.0216 \\ 2.6595 & -1.0865 \end{bmatrix}.$$

From step 5, form the new rational functions; $\underset{\sim}{F} = F + BF_0$ and $\underset{\sim}{B} = zB$; then the singular values of the one step extension Sarason operators $\underset{\sim}{T}$ are

3.9681, 3.9681, 3.9681, 1.0429, 0.1228.

Two linearly independent maximising vectors are

$$v_0(z) = \begin{bmatrix} -0.0260+0.2111z+0.8037z^2 \\ -0.0260+0.3563z-0.4256z^2 \end{bmatrix},$$

$$v_1(z) = \begin{bmatrix} -0.2533-0.3209z+0.3647z^2 \\ -0.2533+0.2390z+0.7606z^2 \end{bmatrix}$$

and

$$T^*v_0(z) = \begin{bmatrix} -0.7478+1.1366z \\ -0.2899+3.6776z+0.5347z^2 \end{bmatrix} = \rho u_0(z),$$

$$T^*v_1(z) = \begin{bmatrix} 2.2283+0.5158z \\ -2.8207-0.1600z+1.5914z^2 \end{bmatrix} = \rho u_1(z).$$

An extremal function with minimal norm is

$$G(z) = [v_0(z), v_1(z)][u_0(z), u_1(z)]^{-1}$$

$$= \begin{bmatrix} \dfrac{z-6.0619z^2+5.3645z^3+8.7616z^4}{1-4.0402z-1.6189z^2+0.5564z^3}, & \dfrac{3z-12.1424z^2-4.3808z^3}{1-4.0402z-1.6189z^2+0.5564z^3} \\ \dfrac{z-1.3808z-13.1424z^3}{1-4.0402z-1.6189z^2+0.5564z^3}, & \dfrac{2-7.0805z-8.3645z^2-4.3808z^3}{1-4.0402z-1.6189z^2+0.5564z^3} \end{bmatrix}.$$

Since $G(z) \in F(z)+B(z)H^\infty_{m\times m}$, and $B(0)=0$, so

$$G(0) = F(0) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

Let $\theta = 0, \frac{\pi}{6}, \frac{2\pi}{6}, \frac{3\pi}{6}, \frac{4\pi}{6}, \frac{5\pi}{6}$, and $z_\theta = \cos\theta + i\sin\theta$, which is on the unit circle. Then $\| G(z_\theta) \| = 3.9681$. Constancy is observed up to 12 decimal places.

ρ has multiplicity $3 \geq 2$. This indicates that $G(z)$ is not in its lowest terms. By using the Cohn algorithm the numerator and denominator of every entry of $G(z)$ have one zero inside the unit circle. By using "co2adb" from ALGOL 68 NAG, the linear term is $z-0.2283$.

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