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Mori Dream Spaces as Fine Moduli of Quiver Representations

by

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Chapter 1

Introduction

1.1 Introduction

Mori Dream Spaces and their Cox rings have been the subject of a great deal of interest since their introduction by Hu–Keel [19] over a decade ago. From the geometric side, these varieties enjoy the property that all operations of the Mori programme can be carried out by variation of GIT quotient, while from the algebraic side, obtaining an explicit presentation of the Cox ring is an interesting problem in itself. Examples include $\mathbb{Q}$-factorial projective toric varieties, spherical varieties and log Fano varieties of arbitrary dimension. In this thesis we use the representation theory of quivers to study multigraded linear series on Mori Dream Spaces. Our main results construct Mori Dream Spaces as fine moduli spaces of ϑ-stable representations of bound quivers for a special stability condition ϑ, thereby extending results of Craw–Smith [10] for projective toric varieties.

Let X be a Mori Dream Space and let $\mathscr{L} = (L_0, L_1, \dots, L_r)$ be a collection of effective line bundles on X with $L_0 = \mathcal{O}_X$. In Chapter 3 we show how to construct a quiver of sections for $\mathscr{L}$. We would like this quiver to encode the sections of $L_j \otimes L_i^{-1}$ for every L_i and L_j in $\mathscr{L}$, but we are obstructed by the lack of a canonical basis for the space $H^0(X, L_j \otimes L_i^{-1})$. However, every Mori Dream Space admits a natural embedding into a projective toric variety $\widetilde{X}$, whose class group is isomorphic to that of X. We harness a key property of this ambient toric variety, or more precisely of the collection $\widetilde{\mathscr{L}} = (E_0, \dots, E_r)$ on $\widetilde{X}$ obtained by lifting $\mathscr{L}$ from X. While the spaces $H^0(X, L_j \otimes L_i)$ have no canonical basis, $H^0(\widetilde{X}, E_j \otimes E_i^{-1})$ certainly does: the torus invariant sections. We define the quiver of sections for $\mathscr{L}$ on X to be the quiver of sections for $\widetilde{\mathscr{L}}$ on $\widetilde{X}$, as given in Craw–Smith [10].

The key difference in the Mori Dream Space case lies in the ideal of relations in the path algebra. We define an ideal of relations R in the path algebra which encodes not only the "toric relations" given in [10], but also all the relations in the Cox ring of X. Indeed, the bound quiver of sections Q for $\mathscr{L}$ is finite, acyclic and the quotient $\Bbbk Q/R$ is isomorphic to the endomorphism algebra

$A_{\mathscr{L}} = \mathrm{End}(\bigoplus_{0\leq i\leq r} L_i)$. Setting aside the ideal of relations for now, we define the *multigraded linear series* of the collection $\mathscr{L}$ to be the toric quiver variety $|\mathscr{L}| = \mathcal{M}_\vartheta(Q)$ obtained as the fine moduli space of ϑ-stable representations of Q with dimension vector $(1,\dots,1)$ for the special weight vector $\vartheta = (-r, 1, \dots, 1)$. This fine moduli space carries a collection of tautological line bundles $(\mathscr{W}_0, \dots, \mathscr{W}_r)$ with $\mathscr{W}_0 = \mathcal{O}_{|\mathscr{L}|}$. Since paths in the quiver arise from sections of line bundles of the form $L_j \otimes L_i^{-1}$ on X, evaluating these sections defines a rational map $\varphi_{|\mathscr{L}|}\colon X \dashrightarrow |\mathscr{L}|$. Our first main result (which we prove on page 42) describes the geometry of this map.

Theorem 1.1.1. *For a collection $\mathscr{L} = (\mathcal{O}_X, L_1, \dots, L_r)$ of effective line bundles on X, the map $\varphi_{|\mathscr{L}|}\colon X \dashrightarrow |\mathscr{L}|$ is a morphism if and only if each L_i is basepoint-free, in which case the image is presented explicitly as a geometric quotient and the tautological bundles satisfy $\varphi^*_{|\mathscr{L}|}(\mathscr{W}_i) = L_i$.*

If each L_i on X is the restriction of a basepoint-free line bundle on $\widetilde{X}$ then this morphism is simply the restriction of the morphism from [10, Theorem 1.1]. This is typically not the case, however, because the nef cone of X may be the union of the nef cones of a finite collection of ambient toric varieties.

We provide a necessary and sufficient criterion for $\varphi_{|\mathscr{L}|}\colon X \dashrightarrow |\mathscr{L}|$ to be a closed immersion, and a straightforward application of multigraded regularity due to Hering–Schenck–Smith [17] (see also Maclagan–Smith [22]) provides an efficient way to exhibit many collections that give rise to closed immersions. The resulting geometric quotient constructions of X are new, and while they cannot improve upon the Hu–Keel construction from the birational point of view, it is sometimes possible to encode more refined information on X via $\mathscr{L}$, such as its bounded derived category of coherent sheaves on X.

In Chapter 4 we give our second main result. This is more algebraic, and provides a fine moduli description of X. The ideal of relations R in the path algebra $\Bbbk Q$ defines an ideal I_R in the Cox ring of $|\mathscr{L}|$ that cuts out $\mathcal{M}_\vartheta(\mathrm{mod}(A_{\mathscr{L}}))$, the fine moduli space of ϑ-stable $A_{\mathscr{L}}$-modules with dimension vector $(1,\dots,1)$. This subscheme contains the image of the morphism $\varphi_{|\mathscr{L}|}$ from Theorem 1.1.1, and in general this inclusion is proper. Nevertheless, by saturating I_R with the irrelevant ideal for the GIT quotient construction of the multigraded linear series, and by comparing the result with the ideal I_Q that cuts out the image of $\varphi_{|\mathscr{L}|}$, we obtain the following algebraic result.

Theorem 1.1.2. *For any Mori Dream Space X, there exist (many) collections $\mathscr{L}$ on X such that the morphism $\varphi_{|\mathscr{L}|}\colon X \to |\mathscr{L}|$ identifies X with the fine moduli space $\mathcal{M}_\vartheta(\mathrm{mod}(A_{\mathscr{L}}))$, and the tautological line bundles on $\mathcal{M}_\vartheta(\mathrm{mod}(A_{\mathscr{L}}))$ coincide with the line bundles of $\mathscr{L}$.*

Our proof of this result uses as far as possible the analogous result from [10, Theorem 1.2] for the ambient toric variety, though much remains to be done because I_R can be rather complicated.

More generally, when the morphism $\varphi_{|\mathscr{L}|}\colon X \to |\mathscr{L}|$ is a closed immersion it identifies X with $\mathcal{M}_\vartheta(\mathrm{mod}(A_{\mathscr{L}}))$ precisely when the saturation of I_R by the irrelevant ideal coincides with the ideal

I_Q. These ideals can be computed explicitly in any given example (see Chapter 5), so it is possible to check directly whether Theorem 1.1.2 holds (subject to computational limitations).

The final two chapters of this thesis are more computational in nature. In Chapter 5 we give a computational method to find the ideals which cut out $\varphi_{|\mathscr{L}|}(X)$ and $\mathcal{M}_\vartheta(Q,R)$ in both the Mori Dream Space and toric cases. In Chapter 6 we use these computations to verify the results of Chapter 4 in three cases: for two non-toric del Pezzo surfaces and for the Grassmannian $\mathrm{Gr}(2,4)$.

For a list of line bundles $\mathscr{L}$, we wish to check whether X is isomorphic to the moduli space of bound quiver representations of the quiver of sections for $\mathscr{L}$. We will see that this amounts to checking whether $I_Q = \big(I_R : B_{|\mathscr{L}|}^\infty\big)$. In Chapter 5 we present a method for computing $\widetilde{I_R}, I_R, \widetilde{I_Q}$ and I_Q for a given quiver Q, and as an application we show that $I_Q = \big(I_R : B_{|\mathscr{L}|}^\infty\big)$ for certain collections of line bundles on X_4, X_5 and $\mathrm{Gr}(2,4)$.

We give code which, given a quiver Q, outputs a list of all paths in Q. To find $\widetilde{I_R}$ as defined in [10], we must simply check through all pairs of paths to find all those with the same head, tail and label. Finding generators for I_R is more complicated. In Lemma 5.1.7, we give a generating set for I_R conducive to calculations. We give an algorithm for computing such a generating set.

We show in Proposition 5.2.1 that the ideals I_Q and $\widetilde{I_Q}$ can be written as kernels of $\Bbbk$-algebra homomorphisms. They can therefore be computed using Elimination theory. We give Macaulay 2 code for computing both $\widetilde{I_Q}$ and I_Q in section 5.2.2.

In Chapter 6, we illustrate the method for a pair of del Pezzo surfaces and the Grassmannian $\mathrm{Gr}(2,4)$. For the del Pezzo surfaces, we choose $\mathscr{L}$ to be a full, strongly exceptional collection of line bundles. Such collections are of particular interest because they freely generate the bounded derived category of coherent sheaves on X, that is, the functor

$$\mathbf{R}\mathrm{Hom}(\mathscr{T},-)\colon D^b\big(\mathrm{coh}(X)\big) \longrightarrow D^b\big(\mathrm{mod}(A_{\mathscr{L}})\big)$$

is an equivalence of bounded derived categories. A result of Bergman–Proudfoot [3] establishes that the del Pezzo surface X is isomorphic to a connected component of $\mathcal{M}_\vartheta(\mathrm{mod}(A_{\mathscr{L}}))$ in each case, and our computations demonstrate that in fact X is isomorphic to the moduli space. For the Grassmannian $X = \mathrm{Gr}(2,4)$, we show that $X \cong \mathcal{M}_\vartheta(\mathrm{mod}(A_{\mathscr{L}}))$ when $\mathscr{L} = (\mathcal{O}_X, \mathcal{O}(2), \mathcal{O}(4))$.

1.2 Acknowledgements

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1.3 Declaration

Chapters 1, 3 and 4 and section 6.1 are revised versions of material from Craw–Winn [11]. Chapter 2 is expository. To the best of my knowledge, the remainder of this thesis is original work by the author, except where explicitly stated otherwise.

Chapter 2

Background

In this chapter we summarise necessary background material. In section 2.1 we first consider toric varieties. We show how toric varieties are constructed from fans (see Fulton [15], Cox–Little–Schenck [6]) and describe the construction of toric varieties as GIT quotients (see Cox [7], Mukai [24], Dolgachev [13]). Secondly, we introduce a generalisation of projective $\mathbb{Q}$-factorial toric varieties: Mori Dream Spaces. These will be the primary objects of interest in this thesis. We give background material on Mori Dream Spaces, including their construction as GIT quotients after Hu–Keel [19], Hassett-Tschinkel [16], Laface–Velasco [21]. In section 2.2, we consider two important families of Mori Dream Spaces. In section 2.2.1 we give background information on Grassmannians (see Mukai [24]) and describe their Cox Rings. In section 2.2.2 we summarise material from Batyrev–Popov [2] and Manin [23] on del Pezzo surfaces. These will be our main source of examples of Mori Dream Spaces. We summarise results due to Batyrev-Popov giving generators and relations for the del Pezzo Surfaces of degree 3, 4 and 5. In section 2.2.3 we give an explicit computation of $\mathrm{Cox}(X_5)$, following Derenthal [12].

In section 2.3 we introduce the notion of multigraded regularity for projective toric varieties (see Maclagan–Smith [22], Hering–Schenck–Smith [17]), which will be a crucial component of the proofs in Chapter 4. In section 2.4 we give background information on quivers from Craw–Smith [10].

This thesis continues the programme begun by Craw–Smith in [10], extending from projective toric varieties to the Mori Dream Space case. In section 2.5, we summarise the results of [10]. For a collection of line bundles on a toric variety X, we introduce quivers of sections for toric varieties. We show how this quiver allows us to define a new ambient space for the toric varieties, the multilinear series, and give necessary and sufficient conditions for the existence of a morphism from X to this ambient variety. If the morphism exists, its image is a GIT quotient. It is almost always possible to find a list of line bundles $\mathscr{L}$ such that the morphism is a closed immersion, and the image of X is a moduli space of bound representations of the quiver of sections for $\mathscr{L}$.

We will assume throughout that $\Bbbk$ is an algebraically closed field of characteristic zero.

2.1 Mori Dream Spaces

In this section we give background information on our objects of study: Mori Dream Spaces. We first examine a special case, projective toric varieties, paying particular attention to their construction as GIT quotients.

2.1.1 Toric Varieties

We summarise material from Fulton [15] and Cox–Little–Schenck [6].

A projective toric variety X is an irreducible projective variety containing an algebraic torus as a dense Zariski open set where the action of the algebraic torus on itself extends to an action of the torus on X. We show how to construct toric varieties from fans and as GIT quotients.

Let V be a real vector space. A *strongly convex polyhedral cone* in V is the span over $\mathbb{R}^+$ of a finite collection of vectors which does not contain a line through 0. Let σ be a strongly convex polyhedral cone. We say a hyperplane H is a supporting hyperplane of σ if σ is contained in a halfspace defined by H and $\sigma \cap H \neq \{0\}$. A *face* of σ is the intersection of σ with a supporting hyperplane. Given a cone σ, its *dual cone* $\sigma^\vee$ is defined to be $\sigma^\vee := \{v \in V^* | \langle u, v\rangle \geq 0$ for all $u \in \sigma\}$.

Let $N \cong \mathbb{Z}^r$ be a lattice, let $M := \mathrm{Hom}(N, \mathbb{Z})$ be its dual lattice and let $N_\mathbb{R} := N \otimes \mathbb{R}$. We define a *rational strongly convex polyhedral* cone in $N_\mathbb{R}$ to be a strongly convex polyhedral cone which is the span of a finite collection of vectors in N. A fan Σ is a collection of rational strongly convex polyhedral cones in $N_\mathbb{R}$ such that the faces of every cone in Σ are also in Σ, and such that every pair of cones in Σ intersects in a common face. We also assume that Σ is non-degenerate in the sense that it is not contained in any vector subspace of $N_\mathbb{R}$.

We define a *toric variety* $X = X(\Sigma)$ as follows. For every cone $\sigma \in \Sigma$ we define an affine variety.

$$U_\sigma := \mathrm{Spec}(\Bbbk[\sigma^\vee \cap M]).$$

Explicitly, if $\sigma^\vee$ is generated by $\mathbf{m}_1, \ldots, \mathbf{m}_r \in \mathbb{Z}^n$ then $\Bbbk[\sigma^\vee \cap M] = \Bbbk[x^{\mathbf{m}_1}, \ldots, x^{\mathbf{m}_r}]$, where if $\mathbf{m} = (m_1, \ldots, m_n)$ then $x^{\mathbf{m}} := x_1^{m_1} \cdots x_n^{m_n}$. Also, if $\Bbbk[x^{\mathbf{m}_1}, \ldots, x^{\mathbf{m}_r}] \cong \Bbbk[y_1, \ldots, y_r]/J$ then

$$\mathrm{Spec}\left(\Bbbk[x^{\mathbf{m}_1}, \ldots, x^{\mathbf{m}_r}]\right) \cong \mathbb{V}(J) \subseteq \mathbb{A}^r$$

where $\mathbb{V}(J)$ denotes the common zero locus of all polynomials in J.

If τ is a face of a cone σ in Σ, then there is a natural embedding

$$U_\tau \hookrightarrow U_\sigma$$

If we consider any two cones $\sigma, \sigma' \in \Sigma$, then their intersection $\tau := \sigma \cap \sigma'$ is a common face of both. Hence U_τ embeds into both U_σ and $U_{\sigma'}$. The toric variety X is defined to be the variety obtained by gluing each pair of affine varieties U_σ and $U_{\sigma'}$ along the open subset of each isomorphic to U_τ.

Example 2.1.1. We describe the construction of $\mathbb{P}^2$ as a toric variety obtained from the fan shown below. The fan comprises seven cones, $\sigma_1, \ldots, \sigma_7$. We define $\sigma_1, \sigma_2, \sigma_3$ as shown in the fan. We

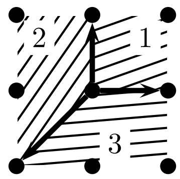


Figure 2.1: Fan for $\mathbb{P}^2$

define $\sigma_4 = \sigma_1 \cap \sigma_3$, $\sigma_5 = \sigma_1 \cap \sigma_2$ and $\sigma_6 = \sigma_2 \cap \sigma_3$. We define σ_7 to be the intersection of $\sigma_1, \ldots, \sigma_6$, i.e. the cone generated by $0 \in \mathbb{Z}^2$. We present a table which gives cones σ in the fan, generators of their dual cones $\sigma^\vee$ and the corresponding $\mathbb{k}$-algebra, $\mathbb{k}[\sigma^\vee \cap M]$.

Cone	Generators of Dual Cone	$\Bbbk$-Algebra
σ_1	$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$	$\Bbbk[x, y]$
σ_2	$\left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$	$\Bbbk[x^{-1}, x^{-1}y]$
σ_3	$\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}$	$\Bbbk[xy^{-1}, y^{-1}]$
σ_4	$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}$	$\Bbbk[x, y, y^{-1}]$
σ_5	$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$	$\Bbbk[x, x^{-1}, y]$
σ_6	$\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$	$\Bbbk[x^{-1}y, y^{-1}, x^{-1}y]$
σ_7	$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}$	$\Bbbk[x, x^{-1}, y, y^{-1}]$

If we change variables $x \mapsto \frac{x_1}{x_0}, y \mapsto \frac{x_2}{x_0}$ we obtain the usual cover of $\mathbb{P}^2$ with coordinates x_0, x_1, x_2 by three copies of $\mathbb{A}^2$:

$$\begin{array}{lllll}
\text{Spec}(\Bbbk[\sigma_1 \cap M]) & = & \text{Spec}(\Bbbk[\frac{x_1}{x_0}, \frac{x_2}{x_0}]) & = & \{(x_0 : x_1 : x_2) \in \mathbb{P}^2 | x_0 \neq 0\} \\
\text{Spec}(\Bbbk[\sigma_2 \cap M]) & = & \text{Spec}(\Bbbk[\frac{x_0}{x_1}, \frac{x_2}{x_1}]) & = & \{(x_0 : x_1 : x_2) \in \mathbb{P}^2 | x_1 \neq 0\} \\
\text{Spec}(\Bbbk[\sigma_3 \cap M]) & = & \text{Spec}(\Bbbk[\frac{x_0}{x_2}, \frac{x_1}{x_2}]) & = & \{(x_0 : x_1 : x_2) \in \mathbb{P}^2 | x_2 \neq 0\} \\
\text{Spec}(\Bbbk[\sigma_4 \cap M]) & = & \text{Spec}(\Bbbk[\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_0}{x_2}]) & = & \{(x_0 : x_1 : x_2) \in \mathbb{P}^2 | x_0 \neq 0 \text{ and } x_2 \neq 0\} \\
\text{Spec}(\Bbbk[\sigma_5 \cap M]) & = & \text{Spec}(\Bbbk[\frac{x_0}{x_1}, \frac{x_0}{x_1}, \frac{x_2}{x_0}]) & = & \{(x_0 : x_1 : x_2) \in \mathbb{P}^2 | x_0 \neq 0 \text{ and } x_1 \neq 0\} \\
\text{Spec}(\Bbbk[\sigma_6 \cap M]) & = & \text{Spec}(\Bbbk[\frac{x_0}{x_2}, \frac{x_1}{x_2}, \frac{x_2}{x_1}]) & = & \{(x_0 : x_1 : x_2) \in \mathbb{P}^2 | x_1 \neq 0 \text{ and } x_2 \neq 0\} \\
\text{Spec}(\Bbbk[\sigma_7 \cap M]) & = & \text{Spec}(\Bbbk[\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_0}{x_1}, \frac{x_0}{x_2}]) & = & \{(x_0 : x_1 : x_2) \in \mathbb{P}^2 | x_0 \neq 0, x_1 \neq 0 \text{ and } x_2 \neq 0\}
\end{array}$$

Let U_i denote the affine open set in $\mathbb{P}^2$ where $x_i \neq 0$. The cones σ_1, σ_2 and σ_3 correspond to U_0, U_1 and U_2 respectively; σ_4, σ_5 and σ_6 correspond to $U_0 \cap U_2, U_0 \cap U_1$ and $U_1 \cap U_2$ respectively; and σ_7 corresponds to $U_0 \cap U_1 \cap U_2$. We also note that points of $U_0 \cap U_1 \cap U_2$ can be written as $(1 : a : b)$ where a and b are nonzero and hence $U_0 \cap U_1 \cap U_2$ is isomorphic to the algebraic torus $(\Bbbk^*)^2$.

Indeed, the subvariety U_0, where 0 is the cone generated by $0 \in N$ is always an algebraic torus. Its natural action on itself extends to an action of the torus on X as follows. Say a cone $\sigma \subseteq N \cong \mathbb{Z}^r$ has dual cone generated by vectors $\mathbf{m}_1, \dots, \mathbf{m}_r$. This gives an affine variety

$$U_\sigma = \text{Spec}\left(\Bbbk[M \cap \sigma^\vee]\right) = \text{Spec}\left(\Bbbk[x^{\mathbf{m}_1}, \dots, x^{\mathbf{m}_r}]\right) \subseteq \mathbb{A}^r$$

The torus $(\Bbbk^*)^n$ acts on $\mathbb{A}^r$ via

$$(\lambda_1, \dots, \lambda_n) \cdot (a_1, \dots, a_r) = (\lambda^{\mathbf{m}_1} a_1, \dots, \lambda^{\mathbf{m}_r} a_r).$$

This restricts to an action of $(\Bbbk^*)^n$ on U_σ. The affine variety U_σ is stable under this action and the action respects the gluing construction so the action extends to the entire toric variety. We illustrate this with an example.

Example 2.1.2. Let σ be the cone in $\mathbb{Z}^3$ generated by

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

The dual cone $\sigma^\vee$ is given by

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

So

$$U_\sigma = \operatorname{Spec}\left(\mathbb{k}\left[x_1, x_2, x_3, \frac{x_1x_2}{x_3}\right]\right) = \operatorname{Spec}\left(\mathbb{k}[y_1, y_2, y_3, y_4]/(y_1y_2 - y_3y_4)\right) = \mathbb{V}(y_1y_2 - y_3y_4) \subseteq \mathbb{A}^4,$$

Where $\mathbb{V}(y_1y_2 - y_3y_4)$ denotes the zero locus of the polynomial $y_1y_2 - y_3y_4$ in $\mathbb{A}^4$. The action of $(\mathbb{k}^*)^3$ on $\mathbb{A}^4$ is given by

$$(\lambda_1, \lambda_2, \lambda_3) \cdot (a_1, a_2, a_3, a_4) = (\lambda_1 a_1, \lambda_2 a_2, \lambda_3 a_3, \frac{\lambda_1\lambda_2}{\lambda_3} a_4).$$

For any $(a_1, a_2, a_3, a_4) \in \mathbb{V}(y_1y_2 - y_3y_4)$ we see that $(\lambda_1, \lambda_2, \lambda_3) \cdot (a_1, a_2, a_3, a_4)$ is also in $\mathbb{V}(y_1y_2 - y_3y_4)$, so $\mathbb{V}(y_1y_2 - y_3y_4)$ is stable under this action.

We say an n dimensional cone is simplicial if it has precisely n generators. We say a fan is simplicial if every cone in the fan is simplicial.

Proposition 2.1.3. *(Theorem 3.1.19 [6]) A toric variety $X(\Sigma)$ is $\mathbb{Q}$-factorial, i.e. some multiple of every Weil divisor is Cartier, if and only if Σ is simplicial.*

Let $\Sigma(1)$ denote the set of one dimensional cones (or rays) of the fan Σ. We assume this set has cardinality d and denote the jth element of this set by τ_j. The elements of $\Sigma(1)$ determine irreducible codimension one torus invariant subvarieties of $X(\Sigma)$. These subvarieties generate the free group of torus invariant Weil divisors, $\mathbb{Z}^d$. Let $\mathrm{Cl}(X)$ denote the class group of X: the group of Weil divisors modulo linear equivalence. We obtain a map $\deg : \mathbb{Z}^d \longrightarrow \mathrm{Cl}(X)$ which maps a Weil divisor to its equivalence class. The map deg fits into a short exact sequence:

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^d \longrightarrow \mathrm{Cl}(X) \longrightarrow 0 \tag{2.1.1}$$

called the *Cox Sequence*. The first map sends $u \in M$ to $\sum_{j=1}^d \langle u, v(j)\rangle D_j$, where $v(j)$ is the lattice point closest to $0 \in N$ which generates τ_j, and D_j is the Weil divisor corresponding to τ_j.

As well as constructing projective toric varieties using a gluing construction given by a fan, we can construct them as GIT quotients. We give an overview of this second construction, summarising material from e.g. Cox [7], King [20], Mukai [24], Dolgachev [13], Craw [9].

We define the *Cox ring* of X to be

$$\mathrm{Cox}(X) := \mathbb{k}[x_1, \dots, x_d],$$

where we recall that d is the number of rays in the fan of X. This is the semigroup algebra of the effective cone of Weil divisors, $\Bbbk[\mathbb{N}^d]$. The map deg induces a $\mathrm{Cl}(X)$ grading of $\mathrm{Cox}(X)$: we set the degree of x_i to be $\deg(\mathbf{e}_i)$. We define $\mathrm{Cox}(X)_D$ to be the Dth graded part of $\mathrm{Cox}(X)$. By Proposition 1.1 of Cox [7],

$$\mathrm{Cox}(X)_D \cong H^0(X, D) \tag{2.1.2}$$

for any $D \in \mathrm{Cl}(X)$. This grading of $\mathrm{Cox}(X)$ induces a $G := \mathrm{Hom}(\mathrm{Cl}(X), \Bbbk^*)$ action on $\mathrm{Spec}(\mathrm{Cox}(X)) \cong \mathbb{A}^d$.

Remark 2.1.4. In the case that we will be interested in, when $\mathrm{Cl}(X)$ is free of rank ρ (and hence $G = (\Bbbk^*)^\rho$), if the lattice map

$$\deg : \mathbb{Z}^d \longrightarrow \mathrm{Cl}(X)$$

is given by a matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{\rho 1} & \cdots & a_{\rho d} \end{pmatrix}$$

then the action of $(\lambda_1, \dots, \lambda_\rho) \in (\Bbbk^*)^\rho$ on a point $(p_1, \dots, p_d) \in \mathrm{Spec}(\mathrm{Cox}(X))$ is given by

$$(\lambda_1, \dots, \lambda_\rho) \cdot (p_1, \dots, p_d) = (\lambda_1^{a_{11}} \cdots \lambda_\rho^{a_{\rho 1}} p_1, \dots, \lambda_1^{a_{1d}} \cdots \lambda_\rho^{a_{\rho d}} p_d).$$

We can construct X as the GIT quotient of $\mathrm{Spec}(\mathrm{Cox}(X))$ by this action. The *character group* for the action of G on $\mathrm{Spec}(\mathrm{Cox}(X))$ is the finitely-generated abelian group $\mathrm{Cl}(X)$. We pick a character $L \in \mathrm{Cl}(X)$ with the additional assumption that L is a very ample line bundle. By (2.1.2) the $\Bbbk$-algebra of semi-invariant functions $\bigoplus_{m \geq 0} \mathrm{Cox}(X)_{mL}$ satisfies

$$\bigoplus_{m \geq 0} H^0(X, L^m) \cong \bigoplus_{m \geq 0} \mathrm{Cox}(X)_{mL}$$

since their ring structures agree. Hence

$$\mathrm{Proj}(\bigoplus_{m \geq 0} H^0(X, L^m)) \cong \mathrm{Proj}(\bigoplus_{m \geq 0} \mathrm{Cox}(X)_{mL}),$$

which are also isomorphic to X since L is very ample. Let B_X be the ideal in the Cox ring $\Bbbk[x_1, \dots, x_d]$ given by

$$B_X = \big(x^{\widehat{\sigma}} \in \mathrm{Cox}(X) | \sigma \text{ is a top-dimensional cone in } \Sigma\big) \tag{2.1.3}$$

where $x^{\widehat{\sigma}} = \prod_{\substack{1 \leq j \leq d \\ \tau_j \not\subseteq \sigma}} x_j$, where we recall d is the number of generators of $\mathrm{Cox}(X)$. Cox [7] gives a description of the L-unstable locus.

Proposition 2.1.5. *([7]) The unstable locus of* $\mathrm{Spec}(\mathrm{Cox}(X))$ *for the action of* $(\Bbbk^*)^r$ *is*

$$\mathbb{V}(\big(H^0(X,L)\big)) = \mathbb{V}(B_X).$$

Suppose sections $s_0, \ldots, s_N$ generate $\bigoplus_{m\in\mathbb{N}} \mathrm{Cox}(X)_{mL}$, then the map

$$\pi : \mathrm{Spec}(\mathrm{Cox}(X)) \setminus \mathbb{V}(B_X) \longrightarrow \mathrm{Proj}(\bigoplus_{m\geq 0} \mathrm{Cox}(X)_{mL})$$

$$p \mapsto (s_0(p) : \ldots : s_N(p))$$

is in fact a morphism (the unstable locus is precisely those points where the morphism would be undefined). It has the property that $\pi(p) = \pi(q)$ if and only if p and q are in the same G orbit since L is very ample, and is thus a *good geometric quotient* — a morphism with the property that the preimage of a point is a G-orbit. We define $\mathrm{Proj}(\bigoplus_{m\geq 0} \mathrm{Cox}(X)_{mL}) \cong X$ to be the GIT quotient of $\mathrm{Spec}(\mathrm{Cox}(X))$.

2.1.2 Mori Dream Spaces

Let X be a projective $\mathbb{Q}$-factorial variety. In this thesis we will assume that the divisor class group of $X, \mathrm{Cl}(X)$, is finitely generated and free of rank ρ. Let $D_1, \ldots, D_\rho$ be Weil divisors whose classes provide an integral basis of $\mathrm{Cl}(X)$.

Definition 2.1.6. The *Cox ring* of X is defined to be the $\mathrm{Cl}(X)$ graded ring

$$\mathrm{Cox}(X, D_1, \ldots, D_\rho) := \bigoplus_{(m_1,\ldots,m_\rho)\in\mathbb{Z}^\rho} H^0(X, D_1^{m_1} \otimes \cdots \otimes D_\rho^{m_\rho}).$$

Mori Dream Spaces are defined in Hu–Keel [19] Definition 1.10. However, we state the main theorem of [19] which gives a much simpler necessary and sufficient condition for X to be a Mori Dream Space:

Theorem 2.1.7. *(Prop 2.9, [19]) A projective $\mathbb{Q}$-factorial variety X is a Mori Dream Space if and only if* $\mathrm{Cox}(X, D_1, \ldots, D_\rho)$ *is finitely generated as a $\Bbbk$-algebra.*

Projective $\mathbb{Q}$-factorial toric varieties are also Mori Dream Spaces:

Theorem 2.1.8. *(Cor 2.10, [19]) X is a projective $\mathbb{Q}$–factorial toric variety if and only if* $\mathrm{Cox}(X)$ *is a polynomial ring.*

Remark 2.1.9. By Hu-Keel [19] and Hassett-Tschinkel [16], for any two bases $D_1, \ldots, D_\rho$ and $E_1, \ldots, E_\rho$ of $\mathrm{Cl}(X)$, the rings $\mathrm{Cox}(X, D_1, \ldots, D_\rho)$ and $\mathrm{Cox}(X, E_1, \ldots, E_\rho)$ are isomorphic. Therefore X being a Mori Dream Space does not depend on the choice of basis for $\mathrm{Cl}(X)$.

From now on, we will assume X is a Mori Dream Space and pick a presentation

$$\mathrm{Cox}(X) = \Bbbk[x_1, \dots .x_d]/I_X \tag{2.1.4}$$

Since this does not depend on the choice of basis for $\mathrm{Cl}(X)$, we will refer to $\mathrm{Cox}(X, D_1, \dots, D_\rho)$ as simply $\mathrm{Cox}(X)$. We will assume that the number of generators in this presentation is as small as possible.

Since we assume that $\mathrm{Cl}(X)$ is finitely generated and free, the ideal I_X is prime by the following theorem due to Elizondo–Kurano–Watanabe:

Theorem 2.1.10. *([14]) Let X be a Mori Dream Space whose class group is finitely generated and free. Then $\mathrm{Cox}(X)$ is a factorial $\Bbbk$-algebra.*

In particular, if we pick a presentation

$$\mathrm{Cox}(X) = \Bbbk[x_1, \dots, x_d]/I_X$$

then I_X is a prime ideal.

Remark 2.1.11. We note that Theorem 2.1.10 implies that I_X does not contain any monomials. If it did, then it would also contain a variable, since I_X is prime. This would contradict our assumption that the number of generators d is as small as possible.

We summarise material from Hu–Keel [19] and Laface–Velsaco [21] on the construction of Mori Dream Spaces as GIT quotients. The Cox ring $\mathrm{Cox}(X)$ is naturally graded by $\mathrm{Cl}(X)$. This grading induces a $G := \mathrm{Hom}(\mathrm{Cl}(X), \Bbbk^*) = (\Bbbk^*)^\rho$ action on $\mathrm{Spec}(\mathrm{Cox}(X)) = \mathbb{V}(I_X) \subseteq \mathbb{A}^d$ (see remark 2.1.4). We construct X as a GIT quotient of $\mathrm{Spec}(\mathrm{Cox}(X))$ under this action as follows. The abelian group $\mathrm{Cl}(X)$ is the *character group* of X. We pick a character $L \in \mathrm{Cl}(X)$, with the additional assumption that L is a very ample line bundle on X. The $\Bbbk$-algebra of L-semi-invariant functions $\bigoplus_{m\geq 0} \mathrm{Cox}(X)_{mL}$ satisfies

$$\bigoplus_{m\geq 0} H^0(X, L^m) = \bigoplus_{m\geq 0} \mathrm{Cox}(X)_{mL}$$

and hence

$$\mathrm{Proj}(\bigoplus_{m\geq 0} H^0(X, L^m)) \cong \mathrm{Proj}(\bigoplus_{m\geq 0} \mathrm{Cox}(X)_{mL}),$$

which are also isomorphic to X since L is very ample. The unstable locus of $\mathrm{Spec}(\mathrm{Cox}(X))$ for the action of G is $\mathbb{V}(\big(H^0(X, L)\big))$.

If sections $s_0, \dots, s_N$ generate $\bigoplus_{m\geq 0} \mathrm{Cox}(X)_{mL}$ then the map

$$\pi : \mathrm{Spec}(\mathrm{Cox}(X)) \setminus \mathbb{V}(B_X) \longrightarrow \mathrm{Proj}(\bigoplus_{m\geq 0} \mathrm{Cox}(X)_{mL})$$

$$p \mapsto (s_0(p) : \ldots : s_N(p))$$

is in fact a morphism (the unstable locus $\mathbb{V}(B_X)$ is precisely the locus where it would be undefined). It has the property that $\pi(p) = \pi(q)$ if and only if p and q are in the same G orbit. Hence, after removal of the unstable locus, π is a good geometric quotient. We define $\mathrm{Proj}(\bigoplus_{m\geq 0} \mathrm{Cox}(X)_{mL}) \cong X$ to be the GIT quotient of $\mathrm{Spec}(\mathrm{Cox}(X))$ under the G action induced by the $\mathrm{Cl}(X)$-grading of $\mathrm{Cox}(X)$. We define a line bundle on a Mori Dream Space X to be *basepoint free* if the common zero locus of its sections $\mathbb{V}\big((H^0(X,L)\big) \subseteq \mathrm{Spec}\,\big(\,\mathrm{Cox}(X)\big)$ is contained in the unstable locus $\mathbb{V}(B_X)$.

We define

$$\tau : \Bbbk[x_1,\ldots,x_d] \longrightarrow \Bbbk[x_1,\ldots,x_d]/I_X \tag{2.1.5}$$

to be the canonical $\Bbbk$-algebra epimorphism mapping x_i to x_i. This induces a $\mathbb{Z}^\rho$ grading of the polynomial ring $\Bbbk[x_1,\ldots,x_d]$ by defining the degree of $x_i \in \Bbbk[x_1,\ldots,x_d]$ to be that of $\tau(x_i)$. This grading induces a $(\Bbbk^*)^\rho$ action on $\mathrm{Spec}(\Bbbk[x_1,\ldots,x_d]) = \mathbb{A}^d$ which restricts to the action on $\mathrm{Spec}(\Bbbk[x_1,\ldots,x_d]/I_X) = \mathbb{V}(I_X) \subseteq \mathbb{A}^d$.

Example 2.1.12. We consider the Grassmannian $\mathrm{Gr}(2,4)$ (for a more complete discussion of Grassmannians we refer to Section 2.2.1). This has Cox ring

$$\Bbbk[x_1,\ldots,x_6]/\big(x_1x_6 - x_2x_5 + x_3x_4\big).$$

The class group of $\mathrm{Gr}(2,4)$ is isomorphic to $\mathbb{Z}$. We pick a character $\chi := \mathcal{O}(1)$ which is also ample line bundle on $\mathrm{Gr}(2,4)$. The $\Bbbk$-algebra of χ semi–invariant functions is $\mathrm{Cox}(\mathrm{Gr}(2,4))$, since the sections of $\mathcal{O}(1)$ are precisely the generators of $\mathrm{Cox}(\mathrm{Gr}(2,4))$. Therefore

$$\mathrm{Gr}(2,4) \cong \mathrm{Proj}\,\big(\,\mathrm{Cox}(\mathrm{Gr}(2,4)\big) = \mathbb{V}(x_1x_6 - x_2x_5 + x_3x_4) \subseteq \mathbb{P}^5.$$

We can also pick an ample character $\chi' := \mathcal{O}(2)$. The $\Bbbk$–algebra of χ' semi–invariant functions is generated by all monomials in six variables of degree two, hence it is:

$$\Bbbk[x_1^2, x_1x_2, x_2^2, \ldots, x_6^2]/\big(x_1x_6 - x_2x_5 + x_3x_4\big) \cong \Bbbk[y_1,\ldots,y_{21}]/J$$

where

$$J = \left(\begin{array}{c} y_9 - y_{12} + y_{16}, y_{20}^2 - y_{15}y_{21}, y_{19}y_{20} - y_{14}y_{21}, y_{18}y_{20} - y_{13}y_{21}, y_{17}y_{20} - y_{12}y_{21}, \\ y_{16}y_{20} - y_{11}y_{21}, y_{19}^2 - y_{10}y_{21}, y_{18}y_{19} - y_{12}y_{21} + y_{16}y_{21}, y_{17}y_{19} - y_8y_{21}, \\ y_{16}y_{19} - y_7y_{21}, y_{15}y_{19} - y_{14}y_{20}, y_{14}y_{19} - y_{10}y_{20}, y_{13}y_{19} - y_{12}y_{20} + y_{11}y_{21}, \\ y_{12}y_{19} - y_8y_{20}, y_{11}y_{19} - y_7y_{20}, y_6y_{19} - y_5y_{20} + y_4y_{21}, y_5y_{19} - y_3y_{20} + y_2y_{21}, \\ y_4y_{19} - y_2y_{20} + y_1y_{21}, y_{18}^2 - y_6y_{21}, y_{17}y_{18} - y_5y_{21}, y_{16}y_{18} - y_4y_{21}, y_{15}y_{18} - y_{13}y_{20}, \\ y_{14}y_{18} - y_{12}y_{20} + y_{11}y_{21}, y_{13}y_{18} - y_6y_{20}, y_{12}y_{18} - y_5y_{20}, y_{11}y_{18} - y_4y_{20}, \\ y_{10}y_{18} - y_8y_{20} + y_7y_{21}, y_8y_{18} - y_3y_{20} + y_2y_{21}, y_7y_{18} - y_2y_{20} + y_1y_{21}, y_{17}^2 - y_3y_{21}, \\ y_{16}y_{17} - y_2y_{21}, y_{15}y_{17} - y_{12}y_{20}, y_{14}y_{17} - y_8y_{20}, y_{13}y_{17} - y_5y_{20}, y_{12}y_{17} - y_3y_{20}, \\ y_{11}y_{17} - y_2y_{20}, y_{10}y_{17} - y_8y_{19}, y_8y_{17} - y_3y_{19}, y_7y_{17} - y_2y_{19}, y_6y_{17} - y_5y_{18}, \\ y_5y_{17} - y_3y_{18}, y_4y_{17} - y_2y_{18}, y_{16}^2 - y_1y_{21}, y_{15}y_{16} - y_{11}y_{20}, y_{14}y_{16} - y_7y_{20}, \\ y_{13}y_{16} - y_4y_{20}, y_{12}y_{16} - y_2y_{20}, y_{11}y_{16} - y_1y_{20}, y_{10}y_{16} - y_7y_{19}, y_8y_{16} - y_2y_{19}, \\ y_7y_{16} - y_1y_{19}, y_6y_{16} - y_4y_{18}, y_5y_{16} - y_2y_{18}, y_4y_{16} - y_1y_{18}, y_3y_{16} - y_2y_{17}, \\ y_2y_{16} - y_1y_{17}, y_{14}^2 - y_{10}y_{15}, y_{13}y_{14} - y_{12}y_{15} + y_{11}y_{20}, y_{12}y_{14} - y_8y_{15}, \\ y_{11}y_{14} - y_7y_{15}, y_6y_{14} - y_5y_{15} + y_4y_{20}, y_5y_{14} - y_3y_{15} + y_2y_{20}, y_4y_{14} - y_2y_{15} + y_1y_{20}, \\ y_{13}^2 - y_6y_{15}, y_{12}y_{13} - y_5y_{15}, y_{11}y_{13} - y_4y_{15}, y_{10}y_{13} - y_8y_{15} + y_7y_{20}, \\ y_8y_{13} - y_3y_{15} + y_2y_{20}, y_7y_{13} - y_2y_{15} + y_1y_{20}, y_{12}^2 - y_3y_{15}, y_{11}y_{12} - y_2y_{15}, \\ y_{10}y_{12} - y_8y_{14}, y_8y_{12} - y_3y_{14}, y_7y_{12} - y_2y_{14}, y_6y_{12} - y_5y_{13}, y_5y_{12} - y_3y_{13}, \\ y_4y_{12} - y_2y_{13}, y_{11}^2 - y_1y_{15}, y_{10}y_{11} - y_7y_{14}, y_8y_{11} - y_2y_{14}, y_7y_{11} - y_1y_{14}, \\ y_6y_{11} - y_4y_{13}, y_5y_{11} - y_2y_{13}, y_4y_{11} - y_1y_{13}, y_3y_{11} - y_2y_{12}, y_2y_{11} - y_1y_{12}, \\ y_6y_{10} - y_3y_{15} + 2y_2y_{20} - y_1y_{21}, y_5y_{10} - y_3y_{14} + y_2y_{19}, y_4y_{10} - y_2y_{14} + y_1y_{19}, \\ y_8^2 - y_3y_{10}, y_7y_8 - y_2y_{10}, y_6y_8 - y_3y_{13} + y_2y_{18}, y_5y_8 - y_3y_{12} + y_2y_{17}, \\ y_4y_8 - y_2y_{12} + y_1y_{17}, y_7^2 - y_1y_{10}, y_6y_7 - y_2y_{13} + y_1y_{18}, y_5y_7 - y_2y_{12} + y_1y_{17}, \\ y_4y_7 - y_1y_{12} + y_1y_{16}, y_3y_7 - y_2y_8, y_2y_7 - y_1y_8, y_5^2 - y_3y_6, y_4y_5 - y_2y_6, \\ y_4^2 - y_1y_6, y_3y_4 - y_2y_5, y_2y_4 - y_1y_5, y_2^2 - y_1y_3 \end{array} \right).$$

This gives an embedding of the Grassmannian $\mathrm{Gr}(2,4)$ into $\mathbb{P}^{20}$.

The ample divisors on X form a cone, $\mathrm{Amp}(X)$, with a decomposition into chambers. Picking a very ample character L in the interior of a chamber, we obtain a toric variety

$$\widetilde{X_L} := \mathrm{Proj}\left(\bigoplus_{m \geq 0} \Bbbk[x_1, \ldots, x_d]_{m\widetilde{L}} \right).$$

The class group of $\widetilde{X_L}$ is $\mathbb{Z}^\rho$ by construction. Hence there exists an isomorphism

$$\psi : \mathrm{Cl}(\widetilde{X_L}) \longrightarrow \mathrm{Cl}(X). \tag{2.1.6}$$

We define the line bundle $\widetilde{L}$ on $\widetilde{X_L}$ to be the inverse image of our choice of line bundle L on X under the map ψ. Picking different characters in the same chamber results in isomorphic toric varieties. The chambers in the decomposition correspond to the ample cones for the toric varieties obtained. The toric variety $\widetilde{X_L}$ has $\mathrm{Cox}(\widetilde{X}_L) = \Bbbk[x_1, \ldots, x_d]$ and is obtained as the GIT quotient of $\mathbb{A}^d$ under the action of G with unstable locus $\mathbb{V}\big(H^0(\widetilde{X}_L, \widetilde{L})\big)$.

Remark 2.1.13. We note that the unstable locus of X is the intersection of the unstable locus of $\widetilde{X}$ and $\mathrm{Spec}(\mathrm{Cox}(X))$. Hence the embedding $\mathrm{Spec}(\mathrm{Cox}(X)) \hookrightarrow \mathrm{Spec}(\mathrm{Cox}(\widetilde{X}_L))$ descends to an embedding $X \hookrightarrow \widetilde{X}_L$.

Remark 2.1.14. We note that $\widetilde{X}_L$ does in general depend on the choice of L, or more precisely, on the chamber containing L. However, in what follows it will not matter what choice of L we make, and hence we will refer to $\widetilde{X}_L$ as simply $\widetilde{X}$ from now on.

We present an example which illustrates the concepts introduced in this section, further details on del Pezzo surfaces can be found in section 2.2.2.

Example 2.1.15. Let X_4 be the del Pezzo surface obtained as the blow-up of $\mathbb{P}^2_{\Bbbk}$ at four points in general position. The Picard group $\mathrm{Pic}(X_4) \cong \mathbb{Z}^5$ has a basis given by l_0, the pullback to X_4 of the hyperplane class on $\mathbb{P}^2_{\Bbbk}$, together with the four exceptional curves l_1, l_2, l_3, l_4. The semigroup homomorphism $\deg\colon \mathbb{N}^{10} \to \mathrm{Pic}(X_4)$ obtained as multiplication by the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

induces a $\mathrm{Pic}(X)$ grading of $\Bbbk[x_1, \ldots, x_{10}]$. The $\mathrm{Pic}(X)$-homogeneous ideal

$$I_{X_4} := \begin{pmatrix} x_2x_5 - x_3x_6 + x_4x_7, \;\; x_1x_5 - x_3x_8 + x_4x_9, \\ x_1x_6 - x_2x_8 + x_4x_{10}, \;\; x_1x_7 - x_2x_9 + x_3x_{10}, \;\; x_5x_{10} - x_6x_9 + x_7x_8 \end{pmatrix}$$

determines $\mathrm{Cox}(X_4) = \Bbbk[x_1, \ldots, x_{10}]/I_{X_4}$ following Batyrev–Popov [2]. We construct an ambient toric variety as follows. Following Example 2.11 of Laface–Velasco [21] we pick a character $\chi = 11l_0 - 5l_1 - 3l_2 - 2l_3 - l_4$. The line bundle χ is ample on X_4 and we show $\widetilde{X_\chi} := \mathbb{A}^{10} /\!\!/_{\chi} T$ is $\mathbb{Q}$-factorial as follows. The unstable locus for $\chi, \mathrm{rad}\big(H^0(\widetilde{X_\chi}, \chi)\big)$ is the common zero locus of the following ideal:

$$\left(\begin{array}{c} x_1x_2x_5x_6x_7, x_1x_3x_5x_6x_7, x_1x_4x_5x_6x_7, x_1x_3x_4x_7x_8, x_3x_4x_7x_9x_{10}, \\ x_2x_3x_4x_7x_8, x_1x_2x_5x_7x_8, x_2x_3x_5x_7x_8, x_1x_4x_5x_7x_8, x_1x_3x_6x_7x_8, \\ x_2x_3x_6x_7x_8, x_1x_3x_4x_6x_9, x_2x_3x_4x_6x_9, x_1x_2x_5x_6x_9, x_1x_3x_5x_6x_9, \\ x_2x_4x_5x_6x_9, x_1x_4x_6x_7x_9, x_2x_4x_6x_7x_9, x_1x_2x_5x_8x_9, x_2x_3x_5x_8x_9, \\ x_2x_4x_5x_8x_9, x_1x_3x_6x_8x_9, x_2x_3x_6x_8x_9, x_1x_4x_7x_8x_9, x_2x_4x_7x_8x_9, \\ x_1x_2x_4x_5x_{10}, x_2x_3x_4x_5x_{10}, x_1x_2x_5x_6x_{10}, x_1x_3x_5x_6x_{10}, \\ x_3x_4x_5x_6x_{10}, x_1x_4x_5x_7x_{10}, x_3x_4x_5x_7x_{10}, x_1x_2x_5x_8x_{10}, \\ x_2x_3x_5x_8x_{10}, x_1x_3x_6x_8x_{10}, x_2x_3x_6x_8x_{10}, x_3x_4x_6x_8x_{10}, \\ x_3x_4x_7x_8x_{10}, x_2x_4x_5x_9x_{10}, x_3x_4x_6x_9x_{10}, x_1x_4x_7x_9x_{10}, x_2x_4x_7x_9x_{10} \end{array}\right).$$

This ideal is also defined in (2.1.3) as

$$B_X = \left(x^{\widehat{\sigma}} \in \mathrm{Cox}(X) | \sigma \text{ is a top-dimensional cone in } \Sigma\right)$$

where $x^{\widehat{\sigma}} = \prod_{\substack{1 \le j \le 10 \\ \tau_j \not\subseteq \sigma}} x_j$. Since each monomial in $\mathrm{rad}\left(H^0(\widetilde{X_\chi}, \chi)\right)$ is a product of five generators, each top dimensional cone in the fan defining $\widetilde{X_\chi}$ has 10-5=5 generators. This follows since the Cox ring of $\widetilde{X_\chi}$ has 10 generators which correspond to the rays of the fan defining $\widetilde{X_\chi}$, and because the lattice M which contains the fan is isomorphic to $\mathbb{Z}^5$ (this follows from the fact that the Cox sequence (2.1.1) is a short exact sequence). Therefore by Proposition 2.1.3, the toric variety $\widetilde{X_\chi}$ is $\mathbb{Q}$-factorial. Thus $\chi \in \mathrm{Pic}(X)$ lies in an open GIT chamber for the action of T on $\mathbb{A}^{10}_{\Bbbk}$, and we set $\widetilde{X_4} := \mathbb{A}^{10} /\!\!/_\chi T$.

Laface–Velasco note further that the ample bundle $-K_{X_4} = 3l_0 - l_1 - l_2 - l_3 - l_4$ defines a non-$\mathbb{Q}$-factorial toric quotient $\widetilde{X}_{-K_{X_4}}$ The ideal $\mathrm{rad}\left(H^0(\widetilde{X}_{-K_{X_4}}, -K_{X_4})\right)$ is

$$\left(\begin{array}{c} x_4x_7x_9x_{10}, x_3x_6x_8x_{10}, x_3x_4x_5x_{10}, x_1x_2x_5x_{10}, x_2x_5x_8x_9, \\ x_2x_4x_6x_9, x_1x_3x_6x_9, x_1x_4x_7x_8, x_2x_3x_7x_8, x_1x_5x_6x_7, x_3x_4x_6x_9x_{10}, \\ x_2x_4x_5x_9x_{10}, x_3x_4x_7x_8x_{10}, x_2x_3x_5x_8x_{10}, x_1x_4x_5x_7x_{10}, \\ x_1x_3x_5x_6x_{10}, x_2x_4x_7x_8x_9, x_2x_3x_6x_8x_9, x_1x_4x_6x_7x_9, x_1x_2x_5x_6x_9, x_1x_3x_6x_7x_8, x_1x_2x_5x_7x_8 \end{array}\right)$$

We see that there exist monomial generators which are a product of 4 variables. By the same logic as above there exist top dimensional cones in the fan defining $\widetilde{X}_{-K_{X_4}}$ with 6 generators which are therefore not simplicial. Hence by Proposition 2.1.3, $\widetilde{X}_{-K_{X_4}}$ is not $\mathbb{Q}$–factorial. So $-K_{X_4}$ lies in a GIT wall for the action on $\mathbb{A}^{10}_{\Bbbk}$.

2.2 Cox Rings of Mori Dream Spaces

In this section we introduce two families of del Pezzo surfaces: Grassmannians and del Pezzo surfaces. We study their Cox rings.

2.2.1 Grassmannians

The *Grassmannian* $\mathrm{Gr}(r, n)$ is the scheme which represents the functor mapping a scheme X to the rank d vector subbundles of the trivial rank n vector bundle on X. A rank r element of $\mathrm{Mat}(r, n)$ determines an r–dimensional vector subspace of n–dimensional space, up to change of basis, i.e. up to multiplication by an element of $\mathrm{GL}(r)$. Hence $\mathrm{Gr}(r, n)$ is the space of $\mathrm{GL}(r)$ orbits of rank r matrices in $\mathrm{Mat}(r, n)$, where the group action is multiplication on the left.

An embedding of $\mathrm{Gr}(r, n)$ into projective space $\mathbb{P}^s$, where $s = \binom{n}{r} - 1$, is given by the determinantal line bundle on $\mathrm{Gr}(r, n)$. Explicitly, this maps an element of $\mathrm{Mat}(r, n)$ to its $r \times r$ minors (there are $\binom{n}{r}$ such). We note that the action of $\mathrm{GL}(r)$ only changes the $r \times r$ minors by a nonzero scalar multiple, hence this map is well defined. We also note that since each matrix has rank r, at least one of the minors will be nonzero. The image of this map is cut out by the ideal of Plücker relations. We consider the map

$$D : \mathbb{k}[z_0, \ldots, z_s] \longrightarrow \mathbb{k}[a_{ij}]$$

mapping z_i to the ith $r \times r$ minor of the generic matrix (a_{ij}). The ideal of Plücker relations is the kernel of D. Hence $\mathbb{k}[z_1, \ldots, z_n]/\ker(D)$ is the homogeneous coordinate ring of the Grassmannian $\mathrm{Gr}(r, n)$. By Remark 3.9 of Castravet–Tevelev [4], this ring coincides with $\mathrm{Cox}(\mathrm{Gr}(r, n))$.

2.2.2 Del Pezzo Surfaces

In this section, we give essential background information on del Pezzo surfaces. We describe results of Batyrev–Popov [2] giving generators and relations for the Cox rings of certain non-toric del Pezzo surfaces. We conclude by calculating the Cox ring of a del Pezzo surface of degree 4 following the method of Batyrev–Popov [2] and Derenthal [12].

We summarise material on del Pezzo surfaces found in Manin [23] and Batyrev–Popov [2].

A *del Pezzo surface of degree* $9 - r$ is the blow up of $\mathbb{P}^2$ at $0 \leq r \leq 8$ points $p_1, \ldots, p_r$ in general position. This is a smooth surface which we denote by X_r. We say r points are in *general position* if no three points lie on a line, no six points lie on a conic, and no cubic with a double point which contains seven of the points contains the eighth.

We denote the blow-up map by $\pi_r : X_r \longrightarrow \mathbb{P}^2$. The Picard group of X_r satisfies $\mathrm{Pic}(X_r) \cong \mathbb{Z}^{r+1}$, with a basis given by $l_0 := \pi_r^*(\mathcal{O}(1)), l_1 := \pi_r^{-1}(p_1), \ldots, l_r := \pi_r^{-1}(p_r)$. To harmonize with the literature, we denote multiplication in the Picard group using additive (rather than tensor) notation. The intersection form is given by the following matrix

$$
\left(\begin{array}{c|cccc}
 & l_0 & l_1 & \cdots & l_r \\
\hline
l_0 & 1 & 0 & \cdots & 0 \\
l_1 & 0 & -1 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
l_r & 0 & \cdots & 0 & -1
\end{array}\right)
$$

We denote the intersection of two curves l and l' by $l \cdot l'$. We say a curve l on a surface is a (-1)-curve if its selfintersection number $l^2 := l \cdot l$ is equal to -1.

The *strict transform* of a curve C under the blow up map $\pi : X_r \longrightarrow \mathbb{P}^2$ is $\overline{\pi^{-1}(C \cap (\mathbb{P}^2 \setminus \{p_1, \ldots, p_r\}))}$. The (-1)-curves on X_r are the inverse images of blown up points and the strict transforms of the following curves on $\mathbb{P}^2$:

(i) Lines between pairs of blown up points;

(ii) Conics containing five blown up points;

(iii) Cubics with a double point containing seven blown up points;

(iv) Quartics with three double points containing eight blown up points;

(v) Quintics with six double points containing eight blown up points;

(vi) Sextics with seven double points containing eight blown up points.

Every del Pezzo Surface is a Mori Dream Space by Batyrev–Popov [2]. If we blow up $r \leq 3$ points we obtain a smooth projective toric surface. Batyrev–Popov describe generators and relations for the Cox rings of X_4, X_5 and X_6 in [2]. We summarise their results.

The following result can be found in Laface–Velasco [21]

Proposition 2.2.1. *([21]) Let X be a surface. If $E \in \mathrm{Pic}(X)$ is a (-1)-curve then $H^0(X, E)$ is generated by a unique (up to scalar multiplication) section. This section is a generator of* $\mathrm{Cox}(X)$.

Therefore the section of any (-1)-curve is a generator of $\mathrm{Cox}(X_r)$. These are the only generators of $\mathrm{Cox}(X_r)$ by the following Theorem due to Batyev–Popov.

Theorem 2.2.2. *(Theorem 3.2, [2]) For $3 \leq r \leq 7$, the Cox ring of the del Pezzo surface X_r is generated by global sections of $\mathcal{O}(D)$ where D is a (-1)-curve.*

The del Pezzo surfaces X_4, X_5 and X_6 are the blow ups of $\mathbb{P}^2$ at the first four, five and six points respectively from points $p_1, \ldots, p_6$ in general position. We can always pick $p_1 = (1, 0, 0), p_2 = (0, 1, 0), p_3 = (0, 0, 1)$ and $p_4 = (1, 1, 1)$. The (-1)-curves on X_4, X_5 and X_6 are the preimages of

Generator	Degree	Point or Curve in $\mathbb{P}^2$
x_1	l_1	p_1
x_2	l_2	p_2
x_3	l_3	p_3
x_4	l_4	p_4
x_5	$2l_0 - l_1 - l_2$	line between p_1 and p_2
x_6	$2l_0 - l_1 - l_3$	line between p_1 and p_3
x_7	$2l_0 - l_1 - l_4$	line between p_1 and p_4
x_8	$2l_0 - l_2 - l_3$	line between p_2 and p_3
x_9	$2l_0 - l_2 - l_4$	line between p_2 and p_4
x_{10}	$2l_0 - l_3 - l_4$	line between p_3 and p_4

Figure 2.2: X_4 Case

blown up points, the strict transforms of lines in $\mathbb{P}^2$ between pairs of blown up points, and the strict transforms of conics through five blown up points. We describe these explicitly below.

The strict transform of the line containing points p_i and p_j is $l_0 - l_i - l_j$, and the strict transform of the conic containing points $p_{i_1}, \ldots, p_{i_5}$ is $2l_0 - l_{i_1} - \cdots - l_{i_5}$. All the (-1)-curves are either equal to l_i for some $i \in \{1, \ldots, 6\}$, or the strict transforms of lines through pairs of points or conics through five points. Hence we can associate to each (-1)-curve the equation of a line or conic, unless it is the preimage of a blown up point. In that case, it is a useful convention to assign a constant to each l_i. We will always choose that constant to be 1. We denote the homogeneous polynomial (or form) associated to a line bundle l in this way to be $f_l \in \mathbb{k}[z_1, z_2, z_3]$.

We present the generators of X_4, X_5 and X_6 their degree and the curve in $\mathbb{P}^2$ of which they are the strict transform in figures 2.2, 2.3 and 2.4 respectively.

Generator	Degree	Point or Curve in $\mathbb{P}^2$
x_1	l_1	p_1
x_2	l_2	p_2
x_3	l_3	p_3
x_4	l_4	p_4
x_5	l_5	p_5
x_6	$l_0 - l_1 - l_2$	line between p_1 and p_2
x_7	$l_0 - l_1 - l_3$	line between p_1 and p_3
x_8	$l_0 - l_1 - l_4$	line between p_1 and p_4
x_9	$l_0 - l_1 - l_5$	line between p_1 and p_5
x_{10}	$l_0 - l_2 - l_3$	line between p_2 and p_3
x_{11}	$l_0 - l_2 - l_4$	line between p_2 and p_4
x_{12}	$l_0 - l_2 - l_5$	line between p_2 and p_5
x_{13}	$l_0 - l_3 - l_4$	line between p_3 and p_4
x_{14}	$l_0 - l_3 - l_5$	line between p_3 and p_5
x_{15}	$l_0 - l_4 - l_5$	line between p_4 and p_5
x_{16}	$2l_0 - l_1 - l_2 - l_3 - l_4 - l_5$	conic containing p_1, p_2, p_3, p_4 and p_5

Figure 2.3: X_5 Case

Generator	Degree	Point or Curve in $\mathbb{P}^2$
x_1	l_1	p_1
x_2	l_2	p_2
x_3	l_3	p_3
x_4	l_4	p_4
x_5	l_5	p_5
x_6	l_6	p_6
x_7	$l_0 - l_1 - l_2$	line between p_1 and p_2
x_8	$l_0 - l_1 - l_3$	line between p_1 and p_3
x_9	$l_0 - l_1 - l_4$	line between p_1 and p_4
x_{10}	$l_0 - l_1 - l_5$	line between p_1 and p_5
x_{11}	$l_0 - l_1 - l_6$	line between p_1 and p_6
x_{12}	$l_0 - l_2 - l_3$	line between p_2 and p_3
x_{13}	$l_0 - l_2 - l_4$	line between p_2 and p_4
x_{14}	$l_0 - l_2 - l_5$	line between p_2 and p_5
x_{15}	$l_0 - l_2 - l_6$	line between p_2 and p_6
x_{16}	$l_0 - l_3 - l_4$	line between p_3 and p_4
x_{17}	$l_0 - l_3 - l_5$	line between p_3 and p_5
x_{18}	$l_0 - l_3 - l_6$	line between p_3 and p_6
x_{19}	$l_0 - l_4 - l_5$	line between p_4 and p_5
x_{20}	$l_0 - l_4 - l_6$	line between p_4 and p_6
x_{21}	$l_0 - l_5 - l_6$	line between p_5 and p_6
x_{22}	$2l_0 - l_1 - l_2 - l_3 - l_4 - l_5$	conic containing p_1, p_2, p_3, p_4 and p_5
x_{23}	$2l_0 - l_1 - l_2 - l_3 - l_4 - l_6$	conic containing p_1, p_2, p_3, p_4 and p_6
x_{24}	$2l_0 - l_1 - l_2 - l_3 - l_5 - l_6$	conic containing p_1, p_2, p_3, p_5 and p_6
x_{25}	$2l_0 - l_1 - l_2 - l_4 - l_5 - l_6$	conic containing p_1, p_2, p_4, p_5 and p_6
x_{26}	$2l_0 - l_1 - l_3 - l_4 - l_5 - l_6$	conic containing p_1, p_3, p_4, p_5 and p_6
x_{27}	$2l_0 - l_2 - l_3 - l_4 - l_5 - l_6$	conic containing p_2, p_3, p_4, p_5 and p_6

Figure 2.4: X_6 Case

Having found the generators of $\mathrm{Cox}(X_r)$, we turn our attention to finding the ideal of relations I_{X_r}. Batyrev–Popov first computed I_{X_4}, then used induction on the number of blown up points to obtain I_{X_5} and I_{X_6}.

Proposition 2.2.3. *(Prop 4.1, [2]) The Cox ring of X_4 is isomorphic to the homogeneous coordinate ring of* $\mathrm{Gr}(3,5)$*, i.e.*

$$\mathrm{Cox}(X_4) = \Bbbk[x_1, \ldots, x_{10}]/I_{X_4}$$

where

$$I_{X_4} := \left(\begin{array}{c} x_2x_5 - x_3x_6 + x_4x_7, \;\; x_1x_5 - x_3x_8 + x_4x_9, \\ x_1x_6 - x_2x_8 + x_4x_{10}, \;\; x_1x_7 - x_2x_9 + x_3x_{10}, \;\; x_5x_{10} - x_6x_9 + x_7x_8 \end{array} \right)$$

The terms of each relation in I_{X_4} are sections of a single line bundle, each of which is a *ruling.*

Definition 2.2.4. A *ruling* is a line bundle l such that $l = l_1 + l_2$ where l_1, l_2 are (-1)-curves and $l_1 \cdot l_2 = 1$.

Each ruling L can be written in $r-1$ ways as a sum of (-1)-curves, i.e.

$$L = L_1 + L_1' = L_2 + L_2' = \cdots = L_{r-1} + L_{r-1}'$$

where each L_i and L_i' is a (-1)-curve. A relation in $\mathrm{Cox}(X_r)$ arises from a ruling in the following way. We recall that a form f_l is associated to each (-1)-curve on X_r. The forms

$$f_{L_1}f_{L_1'}, \ldots, f_{L_{r-1}}f_{L_{r-1}'}$$

have $r-3$ relations between them. These lift to give relations between the sections of the ruling L, in the sense that if

$$a_1 f_{L_1} f_{L_1'} + \cdots + a_n f_{L_n} f_{L_n'} = 0$$

then

$$a_1 x_{L_1} x_{L_1'} + \cdots + a_n x_{L_n} x_{L_n'}$$

is a relation in the Cox ring, where x_L is the generator of the Cox ring corresponding to the (-1)–curve L. By the following theorem due to Batyrev–Popov [2], these are the only relations.

Proposition 2.2.5. *(Theorem 4.9, [2]). For $r = 4, 5$ or 6, I_{X_R} is the ideal generated by relations between sections of rulings as described above.*

2.2.3 Computing $\mathrm{Cox}(X_5)$

In the rest of this section, we use the theory due to Batyrev–Popov [2] summarised above to calculate the ideal I_{X_5}. This calculation can be found in Derenthal [12] using a generic fifth point $(1, \alpha, \beta)$. For our calculation we pick a specific fifth point and give a little more detail. First we compute the forms associated to the generators of I_{X_5}. We give the rulings, their sections and associated forms for X_5. Once we have this information we calculate the relations between the associated forms, and hence the relations between sections of rulings. This gives us I_{X_5}.

Let X_5 be the blow up of $p_1, \dots, p_5$ where we choose

$$p_1 := (1,0,0), p_2 := (0,1,0), p_3 := (0,0,1), p_4 := (1,1,1), p_5 := (1,2,3).$$

We emphasise that X_5 is not independent of this choice of points. We compute the forms f_l for X_5 as described in Figure 2.3. First, it is possible to compute the equations of lines through pairs of points by inspection. To compute the equations of conics through five points we use a simple Maple procedure, findconics, described in Appendix B. We present the forms in the following table.

Generator of $\mathrm{Cox}(X_5)$	(-1)-curve l	f_l
x_1	l_1	1
x_2	l_2	1
x_3	l_3	1
x_4	l_4	1
x_5	l_5	1
x_6	$l_0 - l_1 - l_2$	z_3
x_7	$l_0 - l_1 - l_3$	z_2
x_8	$l_0 - l_1 - l_4$	$z_2 - z_3$
x_9	$l_0 - l_1 - l_5$	$2z_3 - 3z_2$
x_{10}	$l_0 - l_2 - l_3$	z_1
x_{11}	$l_0 - l_2 - l_4$	$z_1 - z_3$
x_{12}	$l_0 - l_2 - l_5$	$z_3 - 3z_1$
x_{13}	$l_0 - l_3 - l_4$	$z_1 - z_2$
x_{14}	$l_0 - l_3 - l_5$	$2z_1 - z_2$
x_{15}	$l_0 - l_4 - l_5$	$z_1 - 2z_2 + z_3$
x_{16}	$2l_0 - l_1 - l_2 - l_3 - l_4 - l_5$	$3z_1z_2 - 4z_1z_3 + z_2z_3$

We compute the rulings for X_5 and their sections using the code in Appendix B. For each ruling, we calculate the relations between the forms corresponding to its sections using Maple. There are four forms f_1, f_2, f_3 and f_4 corresponding to sections s_1, s_2, s_3 and s_4 and two relations between them. To find the relations, we use the Maple command “solve” to find solutions to the pair of equations:

$$af_1 + bf_2 + cf_3 = 0$$
$$df_1 + ef_2 + hf_4 = 0$$

thus we obtain two generators

$$as_1 + bs_2 + cs_3 \text{ and } ds_1 + es_2 + hs_4$$

of I_{X_5}.

We give the rulings, their sections, associated forms in $\Bbbk[z_1, z_2, z_3]$ and the relation between them in the following table:

Rulings	Sections	Forms in $\Bbbk[z_1, z_2, z_3]$	Relations
$l_0 - l_1$	x_2x_6	z_3	$x_2x_6 - x_3x_7 + x_4x_8$
	x_3x_7	z_2	$2x_2x_6 - 3x_3x_7 - x_5x_9$
	x_4x_8	$z_2 - z_3$	
	x_5x_9	$2z_3 - 3z_2$	
$l_0 - l_2$	x_1x_6	z_3	$x_1x_6 - x_3x_{10} + x_4x_{11}$
	x_3x_{10}	z_1	$x_1x_6 - 3x_3x_{10} - x_5x_{12}$
	x_4x_{11}	$z_1 - z_3$	
	x_5x_{12}	$z_3 - 3z_1$	
$l_0 - l_3$	x_1x_7	z_2	$x_1x_7 - x_2x_{10} + x_4x_{13}$
	x_2x_{10}	z_1	$x_1x_7 - 2x_2x_{10} + x_5x_{14}$
	x_4x_{13}	$z_1 - z_2$	
	x_5x_{14}	$2z_1 - z_2$	
$l_0 - l_4$	x_1x_8	$z_2 - z_3$	$x_1x_8 - x_2x_{11} + x_3x_{13}$
	x_2x_{11}	$z_1 - z_3$	$-2x_1x_8 + x_2x_{11} - x_5x_{15}$
	x_3x_{13}	$z_1 - z_2$	
	x_5x_{15}	$z_1 - 2z_2 + z_3$	
$l_0 - l_5$	x_1x_9	$2z_3 - 3z_2$	$-x_1x_9 + 2x_2x_{12} + 3x_3x_{14}$,
	x_2x_{12}	$z_3 - 3z_1$	$2x_1x_9 + x_2x_{12} + 3x_4x_{15}$
	x_3x_{14}	$2z_1 - z_2$	
	x_4x_{15}	$z_1 - 2z_2 + z_3$	
$2l_0 - l_1 - l_2 - l_3 - l_4$	x_5x_{16}	$3z_1z_2 - 4z_1z_3 + z_2z_3$	$x_5x_{16} + x_6x_{13} - 3x_8x_{10}$,
	x_6x_{13}	$z_3(z_1 - z_2)$	$x_6x_{13} - x_7x_{11} + x_8x_{10}$
	x_7x_{11}	$z_2(z_1 - z_3)$	
	x_8x_{10}	$(z_2 - z_3)z_1$	

$2l_0 - l_1 - l_2 - l_3 - l_5$	x_4x_{16} x_6x_{14} x_7x_{12} x_9x_{10}	$3z_1z_2 - 4z_1z_3 + z_2z_3$ $z_3(2z_1 - z_2)$ $z_2(z_3 - 3z_1)$ $(2z_3 - 3z_2)z_1$	$x_4x_{16} + 2x_6x_{14} + x_7x_{12}$, $x_4x_{16} + x_6x_{14} + x_9x_{10}$
$2l_0 - l_1 - l_2 - l_4 - l_5$	x_3x_{16} x_6x_{15} x_8x_{12} x_9x_{11}	$3z_1z_2 - 4z_1z_3 + z_2z_3$ $z_3(z_1 - 2z_2 + z_3)$ $-(-z_2 + z_3)(z_3 - 3z_1)$ $(2z_3 - 3z_2)(z_1 - z_3)$	$x_3x_{16} + x_6x_{15} + x_8x_{12}$, $x_3x_{16} + 2x_6x_{15} + x_9x_{11}$
$2l_0 - l_1 - l_3 - l_4 - l_5$	x_2x_{16} x_7x_{15} x_8x_{14} x_9x_{13}	$3z_1z_2 - 4z_1z_3 + z_2z_3$ $z_2(z_1 - 2z_2 + z_3)$ $(z_2 - z_3)(2z_1 - z_2)$ $(2z_3 - 3z_2)(z_1 - z_2)$	$x_2x_{16} + x_7x_{15} - 2x_8x_{14}$, $x_2x_{16} + 3x_7x_{15} + 2x_9x_{13}$
$2l_0 - l_2 - l_3 - l_4 - l_5$	x_1x_{16} $x_{10}x_{15}$ $x_{11}x_{14}$ $x_{12}x_{13}$	$3z_1z_2 - 4z_1z_3 + z_2z_3$ $z_1(z_1 - 2z_2 + z_3)$ $(z_1 - z_3)(2z_1 - z_2)$ $(z_3 - 3z_1)(z_1 - z_2)$	$x_1x_{16} + 2x_{10}x_{15} - x_{11}x_{14}$, $x_1x_{16} + 3x_{10}x_{15} + x_{12}x_{13}$

Hence,

$$I_{X_5} = \left(\begin{array}{c} x_5x_{16} + x_6x_{13} - 3x_8x_{10},\ x_4x_{16} + 2x_6x_{14} + x_7x_{12},\ x_4x_{16} + x_6x_{14} + x_9x_{10}, \\ x_3x_{16} + x_6x_{15} + x_8x_{12},\ x_3x_{16} + 2x_6x_{15} + x_9x_{11},\ x_2x_{16} + x_7x_{15} - 2x_8x_{14}, \\ x_2x_{16} + 3x_7x_{15} + 2x_9x_{13},\ x_1x_{16} + 2x_{10}x_{15} - x_{11}x_{14},\ x_1x_{16} + 3x_{10}x_{15} + x_{12}x_{13}, \\ x_2x_6 - x_3x_7 + x_4x_8,\ 2x_2x_6 - 3x_3x_7 - x_5x_9,\ x_1x_6 - x_3x_{10} + x_4x_{11}, \\ x_1x_6 - 3x_3x_{10} - x_5x_{12},\ x_1x_7 - x_2x_{10} + x_4x_{13},\ x_1x_7 - 2x_2x_{10} + x_5x_{14}, \\ x_1x_8 - x_2x_{11} + x_3x_{13},\ -2x_1x_8 + x_2x_{11} - x_5x_{15},\ -x_1x_9 + 2x_2x_{12} + 3x_3x_{14}, \\ -2x_1x_9 + x_2x_{12} + 3x_4x_{15},\ x_6x_{13} - x_7x_{11} + x_8x_{10} \end{array} \right).$$

2.3 Multigraded Regularity for Projective Toric Varieties

Maclagan-Smith introduced the notion of multigraded regularity in [22] as a generalisation of Castelnuovo-Mumford regularity. Let X be a projective toric variety, and let $\mathrm{Cox}(X) = \Bbbk[x_1, \ldots, x_d]$. Multigraded regularity is a useful tool for studying the geometry of X. For example, it gives a bound for the multidegrees of the equations which cut out the subvariety corresponding to an ideal sheaf, and it allows us to test whether an ample line bundle gives a projectively normal embedding of X. In this thesis, we will use multigraded regularity of a line bundle $L = L_1 \otimes \cdots \otimes L_k$ with respect to $L_1, \ldots, L_k$ to ensure surjectivity of certain maps.

We summarise material due to Maclagan–Smith [22] and Hering–Schenck–Smith [17].

Let $\mathscr{F}$ be a coherent sheaf, let B and $M_1, \dots, M_k$ be line bundles on X. For a vector $\mathbf{u} = (u_1, \dots, u_k) \in \mathbb{N}^k$, we denote $M_1^{u_1} \otimes \cdots \otimes M_k^{u_k}$ by $M^{\mathbf{u}}$.

Definition 2.3.1. We say $\mathscr{F}$ is *B-regular* (with respect to $M_1, \dots, M_k$) if $H^i(X, \mathscr{F} \otimes B \otimes M^{-\mathbf{u}}) = 0$ for all $i > 0$ and all $\mathbf{u} \in \mathbb{N}^k$ satisfying $|\mathbf{u}| := u_1 + \cdots + u_k = i$.

The following theorem is due to Maclagan–Smith in the toric case, but was generalised by Hering–Schenck–Smith [17].

Theorem 2.3.2. *([22], [17])*
Let $\mathscr{F}$ be a coherent sheaf that is B-regular with respect to $M_1, \dots, M_k$. For all $\mathbf{u} \in \mathbb{N}^k$ the map

$$H^0(X, \mathscr{F} \otimes B \otimes M^{\mathbf{u}}) \otimes H^0(X, M^{\mathbf{v}}) \longrightarrow H^0(X, \mathscr{F} \otimes B \otimes M^{\mathbf{u}+\mathbf{v}})$$

is surjective for all $\mathbf{v} \in \mathbb{N}^k$.

Corollary 2.3.3. *Let $L_1, \dots, L_k$ be line bundles and suppose $L = L_1^{\beta_1} \otimes \cdots \otimes L_k^{\beta_k}$ be $\mathcal{O}_X$-regular with respect to $L_1, \dots, L_r$ for some $\beta_1, \dots, \beta_k \geq 0$. The multiplication map*

$$H^0(X, L)^{\otimes d} \longrightarrow H^0(X, L^d)$$

is surjective.

Proof. If the multiplication map

$$H^0(X, L)^{\otimes d-1} \longrightarrow H^0(X, L^{d-1})$$

is surjective, then every section of L^{d-1} can be written as a product of sections of L. Hence every section of L^d can be written as a product of sections of L, since the map

$$H^0(X, L^{d-1}) \otimes H^0(X, L) \longrightarrow H^0(X, L^d)$$

is surjective by Theorem 2.3.2. Hence by induction it is true that every section of L^d can be written as a product of sections of L (and hence $H^0(X, L)^{\otimes d} \longrightarrow H^0(X, L^d)$ is surjective) since the map

$$H^0(X, L)^{\otimes 1} \longrightarrow H^0(X, L^1)$$

is clearly surjective.

□

Proposition 2.3.4. *For any nef line bundles $L_1, \dots, L_k \in$ Pic(X), if the sublattice of* Pic(X) *generated by $L_1, \dots, L_k$ contains an ample bundle then there exist $\beta_1, \dots, \beta_k \in \mathbb{N}$ such that $L := L_1^{\beta_1} \otimes \cdots \otimes L_k^{\beta_k}$ is $\mathcal{O}_X$-regular with respect to $L_1, \dots, L_k$.*

Proof. Since the sublattice of $\mathrm{Pic}(X)$ generated by $L_1, \dots, L_k$ contains an ample line bundle, we can pick $\alpha_1, \dots, \alpha_k \in \mathbb{N}$ such that $L_1^{\alpha_1} \otimes \cdots \otimes L_k^{\alpha_k}$ is ample. Suppose X is n dimensional, then $L(\mathbf{u}) := (L_1^{\alpha_1+n} \otimes \cdots \otimes L_k^{\alpha_k+n}) \otimes (L_1^{-u_1} \otimes \cdots \otimes L_k^{-u_k})$ is also ample for any $\mathbf{u} = (u_1, \dots, u_k)$ with $u_1 + \cdots + u_k \leq n$. Therefore by Demazure Vanishing (see e.g. Thm 9.2.3 Cox–Little–Schenck [6]) $H^i(X, L(\mathbf{u})) = 0$ for all $i > 0$ and all $\mathbf{u}$ such that $u_1 + \cdots + u_k = i$, since $H^i(X, L(\mathbf{u})) = 0$ for $i > n$. Hence, letting $\beta_i := \alpha_i + n$, we have the statement of the proposition. $\square$

2.4 Quivers and Quiver Representations

A *quiver* can be defined by giving its vertices, its arrows, and the vertices at the head and tail of each arrow. For a quiver Q, define Q_0 to be its set of vertices, Q_1 to be its set of arrows, and define maps

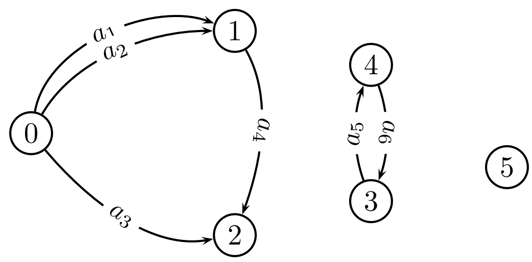
$$\mathsf{h}, \mathsf{t} : Q_1 \longrightarrow Q_0$$

mapping each arrow to its head and tail respectively. A *path* p is a sequence of arrows

$$p = a_n \dots a_1$$

such that $\mathsf{t}(a_i) = \mathsf{h}(a_{i-1})$. We define the support of p to be the set $\{a_1, \dots, a_n\}$. The path algebra $\Bbbk Q$ is defined to be the $\Bbbk$-algebra generated by all paths in Q, including trivial paths e_i for each $i \in Q_0$. The multiplication of two paths is defined to be their concatenation if it exists and zero otherwise. The maps h and t can be extended to $\Bbbk Q$ by defining $\mathsf{h}(p) = \mathsf{h}(a_n)$ and $\mathsf{t}(p) = \mathsf{t}(a_1)$. A *cycle* is a path where $\mathsf{h}(p) = \mathsf{t}(p)$, and Q is said to be *acyclic* if none of its nontrivial paths are cycles. A *walk* in Q is an sequence $v_0 a_0 v_1 a_1 \cdots a_k v_{k+1}$, where v_i's are vertices and a_i is an arrow between v_i and v_{i+1} (in either direction). We say a quiver is *connected* if there is a walk between any two vertices. We say a vertex is a *source* if it is not the head of any arrow. We say a quiver is *rooted* if exactly one vertex is a source. We say a subquiver of a rooted quiver is a *spanning tree* if it consists of precisely one path from the unique source to each vertex $i \in Q_0$.

Example 2.4.1. For the quiver below the maps h and t are:



$$\begin{aligned} \mathsf{t}(a_1) &= 0 & \mathsf{h}(a_1) &= 1 \\ \mathsf{t}(a_2) &= 0 & \mathsf{h}(a_2) &= 1 \\ \mathsf{t}(a_3) &= 0 & \mathsf{h}(a_3) &= 2 \\ \mathsf{t}(a_4) &= 1 & \mathsf{h}(a_4) &= 2 \\ \mathsf{t}(a_5) &= 3 & \mathsf{h}(a_5) &= 4 \\ \mathsf{t}(a_6) &= 4 & \mathsf{h}(a_6) &= 3 \end{aligned}$$

Let Q be a finite, connected quiver. A representation of Q consists of a $\Bbbk$-vector space W_i for $i \in Q_0$ and a $\Bbbk$-linear map $w_a \colon W_{\mathsf{t}(a)} \to W_{\mathsf{h}(a)}$ for $a \in Q_1$. It is convenient to write W as shorthand for $\big((W_i)_{i \in Q_0}, (w_a)_{a \in Q_1}\big)$. The dimension vector of W is the vector $\underline{r} \in \mathbb{Z}^{Q_0}$ with components $r_i = \dim_{\Bbbk}(W_i)$ for $i \in Q_0$. We define a *subrepresentation* of W to be a representation W' where W'_i is a vector subspace of W_i and where $w'_a := w_a$ gives a well-defined map $w'_a : W'_{\mathsf{t}(a)} \longrightarrow W'_{\mathsf{h}(a)}$. A map of representations $\psi \colon W \to W'$ is a family $\psi_i \colon W_i \to W'_i$ of $\Bbbk$-linear maps for $i \in Q_0$ satisfying $w'_a \psi_{\mathsf{t}(a)} = \psi_{\mathsf{h}(a)} w_a$ for $a \in Q_1$. With composition defined componentwise, we obtain the abelian category of finite dimensional representations of Q. For $\theta \in \mathbb{Z}^{Q_0}$, define $\theta(W) := \sum_{0 \leq i \leq \rho} \theta_i \dim_{\Bbbk}(W_i)$. Following King [20], a representation W of Q is θ-semistable if $\theta(W) = 0$ and every subrepresentation $W' \subset W$ satisfies $\theta(W') \geq 0$. Moreover, W is θ-stable if the only subrepresentations W' with $\theta(W') = 0$ are 0 and W.

2.5 Toric Varieties as Fine Moduli of Quiver Representations

In this section we summarise the findings of Craw–Smith in [10]. This paper investigates the link between the existence of an interpretation of a projective toric variety as a fine moduli space of quiver representations and the existence of a strong exceptional collection of line bundles.

We summarise the main results. Let X be a projective toric variety with $\mathrm{Cox}(X) \cong \Bbbk[x_1, \ldots, x_d]$. Given a list of line bundles $\mathscr{L} = (\mathcal{O}_X, L_1, \ldots, L_r)$ on X, Craw-Smith defined the *quiver of sections* for $\mathscr{L}$. They defined $|\mathscr{L}|$ to be the fine moduli space of representations of this quiver. This is a generalisation of the linear series for a single line bundle, so they refer to $|\mathscr{L}|$ as the multilinear series (or multigraded linear series) for $\mathscr{L}$. They showed that there exists a natural map $\varphi_{|\mathscr{L}|} : X \longrightarrow |\mathscr{L}|$, and that this map is a morphism if and only if $L_1, \ldots, L_r$ are basepoint free. If this is the case then the image of X is a GIT quotient. Then, whereas strong exceptional collections are comparatively rare, they showed that it is almost always possible to pick line bundles $\mathscr{L}$ such that $\varphi_{|\mathscr{L}|}$ is a closed embedding, and such that its image is the fine moduli space of bound quiver representations of the complete quiver of sections for $\mathscr{L}$. We will assume throughout this section that X is a projective toric variety.

2.5.1 Multilinear Series

Let $\mathscr{L} = (\mathcal{O}_X, L_1, \dots, L_r)$ be a list of distinct effective line bundles on the projective toric variety X. A torus invariant section $s \in H^0(X, L_j \otimes L_i^{-1}) = \text{Hom}(L_i, L_j)$ is said to be *irreducible* if it does not factor through some L_k with $k \neq i, j$.

Definition 2.5.1. (i) The *quiver of sections* for $\mathscr{L}$ is defined to be the quiver whose vertices are in one to one correspondence with bundles in $\mathscr{L}$. So if Q is the quiver of sections for $\mathscr{L}$ then $Q_0 = \{0, \dots, r\}$. We define the arrows from vertex i to vertex j to be in one to one correspondence with irreducible torus-invariant sections of $L_j \otimes L_i^{-1}$. We can think of Q as being a labelled quiver, where each arrow is labelled by the section it corresponds to. For a path p in Q, we will say that the label of p is the product of the labels of the arrows in the support of p. It is possible to assume that the elements in $\mathscr{L}$ are ordered such that if $j < i$ then $L_j \otimes L_i^{-1}$ is not effective.

(ii) Define

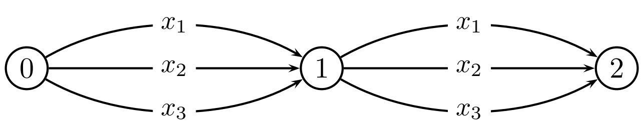
$$\text{div} : Q_1 \longrightarrow \mathbb{Z}^d$$

to be the map which sends an arrow a to the *divisor of zeros* of the torus-invariant section labelling a. Explicitly if the torus invariant section of a is $x_1^{m_1} \cdots x_d^{m_d} \in \text{Cox}(X)$ then $\text{div}(a) = (m_1, \dots, m_d) \in \mathbb{N}^d$.

Lemma 2.5.2. *The quiver of sections Q is connected, acyclic, and $0 \in Q_0$ is the unique source.*

Proof. Projectivity of $\widetilde{X}$ ensures that at most one of $\text{Hom}(E_i, E_j)$ and $\text{Hom}(E_j, E_i)$ is nonzero for $i \neq j$, so Q is acyclic since there cannot be paths from i to j and from j to i . For $i \in Q_0$, the space $\text{Hom}(E_0, E_i)$ has a torus-invariant element since $E_1, \dots, E_r$ are effective and $E_0 \cong \mathcal{O}_{\widetilde{X}}$, giving rise to a path in Q from 0 to $i \in Q_0$ so 0 is the unique source. □

Example 2.5.3. Let $X = \mathbb{P}^2$ and let $\mathscr{L} = (\mathcal{O}_X, \mathcal{O}(1), \mathcal{O}(2))$. The quiver of sections for $\mathscr{L}$ is:



The map div can be extended to the path algebra. If a path p has support $\{a_1, \dots, a_n\}$, define

$$\mathrm{div}(p) := \mathrm{div}(a_1) + \cdots + \mathrm{div}(a_n).$$

Define the ideal of relations R to be a two sided ideal in the path algebra $\Bbbk Q$ generated by all differences $p - p'$ where $\mathsf{t}(p) = \mathsf{t}(p'), \mathsf{h}(p) = \mathsf{h}(p')$ and $\mathrm{div}(p) = \mathrm{div}(p')$. The pair (Q, R) is called a *bound quiver of sections*, or a quiver of sections with relations.

Proposition 2.5.4. *(Proposition 3.3, [10]) If (Q, R) is the complete bound quiver of sections for $\mathscr{L} = (\mathcal{O}_X, L_1, \dots, L_r)$, then the quotient algebra $\Bbbk Q/R$ is isomorphic to* $\mathrm{End}(\bigoplus_{i=0}^{r} L_i)$.

Definition 2.5.5. Let Q be a connected, rooted, acyclic quiver (e.g. the quiver of sections for a collection of line bundles).

(i) We define $\mathrm{Wt}(Q) \subset \mathbb{Z}^{Q_0}$ to be the sublattice of functions $\theta \colon Q_0 \to \mathbb{Z}$ satisfying $\sum_{i \in Q_0} \theta_i = 0$. The vectors $\{\mathbf{e}_i - \mathbf{e}_0 : i \neq 0\}$ form a $\mathbb{Z}$-basis for $\mathrm{Wt}(Q)$.

(ii) We define the incidence map $\mathrm{inc} \colon \mathbb{Z}^{Q_1} \to \mathbb{Z}^{Q_0}$ by setting $\mathrm{inc}(\mathbf{e}_a) = \mathbf{e}_{\mathsf{h}(a)} - \mathbf{e}_{\mathsf{t}(a)}$. The image of inc is $\mathrm{Wt}(Q)$.

(iii) We introduce a $\Bbbk$-algebra, $\Bbbk[y_a | a \in Q_1]$. For a path p in Q we define $y_p := \prod_{a \in \mathrm{Supp}(p)} y_a$. For a spanning tree T in Q we define $y_T := \prod_{a \in \mathrm{Supp}(T)} y_a$.

(iv) We define the map $\mathrm{pic} : \mathrm{Wt}(Q) \longrightarrow \mathrm{Cl}(X)$ by setting $\mathrm{pic}(\mathbf{e}_i - \mathbf{e}_0) = L_i$.

The $\Bbbk$-algebra $\Bbbk[y_a : a \in Q_1]$ has a $\mathrm{Wt}(Q)$-grading. We define the weight of y_a to be $\mathrm{inc}(\mathbf{e}_a)$ for each $a \in Q_1$. This grading induces a faithful action of the algebraic torus $G := \mathrm{Hom}(\mathrm{Wt}(Q), \Bbbk^\times)$ on $\mathbb{A}_\Bbbk^{Q_1} = \mathrm{Spec}\, \Bbbk[y_a : a \in Q_1]$. An element $g = (g_i)_{i \in Q_0} \in (\Bbbk^*)^{r+1}$ acts on $w = (w_a)_{a \in Q_1}$ as $g \cdot w = (g_{\mathsf{h}(a)} w_a g^{-1}_{\mathsf{t}(a)})_{a \in Q_1}$. For $\theta \in \mathrm{Wt}(Q)$, let $\Bbbk[y_a : a \in Q_1]_\theta$ denote the θ-graded piece. We have

$$\mathbb{A}_\Bbbk^{Q_1} /\!\!/_\theta G = \mathrm{Proj}\Big(\bigoplus_{j \geq 0} \Bbbk[y_a : a \in Q_1]_{j\theta}\Big).$$

Let Q be a quiver of sections, and note in particular that Q is acyclic with a unique source $0 \in Q_0$. The *toric quiver flag variety* $|\mathscr{L}|$ is the GIT quotient $\mathbb{A}_\Bbbk^{Q_1} /\!\!/_\vartheta G$ linearised by the special weight $\vartheta := \sum_{i \in Q_0} (\mathbf{e}_i - \mathbf{e}_0) \in \mathrm{Wt}(Q)$. Such varieties, studied initially by Craw–Smith [10] and in greater generality by Craw [8], can be characterised as follows:

Proposition 2.5.6. *(Proposition 3.8, [10]) Let Q be a finite, connected, acyclic quiver with a unique source $0 \in Q_0$ and special weight $\vartheta = \sum_{i \in Q_0} (\mathbf{e}_i - \mathbf{e}_0)$. The toric quiver flag variety $|\mathscr{L}|$ coincides with:*

(i) *the GIT quotient $\mathbb{A}_\Bbbk^{Q_1} /\!\!/_\vartheta G$ linearised by $\vartheta \in \mathrm{Wt}(Q)$;*

(ii) *the geometric quotient of* $\mathbb{A}^{Q_1}_{\Bbbk} \setminus \mathbb{V}(B_Y)$ *by the action of* G*, where the irrelevant ideal is*

$$B_{|\mathscr{L}|} := \left(\prod_{a\in\mathcal{T}} y_a : \mathcal{T} \text{ is a spanning tree of } Q \text{ rooted at } 0\right) = \bigcap_{i\in Q_0\setminus\{0\}} \big(y_a : \mathsf{h}(a) = i\big)\,;$$

(iii) *the fine moduli space* $\mathcal{M}_\vartheta(Q)$ *of* ϑ*-stable representations of the quiver* Q *of dimension vector* $\underline{r} = (1,\ldots,1) \in \mathbb{Z}^{Q_0}$.

Moreover, $|\mathscr{L}|$ *is a smooth projective toric variety obtained as a tower of projective space bundles over* $\operatorname{Spec}(\Bbbk)$

Definition 2.5.7. We say a scheme $\mathcal{M}$ is a *fine moduli space* for some class of objects if there is a one–to–one correspondence between families of those objects over any scheme S and morphisms from S to $\mathcal{M}$. A *tautological family* over $\mathcal{M}$ is a family $\mathscr{T}$ over $\mathcal{M}$ for which any family of objects over S is a pullback of $\mathscr{T}$ under a unique map $\varphi : S \longrightarrow \mathcal{M}$.

Remark 2.5.8. The description of $|\mathscr{L}| = \mathcal{M}_\vartheta(Q)$ as a fine moduli space of representations ensures that it carries a tautological vector bundle $\bigoplus_{i\in Q_0} \mathscr{W}_i$ with $\mathscr{W}_0 \cong \mathscr{O}_{|\mathscr{L}|}$ and sheaf homomorphisms $\{\mathscr{W}_{\mathsf{t}(a)} \to \mathscr{W}_{\mathsf{h}(a)} : a \in Q_1\}$ whose restriction to the fibre over $\mathcal{M}_\vartheta(Q)$ encodes the corresponding representation $\{W_{\mathsf{t}(a)} \to W_{\mathsf{h}(a)} : a \in Q_1\}$. Moreover, the abelian group homomorphism $\mathrm{Wt}(Q) \to \operatorname{Pic}(|\mathscr{L}|)$ sending $(\theta_0,\ldots,\theta_r)$ to $\mathscr{W}_1^{\theta_1}\otimes\cdots\otimes\mathscr{W}_r^{\theta_r}$ is an isomorphism. For more details, see [8, Sections 2-3].

2.5.2 Bound quiver representations

Let Q be a quiver. For any representation W of Q, define $w_p \colon W_{\mathsf{t}(p)} \to W_{\mathsf{h}(p)}$ to be the $\Bbbk$-linear map $w_p = w_{a_k}\cdots w_{a_1}$ obtained by composition. Let $J \subset \Bbbk Q$ be a two-sided ideal of relations with generators of the form $\sum_{p\in\Gamma} c_p p$, where each Γ is a finite set of paths that share the same head and the same tail. A representation W of Q is a representation of the bound quiver (Q, J) if and only if $\sum_{p\in\Gamma} c_p w_p = 0$ for each Γ arising in the definition of J. A point in representation space $(w_a) \in \mathbb{A}^{Q_1}_{\Bbbk}$ defines a representation of (Q, J) if and only it lies in the subscheme $\mathbb{V}(I_J)$ cut out by the ideal

$$I_J := \left(\sum_{p\in\Gamma} c_p y_p \in \Bbbk[y_a : a \in Q_1] \mid \sum_{p\in\Gamma} c_p p \text{ is a generator of } J\right)$$

of relations in $\Bbbk[y_a : a \in Q_1]$. The ideal I_J is $\mathrm{Wt}(Q)$-homogeneous, since J is generated by sums $\sum_{p\in\Gamma} c_p p$ where the p's have the same heads and tails. Hence $\mathbb{V}(I_J)$ is G-invariant and the GIT quotient

$$\mathcal{M}_\vartheta(Q, J) := \mathbb{V}(I_J)/\!\!/_\vartheta G = \operatorname{Proj}\left(\bigoplus_{j\geq 0} \left(\Bbbk[y_a : a \in Q_1]/I_J\right)_{j\vartheta}\right) \tag{2.5.1}$$

is the fine moduli space of ϑ-stable representations of (Q, J) with dimension vector $(1,\ldots,1)$. The tautological bundles on $\mathcal{M}_\vartheta(Q, J)$ are obtained from those on $\mathcal{M}_\vartheta(Q)$ by restriction.

Remark 2.5.9. The abelian category of finite-dimensional representations of (Q, J) is equivalent to the category of finitely-generated $\Bbbk Q/J$-modules, so $\mathcal{M}_\vartheta(Q, J)$ is equivalently the fine moduli space of ϑ-stable modules over $\Bbbk Q/J$ that are isomorphic as $\big(\bigoplus_{i\in Q_0} \Bbbk e_i\big)$-modules to $\bigoplus_{i\in Q_0} \Bbbk e_i$.

2.5.3 Morphism to the Multigraded Linear Series

Consider the $\Bbbk$-algebra homomorphism $\widetilde{\Phi}\colon \Bbbk[y_a : a \in Q_1] \to \Bbbk[x_1, \dots, x_d]$ sending y_a to $x^{\mathrm{div}(a)}$ for $a \in Q_1$. This induces a map $\mathbb{A}^d \longrightarrow \mathbb{A}^{Q_1}$ which descends to a rational map $\varphi_{|\mathscr{L}|} : X \longrightarrow |\mathscr{L}|$.

Proposition 2.5.10. *(Proposition 4.1 [10]) The rational map $\varphi_{|\mathscr{L}|}$ is a morphism if and only if the preimage of the unstable locus $\mathbb{V}(B_{|\mathscr{L}|})$ in $\mathbb{A}^{Q_1}$ is contained in the unstable locus $\mathbb{V}(B_X)$.*

Proof. The actions of the groups $G = \mathrm{Hom}(\mathrm{Wt}(Q), \Bbbk^*)$ and $T = \mathrm{Hom}(\mathrm{Cl}(\widetilde{X}), \Bbbk^*)$ on $\Bbbk[y_a : a \in Q_1]$ and $\Bbbk[x_1, \dots, x_d]$ respectively arise from the horizontal semigroup homomorphisms in the diagram

$$\begin{array}{ccc} \mathbb{N}^{Q_1} & \xrightarrow{\text{inc}} & \mathrm{Wt}(Q) \\ {\scriptstyle\mathrm{div}}\downarrow & & \downarrow{\scriptstyle\mathrm{pic}} \\ \mathbb{N}^d & \xrightarrow{\deg} & \mathrm{Cl}(\widetilde{X}) \end{array} \tag{2.5.2}$$

where the vertical maps satisfy $\mathrm{div}(\mathbf{e}_a) = \mathrm{div}(a)$ for $a \in Q_1$ and $\mathrm{pic}(\mathbf{e}_i) = L_i$ for $i \in Q_0$. We recall that deg is the map giving the $\mathrm{Cl}(X)$ grading of $\mathrm{Cox}(X)$ and that pic is defined in Definition 2.5.5. The map $\widetilde{\Phi}$ respects gradings precisely because (2.5.2) commutes. We explain why in more depth as follows. The map inc sends $\mathbf{e}_a$ to $\mathbf{e}_{\mathsf{h}(a)} - \mathbf{e}_{\mathsf{t}(a)} \in \mathrm{Wt}(Q)$. If $\mathsf{h}(a) = i$ and $\mathsf{t}(a) = j$ then pic maps $\mathbf{e}_{\mathsf{h}(a)} - \mathbf{e}_{\mathsf{t}(a)}$ to $L_j \otimes L_i^{-1}$. Furthermore, div maps $\mathbf{e}_a$ to $\mathrm{div}(a)$, which is mapped by deg to $L_j \otimes L_i^{-1}$. This holds since the label of an arrow from i to j is a section of $L_j \otimes L_i^{-1}$.

Since the map $\widetilde{\Phi}$ respects gradings, the induced map of affine spaces $\mathbb{A}^d \longrightarrow \mathbb{A}^{Q_1}$ maps orbits to orbits. Hence the rational map $\varphi_{|\mathscr{L}|}$ is a morphism if and only if every semistable point in $\mathbb{A}^d$ maps to a semistable point in $\mathbb{A}^{Q_1}$. This holds if and only if the preimage of the unstable locus $\mathbb{V}(B_{|\mathscr{L}|})$ in $\mathbb{A}^{Q_1}$ is contained in the unstable locus $\mathbb{V}(B_X)$. ☐

Theorem 2.5.11. *(Cor 4.2, [10]) We obtain a morphism $\varphi_{|\mathscr{L}|} : X \longrightarrow |\mathscr{L}|$ if and only if each line bundle in the list $\mathscr{L}$ is basepoint free.*

If each line bundle in $\mathscr{L}$ is basepoint free, then we say the quiver of sections for $\mathscr{L}$ is a *basepoint free quiver of sections.* If this is the case, by Proposition 4.3 of [10], the image of X is given as a GIT quotient:

$$\varphi(X) = \mathbb{V}(I_Q) /\!\!/_\vartheta G$$

where I_Q is the prime ideal

$$I_Q = \big(f \in \Bbbk[y_a | a \in Q_1] | f \text{ is homogeneous and } f \in \ker(\Phi)\big).$$

Remark 2.5.12. The ideal I_Q is also the kernel of the semigroup homomorphism

$$\mathrm{inc} \oplus \mathrm{div} : \mathbb{Z}^{Q_1} \longrightarrow \mathrm{Wt}(Q) \bigoplus \mathbb{N}^d.$$

Craw–Smith gave necessary and sufficient conditions for the morphism φ to be a closed embedding.

Proposition 2.5.13 (Proposition 4.9, [10])**.** *Let Q be a basepoint free quiver of sections, and let $\vartheta = \sum_{i \in Q_0} (\mathbf{e}_i - \mathbf{e}_0)$. The map $\varphi : X \longrightarrow |\mathscr{L}|$ is a closed embedding if and only if the line bundle $L := L_0^{\vartheta_0} \otimes \cdots \otimes L_r^{\vartheta_r}$ is ample and $((\mathrm{Cox}(|\mathscr{L}|)/I_Q)[y^{-\hat{\sigma}}])_{[0]} \cong ((\mathrm{Cox}(X)[x^{-\hat{\sigma}}])_{[0]}$ for all top dimensional cones σ in the fan defining X.*

We say a quiver of sections Q is *very ample* if it is basepoint free and $\varphi_Q : X \longrightarrow |\mathscr{L}|$ is a closed embedding.

Corollary 2.5.14. *(Cor 4.10 [10]) Let $\mathscr{L}$ be a list of basepoint free line bundles and define $L := \bigotimes_{i \in Q_0} L_i$. Assume that the multiplication map $H^0(X, L_1) \otimes \cdots \otimes H^0(X, L_r) \longrightarrow H^0(X, L)$ is surjective. Then $\varphi_Q : X \longrightarrow |\mathscr{L}|$ is a closed embedding if and only if L is very ample.*

2.5.4 Projective Toric Varieties as Fine Moduli

Recall the ideal of relations R in $\Bbbk Q$ is generated by all differences of paths $p - p'$ where $\mathsf{h}(p) = \mathsf{h}(p'), \mathsf{t}(p) = \mathsf{t}(p')$ and $\mathrm{div}(p) = \mathrm{div}(p')$. A representation of the bound quiver (Q, R) is a representation $W = (W_i, w_a)$ of Q where $w_p - w_{p'} = 0$ whenever $p - p' \in R$. The fine moduli space of representations of (Q, R), $\mathcal{M}_\vartheta(Q, R)$, is the GIT quotient of $\mathbb{V}(I_R)$ under the action of G, where

$$I_R = \big(y_p - y_{p'} \,|\, \mathsf{h}(p) = \mathsf{h}(p'), \mathsf{t}(p) = \mathsf{t}(p') \text{ and } \mathrm{div}(p) = \mathrm{div}(p')\big).$$

The ideal I_R is homogeneous with respect to the $\mathrm{Wt}(Q)$ grading, and hence $\mathbb{V}(I_R)$ is a G-invariant subset of $\mathbb{A}^{Q_1}$.

If Q is a very ample quiver of sections, then $\mathcal{M}_\vartheta(Q, R) \cong X$ if and only if $\overline{\mathbb{V}(I_Q) \setminus \mathbb{V}(B_{|\mathscr{L}|})} = \overline{\mathbb{V}(I_R) \setminus \mathbb{V}(B_{|\mathscr{L}|})}$. Furthermore

$$X \cong \mathcal{M}_\vartheta(Q, R) \text{ if } I_Q = \big(I_R : B_{|\mathscr{L}|}^\infty\big),$$

where for ideals I and J in a ring R,

$$\big(I : J^\infty\big) := \{f \in R | \text{ for every } j \in J \text{ there exists } n \in \mathbb{N} \text{ such that} f \cdot j^n \in I\}.$$

If this is the case, then we say that Q is *fine*. The next theorem gives conditions that guarantee that we can find a list of line bundles $\mathscr{L}$ such that the complete quiver of sections for $\mathscr{L}$ is fine.

Theorem 2.5.15. *(Theorem 5.5, [10]) Let $L_1, \ldots, L_{r-2}$ be basepoint free line bundles on X. If the subsemigroup of $\mathrm{Pic}(X)$ generated by $L_1, \ldots, L_{r-2}$ contains an ample line bundle, then there exist line bundles L_{r-1} and L_r such that the quiver of sections of $\mathscr{L} := \{\mathcal{O}_X, L_1, \ldots, L_r\}$ is fine.*

Chapter 3

Geometric Results

3.1 Quivers of Sections on Mori Dream Spaces

In this section we introduce the bound quiver of sections for a collection of line bundles on a Mori Dream Space. These bound quivers encode the endomorphism algebra of the direct sum of the sheaves in the collection. For $r \geq 0$, consider a collection of distinct line bundles

$$\mathscr{L} := (L_0, L_1, \ldots, L_r) \subset \mathrm{Cl}(X)$$

on the Mori Dream Space X, where $L_0 = \mathscr{O}_X$ and $L_1, \ldots, L_r$ are effective. For $0 \leq i \leq r$, define $E_i := \psi^{-1}(L_i)$ where ψ is the isomorphism from $\mathrm{Cl}(\widetilde{X})$ to $\mathrm{Cl}(X)$ to obtain a collection

$$\widetilde{\mathscr{L}} := (E_0, E_1, \ldots, E_r)$$

of distinct rank one reflexive sheaves on an ambient toric variety $\widetilde{X}$. For $0 \leq i, j \leq r$, we say that a torus–invariant section $s \in H^0(X, E_j \otimes E_i^{-1}) = \mathrm{Hom}(E_i, E_j)$ is *irreducible* if it does not factor through some E_k with $k \neq i, j$. The following definition extends the notion of a quiver of sections for a collection of line bundles on a projective toric variety due to Craw-Smith [10] introduced in Section 2.5.

Definition 3.1.1. The *quiver of sections* of the collection $\mathscr{L}$ on X is defined to be the quiver of sections of the collection $\widetilde{\mathscr{L}}$ on $\widetilde{X}$, that is, the quiver Q with vertex set $Q_0 = \{0, \ldots, r\}$, and where the arrows from i to j correspond to the irreducible torus–invariant sections of $E_j \otimes E_i^{-1}$.

Remark 3.1.2.

1. Definition 3.1.1 depends a priori on the choice of ambient toric variety $\widetilde{X}$. However, any two are isomorphic in codimension–one, so they have isomorphic class groups and their fans have the same rays. This implies that the Cox sequence (2.1.1) is the same for any choice of ambient toric variety, and hence Q is independent of the choice.

2. We abuse terminology by calling Q the 'quiver of sections of $\mathscr{L}$' because paths in Q from i to j are not constructed directly from a basis of $\mathrm{Hom}(L_i, L_j)$ as in the literature, see [8, 10]. We justify this abuse by recovering the Hom spaces in Proposition 3.1.4 below.

Definition 3.1.3. Consider the two-sided ideal

$$R := \left(\sum_i c_i p_i \in \Bbbk Q \mid \begin{array}{c} \mathsf{h}(p_i) = \mathsf{h}(p_j), \mathsf{t}(p_i) = \mathsf{t}(p_j) \text{ for all } i, j \\ \text{and } \sum_i c_i x^{\mathrm{div}(p_i)} \in I_X \end{array} \right)$$

in the path algebra $\Bbbk Q$. The pair (Q, R) is the *bound quiver of sections* of the collection $\mathscr{L}$.

Proposition 3.1.4. *The quotient algebra $\Bbbk Q/R$ is isomorphic to $\mathrm{End}_{\mathcal{O}_X}\big(\bigoplus_{i\in Q_0} L_i\big)$, and each vertex $i \in Q_0$ satisfies $e_i(\Bbbk Q/R)e_0 \cong H^0(X, L_i)$ where e_i is the trivial path at vertex i.*

Proof. The endomorphism algebra $\mathrm{End}_{\mathcal{O}_{\widetilde{X}}}\big(\bigoplus_{i\in Q_0} E_i\big)$ is constructed as a direct sum of $\Bbbk$–vector spaces

$$\mathrm{End}_{\mathcal{O}_{\widetilde{X}}}\big(\bigoplus_{i\in Q_0} E_i\big) = \bigoplus_{i,j\in Q_0} H^0(\widetilde{X}, E_j \otimes E_i^{-1}).$$

A basis for each direct summand $H^0(\widetilde{X}, E_j \otimes E_i^{-1})$ is given by torus invariant sections. Multiplication of two sections $x_1 \in H^0(\widetilde{X}, E_{j_1} \otimes E_{i_1}^{-1})$ and $x_2 \in H^0(\widetilde{X}, E_{j_2} \otimes E_{i_2}^{-1})$ is defined to be the product $x_1x_2 \in H^0(\widetilde{X}, E_{j_2} \otimes E_{i_1}^{-1})$ if $i_2 = j_1$ and zero otherwise.

For each $i, j \in Q_0$, there exists a map of $\Bbbk$-vector spaces from the vector subspace of $\Bbbk Q$ spanned by paths from i to j to $H^0(\widetilde{X}, E_j \otimes E_i^{-1})$ which maps a path to its label. This induces a map of $\Bbbk$-vector spaces

$$\widetilde{\nu} : \Bbbk Q \longrightarrow \mathrm{End}_{\mathcal{O}_{\widetilde{X}}}\big(\bigoplus_{i\in Q_0} E_i\big)$$

defined to be the direct sum of the maps described above. The map $\widetilde{\nu}$ is also a $\Bbbk$-algebra homomorphism since the product of a pair of paths is defined to be their concatenation if it exists and zero otherwise, and their concatenation is labelled by the product of the labels of each path.

The endomorphism algebra $\mathrm{End}_{\mathcal{O}_X}\big(\bigoplus_{i\in Q_0} L_i\big)$ is also given as a direct sum of $\Bbbk$–vector spaces

$$\mathrm{End}_{\mathcal{O}_X}\big(\bigoplus_{i\in Q_0} L_i\big) = \bigoplus_{i,j\in Q_0} H^0(X, L_j \otimes L_i^{-1}).$$

By picking a basis for each space of sections $H^0(X, L_j \otimes L_i)$, we can define multiplication in the endomorphism algebra as in the toric case. The natural map $\tau : \mathrm{Cox}(\widetilde{X}) \longrightarrow \mathrm{Cox}(X)$ induces maps

$$\tau_{ij} : H^0(\widetilde{X}, E_j \otimes E_i^{-1}) \longrightarrow H^0(X, L_j \otimes L_i^{-1}).$$

The direct sum of these maps over $i, j \in Q_0$ gives a $\Bbbk$-algebra homomorphism

$$\hat{\tau} : \mathrm{End}_{\mathcal{O}_{\widetilde{X}}}\big(\bigoplus_{i\in Q_0} E_i\big) \longrightarrow \mathrm{End}_{\mathcal{O}_X}\big(\bigoplus_{i\in Q_0} L_i\big).$$

The kernel of $\hat{\tau}$ is the direct sum of the kernels of the τ_{ij}'s.

For each $i, j \in Q_0$, there also exists a map ν of $\Bbbk$-vector spaces from the vector subspace of $\Bbbk Q$ spanned by paths from i to j to $H^0(X, L_j \otimes L_i^{-1})$ mapping a path to its label modulo I_X. This induces a map of $\Bbbk$-vector spaces

$$\nu : \Bbbk Q \longrightarrow \operatorname{End}_{\mathcal{O}_X}\big(\bigoplus_{i \in Q_0} L_i\big)$$

defined to be the direct sum of the maps described above. The map ν is also a $\Bbbk$-algebra homomorphism for the same reason as above.

These maps fit into a commutative diagram:

$$\begin{array}{ccc} \Bbbk Q & \xrightarrow{\widetilde{\nu}} & \operatorname{End}_{\mathcal{O}_{\widetilde{X}}}\big(\bigoplus_{i \in Q_0} E_i\big) \\ \Big\| & & \Big\downarrow \hat{\tau} \\ \Bbbk Q & \xrightarrow{\nu} & \operatorname{End}_{\mathcal{O}_X}\big(\bigoplus_{i \in Q_0} L_i\big) \end{array} \tag{3.1.1}$$

The map $\widetilde{\nu}$ is surjective, since for each $i, j \in Q_0$ there exist paths from i to j labelled by every section of $H^0(\widetilde{X}, E_j \otimes E_i^{-1})$. The map $\hat{\tau}$ is surjective since each τ_{ij} is. Finally, ν is surjective since the diagram commutes. Therefore by the first isomorphism theorem $\operatorname{End}_{\mathcal{O}_X}\big(\bigoplus_{i \in Q_0} L_i\big) \cong \Bbbk Q / \ker(\nu)$. This is the preimage under $\widetilde{\nu}$ of the kernel of $\hat{\tau}$, i.e. the ideal generated by linear combinations of paths $\sum a_i p_i$ such that the paths p_i have the same heads and tails and such that $\sum a_i x^{\operatorname{div}(p_i)} \in I_X$.

The second statement follows from the first since we have $L_0 = \mathcal{O}_X$ and we compose arrows and maps from right to left. □

3.2 Multilinear Series

In this section we use the quiver of sections of a collection $\mathscr{L}$ of line bundles on a Mori Dream Space X to define the corresponding multilinear series $|\mathscr{L}|$. This variety generalises the classical linear series of a single line bundle in that one obtains a natural map from X to $|\mathscr{L}|$ by evaluating sections of line bundles. We give necessary and sufficient conditions for this map to be a morphism and to be a closed embedding. In the case that the map is a morphism we describe its image as a GIT quotient.

Let $\mathscr{L} = (\mathcal{O}_X, L_1, \dots, L_r)$ be a collection of effective line bundles on a Mori Dream Space X. Lemma 2.5.2 guarantees that the corresponding quiver of sections Q is finite, connected, acyclic and has a unique source $0 \in Q_0$.

Definition 3.2.1. The *multilinear series* for $\mathscr{L}$ is the toric quiver flag variety $|\mathscr{L}|$ of Q from Proposition 2.5.6.

Remark 3.2.2. Just as Q is not precisely the quiver of sections of $\mathscr{L}$ (see Remark 3.1.2), it is perhaps an abuse of terminology to call $|\mathscr{L}|$ the multilinear series of $\mathscr{L}$. Indeed, for the special case $\mathscr{L} = (\mathcal{O}_X, L_1)$ we have that $|\mathscr{L}| \cong \mathbb{P}(H^0(E_1))$ is a projective space, but it need not coincide with the classical linear series $|L_1|$ because the epimorphism $\tau|_{H^0(\widetilde{X},E_i)} \colon H^0(\widetilde{X}, E_1) \to H^0(X, L_1)$ from diagram (2.1.5) need not be an isomorphism.

In order to study morphisms from X to the multigraded linear series $|\mathscr{L}|$, define

$$\widetilde{\Phi} \colon \Bbbk[y_a : a \in Q_1] \to \Bbbk[x_1, \dots, x_d]$$

to be the $\Bbbk$-algebra homomorphism sending y_a to a's label for $a \in Q_1$. We recall that the map inc and pic are defined in Definition 2.5.5, div is defined in Definition 2.5.1 and $\widetilde{\deg}$ is defined to be the map deg given by (2.1.1). The actions of the groups $G = \mathrm{Hom}(\mathrm{Wt}(Q), \Bbbk^*)$ and $T = \mathrm{Hom}(\mathrm{Cl}(X), \Bbbk^*)$ on $\Bbbk[y_a : a \in Q_1]$ and $\Bbbk[x_1, \dots, x_d]$ respectively arise from the horizontal semigroup homomorphisms in the diagram

$$\begin{array}{ccc} \mathbb{N}^{Q_1} & \xrightarrow{\text{inc}} & \mathrm{Wt}(Q) \\ {\scriptstyle \mathrm{div}}\downarrow & & \downarrow{\scriptstyle \mathrm{pic}} \\ \mathbb{N}^d & \xrightarrow{\widetilde{\deg}} & \mathrm{Cl}(\widetilde{X}) \end{array} \tag{3.2.1}$$

where the vertical maps satisfy $\mathrm{div}(\chi_a) = \mathrm{div}(a)$ for $a \in Q_1$ and $\mathrm{pic}(\chi_i) = E_i$ for $i \in Q_0$. The map $\widetilde{\Phi}$ is a graded ring homomorphism precisely because (3.2.1) commutes (see the proof of Proposition 2.5.10). Under the identification of $\mathrm{Wt}(Q)$ with the Picard group of $|\mathscr{L}|$, the subspace of the Cox ring $\Bbbk[y_a | a \in Q_1]$ of $|\mathscr{L}|$ spanned by monomials of weight $\theta \in \mathrm{Wt}(Q)$ coincides with $H^0(\mathscr{W}_1^{\theta_1} \otimes \cdots \otimes \mathscr{W}_r^{\theta_r})$.

Recall that τ is the canonical surjection $\mathrm{Cox}(\widetilde{X}) \longrightarrow \mathrm{Cox}(X)$. Since the T-action on $\mathrm{Cox}(X)$ is compatible with that on $\Bbbk[x_1, \dots, x_d]$, the map

$$\Phi := \tau \circ \widetilde{\Phi} \colon \Bbbk[y_a : a \in Q_1] \longrightarrow \mathrm{Cox}(X)$$

is a graded ring homomorphism. The induced equivariant morphism $\Phi^* \colon \mathbb{V}(I_X) \to \mathbb{A}_{\Bbbk}^{Q_1}$ descends to a rational map $\varphi_{|\mathscr{L}|} \colon X \dashrightarrow |\mathscr{L}|$.

Proposition 3.2.3. *Let $\mathscr{L} = (\mathcal{O}_X, L_1, \dots, L_r)$ be a collection of effective line bundles on X. The rational map $\varphi_{|\mathscr{L}|} \colon X \dashrightarrow |\mathscr{L}|$ is a morphism if and only if L_i is basepoint-free for $1 \leq i \leq r$.*

Proof. For $x \in X$ choose any lift $\widetilde{x} \in \mathbb{V}(I_X) \setminus \mathbb{V}(B_X)$. We will show that the rational map $\varphi_{|\mathscr{L}|}$ is well defined at $\widetilde{x}$ if and only if none of the line bundles in $\mathscr{L}$ has a basepoint at $\widetilde{x}$. The G-orbit of the quiver representation $\Phi^*(\widetilde{x}) \in \mathbb{A}_{\Bbbk}^{Q_1}$, which is independent of the choice of lift, is obtained by evaluating the labels on arrows at $\widetilde{x}$, that is, by evaluating sections of the bundles $L_{\mathsf{h}(a)} \otimes L_{\mathsf{t}(a)}^{-1}$

at x. The rational map $\varphi_{|\mathscr{L}|}\colon X \dashrightarrow |\mathscr{L}|$ is a morphism if and only if every such $\Phi^*(\widetilde{x}) \in \mathbb{A}_{\Bbbk}^{Q_1}$ is ϑ-stable.

Let $W' = \big((W'_i)_{i\in Q_0}, (w'_a)_{a\in Q_1}\big)$ be a proper subrepresentation of $\Phi^*(\widetilde{x})$. We recall that $\Phi^*(\widetilde{x})$ is ϑ stable if and only if $\sum_{i\in Q_0} \vartheta_i \dim(W'_i) < 0$. Since $\vartheta = (-r, 1, \ldots, 1)$ where $r = |Q_0| - 1$ and $\dim(W'_i)$ is either 0 or 1, this is the case if and only $\dim(W'_0) = 1$ and there exists $i > 0$ such that $\dim(W_i) = 0$. For each path p from 0 to i, the map w'_p is given by evaluating the label of p at $\widetilde{x}$. This means that it is possible for $\dim(W'_i)$ to be 0 if and only if the map $w'_p = 0$ for each path p from 0 to i, but this can happen if and only if L_i has a basepoint at $\widetilde{x}$. Hence, $\Phi^*(\widetilde{x}) \in \mathbb{A}_{\Bbbk}^{Q_1}$ is ϑ-unstable if and only if there exists $i > 0$ such that the evaluation of every section of L_i at x equals zero. Equivalently, $\Phi^*(\widetilde{x}) \in \mathbb{A}_{\Bbbk}^{Q_1}$ is ϑ-semistable if and only if L_i is basepoint-free for $1 \leq i \leq r$. □

The Cox ring of X is a unique factorisation domain by Theorem 2.1.10, so $\ker(\Phi)$ is prime and hence so is the ideal

$$I_Q := \Big(f \in \Bbbk[y_a : a \in Q_1] : f \in \ker(\Phi) \text{ is } \operatorname{Wt}(Q)\text{-homogeneous}\Big) \tag{3.2.2}$$

generated by its $\operatorname{Wt}(Q)$-homogeneous elements. This ideal can be computed explicitly as the kernel of the $\Bbbk$-algebra homomorphism

$$\Psi\colon \Bbbk[y_a : a \in Q_1] \to \Bbbk[x_1, \ldots, x_d, h_i, t_i | i \in Q_0]/\big(I_X + K\big) \tag{3.2.3}$$

satisfying $\Psi(y_a) = t_{\mathsf{t}(a)} x^{\operatorname{div}(a)} h_{\mathsf{h}(a)}$ for $a \in Q_1$ and where K is the ideal generated by $\{h_i t_i - 1 | i \in Q_0\}$; see Chapter 5 for details. This ideal cuts out the image of the morphism constructed in Proposition 3.2.3 as follows.

Proposition 3.2.4. *Let $\mathscr{L} = (\mathcal{O}_X, L_1, \ldots, L_r)$ be a collection of basepoint-free line bundles on X with quiver of sections Q. Then*

(i) *the image of the morphism $\varphi_{|\mathscr{L}|}\colon X \to |\mathscr{L}|$ is $\mathbb{V}(I_Q)/\!\!/_\vartheta G$; and*

(ii) *the tautological line bundles on $|\mathscr{L}|$ satisfy $\varphi^*_{|\mathscr{L}|}(\mathscr{W}_i) = L_i$ for $i \in Q_0$.*

Proof. Since X is complete, the image of $\varphi_{|\mathscr{L}|}$ is a closed subscheme of $|\mathscr{L}|$. The geometric quotient construction of $|\mathscr{L}|$ from Proposition 2.5.6(i) implies that the image is therefore the geometric quotient of a G-invariant closed subscheme of $\mathbb{A}_{\Bbbk}^{Q_1} \setminus \mathbb{V}(B_{|\mathscr{L}|})$. The affine variety $\mathbb{V}(\ker(\Phi))$ is the image of the equivariant morphism $\operatorname{Spec}(\operatorname{Cox}(X)) \to \mathbb{A}_{\Bbbk}^{Q_1}$ induced by Φ, and the variety $\mathbb{V}(I_Q)$ cut out by the $\operatorname{Wt}(Q)$-homogeneous part of $\ker(\Phi)$ is the minimal G-invariant algebraic set in $\mathbb{A}^{Q_1}$ containing all G-orbits from $\mathbb{V}(\ker(\Phi))$. The image of $\varphi_{|\mathscr{L}|}$ is therefore the geometric quotient of $\mathbb{V}(I_Q) \setminus \mathbb{V}(B_{|\mathscr{L}|})$ by the action of G. This coincides with the GIT quotient $\mathbb{V}(I_Q)/\!\!/_\vartheta G$ by Proposition 2.5.6, so (i) holds. For part (ii), the tautological bundle $\mathscr{W}_i$ on $|\mathscr{L}|$ corresponds to the weight $\chi_i - \chi_0 \in \operatorname{Wt}(Q)$ under the isomorphism from Remark 2.5.8. Since the equivariant morphism

$\mathrm{Spec}(\mathrm{Cox}(X)) \to \mathbb{A}^{Q_1}_{\Bbbk}$ factors through $\mathbb{A}^d_{\Bbbk}$, examining the diagrams (2.1.6) and (3.2.1) shows that $\varphi^*_{|\mathscr{L}|}(\mathscr{W}_i) = (\psi \circ \mathrm{pic})(\chi_i - \chi_0) = \psi(E_i) = L_i$ for $i \in Q_0$. □

We recall that Theorem 1.1 states that for a collection $\mathscr{L} = (\mathcal{O}_X, L_1, \dots, L_r)$ of effective line bundles on X, the map $\varphi_{|\mathscr{L}|} \colon X \dashrightarrow |\mathscr{L}|$ is a morphism if and only if each L_i is basepoint-free, in which case the image is presented explicitly as a geometric quotient and the tautological bundles satisfy $\varphi^*_{|\mathscr{L}|}(\mathscr{W}_i) = L_i$.

Proof of Theorem 1.1.1. Proposition 3.2.3 establishes that $\varphi_{|\mathscr{L}|} \colon X \dashrightarrow |\mathscr{L}|$ is a morphism if and only if L_i is basepoint-free for $1 \leq i \leq r$. Proposition 3.2.4 then presents the image explicitly as a geometric quotient, and establishes that the tautological line bundles on $|\mathscr{L}|$ satisfy $\varphi^*_{|\mathscr{L}|}(\mathscr{W}_i) = L_i$ for $i \in Q_0$ as required. □

Remark 3.2.5. The list of reflexive sheaves $\widetilde{\mathscr{L}}$ on $\widetilde{X}$ determines the ideal

$$\widetilde{I_Q} = \Big(f \in \Bbbk[y_a : a \in Q_1] : f \in \ker(\widetilde{\Phi}) \text{ is } \mathrm{Wt}(Q)\text{-homogeneous} \Big) \tag{3.2.4}$$

obtained as the toric ideal of the semigroup homomorphism $\mathrm{inc} \oplus \mathrm{div} \colon \mathbb{N}^{Q_1} \to \mathrm{Wt}(Q) \oplus \mathbb{N}^d$. If each reflexive sheaf in $\widetilde{\mathscr{L}}$ is a basepoint-free line bundle on $\widetilde{X}$, then Theorem 1.1 of [10] gives a morphism $\varphi_{|\widetilde{\mathscr{L}}|} \colon \widetilde{X} \to \mathbb{V}(I_{\widetilde{Q}}) /\!\!/_\vartheta G$ whose restriction to X is the morphism $\varphi_{|\mathscr{L}|} \colon X \to \mathbb{V}(I_Q) /\!\!/_\vartheta G$ from Proposition 3.2.4. However, this is typically not the case as Example 3.3.4 shows.

3.3 Criteria for Closed Immersion

A collection $\mathscr{L}$ is said to be *very ample* if the morphism $\varphi_{|\mathscr{L}|}$ from Proposition 3.2.3 is a closed immersion. We now introduce a necessary and sufficient condition for $\mathscr{L}$ to be very ample. We (enhance and) adapt the proofs of Proposition 5.7 of [8] and Corollary 4.10 of [10] to our situation because Q is not precisely the quiver of sections for $\mathscr{L}$ (see Remarks 3.1.2 and 3.2.2). We recall that a subspace of $H^0(X, L)$ is a very ample linear series if a basis gives a closed embedding of X into $\mathbb{P}^*(H^0(X, L)$.

Theorem 3.3.1. *Let $\mathscr{L} = (\mathcal{O}_X, L_1, \dots, L_r)$ be a collection of line bundles on X where we assume each L_i is basepoint free. The following are equivalent:*

(i) *the morphism $\varphi_{|\mathscr{L}|} \colon X \to |\mathscr{L}|$ is a closed immersion;*

(ii) *the image of the multiplication map*

$$H^0(L_1) \otimes \cdots \otimes H^0(L_r) \longrightarrow H^0(L_1 \otimes \cdots \otimes L_r). \tag{3.3.1}$$

is a very ample linear series;

(iii) *the map* $\prod_{1\le i\le r}\varphi_{|L_i|}\colon X \to |L_1|\times\cdots\times|L_r|$ *is a closed immersion.*

Proof. The bundle $\vartheta = \mathscr{W}_1\otimes\cdots\otimes\mathscr{W}_r$ is very ample by Proposition 2.5.6. The toric variety $|\mathscr{L}|$ is smooth, so the ample bundle ϑ determines the closed immersion $\varphi_{|\vartheta|}\colon |\mathscr{L}| \longrightarrow \mathbb{P}^*\big(H^0(|\mathscr{L}|,\vartheta)\big)$. The composition $\varphi_{|\vartheta|}\circ\varphi_{|\mathscr{L}|}\colon X\to\mathbb{P}^*(H^0(|\mathscr{L}|,\vartheta))$ is determined by the line bundle $(\varphi_{|\vartheta|}\circ\varphi_{|\mathscr{L}|})^*(\vartheta) = (\psi\circ\mathrm{pic})(\theta) = L_1\otimes\cdots\otimes L_r$ and the subspace of sections $\Phi(H^0(|\mathscr{L}|,\vartheta))\subseteq H^0(X, L_1\otimes\cdots\otimes L_r)$. We claim that $\Phi(H^0(|\mathscr{L}|,\vartheta))$ coincides with the image V of the multiplication map (3.3.1), in which case $\varphi_{|\vartheta|}\circ\varphi_{|\mathscr{L}|}$ coincides with the (a priori rational) map $\varphi_V\colon X\to\mathbb{P}^*(V)$ to the classical linear series. Indeed, for $\theta=(\theta_0,\ldots,\theta_r)\in\mathrm{Wt}(Q)$, the restriction of Φ to the subspace spanned by monomials of weight θ defines a $\Bbbk$-linear map

$$\Phi_\theta\colon H^0(|\mathscr{L}|,\mathscr{W}_1^{\theta_1}\otimes\cdots\otimes\mathscr{W}_r^{\theta_r})\to H^0(X, L_1^{\theta_1}\otimes\cdots\otimes L_r^{\theta_r})$$

because $(\psi\circ\mathrm{pic})(\theta) = L_1^{\theta_1}\otimes\cdots\otimes L_r^{\theta_r}$. In particular, the map Φ_ϑ for $\vartheta=\sum_{1\le i\le r}(\chi_i-\chi_0)$ and the product $\otimes_{1\le i\le r}\Phi_{(\chi_i-\chi_0)}$ fit in to a commutative diagram of $\Bbbk$-vector spaces

$$\begin{array}{ccc} H^0(|\mathscr{L}|,\mathscr{W}_1)\otimes\cdots\otimes H^0(|\mathscr{L}|,\mathscr{W}_r) & \longrightarrow & H^0(|\mathscr{L}|,\mathscr{W}_1\otimes\cdots\otimes\mathscr{W}_r) \\ {\scriptstyle\bigotimes_{1\le i\le r}\Phi_{(\chi_i-\chi_0)}}\big\downarrow & & \big\downarrow{\scriptstyle\Phi_\vartheta} \\ H^0(X,L_1)\otimes\cdots\otimes H^0(X,L_r) & \longrightarrow & H^0(X,L_1\otimes\cdots\otimes L_r) \end{array} \tag{3.3.2}$$

in which the horizontal maps are given by multiplication. For $1\le i\le r$, the map $\Phi_{(\chi_i-\chi_0)}$: $H^0(|\mathscr{L}|,\mathscr{W}_i)\longrightarrow H^0(X,L_i)$ can be obtained by composing two surjective maps $H^0(|\mathscr{L}|,\mathscr{W}_i)\longrightarrow H^0(\widetilde{X},E_i)$ and $H^0(\widetilde{X},E_i)\longrightarrow H^0(X,L_i)$. First the map $H^0(|\mathscr{L}|,\mathscr{W}_i)\longrightarrow H^0(\widetilde{X},E_i)$ is surjective, since a basis of the space of sections of $\mathscr{W}_i$ is given by $\{y_p | p \text{ is a path from } 0 \text{ to } i\}$, this map sends y_p to the label of p. By definition of the quiver of sections, there exists a path p from 0 to i labelled by every torus invariant section of E_i. Hence this first map is surjective. The second map $H^0(\widetilde{X},E_i)\longrightarrow H^0(X,L_i)$ is the restriction of the canonical surjection $\tau:\mathrm{Cox}(\widetilde{X})\longrightarrow\mathrm{Cox}(X)$ and is hence also surjective.

Every monomial of weight ϑ in $\Bbbk[y_a | a\in Q_1]$ can be decomposed as a product of monomials of weight $\mathbf{e}_i-\mathbf{e}_0$ for each $i\in Q_0$ (see Remark 4.2.3 (ii)) therefore the top map in the diagram is surjective. Hence commutativity of the diagram implies that the image of Φ_ϑ coincides with the image V of (3.3.1). This proves the claim.

Since V is the image of the multiplication map (3.3.1), the morphism $\varphi_V\colon X\to\mathbb{P}^*(V)$ is the composition of the product $\prod_{1\le i\le r}\varphi_{|L_i|}\colon X\longrightarrow|L_1|\times\cdots\times|L_r|$ of morphisms to the classical linear series and the appropriate Segre embedding to $\mathbb{P}^*(V)$. This is because the map $\prod_{1\le i\le r}\varphi_{|L_i|}\colon X\longrightarrow |L_1|\times\cdots\times|L_r|$ composed with the Segre embedding is given by every possible product of one section from each of $L_1,\ldots,L_r$. The claim implies that the diagram

$$
\begin{array}{ccccc}
|L_1| \times \cdots \times |L_r| & \xrightarrow{\text{Segre}} & \mathbb{P}^*(V) & \xrightarrow{\iota} & \mathbb{P}^*(H^0(\vartheta)) \\
\Big\uparrow {\scriptstyle \prod_{1\leq i\leq r}\varphi_{|L_i|}} & & & & \Big\uparrow {\scriptstyle \varphi_{|\vartheta|}} \\
X & & \xrightarrow{\quad\quad \varphi_{|\mathscr{L}|} \quad\quad} & & |\mathscr{L}|
\end{array}
$$

commutes, where ι is the closed immersion of projective spaces induced by Φ_ϑ. Three maps in the diagram are closed immersions, so $\varphi_{|\mathscr{L}|}$ is a closed immersion if and only if $\prod_{1\leq i\leq r}\varphi_{|L_i|}$ is a closed immersion if and only if the linear series V is very ample as required □

Remark 3.3.2. Neither of the maps from statements (i) and (iii) of Theorem 3.3.1 factors through the other. Typically $|\mathscr{L}|$ has much lower dimension than $|L_1| \times \cdots \times |L_r|$, so the multigraded linear series is a more efficient multigraded ambient space than the product.

Corollary 3.3.3. *Let $L_1, \dots, L_{r-1}$ be basepoint-free line bundles on X. If the subsemigroup of* $\mathrm{Pic}(X)$ *generated by $L_1, \dots, L_{r-1}$ contains an ample bundle, then there exists a line bundle L_r such that the quiver of sections for $\mathscr{L} = (\mathcal{O}_X, L_1, \dots, L_r)$ is very ample.*

Proof. Theorem 3.3.1 implies that $\varphi_{|\mathscr{L}|}$ is a closed immersion if $L_1 \otimes \cdots \otimes L_r$ is very ample and the map (3.3.1) is surjective. The proof of [10, Proposition 4.14] now applies verbatim. □

Example 3.3.4. Continuing Example 2.1.15, let X_4 be the del Pezzo surface for which the ample linearisation $\chi = 11l_0 - 5l_1 - 3l_2 - 2l_3 - l_4$ defines $\widetilde{X_4} := \mathbb{A}^{10} /\!\!/_\chi T$. We compute using the intersection pairing on X_4 (See Section 2.2.2) that each line bundle in the list

$$\mathscr{L} = (\mathscr{O}_{X_4}, l_0, 2l_0 - l_1, 2l_0 - l_2, 2l_0 - l_3, 2l_0 - l_4, 2l_0) \tag{3.3.3}$$

is nef and therefore basepoint-free but not ample. Write $\widetilde{\mathscr{L}} = (E_0, E_1, \dots, E_6)$. Since the nef cone of any Mori Dream Space has a chamber decomposition into the nef cones of ambient toric varieties, each E_i is basepoint-free on some ambient toric variety. This implies that the code from [21, Example 2.11] computes the irrelevant ideal for the GIT quotient $\mathbb{A}^d_{\Bbbk} /\!\!/_{E_i} T$ determined by the corresponding linearisation $E_i \in \mathrm{Cl}(X)$. By comparing each with the irrelevant ideal of $\chi \in \mathrm{Cl}(X)$ we see that E_3, E_4, E_5 are not basepoint-free line bundles on $\widetilde{X}_4$ as follows. Let J be the radical of the ideal generated by sections of $\psi^{-1}(\chi) \in \mathrm{Cox}(\widetilde{X})$ (where we recall ψ is the isomorphism from $\mathrm{Cl}(\widetilde{X})$ to $\mathrm{Cl}(X)$) and let J_i be the radical of the ideal generated by all sections of E_i. Explicitly, these are:

$$J = \left(\begin{array}{c} x_3x_4x_7x_9x_{10}, x_2x_4x_7x_9x_{10}, x_1x_4x_7x_9x_{10}, x_3x_4x_6x_9x_{10}, \\ x_2x_4x_5x_9x_{10}, x_3x_4x_7x_8x_{10}, x_3x_4x_6x_8x_{10}, x_3x_4x_5x_7x_{10}, \\ x_1x_4x_5x_7x_{10}, x_3x_4x_5x_6x_{10}, x_2x_3x_4x_5x_{10}, x_1x_2x_4x_5x_{10}, \\ x_2x_4x_7x_8x_9, x_1x_4x_7x_8x_9, x_2x_4x_5x_8x_9, x_2x_4x_6x_7x_9, x_1x_4x_6x_7x_9, \\ x_2x_4x_5x_6x_9, x_2x_3x_4x_6x_9, x_1x_3x_4x_6x_9, x_1x_4x_5x_7x_8, x_2x_3x_4x_7x_8, \\ x_1x_3x_4x_7x_8, x_2x_3x_4x_6x_8, x_1x_3x_4x_6x_8, x_2x_3x_4x_5x_8, x_1x_2x_4x_5x_8, \\ x_1x_4x_5x_6x_7, x_1x_3x_4x_5x_6, x_1x_2x_4x_5x_6 \end{array} \right)$$

$$J_1 = \left(\; x_3x_4x_{10}, x_2x_4x_9, x_2x_3x_8, x_1x_4x_7, x_1x_3x_6, x_1x_2x_5 \; \right)$$

$$J_2 = \left(\begin{array}{c} x_1x_4x_7, x_1x_3x_6, x_1x_2x_5, x_3x_4x_7x_{10}, x_3x_4x_6x_{10}, \\ x_2x_4x_7x_9, x_2x_4x_5x_9, x_2x_3x_6x_8, x_2x_3x_5x_8, x_2x_3x_4x_5x_{10}, \\ x_2x_3x_4x_6x_9, x_2x_3x_4x_7x_8 \end{array} \right)$$

$$J_3 = \left(\begin{array}{c} x_2x_4x_9, x_2x_3x_8, x_1x_2x_5, x_3x_4x_9x_{10}, x_3x_4x_8x_{10}, \\ x_1x_4x_7x_9, x_1x_3x_6x_8, x_1x_4x_5x_7, x_1x_3x_5x_6, \\ x_1x_3x_4x_5x_{10}, x_1x_3x_4x_6x_9, x_1x_3x_4x_7x_8 \end{array} \right)$$

$$J_4 = \left(\begin{array}{c} x_3x_4x_{10}, x_2x_3x_8, x_1x_3x_6, x_2x_4x_9x_{10}, \\ x_1x_4x_7x_{10}, x_2x_4x_8x_9, x_1x_2x_5x_8, x_1x_4x_6x_7, x_1x_2x_5x_6, \\ x_1x_2x_4x_5x_{10}, x_1x_2x_4x_6x_9, x_1x_2x_4x_7x_8 \end{array} \right)$$

$$J_5 = \left(\begin{array}{c} x_3x_4x_{10}, x_2x_4x_9, x_1x_4x_7, x_2x_3x_8x_{10}, x_1x_3x_6x_{10}, \\ x_2x_3x_8x_9, x_1x_2x_5x_9, x_1x_3x_6x_7, x_1x_2x_5x_7, \\ x_1x_2x_3x_5x_{10}, x_1x_2x_3x_6x_9, x_1x_2x_3x_7x_8 \end{array} \right)$$

$$J_6 = \left(\; x_3x_4x_{10}, x_2x_4x_9, x_2x_3x_8, x_1x_4x_7, x_1x_3x_6, x_1x_2x_5 \; \right)$$

Using the Macaulay 2 command "isSubset", we see that J_1, J_2 and J_6 contain J, therefore the common zero loci of the sections of $l_0, 2l_0 - l_1$ and $2l_0$ are contained in the common zero locus of the sections of χ–the unstable locus. However, the common zero loci of $2l_0 - l_2$, $2l_0 - l_3$ and $2l_0 - l_4$ are not contained in the unstable locus, and therefore are not basepoint free.

In particular, whilst it would be possible to restrict ourselves to lists $\mathscr{L}$ of line bundles on X which lift to basepoint free line bundles on $\widetilde{X}$, we will show that in this example $\varphi_{|\mathscr{L}|}$ is a morphism which is not the restriction of a morphism on the ambient toric variety. Indeed, since not all the E_i's are basepoint-free the rational map from the toric variety is not a morphism by Theorem 2.5.11. This shows that we cannot deduce that $\varphi_{|\mathscr{L}|}$ is a morphism simply by restriction from the toric case (compare Remark 3.2.5).

We now show $\varphi_{|\mathscr{L}|}$ is a morphism directly. In this case, the quiver of sections Q is shown in Figure 6.1, where each arrow is labelled by the torus-invariant section of the relevant reflexive sheaf on $\widetilde{X_4}$. We list arrows with tail at 0 as $a_1, \ldots, a_6$ from the top of Figure 6.1 to the bottom; list

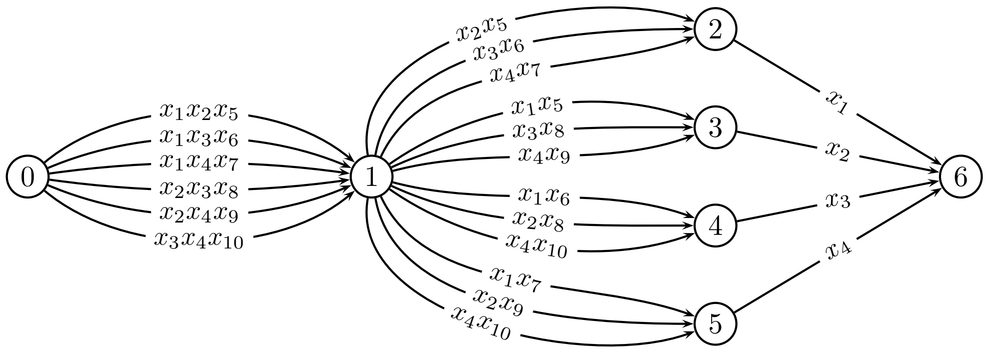


Figure 3.1: A quiver of sections for a collection on X_4

those with tail at 1 as $a_7, \ldots, a_{18}$ from the top of the figure to the bottom; and list those with head at 6 as $a_{19}, \ldots, a_{22}$ from the top to the bottom. Likewise, list the coordinates of $\mathbb{A}_{\Bbbk}^{Q_1}$ as $y_1, \ldots, y_{22}$, and compute the kernel of (3.2.3) to obtain the ideal

$$I_Q = \left(\begin{array}{c} y_{16} - y_{17} + y_{18},\ y_{13} - y_{14} + y_{15},\ y_{10} - y_{11} + y_{12},\ y_7 - y_8 + y_9,\ y_3 - y_5 + y_6, \\ y_2 - y_4 + y_6,\ y_1 - y_4 + y_5,\ y_{15}y_{21} - y_{18}y_{22},\ y_{12}y_{20} - y_{17}y_{22},\ y_{11}y_{20} - y_{14}y_{21}, \\ y_9y_{19} - y_{17}y_{22} + y_{18}y_{22},\ y_8y_{19} - y_{14}y_{21} + y_{18}y_{22},\ y_6y_{17} - y_5y_{18}, y_6y_{14} - y_4y_{15}, \\ y_5y_{11} - y_4y_{12},\ y_5y_8 - y_6y_8 - y_4y_9 + y_6y_9,\ y_8y_{15}y_{17} - y_9y_{14}y_{18} - y_8y_{15}y_{18} + y_9y_{15}y_{18}, \\ y_{11}y_{15}y_{17} - y_{12}y_{14}y_{18},\ y_9y_{11}y_{17} - y_8y_{12}y_{17} + y_8y_{12}y_{18} - y_9y_{12}y_{18}, \\ y_9y_{11}y_{14} - y_8y_{12}y_{14} + y_8y_{11}y_{15} - y_9y_{11}y_{15} \end{array} \right)$$

that cuts out the image of $\varphi_{|\mathscr{L}|}\colon X_4 \to |\mathscr{L}|$. We claim that $\varphi_{|\mathscr{L}|}$ is a closed immersion, and hence $X_4 \cong \mathbb{V}(I_Q)/\!\!/_\vartheta G$. Indeed, for $1 \leq i \leq 4$ we have $L_{i+1} = 2l_0 - l_i$, and the intersection pairing shows that $\varphi_{|L_{i+1}|}\colon X_4 \to \mathbb{F}_1$ contracts the (-1)-curves $\{l_j : j \neq i\}$ but not l_i. A simple case-by-case analysis shows that the morphism $\prod_{2\leq i\leq 5} \varphi_{|L_i|}$ separates all points and tangent vectors of X_4: a pair of distinct points on X_4 must either both lie on the same exceptional curve, lie on different (non-intersecting) exceptional curves, have one point on an exceptional curve and one off an exceptional curve or have neither lying on an exceptional curve. In each of the above cases, there is an exceptional curve L_j which has neither point on it, and hence the map $\varphi_{|L_i|}$ separates the two points and their tangent vectors. Therefore $\prod_{2\leq i\leq 5} \varphi_{|L_i|}$ must also separate the points and their tangent vectors, hence so does $\prod_{1\leq i\leq 6} \varphi_{|L_i|}$. We deduce from Theorem 3.3.1 that $\varphi_{|\mathscr{L}|}\colon X_4 \to |\mathscr{L}|$ is a closed immersion.

Chapter 4

Algebraic Results

4.1 Fine Moduli of Bound Quiver Representations

This chapter establishes when the morphism $\varphi_{|\mathscr{L}|}\colon X \to |\mathscr{L}|$ induces an isomorphism between the Mori Dream Space X and a fine moduli space $\mathcal{M}_\vartheta(Q,R)$ of ϑ-stable modules over the endomorphism algebra of $\bigoplus_{i\in Q_0} L_i$. Our main algebraic result is an efficient construction for collections of line bundles with this property. We recall that for a path p, $y_p := \prod_{a\in\mathrm{Supp}(p)} y_a$, that $\widetilde{\Phi} : \Bbbk[y_a|a \in Q_1] \longrightarrow \Bbbk[x_1,\dots,x_d]$ maps y_a to $x^{\mathrm{div}(a)}$, and that $\Phi : \Bbbk[y_a|a\in Q_1] \longrightarrow \mathrm{Cox}(X)$ is the composition $\tau\circ\widetilde{\Phi}$ where τ is the canonical surjection $\mathrm{Cox}(\widetilde{X}) \longrightarrow \mathrm{Cox}(X)$.

A list $\mathscr{L}$ of line bundles on X defines a pair of two-sided ideals in $\Bbbk Q$ and hence a pair of ideals of relations in $\Bbbk[y_a : a \in Q_1]$. First, the ideal R from Definition 3.1.3 determines the ideal of relations

$$I_R = \left(\sum_{p\in\Gamma} c_p y_p \in \Bbbk[y_a : a\in Q_1] \;\middle|\; \begin{array}{c}\Gamma \text{ is any set of paths sharing head and}\\ \text{tail for which } \sum_{p\in\Gamma} c_p x^{\mathrm{div}(p)} \in I_X\end{array} \right). \tag{4.1.1}$$

Each generator of I_R is $\mathrm{Wt}(Q)$-homogeneous and lies in $\ker(\Phi)$, so I_R is contained in the prime ideal of equations I_Q from (3.2.2). In Chapter 6 we present code that allows us to compute I_R explicitly. i In addition, the kernel $\widetilde{R}$ of the epimorphism $\Bbbk Q \to \mathrm{End}_{\mathcal{O}_{\widetilde{X}}}(\bigoplus_{i\in Q_0} E_i)$ obtained by sending p to $x^{\mathrm{div}(p)}$ determines the ideal of relations

$$\widetilde{I_R} := I_{\widetilde{R}} = \left(\sum_{p\in\widetilde{\Gamma}} c_p y_p \in \Bbbk[y_a : a\in Q_1] \;\middle|\; \begin{array}{c}\widetilde{\Gamma} \text{ is any set of paths sharing head and}\\ \text{tail for which } \sum_{p\in\widetilde{\Gamma}} c_p x^{\mathrm{div}(p)} = 0\end{array} \right). \tag{4.1.2}$$

We have that $\widetilde{I_R}$ is contained in I_R and $\widetilde{I_Q}$ (see (3.2.4)) is contained in I_Q since $\ker(\widetilde{\Phi})$ is contained in $\ker(\Phi)$. It also holds that $\widetilde{I_R}$ is contained in $\widetilde{I_Q}$ since I_R is generated by homogeneous polynomials in $\Bbbk[y_a | a \in Q_1]$ in the kernel of $\widetilde{\Phi}$. Therefore we have the following inclusions:

$$\begin{array}{ccc} I_R & \subset & I_Q \\ \cup & & \cup \\ \widetilde{I_R} & \subset & \widetilde{I_Q} \end{array}$$

Compute the affine varieties in $\mathbb{A}_{\Bbbk}^{Q_1}$ cut out by the ideals $\widetilde{I_R}, \widetilde{I_Q}, I_R, I_Q \subset \Bbbk[y_a : a \in Q_1]$, remove from each the ϑ-unstable locus $\mathbb{V}(B_{|\mathscr{L}|})$, and compute the geometric quotient by the action of G to obtain the left-hand square in the commutative diagram of GIT quotients

$$\begin{array}{ccccc} \mathbb{V}(\widetilde{I_Q})/\!\!/_\vartheta G & \longrightarrow & \mathcal{M}_\vartheta(Q, \widetilde{R}) & \longrightarrow & \mathbb{A}_{\Bbbk}^{Q_1}/\!\!/_\vartheta G \\ \uparrow & & \uparrow & & \| \\ \mathbb{V}(I_Q)/\!\!/_\vartheta G & \longrightarrow & \mathcal{M}_\vartheta(Q, R) & \longrightarrow & |\mathscr{L}| \end{array} \tag{4.1.3}$$

in which each morphism is a closed immersion.

Theorem 4.1.1. *If $\mathscr{L}$ is a list of basepoint-free line bundles on X, then the induced morphism*

$$\varphi_{|\mathscr{L}|} \colon X \longrightarrow \mathcal{M}_\vartheta(Q, R) \tag{4.1.4}$$

is surjective if I_Q coincides with the saturation

$$(I_R : B_{|\mathscr{L}|}^\infty) := \{f \in \Bbbk[y_a | a \in Q_1] \mid \text{for every spanning tree } T \text{ of } Q \text{ there exists } n \in \mathbb{N} \text{ such that } f \cdot y_T^n \in I_R\},$$

where we recall that for a spanning tree $T, y_T := \prod_{a \in \mathrm{Supp}(T)} y_a$. In particular, if $\mathscr{L}$ is very ample and $I_Q = (I_R : B_{|\mathscr{L}|}^\infty)$ then (4.1.4) *is an isomorphism.*

Proof. The ideal I_Q is prime since it is the homogeneous part of the kernel of map from a ring to a domain. It suffices by Theorem 3.2.4 to show that the closed immersion $\mathbb{V}(I_Q)/\!\!/_\vartheta G \to \mathbb{V}(I_R)/\!\!/_\vartheta G$ is an isomorphism. Proposition 2.5.6 shows that the ideal $B_{|\mathscr{L}|}$ cuts out the ϑ-unstable locus in $\mathbb{A}_{\Bbbk}^{Q_1}$, so we need only show that $\mathbb{V}(I_Q) \setminus \mathbb{V}(B_{|\mathscr{L}|})$ is isomorphic to $\mathbb{V}(I_R) \setminus \mathbb{V}(B_{|\mathscr{L}|})$. Since I_Q is prime, this holds if $I_Q = (I_R : B_{|\mathscr{L}|}^\infty)$. The second statement is immediate. □

Remark 4.1.2. In light of Proposition 3.1.4 and Remark 2.5.9, when the map (4.1.4) is an isomorphism then we describe the Mori Dream Space X as the fine moduli space $\mathcal{M}_\vartheta(Q, R)$ of ϑ-stable modules over $\mathrm{End}(\bigoplus_{i \in Q_0} L_i)$ that are isomorphic as $\left(\bigoplus_{i \in Q_0} \Bbbk e_i\right)$-modules to $\bigoplus_{i \in Q_0} \Bbbk e_i$.

4.2 Main Algebraic Result

We now work towards our main algebraic result which exhibits many collections of line bundles on X for which the morphism from (4.1.4) is an isomorphism, thereby providing a noncommutative algebraic construction of X as in Remark 4.1.2.

We first introduce the collections of interest. Choose generators $g_1, \dots, g_m \in \Bbbk[x_1, \dots, x_d]$ of the ideal I_X, set $\delta_0 := \max_{1\le j\le m}\{\text{total degree of } g_j\}$ (where total degree is as defined in Definition 1.1.3 of Cox–Little–O'Shea [5]) and define

$$\delta := \begin{cases} \delta_0/2 & \text{if } \delta_0 \text{ is even;} \\ (\delta_0+1)/2 & \text{otherwise.} \end{cases} \tag{4.2.1}$$

We recall that ψ is the isomorphism from $\mathrm{Cl}(\widetilde{X})$ to $\mathrm{Cl}(X)$. Consider line bundles $L_1, \dots, L_{r-2}$ on X for which the corresponding rank one reflexive sheaves $E_1 := \psi^{-1}(L_1), \dots, E_{r-2} := \psi^{-1}(L_{r-2})$ on $\widetilde{X}$ are basepoint-free line bundles such that the subsemigroup of $\mathrm{Pic}(\widetilde{X})$ generated by $E_1, \dots, E_{r-2}$ contains an ample line bundle. Choose sufficiently large integers $\beta_1, \dots, \beta_{r-2}$ to ensure that $E := E_1^{\beta_1} \otimes \cdots \otimes E_{r-2}^{\beta_{r-2}}$ is $\mathscr{O}_X$–regular with respect to $E_1, \dots, E_{r-2}$ and, moreover, that $E^{2\delta}$ is very ample. We can always find such $\beta_1, \dots, \beta_{r-2}$ by Propositon 2.3.4. Define $E_{r-1} := E^{\delta}$ and $E_r := E^{2\delta}$. Augment the list $L_1, \dots, L_{r-2}$ on X with $L_0 := \mathscr{O}_X$, $L_{r-1} := \psi(E_{r-1})$ and $L_r := \psi(E_r)$ to obtain a collection

$$\mathscr{L} = (\mathscr{O}_X, L_1, \dots, L_r) \tag{4.2.2}$$

of basepoint-free line bundles on X. Let Q denote the quiver of sections of $\mathscr{L}$. The corresponding collection of line bundles $\widetilde{\mathscr{L}} := (\mathscr{O}_{\widetilde{X}}, E_1, \dots, E_r)$ on $\widetilde{X}$ satisfies the conditions of Theorem 2.5.15, so

$$\widetilde{I_Q} = (\widetilde{I_R} : B^{\infty}_{|\mathscr{L}|}). \tag{4.2.3}$$

Thus, the induced morphism $\varphi_{|\widetilde{\mathscr{L}}|} \colon \widetilde{X} \to \mathbb{A}^{Q_1} /\!\!/_{\vartheta} G$ is a closed immersion whose image $\mathbb{V}(\widetilde{I_Q}) /\!\!/_{\vartheta} G$ is isomorphic to $\mathcal{M}_{\vartheta}(Q, \widetilde{R})$.

Remark 4.2.1. 1. It follows that each collection (4.2.2) determines a commutative diagram

$$\begin{array}{ccccccc} \widetilde{X} & \xrightarrow{\varphi_{|\widetilde{\mathscr{L}}|}} & \mathbb{V}(\widetilde{I_Q}) /\!\!/_{\vartheta} G & \xrightarrow{\cong} & \mathcal{M}_{\vartheta}(Q, \widetilde{R}) & \longrightarrow & \mathbb{A}^{Q_1}_{\Bbbk} /\!\!/_{\vartheta} G \\ \uparrow & & \uparrow & & \uparrow & & \| \\ X & \xrightarrow{\varphi_{|\mathscr{L}|}} & \mathbb{V}(I_Q) /\!\!/_{\vartheta} G & \longrightarrow & \mathcal{M}_{\vartheta}(Q, R) & \longrightarrow & |\mathscr{L}| \end{array} \tag{4.2.4}$$

in which every morphism is a closed immersion.

2. Since E is $\mathscr{O}_{\widetilde{X}}$-regular with respect to $E_1, \dots, E_{r-2}$ and each $\beta_i > 0$, Theorem 2.3.2 shows that the multiplication map $H^0(E_{r-1} \otimes E_i^{-1}) \otimes_{\Bbbk} H^0(E_{r-1}) \to H^0(E_r \otimes E_i^{-1})$ is surjective for all $1 \le i \le r-1$. This means that for any i, every path from vertex i to r can be decomposed into a path from i to $r-1$ and a path from $r-1$ to r since $E_{r-1} = E_r \otimes E_{r-1}^{-1}$. In particular, every path in Q from 0 to r passes through $r-1$.

3. For clarity in what follows, we work with elements of $\Bbbk[y_a : a \in Q_1]$ modulo the relation $\sim$ in which polynomials are equivalent when their difference lies in $\widetilde{I_Q}$. Since $\widetilde{I_Q}$ is the toric ideal of

the semigroup homomorphism $\mathrm{inc}\oplus\mathrm{div}\colon \mathbb{N}^{Q_1}\to \mathrm{Wt}(Q)\oplus\mathbb{N}^d$ (see Remark 2.5.12) monomials satisfy $y^{\mathbf{m}}\sim y^{\mathbf{m}'}$ if and only if $\mathrm{inc}(\mathbf{m}-\mathbf{m}')=0$ and $\mathrm{div}(\mathbf{m}-\mathbf{m}')=0$, that is, $y^{\mathbf{m}}\sim y^{\mathbf{m}'}$ if and only if $y^{\mathbf{m}}$ and $y^{\mathbf{m}'}$ share the same weight in $\mathrm{Wt}(Q)$ and have the same image under $\widetilde{\Phi}$.

Before introducing the main result, we present a technical lemma for any list $\mathscr{L}=(\mathcal{O}_X,L_1,\dots,L_r)$ as in (4.2.2). Write $\chi=\sum_i\chi_i\mathbf{e}_i\in\mathbb{Z}^{Q_0}$ as $\chi=\chi^+-\chi^-$ where $\chi^\pm=\sum_i\chi_i^\pm\mathbf{e}_i\in\mathbb{N}^{Q_0}$ have disjoint supports $I_\chi^+=\{i\in Q_0:\chi_i>0\}$ and $I_\chi^-=\{i\in Q_0:\chi_i<0\}$. In particular, $\chi\in\mathrm{Wt}(Q)$ gives

$$n_\chi:=\sum_{i\in I_\chi^+}\chi_i^+=\sum_{i\in I_\chi^-}\chi_i^-.$$

For any spanning tree $\mathcal{T}$ in Q, set $y_{\mathcal{T}}:=\prod_{a\in\mathsf{supp}(\mathcal{T})}y_a$. We recall that the map $\mathrm{inc}:\mathbb{N}^{Q_1}\longrightarrow\mathbb{Z}^{r+1}$ maps $\mathbf{e}_a$ to $\mathbf{e}_{\mathsf{h}(a)}-\mathbf{e}_{\mathsf{t}(a)}$ and that the image of inc is $\mathrm{Wt}(Q)$.

Lemma 4.2.2. *Assume $\mathscr{L}$ is a list of line bundles as in* (4.2.2)*, and let Q be the quiver of sections for $\mathscr{L}$. Let $\mathcal{T}$ be a spanning tree in Q and let $\chi\in\mathrm{inc}(\mathbb{N}^{Q_1})\setminus\{0\}$. There exists $\mathbf{m}\in\mathbb{N}^{Q_1}$ such that for any monomial $y^{\mathbf{v}}\in\Bbbk[y_a:a\in Q_1]$ of weight χ, we have*

$$(y_{\mathcal{T}})^{2n_\chi}y^{\mathbf{v}}\sim y^{\mathbf{m}}\prod_{\alpha=1}^{n_\chi}y_{\gamma_\alpha}\tag{4.2.5}$$

where $\gamma_1,\dots,\gamma_{n_\chi}$ are paths in Q, each with tail at 0 and head at r, where we recall that r is the number of vertices in Q excluding the source vertex 0. Also, $y^{\mathbf{v}}$ divides $\prod_{\alpha=1}^{n_\chi}y_{\gamma_\alpha}$, and the resulting quotient $\widetilde{\Phi}(\prod_\alpha y_{\gamma_\alpha})/\widetilde{\Phi}(y^{\mathbf{v}})$ depends only on $\mathcal{T}$ and χ.

Proof. We begin by constructing the relevant $\mathbf{m}\in\mathbb{N}^{Q_1}$. The spanning tree $\mathcal{T}$ supports a path q_i from $0\in Q_0$ to each vertex $i\in Q_0$ and hence to each vertex in I_χ. We may therefore write

$$(y_{\mathcal{T}})^{n_\chi}=y^{\mathbf{m}_1}\prod_{i\in I_\chi^-}(y_{q_i})^{\chi_i^-}.\tag{4.2.6}$$

where $\mathbf{m}_1\in\mathbb{N}^{Q_1}$ depends only on $\mathcal{T}$ and χ. The tree $\mathcal{T}$ supports a path γ from 0 to r whose label is a torus-invariant section $s\in H^0(E_r)$. Since E_{r-1} is $\mathcal{O}_X$-regular with respect to $E_1,\dots,E_{r-2}$ and each $\beta_i>0$, Theorem 2.3.2 implies that the multiplication map

$$H^0(E_{r-1}\otimes E_1^{\beta_1}\otimes\cdots\otimes E_{r-2}^{\beta_{r-2}}\otimes E_j^{-1})\otimes_\Bbbk H^0(E_j)\to H^0(E_r)\tag{4.2.7}$$

is surjective. In particular, for each $j\leq r-2$ there exist sections of $E_r\otimes E_j^{-1}$ and E_j whose product is s. By definition of the quiver of sections, there exists a pair of paths in Q labelled by these sections, one from 0 to j denoted q_j'', and the other from j to r denoted q_j'. Concatenating gives a path $q_j'q_j''$ from 0 to r that passes via j and, by Remark 4.2.1(2), through $r-1$ such that $y_\gamma\sim y_{q_j'q_j''}$. Multiply by $y_{\mathcal{T}}/y_\gamma$ to obtain $y_{\mathcal{T}}\sim y_{q_j'}y^{\mathbf{m}(j)}$ for some $\mathbf{m}(j)\in\mathbb{N}^{Q_1}$ that depends only on $\mathcal{T}$ and j (and on the lift of s via (4.2.7), but we fix one such lift for $\mathcal{T}$ and i). Applying this χ_j^+-times for each $j \in I_\chi^+$ and multiplying gives

$$(y_{\mathcal{T}})^{n_\chi} \sim y^{\mathbf{m}_2} \prod_{j \in I_\chi^+} (y_{q'_j})^{\chi_j^+}.$$

where $\mathbf{m}_2 \in \mathbb{N}^{Q_1}$ depends only on $\mathcal{T}$ and χ. Multiply by (4.2.6) to see that

$$(y_{\mathcal{T}})^{2n_\chi} \sim y^{\mathbf{m}} \prod_{i \in I_\chi^-} (y_{q_i})^{\chi_i^-} \prod_{j \in I_\chi^+} (y_{q'_j})^{\chi_j^+}. \tag{4.2.8}$$

where $\mathbf{m} := \mathbf{m}_1 + \mathbf{m}_2 \in \mathbb{N}^{Q_1}$ depends only on $\mathcal{T}$ and χ.

To complete the proof, write $\mathbf{v} = \sum_{a \in Q_1} v_a \mathbf{e}_a \in \mathbb{N}^{Q_1}$ where $\mathrm{inc}(\mathbf{v}) = \chi$. Since $\chi \neq 0$ there exists $i \in I_\chi^-$, so there exists $a_1 \in Q_1$ with $\mathsf{t}(a_1) = i$ such that $v_{a_1} > 0$. There are two cases. If $\chi_{\mathsf{h}(a_1)} < 0$ then $\mathsf{h}(a_1) \in I_\chi^+$, in which case we define $p_1 := a_1$ and repeat the above for $\mathbf{v}' := \mathbf{v} - \mathbf{e}_a$. Otherwise, $\chi_{\mathsf{h}(a_1)} \leq 0$ in which case there exists $a_2 \in Q_1$ with $\mathsf{t}(a_2) = \mathsf{h}(a_1)$ such that $v_{a_2} > 0$. Since Q is acyclic we can continue in this way, obtaining a path p_1 that traverses the arrows $a_1, a_2, \dots$ and satisfies $\chi_{\mathsf{h}(p_1)} > 0$, that is, $\mathsf{h}(p_1) \in I_\chi^+$. As in the first case, we may repeat the above for $\mathbf{v}' := \mathbf{v} - \sum_{a \in \mathrm{supp}(p_1)} \mathbf{e}_a$. In either case, the weight $\chi' := \mathrm{inc}(\mathbf{v}')$ satisfies $n_{\chi'} = n_\chi - 1$, and we obtain by induction a set of paths $p_1, \dots, p_{n_\chi}$ satisfying $y^{\mathbf{v}} = \prod_{\alpha=1}^{n_\chi} y_{p_\alpha}$, where precisely χ_i^- of these paths have tail at $i \in I_\chi^-$ and χ_i^+ have head at $i \in I_\chi^+$. Thus, for $1 \leq \alpha \leq n_\chi$, there exists $i \in I_\chi^-, j \in I_\chi^+$ such that $\gamma_\alpha := q'_j p_\alpha q_i$ is a path in Q from 0 to r and

$$\prod_{\alpha=1}^{n_\chi} y_{\gamma_\alpha} = \prod_{i \in I_\chi^-} (y_{q_i})^{\chi_i^-} \prod_{\alpha=1}^{n_\chi} y_{p_\alpha} \prod_{i \in I_\chi^+} (y_{q'_i})^{\chi_i^+}.$$

Note that $y^{\mathbf{v}}$ divides $\prod_{\alpha=1}^{n_\chi} y_{\gamma_\alpha}$. Moreover, multiplying (4.2.8) by $y^{\mathbf{v}}$ gives (4.2.5). The quotient $\widetilde{\Phi}(\prod_\alpha y_{\gamma_\alpha})/\widetilde{\Phi}(y^{\mathbf{v}})$ equals $\widetilde{\Phi}\big((y_{\mathcal{T}})^{2n_\chi}\big)/\widetilde{\Phi}(y^{\mathbf{m}})$, so depends only on $\mathcal{T}$ and χ as required. □

Remark 4.2.3. (i) Applying $\widetilde{\Phi}(-)$ to (4.2.5) and dividing the resulting equality by $\widetilde{\Phi}(y^{\mathbf{v}})$ shows in addition that the monomial $\widetilde{\Phi}(y^{\mathbf{m}})$ divides $\widetilde{\Phi}\big((y_{\mathcal{T}})^{2n_\chi}\big)$.

(ii) We draw the reader's attention to the fact that we have also constructed a set of paths $p_1, \dots, p_{n_\chi}$ satisfying $y^{\mathbf{v}} = \prod_{\alpha=1}^{n_\chi} y_{p_\alpha}$, where precisely χ_i^- of these paths have tail at $i \in I_\chi^-$ and χ_i^+ have head at $i \in I_\chi^+$.

We are now in a position to state and prove our main algebraic result.

Theorem 4.2.4. *Let $L_1, \dots, L_{r-2}$ be basepoint-free line bundles on a Mori Dream Space X. If the corresponding rank one reflexive sheaves $E_1 := \psi^{-1}(L_1), \dots, E_{r-2} = \psi^{-1}(L_{r-2})$ on $\widetilde{X}$ are basepoint-free line bundles such that the subsemigroup of* $\mathrm{Pic}(\widetilde{X})$ *generated by $E_1, \dots, E_{r-2}$ contains an ample* *line bundle, then there exist line bundles L_{r-1}, L_r such that the induced morphism*

$$\varphi_{|\mathscr{L}|}\colon X \longrightarrow \mathcal{M}_\vartheta(Q,R) \tag{4.2.9}$$

is an isomorphism for $\mathscr{L} = (\mathcal{O}_X, L_1, \dots, L_r)$.

Proof. Define the line bundles L_{r-1} and L_r as described at the start of this section to produce a collection $\mathscr{L}$ of the form (4.2.2). Remark 4.2.1(1) shows that $\mathscr{L}$ is very ample, so by Theorem 4.1.1 it suffices to prove that $I_Q = (I_R : B_{|\mathscr{L}|}^\infty)$. To establish one inclusion, let $f \in (I_R : B_{|\mathscr{L}|}^\infty)$. Since $I_R \subseteq I_Q$ and hence $(I_R : B_{|\mathscr{L}|}^\infty) \subseteq (I_Q : B_{|\mathscr{L}|}^\infty)$, we have that $(y_{\mathcal{T}})^N f \in I_Q$ for any spanning tree $\mathcal{T}$ and $N \in \mathbb{N}$. Since I_Q is prime, we have either $I_Q = (I_Q : B_{|\mathscr{L}|}^\infty)$ as required, or $B_{|\mathscr{L}|} \subseteq I_Q$. The ideal $B_{|\mathscr{L}|}$ is generated by monomials, and since I_Q is prime, this would imply that I_Q contained a variable. The map $\widetilde{\Phi}$ maps variables to nonzero monomials, and since the image of I_Q under $\widetilde{\Phi}$ is contained in I_X this would imply that I_X contains a monomial. Since I_X is also prime, this in turn would imply that I_X contained a variable, contradicting our assumption that the number of generators d of $\operatorname{Cox}(X)$ is as small as possible. Therefore $(I_R : B_{|\mathscr{L}|}^\infty) \subseteq I_Q$. For the opposite inclusion, let $f \in I_Q$ be a homogeneous generator of weight $\chi \in \operatorname{inc}(\mathbb{N}^{Q_1}) \setminus \{0\}$ and let $\mathcal{T}$ be a spanning tree in Q. If we can show that $(y_{\mathcal{T}})^N f \in \widetilde{I_Q} + I_R$ for some $N \in \mathbb{N}$, then by increasing N if necessary and applying the equality $\widetilde{I_Q} = (\widetilde{I_R} : B_{|\mathscr{L}|}^\infty)$ from (4.2.3), we deduce that $(y_{\mathcal{T}})^N f \in \widetilde{I_R} + I_R = I_R$ and hence $f \in (I_R : B_{|\mathscr{L}|}^\infty)$ as required.

In fact we show that $(y_{\mathcal{T}})^N f \in \widetilde{I_Q} + I_R$ for $N = 2n_\chi$. We proceed in four steps:

STEP 1: *Introduce a set of paths $\{\gamma_{\alpha,\beta}\}$ in Q such that*

$$(y_{\mathcal{T}})^{2n_\chi} f \sim y^{\mathbf{m}} \Big(\sum_\beta c_\beta \prod_{\alpha=1}^{n_\chi} y_{\gamma_{\alpha,\beta}} \Big) \tag{4.2.10}$$

for some $\mathbf{m} \in \mathbb{N}^{Q_1}$ and $c_\beta \in \Bbbk$, where in addition we have $\widetilde{\Phi}\big(\sum_\beta c_\beta \prod_{1\leq\alpha\leq n_\chi} y_{\gamma_{\alpha,\beta}}\big) \in I_X$.

Decompose f as a sum of terms $f = \sum_\beta c_\beta y^{\mathbf{v}_\beta}$ for $c_\beta \in \Bbbk$ and $\mathbf{v}_\beta \in \mathbb{N}^{Q_1}$ satisfying $\chi = \operatorname{inc}(\mathbf{v}_\beta)$. Since $\chi \neq 0$ we apply Lemma 4.2.2 to each monomial $y^{\mathbf{v}_\beta}$ to obtain $(y_{\mathcal{T}})^{2n_\chi} y^{\mathbf{v}_\beta} \sim y^{\mathbf{m}} \prod_{\alpha=1}^{n_\chi} y_{\gamma_{\alpha,\beta}}$, where $\mathbf{m}$ depends only on $\mathcal{T}$ and χ (not on β) and where each $\gamma_{\alpha,\beta}$ is a path in Q with tail at 0 and head at r. This gives (4.2.10). Also, the quotient $x^{\mathbf{q}} := \widetilde{\Phi}(\prod_\alpha y_{\gamma_{\alpha,\beta}})/\widetilde{\Phi}(y^{\mathbf{v}_\beta}) \in \Bbbk[x_1, \dots, x_d]$ depends only on $\mathcal{T}$ and χ (not on β). Since $f \in I_Q$, we have $\widetilde{\Phi}(f) \in I_X$ and hence we deduce that $\widetilde{\Phi}\big(\sum_\beta c_\beta \prod_{\alpha=1}^{n_\chi} y_{\gamma_{\alpha,\beta}}\big) = x^{\mathbf{q}}\big(\sum_\beta c_\beta \widetilde{\Phi}(y^{\mathbf{v}_\beta})\big) = x^{\mathbf{q}} \widetilde{\Phi}(f) \in I_X$ as required.

STEP 2: *We fix generators $g_1, \dots g_m$ of I_X and introduce a second set of paths $\{p_{i,j,k,\ell}\}$ in Q such that*

$$\sum_\beta c_\beta \prod_{\alpha=1}^{n_\chi} y_{\gamma_{\alpha,\beta}} \sim \sum_{i,j,k} c_{i,j,k} \prod_{\ell=1}^{n_\chi} y_{p_{i,j,k,\ell}}$$

for some $c_{i,j,k} \in \Bbbk$, where for each i, j we have $\widetilde{\Phi}\big(\sum_k c_{i,j,k} \prod_{1\leq \ell \leq n_\chi} y_{p_{i,j,k,\ell}}\big)$ is a term in $\Bbbk[x_1, \dots, x_d]$ multiplied by a generator of I_X.

In light of Step 1, expand $\widetilde{\Phi}\big(\sum_\beta c_\beta \prod_{\alpha=1}^{n_\chi} y_{\gamma_{\alpha,\beta}}\big) = \sum_{i,j} h_{i,j} g_i$ in terms of generators of I_X, where each $h_{i,j} \in \Bbbk[x_1, \dots, x_d]$ is a nonzero term. Since $\widetilde{\Phi}$ is graded and $y_{\gamma_{\alpha,\beta}}$ has weight $\mathbf{e}_r - \mathbf{e}_0 \in \mathrm{Wt}(Q)$, we may assume that each term in this expansion has degree $\widetilde{\mathrm{pic}}(n_\chi(\mathbf{e}_r - \mathbf{e}_0)) = E_r^{n_\chi}$. Thus, expanding each $g_i := g_{i,1} + \cdots + g_{i,t_i}$ as a sum of terms for some $t_i \in \mathbb{N}$ gives $h_{i,j} g_{i,k} \in H^0(E_r^{n_\chi})$ for all i, j, k. Since E_{r-1} is $\mathcal{O}_X$-regular with respect to $E_1, \dots, E_{r-2}$ and $E_r = E_{r-1}^2$, Proposition 2.3.3 implies that the multiplication map $H^0(E_r) \otimes_\Bbbk \cdots \otimes_\Bbbk H^0(E_r) \to H^0(E_r^{n_\chi})$ is surjective, so for each i, j, k there exists $c_{i,j,k} \in \Bbbk$ and torus-invariant sections $s_{i,j,k,\ell} \in H^0(E_r)$ for $1 \leq \ell \leq n_\chi$ such that $h_{i,j} g_{i,k} = c_{i,j,k} \prod_{\ell=1}^{n_\chi} s_{i,j,k,\ell}$. Since Q is a quiver of sections, there exists a path $p_{i,j,k,\ell}$ in Q from 0 to r whose label is the torus-invariant section $s_{i,j,k,\ell}$, that is, $\widetilde{\Phi}(y_{p_{i,j,k,\ell}}) = s_{i,j,k,\ell}$. For fixed i, j, we therefore obtain

$$h_{i,j} g_{i,k} = c_{i,j,k} \widetilde{\Phi}\bigg(\prod_{\ell=1}^{n_\chi} y_{p_{i,j,k,\ell}}\bigg). \tag{4.2.11}$$

Summing over $1 \leq k \leq t_i$ gives $h_{i,j} g_i = \widetilde{\Phi}\big(\sum_k c_{i,j,k} \prod_{1 \leq \ell \leq n_\chi} y_{p_{i,j,k,\ell}}\big)$, and by summing this new expression over all i, j we deduce that

$$\widetilde{\Phi}\bigg(\sum_\beta c_\beta \prod_{\alpha=1}^{n_\chi} y_{\gamma_{\alpha,\beta}}\bigg) = \widetilde{\Phi}\bigg(\sum_{i,j,k} c_{i,j,k} \prod_{\ell=1}^{n_\chi} y_{p_{i,j,k,\ell}}\bigg) \tag{4.2.12}$$

lies in I_X by Step 1. The main statement of Step 2 now follows from Remark 4.2.1(3) because these polynomials also share the same weight in $\mathrm{Wt}(Q)$, namely $n_\chi(\mathbf{e}_r - \mathbf{e}_0)$.

STEP 3: *Introduce a third set of paths $\{q_{i,j,k}\}$ in Q such that*

$$\prod_{\ell=1}^{n_\chi} y_{p_{i,j,k,\ell}} \sim y^{\mathbf{m}'_{i,j}} y_{q_{i,j,k}} \tag{4.2.13}$$

for some $\mathbf{m}'_{i,j} \in \mathbb{N}^{Q_1}$, where for each i, j we have $\widetilde{\Phi}\big(\sum_k c_{i,j,k} y_{q_{i,j,k}}\big)$ equal to a term in $\Bbbk[x_1, \dots, x_d]$ multiplied by a generator of I_X.

Fix i and j and define $y^{\mathbf{v}_{i,j,k}} := \prod_{1 \leq \ell \leq n_\chi} y_{p_{i,j,k,\ell}}$. The map $\widetilde{\Phi}$ is equivariant and sends monomials to monomials, so $\frac{1}{c_{i,j,k}} h_{i,j} g_{i,k} \in H^0(E_r^{n_\chi})$ defines a torus-invariant section. Since $E = E_1^{\beta_1} \otimes \cdots \otimes E_{r-2}^{\beta_{r-2}}$ is $\mathcal{O}_X$-regular with respect to $E_1, \dots, E_{r-2}$ and $E_r = E_{r-1}^2 = E^{2\delta}$, Proposition 2.3.3 implies that the multiplication map

$$H^0(E) \otimes_\Bbbk \cdots \otimes_\Bbbk H^0(E) \to H^0(E_r^{n_\chi})$$

is surjective, so $\frac{1}{c_{i,j,k}} h_{i,j} g_{i,k}$ is equal to the product of $2\delta n_\chi$ torus-invariant sections of E. Since $g_{i,k}$ is a term of a generator of I_X, its total degree is at most $\delta_0 \leq 2\delta$ by (4.2.1), so we may choose

2δ of these sections $s_{i,k,1},\dots,s_{i,k,2\delta}\in H^0(E)$ such that $g_{i,k}$ divides $\prod_{1\leq\mu\leq 2\delta}s_{i,k,\mu}\in H^0(E_r)$. We now apply the above only for $k=1$. Since Q is a quiver of sections, there exists a path $q_{i,j,1}$ in Q from 0 to r satisfying $\widetilde{\Phi}(y_{q_{i,j,1}})=\prod_{1\leq\mu\leq 2\delta}s_{i,1,\mu}$, so the section $h_{i,j}g_{i,1}/c_{i,j,1}\widetilde{\Phi}(y_{q_{i,j,1}})\in H^0(E_r^{n_\chi-1})$ is torus-invariant. Surjectivity of the multiplication map $H^0(E_r)\otimes_{\Bbbk}\cdots\otimes_{\Bbbk}H^0(E_r)\to H^0(E_r^{n_\chi-1})$ determines $n_\chi-1$ sections of E_r and hence paths $q'_{i,j,1},\dots,q'_{i,j,n_\chi-1}$ in Q from 0 to r labelled by these sections such that $\widetilde{\Phi}(y^{\mathbf{m}'_{i,j}})=h_{i,j}g_{i,1}/c_{i,j,1}\widetilde{\Phi}(y_{q_{i,j,1}})$ for $y^{\mathbf{m}'_{i,j}}:=\prod_{1\leq\nu\leq n_\chi-1}y_{q'_{i,j,\nu}}$. In particular,

$$\widetilde{\Phi}(y^{\mathbf{v}_{i,j,1}})=\frac{h_{i,j}g_{i,1}}{c_{i,j,1}}=\widetilde{\Phi}(y^{\mathbf{m_{i,j}}'}y_{q_{i,j,1}}).\tag{4.2.14}$$

Both monomials $y^{\mathbf{v}_{i,j,1}}$ and $y^{\mathbf{m}'_{i,j}}y_{q_{i,j,1}}$ have weight $n_\chi(\mathbf{e}_r-\mathbf{e}_0)\in\mathrm{Wt}(Q)$, hence $y^{\mathbf{v}_{i,j,1}}\sim y^{\mathbf{m}'_{i,j}}y_{q_{i,j,1}}$. This gives us (4.2.13) for the case $k=1$.

For $k>1$, we have $h_{i,j}g_{i,k}=c_{i,j,k}\widetilde{\Phi}(y^{\mathbf{v}_{i,j,k}})$. For $1\leq i\leq m$, the generator g_i of I_X is $\mathrm{Cl}(X)$-homogeneous, so $g_{i,k}$ and $g_{i,1}$ have the same degree in $\mathrm{Cl}(X)$ for any k. Since $g_{i,1}$ divides $\widetilde{\Phi}(y_{q_{i,j,1}})\in H^0(E_r)$, it follows that the term $\widetilde{\Phi}(y_{q_{i,j,1}})g_{i,k}/g_{i,1}$ also has degree E_r. Divide by its coefficient $c_{i,j,k}/c_{i,j,1}\in\Bbbk$ to obtain a torus-invariant section $\widetilde{\Phi}(y_{q_{i,j,1}})c_{i,j,1}g_{i,k}/c_{i,j,k}g_{i,1}\in H^0(E_r)$ which in turn determines a path $q_{i,j,k}$ in Q with tail at 0 and head at r for which $\widetilde{\Phi}(y_{q_{i,j,k}})=\widetilde{\Phi}(y_{q_{i,j,1}})c_{i,j,1}g_{i,k}/c_{i,j,k}g_{i,1}$. Then (4.2.14) gives

$$\widetilde{\Phi}(y^{\mathbf{v}_{i,j,k}})=h_{i,j}g_{i,1}\cdot\frac{g_{i,k}}{c_{i,j,k}g_{i,1}}=c_{i,j,1}\widetilde{\Phi}(y^{\mathbf{m}'_{i,j}})\widetilde{\Phi}(y_{q_{i,j,1}})\cdot\frac{g_{i,k}}{c_{i,j,k}g_{i,1}}=\widetilde{\Phi}(y^{\mathbf{m}'_{i,j}}y_{q_{i,j,k}}).$$

It follows that the monomials $y^{\mathbf{v}_{i,j,k}}$ and $y^{\mathbf{m}'_{i,j}}y_{q_{i,j,k}}$ have weight $n_\chi(\mathbf{e}_r-\mathbf{e}_0)$, hence $y^{\mathbf{v}_{i,j,k}}\sim y^{\mathbf{m}'_{i,j}}y_{q_{i,j,k}}$, and we obtain (4.2.13) for all k. Then

$$\widetilde{\Phi}(y^{\mathbf{m}'_{i,j}})\widetilde{\Phi}\Big(\sum_k c_{i,j,k}y_{q_{i,j,k}}\Big)=\widetilde{\Phi}\Big(\sum_k c_{i,j,k}\prod_{\ell=1}^{n_\chi}y_{p_{i,j,k,\ell}}\Big)\in I_X$$

holds for every i,j by combining (4.2.13) and Step 2. The ideal I_X does not contain the monomial $\widetilde{\Phi}(y^{\mathbf{m}'_{i,j}})$ for any i,j, otherwise it would contain a variable of $\Bbbk[x_1,\dots,x_d]$ because I_X is prime, which would give a contradiction since we assumed that d is as small as possible. Thus, $\widetilde{\Phi}\big(\sum_k c_{i,j,k}y_{q_{i,j,k}}\big)\in I_X$ for every i,j as required.

Step 4: *Establish that $(y_{\mathcal{T}})^{2n_\chi}f\in\widetilde{I_Q}+I_R$ as required by proving that*

$$(y_{\mathcal{T}})^{2n_\chi}f\sim y^{\mathbf{m}}\Big(\sum_{i,j}y^{\mathbf{m}'_{i,j}}\Big(\sum_k c_{i,j,k}y_{q_{i,j,k}}\Big)\Big).\tag{4.2.15}$$

Relation (4.2.15) is immediate from Steps 1-3. For every i,j we also have $\sum_k c_{i,j,k}y_{q_{i,j,k}}\in I_R$ by Step 3, so the right hand side of (4.2.15) also lies in I_R. The definition of $\sim$ given in Remark 4.2.1(3) then implies $(y_{\mathcal{T}})^{2n_\chi}f\in\widetilde{I_Q}+I_R$. This completes the proof of Theorem 4.2.4. ☐

Chapter 5

Computing I_R and I_Q

In this chapter we show how to compute $I_R, \widetilde{I_R}, I_Q$ and $\widetilde{I_Q}$ explicitly using Maple and Macaulay 2. As an application, in the next chapter we show that $I_Q = \left(I_R : B^{\infty}_{|\mathscr{L}|}\right)$ for certain ample quivers of sections Q on X_4, X_5 and $\mathrm{Gr}(2,4)$, therefore each is isomorphic to a moduli space of bound quiver representations by Theorem 4.1.1.

We summarise our method for computing $\widetilde{I_R}$ and I_R using Maple below:

1. In section 5.1.1, we show how to input quivers into Maple. We give pseudocode for finding the set of all paths in Q, along with their heads, tails and labels in section 5.1.2.

2. In section 5.1.3, we give pseudocode for finding the generators of $\widetilde{I_R}$.

3. In section 5.1.4, we prove that there is a choice of generating set for I_R which contains the generating set for $\widetilde{I_R}$, plus certain additional generators in a form conducive to calculations.

4. In 5.1.5, we give pseudocode for finding the additional generators of I_R mentioned above.

We summarise our method for calculating $\widetilde{I_Q}$ and I_Q using Macaulay 2:

1. In Appendix A, we give a method for computing the kernels of $\Bbbk$-algebra homomorphisms using Macaulay 2.

2. In section 5.2.1, we prove $\widetilde{I_Q}$ and I_Q are kernels of certain $\Bbbk$-algebra homomorphisms. Hence we can apply the results of Appendix A to compute them.

3. In section 5.2.2 we give Macaulay 2 code for calculating $\widetilde{I_Q}$ and I_Q.

5.1 Computing $\widetilde{I_R}$ and I_R using Maple

In this section we give a method for calculating $\widetilde{I_R}$ and I_R explicitly.

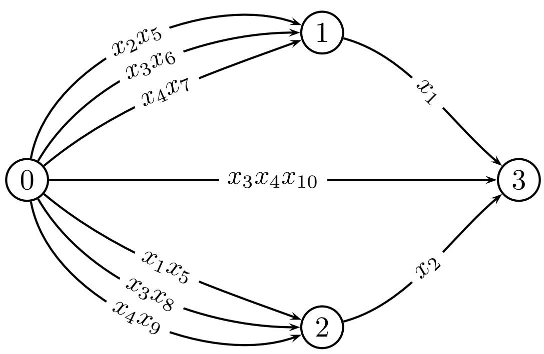
5.1.1 Quivers in Maple

In order to calculate $\widetilde{I_R}$ and I_R, we need to input quivers into Maple. We input quivers as Maple "lists" of all arrows, plus their heads, tails and labels as shown:

$$Q := [[\mathsf{t}(a_i), \mathrm{div}(a_i), \mathsf{h}(a_i), y_{a_i}] | a_i \in Q_1].$$

We will refer to the ith entry in Q (and more generally in any list) as $Q[i]$. We introduce some notation: for a quiver Q, let $\mathsf{t}(Q[i]) := Q[i][1], \mathrm{div}(Q[i]) := Q[i][2], \mathsf{h}(Q[i]) := Q[i][3]$ and $y_{Q[i]} := Q[i][4]$. We write $\mathrm{div}(Q[i])$ as an element of $\mathbb{N}^d$ where d is the number of generators of $\mathrm{Cox}(X)$.

Example 5.1.1. On X_4, $\mathscr{L} = (\mathcal{O}_{X_4}, l_0 - l_1, l_0 - l_2, l_0)$. The quiver of sections for $\mathscr{L}$, Q, is given below:



Arrows 1-3 are those from 0 to 1. Arrows 4-6 are those from 0 to 2. Arrow 7 goes from 0 to 3. Arrow 8 goes from 1 to 3. Arrow 9 goes from 2 to 3. We input Q into Maple as:

$$[[0, [0, 1, 0, 0, 1, 0, 0, 0, 0, 0], 1, y_1], [0, [0, 0, 1, 0, 0, 1, 0, 0, 0, 0], 1, y_2], [0, [0, 0, 0, 1, 0, 0, 1, 0, 0, 0], 1, y_3]$$

$$[0, [1, 0, 0, 0, 1, 0, 0, 0, 0, 0], 2, y_4], [0, [0, 0, 1, 0, 0, 0, 0, 1, 0, 0], 2, y_5], [0, [0, 0, 0, 1, 0, 0, 0, 0, 1, 0], 2, y_6]$$

$$[0, [0, 0, 1, 1, 0, 0, 0, 0, 0, 1], 3, y_7], [1, [1, 0, 0, 0, 0, 0, 0, 0, 0, 0], 3, y_8], [2, [0, 1, 0, 0, 0, 0, 0, 0, 0, 0], 3, y_9]]$$

5.1.2 Finding all Paths in Q

The ideals I_R and $\widetilde{I_R}$ are defined in terms of the paths of Q, therefore we need to consider paths as well as arrows of Q. With that in mind, we wrote a Maple procedure "getpaths" which outputs the list of all paths P for a given quiver Q. More specifically, for every path p in Q the output P lists $\mathsf{t}(p), \mathrm{div}(p), \mathsf{h}(p)$ and y_p. We give pseudocode and a proof of its efficacy.

We define $\mathsf{h}(P[i]), \mathrm{div}(P[i]), \mathsf{t}(P[i])$ and $y_{P[i]}$ to be the first, second, third and fourth terms of $P[i]$ respectively. We denote the number of terms in a list L by $|L|$.

Pseudocode 5.1.2. *Input:* Q

Procedure: $P := Q$, $r\ :\ = \max\{\mathsf{h}(a_i)|a_i \in Q_0\}, a := 0, b := |Q|$

for i *from 1 to* r *do*

for j *from* $|P| - b + 1$ *to* $|P|$ *do*

for k *from 1 to* $|Q|$ *do*

if $\mathsf{t}(Q[k]) = \mathsf{h}(P[j])$ *then* $P := [P, [\mathsf{t}(P[j]), \mathrm{div}(P[j])\,\mathrm{div}(Q[k]), \mathsf{h}(Q[k]), y_{Q[k]}y_{P[j]}]]$ *and* $a := a + 1$.

end if

end do

end do

$b := a, a := 0$

end do.

Output: P

Proof. We begin by defining $P := Q$. We work through all the elements of P and Q, and if say the ith element of Q has tail equal to the head of the jth element of P then we add their concatenation (a path of length 2) to P. Once we have worked through all the elements of P and Q in this way we will have added all the paths of length two to P. We record the number of paths we have added (this is the role of a and b).

Next we consider all arrows in Q and all paths of length 2 (i.e. the last b paths in P). If it is possible to concatenate them to form a path of length 3, they are added to P. Again we record the number of additions to P.

We repeat this process r times, where r is the number of vertices in Q. The paths in Q have length at most r since Q contains no cycles, hence after repeating the process r times P lists the details of every path in Q. ☐

Example 5.1.3. Let Q be as in example 5.1.1. The output for "getpaths(Q)" is:

$$[[0, [0, 1, 0, 0, 1, 0, 0, 0, 0, 0], 1, y_1], [0, [0, 0, 1, 0, 0, 1, 0, 0, 0, 0], 1, y_2], [0, [0, 0, 0, 1, 0, 0, 1, 0, 0, 0], 1, y_3],$$

$$[0, [1, 0, 0, 0, 1, 0, 0, 0, 0, 0], 2, y_4], [0, [0, 0, 1, 0, 0, 0, 0, 1, 0, 0], 2, y_5], [0, [0, 0, 0, 1, 0, 0, 0, 0, 1, 0], 2, y_6],$$

$$[0, [0, 0, 1, 1, 0, 0, 0, 0, 0, 1], 3, y_7], [1, [1, 0, 0, 0, 0, 0, 0, 0, 0, 0], 3, y_8], [2, [0, 1, 0, 0, 0, 0, 0, 0, 0, 0], 3, y_9],$$

$$[0, [1, 1, 0, 0, 1, 0, 0, 0, 0, 0], 3, y_8y_1], [0, [1, 0, 1, 0, 0, 1, 0, 0, 0, 0], 3, y_8y_2], [0, [1, 0, 0, 1, 0, 0, 1, 0, 0, 0], 3, y_8y_3],$$

$$[0, [1, 1, 0, 0, 1, 0, 0, 0, 0, 0], 3, y_9y_4], [0, [0, 1, 1, 0, 0, 0, 0, 1, 0, 0], 3, y_9y_5], [0, [0, 1, 0, 1, 0, 0, 0, 0, 1, 0], 3, y_9y_6]]$$

5.1.3 Calculating $\widetilde{I_R}$

We give pseudocode for finding the generators of $\widetilde{I_R}$ and prove its efficacy.

Pseudocode 5.1.4. *Input: P:= getpaths(Q).*
Procedure: L:=[ ] (the "empty list").
for i from 1 to $|P|$ do
for j from 1 to $|P|$ do
If $\mathsf{h}(P[i]) = \mathsf{h}(P[j]), \mathsf{t}(P[i]) = \mathsf{t}(P[j])$ *and* $\mathrm{div}(P[i]) = \mathrm{div}(P[j])$ *then* $L := [L, y_{P[i]} - y_{P[j]}]$.
end if
end do
end do.
Output: L.

Proof. We check all pairs of paths p_i and p_j. If their heads, tails and labels are equal then $y_{p_i} - y_{p_j}$ is a generator of $\widetilde{I_R}$ so we add $y_{p_i} - y_{p_j}$ to L. All generators are of this form, and since we check all pairs of paths this must give a list of all generators for $\widetilde{I_R}$. □

Remark 5.1.5. Note that while L is a generating set for $\widetilde{I_R}$, it will almost certainly contain many redundancies. In particular, L will probably have many terms equal to zero.

Example 5.1.6. Let Q be the quiver from Example 5.1.1. If our input is the list of all paths in Q, then the output from "zeropart" is

$$[0,0,0,0,0,0,0,0,0,0,y_8y_1 - y_9y_4, 0, 0, y_9y_4 - y_8y_1, 0, 0, 0].$$

Hence

$$\widetilde{I_R} = \big(y_8y_1 - y_9y_4\big).$$

5.1.4 A Generating Set for I_R

We give a technical lemma which describes a generating set for I_R.

Lemma 5.1.7. *Fix a presentation $I_X = \langle g_1, \dots, g_m\rangle$. The ideal I_R is generated by $S_1 \cup S_2$ where*

$$S_1 := \{y_p - y_{p'} \,|\, \mathsf{h}(p) = \mathsf{h}(p'), \mathsf{t}(p) = \mathsf{t}(p'), \mathrm{div}(p) = \mathrm{div}(p')\}$$

and

$$S_2 := \left\{ \sum c_i y_{p_i} \middle| \begin{array}{c} \mathsf{h}(p_i) = \mathsf{h}(p_j), \mathsf{t}(p_i) = \mathsf{t}(p_j) \textit{ for all } i,j \textit{ and } \sum c_i \widetilde{\Phi}(y_{p_i}) = h_{ij} g_i \\ \textit{for some } j \textit{ where } h_{ij} \textit{ is a term in } \Bbbk[x_1, \dots, x_d] \end{array} \right\}.$$

Proof. The ideal generated by S_1 and S_2 is contained in I_R. For the converse, let $f = \sum c_i y_{p_i}$ be a generator of I_R (where paths p_i share same head and same tail). It suffices to show that f lies in the ideal generated by S_1 and S_2.

Since $f \in I_R$, $\widetilde{\Phi}(f) = \sum_{i,j} h_{ij} g_i \in I_X$ for some terms $h_{ij} \in \Bbbk[x_1, \dots, x_d]$. We proceed in two stages: first we show that f can be written as an element of $\widetilde{I_R}$ plus a linear combination of $\mathrm{Wt}(Q)$ homogeneous y_p's mapping to terms of the sum $\sum h_{ij} g_i$. All elements of $\widetilde{I_R}$ lie in the ideal generated by S_1 by Proposition 2.5.4. Secondly, we show that the remaining part lies in the ideal generated by S_2.

Step 1: Identifying vertices of Q with line bundles, let $E = \mathsf{h}(p_i) \otimes \mathsf{t}(p_i)^{-1}$ for any i (note that this is independent of i). Since $\widetilde{\Phi}$ is a toric homomorphism and maps monomials to monomials, each term $c_i y_{p_i}$ maps to a term $c_i x^{\mathbf{m}_i}$ in $(S_{\widetilde{X}})_E$. So, we can write

$$\widetilde{\Phi}(\sum_{i=1}^{n} c_i y_{p_i}) = \sum_{i=1}^{n} c_i x^{\mathbf{m}_i} = c_{i_1} x^{\mathbf{m}_{i_1}} + \cdots + c_{i_t} x^{\mathbf{m}_{i_t}} \text{ after cancelling },$$

where $\{i_1, \dots, i_t\} \subseteq \{1, \dots, n\}$. Hence we can decompose f as

$$f = c_{i_1} y_{p_{i_1}} + \cdots + c_{i_t} y_{p_{i_t}} + (f - (c_{i_1} y_{p_{i_1}} + \cdots + c_{i_t} y_{p_{i_t}})).$$

We note that $f - (c_{i_1} y_{p_{i_1}} + \cdots + c_{i_t} y_{p_{i_t}})$ is homogeneous and in the kernel of $\widetilde{\Phi}$. It is therefore an element of $\widetilde{I_R}$, and lies in the ideal generated by S_1 by Proposition 2.5.4.

Step 2 : Redefine $c_{i_\alpha} y_{p_{i_\alpha}} =: c_\alpha y_{p_\alpha}$ and $x^{\mathbf{m}_{i_\alpha}} =: x^{\mathbf{m}_\alpha}$. We show that $\sum_{\alpha=1}^{t} c_\alpha y_{p_\alpha}$ lies in the ideal generated by S_2. Since $\sum c_\alpha x^{\mathbf{m}_\alpha} \in I_X$, we can write

$$\sum_{\alpha} c_\alpha x^{\mathbf{m}_\alpha} = \sum_{i,j} h_{ij} g_i$$

where g_i is a generator of I_X and h_{ij} is a term in $\Bbbk[x_1, \dots, x_d]$. Since $\widetilde{\Phi}$ is equivariant, $\sum_{i,j} h_{ij} g_i$ is homogeneous of degree E. We can decompose $h_{ij} g_i$ into terms, say $h_{ij} g_i = \sum_k h_{ij} g_{ik}$. For all i, j and k,

$$h_{ij} g_{ik} = c_{ijk} x^{\mathbf{v}_{ijk}} \tag{5.1.1}$$

where $c_{ijk} \in \Bbbk$ and $x^{\mathbf{v}_{ijk}}$ is a torus–invariant section of E. By definition of the quiver of sections there exists a path p_{ijk} from $\mathsf{t}(p_\alpha)$ to $\mathsf{h}(p_\alpha)$ labelled by $x^{\mathbf{v}_{ijk}}$. Additionally, we can ensure that $\mathbf{v}_{ijk} = \mathbf{v}_{i'j'k'}$ if and only if $p_{ijk} = p_{i'j'k'}$, and that $p_{ijk} = p_\alpha$ if and only if $x^{\mathbf{v}_{ijk}} = x^{\mathbf{m}_\alpha}$.

Now we will show that

$$\sum_{ijk} c_{ijk} y_{p_{ijk}} = \sum_{\alpha} c_\alpha y_{p_\alpha}.$$

For each $\mathbf{v} \in \mathbb{N}^d$,

$$\sum_{\substack{i,j,k \\ \text{s.t. } \mathbf{v}_{ijk} = \mathbf{v}}} c_{ijk} x^{\mathbf{v}_{ijk}} = \begin{cases} c_\alpha x^{\mathbf{m}_\alpha} & \text{if } \mathbf{v} = \mathbf{m}_\alpha \\ 0 & \text{otherwise} \end{cases}$$

because the sum of terms in $\sum_{ijk} c_{ijk} x^{\mathbf{v}_{ijk}}$ which are equal modulo constant is either zero or a term in $\sum_\alpha c_\alpha x^{\mathbf{m}_\alpha}$ since $\sum_{ijk} c_{ijk} x^{\mathbf{v}_{ijk}} = \sum_{ijk} h_{ij} g_{ik} = \sum_\alpha c_\alpha x^{\mathbf{m}_\alpha}$ which has no cancelling by construction. Since $p_{ijk} = p_\alpha$ if and only if $\mathbf{v}_{ijk} = \mathbf{m}_\alpha$, for every $\mathbf{v} \in \mathbb{N}^d$ we must also have

$$\sum_{\substack{i,j,k \\ \text{s.t. } \mathbf{v}_{ijk}=\mathbf{v}}} c_{ijk} y_{p_{ijk}} = \begin{cases} c_\alpha y_{p_\alpha} & \text{if } \mathbf{v} = \mathbf{m}_\alpha \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, when we sum over all i, j and k we must have

$$\sum_{\mathbf{v} \in \mathbb{N}^d} \sum_{\substack{i,j,k \\ \text{s.t. } \mathbf{v}_{ijk}=\mathbf{v}}} c_{ijk} y_{p_{ijk}} = \sum_{ijk} c_{ijk} y_{p_{ijk}} = \sum_\alpha c_\alpha y_{p_\alpha} \tag{5.1.2}$$

as claimed.

Crucially, $\sum_k c_{ijk} y_{p_{ijk}}$ is an element of the ideal generated by S_2. This is because each p_{ijk} has the same head and tail, and by (5.1.1)

$$\widetilde{\Phi_Q}\big(\sum_k c_{ijk} y_{p_{ijk}}\big) = \sum_k c_{ijk} x^{\mathbf{v}_{ijk}} = h_{ij} g_i$$

where $h_{ij} g_i$ is a term times g_i. By (5.1.2), $\sum_\alpha c_\alpha y_{p_\alpha}$ is a sum of elements of S_2, therefore $\sum_\alpha c_\alpha y_{p_\alpha}$ also lies in the ideal generated by S_2. □

5.1.5 Finding Generators of I_R

We describe an algorithm for computing I_R. We include pseudocode for the case where I_X is generated by quadratic polynomials with three terms, as in X_4, X_5, X_6 and Grassmannians $\mathrm{Gr}(r, n)$ in Appendix C. We introduce some notation. Let Q be a quiver, and let P denote the list of all paths in Q and suppose $I_X := (g_1, \dots, g_m)$.

Algorithm 5.1.8. *Input: the generators of I_X, the list P of all paths in Q. For $1 \leq i \leq m$, we consider the generator g_i. Suppose g_i has n_i terms:*

$$g_i = c_{i1} x^{\mathbf{m}_{i1}} + \cdots + c_{in_i} x^{\mathbf{m}_{in_i}}$$

where $c_{ij} \in \Bbbk$ and $\mathbf{m}_{ij} \in \mathbb{N}^d$ (recalling that d is the number of generators of $\mathrm{Cox}(\widetilde{X})$).

For each $1 \leq j \leq n_i$ we construct a list L_{ij} of containing information about each path whose label is divisible by $x^{\mathbf{m}_{ij}}$ as follows. For each j define $L_{ij} := [\,]$.
For each j and for $1 \leq k \leq |P|$ we check if $x^{\mathrm{div}(P[k])}$ is divisible by $x^{\mathbf{m}_{ij}}$. If it is, we add

$$[\mathsf{t}(P[k]), x^{\mathrm{div}(P[k])}/x^{\mathbf{m}_{ij}}, \mathsf{h}(P[k]), y_{P[k]}]$$

to L_{ij}. If it isn't, we don't. We define $\operatorname{div}(L_{ij}[n])$ *to be the second entry in* $L_{ij}[n]$.

Let $M_i := [\,]$. *For* $1 \leq k_{i1} \leq |L_{i1}|, \ldots, 1 \leq k_{in_i} \leq |L_{in_i}|$ *we check if*

$$\mathsf{t}(L_{i1}[k_{i1}]) = \cdots = \mathsf{t}(L_{in_i}[k_{in_i}]),$$
$$\mathsf{h}(L_{i1}[k_{i1}]) = \cdots = \mathsf{h}(L_{in_i}[k_{in_i}]),$$
$$\textit{and } \operatorname{div}(L_{i1}[k_{i1}]) = \cdots = \operatorname{div}(L_{in_i}[k_{in_i}]).$$

If this is the case then we add $c_{i1}y_{P[k_{i1}]} + \cdots + c_{in_i}y_{P[k_{in_i}]}$ *to* M_i. *If it isn't the case, we don't. The paths* $P[k_{ij}]$ *have the same head and tail for each* j *and*

$$\widetilde{\Phi}(c_{i1}y_{P[k_{i1}]} + \cdots + c_{in_i}y_{P[k_{in_i}]}) = \left(x^{\operatorname{div}(P[k_ij])}/x^{\mathbf{m}_{ij}}\right)\left(c_{i1}x^{\mathbf{m}_{i1}} + \cdots c_{in_i}x^{\mathbf{m}_{in_i}}\right) \in I_X$$

so $c_{i1}y_{P[k_{i1}]} + \cdots + c_{in_i}y_{P[k_{in_i}]}$ *is an element of* I_R. *By construction,* M_i *contains every sum* $\sum c_i y_{p_i}$ *such that* $\mathsf{t}(p_i) = \mathsf{t}(p_j), \mathsf{h}(p_i) = \mathsf{h}(p_j)$ *for all* i, j *and such that* $\sum c_i \widetilde{\Phi}(y_{p_i})$ *is a term multiplied* g_i. *If we repeat this process for every* g_i, *Lemma 5.1.7 tells us that the union of the* M_i*'s plus the output from Pseudocode 5.1.4 must therefore generate* I_R *(possibly with many redundant terms).*

Example 5.1.9. With Q as in Example 5.1.1, the output from Algorithm 5.1.8 is:

$$[y_1 - y_2 + y_3, y_8y_1 - y_8y_2 + y_8y_3, y_9y_4 - y_8y_2 + y_8y_3, y_4 - y_5 + y_6,$$
$$y_8y_1 - y_9y_5 + y_9y_6, y_9y_4 - y_9y_5 + y_9y_6, y_8y_2 - y_9y_5 + y_7, y_8y_3 - y_9y_6 + y_7].$$

Hence in this case

$$I_R = \left(y_8y_1 - y_9y_4, y_4 - y_5 + y_6, y_1 - y_2 + y_3, y_3y_8 - y_6y_9 + y_7, y_2y_8 - y_5y_9 + y_7\right).$$

5.2 Calculating I_Q and $\widetilde{I_Q}$ Using Macaulay 2

In this section we give a method for computing $\widetilde{I_Q}$ and I_Q explicitly.

5.2.1 I_Q and $\widetilde{I_Q}$ as Kernels

In this section we use the theory from Appendix A to calculate I_Q using Macaulay 2. In order to do this, we show that I_Q is the kernel of a $\Bbbk$-algebra homomorphism ψ:

$$\psi : \Bbbk[y_a | a \in Q_1] \longrightarrow \Bbbk[x_1, \ldots, x_d, t_i, h_i | i \in Q_0]/I_X + A$$
$$y_a \mapsto t_{\mathsf{t}(a)} x^{\operatorname{div}(a)} h_{\mathsf{h}(a)}$$

where

$$A = \left(t_i h_i - 1 | i \in Q_0\right).$$

First we need a technical lemma:

Lemma 5.2.1. *Let $f \in \Bbbk[y_a | a \in Q_1]$ be homogeneous of weight $\chi \in \operatorname{inc}(\mathrm{Wt}(Q)) \setminus \{0\}$ and let $n := n_\chi$. We consider the map:*

$$\overline{\psi} : \Bbbk[y_a | a \in Q_1] \longrightarrow \Bbbk[x_1, \ldots, x_d, t_i, h_i]/A$$

$$y_a \mapsto t_{\mathsf{t}(a)} x^{\operatorname{div}(a)} h_{\mathsf{h}(a)}.$$

The image of f satisfies

$$\overline{\psi}(f) = t_{i_1} \cdots t_{i_n} h_{j_1} \cdots h_{j_n} g(x_1, \ldots, x_d)$$

where $i_1, \ldots, i_n, j_1, \ldots, j_n \in Q_0$.

Proof. Since f is homogeneous, we can decompose f into terms, each of weight χ. By Remark 4.2.3 (ii), for each term we have

$$f = \sum_{\beta=1}^{k} c_\beta \prod_{\alpha=1}^{n} y_{p_{\alpha\beta}}$$

where $c_\beta \in \Bbbk$, the $p_{\alpha\beta}$'s are paths where χ_i^+ of the $p_{\alpha\beta}$'s have head at $i \in Q_0$ and χ_i^- of the $p_{\alpha\beta}$'s have tail at $i \in Q_0$.

For each $y_{p_{\alpha\beta}}$ we have

$$\overline{\psi}(y_{p_{\alpha\beta}}) = t_{\mathsf{t}(p_{\alpha\beta})} x^{\operatorname{div}(p_{\alpha\beta})} h_{\mathsf{h}(p_{\alpha\beta})}$$

since we are working modulo A. So for any β:

$$\overline{\psi}\Big(\prod_{\alpha=1}^{k} y_{p_{\alpha\beta}}\Big) = \prod_{\alpha=1}^{n} t_{\mathsf{t}(p_{\alpha\beta})} h_{\mathsf{h}(p_{\alpha\beta})} x^{\operatorname{div}(p_{\alpha\beta})}.$$

Now since $\prod_{\alpha=1}^{n} t_{\mathsf{t}(p_{\alpha\beta})} h_{\mathsf{h}(p_{\alpha\beta})}$ depends only on χ, this is a common factor for $\overline{\psi}(\prod_{\alpha=1}^{k} y_{p_{\alpha\beta}})$ for each β. Hence, summing over β we have:

$$\overline{\psi}(f) = \prod_{\alpha=1}^{n} \left(t_{\mathsf{t}(p_{\alpha\beta})} h_{\mathsf{h}(p_{\alpha\beta})}\right) \times \left(\sum_{\beta=1}^{k} c_\beta \prod_{\alpha=1}^{n} x^{\operatorname{div}(p_{\alpha\beta})}\right).$$

Letting $g(x) := \sum_{\beta=1}^{k} c_\beta \prod_{\alpha=1}^{n} x^{\operatorname{div}(p_{\alpha\beta})}$ we have the statement of the Lemma. □

Proposition 5.2.2. *The kernel of ψ is equal to I_Q.*

Proof. We note that the kernel of ψ is precisely the set:

$$\ker \psi = \{ f \in \Bbbk[y_a | a \in Q_1] | \overline{\psi}(f) \in I_X \}$$

where I_X is considered as an ideal of $\Bbbk[x_1, \ldots, x_d, t_i, h_i]/A$.

First we show $I_Q \subseteq \ker(\psi)$. Let f be an element of I_Q. We assume f is homogeneous of weight $\chi \in \mathrm{inc}(\mathbb{N}^{Q_1}) \setminus \{0\}$ and let $n := n_\chi$. By Remark 4.2.3 (ii), we have

$$f = \sum_{\beta=1}^{k} c_\beta \prod_{\alpha=1}^{n} y_{p_{\alpha\beta}}$$

and by the proof of Lemma 5.2.1 $\overline{\psi}(f) = t_{i_1} \cdots t_{i_n} h_{j_1} \cdots h_{j_n} g(x)$ where $i_1, \ldots, i_n, j_1, \ldots, j_n \in Q_0$ and where $g(x) = \sum_{\beta=1}^{k} c_\beta \prod_{\alpha=1}^{n} x^{\mathrm{div}(p_{\alpha\beta})}$. Now, since $f \in I_Q$, we have that $\widetilde{\Phi}(f) \in I_X$. This means

$$\widetilde{\Phi}(f) = \widetilde{\Phi}\bigg(\sum_{\beta=1}^{k} c_\beta \prod_{\alpha=1}^{n} y_{p_{\alpha\beta}}\bigg) = \sum_{\beta=1}^{k} c_\beta \prod_{\alpha=1}^{n} \widetilde{\Phi}(y_{p_{\alpha\beta}}) = \sum_{\beta=1}^{k} c_\beta \prod_{\alpha=1}^{n} x^{\mathrm{div}(p_{\alpha\beta})} = g(x) \in I_X.$$

Hence $f \in \ker(\psi)$.

Now to show opposite inclusion let $f \in \ker(\psi)$ be homogeneous of weight χ. So

$$f = \sum_{\beta=1}^{k} c_\beta \prod_{\alpha=1}^{n} y_{p_{\alpha\beta}}$$

where $c_\beta \in \Bbbk$, the $p_{\alpha\beta}$'s are paths where χ_i^+ of the $p_{\alpha\beta}$'s have head at $i \in Q_0$ and χ_i^- of the $p_{\alpha\beta}$'s have tail at $i \in Q_0$. Also, $\overline{\psi}(f) = t_{i_1} \cdots t_{i_n} h_{j_1} \cdots h_{j_n} g(x) \in I_X$ where $i_1, \ldots, i_n, j_1, \ldots, j_n \in Q_0$. I_X is generated by $g_1(x_1, \ldots, x_d), \ldots, g_m(x_1, \ldots, x_d)$, so

$$\begin{gathered} t_{i_1} \cdots t_{i_n} h_{j_1} \cdots h_{j_n} g(\mathbf{x}) = \\ f_1(\mathbf{x}, \mathbf{t}, \mathbf{h}) g_1(\mathbf{x}) + \cdots + f_m(\mathbf{x}, \mathbf{t}, \mathbf{h}) g_m(\mathbf{x}) \end{gathered}$$

for some $f_1, \ldots, f_m \in \Bbbk[x_1, \ldots, x_d, t_i, h_i]/A$, where $\mathbf{x} = (x_1, \ldots, x_d), \mathbf{t} = (t_0, \ldots, t_r)$ and $\mathbf{h} = (h_0, \ldots, h_r)$. Substituting $t_i = 1, h_i = 1$ for all $i \in Q_0$ we obtain:

$$g(x) = f_1(\mathbf{x}, 1, \ldots, 1) g_1(\mathbf{x}) + \cdots + f_m(\mathbf{x}, 1, \ldots, 1) g_m(\mathbf{x})$$

hence $g(\mathbf{x}) \in I_X$.

By the proof of Lemma 5.2.1, we also have

$$g(\mathbf{x}) = \sum_{\beta=1}^{k} c_\beta \prod_{\alpha=1}^{n} x^{\mathrm{div}(p_{\alpha\beta})} = \widetilde{\Phi}(f) \in I_X.$$

Hence $f \in I_Q$ since it is homogeneous by assumption. $\square$

We note that these results also apply to the toric case by setting $I_X = (0)$.

5.2.2 Macaulay 2 Code

We present Macaulay 2 code for computing I_Q and $\widetilde{I_Q}$. Let $q := |Q_1|$ and denote I_Q by IQ, and $\widetilde{I_Q}$ by IQtilde

```
i1:  R = QQ[x_1..x_d,t_0..t_r,h_0..h_r,y_1..y_q,MonomialOrder => Eliminate d+2*(r+1)]
i2:  K = ideal(y_1-x^div(a_1),... y_m-x^div(a_1))
i3:  I = ideal(g_1,...g_m, t_1*h_1-1,..., t_r*h_r-1)
i4:  Itilde = ideal(t_0*h_0-1,..., t_r*h_r-1)
i5:  H = K+I
i5:  G = gens gb H
i6:  J = selectInSubring(1,G)
i7:  IQ = ideal(J)
i8:  Htilde = K+Itilde
i9:  Gtilde = gens gb Htilde
i10:  Jtilde = selectInSubring(1,Gtilde)
i11:  IQtilde = ideal(Jtilde)
```

Chapter 6

Examples of Mori Dream Spaces as Fine Moduli of Quiver Representations

As an application of our main results we illustrate how to reconstruct del Pezzo surfaces directly from the bound quiver of sections of a collection of line bundles whose direct sum is a tilting bundle.

6.1 Tilting bundles on del Pezzo surfaces

Let X be a smooth projective variety over $\Bbbk$ and write $\mathrm{coh}(X)$ for the category of coherent sheaves on X. For any vector bundle $\mathscr{T}$ on X, let $A := \mathrm{End}_{\mathcal{O}_X}(\mathscr{T})$ denote its endomorphism algebra and $\mathrm{mod}(A)$ the abelian category of finitely generated right A-modules. We say that $\mathscr{T}$ is a *tilting bundle* on X if the functor

$$\mathbf{R}\mathrm{Hom}(\mathscr{T}, -)\colon D^b\big(\mathrm{coh}(X)\big) \longrightarrow D^b\big(\mathrm{mod}(A)\big)$$

is an exact equivalence of bounded derived categories. If $\mathscr{T}$ decomposes as a direct sum of line bundles $\mathscr{T} = \bigoplus_{0 \leq i \leq r} L_i$ (we need not assume that each L_i has rank one, but we choose to), then after reordering if necessary, the collection $(L_0, L_1, \ldots, L_r)$ is a full, strongly exceptional sequence on X. That is, the line bundles in the collection generate $D^b(\mathrm{coh}(X))$ and they satisfy appropriate Ext-vanishing conditions, namely, that $\mathrm{Hom}(L_j, L_i) = 0$ for $j > i$ and that $\mathrm{Ext}^k(L_i, L_j) = 0$ for $k > 0$ and all $0 \leq i, j \leq r$.

For $0 \leq k \leq 8$, let X_k denote the del Pezzo surface obtained as the blow-up of $\mathbb{P}^2_{\Bbbk}$ at k points in general position. The Picard group $\mathrm{Cl}(X_k) \cong \mathbb{Z}^{k+1}$ has a basis given by l_0, the pullback to X_d of the hyperplane class on $\mathbb{P}^2_{\Bbbk}$, together with the k exceptional curves $l_1, \ldots, l_k$. Consider the sequence of basepoint-free line bundles

$$\mathscr{L}_k := \big(\mathcal{O}_{X_k}, l_0, 2l_0 - l_1, \dots, 2l_0 - l_k, 2l_0\big) \tag{6.1.1}$$

on X_k, and write $L_0 = \mathcal{O}_{X_k}$, $L_1 = l_0$, $L_{i+1} = 2l_0 - l_i$ for $1 \leq i \leq k$, and $L_{k+2} = 2l_0$. The following result is well known. We guide the reader towards a proof.

Lemma 6.1.1. *The sequence of line bundles* (6.1.1) *on X_k is full and strongly exceptional, so the vector bundle $\mathscr{T}_k := \bigoplus_{0\leq i\leq k+2} L_i$ is tilting.*

Proof. We use the technology of toric systems developed by Hille–Perling [18]. Beginning with the unique toric system l_0, l_0, l_0 on $\mathbb{P}^2_{\Bbbk}$, construct a toric system on each X_k as follows: choose $l_0, l_0 - l_1, l_1, l_0 - l_1$ on X_1, then repeat for $k \geq 2$, introducing l_k in the second-last position while subtracting l_k from each neighbouring divisor to obtain the toric system

$$l_0, l_0 - l_1, l_1 - l_2, l_2 - l_3, \dots, l_{k-1} - l_k, l_k, l_0 - \sum_{1\leq i\leq k} l_i$$

on X_k. List these divisors from left to right as $D_1, \dots, D_{k+3}$. Observe that for $1 \leq i \leq k+2$ we have $L_i = \mathscr{O}(D_1 + \cdots + D_i)$, and $-K_{X_k} = \mathscr{O}(D_1 + \cdots + D_{k+3})$, and Theorem 5.7 of Hille–Perling [18] establishes that the sequence $(L_0, L_1, \dots, L_{k+2})$ is full and strongly exceptional as required. □

Let (Q_k, J_k) denote the bound quiver of sections of the collection $\mathscr{L}_k$ on X_k. For $k \leq 3$, the variety X_k is toric, in which case $\mathscr{L} = \widetilde{\mathscr{L}_k}$ and the method of Craw–Smith [10] shows that the morphism $\varphi_{|\mathscr{L}_k|}\colon X_k \to \mathcal{M}_\vartheta(Q_k, J_k)$ is an isomorphism. We now consider the cases where $k = 4$ and 5. We were unable to compute the case $k = 6$ due to computational complexity.

We also consider a collection of line bundles $\mathscr{L}$ on $\mathrm{Gr}(2,4)$ which gives an isomorphism with the moduli space of bound quiver representations for the quiver of sections of $\mathscr{L}$.

6.2 X_4 Tilting Example

On X_4, a strong exceptional collection of line bundles is $\mathscr{L} := (\mathcal{O}_{X_4}, l_0, 2l_0 - l_1, 2l_0 - l_2, 2l_0 - l_3, 2l_0 - l_4, 2l_0)$ where notation is as in Section 2.2.2. The quiver of sections for $\mathscr{L}$ is given in Figure 6.1. Arrows with tail at 0 are listed $a_1, \dots, a_6$ from the top of Figure 6.1 to the bottom; list those with tail at 1 as $a_7, \dots, a_{18}$ from the top of the figure to the bottom; and list those with head at 6 as $a_{19}, \dots, a_{22}$ from the top to the bottom. Likewise, list the coordinates of $\mathbb{A}^{Q_1}_{\Bbbk}$ as $y_1, \dots, y_{22}$.

Using the methods described in sections 5.1 and 5.2 , we calculated $\widetilde{I_R}, I_R, \widetilde{I_Q}$, and I_Q. We compute B_Y by computing the intersection

$$B_Y = \bigcap_{i\in Q_0} \big(y_{a_j} \in \Bbbk[y_a | a \in Q_1] \,|\, \mathsf{h}(a_j) = i\big).$$

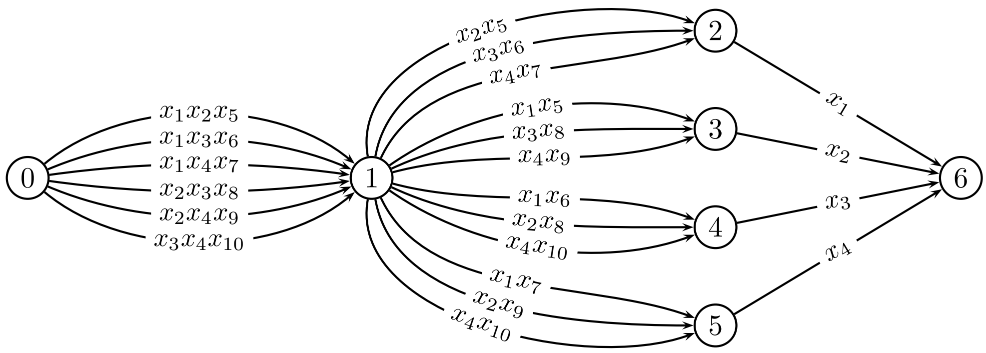


Figure 6.1: A quiver of sections for a collection on X_4

We then compared $\left(\widetilde{I_R} : B_Y^\infty\right)$ to $\widetilde{I_Q}$, and $\left(I_R : B_Y^\infty\right)$ to I_Q. The results were as follows:

$$\widetilde{I_R} = \left(\begin{array}{c} y_{15}y_{21} - y_{18}y_{22}, y_{12}y_{20} - y_{17}y_{22}, y_{11}y_{20} - y_{14}y_{21}, y_9y_{19} - y_{16}y_{22}, \\ y_8y_{19} - y_{13}y_{21}, y_7y_{19} - y_{10}y_{20}, y_6y_{17} - y_5y_{18}, y_6y_{16} - y_3y_{18}, \\ y_5y_{16} - y_3y_{17}, y_6y_{14} - y_4y_{15}, y_6y_{13} - y_2y_{15}, y_4y_{13} - y_2y_{14}, y_5y_{11} - y_4y_{12}, \\ y_5y_{10} - y_1y_{12}, y_4y_{10} - y_1y_{11}, y_3y_8 - y_2y_9, y_3y_7 - y_1y_9, y_2y_7 - y_1y_8, \\ y_3y_{14}y_{21} - y_4y_{16}y_{22}, y_5y_{13}y_{21} - y_2y_{17}y_{22}, y_6y_{10}y_{20} - y_1y_{18}y_{22} \end{array} \right)$$

$$I_R = \left(\begin{array}{c} y_{15}y_{21} - y_{18}y_{22}, y_{12}y_{20} - y_{17}y_{22}, y_{11}y_{20} - y_{14}y_{21}, y_9y_{19} - y_{16}y_{22}, \\ y_8y_{19} - y_{13}y_{21}, y_7y_{19} - y_{10}y_{20}, y_6y_{17} - y_5y_{18}, y_6y_{16} - y_3y_{18}, \\ y_5y_{16} - y_3y_{17}, y_6y_{14} - y_4y_{15}, y_6y_{13} - y_2y_{15}, y_4y_{13} - y_2y_{14}, y_5y_{11} - y_4y_{12}, \\ y_5y_{10} - y_1y_{12}, y_4y_{10} - y_1y_{11}, y_3y_8 - y_2y_9, y_3y_7 - y_1y_9, y_2y_7 - y_1y_8, \\ y_3y_{14}y_{21} - y_4y_{16}y_{22}, y_5y_{13}y_{21} - y_2y_{17}y_{22}, y_6y_{10}y_{20} - y_1y_{18}y_{22}, \\ y_{16} - y_{17} + y_{18}, y_{13} - y_{14} + y_{15}, y_{10} - y_{11} + y_{12}, y_7 - y_8 + y_9, y_3 - y_5 + y_6, \\ y_2 - y_4 + y_6, y_1 - y_4 + y_5, y_{15}y_{21} - y_{18}y_{22}, y_{12}y_{20} - y_{17}y_{22}, y_{11}y_{20} - y_{14}y_{21}, \\ y_9y_{19} - y_{17}y_{22} + y_{18}y_{22}, y_8y_{19} - y_{14}y_{21} + y_{18}y_{22}, y_6y_{17} - y_5y_{18}, \\ y_6y_{14} - y_4y_{15}, y_5y_{11} - y_4y_{12}, y_5y_8 - y_6y_8 - y_4y_9 + y_6y_9 \end{array} \right)$$

$$\widetilde{I_Q} = \left(\begin{array}{c} y_{15}y_{21} - y_{18}y_{22}, y_{12}y_{20} - y_{17}y_{22}, y_{11}y_{20} - y_{14}y_{21}, y_9y_{19} - y_{16}y_{22}, \\ y_8y_{19} - y_{13}y_{21}, y_7y_{19} - y_{10}y_{20}, y_6y_{17} - y_5y_{18}, y_6y_{16} - y_3y_{18}, y_5y_{16} - y_3y_{17}, \\ y_6y_{14} - y_4y_{15}, y_6y_{13} - y_2y_{15}, y_4y_{13} - y_2y_{14}, y_5y_{11} - y_4y_{12}, y_5y_{10} - y_1y_{12}, \\ y_4y_{10} - y_1y_{11}, y_3y_8 - y_2y_9, y_3y_7 - y_1y_9, y_2y_7 - y_1y_8, y_3y_{14}y_{21} - y_4y_{16}y_{22}, \\ y_5y_{13}y_{21} - y_2y_{17}y_{22}, y_6y_{10}y_{20} - y_1y_{18}y_{22}, y_{11}y_{15}y_{17} - y_{12}y_{14}y_{18}, \\ y_8y_{15}y_{16} - y_9y_{13}y_{18}, y_7y_{12}y_{16} - y_9y_{10}y_{17}, y_7y_{11}y_{13} - y_8y_{10}y_{14}, \\ y_8y_{10}y_{15}y_{17} - y_7y_{12}y_{13}y_{18}, y_7y_{11}y_{15}y_{16} - y_9y_{10}y_{14}y_{18}, \\ y_8y_{12}y_{14}y_{16} - y_9y_{11}y_{13}y_{17} \end{array} \right)$$

$$I_Q = \left(\begin{array}{c} y_{15}y_{21} - y_{18}y_{22}, y_{12}y_{20} - y_{17}y_{22}, y_{11}y_{20} - y_{14}y_{21}, y_9y_{19} - y_{16}y_{22}, \\ y_8y_{19} - y_{13}y_{21}, y_7y_{19} - y_{10}y_{20}, y_6y_{17} - y_5y_{18}, y_6y_{16} - y_3y_{18}, y_5y_{16} - y_3y_{17}, \\ y_6y_{14} - y_4y_{15}, y_6y_{13} - y_2y_{15}, y_4y_{13} - y_2y_{14}, y_5y_{11} - y_4y_{12}, y_5y_{10} - y_1y_{12}, \\ y_4y_{10} - y_1y_{11}, y_3y_8 - y_2y_9, y_3y_7 - y_1y_9, y_2y_7 - y_1y_8, y_3y_{14}y_{21} - y_4y_{16}y_{22}, \\ y_5y_{13}y_{21} - y_2y_{17}y_{22}, y_6y_{10}y_{20} - y_1y_{18}y_{22}, y_{11}y_{15}y_{17} - y_{12}y_{14}y_{18}, \\ y_8y_{15}y_{16} - y_9y_{13}y_{18}, y_7y_{12}y_{16} - y_9y_{10}y_{17}, y_7y_{11}y_{13} - y_8y_{10}y_{14}, \\ y_8y_{10}y_{15}y_{17} - y_7y_{12}y_{13}y_{18}, y_7y_{11}y_{15}y_{16} - y_9y_{10}y_{14}y_{18}, \\ y_8y_{12}y_{14}y_{16} - y_9y_{11}y_{13}y_{17}, y_{16} - y_{17} + y_{18}, y_{13} - y_{14} + y_{15}, y_{10} - y_{11} + y_{12}, \\ y_7 - y_8 + y_9, y_3 - y_5 + y_6, y_2 - y_4 + y_6, y_1 - y_4 + y_5, y_{15}y_{21} - y_{18}y_{22}, y_{12}y_{20} - y_{17}y_{22}, \\ y_{11}y_{20} - y_{14}y_{21}, y_9y_{19} - y_{17}y_{22} + y_{18}y_{22}, y_8y_{19} - y_{14}y_{21} + y_{18}y_{22}, \\ y_6y_{17} - y_5y_{18}, y_6y_{14} - y_4y_{15}, y_5y_{11} - y_4y_{12}, y_5y_8 - y_6y_8 - y_4y_9 + y_6y_9, \\ y_{11}y_{15}y_{17} - y_{12}y_{14}y_{18}, y_8y_{15}y_{17} - y_9y_{14}y_{18} - y_8y_{15}y_{18} + y_9y_{15}y_{18}, \\ y_9y_{11}y_{17} - y_8y_{12}y_{17} + y_8y_{12}y_{18} - y_9y_{12}y_{18}, \\ y_9y_{11}y_{14} - y_8y_{12}y_{14} + y_8y_{11}y_{15} - y_9y_{11}y_{15} \end{array}\right)$$

B_Y is the intersection of the ideals:

$$(y_1, \ldots, y_6), (y_7, y_8, y_9), (y_{10}, y_{11}, y_{12}), (y_{13}, y_{14}, y_{15}), (y_{16}, y_{17}, y_{18}) \text{ and } (y_{19}, y_{20}, y_{21}, y_{22}).$$

We present Macaulay 2 code for computing I_Q and $\widetilde{I_Q}$.

```
i1:  R = QQ[x_1..x_10,t_0..t_6,h_0..h_6,y_1..y_22, MonomialOrder => Eliminate 24]
i2:  H = K+I
i4:  G = gens gb H
i5:  J = selectInSubring(1,G)
i6:  IQ = ideal(J)
i6:  Htilde= K +Itilde
i7:  Gtilde = gens gb K
i8:  Jtilde = selectInSubring(1,Gtilde)
i9:  IQtilde = ideal(Jtilde)
```

where

$$K = \left(\begin{array}{c} y_1 - t_0h_1x_1x_2x_5, y_2 - t_0h_1x_1x_3x_6, y_3 - t_0h_1x_1x_4x_7, y_4 - t_0h_1x_2x_3x_8, \\ y_5 - t_0h_1x_2x_4x_9, y_6 - t_0h_1x_3x_4x_{10}, y_7 - t_1h_2x_2x_5, y_8 - t_1h_2x_3x_6, \\ y_9 - t_1h_2x_4x_7, y_{10} - t_1h_3x_1x_5, y_{11} - t_1h_3x_3x_8, y_{12} - t_1h_3x_4x_9, \\ y_{13} - t_1h_4x_1x_6, y_{14} - t_1h_4x_2x_8, y_{15} - t_1h_4x_4x_{10}, y_{16} - t_1h_5x_1x_7, \\ y_{17} - t_1h_5x_2x_9, y_{18} - t_1h_5x_3x_{10}, y_{19} - t_2h_6x_1, y_{20} - t_3h_6x_2, y_{21} - t_4h_6x_3, y_{22} - t_5h_6x_4 \end{array}\right),$$

$$I = \left(\begin{array}{c} x_2x_5 - x_3x_6 + x_4x_7, x_1x_5 - x_3x_8 + x_4x_9, x_1x_6 - x_2x_8 + x_4x_{10}, \\ x_1x_7 - x_2x_9 + x_3x_{10}, x_5x_{10} - x_6x_9 + x_7x_8, t_0h_0 - 1, \dots, t_6h_6 - 1 \end{array} \right)$$

and

$$\widetilde{I} = \left(\; t_0h_0 - 1, \dots, t_6h_6 - 1 \; \right).$$

In Macaulay 2, we calculate the saturation of I_R and I_Q with B_Y using the command "saturate",

```
i1:  IQQ = saturate(IQ,BY)
i2:  IRR = saturate(IR,BY)
i3:  IRR == IQQ
o3:  true
```

In the same way we obtain $\widetilde{I_Q} = \widetilde{I_R} : B_Y^\infty$. Example 3.3.4 showed that $\mathscr{L}_4$ is very ample, so Theorem 4.1.1 implies that $\varphi_{|\mathscr{L}_4|} \colon X_4 \longrightarrow \mathcal{M}_\vartheta(\mathrm{mod}(A_{\mathscr{L}_4}))$ is an isomorphism.

6.3 X_5 Tilting Example

On X_5, a strong exceptional collection of line bundles is $\mathscr{L} := (\mathcal{O}_{X_5}, l_0, 2l_0 - l_1, 2l_0 - l_2, 2l_0 - l_3, 2l_0 - l_4, 2l_0 - l_5, 2l_0)$ where notation is as in section 2.2.2. The quiver of sections Q is shown in Figure 6.2 (in fact we omit one arrow labelled $x_1x_2x_4x_5x_{16}$ with tail at 0 and head at 4 to prevent the figure from becoming illegible). Arrows with tail at 0 and head at 1 are listed $a_1, \dots, a_{10}$ from the top of Figure 6.2 to the bottom; list those with tail at 1 as $a_{11}, \dots, a_{30}$ from top to bottom; list

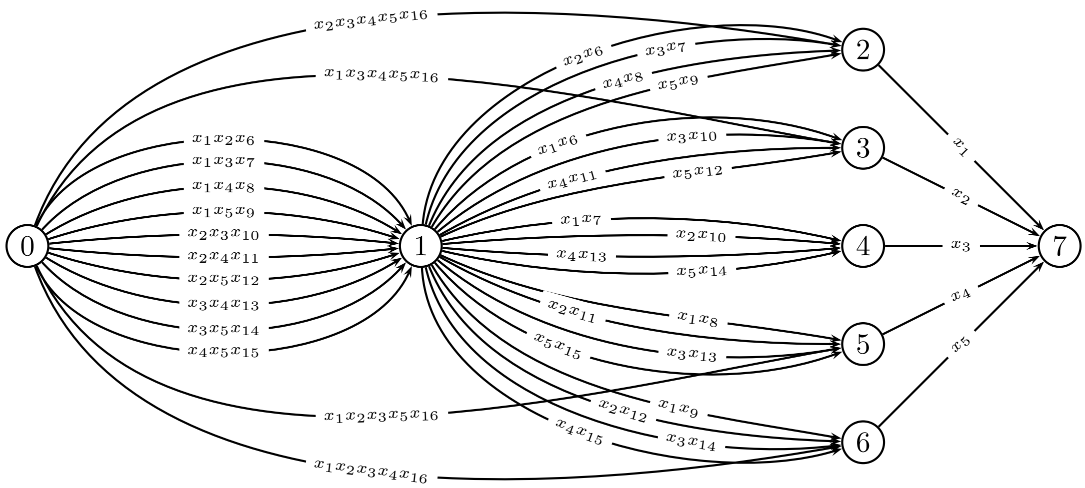


Figure 6.2: A quiver of sections for a full strongly exceptional collection on X_5

those with head at 7 as $a_{31}, \dots, a_{35}$ from top to bottom; and list those with tail at 0 and head at

$i \geq 2$ as $a_{36}, \ldots, a_{40}$ from top to bottom, where the arrow omitted from the figure is a_{38}. List the coordinates of $\mathbb{A}_{\mathbb{k}}^{Q_1}$ as $y_1, \ldots, y_{40}$

Using the methods described in sections 5.1 and 5.2, we calculated $\widetilde{I_R}, I_R, \widetilde{I_Q}, I_Q, B_Y,$ and compared $\left(\widetilde{I_R} : B_Y^{\infty}\right)$ to $\widetilde{I_Q}$, and $\left(I_R : B_Y^{\infty}\right)$ to I_Q. The results were as follows:

$$\widetilde{I_R} = \left(\begin{array}{c} y_{34}y_{39} - y_{35}y_{40}, y_{33}y_{38} - y_{35}y_{40}, y_{32}y_{37} - y_{35}y_{40}, \\ y_{31}y_{36} - y_{35}y_{40}, y_{26}y_{34} - y_{30}y_{35}, y_{22}y_{33} - y_{29}y_{35}, \\ y_{21}y_{33} - y_{25}y_{34}, y_{18}y_{32} - y_{28}y_{35}, y_{17}y_{32} - y_{24}y_{34}, \\ y_{16}y_{32} - y_{20}y_{33}, y_{14}y_{31} - y_{27}y_{35}, y_{13}y_{31} - y_{23}y_{34}, \\ y_{12}y_{31} - y_{19}y_{33}, y_{11}y_{31} - y_{15}y_{32}, y_{10}y_{29} - y_{9}y_{30}, \\ y_{10}y_{28} - y_{7}y_{30}, y_{9}y_{28} - y_{7}y_{29}, y_{10}y_{27} - y_{4}y_{30}, \\ y_{9}y_{27} - y_{4}y_{29}, y_{7}y_{27} - y_{4}y_{28}, y_{10}y_{25} - y_{8}y_{26}, \\ y_{10}y_{24} - y_{6}y_{26}, y_{8}y_{24} - y_{6}y_{25}, y_{10}y_{23} - y_{3}y_{26}, \\ y_{8}y_{23} - y_{3}y_{25}, y_{6}y_{23} - y_{3}y_{24}, y_{9}y_{21} - y_{8}y_{22}, y_{9}y_{20} - y_{5}y_{22}, \\ y_{8}y_{20} - y_{5}y_{21}, y_{9}y_{19} - y_{2}y_{22}, y_{8}y_{19} - y_{2}y_{21}, y_{5}y_{19} - y_{2}y_{20}, \\ y_{7}y_{17} - y_{6}y_{18}, y_{7}y_{16} - y_{5}y_{18}, y_{6}y_{16} - y_{5}y_{17}, y_{7}y_{15} - y_{1}y_{18}, \\ y_{6}y_{15} - y_{1}y_{17}, y_{5}y_{15} - y_{1}y_{16}, y_{4}y_{13} - y_{3}y_{14}, y_{4}y_{12} - y_{2}y_{14}, \\ y_{3}y_{12} - y_{2}y_{13}, y_{4}y_{11} - y_{1}y_{14}, y_{3}y_{11} - y_{1}y_{13}, y_{2}y_{11} - y_{1}y_{12}, \\ y_{7}y_{25}y_{34} - y_{8}y_{28}y_{35}, y_{4}y_{25}y_{34} - y_{8}y_{27}y_{35}, \\ y_{9}y_{24}y_{34} - y_{6}y_{29}y_{35}, y_{4}y_{24}y_{34} - y_{6}y_{27}y_{35}, \\ y_{9}y_{23}y_{34} - y_{3}y_{29}y_{35}, y_{7}y_{23}y_{34} - y_{3}y_{28}y_{35}, \\ y_{10}y_{20}y_{33} - y_{5}y_{30}y_{35}, y_{4}y_{20}y_{33} - y_{5}y_{27}y_{35}, \\ y_{3}y_{20}y_{33} - y_{5}y_{23}y_{34}, y_{10}y_{19}y_{33} - y_{2}y_{30}y_{35}, \\ y_{7}y_{19}y_{33} - y_{2}y_{28}y_{35}, y_{6}y_{19}y_{33} - y_{2}y_{24}y_{34}, \\ y_{10}y_{15}y_{32} - y_{1}y_{30}y_{35}, y_{9}y_{15}y_{32} - y_{1}y_{29}y_{35}, y_{8}y_{15}y_{32} - y_{1}y_{25}y_{34} \end{array}\right)$$

$$I_R = \left(\begin{array}{c} y_7 + 2y_9 - y_{10}, y_6 - 2y_8 + y_{10}, y_5 + y_8 - y_9, y_4 + y_9 - 2y_{10}, y_3 - y_8 + y_{10} \\ y_2 + 2y_8 - y_9, y_1 + 3y_8 - y_9 - y_{10}, y_{28} + 2y_{29} - y_{30}, y_{27} + y_{29} - 2y_{30} \\ y_{24} - 2y_{25} + y_{26}, y_{23} - y_{25} + y_{26}, y_{20} + y_{21} - y_{22}, y_{19} + 2y_{21} - y_{22} \\ 2y_{16} + y_{17} + y_{18}, 2y_{15} + 3y_{17} + y_{18}, y_{12} + 2y_{13} + y_{14}, y_{11} + 3y_{13} + y_{14} \\ y_{10}y_{29} - y_9y_{30}, 2y_8y_{29} + 2y_8y_{30} - 3y_9y_{30} - y_{40}, y_{10}y_{25} - y_8y_{26} \\ 2y_9y_{25} + 2y_8y_{26} - 3y_9y_{26} - y_{39}, 2y_{10}y_{21} + 2y_8y_{22} - 3y_{10}y_{22} - y_{38} \\ y_9y_{21} - y_8y_{22}, 3y_{10}y_{17} - 2y_8y_{18} + 3y_{10}y_{18} - 2y_{37}, 3y_9y_{17} + 2y_8y_{18} - y_{37} \\ 6y_{10}y_{13} - 2y_8y_{14} + 3y_{10}y_{14} - y_{36}, 3y_9y_{13} + y_8y_{14} - y_{36}, y_{26}y_{34} - y_{30}y_{35} \\ y_{22}y_{33} - y_{29}y_{35}, y_{21}y_{33} - y_{25}y_{34}, y_{18}y_{32} + 2y_{29}y_{35} - y_{30}y_{35} \\ y_{17}y_{32} - 2y_{25}y_{34} + y_{30}y_{35}, y_{14}y_{31} + y_{29}y_{35} - 2y_{30}y_{35} \\ y_{13}y_{31} - y_{25}y_{34} + y_{30}y_{35} \end{array}\right)$$

$$\widetilde{I_Q} = \left(\begin{array}{c} 2y_{17}y_{26}y_{29} + 2y_{18}y_{25}y_{30} - y_{17}y_{26}y_{30} - y_{18}y_{26}y_{30}, \\ y_{13}y_{26}y_{29} + y_{14}y_{25}y_{30} - 2y_{13}y_{26}y_{30} - y_{14}y_{26}y_{30}, \\ 2y_{18}y_{21}y_{29} + 2y_{17}y_{22}y_{29} - y_{17}y_{22}y_{30} - y_{18}y_{22}y_{30}, \\ y_{14}y_{21}y_{29} + y_{13}y_{22}y_{29} - 2y_{13}y_{22}y_{30} - y_{14}y_{22}y_{30}, \\ 2y_{14}y_{17}y_{29} - 2y_{13}y_{18}y_{29} - y_{14}y_{17}y_{30} + 4y_{13}y_{18}y_{30} + y_{14}y_{18}y_{30}, \\ 2y_{18}y_{21}y_{25} + 2y_{17}y_{22}y_{25} - y_{17}y_{21}y_{26} - y_{18}y_{21}y_{26}, \\ y_{14}y_{21}y_{25} + y_{13}y_{22}y_{25} - 2y_{13}y_{21}y_{26} - y_{14}y_{21}y_{26}, \\ 2y_{14}y_{17}y_{25} - 2y_{13}y_{18}y_{25} - 3y_{13}y_{17}y_{26} - 2y_{14}y_{17}y_{26} + y_{13}y_{18}y_{26} \\ y_{14}y_{17}y_{21} - 4y_{13}y_{18}y_{21} - y_{14}y_{18}y_{21} - 3y_{13}y_{17}y_{22} - 2y_{14}y_{17}y_{22} + y_{13}y_{18}y_{22} \end{array} \right)$$

$$I_Q = \left(\begin{array}{c} y_7 + 2y_9 - y_{10}, y_6 - 2y_8 + y_{10}, y_5 + y_8 - y_9, y_4 + y_9 - 2y_{10}, y_3 - y_8 + y_1, \\ y_2 + 2y_8 - y_9, y_1 + 3y_8 - y_9 - y_{10}, y_{28} + 2y_{29} - y_{30}, y_{27} + y_{29} - 2y_{30}, \\ y_{24} - 2y_{25} + y_{26}, y_{23} - y_{25} + y_{26}, y_{20} + y_{21} - y_{22}, \\ y_{19} + 2y_{21} - y_{22}, 2y_{16} + y_{17} + y_{18}, 2y_{15} + 3y_{17} + y_{18}, \\ y_{12} + 2y_{13} + y_{14}, y_{11} + 3y_{13} + y_{14}, 10y_{29} - y_9y_{30}, \\ 2y_8y_{29} + 2y_8y_{30} - 3y_9y_{30} - y_{40}, y_{10}y_{25} - y_8y_{26}, \\ 2y_9y_{25} + 2y_8y_{26} - 3y_9y_{26} - y_{39}, 2y_{10}y_{21} + 2y_8y_{22} - 3y_{10}y_{22} - y_{38}, \\ y_9y_{21} - y_8y_{22}, 3y_{10}y_{17} - 2y_8y_{18} + 3y_{10}y_{18} - 2y_{37}, \\ 3y_9y_{17} + 2y_8y_{18} - y_{37}, 6y_{10}y_{13} - 2y_8y_{14} + 3y_{10}y_{14} - y_{36}, \\ 3y_9y_{13} + y_8y_{14} - y_{36}, y_{30}y_{35} - y_{10}, y_{29}y_{35} - y_9, y_{26}y_{34} - y_{10}, \\ y_{25}y_{34} - y_8, y_{22}y_{33} - y_9, y_{21}y_{33} - y_8, y_{18}y_{32} + 2y_9 - y_{10}, y_{17}y_{32} - 2y_8 + y_{10}, \\ y_{14}y_{31} + y_9 - 2y_{10}, y_{13}y_{31} - y_8 + y_{10}, y_{21}y_{26}y_{29} - y_{22}y_{25}y_{30} \end{array} \right)$$

We present Macaulay 2 code for computing I_Q and $\widetilde{I_Q}$.

```
i1:  R = QQ[x_1..x_16,t_0..t_7,h_0..h_7,y_1..y_40,MonomialOrder => Eliminate 32]
i2:  H = I+K
i3:  G = gens gb K+I
i4:  J = selectInSubring(1,G)
i5:  IQ = ideal(J)
i6:  Htilde = K+Itilde
i7:  Gtilde = gens gb Htilde
i8:  Jtilde = selectInSubring(1,Gtilde)
i9:  IQtilde = ideal(Jtilde)
```

where

$$K = \left(\begin{array}{c} y_1 - t_0h_1x_1x_2x_6, y_2 - t_0h_1x_1x_3x_7, y_3 - t_0h_1x_1x_4x_8, y_4 - t_0h_1x_1x_5x_9, y_5 - t_0h_1x_2x_3x_{10}, \\ y_6 - t_0h_1x_2x_4x_{11}, y_7 - t_0h_1x_2x_5x_{12}, y_8 - t_0h_1x_3x_4x_{13}, y_9 - t_0h_1x_3x_5x_{14}, y_{10} - t_0h_1x_4x_5x_{15}, \\ y_{11} - t_1h_2x_2x_6, y_{12} - t_1h_2x_3x_7, y_{13} - t_1h_2x_4x_8, y_{14} - t_1h_2x_5x_9, y_{15} - t_1h_3x_1x_6, \\ y_{16} - t_1h_3x_3x_{10}, y_{17} - t_1h_3x_4x_{11}, y_{18} - t_1h_3x_5x_{12}, y_{19} - t_1h_4x_1x_7, y_{20} - t_1h_4x_2x_{10}, \\ y_{21} - t_1h_4x_4x_{13}, y_{22} - t_1h_4x_5x_{14}, y_{23} - t_1h_5x_1x_8, y_{24} - t_1h_5x_2x_{11}, y_{25} - t_1h_5x_3x_{13}, \\ y_{26} - t_1h_5x_5x_{15}, y_{27} - t_1h_6x_1x_9, y_{28} - t_1h_6x_2x_{12}, y_{29} - t_1h_6x_3x_{14}, y_{30} - t_1h_6x_4x_{15}, \\ y_{31} - t_2h_7x_1, y_{32} - t_3h_7x_2, y_{33} - t_4h_7x_3, y_{34} - t_5h_7x_4, y_{35} - t_6h_7x_5, y_{36} - t_0h_2x_2x_3x_4x_5x_{16}, \\ y_{37} - t_0h_3x_1x_3x_4x_5x_{16}, y_{38} - t_0h_4x_1x_2x_4x_5x_{16}, y_{39} - t_0h_5x_1x_2x_3x_5x_{16}, y_{40} - t_0h_6x_1x_2x_3x_4x_{16} \end{array} \right)$$

$$I = \left(\begin{array}{c} x_5x_{16} + x_6x_{13} - 3x_8x_{10}, x_4x_{16} + 2x_6x_{14} + x_7x_{12}, \\ x_4x_{16} + x_6x_{14} + x_9x_{10}, x_3x_{16} + x_6x_{15} + x_8x_{12}, x_3x_{16} + 2x_6x_{15} + x_9x_{11}, \\ x_2x_{16} + x_7x_{15} - 2x_8x_{14}, x_2x_{16} + 3x_7x_{15} + 2x_9x_{13}, \\ x_1x_{16} + 2x_{10}x_{15} - x_{11}x_{14}, x_1x_{16} + 3x_{10}x_{15} + x_{12}x_{13}, x_2x_6 - x_3x_7 + x_4x_8, \\ 2x_2x_6 - 3x_3x_7 - x_5x_9, x_1x_6 - x_3x_{10} + x_4x_{11}, x_1x_6 - 3x_3x_{10} - x_5x_{12}, \\ x_1x_7 - x_2x_{10} + x_4x_{13}, x_1x_7 - 2x_2x_{10} + x_5x_{14}, x_1x_8 - x_2x_{11} + x_3x_{13}, \\ -2x_1x_8 + x_2x_{11} - x_5x_{15}, -x_1x_9 + 2x_2x_{12} + 3x_3x_{14}, \\ -2x_1x_9 + x_2x_{12} + 3x_4x_{15}, x_6x_{13} - x_7x_{11} + x_8x_{10} \\ t_0h_0 - 1, \dots, t_7h_7 - 1 \end{array} \right)$$

and

$$\widetilde{I} = \left(\; t_0h_0 - 1, \dots, t_7h_7 - 1 \; \right)$$

In Macaulay 2, we calculate the saturation of I_R and I_Q with B_Y using the command "saturate",

```
i1:  IQQ = saturate(IQ,BY)
i2:  IRR = saturate(IR,BY)
i3:  IRR == IQQ
o3:  true
```

where B_Y is the intersection of:

$$\big(y_1, \dots, y_{10}\big), \big(y_{11}, \dots, y_{14}, y_{36}\big), \big(y_{15}, \dots, y_{18}, y_{37}\big), \big(y_{19}, \dots, y_{22}, y_{38}\big),$$

$$\big(y_{23}, \dots, y_{26}, y_{39}\big), \big(y_{27}, \dots, y_{30}, y_{40}\big) \text{ and } \big(y_{31}, \dots, y_{35}\big)$$

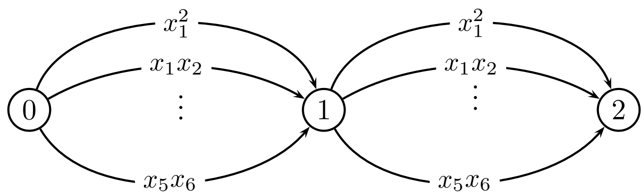
In the same way we obtain $\widetilde{I_Q} = \big(\widetilde{I_R} : B_Y^\infty\big)$. The collection $\mathscr{L}_5$ is very ample, so Theorem 4.1.1 implies that $\varphi_{|\mathscr{L}_5|} \colon X_5 \longrightarrow \mathcal{M}_\vartheta(A_{\mathscr{L}_5})$ is an isomorphism.

6.4 Gr(2, 4) Example

Let $X = \mathrm{Gr}(2,4)$. The Cox ring of X is

$$\mathrm{Cox}(X) = \Bbbk[x_1, \ldots, x_6]/\big(x_1x_6 - x_2x_5 + x_3x_4)\big).$$

We recall that $\mathrm{Pic}(X) \cong \mathbb{Z}$ is generated by the determinantal line bundle on X. Let $\mathscr{L} := (\mathcal{O}_X, \mathcal{O}(2), \mathcal{O}(4))$. The quiver of sections for $\mathscr{L}$ is:



Arrows 1-21 are those from 0 to 1. They are labeled by all monomials in $\Bbbk[x_1, \ldots, x_6]$ of degree 2. Arrows 22-42 are those from 1 to 2. They are also labelled by all monomials of degree 2.

$$
I_R = \left(\begin{array}{c}
y_{32} - y_{35} + y_{37}, y_{11} - y_{14} + y_{16}, y_{21}y_{41} - y_{20}y_{42}, y_{19}y_{41} - y_4y_{42}, \\
y_{17}y_{41} - y_{16}y_{42}, y_{14}y_{41} - y_{13}y_{42}, y_{10}y_{41} - y_9y_{42}, y_5y_{41} - y_{19}y_{42}, \\
y_{21}y_{40} - y_{19}y_{42}, y_{20}y_{40} - y_4y_{42}, y_{18}y_{40} - y_{16}y_{42}, y_{15}y_{40} - y_{13}y_{42}, \\
y_{14}y_{40} - y_{16}y_{40} - y_9y_{42}, y_6y_{40} - y_{20}y_{42}, y_{21}y_{39} - y_{18}y_{42}, y_{20}y_{39} - y_{18}y_{41}, \\
y_{19}y_{39} - y_{16}y_{42}, y_{17}y_{39} - y_3y_{42}, y_{16}y_{39} - y_3y_{41}, y_{14}y_{39} - y_{12}y_{42}, \\
y_{13}y_{39} - y_{12}y_{41}, y_{10}y_{39} - y_8y_{42}, y_9y_{39} - y_8y_{41}, y_5y_{39} - y_{17}y_{42}, \\
y_4y_{39} - y_{16}y_{41}, y_{21}y_{38} - y_{17}y_{42}, y_{20}y_{38} - y_{16}y_{42}, y_{19}y_{38} - y_{17}y_{40}, \\
y_{18}y_{38} - y_3y_{42}, y_{16}y_{38} - y_3y_{40}, y_{15}y_{38} - y_{12}y_{42}, y_{14}y_{38} - y_3y_{40} - y_8y_{42}, \\
y_{13}y_{38} - y_{12}y_{40}, y_9y_{38} - y_8y_{40}, y_6y_{38} - y_{18}y_{42}, y_4y_{38} - y_{16}y_{40}, \\
y_{21}y_{37} - y_{16}y_{42}, y_{20}y_{37} - y_{16}y_{41}, y_{19}y_{37} - y_{16}y_{40}, y_{18}y_{37} - y_3y_{41}, \\
y_{17}y_{37} - y_3y_{40}, y_{16}y_{37} - y_{12}y_{40} + y_8y_{41}, y_{15}y_{37} - y_{12}y_{41}, y_{14}y_{37} - y_{12}y_{40}, \\
y_{13}y_{37} - y_2y_{40} + y_7y_{41}, y_{12}y_{37} - y_2y_{38} + y_7y_{39}, y_{10}y_{37} - y_8y_{40}, \\
y_9y_{37} - y_7y_{40} + y_1y_{41}, y_8y_{37} - y_7y_{38} + y_1y_{39}, y_6y_{37} - y_{18}y_{41}, \\
y_5y_{37} - y_{17}y_{40}, y_4y_{37} - y_{13}y_{40} + y_9y_{41}, y_3y_{37} - y_{12}y_{38} + y_8y_{39}, \\
y_{21}y_{36} - y_{15}y_{42}, y_{20}y_{36} - y_{15}y_{41}, y_{19}y_{36} - y_{13}y_{42}, y_{18}y_{36} - y_{15}y_{39}, \\
y_{17}y_{36} - y_{12}y_{42}, y_{16}y_{36} - y_{12}y_{41}, y_{14}y_{36} - y_2y_{42}, y_{13}y_{36} - y_2y_{41}, \\
y_{12}y_{36} - y_2y_{39}, y_{10}y_{36} - y_7y_{42}, y_9y_{36} - y_7y_{41}, y_8y_{36} - y_7y_{39}, \\
y_5y_{36} - y_{14}y_{42}, y_4y_{36} - y_{13}y_{41}, y_3y_{36} - y_{12}y_{39}, y_{21}y_{35} - y_{14}y_{42}, \\
y_{20}y_{35} - y_{13}y_{42}, y_{19}y_{35} - y_{16}y_{40} - y_9y_{42}, y_{18}y_{35} - y_{12}y_{42}, \\
y_{17}y_{35} - y_3y_{40} - y_8y_{42}, y_{16}y_{35} - y_{12}y_{40}, y_{15}y_{35} - y_2y_{42}, \\
y_{14}y_{35} - y_{12}y_{40} - y_7y_{42}, y_{13}y_{35} - y_2y_{40}, y_{12}y_{35} - y_2y_{38}, y_{10}y_{35} - y_8y_{40} - y_1y_{42}, \\
y_9y_{35} - y_7y_{40}, y_8y_{35} - y_7y_{38}, y_7y_{35} - y_1y_{36} - y_7y_{37}, y_6y_{35} - y_{15}y_{42}, \\
y_5y_{35} - y_{17}y_{40} - y_{10}y_{42}, y_4y_{35} - y_{13}y_{40}, y_3y_{35} - y_{12}y_{38}, y_2y_{35} - y_7y_{36} - y_2y_{37}, \\
y_{21}y_{34} - y_{13}y_{42}, y_{20}y_{34} - y_{13}y_{41}, y_{19}y_{34} - y_{13}y_{40}, y_{18}y_{34} - y_{12}y_{41}, \\
y_{17}y_{34} - y_{12}y_{40}, y_{16}y_{34} - y_2y_{40} + y_7y_{41}, y_{15}y_{34} - y_2y_{41}, y_{14}y_{34} - y_2y_{40}, \\
y_{12}y_{34} - y_2y_{37}, y_{10}y_{34} - y_7y_{40}, y_8y_{34} - y_7y_{37}, y_6y_{34} - y_{15}y_{41}, \\
y_5y_{34} - y_{16}y_{40} - y_9y_{42}, y_3y_{34} - y_2y_{38} + y_7y_{39}, y_{21}y_{33} - y_{12}y_{42}, \\
y_{20}y_{33} - y_{12}y_{41}, y_{19}y_{33} - y_{12}y_{40}, y_{18}y_{33} - y_{12}y_{39}, y_{17}y_{33} - y_{12}y_{38}, \\
y_{16}y_{33} - y_2y_{38} + y_7y_{39}, y_{15}y_{33} - y_2y_{39}, y_{14}y_{33} - y_2y_{38}, y_{13}y_{33} - y_2y_{37}, \\
y_{10}y_{33} - y_7y_{38}, y_9y_{33} - y_7y_{37}, y_6y_{33} - y_{15}y_{39}, y_5y_{33} - y_3y_{40} - y_8y_{42}, \\
y_4y_{33} - y_2y_{40} + y_7y_{41}, y_{21}y_{31} - y_{10}y_{42}, y_{20}y_{31} - y_9y_{42}, y_{19}y_{31} - y_{10}y_{40}, \\
y_{18}y_{31} - y_8y_{42}, y_{17}y_{31} - y_{10}y_{38}, y_{16}y_{31} - y_8y_{40}, y_{15}y_{31} - y_7y_{42}, \\
y_{14}y_{31} - y_8y_{40} - y_1y_{42}, y_{13}y_{31} - y_7y_{40}, y_{12}y_{31} - y_7y_{38}, y_9y_{31} - y_1y_{40}, \\
y_8y_{31} - y_1y_{38}, y_7y_{31} - y_1y_{35}, y_6y_{31} - y_{14}y_{42} + y_{16}y_{42}, y_4y_{31} - y_9y_{40}, \\
y_3y_{31} - y_8y_{38}, y_2y_{31} - y_1y_{36} - y_7y_{37}, y_{21}y_{30} - y_9y_{42}, y_{20}y_{30} - y_9y_{41}, \\
y_{19}y_{30} - y_9y_{40}, y_{18}y_{30} - y_8y_{41}, y_{17}y_{30} - y_8y_{40}, y_{16}y_{30} - y_7y_{40} + y_1y_{41}, \\
y_{15}y_{30} - y_7y_{41}, y_{14}y_{30} - y_7y_{40}, y_{13}y_{30} - y_9y_{34}, y_{12}y_{30} - y_7y_{37},
\end{array} \right.
$$

$$\begin{pmatrix} y_{10}y_{30}-y_1y_{40}, y_8y_{30}-y_1y_{37}, y_7y_{30}-y_1y_{34}, y_6y_{30}+y_{16}y_{41}-y_{13}y_{42}, y_5y_{30}-y_{10}y_{40}, \\ y_3y_{30}-y_7y_{38}+y_1y_{39}, y_2y_{30}-y_7y_{34}, y_{21}y_{29}-y_8y_{42}, y_{20}y_{29}-y_8y_{41}, \\ y_{19}y_{29}-y_8y_{40}, y_{18}y_{29}-y_8y_{39}, y_{17}y_{29}-y_8y_{38}, y_{16}y_{29}-y_7y_{38}+y_1y_{39}, \\ y_{15}y_{29}-y_7y_{39}, y_{14}y_{29}-y_7y_{38}, y_{13}y_{29}-y_7y_{37}, y_{12}y_{29}-y_8y_{33}, y_{10}y_{29}-y_1y_{38}, \\ y_9y_{29}-y_1y_{37}, y_7y_{29}-y_1y_{33}, y_6y_{29}+y_3y_{41}-y_{12}y_{42}, y_5y_{29}-y_{10}y_{38}, \\ y_4y_{29}-y_7y_{40}+y_1y_{41}, y_2y_{29}-y_7y_{33}, y_{21}y_{28}-y_7y_{42}, y_{20}y_{28}-y_7y_{41}, \\ y_{19}y_{28}-y_7y_{40}, y_{18}y_{28}-y_7y_{39}, y_{17}y_{28}-y_7y_{38}, y_{16}y_{28}-y_7y_{37}, \\ y_{15}y_{28}-y_7y_{36}, y_{14}y_{28}-y_1y_{36}-y_7y_{37}, y_{13}y_{28}-y_7y_{34}, y_{12}y_{28}-y_7y_{33}, \\ y_{10}y_{28}-y_1y_{35}, y_9y_{28}-y_1y_{34}, y_8y_{28}-y_1y_{33}, y_6y_{28}+y_{12}y_{41}-y_2y_{42}, \\ y_5y_{28}-y_8y_{40}-y_1y_{42}, y_4y_{28}-y_9y_{34}, y_3y_{28}-y_8y_{33}, y_{21}y_{27}-y_6y_{42}, \\ y_{20}y_{27}-y_6y_{41}, y_{19}y_{27}-y_{20}y_{42}, y_{18}y_{27}-y_6y_{39}, y_{17}y_{27}-y_{18}y_{42}, \\ y_{16}y_{27}-y_{18}y_{41}, y_{15}y_{27}-y_6y_{36}, y_{14}y_{27}-y_{15}y_{42}, y_{13}y_{27}-y_{15}y_{41}, \\ y_{12}y_{27}-y_{15}y_{39}, y_{10}y_{27}-y_{14}y_{42}+y_{16}y_{42}, y_9y_{27}+y_{16}y_{41}-y_{13}y_{42}, \\ y_8y_{27}+y_3y_{41}-y_{12}y_{42}, y_7y_{27}+y_{12}y_{41}-y_2y_{42}, y_5y_{27}-y_{21}y_{42}, y_4y_{27}-y_{20}y_{41}, \\ y_3y_{27}-y_{18}y_{39}, y_2y_{27}-y_{15}y_{36}, y_1y_{27}+y_8y_{41}-y_7y_{42}, y_{21}y_{26}-y_5y_{42}, \\ y_{20}y_{26}-y_{19}y_{42}, y_{19}y_{26}-y_5y_{40}, y_{18}y_{26}-y_{17}y_{42}, y_{17}y_{26}-y_5y_{38}, \\ y_{16}y_{26}-y_{17}y_{40}, y_{15}y_{26}-y_{14}y_{42}, y_{14}y_{26}-y_{17}y_{40}-y_{10}y_{42}, \\ y_{13}y_{26}-y_{16}y_{40}-y_9y_{42}, y_{12}y_{26}-y_3y_{40}-y_8y_{42}, y_{10}y_{26}-y_5y_{31}, y_9y_{26}-y_{10}y_{40}, \\ y_8y_{26}-y_{10}y_{38}, y_7y_{26}-y_8y_{40}-y_1y_{42}, y_6y_{26}-y_{21}y_{42}, y_4y_{26}-y_{19}y_{40}, \\ y_3y_{26}-y_{17}y_{38}, y_2y_{26}-y_{12}y_{40}-y_7y_{42}, y_1y_{26}-y_{10}y_{31}, y_{21}y_{25}-y_4y_{42}, \\ y_{20}y_{25}-y_4y_{41}, y_{19}y_{25}-y_4y_{40}, y_{18}y_{25}-y_{16}y_{41}, y_{17}y_{25}-y_{16}y_{40}, \\ y_{16}y_{25}-y_{13}y_{40}+y_9y_{41}, y_{15}y_{25}-y_{13}y_{41}, y_{14}y_{25}-y_{13}y_{40}, y_{13}y_{25}-y_4y_{34}, \\ y_{12}y_{25}-y_2y_{40}+y_7y_{41}, y_{10}y_{25}-y_9y_{40}, y_9y_{25}-y_4y_{30}, y_8y_{25}-y_7y_{40}+y_1y_{41}, \\ y_7y_{25}-y_9y_{34}, y_6y_{25}-y_{20}y_{41}, y_5y_{25}-y_{19}y_{40}, y_3y_{25}-y_{12}y_{40}+y_8y_{41}, \\ y_2y_{25}-y_{13}y_{34}, y_1y_{25}-y_9y_{30}, y_{21}y_{24}-y_3y_{42}, y_{20}y_{24}-y_3y_{41}, y_{19}y_{24}-y_3y_{40}, \\ y_{18}y_{24}-y_3y_{39}, y_{17}y_{24}-y_3y_{38}, y_{16}y_{24}-y_{12}y_{38}+y_8y_{39}, y_{15}y_{24}-y_{12}y_{39}, \\ y_{14}y_{24}-y_{12}y_{38}, y_{13}y_{24}-y_2y_{38}+y_7y_{39}, y_{12}y_{24}-y_3y_{33}, y_{10}y_{24}-y_8y_{38}, \\ y_9y_{24}-y_7y_{38}+y_1y_{39}, y_8y_{24}-y_3y_{29}, y_7y_{24}-y_8y_{33}, y_6y_{24}-y_{18}y_{39}, y_5y_{24}-y_{17}y_{38}, \\ y_4y_{24}-y_{12}y_{40}+y_8y_{41}, y_2y_{24}-y_{12}y_{33}, y_1y_{24}-y_8y_{29}, y_{21}y_{23}-y_2y_{42}, y_{20}y_{23}-y_2y_{41}, \\ y_{19}y_{23}-y_2y_{40}, y_{18}y_{23}-y_2y_{39}, y_{17}y_{23}-y_2y_{38}, y_{16}y_{23}-y_2y_{37}, y_{15}y_{23}-y_2y_{36}, \\ y_{14}y_{23}-y_7y_{36}-y_2y_{37}, y_{13}y_{23}-y_2y_{34}, y_{12}y_{23}-y_2y_{33}, y_{10}y_{23}-y_1y_{36}-y_7y_{37}, \\ y_9y_{23}-y_7y_{34}, y_8y_{23}-y_7y_{33}, y_7y_{23}-y_2y_{28}, y_6y_{23}-y_{15}y_{36}, \\ y_5y_{23}-y_{12}y_{40}-y_7y_{42}, y_4y_{23}-y_{13}y_{34}, y_3y_{23}-y_{12}y_{33}, y_1y_{23}-y_7y_{28}, \\ y_{21}y_{22}-y_1y_{42}, y_{20}y_{22}-y_1y_{41}, y_{19}y_{22}-y_1y_{40}, y_{18}y_{22}-y_1y_{39}, y_{17}y_{22}-y_1y_{38}, \\ y_{16}y_{22}-y_1y_{37}, y_{15}y_{22}-y_1y_{36}, y_{14}y_{22}-y_1y_{35}, y_{13}y_{22}-y_1y_{34}, y_{12}y_{22}-y_1y_{33}, \\ y_{10}y_{22}-y_1y_{31}, y_9y_{22}-y_1y_{30}, y_8y_{22}-y_1y_{29}, y_7y_{22}-y_1y_{28}, y_6y_{22}+y_8y_{41}-y_7y_{42}, \\ y_5y_{22}-y_{10}y_{31}, y_4y_{22}-y_9y_{30}, y_3y_{22}-y_8y_{29}, y_2y_{22}-y_7y_{28} \end{pmatrix}$$

$$
I_Q = \left(\begin{array}{c}
y_{32} - y_{35} + y_{37}, y_{11} - y_{14} + y_{16}, y_{40}y_{41} - y_{25}y_{42}, y_{38}y_{41} - y_{37}y_{42}, \\
y_{35}y_{41} - y_{34}y_{42}, y_{31}y_{41} - y_{30}y_{42}, y_{26}y_{41} - y_{40}y_{42}, \\
y_{21}y_{41} - y_{20}y_{42}, y_{19}y_{41} - y_{4}y_{42}, y_{17}y_{41} - y_{16}y_{42}, \\
y_{14}y_{41} - y_{13}y_{42}, y_{10}y_{41} - y_{9}y_{42}, y_{5}y_{41} - y_{19}y_{42}, \\
y_{39}y_{40} - y_{37}y_{42}, y_{36}y_{40} - y_{34}y_{42}, y_{35}y_{40} - y_{37}y_{40} - y_{30}y_{42}, \\
y_{27}y_{40} - y_{41}y_{42}, y_{21}y_{40} - y_{19}y_{42}, y_{20}y_{40} - y_{4}y_{42}, \\
y_{18}y_{40} - y_{16}y_{42}, y_{15}y_{40} - y_{13}y_{42}, y_{14}y_{40} - y_{16}y_{40} - y_{9}y_{42}, \\
y_{6}y_{40} - y_{20}y_{42}, y_{38}y_{39} - y_{24}y_{42}, y_{37}y_{39} - y_{24}y_{41}, y_{35}y_{39} - y_{33}y_{42}, \\
y_{34}y_{39} - y_{33}y_{41}, y_{31}y_{39} - y_{29}y_{42}, y_{30}y_{39} - y_{29}y_{41}, y_{26}y_{39} - y_{38}y_{42}, \\
y_{25}y_{39} - y_{37}y_{41}, y_{21}y_{39} - y_{18}y_{42}, y_{20}y_{39} - y_{18}y_{41}, \\
y_{19}y_{39} - y_{16}y_{42}, y_{17}y_{39} - y_{3}y_{42}, y_{16}y_{39} - y_{3}y_{41}, \\
y_{14}y_{39} - y_{12}y_{42}, y_{13}y_{39} - y_{12}y_{41}, y_{10}y_{39} - y_{8}y_{42}, y_{9}y_{39} - y_{8}y_{41}, \\
y_{5}y_{39} - y_{17}y_{42}, y_{4}y_{39} - y_{16}y_{41}, y_{37}y_{38} - y_{24}y_{40}, y_{36}y_{38} - y_{33}y_{42}, \\
y_{35}y_{38} - y_{24}y_{40} - y_{29}y_{42}, y_{34}y_{38} - y_{33}y_{40}, y_{30}y_{38} - y_{29}y_{40}, \\
y_{27}y_{38} - y_{39}y_{42}, y_{25}y_{38} - y_{37}y_{40}, y_{21}y_{38} - y_{17}y_{42}, \\
y_{20}y_{38} - y_{16}y_{42}, y_{19}y_{38} - y_{17}y_{40}, y_{18}y_{38} - y_{3}y_{42}, y_{16}y_{38} - y_{3}y_{40}, \\
y_{15}y_{38} - y_{12}y_{42}, y_{14}y_{38} - y_{3}y_{40} - y_{8}y_{42}, y_{13}y_{38} - y_{12}y_{40}, y_{9}y_{38} - y_{8}y_{40}, \\
y_{6}y_{38} - y_{18}y_{42}, y_{4}y_{38} - y_{16}y_{40}, y_{37}^2 - y_{33}y_{40} + y_{29}y_{41}, \\
y_{36}y_{37} - y_{33}y_{41}, y_{35}y_{37} - y_{33}y_{40}, y_{34}y_{37} - y_{23}y_{40} + y_{28}y_{41}, \\
y_{33}y_{37} - y_{23}y_{38} + y_{28}y_{39}, y_{31}y_{37} - y_{29}y_{40}, y_{30}y_{37} - y_{28}y_{40} + y_{22}y_{41}, \\
y_{29}y_{37} - y_{28}y_{38} + y_{22}y_{39}, y_{27}y_{37} - y_{39}y_{41}, y_{26}y_{37} - y_{38}y_{40}, \\
y_{25}y_{37} - y_{34}y_{40} + y_{30}y_{41}, y_{24}y_{37} - y_{33}y_{38} + y_{29}y_{39}, y_{21}y_{37} - y_{16}y_{42}, \\
y_{20}y_{37} - y_{16}y_{41}, y_{19}y_{37} - y_{16}y_{40}, y_{18}y_{37} - y_{3}y_{41}, y_{17}y_{37} - y_{3}y_{40}, \\
y_{16}y_{37} - y_{12}y_{40} + y_{8}y_{41}, y_{15}y_{37} - y_{12}y_{41}, y_{14}y_{37} - y_{12}y_{40}, \\
y_{13}y_{37} - y_{2}y_{40} + y_{7}y_{41}, y_{12}y_{37} - y_{2}y_{38} + y_{7}y_{39}, y_{10}y_{37} - y_{8}y_{40}, \\
y_{9}y_{37} - y_{7}y_{40} + y_{1}y_{41}, y_{8}y_{37} - y_{7}y_{38} + y_{1}y_{39}, y_{6}y_{37} - y_{18}y_{41}, \\
y_{5}y_{37} - y_{17}y_{40}, y_{4}y_{37} - y_{13}y_{40} + y_{9}y_{41}, y_{3}y_{37} - y_{12}y_{38} + y_{8}y_{39}, \\
y_{35}y_{36} - y_{23}y_{42}, y_{34}y_{36} - y_{23}y_{41}, y_{33}y_{36} - y_{23}y_{39}, y_{31}y_{36} - y_{28}y_{42}, \\
y_{30}y_{36} - y_{28}y_{41}, y_{29}y_{36} - y_{28}y_{39}, y_{26}y_{36} - y_{35}y_{42}, \\
y_{25}y_{36} - y_{34}y_{41}, y_{24}y_{36} - y_{33}y_{39}, y_{21}y_{36} - y_{15}y_{42}, \\
y_{20}y_{36} - y_{15}y_{41}, y_{19}y_{36} - y_{13}y_{42}, y_{18}y_{36} - y_{15}y_{39}, y_{17}y_{36} - y_{12}y_{42}, \\
y_{16}y_{36} - y_{12}y_{41}, y_{14}y_{36} - y_{2}y_{42}, y_{13}y_{36} - y_{2}y_{41}, y_{12}y_{36} - y_{2}y_{39}, \\
y_{10}y_{36} - y_{7}y_{42}, y_{9}y_{36} - y_{7}y_{41}, y_{8}y_{36} - y_{7}y_{39}, y_{5}y_{36} - y_{14}y_{42}, \\
y_{4}y_{36} - y_{13}y_{41}, y_{3}y_{36} - y_{12}y_{39}, y_{35}^2 - y_{33}y_{40} - y_{28}y_{42}, \\
y_{34}y_{35} - y_{23}y_{40}, y_{33}y_{35} - y_{23}y_{38}, y_{31}y_{35} - y_{29}y_{40} - y_{22}y_{42}, \\
y_{30}y_{35} - y_{28}y_{40}, y_{29}y_{35} - y_{28}y_{38}, y_{28}y_{35} - y_{22}y_{36} - y_{28}y_{37}, \\
y_{27}y_{35} - y_{36}y_{42}, y_{26}y_{35} - y_{38}y_{40} - y_{31}y_{42}, y_{25}y_{35} - y_{34}y_{40},
\end{array} \right)
$$

$$
\left(
\begin{array}{c}
y_{24}y_{35} - y_{33}y_{38}, y_{23}y_{35} - y_{28}y_{36} - y_{23}y_{37}, y_{21}y_{35} - y_{14}y_{42}, \\
y_{20}y_{35} - y_{13}y_{42}, y_{19}y_{35} - y_{16}y_{40} - y_{9}y_{42}, y_{18}y_{35} - y_{12}y_{42}, \\
y_{17}y_{35} - y_{3}y_{40} - y_{8}y_{42}, y_{16}y_{35} - y_{12}y_{40}, y_{15}y_{35} - y_{2}y_{42}, y_{14}y_{35} - y_{12}y_{40} - y_{7}y_{42}, \\
y_{13}y_{35} - y_{2}y_{40}, y_{12}y_{35} - y_{2}y_{38}, y_{10}y_{35} - y_{8}y_{40} - y_{1}y_{42}, y_{9}y_{35} - y_{7}y_{40}, \\
y_{8}y_{35} - y_{7}y_{38}, y_{7}y_{35} - y_{1}y_{36} - y_{7}y_{37}, y_{6}y_{35} - y_{15}y_{42}, \\
y_{5}y_{35} - y_{17}y_{40} - y_{10}y_{42}, y_{4}y_{35} - y_{13}y_{40}, y_{3}y_{35} - y_{12}y_{38}, \\
y_{2}y_{35} - y_{7}y_{36} - y_{2}y_{37}, y_{33}y_{34} - y_{23}y_{37}, y_{31}y_{34} - y_{28}y_{40}, \\
y_{29}y_{34} - y_{28}y_{37}, y_{27}y_{34} - y_{36}y_{41}, y_{26}y_{34} - y_{37}y_{40} - y_{30}y_{42}, \\
y_{24}y_{34} - y_{23}y_{38} + y_{28}y_{39}, y_{21}y_{34} - y_{13}y_{42}, y_{20}y_{34} - y_{13}y_{41}, y_{19}y_{34} - y_{13}y_{40}, \\
y_{18}y_{34} - y_{12}y_{41}, y_{17}y_{34} - y_{12}y_{40}, y_{16}y_{34} - y_{2}y_{40} + y_{7}y_{41}, y_{15}y_{34} - y_{2}y_{41}, \\
y_{14}y_{34} - y_{2}y_{40}, y_{12}y_{34} - y_{2}y_{37}, y_{10}y_{34} - y_{7}y_{40}, y_{8}y_{34} - y_{7}y_{37}, \\
y_{6}y_{34} - y_{15}y_{41}, y_{5}y_{34} - y_{16}y_{40} - y_{9}y_{42}, y_{3}y_{34} - y_{2}y_{38} + y_{7}y_{39}, y_{31}y_{33} - y_{28}y_{38}, \\
y_{30}y_{33} - y_{28}y_{37}, y_{27}y_{33} - y_{36}y_{39}, y_{26}y_{33} - y_{24}y_{40} - y_{29}y_{42}, \\
y_{25}y_{33} - y_{23}y_{40} + y_{28}y_{41}, y_{21}y_{33} - y_{12}y_{42}, y_{20}y_{33} - y_{12}y_{41}, \\
y_{19}y_{33} - y_{12}y_{40}, y_{18}y_{33} - y_{12}y_{39}, y_{17}y_{33} - y_{12}y_{38}, \\
y_{16}y_{33} - y_{2}y_{38} + y_{7}y_{39}, y_{15}y_{33} - y_{2}y_{39}, y_{14}y_{33} - y_{2}y_{38}, y_{13}y_{33} - y_{2}y_{37}, \\
y_{10}y_{33} - y_{7}y_{38}, y_{9}y_{33} - y_{7}y_{37}, y_{6}y_{33} - y_{15}y_{39}, y_{5}y_{33} - y_{3}y_{40} - y_{8}y_{42}, \\
y_{4}y_{33} - y_{2}y_{40} + y_{7}y_{41}, y_{30}y_{31} - y_{22}y_{40}, y_{29}y_{31} - y_{22}y_{38}, y_{28}y_{31} - y_{22}y_{35}, \\
y_{27}y_{31} - y_{35}y_{42} + y_{37}y_{42}, y_{25}y_{31} - y_{30}y_{40}, y_{24}y_{31} - y_{29}y_{38}, \\
y_{23}y_{31} - y_{22}y_{36} - y_{28}y_{37}, y_{21}y_{31} - y_{10}y_{42}, y_{20}y_{31} - y_{9}y_{42}, \\
y_{19}y_{31} - y_{10}y_{40}, y_{18}y_{31} - y_{8}y_{42}, y_{17}y_{31} - y_{10}y_{38}, \\
y_{16}y_{31} - y_{8}y_{40}, y_{15}y_{31} - y_{7}y_{42}, y_{14}y_{31} - y_{8}y_{40} - y_{1}y_{42}, \\
y_{13}y_{31} - y_{7}y_{40}, y_{12}y_{31} - y_{7}y_{38}, y_{9}y_{31} - y_{1}y_{40}, y_{8}y_{31} - y_{1}y_{38}, \\
y_{7}y_{31} - y_{1}y_{35}, y_{6}y_{31} - y_{14}y_{42} + y_{16}y_{42}, y_{4}y_{31} - y_{9}y_{40}, y_{3}y_{31} - y_{8}y_{38}, \\
y_{2}y_{31} - y_{1}y_{36} - y_{7}y_{37}, y_{29}y_{30} - y_{22}y_{37}, y_{28}y_{30} - y_{22}y_{34}, \\
y_{27}y_{30} + y_{37}y_{41} - y_{34}y_{42}, y_{26}y_{30} - y_{31}y_{40}, y_{24}y_{30} - y_{28}y_{38} + y_{22}y_{39}, \\
y_{23}y_{30} - y_{28}y_{34}, y_{21}y_{30} - y_{9}y_{42}, y_{20}y_{30} - y_{9}y_{41}, y_{19}y_{30} - y_{9}y_{40}, \\
y_{18}y_{30} - y_{8}y_{41}, y_{17}y_{30} - y_{8}y_{40}, y_{16}y_{30} - y_{7}y_{40} + y_{1}y_{41}, y_{15}y_{30} - y_{7}y_{41}, \\
y_{14}y_{30} - y_{7}y_{40}, y_{13}y_{30} - y_{9}y_{34}, y_{12}y_{30} - y_{7}y_{37}, y_{10}y_{30} - y_{1}y_{40}, \\
y_{8}y_{30} - y_{1}y_{37}, y_{7}y_{30} - y_{1}y_{34}, y_{6}y_{30} + y_{16}y_{41} - y_{13}y_{42}, y_{5}y_{30} - y_{10}y_{40}, \\
y_{3}y_{30} - y_{7}y_{38} + y_{1}y_{39}, y_{2}y_{30} - y_{7}y_{34}, y_{28}y_{29} - y_{22}y_{33}, y_{27}y_{29} + y_{24}y_{41} - y_{33}y_{42}, \\
y_{26}y_{29} - y_{31}y_{38}, y_{25}y_{29} - y_{28}y_{40} + y_{22}y_{41}, y_{23}y_{29} - y_{28}y_{33}, \\
y_{21}y_{29} - y_{8}y_{42}, y_{20}y_{29} - y_{8}y_{41}, y_{19}y_{29} - y_{8}y_{40}, y_{18}y_{29} - y_{8}y_{39}, \\
y_{17}y_{29} - y_{8}y_{38}, y_{16}y_{29} - y_{7}y_{38} + y_{1}y_{39}, y_{15}y_{29} - y_{7}y_{39}, \\
y_{14}y_{29} - y_{7}y_{38}, y_{13}y_{29} - y_{7}y_{37}, y_{12}y_{29} - y_{8}y_{33}, y_{10}y_{29} - y_{1}y_{38}, \\
y_{9}y_{29} - y_{1}y_{37}, y_{7}y_{29} - y_{1}y_{33}, y_{6}y_{29} + y_{3}y_{41} - y_{12}y_{42}, y_{5}y_{29} - y_{10}y_{38}, \\
y_{4}y_{29} - y_{7}y_{40} + y_{1}y_{41}, y_{2}y_{29} - y_{7}y_{33}, y_{27}y_{28} + y_{33}y_{41} - y_{23}y_{42},
\end{array}
\right)
$$

$$
\left(
\begin{array}{c}
y_{26}y_{28} - y_{29}y_{40} - y_{22}y_{42}, y_{25}y_{28} - y_{30}y_{34}, y_{24}y_{28} - y_{29}y_{33}, \\
y_{21}y_{28} - y_{7}y_{42}, y_{20}y_{28} - y_{7}y_{41}, y_{19}y_{28} - y_{7}y_{40}, y_{18}y_{28} - y_{7}y_{39}, \\
y_{17}y_{28} - y_{7}y_{38}, y_{16}y_{28} - y_{7}y_{37}, y_{15}y_{28} - y_{7}y_{36}, \\
y_{14}y_{28} - y_{1}y_{36} - y_{7}y_{37}, y_{13}y_{28} - y_{7}y_{34}, y_{12}y_{28} - y_{7}y_{33}, \\
y_{10}y_{28} - y_{1}y_{35}, y_{9}y_{28} - y_{1}y_{34}, y_{8}y_{28} - y_{1}y_{33}, y_{6}y_{28} + y_{12}y_{41} - y_{2}y_{42}, \\
y_{5}y_{28} - y_{8}y_{40} - y_{1}y_{42}, y_{4}y_{28} - y_{9}y_{34}, y_{3}y_{28} - y_{8}y_{33}, y_{26}y_{27} - y_{42}^{2}, \\
y_{25}y_{27} - y_{41}^{2}, y_{24}y_{27} - y_{39}^{2}, y_{23}y_{27} - y_{36}^{2}, y_{22}y_{27} + y_{29}y_{41} - y_{28}y_{42}, \\
y_{21}y_{27} - y_{6}y_{42}, y_{20}y_{27} - y_{6}y_{41}, y_{19}y_{27} - y_{20}y_{42}, y_{18}y_{27} - y_{6}y_{39}, \\
y_{17}y_{27} - y_{18}y_{42}, y_{16}y_{27} - y_{18}y_{41}, y_{15}y_{27} - y_{6}y_{36}, y_{14}y_{27} - y_{15}y_{42}, \\
y_{13}y_{27} - y_{15}y_{41}, y_{12}y_{27} - y_{15}y_{39}, y_{10}y_{27} - y_{14}y_{42} + y_{16}y_{42}, \\
y_{9}y_{27} + y_{16}y_{41} - y_{13}y_{42}, y_{8}y_{27} + y_{3}y_{41} - y_{12}y_{42}, y_{7}y_{27} + y_{12}y_{41} - y_{2}y_{42}, \\
y_{5}y_{27} - y_{21}y_{42}, y_{4}y_{27} - y_{20}y_{41}, y_{3}y_{27} - y_{18}y_{39}, y_{2}y_{27} - y_{15}y_{36}, \\
y_{1}y_{27} + y_{8}y_{41} - y_{7}y_{42}, y_{25}y_{26} - y_{40}^{2}, y_{24}y_{26} - y_{38}^{2}, y_{23}y_{26} - y_{33}y_{40} - y_{28}y_{42}, \\
y_{22}y_{26} - y_{31}^{2}, y_{21}y_{26} - y_{5}y_{42}, y_{20}y_{26} - y_{19}y_{42}, y_{19}y_{26} - y_{5}y_{40}, \\
y_{18}y_{26} - y_{17}y_{42}, y_{17}y_{26} - y_{5}y_{38}, y_{16}y_{26} - y_{17}y_{40}, y_{15}y_{26} - y_{14}y_{42}, \\
y_{14}y_{26} - y_{17}y_{40} - y_{10}y_{42}, y_{13}y_{26} - y_{16}y_{40} - y_{9}y_{42}, \\
y_{12}y_{26} - y_{3}y_{40} - y_{8}y_{42}, y_{10}y_{26} - y_{5}y_{31}, y_{9}y_{26} - y_{10}y_{40}, y_{8}y_{26} - y_{10}y_{38}, \\
y_{7}y_{26} - y_{8}y_{40} - y_{1}y_{42}, y_{6}y_{26} - y_{21}y_{42}, y_{4}y_{26} - y_{19}y_{40}, y_{3}y_{26} - y_{17}y_{38}, \\
y_{2}y_{26} - y_{12}y_{40} - y_{7}y_{42}, y_{1}y_{26} - y_{10}y_{31}, y_{24}y_{25} - y_{33}y_{40} + y_{29}y_{41}, \\
y_{23}y_{25} - y_{34}^{2}, y_{22}y_{25} - y_{30}^{2}, y_{21}y_{25} - y_{4}y_{42}, y_{20}y_{25} - y_{4}y_{41}, \\
y_{19}y_{25} - y_{4}y_{40}, y_{18}y_{25} - y_{16}y_{41}, y_{17}y_{25} - y_{16}y_{40}, \\
y_{16}y_{25} - y_{13}y_{40} + y_{9}y_{41}, y_{15}y_{25} - y_{13}y_{41}, y_{14}y_{25} - y_{13}y_{40}, \\
y_{13}y_{25} - y_{4}y_{34}, y_{12}y_{25} - y_{2}y_{40} + y_{7}y_{41}, y_{10}y_{25} - y_{9}y_{40}, \\
y_{9}y_{25} - y_{4}y_{30}, y_{8}y_{25} - y_{7}y_{40} + y_{1}y_{41}, y_{7}y_{25} - y_{9}y_{34}, y_{6}y_{25} - y_{20}y_{41}, \\
y_{5}y_{25} - y_{19}y_{40}, y_{3}y_{25} - y_{12}y_{40} + y_{8}y_{41}, y_{2}y_{25} - y_{13}y_{34}, y_{1}y_{25} - y_{9}y_{30}, \\
y_{23}y_{24} - y_{33}^{2}, y_{22}y_{24} - y_{29}^{2}, y_{21}y_{24} - y_{3}y_{42}, y_{20}y_{24} - y_{3}y_{41}, \\
y_{19}y_{24} - y_{3}y_{40}, y_{18}y_{24} - y_{3}y_{39}, y_{17}y_{24} - y_{3}y_{38}, y_{16}y_{24} - y_{12}y_{38} + y_{8}y_{39}, \\
y_{15}y_{24} - y_{12}y_{39}, y_{14}y_{24} - y_{12}y_{38}, y_{13}y_{24} - y_{2}y_{38} + y_{7}y_{39}, \\
y_{12}y_{24} - y_{3}y_{33}, y_{10}y_{24} - y_{8}y_{38}, y_{9}y_{24} - y_{7}y_{38} + y_{1}y_{39}, y_{8}y_{24} - y_{3}y_{29}, \\
y_{7}y_{24} - y_{8}y_{33}, y_{6}y_{24} - y_{18}y_{39}, y_{5}y_{24} - y_{17}y_{38}, y_{4}y_{24} - y_{12}y_{40} + y_{8}y_{41}, \\
y_{2}y_{24} - y_{12}y_{33}, y_{1}y_{24} - y_{8}y_{29}, y_{22}y_{23} - y_{28}^{2}, y_{21}y_{23} - y_{2}y_{42}, \\
y_{20}y_{23} - y_{2}y_{41}, y_{19}y_{23} - y_{2}y_{40}, y_{18}y_{23} - y_{2}y_{39}, y_{17}y_{23} - y_{2}y_{38}, \\
y_{16}y_{23} - y_{2}y_{37}, y_{15}y_{23} - y_{2}y_{36}, y_{14}y_{23} - y_{7}y_{36} - y_{2}y_{37}, y_{13}y_{23} - y_{2}y_{34}, \\
y_{12}y_{23} - y_{2}y_{33}, y_{10}y_{23} - y_{1}y_{36} - y_{7}y_{37}, y_{9}y_{23} - y_{7}y_{34}, y_{8}y_{23} - y_{7}y_{33}, \\
y_{7}y_{23} - y_{2}y_{28}, y_{6}y_{23} - y_{15}y_{36}, y_{5}y_{23} - y_{12}y_{40} - y_{7}y_{42}, y_{4}y_{23} - y_{13}y_{34}, \\
y_{3}y_{23} - y_{12}y_{33}, y_{1}y_{23} - y_{7}y_{28}, y_{21}y_{22} - y_{1}y_{42}, y_{20}y_{22} - y_{1}y_{41}, \\
y_{19}y_{22} - y_{1}y_{40}, y_{18}y_{22} - y_{1}y_{39}, y_{17}y_{22} - y_{1}y_{38}, y_{16}y_{22} - y_{1}y_{37},
\end{array}
\right)
$$

$$\begin{pmatrix} y_{15}y_{22}-y_1y_{36}, y_{14}y_{22}-y_1y_{35}, y_{13}y_{22}-y_1y_{34}, y_{12}y_{22}-y_1y_{33}, \\ y_{10}y_{22}-y_1y_{31}, y_9y_{22}-y_1y_{30}, y_8y_{22}-y_1y_{29}, y_7y_{22}-y_1y_{28}, \\ y_6y_{22}+y_8y_{41}-y_7y_{42}, y_5y_{22}-y_{10}y_{31}, y_4y_{22}-y_9y_{30}, y_3y_{22}-y_8y_{29}, \\ y_2y_{22}-y_7y_{28}, y_{19}y_{20}-y_4y_{21}, y_{17}y_{20}-y_{16}y_{21}, y_{14}y_{20}-y_{13}y_{21}, \\ y_{10}y_{20}-y_9y_{21}, y_5y_{20}-y_{19}y_{21}, y_{18}y_{19}-y_{16}y_{21}, y_{15}y_{19}-y_{13}y_{21}, \\ y_{14}y_{19}-y_{16}y_{19}-y_9y_{21}, y_6y_{19}-y_{20}y_{21}, y_{17}y_{18}-y_3y_{21}, \\ y_{16}y_{18}-y_3y_{20}, y_{14}y_{18}-y_{12}y_{21}, y_{13}y_{18}-y_{12}y_{20}, y_{10}y_{18}-y_8y_{21}, \\ y_9y_{18}-y_8y_{20}, y_5y_{18}-y_{17}y_{21}, y_4y_{18}-y_{16}y_{20}, y_{16}y_{17}-y_3y_{19}, \\ y_{15}y_{17}-y_{12}y_{21}, y_{14}y_{17}-y_3y_{19}-y_8y_{21}, y_{13}y_{17}-y_{12}y_{19}, y_9y_{17}-y_8y_{19}, \\ y_6y_{17}-y_{18}y_{21}, y_4y_{17}-y_{16}y_{19}, y_{16}^2-y_{12}y_{19}+y_8y_{20}, y_{15}y_{16}-y_{12}y_{20}, \\ y_{14}y_{16}-y_{12}y_{19}, y_{13}y_{16}-y_2y_{19}+y_7y_{20}, y_{12}y_{16}-y_2y_{17}+y_7y_{18}, \\ y_{10}y_{16}-y_8y_{19}, y_9y_{16}-y_7y_{19}+y_1y_{20}, y_8y_{16}-y_7y_{17}+y_1y_{18}, \\ y_6y_{16}-y_{18}y_{20}, y_5y_{16}-y_{17}y_{19}, y_4y_{16}-y_{13}y_{19}+y_9y_{20}, \\ y_3y_{16}-y_{12}y_{17}+y_8y_{18}, y_{14}y_{15}-y_2y_{21}, y_{13}y_{15}-y_2y_{20}, y_{12}y_{15}-y_2y_{18}, \\ y_{10}y_{15}-y_7y_{21}, y_9y_{15}-y_7y_{20}, y_8y_{15}-y_7y_{18}, y_5y_{15}-y_{14}y_{21}, y_4y_{15}-y_{13}y_{20}, \\ y_3y_{15}-y_{12}y_{18}, y_{14}^2-y_{12}y_{19}-y_7y_{21}, y_{13}y_{14}-y_2y_{19}, y_{12}y_{14}-y_2y_{17}, \\ y_{10}y_{14}-y_8y_{19}-y_1y_{21}, y_9y_{14}-y_7y_{19}, y_8y_{14}-y_7y_{17}, y_7y_{14}-y_1y_{15}-y_7y_{16}, \\ y_6y_{14}-y_{15}y_{21}, y_5y_{14}-y_{17}y_{19}-y_{10}y_{21}, y_4y_{14}-y_{13}y_{19}, y_3y_{14}-y_{12}y_{17}, \\ y_2y_{14}-y_7y_{15}-y_2y_{16}, y_{12}y_{13}-y_2y_{16}, y_{10}y_{13}-y_7y_{19}, y_8y_{13}-y_7y_{16}, \\ y_6y_{13}-y_{15}y_{20}, y_5y_{13}-y_{16}y_{19}-y_9y_{21}, y_3y_{13}-y_2y_{17}+y_7y_{18}, \\ y_{10}y_{12}-y_7y_{17}, y_9y_{12}-y_7y_{16}, y_6y_{12}-y_{15}y_{18}, y_5y_{12}-y_3y_{19}-y_8y_{21}, \\ y_4y_{12}-y_2y_{19}+y_7y_{20}, y_9y_{10}-y_1y_{19}, y_8y_{10}-y_1y_{17}, y_7y_{10}-y_1y_{14}, \\ y_6y_{10}-y_{14}y_{21}+y_{16}y_{21}, y_4y_{10}-y_9y_{19}, y_3y_{10}-y_8y_{17}, y_2y_{10}-y_1y_{15}-y_7y_{16}, \\ y_8y_9-y_1y_{16}, y_7y_9-y_1y_{13}, y_6y_9+y_{16}y_{20}-y_{13}y_{21}, y_5y_9-y_{10}y_{19}, \\ y_3y_9-y_7y_{17}+y_1y_{18}, y_2y_9-y_7y_{13}, y_7y_8-y_1y_{12}, y_6y_8+y_3y_{20}-y_{12}y_{21}, \\ y_5y_8-y_{10}y_{17}, y_4y_8-y_7y_{19}+y_1y_{20}, y_2y_8-y_7y_{12}, y_6y_7+y_{12}y_{20}-y_2y_{21}, \\ y_5y_7-y_8y_{19}-y_1y_{21}, y_4y_7-y_9y_{13}, y_3y_7-y_8y_{12}, y_5y_6-y_{21}^2, y_4y_6-y_{20}^2, \\ y_3y_6-y_{18}^2, y_2y_6-y_{15}^2, y_1y_6+y_8y_{20}-y_7y_{21}, y_4y_5-y_{19}^2, y_3y_5-y_{17}^2, \\ y_2y_5-y_{12}y_{19}-y_7y_{21}, y_1y_5-y_{10}^2, y_3y_4-y_{12}y_{19}+y_8y_{20}, \\ y_2y_4-y_{13}^2, y_1y_4-y_9^2, y_2y_3-y_{12}^2, y_1y_3-y_8^2, y_1y_2-y_7^2 \end{pmatrix}$$

We present Macaulay 2 code for computing I_Q.

```
i1:  R = QQ[x_1..x_6,t_0..t_2, h_0..h_2, y_1..y_42, MonomialOrder => Eliminate 12 ]
i2:  G = gens gb K+I
i3:  J = selectInSubring(1,G)
i4:  IQ = ideal(J)
```

where

$$K = \left(\begin{array}{c} y_1 - t_1x_1^2h_1, y_2 - t_1x_2^2h_1, y_3 - t_1x_3^2h_1, y_4 - t_1x_4^2h_1, y_5 - t_1x_5^2h_1, \\ y_6 - t_1x_6^2h_1, y_7 - t_1x_1x_2h_1, y_8 - t_1x_1x_3h_1, y_9 - t_1x_1x_4h_1, y_{10} - t_1x_1x_5h_1, \\ y_{11} - t_1x_1x_6h_1, y_{12} - t_1x_2x_3h_1, y_{13} - t_1x_2x_4h_1, y_{14} - t_1x_2x_5h_1, \\ y_{15} - t_1x_2x_6h_1, y_{16} - t_1x_3x_4h_1, y_{17} - t_1x_3x_5h_1, y_{18} - t_1x_3x_6h_1, \\ y_{19} - t_1x_4x_5h_1, y_{20} - t_1x_4x_6h_1, y_{21} - t_1x_5x_6h_1, y_{22} - t_2x_1^2h_2, \\ y_{23} - t_2x_2^2h_2, y_{24} - t_2x_3^2h_2, y_{25} - t_2x_4^2h_2, y_{26} - t_2x_5^2h_2, \\ y_{27} - t_2x_6^2h_2, y_{28} - t_2x_1x_2h_2, y_{29} - t_2x_1x_3h_2, y_{30} - t_2x_1x_4h_2, \\ y_{31} - t_2x_1x_5h_2, y_{32} - t_2x_1x_6h_2, y_{33} - t_2x_2x_3h_2, y_{34} - t_2x_2x_4h_2, \\ y_{35} - t_2x_2x_5h_2, y_{36} - t_2x_2x_6h_2, y_{37} - t_2x_3x_4h_2, y_{38} - t_2x_3x_5h_2, \\ y_{39} - t_2x_3x_6h_2, y_{40} - t_2x_4x_5h_2, y_{41} - t_2x_4x_6h_2, y_{42} - t_2x_5x_6h_2 \end{array}\right)$$

and

$$I = \left(\; x_3x_4 - x_2x_5 + x_1x_6, t_0h_0 - 1, t_1h_1 - 1, t_2h_2 - 1 \; \right).$$

In Macaulay 2, we calculate the saturation of I_R and I_Q with B_Y using the command "saturate",

```
i1:  IQQ = saturate(IQ,BY)
i2:  IRR = saturate(IR,BY)
i3:  IRR == IQQ
o3:  true
```

where B_Y is the intersection of

$$\big(y_1, \dots, y_{21}\big) \text{ and } \big(y_{22}, \dots, y_{42}\big).$$

It is also possible to calculate $\widetilde{I_R}$ and $\widetilde{I_Q}$ and that $\widetilde{I_Q} = \big(\widetilde{I_R} : B_Y^\infty\big)$ but we omit the calculations here. By our Macaulay 2 calculation, the collection $\mathscr{L}$ is very ample, so Theorem 4.1.1 implies that $\varphi_{|\mathscr{L}|} \colon \operatorname{Gr}(2,4) \longrightarrow \mathcal{M}_\vartheta(A_{\mathscr{L}})$ is an isomorphism.

Appendix A

Appendix A: Computing Kernels of $\mathbb{k}$-Algebra Homomorphisms

A.0.1 Kernels of $\mathbb{k}$ -Algebra Homomorphisms.

In order to calculate the Mori Dream Space analogue of I_Q from [10], we will need to be able to compute kernels of $\mathbb{k}$-algebra homomorphisms efficiently. Theorem A.0.2 gives us a way to write kernels. Then using Elimination Theory we can compute kernels using Macaulay 2.

A.0.2 Kernels

Material from this section can be found in Adams–Loustaunau [1]. Let $\varphi : \mathbb{k}[y_1, \ldots, y_m] \longrightarrow \mathbb{k}[x_1, \ldots, x_n]$ be the $\mathbb{k}$ algebra homomorphism mapping y_i to some $f_i(x_1, \ldots, x_n) \in \mathbb{k}[x_1, \ldots, x_n]$ for each i. We want to compute $\ker(\varphi)$. First we need a technical lemma.

Lemma A.0.1. *Let R be a commutative ring. If $a_1, \ldots, a_n, b_1, \ldots, b_n \in R$, then $a_1 \cdots a_n - b_1 \cdots b_n$ is contained in the ideal*

$$\big(a_1 - b_1, \ldots, a_n - b_n\big).$$

Proof. $a_1 \cdots a_n - b_1 \cdots b_n = a_1(a_2 \cdots a_n - b_2 \cdots b_n) + b_2 \cdots b_n(a_1 - b_1)$, hence by induction $a_1 \cdots a_n - b_1 \cdots b_n$ can be written as $\sum g_i(a_i - b_i)$, for $g_i \in R$. ☐

Now we are able to prove Theorem A.0.2.

Theorem A.0.2. *Let the f_i's be as above and let $K = \big(y_1 - f_1, \ldots, y_m - f_m\big) \subseteq \mathbb{k}[x_1, \ldots, x_n, y_1, \ldots, y_m]$. The kernel of φ satisfies*

$$\ker(\varphi) = K \cap \mathbb{k}[y_1, \ldots, y_n].$$

Proof. First let $g \in K \cap \Bbbk[y_1, \dots, y_n]$. We will show $g \in \ker(\varphi)$. Since $g \in K$ and $g \in \Bbbk[y_1, \dots, y_n]$, we must have

$$g(y_1, \dots, y_m) = \sum_{i=1}^{n} (y_i - f_i(x_1, \dots, x_n))h_i(y, x)$$

for some $h_i \in \Bbbk[x_1, \dots, x_m, y_1, \dots, y_m]$. Hence the image of g under φ is

$$g(f_1, \dots, f_m) = \sum_{i=1}^{n} (f_i(x_1, \dots, x_n) - f_i(x_1, \dots, x_n))h_i \ = 0,$$

and therefore $g \in \ker(\varphi)$.

Conversely, let $g \in \ker(\varphi)$. We can write

$$g = \sum c_{\mathbf{v}} y^{\mathbf{v}}$$

for some $\mathbf{v} \in \mathbb{N}^m, c_{\mathbf{v}} \in \Bbbk$. Hence

$$g(f_1, \dots, f_m) = 0 \Rightarrow g(y_1, \dots, y_m) = g(y_1, \dots, y_m) - g(f_1, \dots, f_m) = \sum c_{\mathbf{v}}(y^{\mathbf{v}} - f^{\mathbf{v}})$$

By the lemma, this shows g is in the ideal K. □

Corollary A.0.3. *Let $\varphi : \Bbbk[y_1, \dots, y_m] \longrightarrow \Bbbk[x_1, \dots, x_n]/I$ be the $\Bbbk$-algebra homomorphism mapping y_i to $f_i \in \Bbbk[x_1, \dots, x_n]/I$ for each i. The kernel of φ is*

$$\big(K + I\big) \cap \Bbbk[y_1, \dots, y_m]$$

where we consider

$$K = \big(y_1 - f_1, \dots, y_m - f_m\big)$$

and I to be ideals of $\Bbbk[x_1, \dots, x_n, y_1, \dots, y_m]$.

Proof. Let $g \in (K + I) \cap \Bbbk[y_1, \dots, y_m]$, then $g = h + j$, where $h \in K$ and $j \in I$. We can write $j = \sum c_{\mathbf{uv}j} x^{\mathbf{u}} y^{\mathbf{v}} g_j$, where $g_j \in \Bbbk[x_1, \dots, x_n]$ is a generator of I. Also, since $h \in K$ so $h(y_1, \dots, y_m) = (y_1 - f_1)p_1 + \cdots + (y_m - f_m)p_m$ for some polynomials $p_1, \dots, p_m \in \Bbbk[x_1, \dots, x_n, y_1, \dots, y_m]$. Hence

$$h(f_1, \dots, f_m) = (f_1 - f_1)p_1 + \cdots + (f_m - f_m)p_m = 0.$$

Also, $j(f_1, \dots, f_m) = \sum c_{\mathbf{uv}j} x^{\mathbf{u}} f^{\mathbf{v}} g_j \in I$, so $g \in \ker(\varphi)$.

For the converse, suppose $g \in \ker(\varphi) \subseteq \Bbbk[y_1, \dots, y_m]$. Hence $g(f_1, \dots, f_m) \in I \subseteq \Bbbk[x_1, \dots, x_n]$. We can write

$$g(y_1, \dots, y_m) = g(y_1, \dots, y_m) - g(f_1, \dots, f_m) + g(f_1, \dots, f_m).$$

If we have $g(y_1, \dots, y_m) = \sum c_{\mathbf{v}} y^{\mathbf{v}}$ then

$$g(y_1, \dots, y_m) - g(f_1, \dots, f_m) = \sum c_{\mathbf{v}}(y^{\mathbf{v}} - f^{\mathbf{v}})$$

where $\sum c_{\mathbf{v}}(y^{\mathbf{v}} - f^{\mathbf{v}}) \in K$ by Lemma A.0.1. We also have $g(f_1, \dots, f_m) \in I$, so $g(y_1, \dots, y_m) \in K + I$ as required. □

A.0.3 Elimination Theory and Macaulay 2 Calculations

Theorem A.0.2 and Corollary A.0.3 give us a way of writing kernels in a $\Bbbk$-algebra $\Bbbk[y_1, \ldots, y_m]$ as the intersection of an ideal in a larger ring with $\Bbbk[y_1, \ldots, y_m]$ considered as a subring. In order to compute these intersections, we need the Elimination Theorem. Following Cox–Little–O'Shea [5], we define the *kth elimination ideal* I_k of $I \subseteq \Bbbk[x_1, \ldots, x_n]$ to be

$$I \cap \Bbbk[x_{k+1}, \ldots, x_n].$$

Theorem A.0.4 (The Elimination Theorem). *Let $I \subseteq \Bbbk[x_1, \ldots, x_n]$ be an ideal, and let G be a Groebner basis of I with respect to lex order where $x_1 < x_2 < \cdots < x_n$. Then, for every $k \leq n$, the set*

$$G_k = G \cap \Bbbk[x_1, \ldots, x_n]$$

is a Groebner basis of the kth elimination ideal I_k.

In Macaulay 2, once we have computed the Groebner basis of $K + I$ as in Corollary A.0.3, we can compute the intersection with $\Bbbk[y_1, \ldots, y_m]$ using the command "selectInSubring". Explicitly:

```
i1:  R = QQ[x_1..x_n,y_1..y_m, MonomialOrder => Eliminate n]
i2:  K = ideal(y_1-f_1,...y_m-f_m)
i3:  I = ideal(g_1,...g_k)
i4:  G = gens gb K+I
i5:  J = selectInSubring(1,G)
i6:  kernel = ideal(J)
```

Appendix B

Appendix B: Computing $\mathrm{Cox}(X_5)$ after Batyrev–Popov and Derenthal

We give code used to compute I_{X_5} in section 2.2.3 following the method of Batyrev–Popov [2] and Derenthal [12].

We present code for finding the equation of a conic containing 5 points in $\mathbb{P}^2$.

Pseudocode B.0.5. *Input: coordinates of five points*
$p = (p_1, p_2, p_3), q = (q_1, q_2, q_3), r = (r_1, r_2, r_3), s = (s_1, s_2, s_3), t = (t_1, t_2, t_3)$.
Procedure:
$L := [ap_1^2 + bp_2^2 + cp_3^2 + dp_1p_2 + ep_1p_3 + fp_2p_3,$
$aq_1^2 + bq_2^2 + cq_3^2 + dq_1q_2 + eq_1q_3 + fq_2q_3,$
$ar_1^2 + br_2^2 + cr_3^2 + dr_1r_2 + er_1r_3 + fr_2r_3,$
$as_1^2 + bs_2^2 + cs_3^2 + ds_1s_2 + es_1s_3 + fs_2s_3,$
$at_1^2 + bt_2^2 + ct_3^2 + dt_1t_2 + et_1t_3 + ft_2t_3]$

Let L_i denote the ith term of the list L.
$S := [L_1 = 0, L_2 = 0, L_3 = 0, L_4 = 0, L_5 = 0];$
solve$(S, [a, b, c, d, e, f]);$
Output: solution of S for $a, \ldots, f$.

Proof. A general conic in three variables z_1, z_2, z_3 has the form

$$az_1^2 + bz_2^2 + cz_3^2 + dz_1z_2 + ez_1z_3 + fz_2z_3$$

for $a, b, c, d, e, f \in \Bbbk$. This procedure finds coefficients $a, \ldots, f$ for such a conic which contains $p, \ldots, t$. This is because S contains the equations of the general conic above evaluated at $p, \ldots, t$ set equal to zero, and the Maple command "solve" solves the list of equations S for $a, \ldots, f$. □

We present lattice maps which induce the $\mathrm{Pic}(X_r)$ grading of $\mathrm{Cox}(X_r)$. We use these maps to calculate all monomials in $H^0(X_r, D)$ for a line bundle $D \in \mathrm{Pic}(X_r)$. We give pseudocode for calculating these monomials when $r = 4$, the cases where $r = 5$ or 6 are similar.

For a Mori Dream Space X where $\mathrm{Cox}(X) = \Bbbk[x_1, \ldots, x_d]/I_X$ and $\mathrm{Cl}(X) \cong \mathbb{Z}^\rho$, there exists a lattice map

$$\pi : \mathbb{Z}^d \longrightarrow \mathbb{Z}^\rho$$

which induces the grading of $\mathrm{Cox}(X)$ by $\mathrm{Cl}(X)$. Sections of a line bundle D are therefore elements of $\deg^{-1}(D)$. We give the lattice maps π_4, π_5 and π_6 which induce deg for X_4 and X_5 respectively:

$$\pi_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix}$$

$$\pi_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & -1 \end{pmatrix}$$

For X_6, the matrix π_6 takes up too much space. So we write a list of the degrees in $\mathbb{Z}^7$ of the 27 variables of $\mathrm{Cox}(X_6)$:

$$\{\{0,1,0,0,0,0,0\},\{0,0,1,0,0,0,0\},\{0,0,0,1,0,0,0\},\{0,0,0,0,1,0,0\},\{0,0,0,0,0,1,0\},\{0,0,0,0,0,0,1\}$$

$$\{1,-1,-1,0,0,0,0\},\{1,-1,0,-1,0,0,0\},\{1,-1,0,0,-1,0,0\}\{1,-1,0,0,0,-1,0\},\{1,-1,0,0,0,0,-1\}$$

$$\{1,0,-1,-1,0,0,0\},\{1,0,-1,0,-1,0,0\},\{1,0,-1,0,0,-1,0\},\{1,0,-1,0,0,0,-1\},\{1,0,0,-1,-1,0,0\}$$

$$\{1,0,0,-1,0,-1,0\},\{1,0,0,-1,0,0,-1\},\{1,0,0,0,-1,-1,0\},\{1,0,0,0,-1,0,-1\},\{1,0,0,0,0,-1,-1\}$$

$$\{2,-1,-1,-1,-1,-1,0\},\{2,-1,-1,-1,-1,0,-1\},\{2,-1,-1,-1,0,-1,-1\}$$

$$\{2,-1,-1,0,-1,-1,-1\},\{2,-1,0,-1,-1,-1,-1\},\{2,0,-1,-1,-1,-1,-1\}\}$$

We present pseudocode for computing torus invariant sections of a line bundle L on X_4 (i.e. for finding elements of $\deg^{-1}(L)$) and a proof of efficacy. The code for X_5 and X_6 is similar.

Pseudocode B.0.6. *We show how to compute the sections of a line bundle* $a_0l_0 + a_1l_1 + a_2l_2 + a_3l_3 + a_4l_4 \in \mathrm{Pic}(X_4)$*. Our method is to find all the elements of* $\pi_4^{-1}(a_0, \ldots, a_4)$*. We let* $L[i]$ *denote the* i*th term in a list* L*, and* $L[i][j]$ *denote the* j*th term in* $L[i]$*, and let* $|L|$ *denote the number of elements of* L*.*

Input: $L := [a_0, a_1, a_2, a_3, a_4]$
Procedure:
$L1 := [\,]$: *(i.e. the "empty list")*
for t_5 *from 0 to* a_0 *do*
for t_6 *from 0 to* a_0 *do*
for t_7 *from 0 to* a_0 *do*
for t_8 *from 0 to* a_0 *do*
for t_9 *from 0 to* a_0 *do*
for t_{10} *from 0 to* a_0 *do*
if $t_5 + t_6 + t_7 + t_8 + t_9 + t_{10} = l_0$ *then* $L1 := [L1, [t_5, t_6, t_7, t_8, t_9, t_{10}]]$*:*

$L2 := [\,]$:
for i *from 1 to* $|L1|$ *do*
$c_5 := L1[i][1]$
$c_6 := L1[i][2]$
$c_7 := L1[i][3]$
$c_8 := L1[i][4]$
$c_9 := L1[i][5]$
$c_{10} := L1[i][6]$
$c_1 := a_1 + c_5 + c_6 + c_7$
$c_2 := a_2 + c_5 + c_8 + c_9$
$c_3 := a_3 + c_6 + c_8 + c_{10}$
$c_4 := a_4 + c_7 + c_9 + c_{10}$:
if $c_1 \geq 0$ *and* $c_2 \geq 0$ *and* $c_3 \geq 0$ *and* $c_4 \geq 0$ *then* $L2 := [L2, x_1^{c1} x_2^{c2} x_3^{c3} x_4^{c4} x_5^{c5} x_6^{c6} x_7^{c7} x_8^{c8} x_9^{c9} x_{10}^{c10}]$:
Output: L2.

Proof. By considering the matrix π_4, we see that every torus invariant section in $H^0(X, a_0l_0 + a_1l_1 + a_2l_2 + a_3l_3 + a_4l_4)$ is of the form $x_1^{c1} x_2^{c2} x_3^{c3} x_4^{c4} x_5^{c5} x_6^{c6} x_7^{c7} x_8^{c8} x_9^{c9} x_{10}^{c10}$ where

$$c_5 + c_6 + c_7 + c_8 + c_9 + c_{10} = a_0 \quad \text{(B.0.1)}$$

$$c_1 - c_5 - c_6 - c_7 = a_1 \quad \text{(B.0.2)}$$

$$c_2 - c_5 - c_8 - c_9 = a_2 \quad \text{(B.0.3)}$$

$$c_3 - c_6 - c_8 - c_{10} = a_3 \quad \text{(B.0.4)}$$

$$c_4 - c_7 - c_9 - c_{10} = a_4 \quad \text{(B.0.5)}$$

$$\text{(B.0.6)}$$

We construct $L1$ to contain all the solutions to $(B.0.1)$ with t standing in for c) for non-negative integers $t_5, \ldots, t_{10}$. Then we work through all the possible solutions to (5.3.1) (indexed by i) by defining $c_5, \ldots, c_{10}$ to be the first up to sixth terms respectively in the ith possible solution to $(B.0.1)$. Given $c_5, \ldots, c_{10}$, we define $c_1, \ldots, c_4$ according to $(B.0.2), (B.0.3), (B.0.4)$ and $(B.0.5)$ respectively. We check if this gives $c_1, \ldots, c_4 \geq 0$ and hence a section $x_1^{c1} x_2^{c2} x_3^{c3} x_4^{c4} x_5^{c5} x_6^{c6} x_7^{c7} x_8^{c8} x_9^{c9} x_{10}^{c10}$. We gather all such monomials in $L2$, and hence our output $L2$ contains every point in $\pi_4^{-1}(a_0, \ldots, a_4)$ as required. □

Recall that a ruling is the sum of two (-1)-curves whose intersection number is 1. We give pseudocode for finding rulings on X_4 and a proof of efficacy. The code for X_5 and X_6 is similar.

Pseudocode B.0.7. *First we write a list of all (-1)-curves $a_0 l_0 + a_1 l_1 + a_2 l_2 + a_3 l_3 + a_4 l_4$ on X_4 by listing the corrseponding elements of $\mathbb{Z}^5 : [a_0, a_1, a_2, a_3, a_4]$. In this format, the list of (-1)-curves L is:*

$$L := [[0,1,0,0,0],[0,0,1,0,0],[0,0,0,1,0],[0,0,0,0,1],[1,-1,-1,0,0],[1,-1,0,-1,0],$$

$$[1,-1,0,0,-1],[1,0,-1,-1,0],[1,0,-1,0,-1],[1,0,0,-1,-1]].$$

Again, we denote the ith element of L by $L[i]$, and the number of elements in L by $|L|$. We find all rulings as follows:
Input: the list L.
Procedure:
$S := [\,]:$
$f[1] := l_1$
$f[2] := l_2$
$f[3] := l_3$
$f[4] := l_4$
$f[5] := l_0 - l_1 - l_2$
$f[6] := l_0 - l_1 - l_3$

$f[7] := l_0 - l_1 - l_4$
$f[8] := l_0 - l_2 - l_3$
$f[9] := l_0 - l_2 - l_4$
$f[10] := l_0 - l_3 - l_4$
for i from 1 to $|L| - 1$ *do*
for j from $i + 1$ *to* $|L|$ *do*
if $L[i][1]L[j][1] - L[i][2]L[j][2] - L[i][3]L[j][3] - L[i][4]L[j][4] - L[i][5]L[j][5] = 1$ *then* $S := [S, f[i] + f[j]]$
S:= convert(convert(S,set),list)
Output: S.

Proof. We define the $f[i]$'s to be the (-1)-curves on X_4. We work through all the (-1)-curves, indexed by i and j and compute their intersection number:

$$L[i][1]L[j][i] - L[i][2]L[j][2] - L[i][3]L[j][3] - L[i][4]L[j][4] - L[i][5]L[j][5].$$

By definition, if the intersection number of a pair of (-1)-curves is 1, then their sum is a ruling. We collect the sum of all pairs of generators with intersection number 1 in the list S. To avoid repetitions, we convert S to a set and then back to a list. $\square$

Appendix C

Appendix C: Computing I_R

We present pseudocode to compute I_R for X when I_X is generated by quadratic polynomials with three terms, as in X_4, X_5, X_6 and Grassmannians $\mathrm{Gr}(r,n)$. We give pseudocode and proof of correctness in this case.

We introduce some notation. Fix a presentation for I_X. We denote the jth generator of I_X as $I_X[j]$, and suppose the number of generators is G. We define $r[i], s[i], t[i], u[i], v[i], w[i], l[i], m[i], n[i]$ such that:

$$I_X[i] = l[i]x_{r[i]}x_{s[i]} + m[i]x_{t[i]}x_{u[i]} + n[i]x_{v[i]}x_{w[i]}.$$

We define $D[i], E[i]$ and $F[i]$ to be vectors of length d with 1's in the $r[i]$th and $s[i]$th, $t[i]$th and $u[i]$th, and $v[i]$th and $w[i]$th positions respectively and zeros elsewhere.

Pseudocode C.0.8. *We assume I_X is quadratically generated, and that each generator has three terms.*
Input: To compute I_R for a quiver Q, our input is $P := \mathrm{getpaths}(Q)$.
Procedure:
$S := [\ ]:$
for g from 1 to G do
$L1 := [\ ]:$
$L2 := [\ ]:$
$L3 := [\ ]:$
for i from 1 to $|P|$ do
if $P[i][2][r[g]] > 0$ *and* $P[i][2][s[g]] > 0$ *then* $L1 := [L1[], [P[i][1], P[i][2] - D[g], P[i][3], P[i][4]]]$

for j from 1 to $|P|$ do
if $P[j][2][t[g]] > 0$ *and* $P[j][2][u[g]] > 0$ *then* $L2 := [L2[], [P[j][1], P[j][2] - E[g], p[j][3], p[j][4]]]$

for k from 1 to $|P|$ *do if* $P[k][2][v[g]] > 0$ *and* $P[k][2][w[g]] > 0$ *then* $L3 := [L3[], [P[k][1], P[k][2] - F[g], P[k][3][4]]]$

for a from 1 to $|L1|$ *do*
for b from 1 to $|L2|$ *do*
for c from 1 to $|L3|$ *do*
if $L1[a][1] = L2[b][1] = L3[c][1]$ *and* $L1[a][2] = L2[b][2] = L3[c][2]$*and* $L1[a][3] = L2[b][3] = L3[c][3]$
then $S := [S[], l[g]L1[a][4] + m[g]L2[b][4] + n[g]L3[c][4]]$
Output: S.

Proof. By Lemma 5.1.7, a generating set for I_R consists of the generators of $\widetilde{I_R}$, plus elements of $\Bbbk[y_a|a \in Q_1]$ of the form $\sum_i a_i y_{p_i}$ where the p_i's have the same heads and tails and $\widetilde{\Phi}(\sum_i a_i y_{p_i})$ is a monomial times a generator of I_X.

We have already found generators of $\widetilde{I_R}$, using our Maple procedure "zeropart". Since assume each generator of I_X has three terms, it remains to find all triples of paths p_1, p_2, p_3 with the same heads and tails, where each p_i is labelled by a monomial times a term of a generator of I_X, modulo constant term a_i say. We then have that $a_1 y_{p_1} + a_2 y_{p_2} + a_3 y_{p_3}$ is a generator of I_R, and by Lemma 5.1.7, once we have found all such generators, we will have a generating set for I_R.

We work through all generators of I_X, for the gth generator, we proceed as follows:

First, we define three empty lists $L1, L2$ and $L3$. We find all paths p whose labels are divisible by the first, second or third term of $IX[g]$ mod constant, and record their heads, tails, the remainder when we divide their label by a term of $IX[g]$, and y_p in $L1, L2$ or $L3$ respectively. Explicitly:

1. We work through all paths $P[i]$ and see if their labels $P[i][2]$ are divisible by the first term of $IX[g] : x_{r[g]}x_{s[g]}$. If it is divisible, we record $P[i]$'s tail, the remainder when we divide its label by $x_{r[g]}x_{s[g]}$, head and $y_{P[i]}$ in $L1$.

2. We work through all paths $P[j]$ and see if their labels $P[j][2]$ are divisible by the second term of $IX[g] : x_{t[g]}x_{u[g]}$. If it is divisible, we record $P[j]$'s tail, the remainder when we divide its label by $x_{t[g]}x_{u[g]}$, head and $y_{P[j]}$ in $L2$.

3. We work through all paths $P[k]$ and see if their labels $P[k][2]$ are divisible by the third term of $IX[g] : x_{v[g]}x_{w[g]}$. If it is divisible, we record $P[k]$'s tail, the remainder when we divide its label by $x_{v[g]}x_{w[g]}$, head and $y_{P[k]}$ in $L3$.

Secondly, we work through all entries in $L1, L2$ and $L3$. If $L1[a], L2[b]$ and $L3[c]$ have the same first, second and third entries, then they record information about paths p_1, p_2 and p_3 with the same tails, heads and remainder of their label after division by a term of $IX[g]$ mod constant. Hence, replacing constants, $l[g]y_{p_1} + m[g]y_{p_2} + n[g]y_{p_3}$ is a generator of I_R. $L1[a][4] = y_{p_1}, L2[b][4] = y_{p_2}$

and $L3[c][4] = y_{p_3}$. We record $l[g]L1[a][4] + m[g]L2[b][4] + n[g]L3[c][4]$ in S. After working through all such triples for all generators $IX[g]$, S plus the generators from Pseudocode 5.1.4 will give a generating set for I_R. $\square$

References

[1] William W. Adams and Philippe Loustaunau. *An introduction to Gröbner bases*, volume 3 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1994.

[2] Victor V. Batyrev and Oleg N. Popov. The Cox ring of a del Pezzo surface. In *Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002)*, volume 226 of *Progr. Math.*, pages 85–103. Birkhäuser Boston, Boston, MA, 2004.

[3] Aaron Bergman and Nicholas J. Proudfoot. Moduli spaces for Bondal quivers. *Pacific J. Math.*, 237(2):201–221, 2008.

[4] Ana-Maria Castravet and Jenia Tevelev. Hilbert's 14th problem and Cox rings. *Compos. Math.*, 142(6):1479–1498, 2006.

[5] David Cox, John Little, and Donal O'Shea. *Ideals, varieties, and algorithms*. Undergraduate Texts in Mathematics. Springer, New York, third edition, 2007. An introduction to computational algebraic geometry and commutative algebra.

[6] David Cox, John Little, and Hal Schenck. *Toric Varieties*, 2011. Book in Press.

[7] David A. Cox. The homogeneous coordinate ring of a toric variety. *J. Algebraic Geom.*, 4(1):17–50, 1995.

[8] Alastair Craw. *Quiver Flag Varieties and Multigraded Linear Series* . Duke Mathematical Journal 156 (2011), no. 3., 469-500.

[9] Alastair Craw. *Quiver Representations in Toric Geometry.* Preprint, arXiv:0807.2191.

[10] Alastair Craw and Gregory G. Smith. Projective toric varieties as fine moduli spaces of quiver representations. *Amer. J. Math.*, 130(6):1509–1534, 2008.

[11] Alastair Craw and Dorothy Winn. *Mori Dream Spaces as Fine Moduli of Quiver Representations.* Preprint,arXiv:1104.2490.

[12] Ulrich Derenthal. *On the Cox Ring of del Pezzo Surfaces.* Preprint, arXiv: math/0603111.

[13] Igor Dolgachev. *Lectures on invariant theory*, volume 296 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 2003.

[14] E. Javier Elizondo, Kazuhiko Kurano, and Kei-ichi Watanabe. The total coordinate ring of a normal projective variety. *J. Algebra*, 276(2):625–637, 2004.

[15] William Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

[16] Brendan Hassett and Yuri Tschinkel. Universal torsors and Cox rings. In *Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002)*, volume 226 of *Progr. Math.*, pages 149–173. Birkhäuser Boston, Boston, MA, 2004.

[17] Milena Hering, Hal Schenck, and Gregory G. Smith. Syzygies, multigraded regularity and toric varieties. *Compos. Math.*, 142(6):1499–1506, 2006.

[18] Lutz Hille and Markus Perling. *Exceptional Sequences of Invertible Sheaves on Rational Surfaces.* preprint arXiv:0810.1936.

[19] Yi Hu and Sean Keel. Mori dream spaces and GIT. *Michigan Math. J.*, 48:331–348, 2000. Dedicated to William Fulton on the occasion of his 60th birthday.

[20] Alastair King. Moduli of representations of finite-dimensional algebras. *Quart. J. Math. Oxford Ser. (2)*, 45(180):515–530, 1994.

[21] Antonio Laface and Mauricio Velasco. A survey on Cox rings. *Geom. Dedicata*, 139:269–287, 2009.

[22] Diane Maclagan and Gregory G. Smith. Multigraded Castelnuovo-Mumford regularity. *J. Reine Angew. Math.*, 571:179–212, 2004.

[23] Yu. I. Manin. *Cubic forms*, volume 4 of *North-Holland Mathematical Library.* North-Holland Publishing Co., Amsterdam, second edition, 1986. Algebra, geometry, arithmetic, Translated from the Russian by M. Hazewinkel.

[24] Shigeru Mukai. *An introduction to invariants and moduli*, volume 81 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2003. Translated from the 1998 and 2000 Japanese editions by W. M. Oxbury.