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Sum-of-squares certificates for Vizing's conjecture via determining Gröbner bases [☆]

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ABSTRACT

The famous open Vizing conjecture claims that the domination number of the Cartesian product graph of two graphs G and H is at least the product of the domination numbers of G and H . Recently Gaar, Krenn, Margulies and Wiegele used the graph class \mathcal{G} of all graphs with $n_{\mathcal{G}}$ vertices and domination number $k_{\mathcal{G}}$ and reformulated Vizing's conjecture as the problem that for all graph classes \mathcal{G} and \mathcal{H} the Vizing polynomial is sum-of-squares (SOS) modulo the Vizing ideal. By solving semidefinite programs (SDPs) and clever guessing they derived SOS-certificates for some values of $k_{\mathcal{G}}$, $n_{\mathcal{G}}$, $k_{\mathcal{H}}$, and $n_{\mathcal{H}}$.

In this paper, we consider their approach for $k_{\mathcal{G}} = k_{\mathcal{H}} = 1$. For this case we are able to derive the unique reduced Gröbner basis of the Vizing ideal. Based on this, we deduce the minimum degree $(n_{\mathcal{G}} + n_{\mathcal{H}} - 1)/2$ of an SOS-certificate for Vizing's conjecture, which is the first result of this kind. Furthermore, we present a method to find certificates for graph classes \mathcal{G} and \mathcal{H} with $n_{\mathcal{G}} + n_{\mathcal{H}} - 1 = d$ for general d , which is again based on solving SDPs, but does not depend on guessing and depends on much smaller SDPs. We implement our new method in SageMath and give new SOS-certificates for all graph classes \mathcal{G} and \mathcal{H} with $k_{\mathcal{G}} = k_{\mathcal{H}} = 1$ and $n_{\mathcal{G}} + n_{\mathcal{H}} \leq 15$.

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1. Introduction

A large area of graph theory focuses on the interrelationship of graph invariants. One of these graph invariants is the domination number $\gamma(G)$ of a simple undirected graph G , that is the minimum size of a set of vertices in G , such that each vertex in the graph is either in this set itself or adjacent to a vertex in this set. In 1968, Vizing (1968) made a conjecture regarding the domination number of the Cartesian product $G \square H$ of the graphs G and H . The vertices of $G \square H$ are the Cartesian product of the vertices in G and H , and the subgraphs of $G \square H$ induced by the vertices with same fixed first tuple entry are isomorphic to the graph G and analogously the vertices with the same second tuple entry induce subgraphs isomorphic to H . Vizing conjectured that for any graphs G and H it holds that $\gamma(G \square H) \geq \gamma(G)\gamma(H)$. To date, it is not clear whether this conjecture is true. Nevertheless, for many classes of graphs it has already been shown that Vizing's conjecture holds, see the survey of Brešar et al. (2012) for details.

The first algebraic formulation of Vizing's conjecture has been done by Margulies and Hicks in Margulies and Hicks (2012). An algebraic method to solve combinatorial problems is to encode the problem as a system of polynomial equations and apply the Nullstellensatz or Positivstellensatz. In several areas this and similar approaches have been used to show new results, for example for colorings (Alon and Tarsi, 1992; De Loera, 1995; De Loera et al., 2008; Eliahou, 1992; Hillar and Windfeldt, 2008; Lovász, 1994; Matiyasevich, 2001; Mnuk, 2001), stable sets (De Loera, 1995; De Loera et al., 2009; Gouveia et al., 2010; Li and Li, 1981; Lovász, 1994; Simis et al., 1994), flows (Alon and Tarsi, 1992; Mnuk, 2001; Onn, 2004) and matchings (Fischer, 1988) in graphs. Gaar et al. (2019) used an algebraic method to reformulate Vizing's conjecture as a sum-of-squares (SOS) program. In such a program one asks the question of whether it is possible to represent a non-negative polynomial as the sum of squares of polynomials. SOS are heavily used in the area of polynomial optimization, see for example Blekherman et al. (2013), and also in many other fields like dynamical systems, geometric theorem proving and quantum mechanics, see for example Parrilo (2004).

Such SOS programs can be solved with the help of semidefinite programming (SDP). Roughly speaking, a semidefinite program (SDP) is like a linear program but instead of a non-negative vector variable one has a positive semidefinite matrix variable and the Frobenius inner product is used instead of vector multiplications. As for linear programs, there is also a duality theory for SDPs. They are often used as relaxations of combinatorial optimization problems. The first contribution to this area was the seminal paper of Lovász (1979) in 1979. Around 1990, the interest in SDP exploded. Nowadays there are several off-the-shelf solvers for SDPs, for example MOSEK (2021) and SDPT3 (Toh et al., 1999). Some nice survey papers on SDP are for example Vandenberghe and Boyd (1996) and Todd (2001).

As already mentioned, Gaar et al. (2019, 2021) presented a new approach for proving Vizing's conjecture by finding SOS Positivstellensatz certificates with the help of SDP. In particular, they used SDP in order to prove that the so-called Vizing polynomial is SOS modulo the so-called Vizing ideal I_{viz} . In addition, they provide code to computationally find numeric certificates and check certificates for correctness. Furthermore, they gave certificates for the graph classes \mathcal{G} (all graphs with $n_{\mathcal{G}}$ vertices and domination number $k_{\mathcal{G}}$) and \mathcal{H} (all graphs with $n_{\mathcal{H}}$ vertices and domination number $k_{\mathcal{H}}$) with the property that $n_{\mathcal{G}}, k_{\mathcal{G}}, n_{\mathcal{H}}$ and $k_{\mathcal{H}}$ satisfy $k_{\mathcal{G}} = n_{\mathcal{G}} - 1 \geq 1$ and $k_{\mathcal{H}} = n_{\mathcal{H}} - 1$ for $n_{\mathcal{H}} \in \{2, 3\}$ and the graph classes with $k_{\mathcal{G}} = n_{\mathcal{G}}$ and $k_{\mathcal{H}} = n_{\mathcal{H}} - d$ for $d \leq 4$.

In this paper, we focus on the graph classes \mathcal{G} and \mathcal{H} with domination numbers $k_{\mathcal{G}} = k_{\mathcal{H}} = 1$. Due to this special choice of the parameters, we are able to determine the unique reduced Gröbner basis of the Vizing ideal I_{viz} , which is an important part of finding SOS-certificates. With the help of this Gröbner basis, we can determine the minimum degree of any certificate, which is $(n_{\mathcal{G}} + n_{\mathcal{H}} - 1)/2$.

Furthermore, we show that if SOS-certificates are of a specific form, then the certificates for graph classes with $n_{\mathcal{G}} + n_{\mathcal{H}} - 1 = d$ for a fixed integer $d \geq 3$ depend on d only. Based on that, we introduce a new method to find SOS-certificates for these graph classes. This method again makes use of SDP, but unlike in the approach of Gaar et al. (2019, 2021), no algebraic numbers have to be guessed. Additionally, the SDP that has to be solved is much smaller than the one in the other approach. With the help of our implementation of the new algorithm in SageMath (2021), we give certificates for all graph classes \mathcal{G} and \mathcal{H} with $k_{\mathcal{G}} = k_{\mathcal{H}} = 1$ up to $n_{\mathcal{G}} + n_{\mathcal{H}} \leq 15$. The program code of the

implementation discussed in Section 5 is available as ancillary files from the arXiv page of this paper at arxiv.org/src/2112.04007/anc.

For these specific graph classes with $k_G = k_H = 1$ it is clear that Vizing's conjecture holds, as it simply states that the domination number of the Cartesian product graph is greater or equal to 1, which holds for every graph. Thus, in this paper we do not advance the knowledge on whether Vizing's conjecture is true for some graph classes or not. However, deriving new certificates via an algebraic method is an important step in the area of using conic linear optimization for computer-assisted proofs, because it demonstrates that deriving such proofs is possible for a wider set of graph classes.

The paper is structured as follows. In Section 2 we present all formal definitions and the background on the algebraic method of Gaar et al. we need for our results. In this paper, we focus on the case $k_G = k_H = 1$. First, we determine the reduced Gröbner basis and state the minimum degree of a certificate for $k_G = k_H = 1$ in Section 3. In Section 4 we show how to find 2-SOS-certificates for $n_G = n_H = 2$ and for $n_G = 3$ and $n_H = 2$. Next, derived from the previous examples, we propose a general form of certificates for Vizing's conjecture in the case of $k_G = k_H = 1$ in Section 5. We work out a new general method to prove the correctness of such certificates and we also list certificates for all graph classes with $n_G + n_H \leq 15$, which we found using the newly implemented method. Finally, we conclude and point out some open questions in Section 6.

2. Formal definitions and background

Vizing's conjecture is centered around the domination number defined as follows.

Definition 2.1. Let $G = (V, E)$ be an undirected graph. A subset of vertices $D \subseteq V(G)$ is a *dominating set* of G if for all $u \in V(G) \setminus D$ there exists a vertex $v \in D$ such that u is adjacent to v . In this case we say that vertex v *dominates* vertex u . A dominating set is a *minimum dominating set* of G if there is no dominating set with smaller cardinality. The *domination number* $\gamma(G)$ is the cardinality of a minimum dominating set of G .

Our interest lies in the behavior of the domination number on the product of two graphs, the so-called Cartesian product graph, which is defined as follows.

Definition 2.2. Let G and H be two graphs. The *Cartesian product graph* $G \square H$ is a graph with vertices $V(G \square H) = V(G) \times V(H)$ and edge set

$$E(G \square H) = \left\{ \left\{ (g, h), (g', h') \right\} \mid \begin{array}{l} g = g' \in V(G) \text{ and } \{h, h'\} \in E(H), \text{ or} \\ h = h' \in V(H) \text{ and } \{g, g'\} \in E(G) \end{array} \right\}.$$

For convenience we will further write gh for a vertex $(g, h) \in V(G \square H)$. Fig. 1 shows the Cartesian product graph of the cyclic graph C_4 and the linear graph P_4 .

For any vertex $g \in V(G)$ the vertices $\{(g, h) \mid h \in V(H)\}$ induce a subgraph of $G \square H$ that is isomorphic to H . Such a subgraph is called *H-fiber* and denoted by ${}^g H$. Also for $h \in V(H)$ the subgraph G^h of $G \square H$ induced by $\{(g, h) \mid g \in V(G)\}$ is called a *G-fiber*.

In 1963, Vizing asked the question about the connection between the domination numbers of G and H and the domination number of the Cartesian product graph of G and H in Vizing (1963). Five years later, he published the following conjecture in Vizing (1968).

Conjecture 2.3 (*Vizing's conjecture*). *For any two graphs G and H , the inequality*

$$\gamma(G \square H) \geq \gamma(G)\gamma(H)$$

holds.

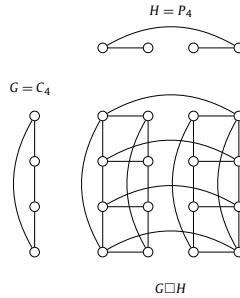


Fig. 1. The Cartesian product graph $C_4 \square P_4$.

To date, there is no answer to the question of whether Vizing’s conjecture is true. The typical approach to attack Vizing’s conjecture is to show that for a specific G Vizing’s conjecture holds for any graph H . Many results are based on the assumption that G can be partitioned into subgraphs of a special kind. The conjecture holds for example whenever G is a cycle, a tree, or has domination number less than or equal to 3 and H may be any graph. Furthermore, Zerbib (2019) proved that

$$\gamma(G \square H) \geq \frac{1}{2} \gamma(G) \gamma(H) + \frac{1}{2} \max\{\gamma(G), \gamma(H)\},$$

a weaker result. To show that Vizing’s conjecture is false, one may try to find a counterexample. Some properties of a minimal counterexample are known, for example it has to be a graph with domination number greater than 3 and for each vertex $g \in V(G)$ there has to exist a minimum dominating set that contains g . We refer to the survey paper of Brešar et al. (2012) for an exceedingly nice and structured overview of the results on Vizing’s conjecture.

In order to present new results for and with the approach of Gaar, Krenn, Margulies and Wiegele introduced in Gaar et al. (2019, 2021), we continue with algebraic basics needed throughout the paper. For more details and, in particular, the definitions of (total degree lexicographical) term orders, the leading term and (reduced) Gröbner bases, which we will need in Section 3, we refer the reader to the book of Cox et al. (2015).

By I we denote an ideal in a polynomial ring $P = \mathbb{K}[z_1, \dots, z_n]$ over a real field $\mathbb{K} \subseteq \mathbb{R}$. By $\overline{\mathbb{K}}$ we denote the algebraic closure of \mathbb{K} . We denote by $\mathcal{V}(I) = \{z^* \in \overline{\mathbb{K}}^n \mid f(z^*) = 0 \text{ for all } f \in I\}$ the variety of the ideal I . The ideal we consider in this paper is proven to be radical (i.e., for any polynomial $f \in P$ and any positive integer m the fact that $f^m \in I$ implies that f is in the ideal I) by Gaar et al. (2019). This allows us to apply the following important theorem.

Theorem 2.4 (Hilbert’s Nullstellensatz for radical ideals). *Let $P = \mathbb{K}[z_1, \dots, z_n]$ be a polynomial ring over a field \mathbb{K} and $I \subseteq P$ a radical ideal. If $f(z^*) = 0$ for all $z^* \in \mathcal{V}(I)$ for some $f \in P$, then f is in the ideal I .*

Note that if the ideal I is finitely generated by the polynomials $f_1, \dots, f_r \in P$, it is enough to check that f vanishes on the common zeros of the generating polynomials (over the algebraic closure $\overline{\mathbb{K}}$).

The main idea of the approach by Gaar et al. is to prove Vizing’s conjecture by showing for a particular constructed ideal that a specific polynomial is non-negative on the variety of the ideal. For this purpose, they use the subsequent definitions.

Definition 2.5. Two polynomials $f, g \in P$ are congruent modulo an ideal I (denoted by $f \equiv g \pmod I$), if $f - g \in I$ or equivalently $f = g + h$ for some $h \in I$.

Let ℓ be a non-negative integer. A polynomial $f \in P$ is ℓ -sum-of-squares modulo I (ℓ -SOS modulo I), if there are polynomials $s_1, \dots, s_t \in P$ of degree at most ℓ such that

$$f \equiv \sum_{i=1}^t s_i^2 \pmod I.$$

We say that the polynomials s_1, \dots, s_ℓ form an SOS-certificate of degree ℓ .

In the approach of Gaar et al., Vizing’s conjecture is investigated for classes of graphs for G and H with fixed number of vertices in the graph and fixed domination number. The graph classes are denoted in the following way.

Definition 2.6. Let n_G and k_G be positive integers with $k_G \leq n_G$ defining the class of graphs \mathcal{G} as the set of graphs with n_G vertices and fixed minimum dominating set D_G of size k_G .

Without loss of generality, the minimum dominating set D_G is fixed. All other graphs can be obtained by relabeling the vertices. For $k_G = k$, we set $D_G = \{g_1, \dots, g_k\}$.

In a next step, Gaar et al. construct an ideal, in which points in the variety correspond to graphs in the graph class \mathcal{G} . The variables in this setting are boolean edge variables $e_{gg'}$ indicating whether there is an edge between the vertices g and g' .

Definition 2.7. Let the set of variables be $e_G = \{e_{gg'} \mid g \neq g' \in V(\mathcal{G})\}$. The ideal $I_G \subseteq P_G = \mathbb{K}[e_G]$ is generated by the polynomials

$$e_{gg'}(e_{gg'} - 1) \quad \text{for all } g \neq g' \in V(\mathcal{G}), \tag{1a}$$

$$\prod_{g' \in D_G} (1 - e_{gg'}) \quad \text{for all } g \in V(\mathcal{G}) \setminus D_G, \tag{1b}$$

$$\prod_{g' \in V(\mathcal{G}) \setminus S} \left(\sum_{g \in S} e_{gg'} \right) \quad \text{for all } S \subseteq V(\mathcal{G}) \text{ with } |S| = k_G - 1. \tag{1c}$$

Note that in the case $k_G = 1$ (1b) simplifies to $(1 - e_{gg_1})$ for all $g \in V(\mathcal{G})$ with $g \neq g_1$ as $D_G = \{g_1\}$ and (1c) is void.

Gaar et al. proved that the following theorem holds.

Theorem 2.8. The points in the variety of I_G are in bijection to the graphs in \mathcal{G} .

We write e_G^* for elements in the variety of I_G , and by $e_{gg'}^*$ we denote the coordinate of e_G^* corresponding to the variable $e_{gg'}$.

Any point e_G^* in the variety of I_G is a common zero of the generating polynomials. Let G be the graph the point e_G^* corresponds to according to Theorem 2.8. Then (1a) ensures that $e_{gg'}^*$ is either 0 or 1 and indicates whether g is adjacent to g' in G , (1b) guarantees that all vertices of G are dominated by D_G and (1c) makes sure that D_G is a minimum dominating set, thus the domination number of G is indeed k_G .

Analogously, Gaar et al. introduce the ideal $I_{\mathcal{H}} \subseteq P_{\mathcal{H}} = \mathbb{K}[e_{\mathcal{H}}]$ corresponding to the class \mathcal{H} , which contains all graphs of size $n_{\mathcal{H}}$ and fixed minimum dominating set $D_{\mathcal{H}}$ of size $k_{\mathcal{H}}$. Moreover, they consider the graph class $\mathcal{G} \square \mathcal{H}$, consisting of all Cartesian product graphs of graphs from \mathcal{G} and \mathcal{H} . The ideal $I_{\mathcal{G} \square \mathcal{H}}$, where the boolean variables x_{gh} indicate whether the vertex $(g, h) \in V(\mathcal{G} \square \mathcal{H})$ is in the dominating set, is constructed as follows.

Definition 2.9. Let $x_{\mathcal{G} \square \mathcal{H}} = \{x_{gh} \mid g \in V(\mathcal{G}), h \in V(\mathcal{H})\}$. The ideal $I_{\mathcal{G} \square \mathcal{H}} \subseteq P_{\mathcal{G} \square \mathcal{H}} = \mathbb{K}[x_{\mathcal{G} \square \mathcal{H}} \cup e_G \cup e_{\mathcal{H}}]$ is generated by the polynomials

$$x_{gh}(x_{gh} - 1) \text{ and} \tag{2a}$$

$$(1 - x_{gh}) \left(\prod_{\substack{g' \in V(\mathcal{G}) \\ g' \neq g}} (1 - e_{gg'} x_{g'h}) \right) \left(\prod_{\substack{h' \in V(\mathcal{H}) \\ h' \neq h}} (1 - e_{hh'} x_{gh'}) \right) \tag{2b}$$

for all $g \in V(\mathcal{G})$ and $h \in V(\mathcal{H})$.

Next, Gaar et al. introduced a final ideal with the following properties.

Definition 2.10. For given graph classes \mathcal{G} and \mathcal{H} the *Vizing ideal* $I_{\text{viz}} \subseteq P_{\mathcal{G} \square \mathcal{H}}$ is defined as the ideal generated by the elements of $I_{\mathcal{G}}$, $I_{\mathcal{H}}$ and $I_{\mathcal{G} \square \mathcal{H}}$.

Lemma 2.11. *The ideal I_{viz} is radical with finite variety.*

Theorem 2.12. *The points in the variety $\mathcal{V}(I_{\text{viz}})$ are in bijection to the triples (G, H, D) , where $G \in \mathcal{G}$, $H \in \mathcal{H}$ and $D \subseteq V(G \square H)$ is any (not necessary minimum) dominating set in $G \square H$.*

We denote the elements from the variety of I_{viz} by z^* , and by x_{gh}^* , $e_{gg'}^*$ and $e_{hh'}^*$, we refer to the different coordinates of z^* for $g, g' \in V(\mathcal{G})$, and $h, h' \in V(\mathcal{H})$. For $z^* \in \mathcal{V}(I_{\text{viz}})$, the polynomial (2a) implies that x_{gh}^* is 0 or 1, which indicates if the vertex (g, h) is in the dominating set D that corresponds to z^* according to Theorem 2.12. Furthermore, (2b) warrants that D is a dominating set.

The polynomial of special interest for Gaar et al. is the so-called Vizing polynomial defined as follows.

Definition 2.13. For given graph classes \mathcal{G} and \mathcal{H} , the *Vizing polynomial* is defined as

$$f_{\text{viz}} = \left(\sum_{gh \in V(\mathcal{G} \square \mathcal{H})} x_{gh} \right) - k_{\mathcal{G}} k_{\mathcal{H}}.$$

With the help of this polynomial, Gaar et al. formulate the following important theorem, that provides a new method to prove Vizing's conjecture.

Theorem 2.14. *Vizing's conjecture is true if and only if for all positive integers $n_{\mathcal{G}}$, $k_{\mathcal{G}}$, $n_{\mathcal{H}}$ and $k_{\mathcal{H}}$ with $k_{\mathcal{G}} \leq n_{\mathcal{G}}$ and $k_{\mathcal{H}} \leq n_{\mathcal{H}}$, there exists a positive integer ℓ such that the Vizing polynomial f_{viz} is ℓ -SOS modulo I_{viz} .*

Note that Theorem 2.14 is based on a result that connects the non-negativity of a polynomial on a variety of an ideal with the fact that this polynomial is ℓ -SOS modulo the ideal for some value of ℓ , see Gaar et al. (2021, Lemma 2.8), and also Laurent (2009, Theorem 2.4), which is based on results by Parrilo (2002), for further details.

We want to point out that with arguments like the ones of Lasserre (2001), one can obtain an upper bound on the ℓ to consider in Theorem 2.14. In particular, due to the generators (1a) and (2a) of I_{viz} , every monomial can be reduced over I_{viz} such that each variable has power at most one. Thus, when setting up the SDP, it suffices to consider all possible monomials that contain each variable with power at most one. As a result, in Theorem 2.14 this gives $\ell \leq n_{\mathcal{G}} n_{\mathcal{H}} + \binom{n_{\mathcal{G}}}{2} + \binom{n_{\mathcal{H}}}{2}$.

To find SOS-certificates for Vizing's conjecture as in Theorem 2.14, Gaar et al. (2021) formulated these problems of finding SOS-certificates as SDPs as described below.

They first fix $n_{\mathcal{G}}$, $n_{\mathcal{H}}$, $k_{\mathcal{G}}$ and $k_{\mathcal{H}}$ and determine I_{viz} . Let B be a Gröbner basis of I_{viz} and fix ℓ to be some positive integer. Let v be the vector of all monomials in $P_{\mathcal{G} \square \mathcal{H}}$ of degree smaller or equal to ℓ , that equal themselves when reduced by B . It is enough to consider these monomials as potential parts of the polynomials s_1, \dots, s_t of an ℓ -SOS-certificate. Let u be the length of v . Furthermore, let S be a real $t \times u$ matrix, where the entries of row i represent the coefficients of the monomials from v in s_i . Then it holds that Sv is the vector $(s_1, \dots, s_t)^\top$. Now, let X be the positive semidefinite matrix $S^\top S$, then

$$\sum_{i=1}^t s_i^2 = (Sv)^\top (Sv) = v^\top X v$$

holds. As a result, the polynomials s_1, \dots, s_t form an ℓ -SOS-certificate if and only if

$$v^\top X v \equiv f_{\text{viz}} \pmod{I_{\text{viz}}}$$

holds, which is the case if for both sides of the equivalence the unique remainder of reduction by B is the same. By equating the coefficients, Gaar et al. obtain linear equations in the entries of the variable matrix X . To find a matrix X , which satisfies these equations and is additionally positive semidefinite, they set up an SDP with the constraints obtained by these equations. The objective function of this SDP can be chosen arbitrarily as any feasible solution gives rise to an ℓ -SOS-certificate.

Note that it is also possible to set up an SDP to decide whether a polynomial is ℓ -SOS without the knowledge of a Gröbner basis, as it is described for example by Laurent (2007, 2009). However, the number of variables and constraints of this alternative SDP may be significantly larger. Thus, using the Gröbner basis of I_{viz} is a useful technical tool to reduce the size of the occurring SDP.

Once an optimal solution X of the SDP is found, the matrix S is derived by computing the eigenvalue decomposition $X = Q^\top \Lambda Q$ and setting $S = \Lambda^{1/2} Q$. Unfortunately, the entries of X are numerical, meaning that the values in S do not represent an exact certificate. The strategy of Gaar et al. is to find an objective function such that one can guess exact values for the entries in X or S and then check whether the obtained certificate is indeed valid with the code provided in Gaar et al. (2021).

The final step is to prove the correctness of the found certificate algebraically. Ideally, one discovers some structures and finds a way to determine a general certificate for further graph classes like Gaar et al. did.

To sum up, the approach presented by Gaar et al. consists of the following steps. First fix n_G, n_H, k_G and k_H and compute a reduced Gröbner basis of I_{viz} , then set up and solve an SDP in order to get a numeric certificate. Next, guess an exact certificate and verify the certificate computationally. Finally, prove the correctness of the certificate and generalize the certificate. In this way, Gaar et al. successfully derived SOS-certificates for $k_G = n_G - 1 \geq 1$ and $k_H = n_H - 1$ where $n_H \in \{2, 3\}$, and for $k_G = n_G$ and $k_H = n_H - d$ where $d \leq 4$.

3. Gröbner basis of the Vizing ideal for $k_G = k_H = 1$

In this paper, we focus on Vizing’s conjecture for graphs G and H with domination number 1, so we consider graph classes \mathcal{G} and \mathcal{H} with $k_G = k_H = 1$ and fixed dominating sets $D_G = \{g_1\}$ and $D_H = \{h_1\}$. In particular, this implies that g_1 and h_1 are adjacent to all other vertices of G and H , respectively. In this section, we first derive some simple statements with similar methods resulting from the work of Gaar et al. (2019, 2021), which we then use to determine the Gröbner basis of the Vizing ideal and to derive the minimum degree of any SOS-certificate.

3.1. Auxiliary results

The statements we derive in this section are based on the proof techniques of Gaar et al. (2019, 2021) and are similar to those in Section 5.1 of Gaar et al. (2021). We introduce the subsets of vertices

$$T_{gh} = \{(g', h') \in V(\mathcal{G} \square \mathcal{H}) \mid g' = g \text{ or } h' = h\} = V(\mathcal{G}^h) \cup V({}^g \mathcal{H})$$

from $V(\mathcal{G} \square \mathcal{H})$, which are potentially adjacent to a vertex $(g, h) \in V(\mathcal{G} \square \mathcal{H})$ in $\mathcal{G} \square \mathcal{H}$. The corresponding variables of the vertices in T_{gh} are exactly those which appear in the polynomial (2b). This leads to the following lemma.

Lemma 3.1. *The polynomial*

$$\prod_{(g', h') \in T_{gh}} (1 - x_{g'h'})$$

is in the Vizing ideal I_{viz} for all $g \in V(\mathcal{G}), h \in V(\mathcal{H})$.

Proof. Let $z^* \in \mathcal{V}(I_{\text{viz}})$ be a common zero of the generating polynomials of I_{viz} . This implies for given $g \in V(\mathcal{G})$ and $h \in V(\mathcal{H})$, that

$$(1 - x_{gh}^*) \left(\prod_{\substack{g' \in V(\mathcal{G}) \\ g' \neq g}} (1 - e_{gg'}^* x_{g'h}^*) \right) \left(\prod_{\substack{h' \in V(\mathcal{H}) \\ h' \neq h}} (1 - e_{hh'}^* x_{gh'}^*) \right) = 0.$$

Moreover, we know that $e_{gg'}^* \in \{0, 1\}$ for all $g, g' \in V(\mathcal{G})$ and $e_{hh'}^* \in \{0, 1\}$ for all $h, h' \in V(\mathcal{H})$. Therefore,

$$(1 - x_{gh}^*) \left(\prod_{\substack{g' \in V(\mathcal{G}) \\ g' \neq g}} (1 - x_{g'h}^*) \right) \left(\prod_{\substack{h' \in V(\mathcal{H}) \\ h' \neq h}} (1 - x_{gh'}^*) \right) = 0$$

holds, which implies that z^* is a zero of $\prod_{(g', h') \in T_{gh}} (1 - x_{g'h'})$. Applying Hilbert's Nullstellensatz for radical ideals (Theorem 2.4) proves the lemma, as I_{viz} is radical (Lemma 2.11). \square

We further define the following polynomials.

Definition 3.2. Let $(g, h) \in V(\mathcal{G} \square \mathcal{H})$ and let $i \leq n_{\mathcal{G}} + n_{\mathcal{H}} - 1$ be a non-negative integer. Then the polynomial ρ_{gh}^i is defined as

$$\rho_{gh}^i = \sum_{\substack{S \subseteq T_{gh} \\ |S|=i}} \prod_{(g', h') \in S} x_{g'h'}.$$

Note that the polynomial ρ_{gh}^i is the sum of all monomials consisting of i distinct variables from T_{gh} .

Lemma 3.3. *It holds that*

$$\prod_{(g', h') \in T_{gh}} (1 - x_{g'h'}) = \sum_{i=0}^{n_{\mathcal{G}} + n_{\mathcal{H}} - 1} (-1)^i \rho_{gh}^i.$$

Proof. By expanding the product we get a sum whose summands are a product of i negative vertex variables and $|T_{gh}| - i = n_{\mathcal{G}} + n_{\mathcal{H}} - 1 - i$ ones for all values of i between 0 and $n_{\mathcal{G}} + n_{\mathcal{H}} - 1$. Since the polynomial ρ_{gh}^i is the sum of all monomials consisting of i distinct variables corresponding to the vertices $(g', h') \in T_{gh}$ and $\rho_{gh}^0 = 1$, the equality holds. \square

By simple reductions and combinatorial reasoning, we obtain the following lemma.

Lemma 3.4. *Let $T \subseteq V(\mathcal{G} \square \mathcal{H})$ be a non-empty set of cardinality d . For any positive integer $k \leq d$ we define the polynomial π^k as*

$$\pi^k = \sum_{\substack{S \subseteq T \\ |S|=k}} \prod_{(g, h) \in S} x_{gh}.$$

Then for all integers i, j with $1 \leq i \leq j \leq d$ it holds that

$$\pi^i \pi^j \equiv \sum_{r=0}^{\min\{i, d-j\}} \binom{i}{r} \binom{j+r}{i} \pi^{j+r} \pmod{I_{\text{viz}}}.$$

Proof. From the generating polynomial (2a) it follows that $x^2 \equiv x \pmod{I_{\text{viz}}}$ for all variables $x \in T$. Furthermore, all monomials in the polynomial π^j have degree j and those in π^i are of degree i . This implies that the monomials in $\pi^i \pi^j$ reduced by the generating polynomials in (2a) have at least degree j and the maximum degree is the minimum of $i + j$ and the maximum number of distinct variables d . Therefore,

$$\pi^i \pi^j \equiv \sum_{r=0}^{\min\{i, d-j\}} \phi_r \pi^{j+r} \pmod{I_{\text{viz}}}$$

for some coefficients $\phi_r \in \mathbb{Z}$.

In order to determine ϕ_r let us take a closer look at the coefficient of π^{j+r} after we reduced $\pi^i \pi^j$ by (2a). All monomials of the polynomial π^{j+r} consist of $j + r$ different variables. When we multiply two monomials m_1 and m_2 with i and j different variables, the resulting reduced monomial m consists of $j + r$ distinct variables, if r variables of m_1 are in m_1 but not in m_2 and $i - r$ variables of m_1 are in both monomials. This can be viewed as dividing $j + r$ variables into 3 groups with $i - r$, r , and $j + r - i$ elements. Hence, for a fixed monomial m in π^{j+r} , this gives us

$$\phi_r = \frac{(j+r)!}{(i-r)!r!(j+r-i)!} = \binom{i}{r} \binom{j+r}{i}$$

different ways to choose the monomials m_1 and m_2 such that $m_1 m_2 \equiv m \pmod{I_{\text{viz}}}$. \square

Lemma 3.4 can be applied to the polynomials ρ_{gh}^i and leads to the following corollary.

Corollary 3.5. For all $(g, h) \in V(\mathcal{G} \square \mathcal{H})$ and for all integers i, j with $1 \leq i \leq j \leq n_{\mathcal{G}} + n_{\mathcal{H}} - 1$ holds

$$\rho_{gh}^i \rho_{gh}^j \equiv \sum_{r=0}^{\min\{i, n_{\mathcal{G}} + n_{\mathcal{H}} - 1 - j\}} \binom{i}{r} \binom{j+r}{i} \rho_{gh}^{j+r} \pmod{I_{\text{viz}}}.$$

One can observe that the form of products of such polynomials solely depends on $|T_{gh}| = n_{\mathcal{G}} + n_{\mathcal{H}} - 1$. Later on, this fact allows us to derive the certificates for all graph classes \mathcal{G} and \mathcal{H} with $k_{\mathcal{G}} = k_{\mathcal{H}} = 1$ and $n_{\mathcal{G}} + n_{\mathcal{H}} - 1 = d$ from the certificate of one of these graph classes for fixed d .

3.2. Gröbner basis of the Vizing ideal

Our first step on the way to compute an SOS-certificate is to determine a Gröbner basis of the Vizing ideal I_{viz} . Note that in our case of $k_{\mathcal{G}} = k_{\mathcal{H}} = 1$ we fix the minimum dominating sets to $D_{\mathcal{G}} = \{g_1\}$ and $D_{\mathcal{H}} = \{h_1\}$. We start with the following lemma.

Lemma 3.6. Let $k_{\mathcal{G}} = 1$, then $1 - e_{gg_1} \in I_{\mathcal{G}} \subseteq I_{\text{viz}}$ holds for all $g \in V(\mathcal{G}) \setminus \{g_1\}$.

Proof. For $n_{\mathcal{G}} = 1$ this holds trivially, for $n_{\mathcal{G}} > 1$ it follows directly from (1b). \square

To describe the elements of the Gröbner basis, we define the following sets of vertices.

Definition 3.7. For some vertex $(g, h) \in V(\mathcal{G} \square \mathcal{H})$ in the Cartesian product graph we define the following subsets of T_{gh} as

$$U_{gh}^r = \begin{cases} \{(g, h') \in V(\mathcal{G} \square \mathcal{H}) \mid h' \notin \{h_1, h\}\}, & \text{for } h \neq h_1 \\ \emptyset, & \text{for } h = h_1 \end{cases}$$

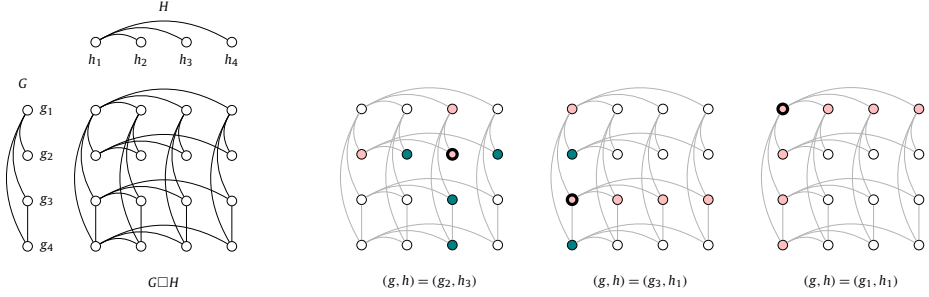


Fig. 2. Illustration of $G \square H$ and U_{gh} and \bar{U}_{gh} for different choices of (g, h) in $G \square H$.

$$U_{gh}^c = \begin{cases} \{(g', h) \in V(G \square H) \mid g' \notin \{g_1, g\}\}, & \text{for } g \neq g_1 \\ \emptyset, & \text{for } g = g_1 \end{cases},$$

$$U_{gh} = U_{gh}^r \cup U_{gh}^c \quad \text{and}$$

$$\bar{U}_{gh} = T_{gh} \setminus U_{gh}.$$

The next example uses different selections of (g, h) to illustrate the rather technical definitions of U_{gh} and \bar{U}_{gh} .

Example 3.8. Fig. 2 shows the vertices in U_{gh} and \bar{U}_{gh} in one graph $G \square H$ of the graph class $\mathcal{G} \square \mathcal{H}$ with $n_{\mathcal{G}} = n_{\mathcal{H}} = 4$ and $k_{\mathcal{G}} = k_{\mathcal{H}} = 1$ for different choices of $(g, h) \in V(G \square H)$. The vertex (g, h) is highlighted with a thicker border, the vertices \circ are in the set \bar{U}_{gh} , whereas the ones marked as \bullet are in U_{gh} .

Note that the vertices in U_{gh} and \bar{U}_{gh} are independent of the choice of $G \square H$ and only depend on \mathcal{G} and \mathcal{H} . More precisely, the vertices in \bar{U}_{gh} are exactly those adjacent to (g, h) in any graph $G \square H$ of the class $\mathcal{G} \square \mathcal{H}$ and the vertex (g, h) itself. This means that the vertices in U_{gh} are those, which are not necessarily adjacent to (g, h) .

Besides the polynomials encountered so far, there is also a new type of polynomial in the Gröbner basis. To express these, we make use of the Iverson notation. In particular, for a statement A the value of the expression $\llbracket A \rrbracket$ is 1 if A is true and 0 otherwise. The following lemma shows that also these new polynomials are in I_{viz} .

Lemma 3.9. For all vertices $(g, h) \in V(\mathcal{G} \square \mathcal{H})$ and for all choices of subsets $M \subseteq U_{gh}$, the polynomial

$$\prod_{(g', h') \in \bar{U}_{gh}} (x_{g'h'} - 1) \prod_{(g, h') \in U_{gh}^r} (\llbracket (g, h') \in M \rrbracket (x_{gh'} - e_{hh'}) + e_{hh'} - 1) \times$$

$$\times \prod_{(g', h) \in U_{gh}^c} (\llbracket (g', h) \in M \rrbracket (x_{g'h} - e_{gg'}) + e_{gg'} - 1) \quad (3)$$

is in the Vizing ideal I_{viz} .

Proof. Let $z^* \in \mathcal{V}(I_{\text{viz}})$. By Theorem 2.12, z^* is in bijection to a triple (G, H, D) with $G \in \mathcal{G}$, $H \in \mathcal{H}$ and D is a dominating set of $G \square H$. Assume that z^* is not a zero of (3) for some vertex $(g, h) \in V(\mathcal{G} \square \mathcal{H})$ and some $M \subseteq U_{gh}$.

Since all edge and vertex variables are boolean, this implies that all variables corresponding to vertices in \bar{U}_{gh} , especially x_{gh}^* , have to be zero. For all other vertices in T_{gh} we have that the vertex variable has to be zero if the vertex is in the set M and the edge variable indicating whether there is an edge between the vertex and (g, h) has to be zero if the vertex is not in M . This implies that

$$(1 - x_{gh}^*) \left(\prod_{\substack{g' \in V(\mathcal{G}) \\ g' \neq g}} (1 - e_{gg'}^* x_{g'h}^*) \right) \left(\prod_{\substack{h' \in V(\mathcal{H}) \\ h' \neq h}} (1 - e_{hh'}^* x_{gh'}^*) \right) = 1$$

and therefore, z^* is not a zero of (2b), thus z^* can not be a common zero of the polynomials generating I_{viz} , which contradicts $z^* \in \mathcal{V}(I_{viz})$. Hence, the polynomial (3) vanishes on $\mathcal{V}(I_{viz})$ and with Hilbert’s Nullstellensatz for radical ideals (Theorem 2.4) the claim follows, as I_{viz} is radical (Lemma 2.11). \square

The next two lemmas will be the main ingredients to prove that the polynomials generating I_{viz} can be generated by the polynomials of the prospective Gröbner basis.

Lemma 3.10. For all vertices $(g, h) \in V(\mathcal{G} \square \mathcal{H})$, the polynomial

$$\prod_{(g,h') \in U_{gh}^r} (e_{hh'} x_{gh'} - 1) \prod_{(g',h) \in U_{gh}^c} (e_{gg'} x_{g'h} - 1) \tag{4}$$

is equal to

$$\sum_{m=0}^{|U_{gh}|} \sum_{\substack{M \subseteq U_{gh} \\ |M|=m}} \prod_{(g',h') \in M} (x_{g'h'} - 1) \prod_{(g,h') \in U_{gh}^r} (e_{hh'} - \llbracket (g,h') \notin M \rrbracket) \prod_{(g',h) \in U_{gh}^c} (e_{gg'} - \llbracket (g',h) \notin M \rrbracket). \tag{5}$$

Proof. Using the fact that

$$\prod_{i=1}^n (y_i - 1) = \sum_{m=0}^n \sum_{\substack{M \subseteq \{1, \dots, n\} \\ |M|=m}} (-1)^{n-m} \prod_{i \in M} y_i \tag{6}$$

holds for any variables y_1, \dots, y_n by expanding the product, we get that (4) is equal to

$$\sum_{m=0}^{|U_{gh}|} \sum_{\substack{M \subseteq U_{gh} \\ |M|=m}} (-1)^{|U_{gh}|-m} \prod_{(g,h') \in U_{gh}^r \cap M} e_{hh'} x_{gh'} \prod_{(g',h) \in U_{gh}^c \cap M} e_{gg'} x_{g'h}.$$

Then, we consider one summand of (5) for a fixed $M \subseteq U_{gh}$, so

$$\prod_{(g',h') \in M} (x_{g'h'} - 1) \prod_{(g,h') \in U_{gh}^r} (e_{hh'} - \llbracket (g,h') \notin M \rrbracket) \prod_{(g',h) \in U_{gh}^c} (e_{gg'} - \llbracket (g',h) \notin M \rrbracket). \tag{7}$$

Applying (6) to the first product in (7) yields that (7) equals

$$\left(\prod_{(g',h') \in M} x_{g'h'} + \sum_{k=0}^{m-1} \sum_{\substack{K \subseteq M \\ |K|=k}} (-1)^{m-k} \prod_{(g',h') \in K} x_{g'h'} \right) \times \\ \times \prod_{(g,h') \in U_{gh}^r} (e_{hh'} - \llbracket (g,h') \notin M \rrbracket) \prod_{(g',h) \in U_{gh}^c} (e_{gg'} - \llbracket (g',h) \notin M \rrbracket).$$

Since all vertices in M are either in U_{gh}^r or in U_{gh}^c , we can rewrite this polynomial as

$$\prod_{(g,h') \in U_{gh}^r \cap M} e_{hh'} x_{gh'} \prod_{(g',h) \in U_{gh}^c \cap M} e_{gg'} x_{g'h} \prod_{(g,h') \in U_{gh}^r \setminus M} (e_{hh'} - 1) \prod_{(g',h) \in U_{gh}^c \setminus M} (e_{gg'} - 1) + \left(\sum_{k=0}^{m-1} \sum_{\substack{K \subseteq M \\ |K|=k}} (-1)^{m-k} \prod_{(g',h') \in K} x_{g'h'} \right) \times \prod_{(g,h') \in U_{gh}^r} (e_{hh'} - \llbracket (g,h') \notin M \rrbracket) \prod_{(g',h) \in U_{gh}^c} (e_{gg'} - \llbracket (g',h) \notin M \rrbracket), \tag{8}$$

which, using the fact that $m = |M|$, can be further rewritten as

$$(-1)^{|U_{gh}|-m} \prod_{(g,h') \in U_{gh}^r \cap M} e_{hh'} x_{gh'} \prod_{(g',h) \in U_{gh}^c \cap M} e_{gg'} x_{g'h} + p_M \tag{9}$$

for some polynomial p_M that depends on the set M and that captures the whole second summand of (8) and every part of the first summand of (8) that does not contain only -1 in the third and fourth factor after expanding the third and the fourth factor.

To finish the proof, it remains to show that the polynomial

$$p = \sum_{m=0}^{|U_{gh}|} \sum_{\substack{M \subseteq U_{gh} \\ |M|=m}} p_M$$

is equal to the zero polynomial.

All monomials in the expanded expression of p have in common that the number of occurring edge variables is greater than the number of occurring vertex variables. Indeed, when expanding the product (7) we get that if a vertex variable is a factor of a monomial, the corresponding edge variable is a factor of this monomial too. Moreover, all monomials that have the same number of edge and vertex variables are captured within the first summand of (9). As a result, all monomials in p have less vertex variables than edge variables.

Now, choose some fixed monomial q in p of degree $k + \ell$ that is a product of k vertex and ℓ edge variables, so $0 \leq k < \ell \leq |U_{gh}|$ holds. To determine the coefficient of q in p we count the number of different choices of the set M such that q is a summand in p_M . Clearly, if $|M| = m$, then $m \geq k$ has to hold. The k vertex variables in q determine k vertices that have to be in M . Note that the corresponding edge variables are in q too. Then, there are $\ell - k$ edge variables in q left that do not correspond to a vertex variable. Of these $\ell - k$ edge variables, $m - k$ correspond to a vertex in M . Therefore, there are $\binom{\ell-k}{m-k}$ different choices of $M \subseteq U_{gh}$ with $|M| = m$ such that q is a summand of p_M with coefficient $(-1)^{m-k+|U_{gh}|-\ell}$. Hence, the coefficient of q in p equals

$$\sum_{m=k}^{\ell} \binom{\ell-k}{m-k} (-1)^{m-k+|U_{gh}|-\ell}.$$

Substituting $i = m - k$ and $n = \ell - k$, this coefficient can be written as

$$\sum_{i=0}^n \binom{n}{i} (-1)^{i+|U_{gh}|-\ell} = (-1)^{|U_{gh}|-\ell} \sum_{i=0}^n (-1)^i \binom{n}{i}.$$

Using the identity

$$\sum_{i=0}^n (-1)^i \binom{n}{i} = 0,$$

we get that all coefficients are zero and therefore p is in fact the zero polynomial, which completes the proof. \square

Lemma 3.11. Let $(g, h) \in V(\mathcal{G} \square \mathcal{H})$, then the equations

$$(e_{g_1 g'} x_{g'h} - 1)p = (x_{g'h} - 1)p + x_{g'h}(e_{g_1 g'} - 1)p \quad \text{and}$$

$$(e_{h_1 h'} x_{gh'} - 1)p = (x_{gh'} - 1)p + x_{gh'}(e_{h_1 h'} - 1)p$$

hold for all $g' \in V(\mathcal{G}) \setminus \{g_1\}$, $h' \in V(\mathcal{H}) \setminus \{h_1\}$ and $p \in P_{\mathcal{G} \square \mathcal{H}}$.

Proof. This is straightforward to check. \square

With all the results so far, we are now able to state the unique reduced Gröbner basis of I_{viz} for the total degree lexicographical ordering.

Theorem 3.12. Let $k_{\mathcal{G}} = k_{\mathcal{H}} = 1$, then the reduced Gröbner basis of I_{viz} with respect to a total degree lexicographical term ordering consists of the polynomials

$$e_{g_1 g} - 1 \quad \text{for all } g \in V(\mathcal{G}) \setminus \{g_1\}, \tag{10a}$$

$$e_{h_1 h} - 1 \quad \text{for all } h \in V(\mathcal{H}) \setminus \{h_1\}, \tag{10b}$$

$$e_{gg'}(e_{gg'} - 1) \quad \text{for all } g \neq g' \in V(\mathcal{G}) \setminus \{g_1\}, \tag{10c}$$

$$e_{hh'}(e_{hh'} - 1) \quad \text{for all } h \neq h' \in V(\mathcal{H}) \setminus \{h_1\}, \tag{10d}$$

$$x_{gh}(x_{gh} - 1) \quad \text{for all } (g, h) \in V(\mathcal{G} \square \mathcal{H}), \tag{10e}$$

and

$$b_{gh, M} = \prod_{(g', h') \in \overline{U}_{gh}} (x_{g'h'} - 1) \prod_{(g, h') \in U_{gh}^r} (\llbracket (g, h') \in M \rrbracket (x_{gh'} - e_{hh'}) + e_{hh'} - 1) \times \\ \times \prod_{(g', h) \in U_{gh}^c} (\llbracket (g', h) \in M \rrbracket (x_{g'h} - e_{gg'}) + e_{gg'} - 1) \tag{10f}$$

for all subsets $M \subseteq U_{gh}$ for all choices of $(g, h) \in V(\mathcal{G} \square \mathcal{H})$.

Before we start with the proof, we want to give combinatorial interpretation to (10f). Let (g, h) be a fixed vertex in the Cartesian product graph $G \square H$. As already mentioned, the vertices in \overline{U}_{gh} are (g, h) and all vertices that are adjacent to (g, h) in all product graphs $G \square H$ of the graph class $\mathcal{G} \square \mathcal{H}$.

Let D be a dominating set (of any size) in $G \square H$. If there is a vertex in $\overline{U}_{gh} \cap D$, the vertex (g, h) is dominated by this vertex in D . If this is not the case, then there has to be a vertex in $U_{gh} \cap D$, that is adjacent to (g, h) . The polynomial $b_{gh, U_{gh}}$ ensures that at least one vertex in U_{gh} is in D . The above choice of M does not assure that a vertex adjacent to (g, h) in U_{gh} is in D . Indeed, assume that there is no vertex in $D \cap U_{gh}$ that is adjacent to (g, h) . Then all vertex variables occurring in $b_{gh, U_{gh} \setminus D}$ are zero because the corresponding vertices are not in D . Hence, there has to be at least one edge between (g, h) and a vertex in $D \cap U_{gh}$. This ensures that the vertex (g, h) is dominated by D .

To sum this up, the polynomials in (10f) in the Gröbner basis guarantee that (g, h) is dominated by a vertex in D . Next, we present the proof of Theorem 3.12.

Proof of Theorem 3.12. Let B be the set of all polynomials in the claimed reduced Gröbner basis. First, we will show that the polynomials in B are indeed in I_{viz} . Then, we will show that the leading term of each polynomial f in I_{viz} is divisible by the leading term of some polynomial in B . The third step in the proof will be to show that the polynomials in B are a generating system of I_{viz} , hence after this step we know that B is a Gröbner basis. The last step will be to show that B is even a reduced Gröbner basis.

From (1a), (2a) and Lemma 3.6 and 3.9 we get that all polynomials in B are in I_{viz} , so the first step of the proof is easily finished.

For the second step, let us consider the divisibility of the leading terms. We show that the desired property holds for each of the polynomials we used to generate I_{viz} , which then implies the property for all polynomials f in I_{viz} . For the polynomials in (1a), (1b) and (2a) this is trivial. Since $k_G = k_{\mathcal{H}} = 1$ in our setting, there are no polynomials in (1c). Furthermore, for all $(g, h) \in V(\mathcal{G} \square \mathcal{H})$ the leading term of (2b), that is

$$(-1)^{n_G+n_{\mathcal{H}}-1} x_{gh} \left(\prod_{\substack{g' \in V(\mathcal{G}) \\ g' \neq g}} e_{gg'} x_{g'h} \right) \left(\prod_{\substack{h' \in V(\mathcal{H}) \\ h' \neq h}} e_{hh'} x_{gh'} \right),$$

is divisible by the leading term of (10f) for $M = U_{gh}$, which equals $\rho_{gh}^{n_G+n_{\mathcal{H}}-1}$.

As a third step, we prove that B is a generating system of I_{viz} by representing the polynomials of Definition 2.7 and 2.9 in terms of the polynomials in B . This is again easy to check, except for the polynomials (2b). For a fixed vertex $(g, h) \in V(\mathcal{G} \square \mathcal{H})$, we will build the polynomial (2b) step by step using polynomials of B . First, we sum up $b_{gh,M}$ multiplied by

$$\prod_{(g',h') \in U_{gh}^r \cap M} e_{hh'} \prod_{(g',h) \in U_{gh}^c \cap M} e_{gg'} \in P_{\mathcal{G} \square \mathcal{H}}$$

for all possible subsets M of U_{gh} . Since $b_{gh,M} \in B$, this sum can be represented by polynomials in B and equals (5) multiplied with

$$\prod_{(g',h') \in \overline{U}_{gh}} (x_{g'h'} - 1).$$

Due to Lemma 3.10, this sum is also equal to

$$\prod_{(g',h') \in \overline{U}_{gh}} (x_{g'h'} - 1) \prod_{(g,h') \in U_{gh}^r} (e_{hh'} x_{gh'} - 1) \prod_{(g',h) \in U_{gh}^c} (e_{gg'} x_{g'h} - 1). \tag{11}$$

Next, we iteratively apply Lemma 3.11 for all vertices in $\overline{U}_{gh} \setminus \{(g, h)\}$ to obtain from (11) the polynomial

$$(x_{gh} - 1) \left(\prod_{\substack{g' \in V(\mathcal{G}) \\ g' \neq g}} (e_{gg'} x_{g'h} - 1) \right) \left(\prod_{\substack{h' \in V(\mathcal{H}) \\ h' \neq h}} (e_{hh'} x_{gh'} - 1) \right)$$

in the following way. First we fix a vertex $(g', h') \in \overline{U}_{gh} \setminus \{(g, h)\}$. Let the polynomial p be such that (11) equals $(x_{g'h'} - 1)p$. By e we denote the edge variable corresponding to (g', h') , that is $e_{g_1g'}$ if $h' = h$ and $e_{h_1h'}$ otherwise. Now, we add to $(x_{g'h'} - 1)p$ the polynomial $(e - 1)x_{g'h'}p$ to obtain $(ex_{g'h'} - 1)p$ by Lemma 3.11. Since we added a polynomial from (10a) or (10b) times a polynomial in $P_{\mathcal{G} \square \mathcal{H}}$ to a polynomial generated by B , the resulting polynomial is again generated by B . Based on this new polynomial, we choose the next vertex in $\overline{U}_{gh} \setminus \{(g, h)\}$ and apply the same arguments as before to this polynomial. This is done for all vertices in $\overline{U}_{gh} \setminus \{(g, h)\}$.

Finally, multiplying by $(-1)^{n_G+n_{\mathcal{H}}-1}$ yields that the requested polynomial (2b) can be generated by B . This finalizes to prove that B is a Gröbner basis.

The last step in the proof is to show that B is a reduced Gröbner basis. It is rather easy to see that there is no monomial in any of the polynomials in (10a)–(10e), which can be represented by the leading terms of the other polynomials in B . Moreover, it holds that the leading term of any polynomial from (10f) is the product of all variables in the polynomial that do not cancel out. The leading terms of these polynomials are of the same degree, square-free and pairwise distinct. Moreover, it holds that the variables in the leading terms of (10f) are not leading term of any polynomial from (10a) or (10b) in B . Therefore, we can not represent a leading term of a polynomial from (10f) by the leading terms

of the other polynomials in B . A monomial m_1 of a polynomial p of type (10f) in B , which is not the leading term, has a smaller degree than the leading term and is the product of pairwise distinct variables, which occur in the polynomials of degree 2 in B . For each leading term of the polynomials of type (10f) it holds that there is a variable which is a factor of the leading term but is no factor of m_1 . Due to these facts, we are not able to represent m_1 by the leading terms of $B \setminus \{p\}$. Together with the fact that all polynomials in B have leading coefficient 1, we conclude that the Gröbner basis is reduced. \square

With the help of the reduced Gröbner basis of I_{viz} obtained in Theorem 3.12, we know that if a polynomial is not representable in terms of the polynomials in this basis, then it can not be in I_{viz} . We use this to get an SDP formulation to computationally find an SOS-certificate. Before doing so, we use the Gröbner basis to determine the minimum degree of an SOS-certificate.

3.3. Minimum degree of a sum-of-squares certificate

The knowledge of the reduced Gröbner basis of I_{viz} allows us to state a lower bound on the degree ℓ of an SOS-certificate for Vizing’s conjecture in the case of $k_{\mathcal{G}} = k_{\mathcal{H}} = 1$.

Theorem 3.13. *Let $k_{\mathcal{G}} = k_{\mathcal{H}} = 1$ and $n_{\mathcal{G}}, n_{\mathcal{H}} > 1$, then there is no ℓ -SOS-certificate of f_{viz} for any integer ℓ less than $(n_{\mathcal{G}} + n_{\mathcal{H}} - 1)/2$.*

Proof. For any set of polynomials $s_1, \dots, s_t \in P_{\mathcal{G} \square \mathcal{H}}$ that forms an ℓ -SOS-certificate of f_{viz} , it needs to hold that

$$\sum_{i=1}^t s_i^2 - f_{\text{viz}} \equiv 0 \pmod{I_{\text{viz}}}.$$

Additionally, the degrees of the polynomials s_1, \dots, s_t have to be at most ℓ .

For all $1 \leq i \leq t$ let p_i be the polynomial that results from s_i by evaluating $e_{g_1g} = 1$ and $e_{h_1h} = 1$ for all $g \in V(\mathcal{G}) \setminus \{g_1\}$ and for all $h \in V(\mathcal{H}) \setminus \{h_1\}$. Lemma 3.6 yields that

$$\sum_{i=1}^t p_i^2 - f_{\text{viz}} \equiv \sum_{i=1}^t s_i^2 - f_{\text{viz}} \equiv 0 \pmod{I_{\text{viz}}}.$$

To show that something is congruent to 0 modulo I_{viz} is the same as proving that it is contained in I_{viz} . This implies that

$$\sum_{i=1}^t p_i^2 - f_{\text{viz}} = \sum_{i=1}^t p_i^2 - \sum_{(g,h) \in V(\mathcal{G} \square \mathcal{H})} x_{gh} + 1 \tag{12}$$

has to be generated by the elements in the Gröbner basis of I_{viz} stated in Theorem 3.12.

Assume that this can be done by using the elements of degree 1 and 2 only. We know that the constant term of the polynomial in (12) is greater or equal to 1. Furthermore, we are not able to represent a polynomial with constant term other than zero by using only the Gröbner basis elements of degree 2. This means that we have to use at least one Gröbner basis element of degree 1. But if we do so, we end up getting an edge variable e_{g_1g} with $g \in V(\mathcal{G}) \setminus \{g_1\}$ or e_{h_1h} with $h \in V(\mathcal{H}) \setminus \{h_1\}$ in the resulting polynomial. Clearly, there is no such edge variable in (12). Therefore, it holds that it is not possible to represent the polynomial in (12) by using the elements of degree 1 and 2 only.

Intuitively, this makes sense, as the polynomials in the Gröbner basis of degree 1 can be used to reduce the variable to 1, and the polynomials of degree 2 can be used to reduce higher powers of the variable to the variable itself, and this is not enough to reduce (12) to zero.

However, all other polynomials in the Gröbner basis have degree $n_{\mathcal{G}} + n_{\mathcal{H}} - 1$. Hence, the degree of the polynomial in (12) is at least $n_{\mathcal{G}} + n_{\mathcal{H}} - 1$. Consequently, there has to be at least one polynomial p_i

such that the degree of p_i^2 is greater or equal to $n_G + n_{\mathcal{H}} - 1$. As the degree of p_i is less or equal to the degree of s_i , we get that two times the degree of s_i is also greater or equal to $n_G + n_{\mathcal{H}} - 1$. This implies that there is no ℓ -SOS-certificate of f_{viz} for $\ell < (n_G + n_{\mathcal{H}} - 1)/2$. \square

As a result of Theorem 3.13, any ℓ -SOS-certificate for $k_G = k_{\mathcal{H}} = 1$ has to be at least of degree $\ell \geq (n_G + n_{\mathcal{H}} - 1)/2$. This is the first result stating the minimum degree of an ℓ -SOS-certificate for any values of $n_G, n_{\mathcal{H}}, k_G$ and $k_{\mathcal{H}}$.

4. New certificates for two subclasses of $k_G = k_{\mathcal{H}} = 1$

In this section, we present SOS-certificates for Vizing’s conjecture obtained with the method of Gaar et al. (2021), i.e., by following the steps in Gaar et al. (2021, Section 4), and recalled in Section 2. To set up the SDP, to solve the SDP and to computationally verify our obtained certificates, we made use of the code provided in Gaar et al. (2021). In particular, we ran the code in SageMath (2021) and in MATLAB using MOSEK (2021). We refrain from detailing the steps and focus on presenting the SOS-certificates and proving their correctness. For details on how they were obtained we refer to the master thesis of Siebenhofer (2021).

4.1. Certificate for $n_G = 3, n_{\mathcal{H}} = 2$ and $k_G = k_{\mathcal{H}} = 1$

We start by presenting a 2-SOS-certificate for the case $n_G = 3, n_{\mathcal{H}} = 2$ and $k_G = k_{\mathcal{H}} = 1$.

Theorem 4.1. For $n_G = 3, n_{\mathcal{H}} = 2$ and $k_G = k_{\mathcal{H}} = 1$, Vizing’s conjecture holds, as for all choices of $(g, h) \in V(\mathcal{G} \square \mathcal{H})$ the polynomials

$$\begin{aligned}
 s_{g^*h^*} &= x_{g^*h^*} && \text{for all } (g^*, h^*) \in V(\mathcal{G} \square \mathcal{H}) \text{ with } g^* \neq g \text{ and } h^* \neq h, \\
 s_1 &= -\alpha + \alpha \sum_{(g', h') \in T_{gh}} x_{g'h'} + \beta \sum_{(g', h') \in T_{gh}} x_{g'h'} \sum_{(g'', h'') \in T_{gh} \setminus \{(g', h')\}} x_{g''h''} \quad \text{and} \\
 s_2 &= \delta \sum_{(g', h') \in T_{gh}} x_{g'h'} \sum_{(g'', h'') \in T_{gh} \setminus \{(g', h')\}} x_{g''h''},
 \end{aligned}$$

form a 2-SOS-certificate of f_{viz} for all (α, β, δ) in

$$\left\{ \left(-\sqrt{3}, \frac{4}{9}\sqrt{3}, -\frac{1}{9}\sqrt{6} \right), \left(-\sqrt{3}, \frac{4}{9}\sqrt{3}, \frac{1}{9}\sqrt{6} \right), \left(\sqrt{3}, -\frac{4}{9}\sqrt{3}, -\frac{1}{9}\sqrt{6} \right), \left(\sqrt{3}, -\frac{4}{9}\sqrt{3}, \frac{1}{9}\sqrt{6} \right) \right\}.$$

Remark 4.2. Theorem 4.1 is true whenever α, β, δ are solutions to the system of equations

$$\begin{aligned}
 \alpha^2 + 1 &= 2\alpha^2 + \beta^2 + 2\alpha\beta + \delta^2, \\
 -(\alpha^2 + 1) &= 6\beta^2 + 6\alpha\beta + 6\delta^2 \quad \text{and} \\
 \alpha^2 + 1 &= 6\beta^2 + 6\delta^2.
 \end{aligned}$$

It can be checked that the pairs (α, β, δ) stated in Theorem 4.1 are all solutions to this system of equations.

Proof of Theorem 4.1. We start by fixing a vertex $(g, h) \in V(\mathcal{G} \square \mathcal{H})$. Next, we write the polynomials s_1 and s_2 using the polynomials ρ_{gh}^1 and ρ_{gh}^2 from Definition 3.2, so

$$\begin{aligned}
 s_1 &= -\alpha + \alpha\rho_{gh}^1 + \beta\rho_{gh}^2 \quad \text{and} \\
 s_2 &= \delta\rho_{gh}^2.
 \end{aligned}$$

For brevity, we denote by ρ^k the polynomial ρ_{gh}^k for $1 \leq k \leq 4$. By Corollary 3.5 we get

$$\begin{aligned} \rho^1 \rho^1 &\equiv \rho^1 + 2\rho^2 \pmod{I_{\text{viz}}}, \\ \rho^2 \rho^2 &\equiv \rho^2 + 6\rho^3 + 6\rho^4 \pmod{I_{\text{viz}}} \text{ and} \\ \rho^1 \rho^2 &\equiv 2\rho^2 + 3\rho^3 \pmod{I_{\text{viz}}}. \end{aligned}$$

These congruences imply that

$$\begin{aligned} s_1^2 &= (-\alpha + \alpha\rho^1 + \beta\rho^2)^2 \\ &= \alpha^2 + \alpha^2\rho^1\rho^1 + \beta^2\rho^2\rho^2 - 2\alpha^2\rho^1 - 2\alpha\beta\rho^2 + 2\alpha\beta\rho^1\rho^2 \\ &\equiv \alpha^2 + \alpha^2(\rho^1 + 2\rho^2) + \beta^2(\rho^2 + 6\rho^3 + 6\rho^4) - 2\alpha^2\rho^1 - 2\alpha\beta\rho^2 + 2\alpha\beta(2\rho^2 + 3\rho^3) = \\ &= \alpha^2 - \alpha^2\rho^1 + (2\alpha^2 + \beta^2 + 2\alpha\beta)\rho^2 + (6\beta^2 + 6\alpha\beta)\rho^3 + 6\beta^2\rho^4 \pmod{I_{\text{viz}}} \end{aligned}$$

and

$$s_2^2 = (\delta\rho^2)^2 \equiv \delta^2\rho^2 + 6\delta^2\rho^3 + 6\delta^2\rho^4 \pmod{I_{\text{viz}}}$$

hold.

The sum of squares of the polynomials in the certificate can be written as

$$\begin{aligned} \sum_{\substack{(g^*, h^*) \in V(\mathcal{G} \square \mathcal{H}) \\ g^* \neq g, h^* \neq h}} s_{g^*h^*}^2 + s_1^2 + s_2^2 &= \sum_{\substack{(g^*, h^*) \in V(\mathcal{G} \square \mathcal{H}) \\ g^* \neq g, h^* \neq h}} x_{g^*h^*}^2 + \alpha^2 - \alpha^2\rho^1 + (2\alpha^2 + \beta^2 + 2\alpha\beta + \delta^2)\rho^2 \\ &\quad + (6\beta^2 + 6\alpha\beta + 6\delta^2)\rho^3 + (6\beta^2 + 6\delta^2)\rho^4. \end{aligned} \tag{13}$$

Using the fact that $x_{g^*h^*}^2 \equiv x_{g^*h^*} \pmod{I_{\text{viz}}}$, we get

$$\begin{aligned} \sum_{\substack{(g^*, h^*) \in V(\mathcal{G} \square \mathcal{H}) \\ g^* \neq g, h^* \neq h}} x_{g^*h^*}^2 + \rho^1 - 1 &\equiv \sum_{\substack{(g^*, h^*) \in V(\mathcal{G} \square \mathcal{H}) \\ g^* \neq g, h^* \neq h}} x_{g^*h^*} + \rho^1 - 1 \\ &= \sum_{(g^*, h^*) \in V(\mathcal{G} \square \mathcal{H}) \setminus T_{gh}} x_{g^*h^*} + \sum_{(g', h') \in T_{gh}} x_{g'h'} - 1 \\ &= f_{\text{viz}} \pmod{I_{\text{viz}}}. \end{aligned}$$

Therefore the sum of squares (13) written as

$$\begin{aligned} \sum_{\substack{(g^*, h^*) \in V(\mathcal{G} \square \mathcal{H}) \\ g^* \neq g, h^* \neq h}} x_{g^*h^*}^2 + \rho^1 - 1 + (\alpha^2 + 1) - (\alpha^2 + 1)\rho^1 \\ + (2\alpha^2 + \beta^2 + 2\alpha\beta + \delta^2)\rho^2 + (6\beta^2 + 6\alpha\beta + 6\delta^2)\rho^3 + (6\beta^2 + 6\delta^2)\rho^4 \end{aligned}$$

is congruent

$$\begin{aligned} f_{\text{viz}} + (\alpha^2 + 1) - (\alpha^2 + 1)\rho^1 + (2\alpha^2 + \beta^2 + 2\alpha\beta + \delta^2)\rho^2 \\ + (6\beta^2 + 6\alpha\beta + 6\delta^2)\rho^3 + (6\beta^2 + 6\delta^2)\rho^4 \end{aligned}$$

modulo I_{viz} . Since α , β and δ satisfy

$$\begin{aligned} \alpha^2 + 1 &= 2\alpha^2 + \beta^2 + 2\alpha\beta + \delta^2, \\ -(\alpha^2 + 1) &= 6\beta^2 + 6\alpha\beta + 6\delta^2 \text{ and} \\ \alpha^2 + 1 &= 6\beta^2 + 6\delta^2, \end{aligned}$$

the sum of squares of the polynomials is congruent

$$f_{\text{viz}} + (\alpha^2 + 1)(1 - \rho^1 + \rho^2 - \rho^3 + \rho^4)$$

modulo I_{viz} . Lemma 3.3 together with Lemma 3.1 yields that

$$\begin{aligned} f_{\text{viz}} + (\alpha^2 + 1)(1 - \rho^1 + \rho^2 - \rho^3 + \rho^4) &= f_{\text{viz}} + (\alpha^2 + 1) \prod_{(g',h') \in T_{gh}} (1 - x_{g'h'}) \\ &\equiv f_{\text{viz}} \pmod{I_{\text{viz}}}, \end{aligned}$$

which completes the proof. \square

To sum up, we found for each of the 6 vertices in $\mathcal{G} \square \mathcal{H}$ 4 different certificates of degree 2 for Vizing's conjecture on the graph class $\mathcal{G} \square \mathcal{H}$ with $n_{\mathcal{G}} = 3$, $k_{\mathcal{G}} = 1$, $n_{\mathcal{H}} = 2$ and $k_{\mathcal{H}} = 1$. In total, these give 24 different 2-SOS-certificates.

4.2. Certificate for $n_{\mathcal{G}} = n_{\mathcal{H}} = 2$ and $k_{\mathcal{G}} = k_{\mathcal{H}} = 1$

Next, we consider the graph class with $n_{\mathcal{G}} = n_{\mathcal{H}} = 2$ and $k_{\mathcal{G}} = k_{\mathcal{H}} = 1$. Here we find the following certificate.

Theorem 4.3. For $n_{\mathcal{G}} = n_{\mathcal{H}} = 2$ and $k_{\mathcal{G}} = k_{\mathcal{H}} = 1$ Vizing's conjecture holds, since for any $(g, h) \in V(\mathcal{G} \square \mathcal{H})$ the two polynomials

$$\begin{aligned} s_{g^*h^*} &= x_{g^*h^*} \qquad \qquad \qquad \text{and} \\ s_1 &= -\alpha + \alpha \sum_{(g',h') \in T_{gh}} x_{g'h'} + \beta \sum_{(g',h') \in T_{gh}} x_{g'h'} \sum_{(g'',h'') \in T_{gh} \setminus \{(g',h')\}} x_{g''h''} \end{aligned}$$

with (g^*, h^*) being the only vertex not in the set T_{gh} , form a 2-SOS-certificate of f_{viz} for all pairs

$$(\alpha, \beta) \in \left\{ (\sqrt{2} + 3, -\sqrt{2} - 2), (-\sqrt{2} + 3, \sqrt{2} - 2), (\sqrt{2} - 3, -\sqrt{2} + 2), (-\sqrt{2} - 3, \sqrt{2} + 2) \right\}.$$

Remark 4.4. In particular, Theorem 4.3 is true whenever α and $\beta \in \mathbb{R}$ are solutions to the system of equations

$$\begin{aligned} \alpha^2 + 1 &= 2\alpha^2 + \beta^2 + 2\alpha\beta \quad \text{and} \\ -(\alpha^2 + 1) &= 6\beta^2 + 6\alpha\beta. \end{aligned}$$

The ones stated in the theorem are all solutions to this system of equations.

Proof of Theorem 4.3. This proof is analogous to that of Theorem 4.1. First, we fix a vertex $(g, h) \in V(\mathcal{G} \square \mathcal{H})$. Next, we rewrite s_1 as $-\alpha + \alpha \rho_{gh}^1 + \beta \rho_{gh}^2$. For the sake of brevity, we denote by ρ^k the polynomial ρ_{gh}^k for $1 \leq k \leq 3$. By Corollary 3.5 we get

$$\begin{aligned} \rho^1 \rho^1 &\equiv \rho^1 + 2\rho^2 \pmod{I_{\text{viz}}}, \\ \rho^2 \rho^2 &\equiv \rho^2 + 6\rho^3 \pmod{I_{\text{viz}}} \quad \text{and} \\ \rho^1 \rho^2 &\equiv 2\rho^2 + 3\rho^3 \pmod{I_{\text{viz}}}. \end{aligned}$$

Hence, we can write s_1^2 as

$$\begin{aligned} s_1^2 &= (-\alpha + \alpha \rho^1 + \beta \rho^2)^2 \\ &= \alpha^2 + \alpha^2 \rho^1 \rho^1 + \beta^2 \rho^2 \rho^2 - 2\alpha^2 \rho^1 - 2\alpha\beta \rho^2 + 2\alpha\beta \rho^1 \rho^2 \\ &\equiv \alpha^2 + \alpha^2 (\rho^1 + 2\rho^2) + \beta^2 (\rho^2 + 6\rho^3) - 2\alpha^2 \rho^1 - 2\alpha\beta \rho^2 + 2\alpha\beta (2\rho^2 + 3\rho^3) = \end{aligned}$$

$$\begin{aligned}
&= \alpha^2 + (\alpha^2 - 2\alpha^2)\rho^1 + (2\alpha^2 + \beta^2 - 2\alpha\beta + 4\alpha\beta)\rho^2 + (6\beta^2 + 6\alpha\beta)\rho^3 = \\
&= \alpha^2 - \alpha^2\rho^1 + (2\alpha^2 + \beta^2 + 2\alpha\beta)\rho^2 + (6\beta^2 + 6\alpha\beta)\rho^3 \pmod{I_{\text{viz}}}.
\end{aligned}$$

Using the fact that $x_{g^*h^*}^2 \equiv x_{g^*h^*} \pmod{I_{\text{viz}}}$ holds, we get that

$$\begin{aligned}
x_{g^*h^*}^2 + \rho^1 - 1 &\equiv x_{g^*h^*} + \rho^1 - 1 \\
&= x_{g^*h^*} + \sum_{(g',h') \in T_{gh}} x_{g'h'} - 1 \\
&= f_{\text{viz}} \pmod{I_{\text{viz}}}.
\end{aligned}$$

Therefore, for the sum of the polynomials squared it holds that

$$\begin{aligned}
x_{g^*h^*}^2 + s_1^2 &\equiv x_{g^*h^*}^2 + \alpha^2 - \alpha^2\rho^1 + (2\alpha^2 + \beta^2 + 2\alpha\beta)\rho^2 + (6\beta^2 + 6\alpha\beta)\rho^3 \\
&= x_{g^*h^*}^2 + \rho^1 - 1 + (\alpha^2 + 1) - (\alpha^2 + 1)\rho^1 + (2\alpha^2 + \beta^2 + 2\alpha\beta)\rho^2 + (6\beta^2 + 6\alpha\beta)\rho^3 \\
&\equiv f_{\text{viz}} + (\alpha^2 + 1) - (\alpha^2 + 1)\rho^1 + (2\alpha^2 + \beta^2 + 2\alpha\beta)\rho^2 + (6\beta^2 + 6\alpha\beta)\rho^3 \pmod{I_{\text{viz}}}.
\end{aligned}$$

Since α and β satisfy

$$\begin{aligned}
\alpha^2 + 1 &= 2\alpha^2 + \beta^2 + 2\alpha\beta \quad \text{and} \\
-(\alpha^2 + 1) &= 6\beta^2 + 6\alpha\beta,
\end{aligned}$$

we can further conclude with Lemma 3.3 and Lemma 3.1 that

$$\begin{aligned}
x_{g^*h^*}^2 + s_1^2 &\equiv f_{\text{viz}} + (\alpha^2 + 1) - (\alpha^2 + 1)\rho^1 + (2\alpha^2 + \beta^2 + 2\alpha\beta)\rho^2 + (6\beta^2 + 6\alpha\beta)\rho^3 \\
&= f_{\text{viz}} + (\alpha^2 + 1)(1 - \rho^1 + \rho^2 - \rho^3) \\
&\equiv f_{\text{viz}} \pmod{I_{\text{viz}}}
\end{aligned}$$

holds, which closes the proof. \square

One may observe the strong parallelism between the two graph classes considered. In the next section we derive a generalized method to find certificates of this special form.

5. General approach to find certificates for $k_{\mathcal{G}} = k_{\mathcal{H}} = 1$

In this section, we first give a general formulation of the previous two SOS-certificates, that could potentially be an SOS-certificate for any graph classes \mathcal{G} and \mathcal{H} with $k_{\mathcal{G}} = k_{\mathcal{H}} = 1$. To really obtain an SOS-certificate one has to determine the coefficients of the polynomials in this specific SOS-certificate by finding a solution of a system of equations. We give an algorithm to find such a solution in the second part of this section.

5.1. General certificate for $k_{\mathcal{G}} = k_{\mathcal{H}} = 1$

The SOS-certificates of the last section have a few things in common. First, a vertex $(g, h) \in V(\mathcal{G} \square \mathcal{H})$ is selected, which determines the set T_{gh} as defined at the beginning of Section 3.1. Then, the polynomials of degree greater than 1 in the certificate contain only vertex variables corresponding to the vertices in T_{gh} . In particular, we use the polynomials ρ_{gh}^i from Definition 3.2 to represent these polynomials in the certificate. Based on computational results and on our knowledge of the Gröbner basis, we propose the following specific form of a possible SOS-certificate. Additionally, we give a condition on the correctness of the certificate in the next theorem.

Theorem 5.1. Let $k_G = k_H = 1$ and let $d = n_G + n_H - 1$. If $c_{w,i} \in \mathbb{R}$ for $1 \leq w \leq \lceil d/2 \rceil$ and $0 \leq i \leq \lceil d/2 \rceil$ is a solution to the system of equations

$$c_{w,0} = -c_{w,1} \quad \forall 1 \leq w \leq \lceil d/2 \rceil \quad (14a)$$

$$(-1)^k \left(\sum_{w=1}^{\lceil d/2 \rceil} c_{w,1}^2 + 1 \right) = \sum_{i=\lceil k/2 \rceil}^{\min\{k, \lceil d/2 \rceil\}} \sum_{w=1}^{\lceil d/2 \rceil} c_{w,i}^2 \binom{i}{k-i} \binom{k}{i} \quad (14b)$$

$$+ 2 \sum_{j=\lceil \frac{k+1}{2} \rceil}^{\min\{k, \lceil d/2 \rceil\}} \sum_{i=k-j}^{j-1} \sum_{w=1}^{\lceil d/2 \rceil} c_{w,i} c_{w,j} \binom{i}{k-j} \binom{k}{i} \quad \forall 2 \leq k \leq d,$$

then for any choice of the vertex $(g, h) \in V(G \square H)$ the polynomials

$$s_{g^*h^*} = x_{g^*h^*} \quad \text{for all } (g^*, h^*) \in V(G \square H) \setminus T_{gh} \text{ and}$$

$$s_w = \sum_{i=0}^{\lceil d/2 \rceil} c_{w,i} \rho_{gh}^i \quad \text{for all } 1 \leq w \leq \lceil d/2 \rceil$$

form a $\lceil d/2 \rceil$ -SOS-certificate of f_{viz} , and therefore Vizing's conjecture holds on the graph classes \mathcal{G} and \mathcal{H} .

Note that the SOS-certificates of Theorem 5.1 are of the smallest possible degree according to Theorem 3.13. Additionally, note that the system of equations (14) depends on $d = n_G + n_H - 1$ and not on n_G or n_H explicitly. This means that if we find a solution for some d , then we have found certificates for all graph classes \mathcal{G} and \mathcal{H} with $n_G + n_H - 1 = d$. Furthermore, it can be observed that the constant terms in the polynomials s_w have to be the negative coefficients of the monomials of degree 1, as $c_{w,0}$ is the coefficient of $\rho_{gh}^0 = 1$ and $c_{w,1}$ is the coefficient of ρ_{gh}^1 in s_w .

Furthermore, observe that the system of equations (14b) coincides with those for $n_G = 3, n_H = 2, k_G = k_H = 1$ (so $d = 4$) in Remark 4.2 and for $n_G = n_H = 2, k_G = k_H = 1$ (so $d = 3$) in Remark 4.4. So for these graph classes we were able to find a solution of (14).

To prove Theorem 5.1, we first consider some useful lemma.

Lemma 5.2. Let $d = n_G + n_H - 1$ and fix some vertex $(g, h) \in V(G \square H)$. Furthermore, let $c_i \in \mathbb{R}$ for $0 \leq i \leq \lceil d/2 \rceil$ define the polynomial $s \in P_{G \square H}$ as

$$s = \sum_{i=0}^{\lceil d/2 \rceil} c_i \rho_{gh}^i.$$

Then s squared is congruent to

$$\sum_{k=0}^d \left(\sum_{i=\lceil k/2 \rceil}^{\min\{k, \lceil d/2 \rceil\}} c_i^2 \binom{i}{k-i} \binom{k}{i} + 2 \sum_{j=\lceil \frac{k+1}{2} \rceil}^{\min\{k, \lceil d/2 \rceil\}} \sum_{i=k-j}^{j-1} c_i c_j \binom{i}{k-j} \binom{k}{i} \right) \rho_{gh}^k$$

modulo I_{viz} .

Proof. By expanding the square of the polynomial s , we get that

$$s^2 = \sum_{i=0}^{\lceil d/2 \rceil} c_i^2 \rho_{gh}^i \rho_{gh}^i + 2 \sum_{i=0}^{\lceil d/2 \rceil} \sum_{j=i+1}^{\lceil d/2 \rceil} c_i c_j \rho_{gh}^i \rho_{gh}^j.$$

Next, we use Corollary 3.5 yielding that

$$\rho_{gh}^i \rho_{gh}^j \equiv \sum_{r=0}^{\min\{i,d-j\}} \binom{i}{r} \binom{j+r}{i} \rho_{gh}^{j+r} = \sum_{k=j}^{\min\{i+j,d\}} \binom{i}{k-j} \binom{k}{i} \rho_{gh}^k \pmod{I_{\text{viz}}}$$

holds for all $0 \leq i \leq j \leq \lceil d/2 \rceil$. We apply this to $\rho_{gh}^i \rho_{gh}^j$ for $0 \leq i \leq j \leq \lceil d/2 \rceil$ and sum up the coefficients of ρ_{gh}^k for each k with $0 \leq k \leq d$. From the product $c_i c_j \rho_{gh}^i \rho_{gh}^j$ we get a contribution of

$$c_i c_j \binom{i}{k-j} \binom{k}{i} \tag{15}$$

to the coefficient of ρ_{gh}^k if k is between j and the minimum of $i + j$ and d . For $i = j$, this means that i has to be between $k/2$ and k and additionally, i is less or equal to $\lceil d/2 \rceil$. In the case of $i < j$, combining the inequalities $k \leq j + i$ and $i \leq j - 1$, we get that the inequalities $j \geq (k + 1)/2$ and $i \geq k - j$ have to hold. Moreover, it holds that $j \leq k$ and $j \leq \lceil d/2 \rceil$. Therefore, collecting all coefficients of ρ_{gh}^k yields the stated result. \square

In the next corollary, we apply Lemma 5.2 to the sum of all s_w^2 for $1 \leq w \leq \lceil d/2 \rceil$.

Corollary 5.3. *Let $d = n_G + n_H - 1$ and fix $(g, h) \in V(\mathcal{G} \square \mathcal{H})$. Furthermore, let $c_{w,i} \in \mathbb{R}$ for $0 \leq i \leq \lceil d/2 \rceil$ and for $1 \leq w \leq \lceil d/2 \rceil$ define the polynomial s_w as*

$$s_w = \sum_{i=0}^{\lceil d/2 \rceil} c_{w,i} \rho_{gh}^i.$$

Then the sum of all polynomials s_w squared is congruent to

$$\sum_{k=0}^d \left(\sum_{i=\lceil k/2 \rceil}^{\min\{k, \lceil d/2 \rceil\}} \sum_{w=1}^{\lceil d/2 \rceil} c_{w,i}^2 \binom{i}{k-i} \binom{k}{i} + 2 \sum_{j=\lceil \frac{k+1}{2} \rceil}^{\min\{k, \lceil d/2 \rceil\}} \sum_{i=k-j}^{j-1} \sum_{w=1}^{\lceil d/2 \rceil} c_{w,i} c_{w,j} \binom{i}{k-j} \binom{k}{i} \right) \rho_{gh}^k$$

modulo I_{viz} .

Finally, we have all ingredients to prove Theorem 5.1.

Proof of Theorem 5.1. The proof is analogous to the ones of Theorem 4.1 and 4.3. First, we fix a vertex $(g, h) \in V(\mathcal{G} \square \mathcal{H})$. For brevity, we write ρ^k for ρ_{gh}^k for all $0 \leq k \leq \lceil d/2 \rceil$. Next, we use the fact that $x_{g^*h^*}^2 \equiv x_{g^*h^*} \pmod{I_{\text{viz}}}$, to get the congruence

$$\begin{aligned} \sum_{(g^*, h^*) \in V(\mathcal{G} \square \mathcal{H}) \setminus T_{gh}} x_{g^*h^*}^2 + \rho^1 - 1 &\equiv \sum_{(g^*, h^*) \in V(\mathcal{G} \square \mathcal{H}) \setminus T_{gh}} x_{g^*h^*} + \rho^1 - 1 \\ &= \sum_{(g^*, h^*) \in V(\mathcal{G} \square \mathcal{H}) \setminus T_{gh}} x_{g^*h^*} + \sum_{(g', h') \in T_{gh}} x_{g'h'} - 1 \\ &= f_{\text{viz}} \pmod{I_{\text{viz}}}. \end{aligned}$$

The above and Corollary 5.3 yield the congruence

$$\begin{aligned} \sum_{(g^*, h^*) \in V(\mathcal{G} \square \mathcal{H}) \setminus T_{gh}} x_{g^*h^*}^2 + \sum_{w=1}^{\lceil d/2 \rceil} s_w^2 &\equiv f_{\text{viz}} - \rho^1 + 1 + \sum_{k=0}^d \left(\sum_{i=\lceil k/2 \rceil}^{\min\{k, \lceil d/2 \rceil\}} \sum_{w=1}^{\lceil d/2 \rceil} c_{w,i}^2 \binom{i}{k-i} \binom{k}{i} \right. \\ &\quad \left. + 2 \sum_{j=\lceil \frac{k+1}{2} \rceil}^{\min\{k, \lceil d/2 \rceil\}} \sum_{i=k-j}^{j-1} \sum_{w=1}^{\lceil d/2 \rceil} c_{w,i} c_{w,j} \binom{i}{k-j} \binom{k}{i} \right) \rho^k \pmod{I_{\text{viz}}}. \tag{16} \end{aligned}$$

By writing down the coefficients of $\rho^0 = 1$ and ρ^1 in (16) explicitly, we get that the sum of squares is congruent to

$$f_{\text{viz}} - \rho^1 + \rho^0 + \left(\sum_{w=1}^{\lceil d/2 \rceil} c_{w,0}^2 \right) \rho^0 + \left(\sum_{w=1}^{\lceil d/2 \rceil} (c_{w,1}^2 + 2c_{w,0}c_{w,1}) \right) \rho^1$$

$$+ \sum_{k=2}^d \left(\sum_{i=\lceil k/2 \rceil}^{\min\{k, \lceil d/2 \rceil\}} \sum_{w=1}^{\lceil d/2 \rceil} c_{w,i}^2 \binom{i}{k-i} \binom{k}{i} \right.$$

$$\left. + 2 \sum_{j=\lceil \frac{k+1}{2} \rceil}^{\min\{k, \lceil d/2 \rceil\}} \sum_{i=k-j}^{j-1} \sum_{w=1}^{\lceil d/2 \rceil} c_{w,i} c_{w,j} \binom{i}{k-j} \binom{k}{i} \right) \rho^k$$

modulo I_{viz} . If the coefficients $c_{w,i}$ satisfy $c_{w,0} = -c_{w,1}$ (and thus also $c_{w,1}^2 + 2c_{w,0}c_{w,1} = -c_{w,1}^2$) and the system of equations (14b), then the above expression equals

$$f_{\text{viz}} + \left(\sum_{w=1}^{\lceil d/2 \rceil} c_{w,1}^2 + 1 \right) \left(\sum_{k=0}^d (-1)^k \rho^k \right),$$

which is congruent to f_{viz} modulo I_{viz} due to Lemma 3.1 and Lemma 3.3. Hence, the polynomials stated in Theorem 5.1 form a $\lceil d/2 \rceil$ -SOS-certificate for the graph classes \mathcal{G} and \mathcal{H} if the coefficients $c_{w,i}$ fulfill (14). \square

To summarize, Theorem 5.1 states that if we find a solution to the system of equations (14), then we obtain an SOS-certificate of minimum degree. In fact, it can also be deduced that if there is an SOS-certificate of the form given by Theorem 5.1, then the system of equations (14) has to hold.

5.2. Finding a solution of the system of equations

Next, we consider the problem of finding such solutions. Towards that end, let the vector c_i be defined as

$$c_i = (c_{w,i})_{1 \leq w \leq \lceil d/2 \rceil}$$

for all $0 \leq i \leq \lceil d/2 \rceil$, so c_i denotes the vector collecting all coefficients of ρ_{gh}^i in the general SOS-certificate for $k_{\mathcal{G}} = k_{\mathcal{H}} = 1$ and $d = n_{\mathcal{G}} + n_{\mathcal{H}} - 1$ stated in Theorem 5.1. It is easy to see that the system of equations (14) has a solution if and only if there are vectors $c_i \in \mathbb{R}^{\lceil d/2 \rceil}$ for $0 \leq i \leq \lceil d/2 \rceil$ that are a solution to the system of equations

$$c_0 = -c_1 \quad \forall 1 \leq w \leq \lceil d/2 \rceil \quad (17a)$$

$$(-1)^k \langle (c_1, c_1) + 1 \rangle = \sum_{i=\lceil k/2 \rceil}^{\min\{k, \lceil d/2 \rceil\}} \langle c_i, c_i \rangle \binom{i}{k-i} \binom{k}{i} \quad (17b)$$

$$+ 2 \sum_{j=\lceil \frac{k+1}{2} \rceil}^{\min\{k, \lceil d/2 \rceil\}} \sum_{i=k-j}^{j-1} \langle c_i, c_j \rangle \binom{i}{k-j} \binom{k}{i} \quad \forall 2 \leq k \leq d.$$

The system of equations (17) can be rewritten using $F_{i,j} = \langle c_i, c_j \rangle$ for $0 \leq i, j \leq \lceil d/2 \rceil$. Let F be the $\lceil d/2 \rceil \times \lceil d/2 \rceil$ -matrix with $F = (F_{i,j})_{1 \leq i, j \leq \lceil d/2 \rceil}$, i.e., F is the Gram matrix of the matrix $C = (c_{w,i})_{1 \leq w, i \leq \lceil d/2 \rceil}$ and $F = C^T C$ holds. As any Gram matrix is positive semidefinite and any positive semidefinite matrix is the Gram matrix of some set of vectors (which can for example be determined using Cholesky decomposition), we obtain the following result.

Observation 5.4. *The system of equations (14) has a real solution if and only if there is a positive semidefinite matrix $F = (F_{i,j})_{1 \leq i, j \leq \lceil d/2 \rceil}$ such that*

$$(-1)^k(F_{1,1} + 1) = \sum_{i=\lceil k/2 \rceil}^{\min\{k, \lceil d/2 \rceil\}} F_{i,i} \binom{i}{k-i} \binom{k}{i} + 2 \sum_{j=\lceil \frac{k+1}{2} \rceil}^{\min\{k, \lceil d/2 \rceil\}} \sum_{i=k-j}^{j-1} F_{i,j} \binom{i}{k-j} \binom{k}{i}, \quad (18)$$

where we substitute $F_{0,j} = -F_{1,j}$, holds for all $2 \leq k \leq d$. In particular, $F = C^T C$ for $C = (c_{w,i})_{1 \leq w, i \leq \lceil d/2 \rceil}$ and $c_{w,0} = -c_{w,1}$ for all $1 \leq w \leq \lceil d/2 \rceil$ holds for corresponding solutions.

With Observation 5.4 we have transformed the task of finding a certificate from solving a system of $\lceil d/2 \rceil$ linear and $d - 1$ quadratic equations (14) in $\lceil d/2 \rceil (\lceil d/2 \rceil + 1)$ variables to solve an SDP with matrix variable of dimension $\lceil d/2 \rceil$ with $d - 1$ linear equality constraints.

The objective function of this SDP can be chosen arbitrarily, as any feasible solution leads to an SOS-certificate. Unfortunately, just solving this SDP with an off-the-shelf SDP solver is not enough because any feasible solution obtained from an SDP solver is numerical, i.e., the system of equations is not fulfilled exactly, but only with small numerical errors. So in order to find a certificate, there is still some lucky guessing required.

Thus, we follow a different road to find a positive semidefinite matrix F that is an exact solution to the system of linear equations (18). In fact, any solution F to the system of equations (18) can be represented as linear expression in some free variables, which are a subset of all variables $F_{i,j}$. We iteratively fix the free variables by solving SDPs in the following way. When we want to fix the free variable $F_{i,j}$, we solve the SDP with matrix variable F , the system of linear equations (18) and the already fixed free variables two times, one time with maximizing and one time with minimizing the value of the free variable $F_{i,j}$ as objective function. Let $F_{i,j}^{min}$ and $F_{i,j}^{max}$ denote the optimal objective function values of these SDPs. We fix the free variable $F_{i,j}$ to an arbitrary rational number in the interval $[F_{i,j}^{min}, F_{i,j}^{max}]$, where we try to set $F_{i,j}$ to a rational number with small denominator in order to obtain “nice” values in F . Then we proceed with the next free variable.

Clearly, the choice of $F_{i,j}$ in the interval $[F_{i,j}^{min}, F_{i,j}^{max}]$ makes sure that we find a positive semidefinite matrix F that is an exact solution to the system of linear equations (18) with this procedure if it exists. If the system of linear equations (18) has no positive semidefinite solution, then we are not able to find a certificate of the specific form stated in Theorem 5.1.

As already mentioned before Observation 5.4, from F we can obtain the coefficients $c_{w,i}$ of the SOS-certificate from Theorem 5.1 with a Cholesky decomposition. We now consider an example to demonstrate our approach to determine an SOS-certificate.

Example 5.5. Let $n_G = 4$, $n_{\mathcal{H}} = 2$ and $k_G = k_{\mathcal{H}} = 1$, so $d = 5$. To find a certificate as stated in Theorem 5.1 we need to find a 3×3 positive semidefinite matrix F such that its entries satisfy the system of equations (18), i.e., the equations

$$\begin{aligned} F_{1,1} + 1 &= 2F_{1,1} + 2F_{2,1} + F_{2,2}, \\ -(F_{1,1} + 1) &= 6F_{2,1} + 6F_{2,2} + 4F_{3,1} + 6F_{3,2} + F_{3,3}, \\ F_{1,1} + 1 &= 6F_{2,2} + 8F_{3,1} + 24F_{3,2} + 12F_{3,3} \quad \text{and} \\ -(F_{1,1} + 1) &= 20F_{3,2} + 30F_{3,3}. \end{aligned} \quad (19)$$

All possible solutions of this system of linear equations can be written as

$$\begin{aligned}
 F_{2,1} &= -\frac{1}{2}F_{1,1} - \frac{1}{2}F_{2,2} + \frac{1}{2}, \\
 F_{3,1} &= \frac{47}{40}F_{1,1} - \frac{3}{4}F_{2,2} - \frac{133}{40}, \\
 F_{3,2} &= -\frac{1}{2}F_{1,1} + \frac{7}{4} \quad \text{and} \\
 F_{3,3} &= \frac{3}{10}F_{1,1} - \frac{6}{5},
 \end{aligned} \tag{20}$$

where $F_{1,1}$ and $F_{2,2}$ are free parameters. Thus, we can write any matrix F , which fulfills (20) and hence (19), as

$$\begin{aligned}
 F_{1,1} \begin{pmatrix} 1 & -1/2 & 47/40 \\ -1/2 & 0 & -1/2 \\ 47/40 & -1/2 & 3/10 \end{pmatrix} + F_{2,2} \begin{pmatrix} 0 & -1/2 & -3/4 \\ -1/2 & 1 & 0 \\ -3/4 & 0 & 0 \end{pmatrix} \\
 + \begin{pmatrix} 0 & 1/2 & -133/40 \\ 1/2 & 0 & 7/4 \\ -133/40 & 7/4 & -6/5 \end{pmatrix} \tag{21}
 \end{aligned}$$

for the free variables $F_{1,1}$ and $F_{2,2}$. Next, we need to find exact values for $F_{1,1}$ and $F_{2,2}$ such that the resulting matrix F is positive semidefinite.

Let $F_{1,1}^{min}$ be the result of the SDP which minimizes $F_{1,1}$ under the constraint that (21) is positive semidefinite. Furthermore, let $F_{1,1}^{max}$ be the optimal solution of the same SDP with an objective that maximizes $F_{1,1}$. For this example we get the (numerical) optimal solutions $F_{1,1}^{min} = 4.68455$ and $F_{1,1}^{max} = 38.41658$. We can set $F_{1,1}$ to be any rational value in the interval $[F_{1,1}^{min}, F_{1,1}^{max}]$ and choose $F_{1,1} = 6$.

As a consequence, we get that F has to be of the form

$$F_{2,2} \begin{pmatrix} 0 & -1/2 & -3/4 \\ -1/2 & 1 & 0 \\ -3/4 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 6 & -5/2 & 149/40 \\ -5/2 & 0 & -5/4 \\ 149/40 & -5/4 & 3/5 \end{pmatrix}.$$

To find a rational value for $F_{2,2}$, we follow the same strategy. We determine $F_{2,2}^{min} = 2.64289$ and $F_{2,2}^{max} = 3.26414$ and choose $F_{2,2} = 3$ and finally obtain the matrix

$$F = \begin{pmatrix} 6 & -4 & 59/40 \\ -4 & 3 & -5/4 \\ 59/40 & -5/4 & 3/5 \end{pmatrix},$$

which is positive semidefinite and fulfills the system of equations (19) exactly. To determine the solution of the system of equations (14), i.e., the coefficient matrix C , we compute the Cholesky factorization of $F = C^T C$ and obtain

$$C = \begin{pmatrix} \sqrt{6} & -2/3\sqrt{6} & 59/240\sqrt{6} \\ 0 & 1/3\sqrt{3} & -4/15\sqrt{3} \\ 0 & 0 & 1/80\sqrt{154} \end{pmatrix}.$$

As a consequence of Example 5.5 and Theorem 5.1, we have found the following 3-SOS-certificate for $n_G = 4$ and $n_H = 2$ as well as for $n_G = n_H = 3$ and $k_G = k_H = 1$.

Corollary 5.6. *Let \mathcal{G} and \mathcal{H} be two graph classes with $d = n_G + n_H - 1 = 5$ and $k_G = k_H = 1$, then Vizing's conjecture is true for these graph classes, as for any vertex $(g, h) \in V(\mathcal{G} \square \mathcal{H})$ the polynomials*

$$S_{g^*h^*} = X_{g^*h^*}$$

for all $(g^*, h^*) \in V(\mathcal{G} \square \mathcal{H}) \setminus T_{gh}$

$$s_1 = -\sqrt{6} + \sqrt{6}\rho_{gh}^1 - \frac{2}{3}\sqrt{6}\rho_{gh}^2 + \frac{59}{240}\sqrt{6}\rho_{gh}^3,$$

$$s_2 = \frac{1}{3}\sqrt{3}\rho_{gh}^2 - \frac{4}{15}\sqrt{3}\rho_{gh}^3 \quad \text{and}$$

$$s_3 = \frac{1}{80}\sqrt{154}\rho_{gh}^3$$

form a 3-SOS-certificate of f_{viz} .

5.3. Theoretical properties of certificates

It turns out that for even values of d we can say more about the system of equations (18), in particular $F_{1,1}$ is fixed as stated in the following corollary.

Corollary 5.7. *Let $d \geq 4$ be even, then $F_{1,1} = d - 1$ holds.*

Proof. We show that $F_{1,1} = d - 1$ holds by adding up the equations of (18) multiplied by $(-1)^k \frac{d}{k(k-1)}$ for all k with $2 \leq k \leq d$.

On the left-hand side of the resulting equation we get

$$d(F_{1,1} + 1) \sum_{k=2}^d \frac{1}{k(k-1)} = d(F_{1,1} + 1) \frac{d-1}{d} = (F_{1,1} + 1)(d-1).$$

The right-hand side is

$$\sum_{k=2}^d \frac{(-1)^k d}{k(k-1)} \left(\sum_{i=\lceil k/2 \rceil}^{\min\{k, \lceil d/2 \rceil\}} F_{i,i} \binom{i}{k-i} \binom{k}{i} + 2 \sum_{j=\lceil \frac{k+1}{2} \rceil}^{\min\{k, \lceil d/2 \rceil\}} \sum_{i=k-j}^{j-1} F_{i,j} \binom{i}{k-j} \binom{k}{i} \right). \quad (22)$$

It is enough to show that (22) equals $dF_{1,1}$. The variable $F_{1,1}$ appears only once in (22), namely for $k=2$ with the coefficient $\frac{d}{2} \binom{1}{1} \binom{2}{1} = d$. Thus, it remains to show that the coefficients of all other variables in (22) sum up to zero.

We start with the variables $F_{i,j}$ for $1 < i \leq j \leq d/2$. With the same arguments as in the proof of Lemma 5.2 to obtain the bounds on k for (15), $F_{i,j}$ appears in all summands with k between j and $\min\{d, i+j\} = i+j$. Hence, the coefficient of $F_{i,j}$ in (22) for $i=j$ is

$$\sum_{k=j}^{i+j} \frac{(-1)^k d \binom{i}{k-j} \binom{k}{i}}{(k-1)k} \quad (23)$$

and for $i < j$ it is two times (23). It can be shown that (23) is equal to zero.

The variables left to consider are $F_{1,j}$ for $1 < j \leq d/2$. The variable $F_{1,j}$ appears only in the summands of (22) for $k=j$ and $k=j+1$. Moreover, the variable $F_{0,j}$, which is equal to $-F_{1,j}$, appears in the summand with $k=j$ only. Therefore, when we substitute $F_{0,j} = -F_{1,j}$, the coefficient of $F_{1,j}$ in (22) is

$$(-1)^j \frac{2d}{j(j-1)} \left(\binom{1}{0} \binom{j}{1} - \binom{0}{0} \binom{j}{0} \right) + (-1)^{j+1} \frac{2d}{(j+1)j} \binom{1}{1} \binom{j+1}{1} = 0,$$

which completes the proof. \square

Corollary 5.7 shows that for all even $d \geq 4$, the left-hand sides of (18) are fixed to $(-1)^k d$. This implies that for all certificates of the form stated in Theorem 5.1 for any fixed vertex gh , the sum of the polynomials squared and then reduced by the polynomials of degree 2 in the Gröbner basis stated in Theorem 3.12 equals

$$f_{viz} + d \sum_{k=0}^d (-1)^k \rho_{gh}^k,$$

which is congruent to f_{viz} modulo I_{viz} .

Moreover, the fact that $F_{1,1}$ is fixed implies that $F_{\frac{d}{2}, \frac{d}{2}}$ and $F_{\frac{d}{2}-1, \frac{d}{2}}$ are fixed too.

Observation 5.8. For all even d in (18) the equation for $k = d$ is

$$F_{1,1} = \binom{d}{d/2} F_{\frac{d}{2}, \frac{d}{2}} - 1$$

and the equation for $k = d - 1$ is

$$F_{1,1} = -\frac{d}{2} \binom{d-1}{d/2} F_{\frac{d}{2}, \frac{d}{2}} - 2 \binom{d-1}{d/2-1} F_{\frac{d}{2}-1, \frac{d}{2}} - 1$$

Since $F_{1,1} + 1 = d$ by Corollary 5.7, this implies that

$$F_{\frac{d}{2}, \frac{d}{2}} = \frac{F_{1,1} + 1}{\binom{d}{d/2}} = \frac{d}{\binom{d}{d/2}} \text{ and}$$

$$F_{\frac{d}{2}-1, \frac{d}{2}} = -\frac{d + d^2 \binom{d-1}{d/2} / (2 \binom{d}{d/2})}{2 \binom{d-1}{d/2-1}} = -\frac{d + d^2/4}{\binom{d}{d/2}} = -F_{\frac{d}{2}, \frac{d}{2}} (1 + d/4)$$

holds.

5.4. Further certificates for $k_G = k_{\mathcal{H}} = 1$

We implemented the above-described procedure to find an SOS-certificate as stated in Theorem 5.1 for Vizing’s conjecture for the graph classes \mathcal{G} and \mathcal{H} satisfying $d = n_G + n_{\mathcal{H}} - 1$ and $k_G = k_{\mathcal{H}} = 1$ in SageMath (2021).

The implementation is available as ancillary files from the arXiv page of this paper at arxiv.org/src/2112.04007/anc. In particular, the method `find_certificate(d)` returns for a given integer d the coefficient matrix C of a $\lceil d/2 \rceil$ -SOS-certificate for Vizing’s conjecture.

With the help of this code, we were able to find SOS-certificates for Vizing’s conjecture on the graph classes \mathcal{G} and \mathcal{H} with $k_G = k_{\mathcal{H}} = 1$ and $d = n_G + n_{\mathcal{H}} - 1$ for $6 \leq d \leq 14$.

Corollary 5.9. For all graph classes \mathcal{G} and \mathcal{H} with $k_G = k_{\mathcal{H}} = 1$ and $d = n_G + n_{\mathcal{H}} - 1$ with $6 \leq d \leq 14$ Vizing’s conjecture is true, because the polynomials

$$s_{g^*h^*} = x_{g^*h^*} \quad \text{for all } (g^*, h^*) \in V(\mathcal{G} \square \mathcal{H}) \setminus T_{gh} \text{ and}$$

$$s_w = \sum_{i=0}^{\lceil d/2 \rceil} c_{w,i} \rho_{gh}^i \quad \text{for all } 1 \leq w \leq \lceil d/2 \rceil$$

form a $\lceil d/2 \rceil$ -SOS-certificate of f_{viz} for every choice of $(g, h) \in V(\mathcal{G} \square \mathcal{H})$. Here $c_{w,i} = 0$ for all $1 \leq i < w \leq \lceil d/2 \rceil$ and $c_{w,0} = -c_{w,1}$ for all $1 \leq w \leq \lceil d/2 \rceil$ hold for all values of d . Furthermore, for $d = 6$ we have

$$c_{1,1} = \sqrt{5}, \quad c_{1,2} = -\frac{3}{5}\sqrt{5}, \quad c_{1,3} = \frac{21}{100}\sqrt{5},$$

$$c_{2,2} = \frac{1}{5}\sqrt{5}, \quad c_{2,3} = -\frac{3}{25}\sqrt{5} \text{ and}$$

$$c_{3,3} = \frac{1}{20}\sqrt{3};$$

for $d = 7$ we have

$$c_{1,1} = \sqrt{7}, \quad c_{1,2} = -\frac{5}{7}\sqrt{7}, \quad c_{1,3} = \frac{9}{28}\sqrt{7}, \quad c_{1,4} = -\frac{17}{245}\sqrt{7},$$

$$c_{2,2} = \frac{1}{7}\sqrt{21}, \quad c_{2,3} = -\frac{179}{1260}\sqrt{21}, \quad c_{2,4} = \frac{109}{2205}\sqrt{21},$$

$$c_{3,3} = \frac{1}{90}\sqrt{429}, \quad c_{3,4} = -\frac{53}{6435}\sqrt{429} \text{ and}$$

$$c_{4,4} = \frac{1}{5005}\sqrt{4147};$$

for $d = 8$ we have

$$c_{1,1} = \sqrt{7}, \quad c_{1,2} = -\frac{5}{7}\sqrt{7}, \quad c_{1,3} = \frac{31}{98}\sqrt{7}, \quad c_{1,4} = -\frac{8}{108}\sqrt{7},$$

$$c_{2,2} = \frac{1}{7}\sqrt{21}, \quad c_{2,3} = -\frac{41}{294}\sqrt{21}, \quad c_{2,4} = \frac{16}{315}\sqrt{21},$$

$$c_{3,3} = \frac{1}{21}\sqrt{15}, \quad c_{3,4} = -\frac{8}{225}\sqrt{15} \text{ and}$$

$$c_{4,4} = \frac{2}{525}\sqrt{35};$$

for $d = 9$ we have

$$c_{1,1} = 4, \quad c_{1,2} = -\frac{27}{8}, \quad c_{1,3} = \frac{115}{48}, \quad c_{1,4} = -\frac{103}{80}, \quad c_{1,5} = \frac{11}{28}, \quad c_{2,2} = \frac{1}{8}\sqrt{39}, \quad c_{2,3} = -\frac{925}{5616}\sqrt{39},$$

$$c_{2,4} = \frac{607}{5616}\sqrt{39}, \quad c_{2,5} = -\frac{355}{9828}\sqrt{39}, \quad c_{3,3} = \frac{1}{351}\sqrt{24882}, \quad c_{3,4} = -\frac{34253}{8957520}\sqrt{24882},$$

$$c_{3,5} = \frac{47513}{25081056}\sqrt{638 \cdot 210409}, \quad c_{4,4} = \frac{1}{76560}\sqrt{638 \cdot 210409},$$

$$c_{4,5} = -\frac{586549}{4510495612}\sqrt{210409 \cdot 638} \text{ and } c_{5,5} = \frac{5}{17674356}\sqrt{1262454 \cdot 2417};$$

for $d = 10$ we have

$$c_{1,1} = 3, \quad c_{1,2} = -\frac{7}{3}, \quad c_{1,3} = \frac{17}{12}, \quad c_{1,4} = -\frac{7}{15}, \quad c_{1,5} = \frac{71}{1512}, \quad c_{2,2} = \frac{1}{3}\sqrt{5}, \quad c_{2,3} = -\frac{5}{12}\sqrt{5},$$

$$c_{2,4} = \frac{251}{1200}\sqrt{5}, \quad c_{2,5} = -\frac{379}{30240}\sqrt{5}, \quad c_{3,3} = \frac{1}{4}\sqrt{2}, \quad c_{3,4} = -\frac{33}{160}\sqrt{2}, \quad c_{3,5} = \frac{193}{4032}\sqrt{2},$$

$$c_{4,4} = \frac{1}{800}\sqrt{146170}, \quad c_{4,5} = -\frac{135673}{2016\sqrt{146170}} \text{ and } c_{5,5} = \frac{1}{504\sqrt{146170}}\sqrt{4176691};$$

for $d = 11$ we have

$$c_{1,1} = \sqrt{87}, \quad c_{1,2} = -\frac{28}{29}\sqrt{87}, \quad c_{1,3} = \frac{215}{261}\sqrt{87}, \quad c_{1,4} = -\frac{279446473}{495701640}\sqrt{87}, \quad c_{1,5} = \frac{1345}{4959}\sqrt{87},$$

$$c_{1,6} = -\frac{22906823}{330467760}\sqrt{87}, \quad c_{2,2} = \sqrt{\frac{26}{29}}, \quad c_{2,3} = -\frac{157}{78}\sqrt{\frac{26}{29}}, \quad c_{2,4} = \frac{2674088}{1322685}\sqrt{\frac{26}{29}},$$

$$c_{2,5} = -\frac{5485}{4446}\sqrt{\frac{26}{29}}, \quad c_{2,6} = \frac{1834513}{4702880}\sqrt{\frac{26}{29}}, \quad c_{3,3} = \frac{1}{3}\sqrt{\frac{467}{78}}, \quad c_{3,4} = -\frac{3554462909}{5321670480}\sqrt{\frac{467}{78}},$$

$$\begin{aligned}
 c_{3,5} &= \frac{282916}{452523} \sqrt{\frac{467}{78}}, c_{3,6} = -\frac{23866165}{88694508} \sqrt{\frac{467}{78}}, c_{4,4} = \frac{1}{759696} \sqrt{\frac{127230362521319}{14010}}, \\
 c_{4,5} &= -\frac{82275318718}{41095407094386037} \sqrt{\frac{127230362521319}{14010}}, c_{4,6} = \frac{21037454688547}{20136749476249158130} \sqrt{\frac{127230362521319}{14010}}, \\
 c_{5,5} &= \frac{1}{2907} \sqrt{\frac{132029134219450005907}{8906125376492330}}, c_{5,6} = -\frac{3893456665881898045477}{10234898484691764457910640} \sqrt{\frac{132029134219450005907}{8906125376492330}} \\
 \text{and } c_{6,6} &= \frac{1}{13328} \sqrt{\frac{110317821367843091833849}{1980437013291750088605}};
 \end{aligned}$$

for $d = 12$ we have

$$\begin{aligned}
 c_{1,1} &= \sqrt{11}, c_{1,2} = -\frac{35}{44} \sqrt{11}, c_{1,3} = \frac{1}{2} \sqrt{11}, c_{1,4} = -\frac{655199}{2676520} \sqrt{11}, c_{1,5} = \frac{1}{11} \sqrt{11}, \\
 c_{1,6} &= -\frac{110207}{5353040} \sqrt{11}, c_{2,2} = \frac{1}{4} \sqrt{\frac{95}{11}}, c_{2,3} = -\frac{11}{38} \sqrt{\frac{95}{11}}, c_{2,4} = \frac{147757}{660440} \sqrt{\frac{95}{11}}, c_{2,5} = -\frac{2617}{22515} \sqrt{\frac{95}{11}}, \\
 c_{2,6} &= \frac{4172389}{138692400} \sqrt{\frac{95}{11}}, c_{3,3} = \sqrt{\frac{1}{38}}, c_{3,4} = -\frac{38345}{36498} \sqrt{\frac{1}{38}}, c_{3,5} = \frac{424}{1185} \sqrt{\frac{1}{38}}, c_{3,6} = \frac{8117}{729960} \sqrt{\frac{1}{38}}, \\
 c_{4,4} &= \frac{1}{182490} \sqrt{\frac{176558597}{2}}, c_{4,5} = -\frac{496865}{83688774978} \sqrt{\frac{176558597}{2}}, c_{4,6} = \frac{274043219}{128880713466120} \sqrt{\frac{176558597}{2}}, \\
 c_{5,5} &= \frac{1}{2370} \sqrt{\frac{3003702364301}{2471820358}}, c_{5,6} = -\frac{18498609557237}{62645216509861656} \sqrt{\frac{3003702364301}{2471820358}} \text{ and} \\
 c_{6,6} &= \frac{1}{3080} \sqrt{\frac{1479419046289663}{237292486779790}};
 \end{aligned}$$

for $d = 13$ we have

$$\begin{aligned}
 c_{1,1} &= 4\sqrt{\frac{41}{7}}, c_{1,2} = -\frac{643}{164} \sqrt{\frac{41}{7}}, c_{1,3} = \frac{2285}{656} \sqrt{\frac{41}{7}}, c_{1,4} = -\frac{25057169756187379}{9706601210394348} \sqrt{\frac{41}{7}}, \\
 c_{1,5} &= \frac{223240166469743567}{155305619366309568} \sqrt{\frac{41}{7}}, c_{1,6} = -\frac{133}{260} \sqrt{\frac{41}{7}}, c_{1,7} = \frac{25638546175376171}{328531117890270240} \sqrt{\frac{41}{7}}, \\
 c_{2,2} &= \frac{1}{4} \sqrt{\frac{4423}{287}}, c_{2,3} = -\frac{36921}{70768} \sqrt{\frac{4423}{287}}, c_{2,4} = \frac{648304603385756183}{1047129198867663444} \sqrt{\frac{4423}{287}}, \\
 c_{2,5} &= \frac{8656079938938263059}{16754067181882615104} \sqrt{\frac{4423}{287}}, c_{2,6} = \frac{336651}{1149980} \sqrt{\frac{4423}{287}}, c_{2,7} = -\frac{3101604353052717839}{35441295961674762720} \sqrt{\frac{4423}{287}}, \\
 c_{3,3} &= \frac{1}{4} \sqrt{\frac{52445}{4423}}, c_{3,4} = -\frac{876978256134844544}{1671406615081576485} \sqrt{\frac{52445}{4423}}, c_{3,5} = \frac{139692919173390267203}{248323268526405649200} \sqrt{\frac{52445}{4423}}, \\
 c_{3,6} &= -\frac{414247923}{1172670200} \sqrt{\frac{52445}{4423}}, c_{3,7} = \frac{56813568067103416}{525299221882781181} \sqrt{\frac{52445}{4423}}, \\
 c_{4,4} &= \frac{2}{3541078198497} \sqrt{\frac{278022966275031721230952411511}{681785}}, \\
 c_{4,5} &= -\frac{16842806625507324664137697618319}{15988889469513083235667255314875525722629840} \sqrt{\frac{278022966275031721230952411511}{681785}}, \\
 c_{4,6} &= \frac{7494161056453719279901}{8607591035874982089310286660380560} \sqrt{\frac{278022966275031721230952411511}{681785}}, \\
 c_{4,7} &= -\frac{115260002843756514570159114279703}{364546679904898297773213421179161986475960352} \sqrt{\frac{278022966275031721230952411511}{681785}}, \\
 c_{5,5} &= \frac{1}{14204782259113680} \sqrt{\frac{17451872904879613634434062169896872769038117850574344762823}{278022966275031721230952411511}}, \\
 c_{5,6} &= -\frac{151181192623630707882481830209965662197243013451}{1404805961351189379117404268428018670416492334499832456028200208} \times \\
 &\times \sqrt{\frac{17451872904879613634434062169896872769038117850574344762823}{278022966275031721230952411511}};
 \end{aligned}$$

$$\begin{aligned}
 C_{5,7} &= \frac{65110165541673840570849188883311271190521123019507743585023}{1258569508109052577461744053186022770733948051592944510245658946785111648480} \times \\
 &\quad \times \sqrt{\frac{17451872904879613634434062169896872769038117850574344762823}{278022966275031721230952411511}}, \\
 C_{6,6} &= \sqrt{\frac{25352491093848053461206176651408840063078575112714792491}{680623043290304931742928424625978037992486596172399445750097}}, \\
 C_{6,7} &= -\frac{116354006097743659300270866764317970081269530370009757503847822914109}{2373228356611870629873418754566765279954894894468648815804099558215400} \times \\
 &\quad \times \sqrt{\frac{25352491093848053461206176651408840063078575112714792491}{680623043290304931742928424625978037992486596172399445750097}} \\
 \text{and } C_{7,7} &= \frac{1}{46804638404700} \times \\
 &\quad \times \sqrt{\frac{4197123267905854359177826832232722232187999636495894239101559440686690856571}{1064804625941618245370659419359171282649300154734021284622}};
 \end{aligned}$$

and for $d = 14$ we have

$$\begin{aligned}
 C_{1,1} &= \sqrt{13}, \quad C_{1,2} = -\frac{32}{39}\sqrt{13}, \quad C_{1,3} = \frac{7}{13}\sqrt{13}, \quad C_{1,4} = -\frac{1581}{5005}\sqrt{13}, \quad C_{1,5} = \frac{186021079786121}{1151722559447460}\sqrt{13}, \\
 C_{1,6} &= -\frac{53}{897}\sqrt{13}, \quad C_{1,7} = \frac{3286004765171}{309987438360600}\sqrt{13}, \quad C_{2,2} = \frac{2}{3}\sqrt{\frac{17}{13}}, \quad C_{2,3} = -\frac{59}{68}\sqrt{\frac{17}{13}}, \\
 C_{2,4} &= \frac{1293}{2618}\sqrt{\frac{17}{13}}, \quad C_{2,5} = -\frac{185610347559698}{1882623414481425}\sqrt{\frac{17}{13}}, \quad C_{2,6} = -\frac{12097}{333132}\sqrt{\frac{17}{13}}, \\
 C_{2,7} &= \frac{153739557217}{7871226963600}\sqrt{\frac{17}{13}}, \quad C_{3,3} = \frac{1}{4}\sqrt{\frac{67}{17}}, \quad C_{3,4} = -\frac{19441}{51590}\sqrt{\frac{67}{17}}, \quad C_{3,5} = \frac{64643778225307}{2473250368044225}\sqrt{\frac{67}{17}}, \\
 C_{3,6} &= -\frac{41037}{437644}\sqrt{\frac{67}{17}}, \quad C_{3,7} = \frac{127375771980323}{9585765401612400}\sqrt{\frac{67}{17}}, \quad C_{4,4} = \frac{1}{385}\sqrt{\frac{59509}{67}}, \\
 C_{4,5} &= -\frac{56914455235667}{11576949729765700}\sqrt{\frac{59509}{67}}, \quad C_{4,6} = \frac{2624828}{680247379}\sqrt{\frac{59509}{67}}, \\
 C_{4,7} &= -\frac{51012398758058}{40932786544528725}\sqrt{\frac{59509}{67}}, \\
 C_{5,5} &= \frac{1}{14765673839070}\sqrt{\frac{9222821132677377658501193273}{297545}}, \\
 C_{5,6} &= -\frac{28381206876870420343}{316278205102905312042981420910989}\sqrt{\frac{9222821132677377658501193273}{297545}}, \\
 C_{5,7} &= \frac{800180049472559360738859359}{21631465553921823566411244940446165459600}\sqrt{\frac{9222821132677377658501193273}{297545}}, \\
 C_{6,6} &= \frac{2}{34293}\sqrt{\frac{32861993291352160021413495104861}{46114105663386888292505966365}}, \\
 C_{6,7} &= -\frac{465798003932716733895505153185993057604207}{9403204725977980081537461590679792988551578400}\sqrt{\frac{32861993291352160021413495104861}{46114105663386888292505966365}} \text{ and} \\
 C_{7,7} &= \frac{1}{95380750264800}\sqrt{\frac{27167242806224591574440191807735163931063797448111661}{98585979874056480064240485314583}}.
 \end{aligned}$$

Due to the fact that we were able to find a feasible solution to the SDP derived in Observation 5.4 for any $d \leq 14$ we have the following conjecture.

Conjecture 5.10. *Let $k_G = k_{\mathcal{H}} = 1$ and let $d = n_G + n_{\mathcal{H}} - 1$. Then a $\lceil d/2 \rceil$ -SOS-certificate of f_{viz} of the form presented in Theorem 5.1 exists, as there is a positive semidefinite matrix F fulfilling the system of equations (18) in Observation 5.4.*

Concerning the value of $F_{1,1}$, we know from Corollary 5.7 that $F_{1,1} = d - 1$ for even d , for odd d we make the following observation.

Observation 5.11. For the certificates above with $d \leq 13$ and d odd it turns out that $F_{1,1}$ is not fixed. Moreover, for these certificates the choice of $F_{1,1} = d - 1$ is not possible.

For $d > 14$ we did not derive certificates because of numerical difficulties with off-the-shelf SDP solvers.

6. Conclusion and open questions

In this paper, we extended the approach of Gaar et al. (2019, 2021) to prove Vizing's conjecture via an algebraic method for graph classes \mathcal{G} and \mathcal{H} , where the graph classes \mathcal{G} and \mathcal{H} are defined as all graphs with $n_{\mathcal{G}}$ and $n_{\mathcal{H}}$ vertices and a minimum dominating set of size $k_{\mathcal{G}}$ and $k_{\mathcal{H}}$, respectively. We applied their technique to the case where both minimum dominating sets in \mathcal{G} and \mathcal{H} are of size 1. A bottleneck in their computations is the time-consuming intermediate step to determine a Gröbner basis of I_{viz} . We were able to overcome this obstacle by determining the unique reduced Gröbner basis of I_{viz} for this case. This allowed us to conclude that if an ℓ -SOS-certificate exists, it must be at least of degree $\ell = \lceil (n_{\mathcal{G}} + n_{\mathcal{H}} - 1)/2 \rceil$.

We further presented a procedure to find $\lceil (n_{\mathcal{G}} + n_{\mathcal{H}} - 1)/2 \rceil$ -SOS-certificates of a special form for Vizing's conjecture on these graph classes \mathcal{G} and \mathcal{H} . This new approach is based on our knowledge of the Gröbner basis, and assumes that in addition to the polynomials of degree 2, only one polynomial of higher degree of the Gröbner basis is sufficient to prove correctness of the SOS-certificate. Assuming a specific form of the SOS-certificate, the coefficients of the polynomials of this certificate can be obtained by solving a system of quadratic equations. We presented a method how to obtain an exact solution to this using SDPs, that avoids clever guessing as usually needed in the approach from Gaar et al. (2019, 2021). The specific form of the certificates yields that certificates of classes with $n_{\mathcal{G}} + n_{\mathcal{H}} - 1 = d$ depend only on d and not on $n_{\mathcal{G}}$ or $n_{\mathcal{H}}$. We implemented this new method in SageMath (2021) and used it to find certificates for all graph classes \mathcal{G} and \mathcal{H} with $n_{\mathcal{G}} + n_{\mathcal{H}} \leq 15$ and domination numbers $k_{\mathcal{G}} = k_{\mathcal{H}} = 1$. Even though this does not advance what is known with respect to Vizing's conjecture, deriving this new certificates is an important step in the area of using conic linear optimization for computer-assisted proofs because it demonstrates that deriving such proofs is possible.

We were not able to derive certificates for $n_{\mathcal{G}} + n_{\mathcal{H}} > 15$ due to numerical difficulties with off-the-shelf SDP solvers. This needs to be dealt with in more detail. In future work, another topic for further investigation is whether the system of linear equation, which has to be solved in our new approach, is solvable for any size $d = n_{\mathcal{G}} + n_{\mathcal{H}} - 1$.

Most of all, the question of a general certificate depending on the size d arises. We know that $c_{1,1} = \sqrt{n_{\mathcal{G}} + n_{\mathcal{H}} - 2}$ holds in the case of odd $n_{\mathcal{G}} + n_{\mathcal{H}}$. For $n_{\mathcal{G}} + n_{\mathcal{H}}$ even, however, this is not the case. This coefficient as well as all other coefficients are among the most obvious future topics to work on to find a general certificate.

With our work, we know there are SOS-certificates for Vizing's conjecture for all graph classes \mathcal{G} and \mathcal{H} with $k_{\mathcal{G}} = n_{\mathcal{G}} - 1 \geq 1$ and $k_{\mathcal{H}} = n_{\mathcal{H}} - 1$ for $n_{\mathcal{H}} \in \{2, 3\}$, with $k_{\mathcal{G}} = n_{\mathcal{G}}$ and $k_{\mathcal{H}} = n_{\mathcal{H}} - d$ for $d \leq 4$, and now also with $k_{\mathcal{G}} = k_{\mathcal{H}} = 1$ and $n_{\mathcal{G}} + n_{\mathcal{H}} \leq 15$. Clearly, it would be interesting to derive SOS-certificates also for other graph classes \mathcal{G} and \mathcal{H} .

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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