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A characterization of graphs with regular distance-2 graphs

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ABSTRACT

For non-negative integers k, we consider graphs in which every vertex has exactly k vertices at distance 2, i.e., graphs whose distance-2 graphs are k-regular. We call such graphs k-metamour-regular motivated by the terminology in polyamory.

While constructing k-metamour-regular graphs is relatively easy – we provide a generic construction for arbitrary k – finding all such graphs is much more challenging. We show that only k-metamour-regular graphs with a certain property cannot be built with this construction. Moreover, we derive a complete characterization of k-metamour-regular graphs for each $k = 0$, $k = 1$ and $k = 2$. In particular, a connected graph with n vertices is 2-metamour-regular if and only if $n \geq 5$ and the graph is

- a join of complements of cycles (equivalently every vertex has degree $n - 3$),
- a cycle, or
- one of 17 exceptional graphs with $n \leq 8$.

Moreover, a characterization of graphs in which every vertex has at most one metamour is acquired. Each characterization is accompanied by an investigation of the corresponding counting sequence of unlabeled graphs.

1. Introduction

For a given graph, let us construct its distance-2 graph as follows: It has the same vertices as the original graph, and there is an edge between two vertices if these vertices are at distance 2 in the original graph. Here, distance 2 means that such two vertices are different, not adjacent, and have a common neighbor. An example is shown in Fig. 1.1. Distance-2 graphs and properties of vertices at distance 2 have been heavily studied in the literature; see the survey in Section 1.2. We pursue the theme of characterizing all graphs whose distance-2 graphs are in a given graph class. We specifically set our focus on the graph class of regular graphs.

The research is motivated by the relationship concept *polyamory*,[3] where every person might be in a relationship with any number of other persons. Naturally, this can be modeled as a graph, where each vertex represents a person, and two

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3 Hyde and DeLamater [17] describe *polyamory* as "the non-possessive, honest, responsible, and ethical philosophy and practice of loving multiple people simultaneously". Other descriptions of polyamory are around; see for example Haritaworn, Lin and Klesse [16] or Sheff [24]. The word polyamory appeared in an article by Zell-Ravenheart [23] in 1990 and is itself a combination of the Greek word "πολύ" (poly) meaning "many" and of the Latin word "amor" meaning "love".

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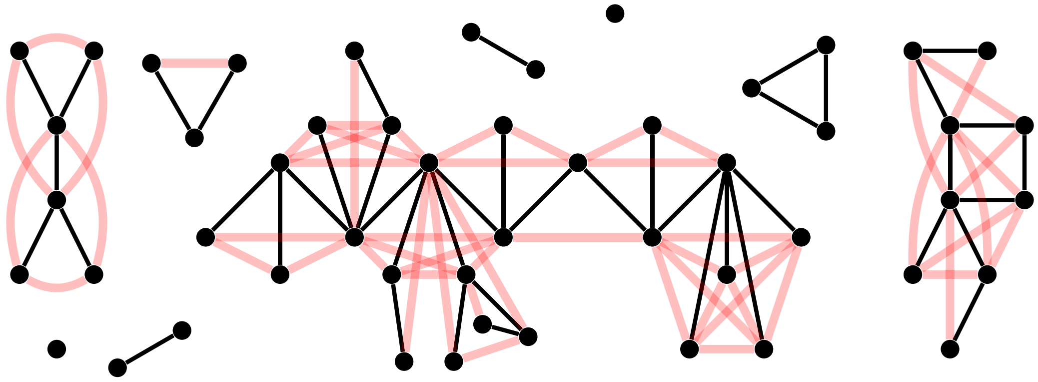


Fig. 1.1. A graph and its distance-2 graph A line presents an edge in the graph and an edge in its distance-2 graph.

vertices are adjacent if the corresponding persons are in a relationship. In polyamory, two persons that are both in a relationship with the same third person, but are not in a relationship with each other, are called *metamours*[4] We adopt this terminology and use the term *metamour of a vertex* for a vertex at distance 2. Correspondingly, we call the distance-2 graph of a graph G, the *metamour graph* of G.

We have discussed the context and set-up the necessary vocabulary, so we are now ready to talk about the content and results of this article. We investigate graphs where each vertex has the same number of metamours. If this number is k, we say that the graph is *k-metamour-regular*. Reformulated, a k-metamour-regular graph is a graph whose distance-2 graph is k-regular. The leftmost connected component of the graph in Fig. 1.1 shows an example of a 2-metamour-regular (sub)graph with six vertices. We ask:

Question. Can we find all k-metamour-regular graphs and give a precise description of how they look like?

Certainly not every graph satisfies this property, but some do. For example, for $k = 2$ it is not hard to check that in (connected) graphs with at least five vertices, where

- every vertex has two neighbors, i.e., cyclic graphs, or
- every vertex is adjacent to every other vertex but two, i.e., complements of cyclic graphs,

each vertex indeed has exactly two metamours. Hence, these graphs are 2-metamour-regular.

The second construction above can be generalized, and in this article we provide a generic construction that allows to create k-metamour-regular graphs for any number k. One of our key results is that the vast majority of these graphs can indeed be built by this generic construction. To be more precise, only k-metamour-regular graphs whose metamour graph consists of at most two connected components cannot necessarily be constructed this way.

This key result lays the foundations for another main result of this article, namely the identification of all 2-metamour-regular graphs, so we answer the question above for $k = 2$. Our findings are as follows: Every 2-metamour-regular graph of any size falls either into one of the two groups (cyclic or complements of cyclic graphs) above or into the third group of

- 17 exceptional graphs with at least six and at most eight vertices.

We provide a systematic and explicit description of the graphs in the first two groups. All 2-metamour-regular graphs with at most nine vertices – this includes the 17 exceptional graphs of the third group – are shown in Figs. 3.2 to 3.6. Summarized, we present a complete characterization of all 2-metamour-regular graphs. Note that as a consequence, exceptional graphs exist only for up to eight vertices.

In addition to the above result we derive several structural properties of k-metamour-regular graphs for any number k. We also characterize all graphs in which every vertex has no metamour ($k = 0$), exactly one metamour ($k = 1$), and at most one metamour. As a byproduct of every characterization including the one for $k = 2$, we are able to count the number of graphs with these properties. Moreover – and this might be the one sentence take-away message of this article – our findings imply that besides the graphs that are simple to discover (i.e., can be built by the generic construction), only a few (if any) small exceptional graphs are 0-metamour-regular, 1-metamour-regular, graphs where every vertex has most one metamour and 2-metamour-regular.

[4] For *metamour* see for example Hardy and Easten [15, p. 219ff, 298] or Veaux and Rickert [30, p. 397ff, 455], or online at https://www.morethantwo.com/polyglossary.html#metamour.

1.1. Outline

We now provide a short overview on the structure of this paper. The terms discussed so far are formally defined in Section 2. Moreover, there we introduce joins of graphs which are used in the systematic and explicit description of the graphs in the characterizations of metamour-regular graphs. The section also includes some basic properties related to those concepts.

Section 3 is a collection of all results derived in this article. This is accompanied by plenty of consequences of these results and discussions. The proofs of all results are given in Sections 4 to 9. We conclude in Section 10 and provide many questions, challenges and open problems for future work.

So far not mentioned is the next section. There, literature related to this article is discussed.

1.2. Related literature

In this section we discuss concepts that have already been examined in literature and that are related to metamours (i.e., vertices having distance 2, or "neighbors of neighbors") and the metamour graphs induced by their relations. Metamour graphs are called *distance-*2 *graphs* in Iqbal, Koolen, Park and Rehman [18], and some authors also call them 2-*distance graphs* (e.g., Azimi and Farrokhi [3]) or 2*nd distance graphs* (e.g., Simić [25]). This notion also appears in the book by Brouwer, Cohen and Neumaier [9, p. 437].

The overall question is to characterize all graphs whose distance-n graph equals some graph of a given graph class. Simić [25] answers the question when the distance-n graph of a graph equals the line graph of this graph. Characterizing when the distance-2 graph is a path or a cycle is done by Azimi and Farrokhi [3], and when it is a union of short paths or and a union of two complete graphs by Ching and Garces [12]. Azimi and Farrokhi [4] also tackled the question when the distance-2 graph of a graph equals the graph itself. This question is also topic of the online discussion [29]. Bringing the context to our article, we investigate the above question for the graph class of regular graphs.

Moreover, vertices in a graph that have distance two, i.e., metamours, or more generally vertices that have a given specific distance, are discussed in the existing literature in many different contexts. The persons participating in the exchange [11] discuss algorithms for efficiently finding vertices having specific distance on trees. Moreover, the notion of dominating sets is extended to vertices at specific distances in Zelinka [32] and in particular to distance two in Kiser and Haynes [20].

Also various kinds of colorings of graphs with respect to vertices of given distance are studied. Typically, the corresponding chromatic number is analyzed, for instance Bonamy, Lévêque and Pinlou [5], Borodin, Ivanova and Neustroeva [7], and Bu and Wang [10] provide such results for vertices at distance two. Algorithms for finding such colorings are also investigated. We mention here Bozdağ, Çatalyürek, Gebremedhin, Manne, Boman and Özgüner [8] as an example. Kamga, Wang, Wang and Chen [19] study variants of so-called vertex distinguishing colorings, i.e., edge colorings where additionally vertices at distance two have distinct sets of colors. Their motivation comes from network problems. The concept is studied more generally but for more specific graph classes in Zhang, Li, Chen, Cheng and Yao [33].

Many of the mentioned results also investigate vertices at distance at most two (compared to exactly two). This is closely related to the concept of the square of a graph, i.e., graphs with the same vertex set as the original graph and two vertices are adjacent if they have distance at most two in the original graph. More generally, this concept is known as powers of graphs; see Bondy and Murty [6, p. 82]. The overall question to characterize all graphs whose nth distance graph equals some graph of a given graph class is studied also for powers of graphs instead of distance-n graphs; see Akiyama, Kaneko and Simić [1]. Colorings are studied for powers of graphs by a motivation coming, among others, from wireless communication networks or graph drawings. The corresponding chromatic number is analyzed, for example, in Kramer and Kramer [21], Alon and Mohar [2] and Molloy and Salavatipour [22]. Results on the hamiltonicity of powers of graphs are studied in Bondy and Murty [6, p. 105] and Underground [28].

Finally, there are distance-regular graphs. Even though the name might suggest that these graphs are closely related to metamour graphs, this is not the case: A graph is distance-regular if it is regular and for any two vertices v and w, the number of vertices at distance j from v and at distance k from w depend only upon j, k and the distance of u and v. The book by Brouwer, Cohen and Neumaier [9] is a good starting point for this whole research area. Plenty of publications related to distance-regular graphs are available, in particular recently Iqbal, Koolen, Park and Rehman [18] considered distance-regular graphs whose distance-2 graphs are strongly regular.

2. Definitions, notation & foundations

This section is devoted to definitions and some simple properties. Moreover, we state (graph-theoretic) conventions and set up the necessary notation that will be used in this article. The proofs of the properties of this section are postponed to Section 4.

2.1. Graph-theoretic definitions, notation & conventions

In this graph-theoretic article we use standard graph-theoretic definitions and notation; see for example Diestel [13]. We use the convention that all graphs in this article contain at least one vertex, i.e., we do not talk about the empty graph. Moreover, we use the following convention for the sake of convenience.

Convention 2.1. *If two graphs are isomorphic, we will call them* equal *and use the equality-sign.*

In many places it is convenient to extend adjacency to subsets of vertices and subgraphs. We give the following definition that is used heavily in Sections 3.5 and 5.

Definition 2.2. Let G be a graph, and let W_1 and W_2 be disjoint subsets of the vertices of G.

- We say that W_1 is *adjacent* in G to W_2 if there is a vertex $v_1 \in W_1$ adjacent in G to some vertex $v_2 \in W_2$.
- We say that W_1 is *completely adjacent* in G to W_2 if every vertex $v_1 \in W_1$ is adjacent in G to every vertex $v_2 \in W_2$.

By identifying a vertex $v \in V(G)$ with the subset $\{v\} \subseteq V(G)$, we may also use (complete) adjacency between v and a subset of $V(G)$. Moreover, for simplicity, whenever we say that subgraphs of G are (completely) adjacent, we mean that the underlying vertex sets are (completely) adjacent.

We explicitly state the negation of adjacent: We say that W_1 is *not adjacent* in G to W_2 if no vertex $v_1 \in W_1$ is adjacent in G to any vertex $v_2 \in W_2$. We will not need the negation of completely adjacent.

We recall the following standard concepts and terminology to fix their notation.

- For a set $W \subseteq V(G)$ of vertices of a graph G, the *induced subgraph* $G[W]$ is the subgraph of G with vertices W and all edges of G that are subsets of W, i.e., edges incident only to vertices of W.
- A set $\mu \subseteq E(G)$ of edges of a graph G is called *matching* if no vertex of G is incident to more than one edge in μ. In particular, the empty set is a matching. The set μ is called *perfect matching* if every vertex of G is incident to exactly one edge in μ.
- For a set $\nu \subseteq E(G)$ of edges of a graph G, we denote by $G - \nu$ the graph with vertices $V(G - \nu) = V(G)$ and edges $E(G - \nu) = E(G) \setminus \nu$.
- The *union* of graphs G_1 and G_2, written as $G_1 \cup G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.
- A *path* π *of length* n in a graph G is written as $\pi = (v_1, \dots, v_n)$ for vertices $v_i \in V(G)$ that are pairwise distinct.
- A *cycle* γ *of length* n or n*-cycle* γ in a graph G is written as $\gamma = (v_1, \dots, v_n, v_1)$ for vertices $v_i \in V(G)$ that are pairwise distinct.
- For a graph G, we write $\mathcal{C}(G)$ for the set of connected components of G. Note that each element of $\mathcal{C}(G)$ is a subgraph of G.
- We write $\overline{G}$ for the *complement* of a graph G, i.e., the graph with the same vertices as G but with an edge between vertices exactly where G has no edge.
- We use the complete graph K_n for $n \geq 1$, the complete t-partite graph $K_{n_1,\dots,n_t}$ for $n_i \geq 1$, $i \in \{1, \dots, t\}$, the path graph P_n for $n \geq 1$, and the cycle graph C_n for $n \geq 3$.

Remark 2.3. We will frequently use the complement $\overline{C_n}$ of the cycle graph C_n for $n \geq 3$. Note that

- $\overline{C_3}$ is the graph with 3 isolated vertices,
- $\overline{C_4}$ is the graph with 2 disjoint single edges, and
- $\overline{C_5}$ equals C_5 (see also Fig. 3.2).

We close this section and continue with definitions and concepts that are specific for this article.

2.2. Metamours

We now formally define the most fundamental concept of this article, namely metamours.

Definition 2.4. Let G be a graph.

- A vertex v of the graph G is a *metamour* of a vertex w of G if the distance of v and w on the graph G equals 2.
- The *metamour graph* M of G is the graph with the same vertex set as G and an edge between the vertices v and w of M whenever v is a metamour of w in G.

We can slightly reformulate the definition of metamours: A vertex v having a different vertex w as metamour, i.e., having distance 2 on a graph, is equivalent to saying that v and w are not adjacent and there is a vertex u such that both v and w have an edge incident to this vertex u, i.e., u is a common neighbor of v and w.

Clearly, there is no edge in a graph between two vertices that are metamours of each other. This is reflected in the relation between the metamour graph and the complement of a graph, and put into writing as the following observation.

Observation 2.5. *Let G be a graph. Then the metamour graph of G is a subgraph of the complement of G.*

The question whether the metamour graph equals the complement will appear in many statements of this article. The following simply equivalence is useful.

Proposition 2.6. *Let G be a connected graph with n vertices. Then the following statements are equivalent:*

(a) The metamour graph of G equals $\overline{G}$.
(b) The graph G has diameter 2 or $G = K_n$.

2.3. Metamour-degree & metamour-regularity

Having the concept of metamours, it is natural to investigate the number of metamours of a vertex. We formally define this "degree" and related concepts below.

Definition 2.7. Let G be a graph.

- The *metamour-degree* of a vertex of G is the number of metamours of this vertex.
- The *maximum metamour-degree* of the graph G is the maximum over the metamour-degrees of its vertices.
- For $k \geq 0$ the graph G is called *k-metamour-regular* if every vertex of G has metamour-degree k, i.e., has exactly k metamours.

We finally have k-metamour-regularity at hand and can now start to relate it to other existing terms. We begin with the following two observations.

Observation 2.8. *Let $k \geq 0$, and let G be a graph and M its metamour graph. Then G is k-metamour-regular if and only if the metamour graph M is k-regular.*

The number of vertices with odd degree is even by the handshaking lemma. Therefore, we get the following observation.

Observation 2.9. *Let $k \geq 1$ be odd. Then the number of vertices of a k-metamour-regular graph is even.*

Proposition 2.10. *Let G be a connected graph with n vertices. Then the following statements are equivalent:*

(a) The metamour graph of G equals $\overline{G}$.
(b) For every vertex of G, the sum of its degree and its metamour-degree equals $n - 1$.

Note that if $k \geq 0$ and G is a connected k-metamour-regular graph with n vertices, then (b) states that the graph G is $(n - 1 - k)$-regular. We use this in Proposition 2.13.

2.4. Joins of graphs

Given two graphs, we already have defined the union of these graphs in Section 2.1. A join of graphs is a variant of that. We will introduce this concept now, see also Harary [14, p. 21], and then discuss a couple of simple properties of joins, also in conjunction with metamour graphs.

Definition 2.11. Let G_1 and G_2 be graphs with disjoint vertex sets $V(G_1)$ and $V(G_2)$. The *join of G_1 and G_2* is the graph denoted by $G_1 \nabla G_2$ with vertices $V(G_1) \cup V(G_2)$ and edges

$$E(G_1) \cup E(G_2) \cup \bigl\{\{g_1, g_2\} \bigm| g_1 \in V(G_1) \text{ and } g_2 \in V(G_2)\bigr\}.$$

Some graphs in Figs. 3.3 to 3.6 are joins of complements of cycle graphs. All of the joins of graphs in this paper are "disjoint joins". We use the convention that if the vertex sets $V(G_1)$ and $V(G_2)$ are not disjoint, then we make them disjoint before the join. We point out that the operator ∇ is associative and commutative.

Let us get to know joins of graphs in form of a supplement to Remark 2.3. We have

$$K_{3,3,\dots,3} = \overline{C_3} \nabla \overline{C_3} \nabla \cdots \nabla \overline{C_3}$$

for the complete multipartite graph $K_{3,3,\dots,3}$.

There are connections between joins of graphs and metamour graphs that will appear frequently in the statements and results of this article. We now present first such relations.

Proposition 2.12. *Let G be a connected graph and M its metamour graph. Then the following statements are equivalent:*

(a) The metamour graph M equals $\overline{G}$ and $|\mathcal{C}(M)| \geq 2$.
(b) The graph G equals $G = \overline{M_1} \nabla \cdots \nabla \overline{M_t}$ with $\{M_1, \ldots, M_t\} = \mathcal{C}(M)$ and $t \geq 2$.
(c) There are graphs G_1 and G_2 with $G = G_1 \nabla G_2$.

Proposition 2.13. *Let $k \geq 0$ and G be a connected k-metamour-regular graph with n vertices. Let M be the metamour graph of G. Then the following statements are equivalent:*

(a) The metamour graph M equals $\overline{G}$.
(b) The graph G has diameter 2 or $G = K_n$.
(c) The graph G equals $G = \overline{M_1} \nabla \cdots \nabla \overline{M_t}$ with $\{M_1, \ldots, M_t\} = \mathcal{C}(M)$.
(d) The graph G is $(n - 1 - k)$-regular.

Note that we have $G = K_n$ in (b) if and only if $k = 0$.

3. Characterizations & properties of metamour-regular graphs

It is now time to present the main results of this article and their implications. In this section, we will do this in a formal manner using the terminology introduced in Section 2. This section also includes brief sketches of the proofs of the main results. The actual and complete proofs of the results follow later, from Section 5 on to Section 9. Proof-wise the results on k-metamour-regular graphs for $k \in \{0, 1, 2\}$ build upon the result for arbitrary $k \geq 0$; this determines the order of the sections containing the proofs. We will in this section, however, start with $k = 0$, followed by $k = 1$ and $k = 2$ and only deal with general k later on.

3.1. 0-metamour-regular graphs

As a warm-up, we start with graphs in which no vertex has a metamour. The following theorem is not very surprising; the only graphs satisfying this property are complete graphs.

Theorem 3.1. *Let G be a connected graph with n vertices. Then G is 0-metamour-regular if and only if $G = K_n$.*

An alternative point of view is that of the metamour graph. The theorem simply implies that in the case of 0-metamour-regularity, the metamour graph is empty and also equals the complement of the graph itself. The latter property will occur frequently later on which also motivates its formulation in the following corollary.

Corollary 3.2. *A connected graph is 0-metamour-regular if and only if its complement equals its metamour graph and this graph has no edges.*

The characterization provided by Theorem 3.1 makes it also easy to count how many different 0-metamour-regular graphs there are and leads to the following corollary.

Corollary 3.3. *The number $m_{=0}(n)$ of unlabeled connected 0-metamour-regular graphs with n vertices is*

$$m_{=0}(n) = 1.$$

The Euler transform, see Sloane and Plouffe [26], of this sequence gives the numbers $m'_{=0}(n)$ of unlabeled but not necessarily connected 0-metamour-regular graphs with n vertices. The number $m'_{=0}(n)$ equals the partition function $p(n)$, i.e., the number of integer partitions[5] of n. The corresponding sequence starts with

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$m'_{=0}(n)$	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176

and is A000041 in The On-Line Encyclopedia of Integer Sequences [27].

This completes the properties of 0-metamour-regular graphs that we bring here. We will, however, see in the following sections how these properties behave in context of other graph classes.

[5] An *integer partition* of a positive integer n is a way of representing n as a sum of positive integers; the order of the summands is irrelevant. The *parts* of a partition are the summands.

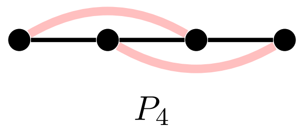


Fig. 3.1. The path graph P_4 where each vertex has exactly 1 metamour.

3.2. 1-metamour-regular graphs

The next easiest case is that of graphs in which every vertex has exactly one other vertex as metamour. As the metamour relation is symmetric, these vertices always come in pairs. We write this fact down in the following proposition.

Proposition 3.4. *Let G be a graph with n vertices. Then the following statements hold:*

(a) If every vertex of G has at most one metamour, then the edges of the metamour graph of G form a matching, i.e., the vertices of G having exactly one metamour come in pairs such that the two vertices of a pair are metamours of each other.
(b) If G is 1-metamour-regular, then n is even and the edges of the metamour graph of G form a perfect matching.

By this connection of 1-metamour-regular graphs to perfect matchings, we can divine the underlying behavior. This leads to our main result of this section, a characterization of 1-metamour-regular graphs; see the theorem below. It turns out that one exceptional case, namely the graph P_4 (Fig. 3.1), occurs.

Theorem 3.5. *Let G be a connected graph with n vertices. Then G is 1-metamour-regular if and only if $n \geq 4$ is even and either*

(a) $G = P_4$ or
(b) $G = K_n - \mu$ for some perfect matching μ of K_n

holds.

When excluding $G = P_4$, then the graphs in the theorem are exactly the cocktail party graphs [31].

Let us again view this from the angle of metamour graphs. As soon as we exclude the exceptional case P_4, the metamour graph and the complement of a 1-metamour-regular graph coincide; see the following corollary.

Corollary 3.6. *A connected graph with $n \geq 5$ vertices is 1-metamour-regular if and only if its complement equals its metamour graph and this graph is 1-regular.*

Note that a 1-regular graph with n vertices is a graph induced by a perfect matching of K_n. In view of Proposition 2.13, we can extend the two equivalent statements.

As we have a characterization of 1-metamour-regular graphs (provided by Theorem 3.5) available, we can determine the number of different graphs in this class. Clearly, this is strongly related to the existence of a perfect matching; details are provided below and also in Section 7, where proofs are given.

Corollary 3.7. *The sequence of numbers $m_{=1}(n)$ of unlabeled connected 1-metamour-regular graphs with n vertices starts with*

n	1	2	3	4	5	6	7	8	9	10
$m_{=1}(n)$	0	0	0	2	0	1	0	1	0	1

and we have

$$m_{=1}(n) = \begin{cases} 0 & \text{for odd } n, \\ 1 & \text{for even } n \geq 6. \end{cases}$$

The Euler transform, see [26], of the sequence of numbers $m_{=1}(2n)$ gives the numbers $m'_{=1}(2n)$ of unlabeled but not necessarily connected 1-metamour-regular graphs with $2n$ vertices. The sequence of these numbers starts with

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$m'_{=1}(2n)$	0	2	1	4	3	8	7	15	15	27	29	48	53	82	94	137	160	225

.

This sequence also counts how often a part 2 appears in all integer partitions[6] of $n+2$ with parts at least 2. The underlying bijection is formulated as the following corollary.

[6] For integer partitions, see footnote 5.

Corollary 3.8. *Let $n \geq 0$. Then the set of unlabeled 1-metamour-regular graphs with $2n$ vertices is in bijective correspondence to the set of partitions of $n+2$ with smallest part equal to 2 and one part 2 of each partition marked.*

3.3. Graphs with maximum metamour-degree 1

Let us now slightly relax the metamour-regularity condition and consider graphs in which every vertex of G has at most one metamour. In view of Proposition 3.4, matchings play an important role again. Formally, the following theorem holds.

Theorem 3.9. *Let G be a connected graph with n vertices. Then the maximum metamour-degree of G is 1 if and only if either*

(a) $G \in \{K_1, K_2, P_4\}$ or
(b) $n \geq 3$ and $G = K_n - \mu$ for some matching μ of K_n

holds.

As in the sections above, the obtained characterization leads to the following equivalent statements with respect to metamour graph and complement.

Corollary 3.10. *A connected graph with $n \geq 5$ vertices has the property that every vertex has at most one metamour if and only if its complement equals its metamour graph and this graph has maximum degree 1.*

Note that graphs with maximum degree 1 and n vertices are graphs induced by a (possibly empty) matching of K_n. In view of Proposition 2.6, we can extend the two equivalent statements by a third saying that G has diameter 2 or $G = K_n$.

Counting the graphs with maximum metamour-degree 1 relies on the number of matchings; see the relevant proofs in Section 8 for details. We obtain the following corollary.

Corollary 3.11. *The sequence of numbers $m_{\leq 1}(n)$ of unlabeled connected graphs with n vertices where every vertex has at most one metamour starts with*

n	1	2	3	4	5	6	7	8	9	10
$m_{\leq 1}(n)$	1	1	2	4	3	4	4	5	5	6

and we have

$$m_{\leq 1}(n) = \lfloor \tfrac{n}{2} \rfloor + 1$$

for $n \geq 5$.

The Euler transform, see [26], gives the sequence of numbers $m'_{\leq 1}(n)$ of unlabeled but not necessarily connected graphs with maximum metamour-degree 1 and n vertices which starts with

n	1	2	3	4	5	6	7	8	9	10	11	12	13
$m'_{\leq 1}(n)$	1	2	4	9	14	26	43	76	122	203	322	523	814

.

3.4. 2-metamour-regular graphs

We now come to the most interesting graphs in this article, namely graphs in which every vertex has exactly two other vertices as metamours. Also in this case a characterization of the class of graphs is possible. We first consider Observation 2.8 in view of 2-metamour-regularity. As a graph is 2-regular if and only if it is a union of cycles, the following observation is easy to verify.

Observation 3.12. *Let G be a graph and M its metamour graph. Then G is 2-metamour-regular if and only if every connected component of the metamour graph M is a cycle.*

We are ready to fully state the mentioned characterization formally as the theorem below. We comment this result and discuss implications afterwards. Note that Theorem 3.13 generalizes the main result of Azimi and Farrokhi [3, Theorem 2.3] which only deals with metamour graphs being connected.

Theorem 3.13. *Let G be a connected graph with n vertices. Then G is 2-metamour-regular if and only if $n \geq 5$ and one of*

(a) $G = \overline{C_{n_1}} \mathbin{\nabla} \cdots \mathbin{\nabla} \overline{C_{n_t}}$ with $n = n_1 + \cdots + n_t$ for some $t \geq 1$ and $n_i \geq 3$ for all $i \in \{1, \dots, t\}$,
(b) $G = C_n$, or
(c) $G \in \left\{ \begin{array}{l} H^a_{4,4}, H^b_{4,4}, H^c_{4,4}, \\ H^a_7, H^b_7, H^a_{4,3}, H^b_{4,3}, H^c_{4,3}, H^d_{4,3}, \\ H^a_6, H^b_6, H^c_6, H^a_{3,3}, H^b_{3,3}, H^c_{3,3}, H^d_{3,3}, H^e_{3,3} \end{array} \right\}$ with graphs defined by Figs. 3.3, 3.4 and 3.5

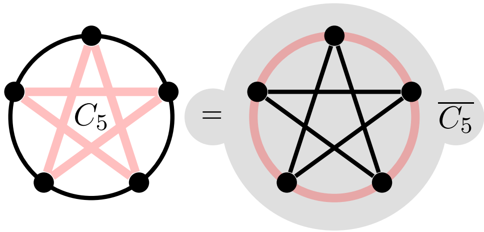


Fig. 3.2. The only graph of order 5 (+ one differently drawn copy) where each vertex has exactly 2 metamours.

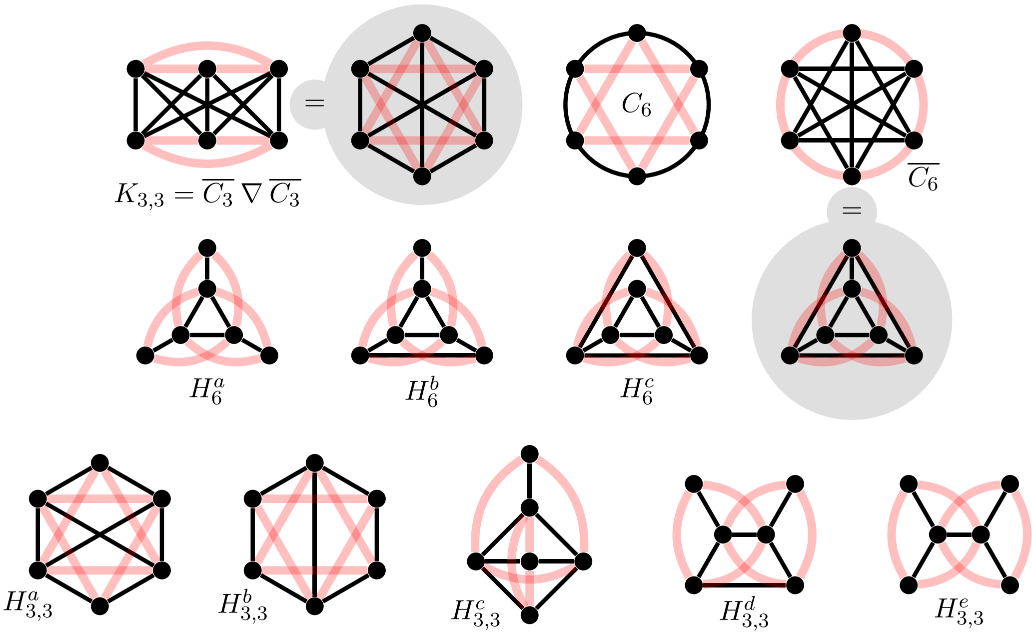


Fig. 3.3. All 11 graphs (+ 2 differently drawn copies) of order 6 where each vertex has exactly 2 metamours.

holds.

A representation of every 2-metamour-regular graph with at most 9 vertices can be found in Figs. 3.2, 3.3, 3.4, 3.5 and 3.6. For 10 vertices, all 2-metamour-regular graphs – there are 6 of them – are C_{10}, $\overline{C_{10}}$, $\overline{C_7} \nabla \overline{C_3}$, $\overline{C_6} \nabla \overline{C_4}$, $\overline{C_5} \nabla \overline{C_5}$, $\overline{C_4} \nabla \overline{C_3} \nabla \overline{C_3}$.

For rounding out Theorem 3.13, we have collected a couple of remarks and bring them now.

Remark 3.14.

1. The smallest possible 2-metamour-regular graph has 5 vertices, and there is exactly one with five vertices, namely C_5; see Fig. 3.2. This graph is covered by Theorem 3.13(a) as well as (b) because $C_5 = \overline{C_5}$.
2. Theorem 3.13(a) can be replaced by any other equivalent statement of Proposition 2.13.
3. For $t = 1$, Theorem 3.13(a) condenses to $G = \overline{C_n}$. Implicitly we get $n = n_1 \geq 5$.
4. The graphs $C_{n_1}, \ldots, C_{n_t}$ of Theorem 3.13(a) satisfy

$$C_{n_i} = M_i$$

   with $\{M_1, \ldots, M_t\} = \mathcal{C}(M)$. This means that the decomposition of the graph $G = \overline{C_{n_1}} \nabla \cdots \nabla \overline{C_{n_t}}$ reveals the metamour graph of G and vice versa.
5. For Theorem 3.13(a) as well as for (b) with $n = 5$, every graph satisfies that its complement equals its metamour graph. For all other cases, this is not the case. A full formulation of this fact is stated as Corollary 3.17.

The proof of Theorem 3.13 is quite extensive and we refer to Section 9 for the complete proof; at this point, we only sketch it.

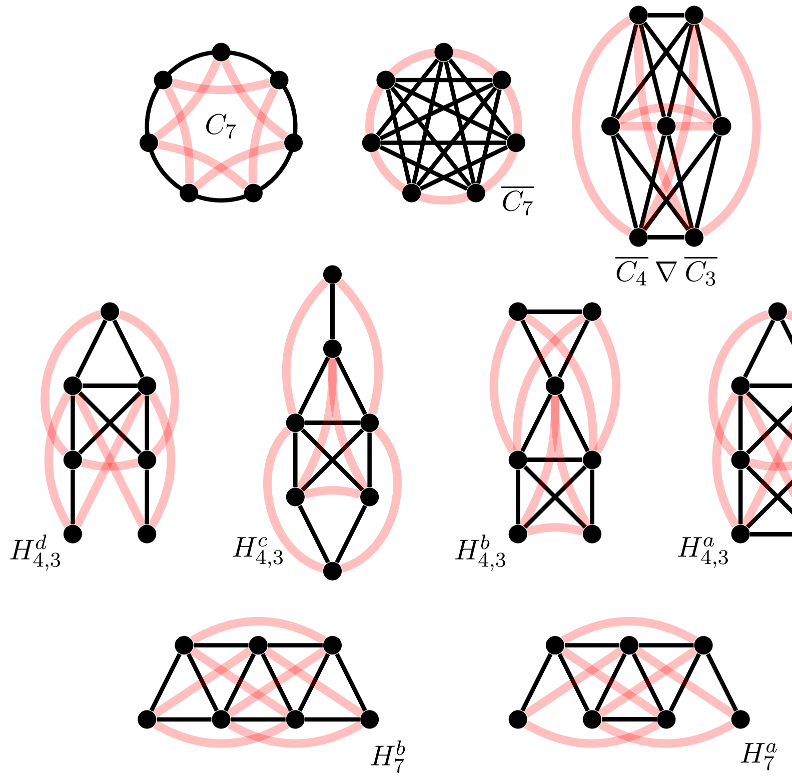


Fig. 3.4. All 9 graphs of order 7 where each vertex has exactly 2 metamours.

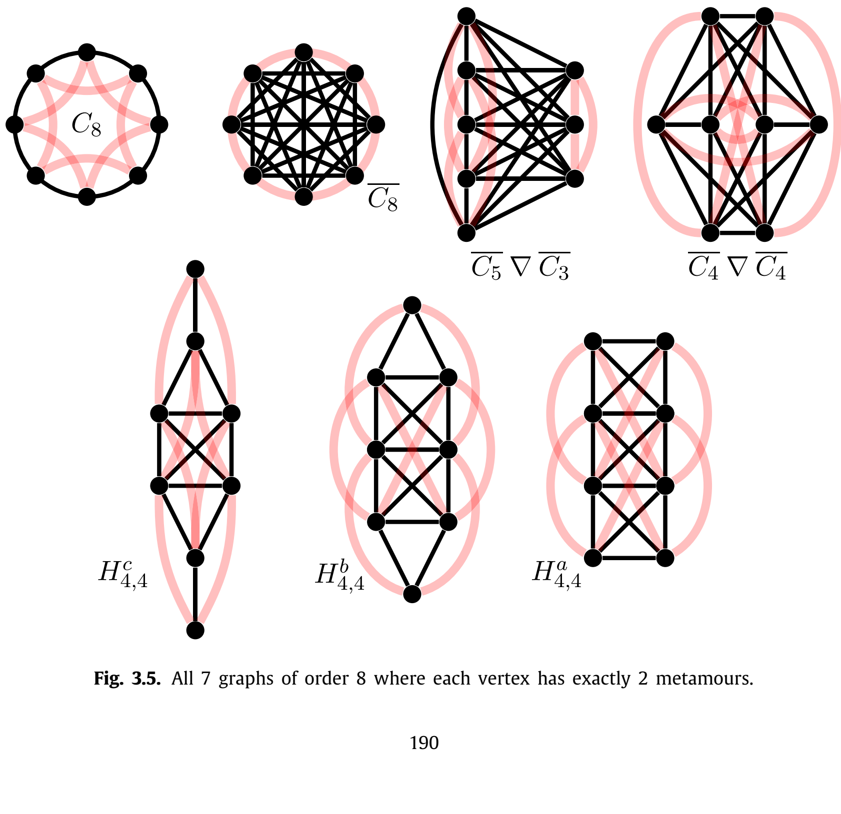


Fig. 3.5. All 7 graphs of order 8 where each vertex has exactly 2 metamours.

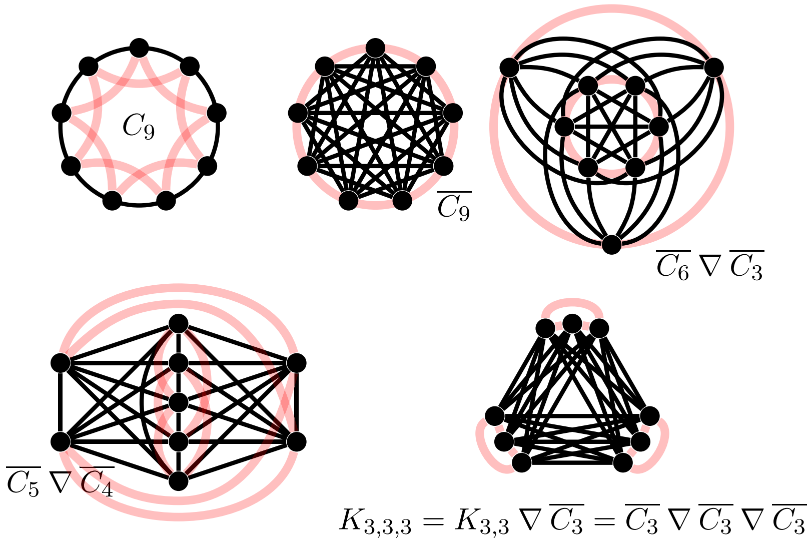


Fig. 3.6. All 5 graphs of order 9 where each vertex has exactly 2 metamours.

Sketch of Proof of Theorem 3.13. Let G be 2-metamour-regular. We apply Theorem 3.20 (to be presented in Section 3.5) with $k = 2$, which leads us to one of three cases. The generic case is already covered by Theorem 3.20 combined with Observation 3.12.

If the metamour graph of G is connected, then we first rule out that G is a tree. If not, then the graph contains a cycle, and depending on whether G contains a cycle of length n or not, we get $G = H_7^b$ or $G \in \{H_6^a, H_6^b, H_7^a\}$, respectively. (This parts of the proof are formulated as Proposition 9.6, Lemmas 9.2 and 9.1.) We prove these parts by studying the longest cycle in the graph G and step-by-step obtaining information between which vertices edges, non-edges and metamour relations need to be. Knowing enough, 2-metamour-regularity forces the graph to be bounded in the number of vertices. As this number is quite small, further case distinctions lead to the desired graphs. Furthermore, a separate investigation is needed when every vertex has degree 2 or $n - 3$ or a mixture of these to degrees; here n is the number of vertices of G. This leads to $G = C_n$ and n odd, $G = \overline{C_n}$ and $G \in \{C_5, H_6^c\}$, respectively. (These parts of the proof are formulated as Lemmas 9.4, 9.3 and 9.5.)

If the metamour graph of G is not connected, it consists of exactly two connected components (by Theorem 3.20), and the graph is split respecting the connected components of the metamour graph of G. Due to 2-metamour-regularity, Theorem 3.20 guarantees that the number of vertices of each piece is a most 2. Studying each configuration separately lead to the graphs $G = C_n$ and n even, or $G \in \{H_{4,4}^a, H_{4,4}^b, H_{4,4}^c, H_{4,3}^a, H_{4,3}^b, H_{4,3}^c, H_{4,3}^d, H_{3,3}^a, H_{3,3}^b, H_{3,3}^c, H_{3,3}^d, H_{3,3}^e\}$. (This part of the proof is formulated as Proposition 9.7.) $\square$

The characterization provided by Theorem 3.13 has many implications. We start with the following easy corollaries.

Corollary 3.15. *Let G be a connected graph with $n \geq 9$ vertices. Then G is 2-metamour-regular if and only if G is either C_n or $\overline{C_{n_1}} \nabla \cdots \nabla \overline{C_{n_t}}$ with $n = n_1 + \cdots + n_t$ for some $t \geq 1$ and $n_i \geq 3$ for all $i \in \{1, \ldots, t\}$.*

Corollary 3.16. *Let G be a connected graph with $n \geq 9$ vertices. Then G is 2-metamour-regular if and only if G is either 2-regular or $(n - 3)$-regular.*

As before, we consider the relation of metamour graphs and complement more closely; see the following corollary. Again, we feel the spirit of Proposition 2.13.

Corollary 3.17. *Let G be a connected 2-metamour-regular graph with n vertices. Then the following statements are equivalent:*

(a) The metamour graph of G is a proper subgraph of $\overline{G}$.
(b) We have either $G = C_n$ and $n \geq 6$ (Theorem 3.13(b)) or G is one of the graphs in Theorem 3.13(c).
(c) The graph G has diameter larger than 2.

Theorem 3.13 makes it also possible to count how many different 2-metamour-regular graphs with n vertices there are. We provide this in the following corollary.

Corollary 3.18. *The sequence of numbers $m_{=2}(n)$ of unlabeled connected 2-metamour-regular graphs with n vertices starts with*

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$m_{=2}(n)$	0	0	0	0	1	11	9	7	5	6	7	10	11	14	18	22	26	34	40	50

and for $n \geq 9$ we have

$$m_{=2}(n) = p_3(n) + 1,$$

where $p_3(n)$ is the number of integer partitions[7] *of n with parts at least 3.*

The sequence of numbers $m_{=2}(n)$ is A334275 in The On-Line Encyclopedia of Integer Sequences [27].

We apply the Euler transform, see Sloane and Plouffe [26], on this sequence and obtain the numbers $m'_{=2}(n)$ of unlabeled but not necessarily connected 2-metamour-regular graphs with n vertices. The sequence of these numbers starts with

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$m'_{=2}(n)$	0	0	0	0	1	11	9	7	5	7	18	85	117	141	143	179	277	667

.

3.5. *k*-metamour-regular graphs

In this section, we present results that are valid for graphs with maximum metamour-degree k and k-metamour-regular graphs for any non-negative number k.

We start with Proposition 3.19 stating that the join of complements of k-regular graphs is a k-metamour-regular graph.

Proposition 3.19. *Let M be a graph having $t \geq 2$ connected components $M_1, \ldots, M_t$. Set*

$$G = \overline{M_1} \nabla \cdots \nabla \overline{M_t}.$$

Then G has metamour-graph M. In particular, if M is k-regular for some $k \geq 0$, then G is k-metamour-regular.

We call this construction of a graph with given metamour graph *generic construction*. In particular this generic construction allows us to build k-metamour-regular graphs. We will not investigate further options to construct k-metamour-regular graphs (as, for example, with circulant graphs), as the above construction suffices for the ultimate goal of this paper to characterize all k-metamour-regular graphs for $k \leq 2$. Last in our discussion of Proposition 3.19, we note that, as we have $G = \overline{M_1} \nabla \cdots \nabla \overline{M_t}$, we can also extend Proposition 3.19 by the equivalent statements of Propositions 2.12 and 2.13.

Next we state the main structural result about graphs with maximum metamour-degree k as Theorem 3.20. A consequence of this main statement is that every k-metamour-regular graph is the join of complements of k-regular graphs as in Proposition 3.19, or has in its metamour graph only one or two connected components; see Corollary 3.21 for a full formulation.

Theorem 3.20. *Let $k \geq 0$. Let G be a connected graph with maximum metamour-degree k and M its metamour graph. Then exactly one of the following statements is true:*

(a) *The metamour graph M is connected.*
(b) *The metamour graph M is not connected and the induced subgraph $G[V(M_i)]$ is connected for some $M_i \in \mathcal{C}(M)$.*
   In this case we have $G = \overline{M_1} \nabla \cdots \nabla \overline{M_t}$ with $\{M_1, \ldots, M_t\} = \mathcal{C}(M)$ and $t \geq 2$, and any other equivalent statement of Proposition 2.12.
(c) *The metamour graph M is not connected and no induced subgraph $G[V(M_i)]$ is connected for any $M_i \in \mathcal{C}(M)$.*
   In this case the metamour graph M has exactly two connected components and the following holds. Set $G^M = G[V(M_1)] \cup G[V(M_2)]$ with $\{M_1, M_2\} = \mathcal{C}(M)$, i.e., G^M is the graph G after deleting every edge between two vertices that are from different connected components of the metamour graph M. Then we have:
   (i) *Every connected component of G^M has at most k vertices.*
   (ii) *If G is k-metamour-regular, then every connected component of G^M is a regular graph.*
   (iii) *Every connected component G_i of G^M satisfies $\overline{G_i} = M[V(G_i)]$.*
   (iv) *If two different connected components of G^M are adjacent in G, then these connected components are completely adjacent in G.*
   (v) *If two vertices of different connected components G_i and G_j of G^M have a common neighbor in G, then every vertex of G_i is a metamour of every vertex of G_j.*

7 For integer partitions, see footnote 5 on page 8. The function $p_3(n)$ is A008483 in [27].

(vi) *If a connected component G_i of G^M is adjacent in G to another connected component G_j consisting of $k-d$ vertices for some $d \geq 0$, then the neighbors (in G) of vertices of G_i are in at most $d+2$ (including G_j) connected components.*

The complete and extensive proof of Theorem 3.20 can be found in Section 5; we again only sketch it at this point.

Sketch of Proof of Theorem 3.20. Suppose the metamour graph of G is not connected. We split the graph respecting the connected components of the metamour graph of G.

First, we lift adjacency of some vertices of different components of this split to all vertices and likewise, metamour relations between vertices of different components. By this structural property, we show on the one hand that the maximum metamour-degree k implies that each of the components has at most k vertices. On the other hand, by investigating an alternating behavior on shortest paths between different components, we deduce that the metamour graph of G has exactly two connected components.

Before gluing everything together, two more properties need to be derived: We show that within each of the components, the non-edges correspond exactly to metamour relations, and we bound the number of neighboring components of a component. This provides enough structure completing the proof. □

Let us discuss the three outcomes of Theorem 3.20 in view of the characterizations provided in Sections 3.1 to 3.4. Toward this end note that in case (b), the graph G is obtained by the generic construction.

For 0-metamour-regular graphs due to Theorem 3.1 there is no graph that is not obtained by the generic construction. So only (b) happens, except if the graph consists of only one vertex in which case a degenerated (a) happens. For 1-metamour-regular graphs we know that there is only one graph not obtained by the generic construction by Theorem 3.5. This exceptional case is associated to (c), otherwise we are in (b). Finally Theorem 3.13 states that beside the generic case associated to (b), there is only the class with graphs C_n and 17 exceptional cases of 2-metamour-regular graphs associated to (a) and (c).

At last in this section, we bring and discuss the full formulation of a statement mentioned earlier.

Corollary 3.21. *Let $k \geq 0$. Let G be a connected graph with maximum metamour-degree k and M its metamour graph. Let $\{M_1, \dots, M_t\} = \mathcal{C}(M)$. If $t \geq 3$, then we have*

$$G = \overline{M_1} \mathbin{\nabla} \cdots \mathbin{\nabla} \overline{M_t}.$$

By this corollary every k-metamour-regular graph that has at least three connected components in its metamour graph is a join of complements of k-regular graphs and therefore can be built by the generic construction. As a consequence, it is only possible that a 2-metamour-regular graph is not obtained by the generic construction if its metamour graph has at most two connected components.

4. Proofs regarding foundations

We start by proving Proposition 2.6 which relates the metamour graph, the complement and the diameter of a graph.

Proof of Proposition 2.6. Suppose (a) holds. Let v and w be vertices in G. If $v = w$, then their distance is 0. If v and w are adjacent in G, then their distance is 1. Otherwise, there is no edge $\{v, w\}$ in G. Therefore, this edge is in $\overline{G} = M$, where M is the metamour graph of G. This implies that v and w are metamours. Thus, the distance between v and w is 2. Consequently, no distance in G is larger than 2, which implies that the diameter of G is at most 2. As a result, either the diameter of G is exactly 2, or it is at most 1. If the diameter is equal to 1, then all vertices of G are pairwise adjacent and therefore $G = K_n$. Furthermore, there are at least two vertices at distance 1 and hence $n \geq 2$. If the diameter is 0, then $G = K_1$.

Now suppose (b) holds. If $G = K_n$, then its diameter is either equal to 0 if $n = 1$ or equal to 1 if $n \geq 2$. Therefore, in this case the diameter of G is at most 2. Now let $\{v, w\}$ be an edge in $\overline{G}$. Then $v \neq w$ and these vertices are not adjacent in G, so their distance is at least 2. As the diameter is at most 2, the distance of v and w is at most 2. Consequently their distance is exactly 2 implying that they are metamours. So the edge $\{v, w\}$ is in M, and hence the complement of G is a subgraph of the metamour graph of G. Due to Observation 2.5, the metamour graph of G is a subgraph of the complement of G, hence the metamour graph of G and $\overline{G}$ coincide, so we have shown (a). □

Next we prove Proposition 2.10 which relates the metamour graph, the complement, and the degree and metamour-degree of the vertices of a graph.

Proof of Proposition 2.10. We use that for a graph G with n vertices, the sum of the degrees of a vertex in G and in the complement $\overline{G}$ always equals $n-1$.

If (a) holds, then the degree of a vertex in the metamour graph equals the degree in the complement $\overline{G}$. This yields (b) by using the statement at the beginning of this proof.

If (b) holds, then the sum of the degree and the metamour-degree of a vertex equals the sum of the degree in G and the degree in $\overline{G}$ of this vertex by the statement at the beginning of this proof. Therefore, the metamour-degree of a vertex is equal to the degree in $\overline{G}$ of this vertex. Due to Observation 2.5, the metamour graph of G is a subgraph of $\overline{G}$, and therefore the metamour graph of G equals $\overline{G}$. □

Finally we prove Propositions 2.12 and 2.13 that relate the metamour graph and joins.

Proof of Proposition 2.12. We start by showing that (a) implies (b). In M there are no edges between its different connected components. Therefore, in the complement $\overline{M} = G$, there are all possible edges between the vertices of different components. This is equivalent to the definition of the join of graphs; the individual graphs in $\mathcal{C}(M)$ are complemented, and consequently (b) follows.

For proving that (b) implies (c), we simply set $G_1 = \overline{M_1}$ and $G_2 = \overline{M_2} \nabla \cdots \nabla \overline{M_t}$. By the definition of the operator ∇, (c) follows.

We now show that (c) implies (a). Every pair of vertices of G_1 has a common neighbor in G_2. This implies that the vertices of this pair are metamours if and only if they are not adjacent in G_1. The same holds for any pair of vertices of G_2 by symmetry or due to commutativity of the operator ∇. As a consequence of this and because $G = G_1 \nabla G_2$ and the definition of the operator ∇, the metamour graph M and the complement of G coincide. As every possible edge from a vertex of G_1 to a vertex of G_2 exists in G, this complement $\overline{G}$ has at least two connected components, hence $t \geq 2$. $\square$

Proof of Proposition 2.13. (a) and (b) are equivalent by Proposition 2.6.

If $|\mathcal{C}(M)| = t = 1$, then (a) and (c) are trivially equivalent. If $|\mathcal{C}(M)| = t \geq 2$, then this equivalence is part of Proposition 2.12.

Finally, the equivalence of (a) and (d) follows from Proposition 2.10. $\square$

At this point we have shown all results form Section 2 and therefore have laid the foundations of the subsequent results.

5. Proofs regarding k-metamour-regular graphs

Next we give the proofs of results from Section 3.5 concerning graphs with maximum metamour-degree k and k-metamour-regular graphs that are valid for arbitrary $k \geq 0$.

Proof of Proposition 3.19. Let v be a vertex of G. Then v is in $\overline{M_i}$ and therefore in M_i for some $i \in \{1, \dots, t\}$.

Let $u \neq v$ be a vertex of G and $j \in \{1, \dots, t\}$ such that u is in $\overline{M_j}$. If $j \neq i$, then u and v are adjacent in G by construction. Therefore, they are not metamours. If $j = i$, then any vertex not in M_i, i.e., in any of $\overline{M_1}, \dots, \overline{M_{i-1}}, \overline{M_{i+1}}, \dots, \overline{M_t}$, is a common neighbor of u and v. Therefore, u and v are metamours if and only if they are not adjacent in $\overline{M_i}$, and this is the case if and only if they are adjacent in M_i.

Summarized, we have that u is a metamour of v if and only if u is in M_i and adjacent to v in M_i. This yields that the metamour graph of G is $M = M_1 \cup \cdots \cup M_t$ which was to show.

The k-metamour-regularity follows directly from Observation 2.8. $\square$

Proof of Theorem 3.20. If the metamour graph M is connected, then we are in case (a) and nothing is to show. So suppose that the metamour graph is not connected.

We partition the vertices of M (and therefore the vertices of G) into two parts $V' \uplus V^*$ such that the vertex set of each connected component of M is a subset of either V' or V^* and such that neither V' nor V^* is empty, i.e., we partition by the connected components $\mathcal{C}(M)$ of the metamour graph M. As M is not connected, it consists of at least two connected components, and therefore such a set-partition of the vertices of M is always possible. We now split up the graph G into the two subgraphs $G' = G[V']$ and $G^* = G[V^*]$.

Rephrased, we obtain G' and G^* from G by cutting it (by deleting edges) in two, but respecting and not cutting the connected components of its metamour graph M. Note that the formulation is symmetric with respect to G' and G^*, therefore, we might switch the two without loss of generality during the proof. This also implies that in the statements of the following claims we may switch the roles of G' and G^*.

A. *If $G'_1 \in \mathcal{C}(G')$ is adjacent in G to $G^*_1 \in \mathcal{C}(G^*)$, then G'_1 is completely adjacent in G to G^*_1.*

Proof of A. Suppose $u' \in V(G'_1)$ and $v^* \in V(G^*_1)$ are adjacent in G. Let $u \in V(G'_1)$ and $v \in V(G^*_1)$. We have to prove that $\{u, v\} \in E(G)$. There is a path $\pi_{u',u} = (u_1, u_2, \dots, u_r)$ from $u_1 = u'$ to $u_r = u$ in G'_1 because G'_1 is connected. Furthermore, there is a path $\pi_{v^*,v} = (v_1, v_2, \dots, v_s)$ from $v_1 = v^*$ to $v_s = v$ in G^*_1 because G^*_1 is connected.

We use induction to prove that $\{u_1, v_\ell\} \in E(G)$ for all $\ell \in \{1, \dots, s\}$. Indeed, this is true for $\ell = 1$ by assumption. So assume $\{u_1, v_\ell\} \in E(G)$. We have $\{v_\ell, v_{\ell+1}\} \in E(G)$ because this is an edge of the path $\pi_{v^*,v}$. If $\{u_1, v_{\ell+1}\} \notin E(G)$, then u_1 and $v_{\ell+1}$ are metamours. But u_1 has all its metamours in G', a contradiction. Hence, $\{u_1, v_{\ell+1}\} \in E(G)$, which finishes the induction. In particular, we have proven that $\{u_1, v_s\} \in E(G)$.

Now we prove that $\{u_\ell, v_s\} \in E(G)$ holds for all $\ell \in \{1, \dots, r\}$ by induction. This holds for $\ell = 1$ by the above. Now assume $\{u_\ell, v_s\} \in E(G)$. We have $\{u_\ell, u_{\ell+1}\} \in E(G)$ because this edge is a part of the path $\pi_{u',u}$. If $\{u_{\ell+1}, v_s\} \notin E(G)$, then $u_{\ell+1}$ and v_s are metamours, a contradiction since every metamour of $u_{\ell+1}$ is in G'. Therefore, $\{u_{\ell+1}, v_s\} \in E(G)$ holds and the induction is completed. As a result, we have $\{u, v\} = \{u_r, v_s\} \in E(G)$. $\triangleleft$

B. *Let $\mathcal{C}(M) = \{M_1, \ldots, M_t\}$. If the graph G' is connected, then $G = G' \nabla G^* = \overline{M_1} \nabla \cdots \nabla \overline{M_t}$ with $t \geq 2$.*

Proof of B. The graph G is connected, so every connected component of G^* is adjacent in G to G'. By **A** this implies that G' is completely adjacent in G to G^*, i.e., all possible edges between G' and G^* exist. Therefore, $G = G' \nabla G^*$ by the definition of the join of graphs.

By Proposition 2.12, the full decomposition into the components $\mathcal{C}(M)$ follows. $\triangleleft$

As a consequence of **B**, we are finished with the proof in the case that $G[V(M_i)]$ is connected for some $M_i \in \mathcal{C}(M)$ because statement (b) follows by setting $G' = G[V(M_i)]$.

So from now on we consider the case that every $G[V(M_i)]$ with $M_i \in \mathcal{C}(M)$ has at least two connected components. This is the set-up of statement (c).

C. *Suppose we are in the set-up of (c). Let $G'_1 \in \mathcal{C}(G')$ and $\{G^*_1, G^*_2\} \subseteq \mathcal{C}(G^*)$. If both G^*_1 and G^*_2 are adjacent in G to G'_1, then every vertex of G^*_1 is a metamour of every vertex of G^*_2.*

Proof of C. As both G^*_1 and G^*_2 are adjacent in G to G'_1, they are both completely adjacent in G to G'_1 due to **A**. Furthermore, G^*_1 is not adjacent in G to G^*_2, i.e., no vertex of G^*_1 is adjacent in G to any vertex of G^*_2, because they are in different connected components of G^*. Hence, every vertex of G^*_1 is a metamour of every vertex of G^*_2. $\triangleleft$

D. *Suppose we are in the set-up of (c). Then every connected component of G' has at most k vertices and this connected component's vertex set is a subset of the vertex set of one connected component of the metamour graph M.*

Proof of D. Let $G'_1 \in \mathcal{C}(G')$. As G' is not connected but the graph G is connected, there is a path π from a vertex of G'_1 to some vertex in some connected component of $\mathcal{C}(G')$ other than G'_1. By construction of G' and G^*, the path π splits from start to end into vertices of G'_1, followed by vertices of some $G^*_1 \in \mathcal{C}(G^*)$, followed by some vertices of $G'_2 \in \mathcal{C}(G')$, and remaining vertices. Therefore, we have connected components G'_2 and G^*_1 such that at least one vertex of G'_1 is connected to some vertex of G^*_1 and at least one vertex of G'_2 is connected to some vertex of G^*_1.

Then, due to **C** every vertex of G'_1 is a metamour of every vertex of G'_2. From this, we now deduce two statements.

First, if we assume that G'_1 contains at least $k+1$ vertices, then every vertex of G'_2 has at least $k+1$ metamours, a contradiction to k being the maximum metamour-degree of G. Therefore, G'_1 contains at most k vertices.

Second, every vertex of G'_1 is adjacent in the metamour graph M to every vertex of G'_2, so G'_1 is completely adjacent in M to G'_2. Therefore, all these vertices are in the same connected component of the metamour graph M. In particular, this is true for the set of vertices of G'_1 as claimed, and so the proof is complete. $\triangleleft$

E. *Suppose we are in the set-up of (c). Let v_1 and v_2 be two vertices of different connected components of G'. Then every shortest path from v_1 to v_2 in G consists of vertices alternating between G' and G^*.*

Proof of E. Let $\{G'_1, G'_2\} \subseteq \mathcal{C}(G')$ such that $v_1 \in G'_1$ and $v_2 \in G'_2$. Let $\pi = (u_1, u_2, \ldots, u_r)$ be a shortest path from $u_1 = v_1$ to $u_r = v_2$ in G. Note that u_1 and u_r are both from G' but from different connected components. Hence, π consists of at least two vertices from G' and at least one vertex from G^*.

Assume that the vertices of π are not alternating between G' and G^*. Then, without loss of generality (by reversing the enumeration of the vertices in the path π), there exist indices $i < j$ and graphs $\{\widetilde{G}, \widehat{G}\} = \{G', G^*\}$ with the following properties: Every vertex of the subpath $\pi_{i,j} = (u_i, u_{i+1}, \ldots, u_j)$ of π is of $\widetilde{G}$, and the vertex u_{j+1} exists and is in $\widehat{G}$.

As $\pi_{i,j}$ is a path, all of its vertices are of the same connected component of $\widetilde{G}$. As u_j is adjacent to u_{j+1} and due to **A**, the vertex u_{j+1} is adjacent to every vertex of $\pi_{i,j}$, in particular, adjacent to u_i. But then u_1, …, u_i, u_{j+1}, …, u_r is a shorter path between v_1 and v_2, a contradiction. Hence, our initial assumption was wrong, and the vertices of π are alternating between G' and G^*. $\triangleleft$

F. *Suppose we are in the set-up of (c). Then the metamour graph M has exactly two connected components.*

Proof of F. The metamour graph M is not connected, therefore it has at least two connected components. Assume it has at least three components. Then, without loss of generality (by switching G' and G^*), the graph G' contains vertices of at least two different connected components of M. Let v_1 and v_2 be two vertices of G' that are in different connected components of M. If follows from **D** that v_1 and v_2 are in different connected components of G'. Let $\{G'_1, G'_2\} \subseteq \mathcal{C}(G')$ such that $v_1 \in G'_1$ and $v_2 \in G'_2$. Let $\pi = (u_1, \ldots, u_r)$ be a shortest path from $u_1 = v_1$ to $u_r = v_2$ in G.

Now consider two vertices u_{2i-1} and u_{2i+1} for $i \geq 1$ of π. Both u_{2i-1} and u_{2i+1} are from G' because π consists of alternating vertices from G' and G^* due to **E** and u_1 is from G'. If both u_{2i-1} and u_{2i+1} are from the same connected component of G', then they are in the same connected component of the metamour graph M by **D**. If u_{2i-1} and u_{2i+1} are in different connected components of G', then they are metamours because they have the common neighbor u_{2i} and they are not adjacent. Therefore, they are in the same connected component of M as well. Hence, in any case u_{2i-1} and u_{2i+1} are in the same connected component of M. By induction this implies that $u_1 = v_1$ is in the same connected component of M as $u_r = v_2$, a contradiction to v_1 and v_2 being from different connected components of M. Hence, our assumption was wrong and the metamour graph consists of exactly two connected components. $\triangleleft$

G. *Suppose we are in the set-up of (c). Then every connected component G'_1 of G' satisfies $\overline{G'_1} = M[V(G'_1)]$. If G is k-metamour-regular, then the connected component is a regular graph.*

Proof of G. Let G'_1 be a connected component of G'. As G is connected, there is an edge from a vertex v_1 of G'_1 to G^* and this is extended to every vertex of G'_1 by **A**. Therefore, two different vertices of G'_1 are metamours if and only if they are not adjacent in G'. Restricting this to the subgraph G'_1 yields the first statement.

Furthermore, by construction of G' and G^*, every metamour of v_1 is in G'. Let v' be such a metamour, and suppose that v' is not in G'_1. The vertices v_1 and v' have a common neighbor u that has to be in G^* as v_1 and v' are in different connected components of G'. The vertex u is completely adjacent in G to G'_1 because of **A**. Hence, v' is a metamour of every vertex of G'_1. As a consequence, every vertex of G'_1 has the same number of metamours outside of G'_1, i.e., in G' but not in G'_1. If no such pair of vertices v_1 of G'_1 and v' not of G'_1 that are metamours exists, then every vertex of G'_1 still has the same number of metamours outside of G'_1, namely zero.

Now let us assume that G is k-metamour-regular. As every vertex of G'_1 has the same number of metamours outside of G'_1, this implies that every vertex of G'_1 must also have the same number of metamours inside G'_1. We combine this with the results of first paragraph and conclude that every vertex of G'_1 is adjacent to the same number of vertices of G'_1, and hence G'_1 is a regular graph. $\triangleleft$

H. *Suppose we are in the set-up of (c). If a connected component $G'_1 \in \mathcal{C}(G')$ is adjacent in G to a connected component $G^*_1 \in \mathcal{C}(G^*)$ consisting of $k-d$ vertices for some $d \geq 0$, then the neighbors (in G) of vertices of G'_1 that are in G^* are in at most $d+2$ connected components of G^* (including G^*_1).*

Proof of H. Let $\mathcal{G}^* \subseteq \mathcal{C}(G^*)$ be such that a connected components of G^* is in $\mathcal{G}^*$ if and only if it is adjacent in G to G'_1. G^*_1 consists of $k-d$ vertices and there is some vertex v' of G'_1 that is adjacent to some vertex of G^*_1.

We have to prove that $|\mathcal{G}^*| \leq d+2$ in order to finish the proof. So assume $|\mathcal{G}^*| > d+2$. Let v^* be a vertex of some connected component $G^*_2 \in \mathcal{G}^*$ other than G^*_1. Then v^* is adjacent to v' due to **A**. Because of **C**, every vertex in any connected component in $\mathcal{G}^*$ except G^*_2 is a metamour of v^*. The component G^*_1 contains $k-d$ vertices and there are at least $d+1$ other components each containing at least one vertex. In total, v^* has at least $(k-d)+(d+1) = k+1$ metamours, a contradiction to k being the maximum metamour-degree of G. Therefore, our assumption was wrong and $|\mathcal{G}^*| \leq d+2$ holds. $\triangleleft$

Now we are able to collect everything we have proven so far and finish the proof of statement (c). Due to **F**, the metamour graph M of G consists of exactly two connected components, and consequently the connected components of G^M coincide with the union of the connected components of G' and G^*. Then **D** implies (i), **G** implies (ii), **G** implies (iii), **A** implies (iv), **C** implies (v) and **H** implies (vi). This completes the proof. $\square$

Proof of Corollary 3.21. As the metamour graph consists of at least 3 connected components, we cannot land in the cases (a) and (c) of Theorem 3.20. But then the statement of the corollary follows from (b). $\square$

Now we have proven everything we need to know about graphs with maximum metamour-degree k and k-regular-metamour graphs for general k and can use this knowledge to derive the results we need in order to characterize all k-regular-metamour graphs for $k \in \{0, 1, 2\}$.

6. Proofs regarding 0-metamour-regular graphs

We are now ready to prove all results concerning 0-metamour-regular graphs from Section 3.1. In order to do so we first need the following lemma.

Lemma 6.1. *Let G be a connected graph. If a vertex has no metamour, then it is adjacent to all other vertices of G.*

Proof. Let n be the number of vertices of G. Clearly the statement is true for $n = 1$ as no other vertices are present and for $n = 2$ as the two vertices are adjacent due to connectedness. So let $n \geq 3$, and let v be a vertex of G that has no metamour.

Assume there is a vertex $w \neq v \in V(G)$ such that $\{v, w\} \notin E(G)$. G has a spanning tree T because G is connected. Let $v = u_1, u_2, \ldots, u_r = w$ be the vertices on the unique path from v to w in T. Then $\{u_i, u_{i+1}\} \in E(G)$ for all $i \in \{1, \ldots, r-1\}$, so due to our assumption $r \geq 3$ holds. In particular, $\{u_1, u_2\} \in E(G)$. If $\{u_1, u_3\} \notin E(G)$, then both u_1 and u_3 are adjacent to u_2, but not adjacent to each other and therefore would be metamours. But $u_1 = v$ does not have a metamour, hence $\{u_1, u_3\} \in E(G)$. By induction $\{u_1, u_i\} \in E(G)$ for all $i \in \{1, \ldots, r\}$ and thus $\{v, w\} = \{u_1, u_r\} \in E(G)$, a contradiction. $\square$

Now we are able to prove Theorem 3.1 that provides a characterization of 0-metamour-regular graphs.

Proof of Theorem 3.1. If G is 0-metamour-regular then every vertex of G has no metamour and hence $G = K_n$ due to Lemma 6.1. Furthermore, K_n is 0-metamour-regular as every vertex is adjacent to all other vertices. $\square$

Next we prove the corollaries which yield an alternative characterization of 0-metamour-regular graphs and allow to count 0-metamour-regular graphs.

Proof of Corollary 3.2. Due to Theorem 3.1, a connected graph with n vertices is 0-metamour-regular if and only if is equal to K_n. This is the case if and only if its complement has no edges. In this case the complement also equals the metamour graph. □

Proof of Corollary 3.3. This is an immediate and easy consequence of the characterization provided by Theorem 3.1. □

7. Proofs regarding 1-metamour-regular graphs

In this section we present the proofs of the results from Section 3.2. They lead to a characterization of 1-metamour-regular graphs.

Proof of Proposition 3.4. Whenever a vertex $v \in V(G)$ is a metamour of a vertex $w \neq v \in V(G)$ then also w is a metamour of v. Therefore, supposing that v has exactly one metamour, so has w and the vertices v and w form a pair such that the two vertices of the pair are metamours of each other. This also leads to an edge from v to w in the metamour graph of G. As every vertex has at most one metamour, the edge from v to w is isolated in the metamour graph of G, so v and w have no other adjacent vertices in the metamour graph of G. Therefore, the edges of the metamour graph form a matching, which yields (a).

Suppose now additionally that G is 1-metamour-regular. Then we can partition the vertices of G into pairs of metamours. Hence, n is even and the edges of the metamour graph form a perfect matching, so (b) holds. □

For proving Theorem 3.5, we need some auxiliary results. We start by showing that the graphs mentioned in the theorem are indeed 1-metamour-regular.

Proposition 7.1. *The graph P_4 depicted in Fig. 3.1 is 1-metamour-regular.*

Proof. This is checked easily. □

The following proposition is slightly more general than needed in the proof of Theorem 3.5 and will be used later on.

Proposition 7.2. *Let $n \geq 3$. In the graph $G = K_n - \mu$ with a matching μ of K_n, every vertex has at most one metamour.*

Proof. Let v be an arbitrary vertex of G. Suppose v is not incident to any edge in μ, then v is adjacent to all other vertices. Thus, v has no metamour.

Now suppose that v is incident to some edge in μ, and let the vertex v' be the other vertex incident to this edge. Then clearly $\{v, v'\} \notin E(G)$, so both v as well as v' have to be adjacent to all other vertices of G by construction of G. Due to the assumption $n \geq 3$, there is at least one other vertex besides v and v', and this vertex is a common neighbor of them. Hence, v and v' are metamours of each other. Both v and v' do not have any other metamour because they are adjacent to all other vertices. As a result, v has exactly one metamour. □

Proposition 7.3. *Let $n \geq 4$ be even. The graph $G = K_n - \mu$ with a perfect matching μ of K_n is 1-metamour-regular.*

Proof. As the matching μ is perfect, every vertex v of G is incident to one edge in μ. Thus, by the proof of Proposition 7.2, every v has exactly one metamour. □

We are now ready for proving Theorem 3.5.

Proof of Theorem 3.5. The one direction of the equivalence follows directly from Propositions 7.1 and 7.3, so only the other direction is left to prove.

Suppose we have a graph G with n vertices that is 1-metamour-regular. Due to Proposition 3.4(b), n is even, and the set of edges of the metamour graph of G forms a perfect matching. In particular, each connected component of the metamour graph consists of two adjacent vertices.

If the metamour graph is connected, then it consists of only two adjacent vertices and $n = 2$. This can be ruled out easily, so we have $n \geq 4$ and the metamour graph is not connected. Now we can use Theorem 3.20 and see that one of the two cases (b) and (c) applies.

In the first case (b) we have $G = \overline{M_1} \nabla \cdots \nabla \overline{M_t}$ with $n = |M_1| + \cdots + |M_t|$ for some $t \geq 2$, where M_i is a connected 1-regular graph for all $i \in \{1, \ldots, t\}$. The only connected 1-regular graph is P_2, therefore $|V(M_i)| = 2$ and $M_i = P_2$ for all $i \in \{1, \ldots, t\}$. Hence, we have $G = \overline{P_2} \nabla \cdots \nabla \overline{P_2}$, which means nothing else than $G = K_n - \mu$ for a perfect matching μ of K_n.

In the second case (c) the metamour graph of G consists of two connected components, so the metamour graph consists of $n = 4$ vertices with two edges that form a perfect matching. It is easy to see that $G = P_4$ or $G = C_4 = K_4 - \mu$ for some perfect matching μ of K_4 are the only two possibilities in this case.

As a result, we obtain in any case $G = P_4$ or $G = K_n - \mu$ for a perfect matching μ of K_n, which is the desired result. □

To finish this section we prove the three corollaries of Theorem 3.5.

Proof of Corollary 3.6. Due to the characterization of 1-metamour-regular graphs of Theorem 3.5, we know that a connected graph with $n \geq 5$ vertices is 1-metamour-regular if and only if it is equal to $K_n - \mu$ for some perfect matching μ of K_n. This is the case if and only if the complement is the graph induced by μ. Furthermore, a graph is induced by a perfect matching if and only if it is 1-regular. To summarize, a connected graph with $n \geq 5$ vertices is 1-metamour-regular if and only if its complement is a 1-regular graph. In this case the complement also equals the metamour graph, which implies the desired result. □

Proof of Corollary 3.7. We use the characterization provided by Theorem 3.5. So let us consider 1-metamour-regular graphs. Such a graph has at least $n \geq 4$ vertices, and n is even. Every perfect matching μ of K_n results in the same unlabeled graph $K_n - \mu$; this brings to account 1. For $n = 4$, there is additionally the graph P_4. In total, this gives the claimed numbers. □

Proof of Corollary 3.8. Let G be an unlabeled graph with n pairs of vertices that each are metamours. We first construct a pair $(\lambda_1 + \cdots + \lambda_t, s)$, where $\lambda_1 + \cdots + \lambda_t$ is a partition of n with $\lambda_i \geq 2$ for all $i \in \{1, \ldots, t\}$ and s is a non-negative integer bounded by r_λ which is defined to be the number of $i \in \{1, \ldots, t\}$ with $\lambda_i = 2$.

Let $\{G_1, \ldots, G_t\} = \mathcal{C}(G)$, set $\lambda_i = |V(G_i)|/2$ for all $i \in \{1, \ldots, t\}$, and let us assume that $\lambda_1 \geq \cdots \geq \lambda_t$. Then $n = \lambda_1 + \cdots + \lambda_t$, so this is a partition of n. As there is no graph G_i with only 1 metamour-pair, $\lambda_i \geq 2$ for all $i \in \{1, \ldots, t\}$. We define s to be the number of $i \in \{1, \ldots, t\}$ with $G_i = P_4$. We clearly have $s \leq r_\lambda$.

Conversely, let a pair $(\lambda_1 + \cdots + \lambda_t, s)$ as above be given. For every $i \in \{1, \ldots, t\}$ with $\lambda_i \geq 3$ there is exactly one choice for a 1-metamour-regular graph G_i with $2\lambda_i$ vertices by Theorem 3.5. Now consider parts 2 of $\lambda_1 + \cdots + \lambda_t$. We choose any (the graphs are unlabeled) s indices and set $G_i = P_4$. Then we set $G_i = C_4$ for the remaining $r_\lambda - s$ indices. The graph $G = G_1 \cup \cdots \cup G_t$ is then fully determined. Thus, we have a found a bijective correspondence.

We still need to related our partition of n to the partition of $n + 2$ of Corollary 3.8. A partition of $n + 2$ is either $n + 2 = (n + 2)$, $n \geq 1$, in which case no additional part 2 appears, or $n + 2 = \lambda_1 + \cdots + \lambda_t + 2$ for a partition $n = \lambda_1 + \cdots + \lambda_t$. Here one additional part 2 appears. Therefore, every pair $(\lambda_1 + \cdots + \lambda_t, s)$ from above maps bijectively to a partition $\lambda_1 + \cdots + \lambda_t + 2$ of $n + 2$ together with a marker of one of the $r_\lambda + 1$ parts 2 in this partition that is uniquely determined by s (by some fixed rule that is not needed to be specified explicitly). This completes the proof of Corollary 3.8. □

8. Proofs regarding graphs with maximum metamour-degree 1

Next we prove the results of Section 3.3 on graphs with maximum metamour-degree 1. We start with the proof of the characterization of these graphs.

Proof of Theorem 3.9. It is easy to see that in the graphs K_1 and K_2 no vertex has any metamour, so the condition that each vertex has at most one metamour is satisfied. Furthermore, by Propositions 7.1 and 7.2, every vertex has indeed at most one metamour in the remaining specified graphs. Therefore, one direction of the equivalence is proven, and we can focus on the other direction.

So, let G be a graph in which every vertex has at most one metamour. Due to Theorems 3.1 and 3.5, it is enough to restrict ourselves to graphs G, where at least one vertex of G has no metamour and at least one vertex of G has exactly one metamour. We will show that $n \geq 3$ and that $G = K_n - \mu$ for some matching μ that is not perfect and contains at least one edge.

Let $V_0 \subseteq V(G)$ and $V_1 \subseteq V(G)$ be the set of vertices of G that have no and exactly one metamour, respectively, and let $v \in V_0$. Due to Lemma 6.1, every vertex in V_0, in particular v, is adjacent to all other vertices. Furthermore, by Proposition 3.4, the vertices in V_1 induce a matching μ in both the metamour graph and the complement of G. This matching μ contains at least one edge because V_1 is not empty, and μ is not perfect because V_0 is not empty. Furthermore, this implies that V_1 contains at least two vertices and in total that $n \geq 3$.

Let w and w' be two vertices in V_1 that are not metamours. Since v is a common neighbor of both w and w', this implies that $\{w, w'\} \in E(G)$. Hence, all possible edges except those in μ are present in G and therefore $G = K_n - \mu$. □

Next we prove the two corollaries of Theorem 3.9.

Proof of Corollary 3.10. Due to Theorem 3.9, in a connected graph G with $n \geq 5$ vertices every vertex has at most one metamour if and only if $G = K_n - \mu$ for some matching μ of K_n. This is the case if and only if the complement is the graph induced by μ. Furthermore, a graph is induced by a matching if and only if it has maximum degree 1. To summarize, a connected graph with $n \geq 5$ vertices has maximum metamour-degree 1 if and only if its complement is a graph with maximum degree 1. In this case the complement also equals the metamour graph, which implies the desired result. □

Proof of Corollary 3.11. We use the characterization provided by Theorem 3.9. So let us consider graphs with maximum metamour-degree 1. For $n \in \{1, 2\}$, we only have K_1 and K_2, so $m_{\leq 1}(n) = 1$ in both cases.

So let $n \geq 3$. Every perfect matching μ of K_n having the same number of edges results in the same graph $K_n - \mu$. A matching can contain at most $\lfloor n/2 \rfloor$ edges and each choice in $\{0, \dots, \lfloor n/2 \rfloor\}$ for the number of edges is possible. This brings to account $\lfloor n/2 \rfloor + 1$. For $n = 4$, there is additionally the graph P_4. In total, this gives the claimed numbers. □

9. Proofs regarding 2-metamour-regular graphs

This section is devoted to the proofs concerning 2-metamour-regular graphs from Section 3.4. It is a long way to obtain the final characterization of 2-metamour-regular graphs of Theorem 3.13, so we have outsourced the key parts of the proof into several lemmas and propositions.

For the proofs of Lemmas 9.1, 9.2, 9.5 and Proposition 9.7 we provide many figures. Every proof consists of a series of steps, and in each of the steps vertices and edges of a graph are analyzed: It is determined whether edges are present or not and which vertices are metamours of each other. The figures of the actual situations show subgraphs of the graph (and additional assumptions) in the following way: Between two vertices there is either an edge or a non-edge or nothing drawn. If nothing is drawn, then it is not (yet) clear whether the edge is present or not. A metamour relation might be indicated at a non-edge.

Note that we frequently use the particular graphs defined by Figs. 3.3, 3.4 and 3.5.

9.1. Graphs with connected metamour graph

The proof of the characterization of 2-metamour-regular graphs in Theorem 3.13 is split into two main parts, which represent whether the metamour graph of G is connected or not in order to apply the corresponding case of Theorem 3.20.

If the metamour graph of a graph with n vertices is connected, then according to Observation 3.12 the metamour graph equals C_n. Here we make a further distinction between graphs that do and that do not contain a cycle of length n as a subgraph. First, we characterize all 2-metamour-regular graphs whose metamour graph is connected and that do not contain a cycle of length n.

Lemma 9.1. *Let G be a connected 2-metamour-regular graph with n vertices*

- *whose metamour graph equals the C_n,*
- *that is not a tree, and*
- *that does not contain a cycle of length n.*

Then

$$G \in \{H_6^a, H_6^b, H_7^a\}.$$

Proof. As G is not a tree, let $\gamma = (v_1, v_2, \dots, v_r, v_1)$, $v_i \in V(G)$ for $i \in \{1, \dots, r\}$, be a longest cycle in G. In all the figures accompanying the proof, the longest cycle is marked by . By assumption, we have $r < n$. For proving the lemma, we have to show that $G \in \{H_6^a, H_6^b, H_7^a\}$.

As a cycle has length at least 3, we have $r \geq 3$ for the length of the cycle γ. As we also have $n > r$, we may assume $n \geq 4$.

We start by showing the following claims.

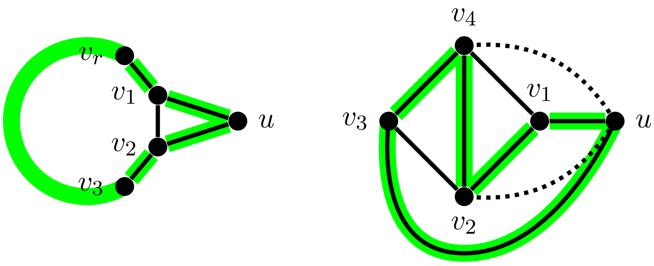
A. *A vertex u in G that is not in the cycle γ is adjacent to at most one vertex in γ. If u is adjacent to a vertex v in γ, then u is a metamour of each neighbor of v in γ.*

Proof of A. Let $u \in V(G)$ be a vertex not in γ. We assume that u is adjacent to v_1 (without loss of generality by renumbering) and some v_j in the cycle γ. We first show that v_1 and v_j are not two consecutive vertices in γ. So let us assume that they are, i.e., $j = 2$ (see Fig. 9.1(a)) or $j = r$ which works analogously. Then $(v_1, u, v_2, \dots, v_r, v_1)$ would be a longer cycle which is a contradiction to γ being a longest cycle. Hence, v_1 and v_j are not consecutive vertices in γ. Then $r \geq 4$ as there need to be at least one vertex between v_1 and v_j on the cycle on each side. If $r = 4$, then $v_j = v_3$ and we are in the situation shown in Fig. 9.1(b). There, $(u, v_1, v_2, v_4, v_3, u)$ is a 5-cycle which contradicts that the longest cycle is of length 4. Therefore, $r = 4$ cannot hold.

If $r > 4$, then u is a metamour of v_2, v_r, v_{j-1} and v_{j+1}, because it has a common neighbor (v_1 or v_j) with these vertices and is not adjacent to them. At least one of v_{j-1} and v_{j+1} is different from v_2 and v_r, so $|\{v_2, v_r, v_{j-1}, v_{j+1}\}| \geq 3$. This contradicts the 2-metamour-regularity of G.

Therefore, we have shown that u is adjacent to at most one vertex in γ. Now suppose u is adjacent to a vertex v in γ. Then u is not adjacent to any neighbor of v in γ and therefore a metamour of every such neighbor. ◁

B. *There exists a vertex w in G but not in γ that is adjacent to (without loss of generality) v_1, but not to any other v_j, $j \neq 1$, in γ.*



(a) consecutive vertices (b) $r = 4$

Fig. 9.1. Subgraphs of the situations in the proof of **A**.

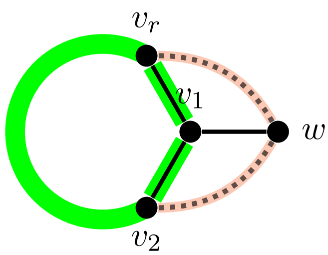


Fig. 9.2. Subgraph of the situation between **B** and **C**.

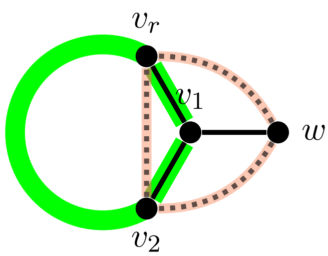


Fig. 9.3. Subgraph of the situation in the proof of **C**.

Proof of B. As $r < n$, there exists a vertex w' not in the cycle γ. The graph G is connected, so there is a path from a vertex of γ to w'. Therefore, there is also a vertex w not in γ which is adjacent to a vertex v_i. By renumbering, we can assume without loss of generality that $i = 1$.

As the vertex w is adjacent to v_1, w is not adjacent to any other v_j by **A**. $\triangleleft$

At this point, we assume to have a vertex w as in **B**; the situation is shown in Fig. 9.2.

C. *The graph G contains the edge $\{v_2, v_r\}$.*

Proof of C. Assume that there is no edge between v_2 and v_r; see Fig. 9.3. Then v_2 and v_r are metamours of each other, and consequently (w, v_2, v_r, w) forms a 3-cycle in the metamour graph of G. This contradicts that the metamour graph of G is C_n and $n \geq 4$. $\triangleleft$

At this point, we have the situation shown in Fig. 9.4. In the next steps we will rule out possible values of r.

D. *If $r = 3$, then $G = H_6^a$.*

Proof of D. Our initial situation is shown in Fig. 9.5(a).

Suppose there is an additional vertex v_1' of G adjacent to v_1; see Fig. 9.5(b). Then by **A**, v_1' has metamours v_2 and v_3. Therefore, (w, v_2, v_1', v_3, w) is a 4-cycle in the metamour graph of G. As this cycle does not cover v_1, we have a contradiction to the metamour graph being the single cycle C_n for $n > r = 3$. Therefore, there is no additional vertex adjacent to v_1.

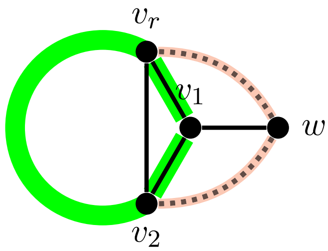


Fig. 9.4. Subgraph of the situation between **C** and **D**.

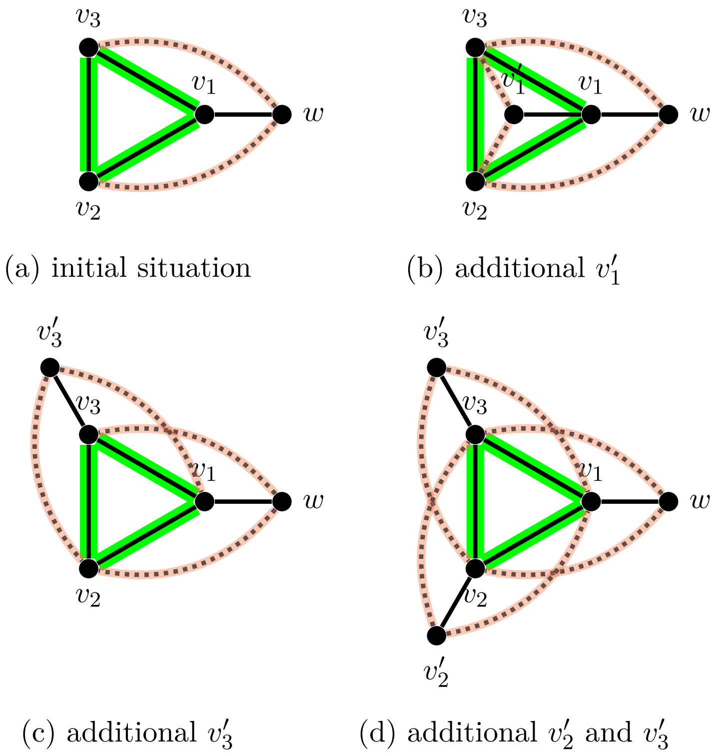


Fig. 9.5. Subgraph of the situation in the proof of **D**.

At this point, we know that w is a metamour of both v_2 and v_3; see again Fig. 9.5(a). We now look for the second metamour of v_2 and v_3, respectively. As we ruled out an additional vertex adjacent to v_1, there need to be an additional vertex adjacent to v_2 or to v_3.

Without loss of generality (by symmetry), suppose there is an additional vertex v_3' of G adjacent to v_3; see Fig. 9.5(c). Then by **A**, v_3' has metamours v_1 and v_2. Therefore, these two vertices are the two metamours of v_3'. There cannot be an additional vertex v_3'' of G adjacent to v_3, because due to the same arguments as for v_3' this vertex would be a metamour of v_2, hence v_2 would have three metamours, and this contradicts the 2-metamour-regularity of G.

Suppose there is no additional vertex adjacent to v_2. Then, in order to close the metamour cycle containing (v_1, v_3', v_2, w, v_3), there needs to be a path from v_3' to w. This implies the existence of a cycle longer than $r = 3$, therefore cannot be. So there is an additional vertex adjacent to v_2'; the situation is shown in Fig. 9.5(d).

By the same argument as above, v_1 and v_3 are the two metamours of v_2'. Therefore, $(w, v_2, v_3', v_1, v_2', v_3, w)$ is a 6-cycle in the metamour graph of G and $n = 6$. This is the graph $G = H_6^a$. We can only add additional edges between the vertices w, v_2' and v_3', but this would lead to a cycle of length larger than 3. So there are no other edges present. There cannot be any additional vertex because this vertex would need to be in a different cycle in the metamour graph, contradicting that the metamour graph is the C_n. $\triangleleft$

As a consequence of **D**, the proof is finished for $r = 3$, because then $G = H_6^a$. What is left to consider is the case $r \geq 4$ and consequently $n \geq 5$. The situation is again as in Fig. 9.4.

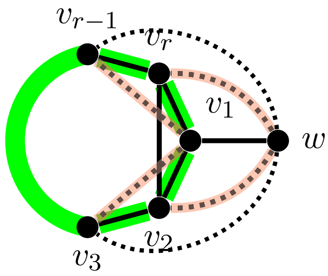


Fig. 9.6. Subgraph of the situation between **E** and **F**.

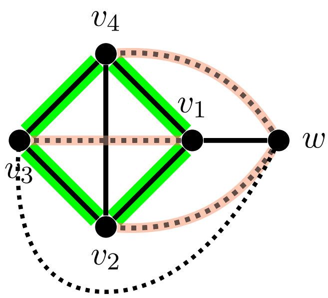


Fig. 9.7. Subgraph of the situation in the proof of **F**.

E. *The only vertices of γ that are adjacent to v_1 are v_2 and v_r. In particular, v_1 is metamour of v_3 and of v_{r-1}.*

Proof of E. Suppose there is a vertex v_i with $i \in \{3, \dots, r-1\}$ adjacent to v_1. Then, as w is not adjacent to v_i by **A** or **B**, v_i is a third metamour of w. This contradicts the 2-metamour-regularity of G.

As v_1 has distance 2 on the cycle γ to both v_3 and v_{r-1}, and is not adjacent to these vertices, the vertices v_3 and v_{r-1} are metamours of v_1. ◁

At this point, we have the situation shown in Fig. 9.6. Note, that it is still possible that $v_3 = v_{r-1}$.

F. *We cannot have $r = 4$.*

Proof of F. As $r = 4$, we have $v_3 = v_{r-1}$. This situation is shown in Fig. 9.7.

Suppose there is an additional vertex v_3' of G adjacent to v_3. Then by **A**, v_3' has metamours v_2 and v_4. Therefore, (w, v_2, v_3', v_4, w) is a 4-cycle in the metamour graph of G. This is a contradiction to the metamour graph being C_n and $n \geq 6$, so there is no additional vertex adjacent to v_3. This implies that we cannot have a vertex at distance 1 from v_3 other than v_2 and v_4.

Now suppose there is an additional vertex v_2' of G adjacent to v_2. Again by **A**, v_2' has metamours v_1 and v_3. Therefore, (v_2', v_1, v_3, v_2') is a 3-cycle in the metamour graph of G. This is again a contradiction to the metamour graph being C_n and $n \geq 6$, so there is no additional vertex adjacent to v_2 either. Likewise, by symmetry, there is also no additional vertex adjacent to v_4.

As v_2 and v_4 are the only neighbors of v_3, we cannot have a vertex at distance 2 from v_3 other than v_1. This means that there is no second metamour of v_3 which contradicts the 2-metamour-regularity of G. ◁

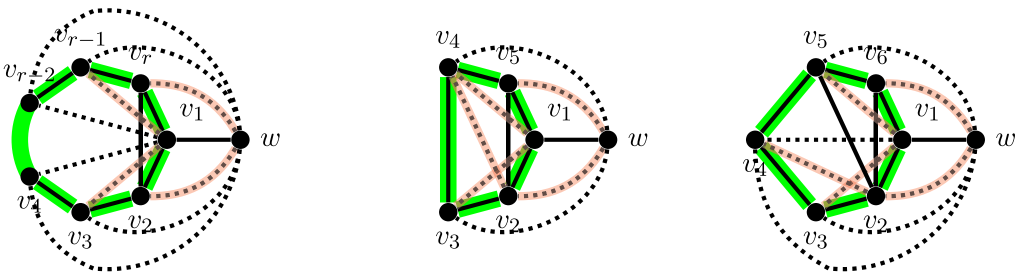
At this point, we can assume that $r \geq 5$ as the case $r = 4$ was excluded by **F**, and consequently also $n \geq 6$. The situation is still as in Fig. 9.6.

G. *We have $r \leq 6$. Specifically, either $r = 5$, or $r = 6$ and there is an edge $\{v_2, v_5\}$ in G. In the second case, the two metamours of the vertex v_2 are w and v_4.*

Proof of G. As $r \geq 5$, the two metamours of v_1 are on the cycle γ, namely the distinct vertices v_3 and v_{r-1}; see Fig. 9.8(a).

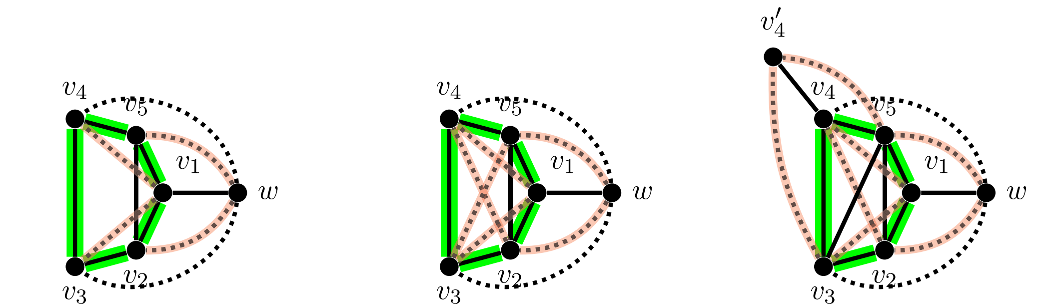
We now consider the neighbors of v_2. Suppose v_2 is adjacent to some v_i with $i \notin \{1, 3, r-1, r\}$. As the vertex v_1 is not connected to v_i by **E**, the vertex v_i is a metamour of v_1 different from v_3 and v_{r-1}. This contradicts the 2-metamour-regularity of G. Furthermore, v_2 is adjacent to v_1, v_3 and v_r. This implies that the neighborhood of v_2 on γ is determined up to v_{r-1}. We will now distinguish whether v_{r-1} is or is not in this neighborhood.

Suppose $\{v_2, v_{r-1}\} \notin E(G)$. If $v_{r-1} \neq v_4$, then $\{v_2, v_4\} \notin E(G)$ because of what is shown in the previous paragraph. But then, the metamours of v_2 would be w, v_{r-1} and v_4. This contradicts the 2-metamour-regularity of G and implies that $v_{r-1} = v_4$ and $r = 5$; see Fig. 9.8(b).



(a) initial situation (b) with $\{v_2, v_{r-1}\} \notin E(G)$ (c) with $\{v_2, v_{r-1}\} \in E(G)$

Fig. 9.8. Subgraphs of the situations in the proof of **G**.



(a) initial situation (b) with none of two edges (c) with one of two edges

Fig. 9.9. Subgraphs of the situations in the proof of **H**.

Suppose $\{v_2, v_{r-1}\} \in E(G)$. We again distinguish between two cases. If $r \geq 6$, then w, v_4 and v_{r-2} are metamours of v_2. In this case, the 2-metamour-regularity of G implies that $v_{r-2} = v_4$ and therefore $r = 6$; see Fig. 9.8(c). If $r < 6$, then by the findings so far, we must have $r = 5$, and therefore we are also done in this case. ◁

By **G** we are left with the two cases $r = 5$ and $r = 6$. One possible situation for $r = 5$ and the situation for $r = 6$ are shown in Fig. 9.8(b) and (c), respectively, and we will deal with these two situations now.

H. *If $r = 5$, then $G \in \{H_6^b, H_7^a\}$.*

Proof of H. The full situation for $r = 5$ is shown in Fig. 9.9(a).

Clearly the situation is symmetric in the potential edges $\{v_2, v_4\}$ and $\{v_3, v_5\}$, so we have to consider the three cases that both, one and none of these two edges are present.

First let us assume that neither $\{v_2, v_4\}$ nor $\{v_3, v_5\}$ is an edge; see Fig. 9.9(b). Then v_2 and v_4 as well as v_3 and v_5 are metamours, so we have the 6-cycle $(w, v_2, v_4, v_1, v_3, v_5, w)$ in the metamour graph of G. This is the graph $G = H_6^b$. There cannot be any additional vertex because this vertex would need to be in a different cycle in the metamour graph contradicting that the metamour graph is the C_n. There also cannot be any additional edges because all edges and non-edges are already determined.

Next let us assume that there is exactly one of the edges $\{v_2, v_4\}$ and $\{v_3, v_5\}$ present in G, without loss of generality let $\{v_3, v_5\} \in E(G)$; see Fig. 9.9(c). At this point we know that v_3 and v_1 as well as v_5 and w are metamours, and we are looking for the second metamours of v_3 and v_5. As the vertices w, v_1 and v_4 already have two metamours each, there need to be additional vertices for these metamours.

Statement **A** implies that there is no additional vertex of G adjacent to v_5 as v_1 has already the two metamours v_3 and v_4. Likewise, by symmetry, there is no additional vertex adjacent to v_2. Moreover, by the same argument, there is also no additional vertex adjacent to v_3 as v_4 has the two metamours v_1 and v_2.

Therefore, there need to be an additional vertex v_4' adjacent to v_4. By **A**, v_4' has metamours v_3 and v_5. This gives the 7-cycle $(w, v_2, v_4, v_1, v_3, v_4', v_5, w)$ in the metamour graph of G and the graph $G = H_7^a$. There cannot be any additional vertex because this vertex would need to be in a different cycle in the metamour graph contradicting that the metamour graph is the C_n. There also cannot be any additional edges because all edges and non-edges are already determined.

At last, let us consider the case that both of the edges $\{v_2, v_4\}$ and $\{v_3, v_5\}$ are present in G. We already know that w is a metamour of v_2 and are now searching for the second metamour of v_2. There does not exist a vertex v_1' adjacent to v_1 in G but not in γ, because this would induce a C_4 in the metamour graph by the same arguments as in the proof of **D**.

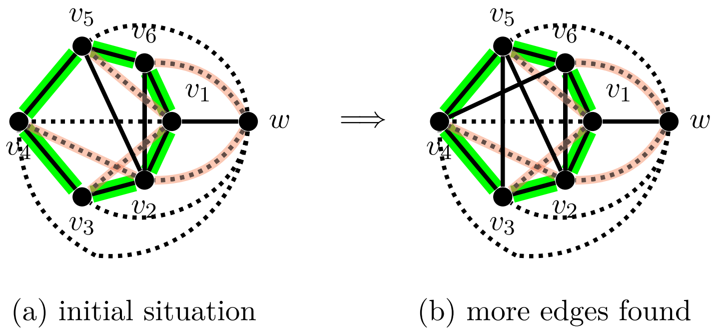


Fig. 9.10. Subgraphs of the situations in the proof of I.

Furthermore, there cannot be a vertex v_2' in G but not in γ that is adjacent to v_2, due to the fact that this vertex would be a third metamour of v_1, a contradiction. By symmetry, there is no vertex of G without γ adjacent to v_5. If there would be a vertex v_3' in G but not in γ which is adjacent to v_3, then due to **A**, this vertex would have v_2, v_4 and v_5 as a metamour, a contradiction to the 2-metamour-regularity of G. Again by symmetry, there is no vertex of G without γ adjacent to v_4. Therefore, v_2 cannot have a second metamour in G and this case cannot happen. $\triangleleft$

Statement **H** finalizes the proof for $r = 5$. Hence, $r = 6$ is the only remaining value for r we have to consider.

I. *We cannot have* $r = 6$.

Proof of I. As $r = 6$, there is an edge $\{v_2, v_5\}$ in G by **G**. The initial situation is shown in Fig. 9.10(a).

Suppose v_3 and v_5 are not adjacent. Then (v_1, v_3, v_5, v_1) is a 3-cycle in the metamour graph of G. This contradicts that the metamour graph is C_n and $n > r = 6$, so we can assume $\{v_3, v_5\} \in E(G)$. Likewise, suppose that v_4 and v_6 are not adjacent. Then (w, v_2, v_4, v_6, w) is a 4-cycle in the metamour graph of G. This contradicts that the metamour graph is C_n and $n > r = 6$, so we can assume $\{v_4, v_6\} \in E(G)$. The current situation is shown in Fig. 9.10(b).

Statement **A** implies that there is no additional vertex of G adjacent to v_2 as v_1 has already the two metamours v_3 and v_5. By symmetry, there is also no additional vertex adjacent to v_6. By the same argumentation as above, there is no additional vertex adjacent to v_1 as well as to v_3 because of vertex v_2 and its metamours. Moreover, we slightly vary the argumentation to show that there cannot be an additional vertex adjacent to v_5. Suppose there is an additional vertex v_5' of G adjacent to v_5. Then, v_5' is not adjacent to v_2 as we have shown above, so v_5' is as well a metamour of v_2. This contradicts the 2-metamour-regularity of G again.

The vertex v_4 has v_2 as metamour. We are now searching for its second metamour. It cannot be w or v_1 as these vertices have already two other metamours each. It cannot be any of v_3, v_5 or v_6 either as all of them are adjacent to v_4. Moreover, the second metamour of v_4 cannot be adjacent to v_3, v_5 or v_6, as we above ruled additional neighbors to these vertices out. Therefore, there has to be an additional vertex v_4' adjacent to v_4. By **A**, this vertex v_4' has metamours v_3 and v_5. This results in the 4-cycle $(v_1, v_3, v_4', v_5, v_1)$ in the metamour graph of G and contradicts our assumption that this graph is C_n and $n > r = 6$. $\triangleleft$

We have now completed the proof of Lemma 9.1 as in all cases we were able to show that $G \in \{H_6^a, H_6^b, H_7^a\}$ holds. $\square$

After characterizing all 2-metamour-regular graphs whose metamour graph is connected and that do not contain a cycle of length n, we can now focus on 2-metamour-regular graphs whose metamour graph is connected and that contain a cycle of length n. Here, we make a further distinction depending on the degree of the vertices and begin with the following lemma.

Lemma 9.2. *Let G be a connected 2-metamour-regular graph with n vertices*

- *whose metamour graph equals the C_n,*
- *that contains a cycle of length n, and*
- *that has a vertex of degree larger than 2 and smaller than $n - 3$.*

Then

$$G = H_7^b.$$

Proof. Let γ be a cycle of length n in G. First, we introduce some notation. Let v be a vertex of G, and let u and u' be the two metamours of v. We explore the vertices on the cycle γ starting with v: The set of vertices on both sides of v strictly before u and u' are called the *fellows* of v. The remaining set of vertices strictly between u and u' on γ is called

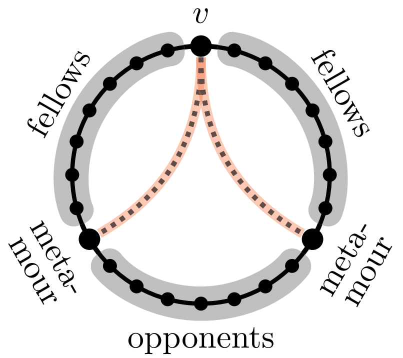


Fig. 9.11. Fellows and opponents of a vertex v in the proof of Lemma 9.2.

the *opponents* of v; see Fig. 9.11. In other words for each vertex v of G the set of vertices of G can be partitioned into v, its fellows, its metamours and its opponents.

We start with the following claims.

A. *Every vertex of G is adjacent to each of its fellows.*

Proof of A. Let v_1 be a vertex of G and $\gamma = (v_1, \ldots, v_n, v_1)$. Suppose v_p is the vertex with smallest index p that is not adjacent to v_1. We have to show that v_p is a metamour of v_1. The index p exists because v_1 is not adjacent to its metamours. Moreover, this index satisfies $p > 2$ as v_2 is adjacent to v_1 because they are consecutive vertices on γ. Thus, v_1 and v_p have v_{p-1} as common neighbor and are therefore metamours.

By symmetry, the vertex v_q with largest index q that is not adjacent to v_1, is also a metamour of v_1. Note that as v_1 has exactly two metamours, v_p and v_q are these metamours, so v_1 is adjacent to each of its fellows. $\lhd$

B. *Every vertex of G is either adjacent to each of its opponents, or not adjacent to any of its opponents.*

Proof of B. It is enough to show that if a vertex v_1 of G is adjacent to at least one opponent of v_1, then it is adjacent to every opponent of v_1. Let $\gamma = (v_1, \ldots, v_n, v_1)$, and let W be a subset of the opponents of v_1 that consists of consecutive vertices of γ, say from v_i to v_j for some $i \le j$, such that each of these vertices is adjacent to v_1, and W is maximal (with respect to inclusion) with this property. Note that the set W is not empty because of our assumption.

Clearly none of the vertices in W is a metamour of v_1. However v_{i-1} and v_{j+1} are metamours of v_1 because of their common neighbors v_i and v_j, and the maximality of W. Therefore, as v_1 has exactly two metamours, W equals the set of opponents of v_1 which was to show. $\lhd$

Now we are ready to start with the heart of the proof of Lemma 9.2. Suppose v_1 is a vertex of G with $2 < \deg(v_1) < n - 3$. In order to complete the proof we have to show that $G = H_7^b$.

Let $\gamma = (v_1, \ldots, v_n, v_1)$ be a cycle of length n, and let v_p and v_q be the metamours of v_1 with $p < q$. In the following claims we will derive several properties of G.

C. *The vertex v_1 is adjacent to its fellows $v_2, \ldots, v_{p-1}, v_{q+1}, \ldots, v_n$ and not adjacent to any metamour or opponent $v_p, \ldots, v_q$. Furthermore, $p + 1 < q$ holds, i.e., there exists at least one opponent of v_1.*

Proof of C. Clearly v_1 is not adjacent to its metamours v_p and v_q. Furthermore, v_1 is adjacent to all its fellows $v_2, \ldots, v_{p-1}, v_{q+1}, \ldots, v_n$ by A. This together with $\deg(v_1) < n - 3$ implies that v_1 has an opponent to which it is not adjacent, so $p + 1 < q$. Then by B, v_1 is not adjacent to any of its opponents. $\lhd$

Now $\deg(v_1) > 2$ together with C imply that v_1 has at least one fellow different from v_2 and v_n. Without loss of generality (by renumbering the vertices in the opposite direction of rotation along γ) assume that v_{q+1} is a fellow of v_1 different from v_n, so in other words we assume $q + 1 < n$. The situation is shown in Fig. 9.12.

We will now prove several claims about edges, non-edges and metamours of G.

D. *No opponent $v_{p+1}, \ldots, v_{q-1}$ is adjacent to any fellow $v_2, \ldots, v_{p-1}, v_{q+1}, \ldots, v_n$.*

Proof of D. Assume that v_j is adjacent to v_i for some $j \in \{p+1, \ldots, q-1\}$ and some $i \in \{2, \ldots, p-1\} \cup \{q+1, \ldots, n\}$. Then v_j and v_1 have the common neighbor v_i because of C. Furthermore, v_j and v_1 are not adjacent by C, so v_j and v_1 are metamours. This is a contradiction to v_p and v_q being the only metamours of v_1, therefore our assumption was wrong. $\lhd$

The known edges and non-edges at this moment are shown in Fig. 9.13(a).

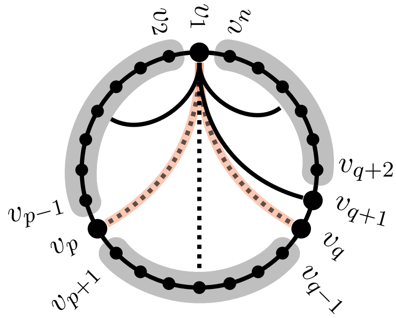
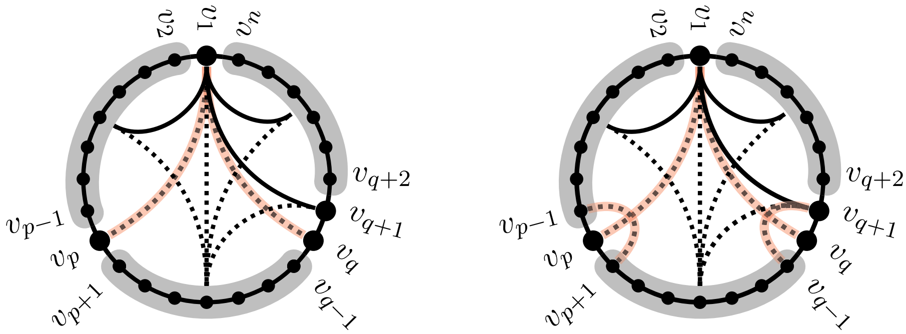


Fig. 9.12. Subgraph of the situation between **C** and **D**.



(a) between **D** and **E** (b) between **E** and **F**

Fig. 9.13. Subgraphs of the situations between **D**, **E**, and **F**.

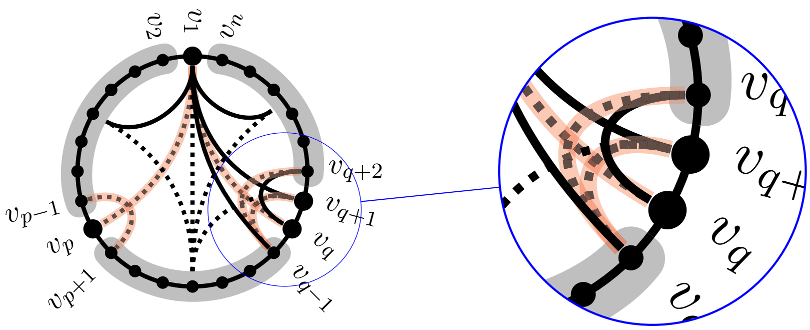


Fig. 9.14. Subgraph of the situation in the proof of **F**.

E. *The vertices v_{q-1} and v_{q+1} are metamours of each other. Also the vertices v_{p-1} and v_{p+1} are metamours of each other.*

Proof of E. The vertices v_{q-1} and v_{q+1} have the common neighbor v_q and are not adjacent due to **D**, so they are metamours. Also v_{p-1} and v_{p+1} are metamours because they have v_p as a common neighbor and are not adjacent because of **D**. ◁

Now we are in the situation shown in Fig. 9.13(b).

F. *The vertices v_q and v_{q+2} are metamours of each other.*

Proof of F. This proof is accompanied by Fig. 9.14. Assume v_q and v_{q+2} are adjacent. Then v_{q-1} and v_{q+2} have the common neighbor v_q and are not adjacent because of **D**. Hence, v_{q+2} is a metamour of v_{q-1}. Due to **E**, v_{q+1} is the second metamour of v_{q-1}. Both metamours are consecutive vertices on the cycle γ, therefore, every other vertex except v_{q-1} is a fellow of v_{q-1}, thus adjacent to v_{q-1} by **A**. In particular, v_1 is adjacent to v_{q-1} which contradicts **C**.

Therefore, v_q and v_{q+2} are not adjacent and because of their common neighbor v_{q+1}, metamours. ◁

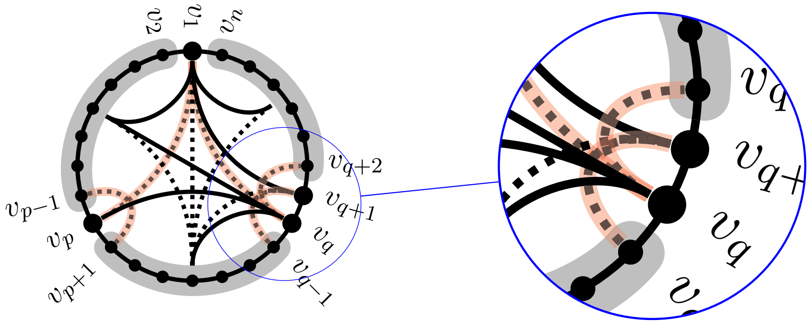


Fig. 9.15. Subgraph of the situation between **G** and **H**.

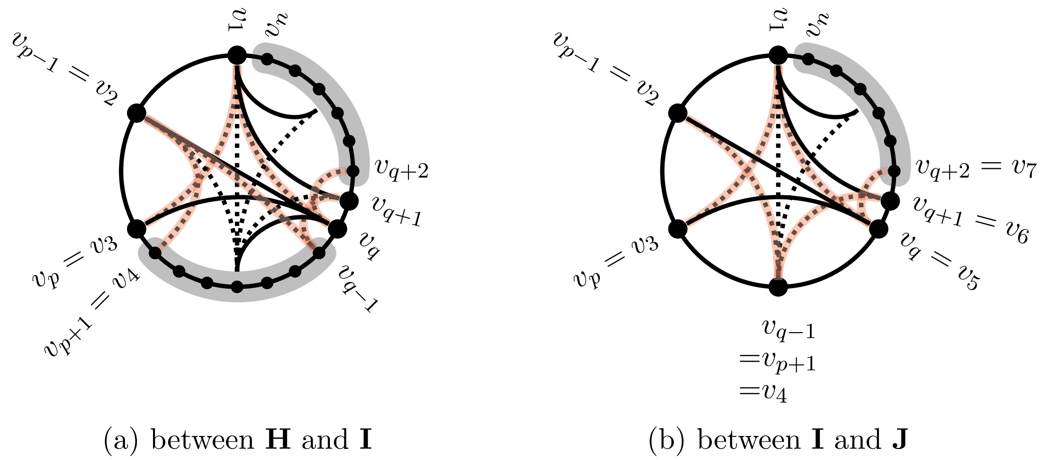


Fig. 9.16. Subgraphs of the situations between **H**, **I** and **J**.

G. *The vertex v_q is adjacent to $v_2, \dots, v_{p-1}, v_p, v_{p+1}, \dots, v_{q-1}$.*

Proof of G. The two metamours of v_q are v_1 and v_{q+2} because of **F**. This implies that $v_2, \dots, v_{q-1}$ are fellows of v_q and therefore adjacent to v_q because of **A**. $\triangleleft$

Fig. 9.15 shows the current situation.

H. *The vertices v_{q-1} and v_{p-1} are metamours of each other. Furthermore, $v_{p-1} = v_2$ holds, so there is exactly one fellow of v_1 on the cycle γ between v_1 and v_p.*

Proof of H. The vertex v_{q-1} is not adjacent to any of $v_2, \dots, v_{p-1}$ due to **D**. Furthermore, v_{q-1} has the common neighbor v_q with each of these vertices because of **G**. So every vertex $v_2, \dots, v_{p-1}$ is a metamour of v_{q-1}. This implies $|\{v_2, \dots, v_{p-1}\}| \leq 1$ because v_{q-1} also has v_{q+1} as metamour and has in total exactly two metamours. Moreover, as v_2 is adjacent to v_1, v_1 and v_2 are not metamours, thus v_2 and v_p cannot coincide. This implies $p > 2$ has to hold. In consequence, we obtain $p = 3$ implying $v_{p-1} = v_2$ has to hold. $\triangleleft$

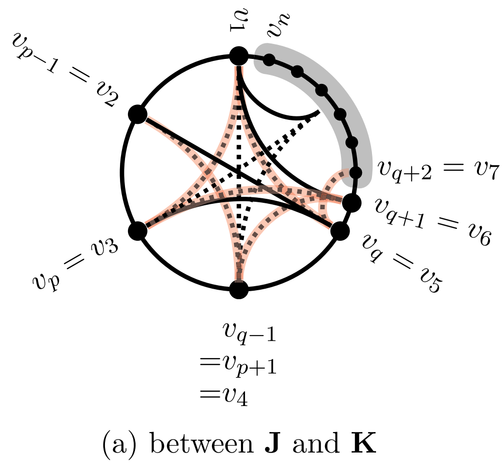
We are now in the situation shown in Fig. 9.16(a).

I. *It holds that $v_{p+1} = v_{q-1}$, so v_1 has exactly one opponent.*

Proof of I. The vertices v_{q+1} and v_{p-1} are metamours of v_{q-1} because of **E** and **H**. Furthermore, v_{p-1} and v_{p+1} are metamours because of **E**.

Now assume $p+1 < q-1$, so the vertices v_{p+1} and v_{q-1} are distinct. Then v_{p+1} and v_{q+1} have the common neighbor v_q because of **G** and they are not adjacent because of **D**, so they are metamours. This implies that $(v_{q-1}, v_{q+1}, v_{p+1}, v_{p-1}, v_{q-1})$ is a cycle in the metamour graph that does not contain all vertices, a contradiction to our assumption. So $p+1 = q-1$. $\triangleleft$

Now we are in the situation shown in of Fig. 9.16(b).



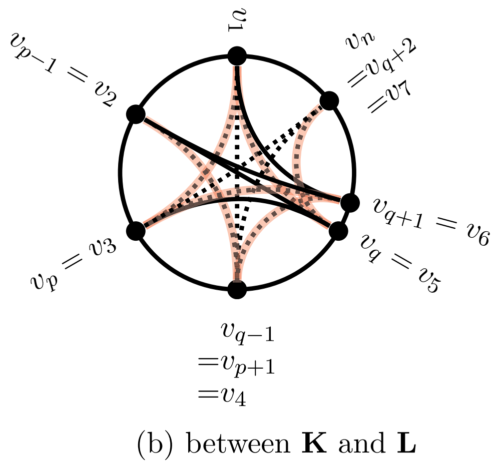


Fig. 9.17. Subgraphs of the situations between J, K and L.

J. *The vertices v_p and v_{q+1} are metamours of each other. Furthermore, v_p is not adjacent to any of the vertices $v_{q+2}, \ldots, v_n$.*

Proof of J. If v_p is adjacent to v_i for $i \in \{q+2, \ldots, n\}$, then v_{p+1} and v_i are metamours because they have v_p as a common neighbor, and they are not adjacent due to D. This is a contradiction as v_{p+1} already has the two metamours v_{p-1} and v_{q+1} because of E and an implication of I. As a result, v_p is not adjacent to any of $v_{q+2}, \ldots, v_n$.

If v_p would be adjacent to v_{q+1}, then v_p and v_{q+2} are metamours because of the common neighbor v_{q+1} and because they are not adjacent by the above. But then, due to F, $(v_p, v_1, v_q, v_{q+2}, v_p)$ is a cycle in the metamour graph which does not contain all vertices, a contradiction to our assumption. Therefore, v_p is not adjacent to v_{q+1}. The vertex v_p is adjacent to v_q due to G, therefore v_q is a common neighbor of v_p and v_{q+1}, and hence these vertices are metamours. ◁

Fig. 9.17(a) shows the situation.

K. *It holds that $q+2 = n$, so there are exactly two fellows of v_1 on the cycle γ between v_q and v_1. Furthermore, the vertices v_{p-1} and v_{q+2} are metamours of each other, and v_{p-1} is adjacent to all vertices except its metamours.*

Proof of K. The vertex v_{p-1} is a metamour of v_{q-1} due to H, and it is adjacent to v_q because of G. This together with $p+1 = q-1$ by I implies that v_{p-1} is adjacent to one of its opponents, namely v_q. Then by B and A, this implies that v_{p-1} is adjacent to all vertices except its metamours.

If v_{p-1} is adjacent to a vertex v_i for $i \in \{q+2, \ldots, n\}$, then v_p and v_i are metamours because they have v_{p-1} as common neighbor and are not adjacent due to J. But v_p already has the two metamours v_1 and v_{q+1} due to J, a contradiction. As a result, v_{p-1} is not adjacent to any vertex of $v_{q+2}, \ldots, v_n$.

Now assume $q+2 < n$, so the vertex v_{q+3} exists. Due to the fact that v_{p-1} is adjacent to all vertices except its metamours and that it has v_{p+1} as metamour by E, it follows that it is adjacent to at least one of v_{q+2} and v_{q+3}. But we showed that v_{p-1} is not adjacent to any of these two vertices, a contradiction. Therefore, $q+2 = n$ holds. Furthermore, v_{p-1} is not adjacent to v_{q+2}, and therefore these two vertices are metamours of each other. ◁

Our final figure is Fig. 9.17(b).

L. *It holds that $G = H_7^b$.*

Proof of L. We have $p-1 = 2$ by H, we have $p+1 = q-1$ by I and $q+2 = n$ by K. This implies that $n = 7$.

The properties we have derived so far fix all edges and non-edges of G except between v_2 and v_7. This has to be a non-edge to close the metamour cycle. The result is $G = H_7^b$. With respect to Fig. 3.4, v_1 is the top left vertex of H_7^b and the vertices are numbered clock-wise. ◁

This completes the proof of Lemma 9.2. □

Next we consider all cases of 2-metamour-regular graphs whose metamour graph is connected, that contain a cycle of length n and whose degrees are not as in the previous lemma.

Lemma 9.3. *Let G be a connected 2-metamour-regular graph with n vertices*

- *whose metamour graph equals the C_n,*
- *that contains a cycle of length n, and*
- *in which every vertex has degree $n-3$.*

Then

$$G = \overline{C_n}.$$

Proof. If a vertex v of G has degree $n-3$, then v is adjacent to all but two vertices. These two vertices are exactly the metamours of v. This implies that G equals the complement of the metamour graph. Hence, $G = \overline{C_n}$ as the metamour graph of G is the C_n. □

Lemma 9.4. *Let G be a connected 2-metamour-regular graph with n vertices*

- *whose metamour graph equals the C_n,*
- *that contains a cycle of length n, and*
- *in which every vertex has degree 2.*

Then

$$G = C_n$$

and n is odd.

Proof. Let γ be a cycle of length n in G. If every vertex of G has degree 2, then every vertex in the induced subgraph $G[\gamma]$ has degree 2 as γ contains every vertex by assumption. As $G[\gamma]$ is connected, it equals C_n. In total this implies $G = G[\gamma] = C_n$.

It is easy to see that if n is even, then the metamour graph consists of exactly two cycles of length $\frac{n}{2}$ which contradicts our assumption. Therefore, n is odd. □

Lemma 9.5. *Let G be a connected 2-metamour-regular graph with n vertices*

- *whose metamour graph equals the C_n,*
- *that contains a cycle of length n,*
- *in which every vertex has degree 2 or $n-3$, and*
- *that has a vertex of degree 2 and a vertex of degree $n-3$.*

Then

$$G \in \{C_5, H_6^c\}.$$

Proof. Let $\gamma = (v_1, \dots, v_n, v_1)$ be a cycle of length n such that $\deg(v_1) = 2$ and $\deg(v_2) = n-3$.

A. *We have $5 \le n \le 7$ and the metamours of v_1 are v_3 and v_{n-1}.*

Proof of A. Clearly v_1 is only adjacent to v_2 and v_n. Hence, v_3 and v_{n-1} have to be the two metamours of v_1 and G contains at least 5 different vertices, so $n \ge 5$.

If v_2 is adjacent to some v_i for $i \in \{4, \dots, n-2\}$, then v_1 is a metamour of v_i due to the common neighbor v_2; see Fig. 9.18. Hence, v_2 is not adjacent to any vertex $v_4, \dots, v_{n-2}$. However, because $\deg(v_2) = n-3$, the vertex v_2 is adjacent to every vertex but its two metamours. This implies that $|\{v_4, \dots, v_{n-2}\}| \le 2$, because v_2 has at most two metamours among $v_4, \dots, v_{n-2}$. As a result, we have $n \le 7$. ◁

This implies that $n = 5$, $n = 6$ and $n = 7$ are the only cases to consider. We do so in the following claims.

B. *If $n = 5$, then $G = C_n$.*

Proof of B. If $n = 5$, then v_3 and $v_{n-1} = v_4$ are the two metamours of v_1; see Fig. 9.19(a). Then v_2 is the only option as second metamour of v_4, and v_5 is the only option as second metamour of v_3. Then v_2 and v_5 have to be metamours in order to close the cycle in the metamour graph; see Fig. 9.19(b). As a result, we have $G = C_5$. ◁

C. *If $n = 6$, then $G = H_6^c$.*

Proof of C. If $n = 6$, then v_3 and $v_{n-1} = v_5$ are the two metamours of v_1.

If v_3 is not adjacent to v_5, then v_3 and v_5 are metamours because of their common neighbor v_4; see Fig. 9.20(a). But then (v_1, v_3, v_5, v_1) is a cycle in the metamour graph that does not contain all vertices, a contradiction to our assumption.

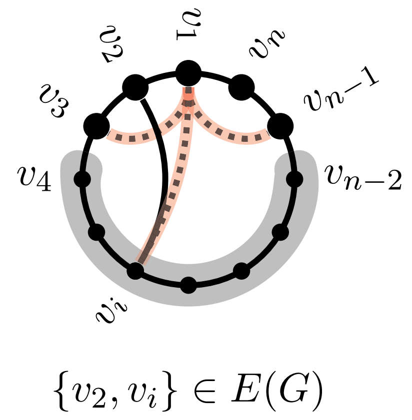


Fig. 9.18. Subgraph of the situation in the proof of A.

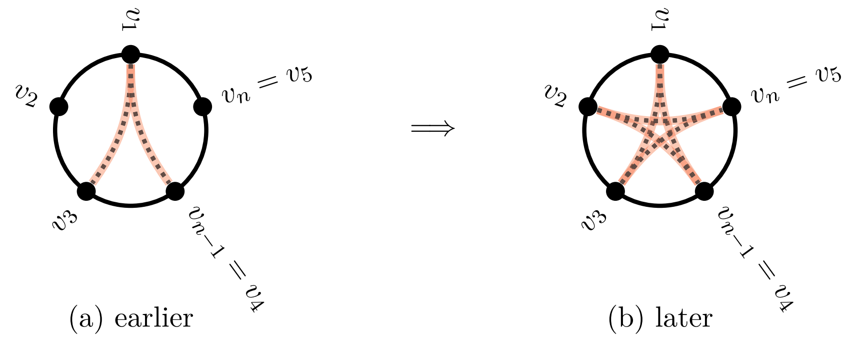


Fig. 9.19. Subgraphs of the situations in the proof of B.

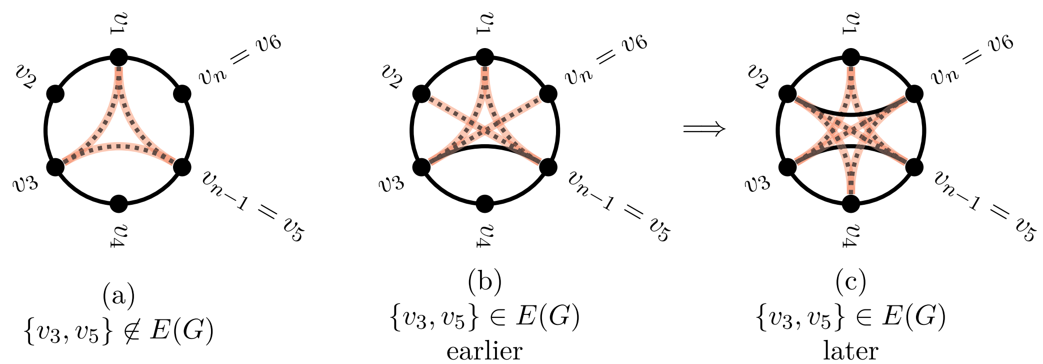


Fig. 9.20. Subgraphs of the situations in the proof of C.

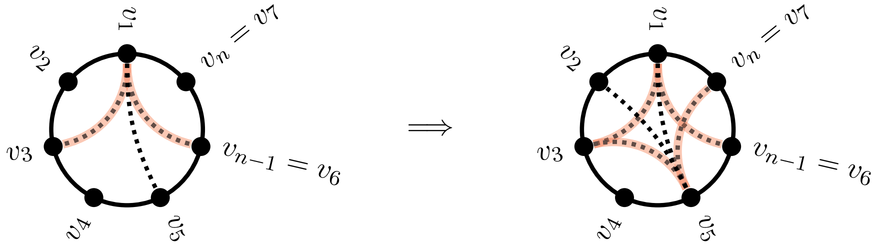
Hence, v_3 and v_5 are adjacent; see Fig. 9.20(b). Then v_6 is the only option left as the second metamour of v_3, and v_2 is the only option left as the second metamour of v_5.

If v_2 and v_6 are not adjacent, then they are metamours because of their common neighbor v_1. But then $(v_1, v_3, v_6, v_2, v_5, v_1)$ is a cycle in the metamour graph that does not contain v_4, a contradiction. So v_2 and v_6 are adjacent; see Fig. 9.20(c).

But then v_4 has to have v_2 and v_6 as metamours, because they are the only options left. Hence, we obtain $G = H_6^c$. $\triangleleft$

D. *We cannot have $n = 7$.*

Proof of D. If $n = 7$, then v_3 and $v_{n-1} = v_6$ are the two metamours of v_1. As $\deg(v_1) = 2$, the vertices v_1 and v_5 are not adjacent, and they are also not metamours; see Fig. 9.21(a). Therefore, $\deg(v_5) < n - 3 = 4$. As the only options are $\deg(v_5) \in \{2, n - 3\}$, we conclude $\deg(v_5) = 2$.



(a) first paragraph (b) second paragraph

Fig. 9.21. Subgraphs of the situations in the proof of **D**.

As a result, v_5 is only adjacent to v_4 and v_6, and the vertices v_3 and v_7 have to be the two metamours of v_5; see Fig. 9.21(b). In particular, v_5 is not adjacent to v_2, and the vertices v_5 and v_2 are not metamours. This implies $\deg(v_2) < n - 3 = 4$ which is a contradiction to $\deg(v_2) = 4$. Hence, $n = 7$ is not possible. ◁

To summarize, in the case that not all vertices of G have the same degree in $\{2, n-3\}$, $G = C_5$ and $G = H_6^c$ are the only possible graphs due to **A**, **B**, **C** and **D**. This finishes the proof of Lemma 9.5. □

Eventually, we can collect all results on 2-metamour-regular graphs that have a connected metamour graph in the following proposition.

Proposition 9.6. *Let G be a connected 2-metamour-regular graph with n vertices whose metamour graph is the C_n. Then $n \geq 5$ and one of*

(a) $G = C_n$ and n is odd,
(b) $G = \overline{C_n}$, or
(c) $G \in \{H_6^a, H_6^b, H_6^c, H_7^a, H_7^b\}$

holds.

Proof. First we derive two properties of G in the following claims.

A. *We have $n \geq 5$.*

Proof of A. The graph G is connected, hence it contains at least $n-1$ edges. Furthermore, the graph G has the metamour graph C_n, so the complement of G contains at least n edges. As the sum of the number of edges of G and of the complement of G is equal to $\binom{n}{2}$, we have $\binom{n}{2} \geq (n-1) + n$. This is only true for $n \geq 5$. ◁

B. *The graph G is not a tree.*

Proof of B. Suppose that G is a tree. We first show that the maximum degree of G is at most 2.

Let v be a vertex and d its degree, and let $v_1, \ldots, v_d$ its neighbors. Then no vertices of a pair in $\{v_1, \ldots, v_d\}$ are adjacent, as otherwise we would have a cycle. Therefore, the vertices of every such pair are metamours.

We cannot have $d \geq 4$, as otherwise one vertex of $\{v_1, \ldots, v_d\}$ would have at least three metamours, and this contradicts the 2-metamour-regularity of the graph G. If $d = 3$, then there is a 3-cycle in the metamour graph of G which contradicts that the metamour graph is C_n and $n \geq 5$. Therefore, $d \leq 2$, and consequently we have indeed shown that the maximum degree of G is at most 2.

This now implies that G has to be the path graph P_n which is again a contradiction to G being 2-metamour-regular, as an end vertex of P_n only has one metamour. ◁

So by **B**, G is not a tree, therefore it contains a cycle. If G does not contain a cycle of length n, then we can apply Lemma 9.1 and conclude that $G \in \{H_6^a, H_6^b, H_7^a\}$. We are finished in this case.

Otherwise, the graph G contains a cycle of length n. If there is a vertex v of G with $2 < \deg(v) < n-3$, then we can use Lemma 9.2, deduce that $G = H_7^b$ and the proof is complete in this case.

Otherwise, every vertex has degree at most 2 or at least $n-3$. Due to the fact that G contains a cycle of length n, the degree of every vertex is at least 2. Because G is 2-metamour-regular, every vertex is not adjacent to at least two vertices, so the degree of every vertex is at most $n-3$. This implies that every vertex has degree 2 or $n-3$.

If all vertices of G have the same degree, then Lemma 9.3 (for degrees $n-3$) implies $G = \overline{C_n}$ and Lemma 9.4 (for degrees 2) implies $G = C_n$ and n odd. Hence, in these cases we are finished with the proof as well.

What is left to consider is the situation that there are two vertices with different degrees in G. This is done in Lemma 9.5, and we conclude $G \in \{C_5, H_6^c\}$ in this case.

This completes the proof. □

9.2. Graphs with disconnected metamour graph

After characterizing all graphs that are 2-metamour-regular and that have a connected metamour graph, we now turn to 2-metamour-regular graphs that do not have a connected metamour graph. In this case either statement (b) or statement (c) of Theorem 3.20 is satisfied. In the case of (b) there is nothing left to do, because it provides a characterization. In the other case we determine all graphs and capture them in the following proposition.

Proposition 9.7. *Let G be a connected 2-metamour-regular graph with n vertices. Suppose statement (c) of Theorem 3.20 is satisfied. Then $n \geq 6$ and one of*

(a) $G = C_n$ and n is even, or
(b) $G \in \{H^a_{4,4}, H^b_{4,4}, H^c_{4,4}, H^a_{4,3}, H^b_{4,3}, H^c_{4,3}, H^d_{4,3}, H^a_{3,3}, H^b_{3,3}, H^c_{3,3}, H^d_{3,3}, H^e_{3,3}\}$

holds.

Proof. Theorem 3.20(c) implies that the metamour graph is not connected. First observe that by Observation 3.12, each connected component of the metamour graph of a 2-metamour-regular graph is a cycle.

The proof is split into several claims. As a first step, we consider the number of vertices of G.

A. *We have $n \geq 6$.*

Proof of A. As the metamour graph of G is not connected, the metamour graph contains at least two connected components, which are cycles. Each cycle has to contain at least three vertices, so $n \geq 6$. ◁

Now we come to the main part of the proof. Theorem 3.20(c) states that the metamour graph consists of exactly two connected components; we denote these by M' and M^*. Set $G' = G[V(M')]$ and $G^* = G[V(M^*)]$. Then G^M (as in Theorem 3.20) equals $G' \cup G^*$. The definitions of G' and G^* are symmetric and we might switch the roles of the two without loss of generality during the proof and in the statements.

We introduce the following notion: The *signature* σ of a graph is the tuple of the numbers of vertices of its connected components,sorted in descending order. If follows from (i) of Theorem 3.20(c) that all connected components of G' and G^* have at most 2 vertices. As a consequence, the signatures $\sigma(G')$ and $\sigma(G^*)$ have entries in $\{1, 2\}$. Note that in case a connected component has two vertices, then these vertices are adjacent, i.e., this component equals P_2.

We perform a case distinction by the signatures of the graphs G' and G^*; this is stated as the following claims. We start with the case that at least one connected component of G' or G^* has two vertices.

B. *If the first (i.e., largest) entry of $\sigma(G')$ is 2, then either $\sigma(G') = (2, 2)$ or $\sigma(G') = (2, 1, 1)$. In the latter case, the two vertices of the two connected components containing only one vertex do not have any common neighbor in G.*

Proof of B. Suppose we have $G'_1 \in \mathcal{C}(G')$ with $|V(G'_1)| = 2$. Let v_1 be one of the two vertices of G'_1. Then v_1 is adjacent to the other vertex of G'_1, therefore it must have its two metamours in another component of G'. Let us assume that a metamour of v_1 is in a connected component G'_2 of G' that consists of two vertices. Then every vertex of G'_1 is a metamour of every vertex of G'_2 due to (v) of Theorem 3.20(c). This implies that the four vertices of G'_1 and G'_2 form a C_4 in the metamour graph and consequently that $M' = C_4$. Therefore, G' cannot contain other vertices and $\sigma(G') = (2, 2)$.

Let us now assume that the two metamours of v_1 are in different connected components G'_2 and G'_3 of G' that consist of only one vertex each. Then also the other vertex of G'_1 is a metamour of the two vertices in G'_2 and G'_3 due to (v) of Theorem 3.20(c). Hence, these four vertices form again a C_4 in the metamour graph and consequently $M' = C_4$. As a result, G' cannot contain any more vertices, so $\sigma(G') = (2, 1, 1)$.

If the two vertices of G'_2 and G'_3 have a common neighbor, then they are metamours of each other because they are not adjacent. This is a contradiction to the fact that the vertex of G'_2 already has two metamours; they are in G'_1. Hence, the vertices of G'_2 and G'_3 do not have any common neighbor. ◁

By **B** we can deduce how the graphs G' and G^* look like, if one of their connected component contains 2 vertices. We will now continue by going through all possible combinations of signatures of G' and G^* implied by **B**. In every case, we have to determine which edges between vertices of G' and G^* exist and which do not exist in order to specify the graph G.

Due to(iv) of Theorem 3.20(c), we know that as soon as there is an edge in G between a connected component of G' and a connected component of G^*, then there are all possible edges between these two components in G. This implies, now rephrased in the language introduced in Section 2.1,adjacency of two connected components of G^M is equivalent to complete adjacency of these components. Therefore, we equip $G^M = G' \cup G^*$ with a graph structure:The set $\mathcal{C}(G^M)$ is the vertex set and the edge set – we simply write it as $E(G^M)$ – is determined by the adjacency relation above. Note that this graph is bipartite.

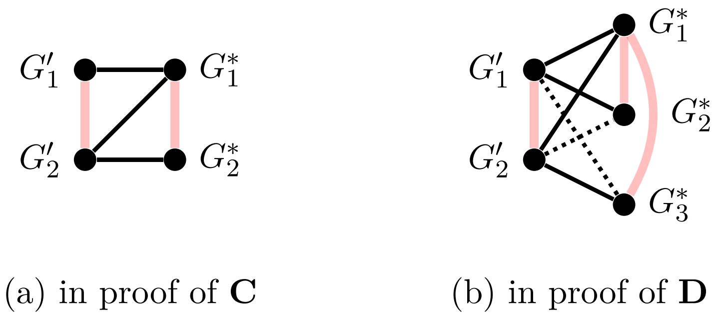


Fig. 9.22. Subgraphs of the situations in the proofs of C and D.

C. *If* $\sigma(G') = (2, 2)$ *and* $\sigma(G^*) = (2, 2)$*, then we have*

$$G \in \{\overline{C_4} \triangledown \overline{C_4}, H^a_{4,4}\}.$$

Proof of C. Let $\{G'_1, G'_2\} = \mathcal{C}(G')$ and $\{G^*_1, G^*_2\} = \mathcal{C}(G^*)$. Each of these four components has size 2, therefore, $n = 8$. This proof is accompanied by Fig. 9.22(a). The components G'_1 and G'_2 need a common neighbor in G^* with respect to G^M because their vertices are metamours; see the proof of **B**. So, we assume without loss of generality (by renumbering the connected components of G^*) that $\{G'_1, G^*_1\} \in E(G^M)$ and $\{G'_2, G^*_1\} \in E(G^M)$. Furthermore, G^*_2 needs to be adjacent to at least one connected component of G' because G is connected, so assume without loss of generality (by renumbering the connected components of G') that $\{G'_2, G^*_2\} \in E(G^M)$. Now there is only the edge between G'_1 and G^*_2 left to consider. If $\{G'_1, G^*_2\} \in E(G^M)$, then $G = \overline{C_4} \triangledown \overline{C_4}$ and if $\{G'_1, G^*_2\} \notin E(G^M)$ then $G = H^a_{4,4}$. This completes the proof. ◃

D. *If* $\sigma(G') = (2, 2)$ *and* $\sigma(G^*) = (2, 1, 1)$*, then we have*

$$G = H^b_{4,4}.$$

Proof of D. Let $\{G'_1, G'_2\} = \mathcal{C}(G')$ and $\{G^*_1, G^*_2, G^*_3\} = \mathcal{C}(G^*)$ such that $|V(G^*_1)| = 2$. The size of each other component is then determined, specifically we have $|V(G'_1)| = 2$, $|V(G'_2)| = 2$, $|V(G^*_2)| = 1$ and $|V(G^*_3)| = 1$. Therefore, $n = 8$. This proof is accompanied by Fig. 9.22(b).

The component G^*_1 need to be adjacent in G^M to at least one connected component of G', so assume without loss of generality (by renumbering the connected components of G') that $\{G'_1, G^*_1\} \in E(G^M)$. It is not possible that both G^*_2 and G^*_3 are adjacent in G^M to G'_1 due to **B**, so assume without loss of generality (by renumbering G^*_2 and G^*_3) that $\{G'_1, G^*_3\} \notin E(G^M)$. But G^*_3 must have a common neighbor in G' with G^*_1 because their vertices are metamours, so this implies $\{G'_2, G^*_3\} \in E(G^M)$ and $\{G'_2, G^*_1\} \in E(G^M)$. Due to **B**, this implies that $\{G'_2, G^*_2\} \notin E(G^M)$. But as G^*_2 needs to be adjacent to at least one connected component of G', we find $\{G'_1, G^*_2\} \in E(G^M)$. As a consequence, we obtain $G = H^b_{4,4}$. ◃

E. *If* $\sigma(G') = (2, 1, 1)$ *and* $\sigma(G^*) = (2, 1, 1)$*, then we have*

$$G = H^c_{4,4}.$$

Proof of E. Let $\{G'_1, G'_2, G'_3\} = \mathcal{C}(G')$ and $\{G^*_1, G^*_2, G^*_3\} = \mathcal{C}(G^*)$ such that $|V(G'_1)| = 2$ and $|V(G^*_1)| = 2$. The size of each other component is then determined. We get $n = 8$.

Because of **B**, not both G^*_2 and G^*_3 can be adjacent in G^M to G'_1, so assume without loss of generality (by renumbering G^*_2 and G^*_3) that $\{G'_1, G^*_3\} \notin E(G^M)$. If $\{G'_1, G^*_2\} \notin E(G^M)$, then G^*_1 is the only possible common neighbor in G^M for G'_1 and G'_2 as well as G'_1 and G'_3. But then G'_2 and G'_3 would have the common neighbor G^*_1 in G^M, a contradiction to **B**. As a result, we obtain $\{G'_1, G^*_2\} \in E(G^M)$.

By symmetric arguments for G^*, we can assume without loss of generality (by renumbering G'_2 and G'_3) that $\{G'_3, G^*_1\} \notin E(G^M)$ and then deduce that $\{G'_2, G^*_1\} \in E(G^M)$. The current situation is shown in Fig. 9.23(a).

As a result, G'_2 is the only possible common neighbor in G^M of G^*_1 and G^*_3, so $\{G'_2, G^*_3\} \in E(G^M)$. Analogously, we obtain $\{G'_3, G^*_2\} \in E(G^M)$. Now due to **B**, we can deduce that $\{G'_2, G^*_2\} \notin E(G^M)$ and $\{G'_3, G^*_3\} \notin E(G^M)$. So far G'_1 and G'_2 do not have a common neighbor, and G^*_1 is the only possibility for that left, so $\{G'_1, G^*_1\} \in E(G^M)$; see Fig. 9.23(b).

This fully determines G and it holds that $G = H^c_{4,4}$. ◃

With **C**, **D** and **E** we have considered all cases in which both G' and G^* have a connected component consisting of two vertices.

So from now on we can assume that at least one of G' and G^* has no connected component consisting of two vertices. Next we will deduce a result in the case that the signature of one of G' and G^* is $(1, \ldots, 1)$ with at least four entries.

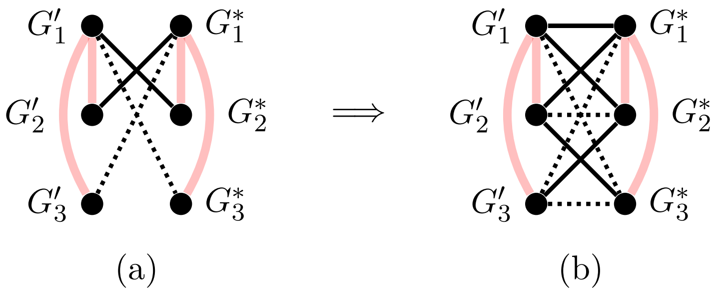


Fig. 9.23. Subgraphs of the situations in the proof of **E**.

F. *If* $\sigma(G') = (1, \ldots, 1)$, r *times with* $r \geq 4$*, then we have*

$$G = C_n$$

and n is even.

Proof of F. Let $\{G_1', \ldots, G_r'\} = \mathcal{C}(G')$. Let without loss of generality (by renumbering the connected components of G') the vertices of G_{i-1}' and G_{i+1}' be the two metamours of the vertex of G_i'. Note that we take the indices modulo r and that we keep doing this for the remaining proof.

Let $i \in \{1, \ldots, r\}$. Then clearly G_i' and G_{i+1}' have a common neighbor $G_i^* \in \mathcal{C}(G^*)$. If G_i^* is adjacent to any other connected component of $\mathcal{C}(G')$, then the vertex of this component together with the two vertices of G_i' and G_{i+1}' form a C_3 in the metamour graph. This is a contradiction, because M' is C_r with $r \geq 4$. Hence, G_i^* is adjacent in G to only G_i' and G_{i+1}' of G'. In particular, this implies that the components G_1^*, ..., G_r^* are pairwise disjoint due to (iv) of Theorem 3.20(c).

Now because of the common neighbors, the vertices of G_i^* have the vertices of G_{i-1}^* and G_{i+1}^* as metamours. Therefore, as we are 2-metamour-regular, every component G_i^* consists of exactly one vertex. As a consequence, G_1^*, ..., G_r^* lead to a C_r in the metamour graph of G, specifically $M^* = C_r$. It is easy to see that the vertices of G_1', G_1^*, G_2', G_2^*, ..., G_r', G_r^* form a C_{2r}. As we have ruled out all other possible edges, this implies that $G = C_{2r}$. Hence, $n = 2r$ and $G = C_n$ for n even. $\triangleleft$

In **F**, we have dealt with signatures $(1, \ldots, 1)$ of length at least 4. We will consider $(1, 1, 1)$ below. There cannot be fewer than three connected components of only single vertices because each connected component of the metamour graph is a cycle and therefore has at least 3 vertices.

So what is left to consider are the two cases that G' contains a connected component with two vertices and G^* has three isolated vertices and the case that both G' and G^* have three isolated vertices. We consider these cases in the following claims.

G. *If* $\sigma(G') = (2, 2)$ *and* $\sigma(G^*) = (1, 1, 1)$*, then we have*

$$G \in \{\overline{C_4} \triangledown \overline{C_3}, H_{4,3}^a, H_{4,3}^b\}.$$

Proof of G. Let $\{G_1', G_2'\} = \mathcal{C}(G')$ and $\{G_1^*, G_2^*, G_3^*\} = \mathcal{C}(G^*)$. The size of each component is then determined, and we get $n = 7$.

The single-vertex components G_1^* and G_2^* need a common neighbor, as well as G_2^* and G_3^*, and G_1^* and G_3^*. At least two of these common neighbors are from the same connected component of G', because G' has only two connected components. Let without loss of generality (by renumbering G_1' and G_2') this connected component be G_1'. As a result, every component G_1^*, G_2^* and G_3^* is adjacent to G_1', and $\{G_1', G_1^*\}$, $\{G_1', G_2^*\}$ and $\{G_1', G_2^*\} \in E(G^M)$.

The connected component G_2' has to be adjacent to some component of G^* because G is connected, so assume without loss of generality (by renumbering G_1^*, G_2^* and G_3^*) that $\{G_2', G_1^*\} \in E(G^M)$. The current situation is shown in Fig. 9.24(a).

Now if both $\{G_2', G_2^*\}$ and $\{G_2', G_3^*\}$ are in $E(G^M)$, then G is fully determined and $G = \overline{C_4} \triangledown \overline{C_3}$. If only one of $\{G_2', G_2^*\}$ and $\{G_2', G_3^*\}$ is in $E(G^M)$, then we have $G = H_{4,3}^a$, and if none of $\{G_2', G_2^*\}$ and $\{G_2', G_3^*\}$ is in $E(G^M)$, then $G = H_{4,3}^b$. As one of these three settings has to occur, this proof is completed. $\triangleleft$

H. *If* $\sigma(G') = (2, 1, 1)$ *and* $\sigma(G^*) = (1, 1, 1)$*, then we have*

$$G \in \{H_{4,3}^c, H_{4,3}^d\}.$$

Proof of H. Let $\{G_1', G_2', G_3'\} = \mathcal{C}(G')$ and $\{G_1^*, G_2^*, G_3^*\} = \mathcal{C}(G^*)$ such that $|V(G_1')| = 2$. The size of each other component is then determined. We get $n = 7$.

As G is connected, let us assume without loss of generality (by renumbering G_1^*, G_2^* and G_3^*) that $\{G_2', G_2^*\}$ and $\{G_3', G_3^*\} \in E(G^M)$. Because of **B**, the vertices of G_2' and G_3' cannot have a common neighbor, therefore $\{G_2', G_3^*\}$ and $\{G_3', G_2^*\} \notin E(G^M)$ holds.

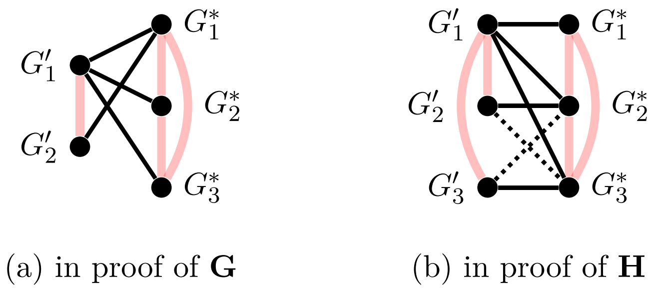


Fig. 9.24. Subgraphs of the situations in the proofs of **G** and **H**.

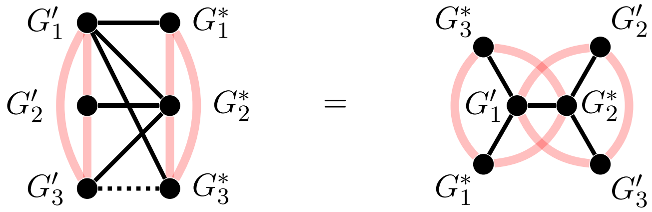


Fig. 9.25. Subgraphs of the situation in the proof of **I**.

The only choice for a common neighbor of G_2^* and G_3^* is G_1', therefore $\{G_1', G_2^*\}$ and $\{G_1', G_3^*\} \in E(G^M)$.

If G_1' is not adjacent in G^M to G_1^*, then we have to have $\{G_2', G_1^*\}$ and $\{G_3', G_1^*\} \in E(G^M)$ so that G_1^* and G_2^* as well as G_1^* and G_3^* have a common neighbor. But then G_2' and G_3' get the common neighbor G_1^* which is a contradiction to **B**. Therefore, we have $\{G_1', G_1^*\} \in E(G^M)$. The current situation is shown in Fig. 9.24(b).

Furthermore, not both of G_2' and G_3' can be adjacent to the vertex of G_1^*, so assume without loss of generality (by renumbering G_2' and G_3') that $\{G_3', G_1^*\} \notin E(G^M)$.

Now if $\{G_2', G_1^*\} \in E(G^M)$, then $G = H^c_{4,3}$. If $\{G_2', G_1^*\} \notin E(G^M)$, then $G = H^d_{4,3}$. This completes the proof. $\triangleleft$

Now the only case left to consider is that both G' and G^* contain three isolated vertices.

I. *If* $\sigma(G') = (1, 1, 1)$ *and* $\sigma(G^*) = (1, 1, 1)$*, then we have*

$$G \in \{C_6, \overline{C_3} \,\nabla\, \overline{C_3}, H^a_{3,3}, H^b_{3,3}, H^c_{3,3}, H^d_{3,3}, H^e_{3,3}\}.$$

Proof of I. Let $\{G_1', G_2', G_3'\} = \mathcal{C}(G')$ and$\{G_1^*, G_2^*, G_3^*\} = \mathcal{C}(G^*)$. Then clearly $n = 6$.

We first consider the case that every connected component has at most degree 2 in G^M. If a vertex has degree 1, then in order to have a common neighbor with its both metamours, the component it is adjacent to has to have degree 3, a contradiction. Therefore, every vertex has degree 2. Due to the fact that G is connected, this implies that $G = C_6$, so in this case we are done.

Now assume there is at least one component that has degree 3 in G^M. Let without loss of generality (by switching G' and G^* and by renumbering the connected components of G') this component be G_1'. Then we have $\{G_1', G_1^*\}$, $\{G_1', G_2^*\}$ and $\{G_1', G_3^*\} \in E(G^M)$.

If every component of G' has degree 3 in G^M, then $G = \overline{C_3} \,\nabla\, \overline{C_3}$, so also in this case we are done. Hence, we can assume without loss of generality (by renumbering G_2' and G_3' and by renumbering the components of G^*) that $\{G_3', G_3^*\} \notin E(G^M)$.

Now G_3' and G_2' need a common neighbor. This cannot be G_3^*. Assume without loss of generality (by renumbering G_1^* and G_2^*) that the common neighbor is G_2^*. Then this implies that $\{G_2', G_2^*\}$ and $\{G_3', G_2^*\} \in E(G^M)$. The current situation is shown in Fig. 9.25.

The potential edges, which status is still undetermined, are $\{G_2', G_1^*\}$, $\{G_2', G_3^*\}$ and $\{G_3', G_1^*\}$. At this stage, if each of these pairs is a non-edge in G^M, then it is easy to see that whenever two vertices should be metamours they are metamours, to be precise all components of G' have the common neighbor G_2^* and all components of G^* have the common neighbor G_1'. Therefore, we have enough edges in G^M, so that additional edges between components of G' and G^* can be included without interfering with the metamours.

Now we first consider all cases where $\{G_2', G_3^*\} \in E(G^M)$. In this case if both of $\{G_2', G_1^*\}$ and $\{G_3', G_1^*\}$ are in $E(G^M)$, then $G = H^a_{3,3}$. If $\{G_2', G_1^*\} \notin E(G^M)$ and $\{G_3', G_1^*\} \in E(G^M)$, then $G = H^b_{3,3}$. If $\{G_2', G_1^*\} \in E(G^M)$ and $\{G_3', G_1^*\} \notin E(G^M)$, then $G = H^c_{3,3}$. And finally, if $\{G_2', G_1^*\} \notin E(G^M)$ and $\{G_3', G_1^*\} \notin E(G^M)$, then $G = H^d_{3,3}$.

Next we consider the case where $\{G_2', G_3^*\} \notin E(G^M)$. If in this case both of $\{G_2', G_1^*\}$ and $\{G_3', G_1^*\}$ are in $E(G^M)$, then $G = H^c_{3,3}$. If one of $\{G_2', G_1^*\}$ and $\{G_3', G_1^*\}$ is in $E(G^M)$, then $G = H^d_{3,3}$. If none of $\{G_2', G_1^*\}$ and $\{G_3', G_1^*\}$ is in $E(G^M)$, then $G = H^e_{3,3}$.

Eventually, we have considered all cases and proven what we wanted to show. ◁

Finally, we are finished in all cases and therefore the proof of Proposition 9.7 is complete. □

9.3. Assembling results & other proofs

With all results above, we are now able to prove the main theorem of this section which provides a characterization of 2-metamour-regular graphs.

Proof of Theorem 3.13. Let G be 2-metamour-regular and M its metamour graph. We apply Theorem 3.20 with $k = 2$. This leads us to one of three cases.

Case (b) of Theorem 3.20 gives

$$G = \overline{M_1} \nabla \cdots \nabla \overline{M_t}$$

with $\{M_1, \ldots, M_t\} = \mathcal{C}(M)$ and $t \geq 2$. By Observation 3.12 every connected component M_i, $i \in \{1, \ldots, t\}$, is a cycle C_{n_i}. This results in (a) of Theorem 3.13 for $t \geq 2$.

If we are in case (a) of Theorem 3.20, then the metamour graph M is connected and we apply Proposition 9.6. If we are in case (c) of Theorem 3.20, then the metamour graph consists of exactly two connected components and we can apply Proposition 9.7. Collecting all graphs coming from these two propositions yields the remaining graphs of (a), (b) and (c) of Theorem 3.13.

For the other direction, Proposition 3.19 implies that the graph G in (a) of Theorem 3.13 is a 2-metamour-regular graph for $t \geq 2$. Furthermore, it is easy to check that all other mentioned graphs are 2-metamour-regular, which proves this side of the equivalence and completes the proof. □

Finally we are able to prove the following corollaries of Theorem 3.13.

Proof of Corollary 3.15. The result is an immediate consequence of Theorem 3.13. □

Proof of Corollary 3.16. Corollary 3.15 provides a characterization of all 2-metamour-regular graphs with $n \geq 9$ vertices. It is easy to see that all of these graphs are either 2-regular (in the case of C_n) or $(n-3)$-regular. This proves one direction of the equivalence.

For the other direction first consider a connected 2-regular graph on n vertices. Clearly, this graph equals C_n, therefore this graph is 2-metamour-regular. If a connected graph is $(n-3)$-regular, then its complement $\overline{G}$ is a 2-regular graph. As a result, each connected component of $\overline{G}$ is a cycle graph. Let $C_{n_1}, \ldots, C_{n_t}$ be the connected components of $\overline{G}$. It is easy to see that then $n_i \geq 3$ and $n = n_1 + \cdots + n_t$ hold. In consequence, $G = \overline{C_{n_1}} \nabla \cdots \nabla \overline{C_{n_t}}$ holds and therefore G is 2-metamour-regular. This completes the proof. □

Proof of Corollary 3.17. The statement of the corollary is a direct consequence of Theorem 3.13. □

Proof of Corollary 3.18. We use the characterization provided by Theorem 3.13. So let us consider 2-metamour-regular graphs. Such a graph has at least $n \geq 5$ vertices.

In case (a), there is one graph per integer partition of n into a sum, where each summand is at least 3. Note that the graph operator ∇ is commutative which coincides with the irrelevance of the order of the summands of the sum. There are $p_3(n)$ many such partitions.

Case (b) gives exactly one graph for each $n \geq 5$. The graph C_5 is counted in both (a) and (b); see first item of Remark 3.14. Case (c) brings in additionally 8 graphs for $n = 6$, 6 graphs for $n = 7$ and 3 graphs for $n = 8$.

In total, this gives the claimed numbers. □

This completes all proofs of the present paper.

10. Conclusions & open problems

In this paper we have introduced the metamour graph M of a graph G: The set of vertices of M is the set of vertices of G and two vertices are adjacent in M if and only if they are at distance 2 in G, i.e., they are metamours. This definition is motivated by polyamorous relationships, where two persons are metamours if they have a relationship with a common partner, but are not in a relationship themselves.

We focused on k-metamour-regular graphs, i.e., graphs in which every vertex has exactly k metamours. We presented a generic construction to obtain k-metamour-regular graphs from k-regular graphs for an arbitrary $k \geq 0$. Furthermore, in our main results, we provided a full characterization of all k-metamour-regular graphs for each $k \in \{0, 1, 2\}$. These characterizations revealed that with a few exceptions, all graphs come from the generic construction. In particular,

- for $k = 0$ every k-metamour-regular graph is obtained by the generic construction.
- For $k = 1$ there is only one exceptional graph that is k-metamour-regular and not obtained by the generic construction.
- In the case of $k = 2$ there are 17 exceptional graphs with at most 8 vertices and a family of graphs, one for each number of vertices at least 6, that are 2-metamour-regular and cannot be created by the generic construction.

Additionally, we were able to characterize all graphs where every vertex has at most one metamour and give properties of the structure of graphs where every vertex has at most k metamours for arbitrary $k \geq 0$. Every characterization is accompanied by counting for each number of vertices how many unlabeled graphs there are.

The obvious unanswered question is clearly the following.

Question 10.1. *What is a characterization of k-metamour-regular graphs for each $k \geq 3$?*

This is of particular interest for $k = 3$. As our generic construction yields k-metamour-regular graphs for every $k \geq 0$, we clearly already have determined a lot of 3-metamour-regular graphs. It would, however, be lovely to determine all remaining graphs. Another interesting question is about fixed maximum metamour-degree.

Question 10.2. *What is a characterization of all graphs that have maximum metamour-degree k?*

We have answered this question for $k \in \{0, 1\}$ and would be delighted to know the answer in general, but as first steps specifically for $k = 2$ and $k = 3$.

It would also be interesting to find some structure in the graphs that are k-metamour-regular and cannot be obtained with our generic construction. In particular, we ask the following.

Question 10.3. *Is it possible to give properties (necessary or sufficient) of the exceptional graphs or graph classes?*

When dealing with metamour graphs, one question to ask is whether it is possible to characterize all graphs whose metamour graph has a certain property. In the present paper we have started to give an answer for the feature that the metamour graph is k-regular. But what about other graph classes? Of course it would be interesting to answer the following questions.

Question 10.4. *Is it possible to characterize all graphs whose metamour graph is in some graph class like planar, bipartite, Eulerian or Hamiltonian graphs or like graphs of a certain diameter, girth, stability number or chromatic number?*

Another question of interest concerns constructing graphs, namely given a graph M, is there a graph G such that M is the metamour graph of G? If M is not connected, then the answer is easy and also provided in this paper, namely $G = \overline{M}$ is such a graph. However, if M is connected this question is still open and an answer more complicated. This give rise to the following question.

Question 10.5. *What is a characterization of the class of graphs with the property that each graph in this class is the metamour graph of some graph?*

Motivated by [29] we ask the following.

Question 10.6. *What is a characterization of the class of graphs, where every graph is isomorphic to its metamour graph?*

Going into another direction, one can also think about random graphs like the graphs from the Erdős–Rényi model $G(n, p)$.

Question 10.7. *Given a random graph of $G(n, p)$, which properties does its metamour graph have? Is there a critical value for p (depending on n) such that the metamour graph is connected?*

In enumerative and probabilistic combinatorics the following question arise.

Question 10.8. *Given a random graph model, for example that all graphs with the same number of vertices are equally likely, what is the expected value of the metamour-degree? What about its distribution?*

Most of the results and open questions focus on the number of vertices of the graph and metamour graph respectively, as these two numbers match. But it would be interesting to know how the number of edges of the metamour graph of a graph relates to the number of edges in this graph. Specifically, we ask the following questions.

Question 10.9. *Given a graph G with m edges, in which range can the number of edges of the metamour graph of G be?*

Question 10.10. *What is the distribution of the number of edges of the metamour graph over all possible graphs with m edges?*

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