On the Hodge conjecture for products of certain surfaces

by

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Dedicado a

esa artista que pinta estrellas en el horizonte, por su coraje y, sobre todo, su risa

Abstract

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In this thesis we prove the Hodge conjecture for products of smooth projective surfaces $S_1 \times S_2$, where $S_2 = A$ is an Abelian surface and S_1 is such that $p_g(S_1) = 1$, $q = 2$. We hereby provide new examples in dimension 4 where the Hodge conjecture holds.

Declaration

The work in this thesis is based on research carried out between October 2001 and May 2003 at the Department of Mathematical Sciences of the University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

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Chapter 1

Introduction

Let X be a smooth projective variety over $\mathbb{C}$, of dimension n. H^i will denote the i-th singular cohomology group with rational coefficients. We have the following (canonical) decomposition, known as the *Hodge decomposition:*

$$H^k(X) \otimes \mathbb{C} = \oplus_{i+j=k} H^i(X, \Omega^j_X).$$

For every $0 \leq p \leq n$ there exists a *cycle map*

$$c_p : CH^p(X)_{\mathbb{Q}} \to H^{2p}(X, \mathbb{Q}) \tag{1.1}$$

whose image falls into $H^{p,p}(X) \cap H^{2p}(X, \mathbb{Q})$, where $CH^p(X)$ denotes the Chow group of codimension-p cycles of X and $H^{p,p}(X) = H^p(\Omega^p_X)$ [35], [20]. The vector space $H^{p,p}(X) \cap H^{2p}(X, \mathbb{Q})$ is called the space of *Hodge cycles* of codimension p.

The Hodge conjecture states the following: the map c_p exhausts every class in $H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X)$ for every $p = 0, \cdots, \dim(X)$.

For general X, it is known since Lefschetz that the Hodge conjecture holds for $p = 1$, even in a stronger form, over $\mathbb{Z}$. This is done by using the equivalence between equivalence classes of divisors and the corresponding line bundle analogues. An example by Atiyah and Hirzebruch, using Godeaux-Serre varieties ([19]pp.189-195) shows that the cycle map with integer coefficients does not exhaust the Hodge classes in the integral cohomology groups $H^{2p}(X, \mathbb{Z})$ in general, for $p \geq 2$; they basically show the existence of non-analytic torsion cycles on $H^{2p}(X, \mathbb{Z})$ for X a certain finite quotient, using the Atiyah-Hirzebruch spectral sequence. More recently, examples of non-torsion non-algebraic integral Hodge cycles have been found; these are examples by Kollár and van Geemen of a conjecture by Griffiths and Harris. The reader can find these in the list of Trento examples [2] pp.134-135.

There are some partial examples which satisfy the conjecture [35], but for most of the examples the ring of Hodge cycles is generated by divisors. The Hodge conjecture is not even known to be true in the case of Abelian varieties - for partial results, we refer to [35] [62]. Even for this case, no examples are known of Abelian varieties that would satisfy the Hodge conjecture whose rings of Hodge cycles would not be generated by the divisor classes, apart from Shioda's examples of Jacobians of curves of Fermat type and the important cases worked out by C. Schoen (see [55], [56]).

The Hodge conjecture is also related to the Tate conjecture: Milne [39] has shown that the Hodge conjecture for Abelian varieties of CM type implies the Tate conjecture for all Abelian varieties over a finite field, by using Tannakian methods.

For a smooth projective surface, we define $q(S) = h^{1,0}(S)$ and $p_g(S) = h^{2,0}(S)$. In this thesis we prove the following theorem:

Theorem 1.1. *The Hodge conjecture holds for $S \times A$, where S is an algebraic surface such that $p_g = 1$, $q = 2$ and A is an Abelian surface.*

This theorem generalises the known result that the Hodge conjecture holds for products of Abelian surfaces $A_1 \times A_2$. The outline of the proof is as follows: first we will see that S is either Abelian or an isotrivial elliptic fibration over a curve B of genus two. S will turn out to be a quasibundle, in the terminology of [57], i.e. an etale quotient of a product of two curves. Next we show that we can restrict ourselves to a particular type of (standard) isotrivial fibrations, namely quotients of products of two curves by a finite cyclic group acting faithfully on the cohomology of the fibre. From then on we use the techniques of Mumford-Tate groups [35], transcendental lattices of K3 surfaces [42] [44], to finally prove that, if S is not Abelian and the transcendental lattices $T(S)$ and $T(A)$ are isomorphic, then there exists a Hodge isomorphism between $T(S)$ and $T(A)$ induced by an algebraic cycle. In the case when S is Abelian the Hodge conjecture is already known (see [35] [66]). In our case, we use a weaker result than the whole of the Hodge conjecture for products of Abelian surfaces $A_1 \times A_2$, namely the Hodge conjecture for $A \times A$, where A is of the form $E_1 \times E_2$, in order to obtain all Hodge cycles in $T(S) \otimes T(A)$ from the construction of one algebraic cycle on $S \times A$, and using the law of composition of correspondences with every (Hodge or) algebraic class in $T(A) \otimes T(A)$; thus we prove that every Hodge cycle in $H^\bullet(S \times A)$ is algebraic. It should be remarked that the Hodge conjecture is already known to hold for all powers A^n of an Abelian surface A (see Ribet [53]).

Throughout this thesis, we assume every variety to be complex, irreducible, projective, and non-singular unless otherwise stated. The cohomology groups will always be seen as rational Hodge structures, and the (iso)morphisms shown between cohomology groups will be assumed to be (iso)morphisms of Hodge structures as well.

Chapter 2

Preliminaries

2.1 The Hodge Conjecture

This chapter is aimed at a full description of the statement of the Hodge (p, p) Conjecture, and also a complete account of the general facts known about this conjecture. We start with an introduction to the Hodge theory of projective smooth manifolds, mainly following Griffiths and Harris [20]. A very readable and pleasant introduction to the subject may be found in the lecture notes by Mark Green at C.I.M.E. 1993, Torino (see [1],pp.1-92). We start with a section devoted to the *Hodge* (also called *Kähler*) *decomposition* on projective manifolds, we include a section on Lefschetz's results on $H^{\bullet}(X)$ involving the *Lefschetz operator* $\mathbf{L}$ defined by cup-product with the Chern class of a hyperplane section; these will allow a reformulation of the Hodge Conjecture into a weaker statement, which will be proven equivalent to the original statement of the conjecture. These first two sections make it natural for us to define a *Hodge structure*, which will be basically a vector space over $\mathbb{Q}$ satisfying analogous properties to the ones stated for rational cohomology groups of a projective manifold X; we mainly follow van Geemen's lecture notes at Banff 1998 [17], whose notation differs slightly from Deligne's exposition (see [8]). The notion of *polarisation* of a Hodge structure will come out naturally, inspired also on the cohomology of projective manifolds – the only Hodge structures that are of interest to us are the ones which admit a polarisation, for these are the only Hodge structures which may be substructures of $H^k(M)$ for some projective manifold

M. The final section of this chapter is the one devoted to the introduction of the *Hodge* and *Mumford-Tate groups*, which are rational algebraic groups attached to a Hodge structure. Every polarisable Hodge structure splits into a direct sum of irreducible Hodge structures; in other words, the category of rational polarisable hodge structures is semisimple abelian. In Tannakian terms, the Mumford-Tate group counterpart is that the Mumford-Tate and Hodge groups (of polarisable Hodge structures) are reductive over $\mathbb{Q}$. The Hodge and Mumford-Tate groups will also turn out to be fundamental in the study of the Hodge structure of the cohomology groups of Abelian varieties. Their study yields the proof of the Hodge Conjecture in some cases, as will be seen in Chapter 10.3. The main reference for this last section will be also van Geemen [17].

2.2 The cycle map

Let X be a smooth projective variety over $\mathbb{C}$, and let $Z^p(X)$ be the free abelian group generated by the integral closed subschemes of codimension p on X. $Z^p(X)$ is called the group of *algebraic cycles of codimension* p on X. $CH^p(X)$ will denote, as usual, the Chow group of codimension p of X, which is the quotient of $Z^p(X)$ modulo rational equivalence. We will construct a graded homomorphism of (commutative) rings

$$c : CH^\bullet(X) \to \oplus_{p=0}^{dim X} H^{2p}(X, \mathbb{Z})/\text{tors}.$$

The first step will be to define a map δ_p from $Z^p(X)$. It will then be seen that it factorises through rational equivalence.

To start with, we suppose that Z is a smooth irreducible subvariety of X of codimension p. We will construct $\delta(Z)$ on these hypotheses. Indeed, the inclusion

$$\iota : Z \hookrightarrow X$$

yields a morphism

$$H^{2(n-p)}(X,\mathbb{Z})/\text{tors} \to H^{2(n-p)}(Z,\mathbb{Z})/\text{tors}$$

which preserves the above defined Hodge decompositions, after complexification. Poincaré duality together with Serre duality and Dolbeault theorem yield in turn that the image of the map

$$\iota_* : H^0(Z,\mathbb{Z}) \to H^{2p}(X,\mathbb{Z})/tors$$

generated by 1_Z (which is the Poincaré dual to $[Z]$ in Z) yields a (p,p) form on X. We define $\delta([Z]) := \iota_*(1_Z)$. If Z is not smooth, we can choose a desingularisation $\nu : \tilde{Z} \to Z$ and compose $f := \iota \circ \nu$, and then proceed analogously. Then the map

$$f_* : H^0(\tilde{Z}) \to H^{2p}(X)$$

satisfies the above hypotheses; we then define $\delta([Z]) := f_*(1_{\tilde{Z}})$. It remains to check that the definition does not depend on the chosen desingularisation. Indeed, suppose that $Z'' \to Z' \to Z$ are two desingularisations of Z, Z'' dominating Z'. Then it is clear (by functoriality) that they both yield the same image in $H^{2p}(X)$. The general case follows from the fact that given two desingularisations Z_1, Z_2 of a projective variety, one can always construct a third desingularisation Z_3 dominating Z_1 and Z_2 – see Proposition 5.5.

We define δ by linear extension, and then define c_p through passage to the quotient from $\delta : Z^p(X) \to H^{2p}(X,\mathbb{Z})/\text{tors}$. It remains to check that δ factors through rational equivalence, in order to see that c_p is well-defined.

Indeed, choose two cycles ζ_i such that there exists a connected algebraic variety T, two points $t_i \in T$ and an algebraic cycle Γ on $X \times T$ such that

$$\zeta_1 - \zeta_2 = proj_{X,*}(\Gamma(X \times t_1 - X \times t_2))$$

Then, because the immersions $u_i : X \to X \times t_i \subset X \times T$ are homotopically equivalent, we get $\delta(\zeta_1 - \zeta_2) = 0$. It suffices to take $T = \mathbb{P}^1$ to obtain the desired factorisation and thus the construction of c_p is well-defined (here we have used the characterisation of rational equivalence given in Kleiman [28] 1.1; see also Fulton [15] 1.6). Now, the linear map

$$c := \oplus c_p : CH^\bullet(X) = \oplus_{p=0}^{\dim X} CH^p(X)_\mathbb{Q} \to H^{\text{even}}(X,\mathbb{Q}) = \oplus H^{2p}(X,\mathbb{Q})$$

defines a morphism of (commutative) rings – this last assertion would follow from the standard fact that cup-product and intersection product can be obtained from the diagonal map $\Delta : X \hookrightarrow X \times X$ through the formulae $Z_1 \cdot Z_2 = \Delta^*(Z_1 \times Z_2)$ and $\alpha \cup \beta = \Delta^*(\alpha \otimes \beta)$. For more details, see Kleiman [28] Sect.1, especially Prop. 1.2.2.; see also Kleiman [29] Lemma 3.1.

For every $0 \leq p \leq n = dimX$ we have thus constructed a *cycle map*

$$c_p : CH^p(X) \to H^{2p}(X, \mathbb{Z})/\text{tors} \tag{2.1}$$

whose image falls into $H^{p,p}(X) \cap H^{2p}(X, \mathbb{Z})/\text{tors}$, where $CH^p(X)$ denotes the Chow group of codimension-p cycles of X and $H^{p,p}(X) = H^p(\Omega_X^p)$. The vector space $H^{p,p}(X) \cap H^{2p}(X, \mathbb{Q})$ is called the space of *Hodge cycles*(or *classes*) of codimension p, and will be denoted in the sequel by $B^p(X)$. The *Hodge ring* of X is the graded ring $\mathcal{B}(X) = \oplus_{p=0}^{dimX} B^p(X)$. (We can define the *integral Hodge ring* $\mathcal{B}_{\mathbb{Z}}(X)$ inside $H^{\text{even}}(X, \mathbb{Z})/\text{tors}$ analogously).

Remark: We outline another procedure to define the cycle map $CH^p(X)_{\mathbb{Q}} \to H^{2p}(X, \mathbb{Q})$ for a complex projective manifold X. Denote the topological and algebraic K^0 by K's in this paragraph, namely $K_{\text{top}}(X)$ and $K_{\text{alg}}(X)$ respectively. We know (see Murre [1]) that the existence of a *topological filtration* on $K_{\text{alg}}(X)$ yields the existence of a *Chern character map*

$$\text{ch}_{alg} : K_{\text{alg}}(X)_{\mathbb{Q}} \to CH^\bullet(X)_{\mathbb{Q}},$$

which is an isomorphism of rings. We also have a topological Chern character map

$$\text{ch}_{top} : K_{\text{top}}(X)_{\mathbb{Q}} \to H^{\text{even}}(X, \mathbb{Q})$$

(see Hirzebruch [26] App.I.24). Let $\vartheta : K_{\text{alg}}(X) \to K_{\text{top}}(X)$ denote the morphism induced by the forgetful functor on the respective categories of vector bundles. Composing, we get the following ring homomorphism

$$\text{ch}_{top} \circ \vartheta \circ \text{ch}_{alg}^{-1} : CH^\bullet(X)_{\mathbb{Q}} \to H^{\text{even}}(X, \mathbb{Q}).$$

It is worth mentioning explicitly how one can conceive an element in $K_{alg}(X)$ given by a closed subscheme Z. In fact, there is a resolution of $\mathcal{O}_Z$ by locally free sheaves

$0 \to \mathcal{E}_1 \to \ldots \to \mathcal{E}_n \to \mathcal{O}_Z \to 0$ (this comes from Hilbert syzygy theorem) – this yields an element $\mathcal{E}_n - \mathcal{E}_{n-1} + \ldots (-1)^{n-1}\mathcal{E}_1$ in $K_{\text{alg}}(X)$.

We also note the following. The Chern classes of a holomorphic vector bundle yield integral (p,p)-cohomology classes (see [20] pp. 416-419); the Chern character map has rational coefficients, which recovers the fact that the image of the cycle map falls into the subring $\oplus_{p=0}^{\dim(X)} (H^{p,p}(X) \cap H^{2p}(X,\mathbb{Q}))$, that is, into the (rational) Hodge ring $\mathcal{B}^\bullet(X)$. Joining this to the last observation, we get that the image of an algebraic cycle Z of X of (pure) codimension p in the cohomology of X yields a rational cohomology class of degree (p,p). Also, the resolution of $\mathcal{O}_Z$ by locally free sheaves implies that this definition of cycle map agrees with the former (see Fulton [15] Ex.3.2.3,Ex.14.4.3, Sect.15.1 for details).

The cycle map is shown to have its image in a subring of the (even) cohomology ring of the projective variety X, namely, in its *ring of Hodge cycles.* We are now ready to formulate the Hodge conjecture.

The Hodge (p,p) Conjecture: The cycle map

$$c_p : CH^p(X)_{\mathbb{Q}} \to B^p(X) = H^{p,p}(X) \cap H^{2p}(X,\mathbb{Q})$$

c_p is surjective for every $p = 0, \cdots, \dim(X)$; equivalently, the ring homomorphism

$$c_X : CH^\bullet(X)_{\mathbb{Q}} \to \mathcal{B}(X)$$

is surjective.

Chapter 3

The Hodge and Lefschetz decompositions

In this chapter we state the Hodge Theorem on harmonic forms on a compact oriented Riemannian manifold. After that we introduce the concept of Kähler manifold, and then we draw important consequences from this condition, namely the *Hodge decomposition* (also known as the *Kähler decomposition* and the *Lefschetz decomposition*. An important case of Kähler manifolds are projective manifolds – it will be shown below that $\mathbb{P}^n$ is a Kähler manifold, and therefore so is every closed submanifold of $\mathbb{P}^n$, i.e. every projective manifold is Kähler. Thus a large list of geometrical properties will hold for a projective manifold X. The main references used are Green [1], Griffiths and Harris [20], Wells [73].

3.1 The Hodge Theorem

We follow M. Green's lecture notes in [1], pp.1-92, and occasionally Warner [71]. We are now going to state the Hodge Theorem on harmonic forms on a compact oriented Riemannian manifold; we define in this section the well-known *Hodge *-operator*, to be used in the sequel. In order to do so, a certain amount of preliminary multilinear algebra is needed.

Let V be an oriented n-dimensional vector space over $\mathbb{R}$ equipped with a positive-definite inner product. Then for any k, $\bigwedge^k V$ may be given a natural positive definite inner product – one way to do this is by seeing $\bigwedge^k V \hookrightarrow \bigotimes^k V$; we will denote this inner product on $\bigwedge^k V$ by $(.,.)$. Thus if M is an oriented Riemannian manifold of dimension n and ω is a smooth k-form on M, we can apply the construction above to $T_p(M)^*$ with the induced inner product to obtain a length $||\omega_p||^2$. If M is an oriented compact Riemannian manifold, then we define

$$||\omega||_M^2 := \int_M ||\omega_p||^2 dV,$$

where dV is the element of volume. This is a positive-definite inner product on the space of smooth k-forms on M. We will denote the corresponding inner product on k-forms by $(.,.)_M$.

Definition A smooth k-form ω on M on a compact Riemannian manifold is *harmonic* if $d\omega = 0$ and

$$||\omega||_M \leq ||\omega + d\tau||_M$$

for all smooth $(k-1)$-forms τ. We denote the set of (real-valued) harmonic k-forms on M by $\mathcal{H}^k_{\mathbb{R}}(M)$ ($\mathcal{H}^k(M)$ will stand for the set of complex-valued harmonic k-forms on M).

There is a slightly different way to describe the inner product on forms. The inner product on a vector space V gives a natural map

$$V \otimes V \to \mathbb{R}.$$

This in turn gives a natural isomorphism $V \to V^*$ of V with its dual. If we take $\bigwedge^k$ of this isomorphism, we obtain an isomorphism $\bigwedge^k V \to \bigwedge^k V^*$. Wedge product gives a map

$$\bigwedge^k V \otimes \bigwedge^{n-k} V \to \bigwedge^n V \cong \mathbb{R}$$

(remember V is already an oriented vector space), and since this is a non-degenerate pairing, it gives a natural isomorphism $\bigwedge^k V \cong \bigwedge^{n-k} V^*$, and now using $\bigwedge^k$ of the isomorphism induced by the inner product, we can identify the factor on the right with $\bigwedge^{n-k} V$. Putting all this together, we obtain a natural map (in the context of oriented Euclidean vector spaces)

$$* : \bigwedge^k V \to \bigwedge^{n-k} V,$$

and this is the Hodge *-operator. For an oriented manifold M, this is defined pointwise and thus extends to a map $* : A^k_{\mathbb{R}}(M) \to A^{n-k}_{\mathbb{R}}(M)$. We now introduce the following operator, which will be frequently used:

Definition We define the operator $w : \bigwedge^k V \to \bigwedge^k V$ to be $w = **$.

The basic facts on the Hodge-*-operator are:

Lemma 3.1. *(Green [1] pp.4-5, Warner [71] Ch.3, Ex. 14)*

1. *For* $\alpha, \beta \in \bigwedge^k V$,

$$\alpha \wedge *\beta = \beta \wedge *\alpha = (\alpha, \beta)\mathrm{Vol},$$

where Vol $\in \bigwedge^n V$ *is the element of volume;*

2. *If* $e_1, \cdots, e_n$ *is an oriented orthonormal basis for* V, *then* $*(e_{i_1} \wedge \ldots \wedge e_{i_k}) = \pm e_{j_1} \wedge \ldots \wedge e_{j_{n-k}}$ *where* $\{j_1, \cdots, j_k\} = \{1, 2, \cdots n\} - \{i_1, \cdots, i_k\}$ *and the sign is chosen so that* $e_{i_1} \wedge \ldots \wedge e_{i_k} \wedge \pm e_{j_1} \wedge \ldots \wedge e_{j_{n-k}} = e_1 \wedge \ldots \wedge e_n$;

3. *For* $\alpha \in \bigwedge^k V$, $*^2\alpha = (-1)^{k(n-k)}\alpha$. *I.e.,* $w|\bigwedge^k V = (-1)^{k(n-k)}$. *If* $n = \dim V$ *is even, then* $w|\bigwedge^k V = (-1)^k id_{\bigwedge^k V}$.

4. *Let* $\omega \in \bigwedge^{2k} V$, *and* $\varepsilon(\omega)$ *be the endomorphism of* $\bigwedge^* V$ *given by left multiplication by* ω, $\varepsilon(\omega)\eta = \omega \wedge \eta$. *Let* Λ *be the adjoint of this endomorphism under the given metric in* $\bigwedge^* V$. *Then*

$$\varepsilon(\omega)^*(v) = *^{-1}(\omega \wedge *v) \quad \text{for every} \quad v \in \bigwedge^{p+2k} V.$$

I.e., $\varepsilon(\omega)^* = *^{-1}\varepsilon(\omega)* = w * \varepsilon(\omega)*$ *for a form* ω *of even degree* $2k$.

Corollary 3.2. *For* $\alpha, \beta \in A^k(M)$,

$$(\alpha, \beta)_M = \int_M \alpha \wedge *\beta.$$

We now construct an adjoint for $d : A^k_{\mathbb{R}}(M) \to A^{k+1}_{\mathbb{R}}(M)$.

Proposition 3.3. *(Green [1] p.5) Let* $d^*\omega = (-1)^{(k+1)(n-k)+1} * d * \omega$ *for all* $\omega \in A^k_{\mathbb{R}}(M)$. *Then*

$$(d^*\omega, \phi)_M = (\omega, d\phi)_M$$

for all $\omega \in A^k_{\mathbb{R}}(M)$, $\phi \in A^{k+1}_{\mathbb{R}}(M)$. *In other words,* $d^* = -w * d *$.

Proof: Follows from Stokes' theorem.

Definition The *Laplace operator* $\Delta : A^k_{\mathbb{R}}(M) \to A^k_{\mathbb{R}}(M)$ is defined by

$$\Delta = dd^* + d^*d.$$

Proposition 3.4. *(Green [1] p.5) For $\omega \in A^k_{\mathbb{R}}(M)$, the following are equivalent:*

1. *ω is harmonic;*
2. *$d\omega = 0$ and $d^*\omega = 0$;*
3. *$\Delta\omega = 0$.*

Proof: The whole proof relies on the following observation of Linear Algebra (applied to $E = \bigwedge^\bullet V$)

L et $f : E \to E$ be an endomorphism of an Euclidean space E. Then the following subspaces are equal:

$$\mathrm{Ker}(ff^* + f^*f) = \mathrm{Ker}\ f \cap\ \mathrm{Ker}\ f^*$$

Indeed, one needs only prove the inclusion "$\subset$", which is an immediate consequence of the following computation

$$\langle v, (ff^* + f^*f)v\rangle = \langle fv, fv\rangle + \langle f^*v, f^*v\rangle.$$

This observation proves $2 \leftrightarrow 3$. To prove $1 \leftrightarrow 2$ it suffices to develop the following quadratic polynomial in the real variable ϵ, for an arbitrary $\tau \in A^{k-1}_{\mathbb{R}}(M)$:

$$\begin{aligned} &||\omega + \epsilon d\tau||^2_M - ||\omega||^2_M \\ &= 2\epsilon(\omega, d\tau) + \epsilon^2||d\tau||^2_M \\ &= 2\epsilon(d^*\omega, \tau) + \epsilon^2||d\tau||^2_M \end{aligned}$$

and the equivalence follows.

Proposition 3.5. *The natural map*

$$\mathcal{H}^k_{\mathbb{R}}(M) \to H^k_{DR}(M)$$

sending a harmonic k-form to its de Rham cohomology class is injective.

Proof: The only thing to be proved is that if a harmonic form is exact, then it is 0. If $\omega \in \mathcal{H}^k(M)$ and 0 belongs to the de Rham class of ω, then by minimality $|\omega|_M \leq |0|_M = 0$, so $\omega = 0$.

The result that the following embedding is an isomorphism is not at all obvious, and it requires non-trivial analytic arguments. The final result, which we quote here, is

Theorem 3.6. (The Hodge Theorem) *For M a compact oriented Riemannian manifold, the natural map*

$$\mathcal{H}^k_{\mathbb{R}}(M) \to H^k_{DR}(M)$$

is an isomorphism, i.e. every de Rham class is represented by a unique harmonic form.

Remark The former results also hold if M is a compact hermitian complex manifold. See Griffiths and Harris [20] pp.81-100 for a complete exposition in the complex Hermitian setting.

3.2 Kähler manifolds

In this section we define the concept of Kähler manifold, and we derive some important consequences obtained from the study of the C^∞ harmonic forms on a compact Kähler manifold, especially the Hodge-Kähler decomposition. We also prove in this section that every projective manifold is Kähler –for results in the other direction, the Kodaira embedding theorem yields a sufficient condition for a compact Kähler manifold to be projective. We follow Mark Green's lecture notes in [1], pp. 6-25, and also use Griffiths and Harris [20], Section 0.7 especially.

As with the Hodge theorem, there is a certain amount of preliminary multilinear algebra needed.

3.2.1 Definitions and Hermitian multilinear algebra

Definition Let V be a vector space over $\mathbb{R}$. An *almost complex structure* **J** on V is an endomorphism $J \in End(V)$ such that $J^2 = -Id$.

Definition Let V, J be a real vector space with almost-complex structure, and $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$, with J extended to $V_{\mathbb{C}}$ in the canonical way. Define $V^{1,0}$ and $V^{0,1}$ respectively as the $+i$ and $-i$ eigenspaces of J on $V_{\mathbb{C}}$. We also define the complex subspaces $V^{p,q}$ of $\bigwedge^\bullet V_{\mathbb{C}}$ as $\bigwedge^p V^{1,0} \otimes \bigwedge^q V^{0,1}$. Then J acts on $V^{p,q}$ by multiplication by i^{p-q}.

Definition In the above hypotheses, the *Weil operator* C is defined by $\bigwedge^\bullet J$ on $\bigwedge^\bullet V_{\mathbb{C}}$, i.e. $C|V^{p,q} = i^{p-q}$.

Remark Then $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$. Further, $\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} V^{1,0} = \dim_{\mathbb{C}} V^{0,1}$.

The next definition appears in Green [1] p.7.

Definition Let V, J be a real vector space with an almost-complex structure. A positive definite inner product $(.,.)$ is (real) *Hermitian* if J is an isometry, i.e. $(Jv, Jw) = (v, w)$ for all $v, w \in V$.

Remark Let V, J be a real vector space with an almost-complex structure. A real, positive definite inner product g is Hermitian (in the sense of last definition) if and only if there exists a (complex-valued, i.e. sesquilinear) Hermitian form h (in the usual sense) with respect to the $\mathbb{C}$-vector space structure sending $i \mapsto J$ such that $g = \mathrm{Re}\, h$. Under these hypotheses, we have a decomposition of h into real and imaginary parts

$$h = g + i \cdot E.$$

It follows that $E(x, y) = g(x, Jy)$ is a (real) alternate bilinear form on V.

Proposition 3.7. *Let V, J be a real vector space with almost-complex structure and (real) Hermitian metric $(.,.)$.*

1. *The map $\omega : V \otimes V \to \mathbb{R}$ defined by*

$$\omega(v, w) = (Jv, w)$$

   is a real alternating form;

2. *If we extend ω to an element of $\bigwedge^2 V_{\mathbb{C}}^*$, then ω is zero when restricted to $V^{1,0} \otimes V^{1,0}$ and $V^{0,1} \otimes V^{0,1}$;*

3. *ω gives a non-degenerate pairing when restricted to $V^{1,0} \otimes V^{0,1}$;*

4. *If we extend $(,)$ to be complex linear in the first variable and conjugate linear in the second variable, then it is a positive definite Hermitian inner product on $V^{1,0}$.*

Proof:

1. $(Jv, w) = (J^2 w, Jw) = -(v, Jw) = -(Jw, v)$.

2. In order to prove **2.** and **3.**, extend $(,)$ to be complex linear in both entries. If $v, w \in V^{1,0}$, then $(Jv, w) = i(v, w) = (v, Jw) = -(Jv, w)$, so $(v, w) = 0$ and thus $\omega(v, w) = 0$. Similarly for $V^{0,1}$.

3. If $v \in V^{1,0}$ then for some $w \in V_{\mathbb{C}}$, $(Jv, w) \neq 0$, as otherwise $Jv = 0$ and hence $v = 0$. If $w = w^{1,0} + w^{0,1}$ is the decomposition of w under the direct sum decomposition $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$, then by **2.**, $(Jv, w) = (Jv, w^{0,1})$, and this proves the pairing is non-degenerate in the first factor. A similar argument works for the second factor.

4. If $v = a + ib \in V^{1,0}$, where $a, b \in V$, then $(v, \overline{v}) = (a, a) + (b, b)$, from which positive-definiteness is clear.

Definition Let V, J be a real vector space with almost-complex structure and real Hermitian metric $(,)$. Let $\omega(x, y) = (Jx, y)$ be as in the preceding Proposition. Then ω is called the *alternating form*(also the *fundamental* 2-*form*) associated to $(,)$. Note that $(Jx, y) = -\text{Im}\ (x, y)$.

To be precise, let $\varphi_k = \alpha_k + \sqrt{-1}\beta_k$ be a unitary coframe for the sesquilinear Hermitian form $(,)$. Then $(,) = \sum \varphi_k \otimes \overline{\varphi}_k$. Thus $\omega = i/2 \sum \varphi_k \wedge \overline{\varphi}_k = \sum_k \alpha_k \wedge \beta_k$.

Definition For a complex manifold M, a C^∞ *form of type* (p, q) is a C^∞-section of the bundle $\bigwedge^p T^{1,0*} \otimes \bigwedge^q T^{0,1*}$; we will denote the set of these by $A^{p,q}(M)$. It is clear that the following decomposition holds:

$$A^k(M) = \oplus_{p+q=k} A^{p,q}(M).$$

Definition Let M be a complex manifold with almost-complex structure $J : T_M \to T_M$. A Riemannian metric on the underlying real manifold of M is *hermitian* if it is hermitian with respect to J on $T_{M,p}$ for every point $p \in M$. The *associated (1,1) form* ω of the hermitian metric is defined by taking ω_p to be the alternating form associated to the metric on $T_{M,p}$ for every $p \in M$.

Corollary 3.8. *The associated* $(1,1)$*-form of a hermitian matrix is a real* 2*-form on the underlying real manifold of* M *and has type* $(1,1)$.

Definition A hermitian metric on a complex manifold M is said to be a *Kähler metric* on M if the associated $(1,1)$ form ω is closed. In this case, ω is called the *Kähler form.* The element of $H^2_{DR}(M)$ determined by ω is called the *Kähler class.* If the Kähler class belongs to the image of $H^2(M,\mathbb{Z})$, the metric is said to be a *Hodge metric.*

Example: *Fubini-Study metric.* On $\mathbb{P}^n = \mathbb{P}(V)$, if we let

$$0 \to S \to V \otimes \mathcal{O}_{\mathbb{P}^n} \to Q \to 0$$

be the tautological sub-bundle sequence (see Fulton [15] Appendix B.5.7 for details), then it is well-known (*loc.cit.*, same paragraph) that

$$T_{\mathbb{P}^n} = Hom(S,Q).$$

If we put a Hermitian metric on the complex vector space V, then it induces natural Hermitian metrics on S and Q by restriction and orthogonal projection. This in turn induces a natural metric on $S^* \otimes Q \cong Hom(S,Q)$. This metric is invariant under the action of the unitary group on V, and this forces the associated $(1,1)$ form ω to be closed (it is even harmonic; this is an old result of E. Cartan and G. de Rham on compact homogeneous spaces, see [6] I.1.6 and Remark II.3.2 for a proof). Since $H^2_{DR}(\mathbb{P}^n)$ is one-dimensional, adjusting the metric by a constant makes ω integral, and thus $\mathbb{P}^n$ has a Hodge metric, the *Fubini-Study metric.*

Another way to define the Fubini-Study metric on $\mathbb{P}^n$ is as follows (see [20] pp.30,31). Consider the standard projection map $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$. Let $U \subset \mathbb{P}^n$

be an open set and $Z : U \to \mathbb{P}^n$ a lifting of U, i.e. a holomorphic map with $\pi \circ Z = id$; consider the differential form

$$\omega = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}log(||Z||^2).$$

If Z' is another lifting, then

$$Z' = f \cdot Z$$

with f a nonzero holomorphic function, so that

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}log(||Z'||^2) = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}log(||Z||^2).$$

Therefore ω is independent of the lifting chosen; since liftings always exist locally, ω is a globally defined differential form in $\mathbb{P}^n$. Clearly ω is of type $(1,1)$. To see that ω is positive-definite, first note that the unitary group $U(n+1)$ acts transitively on $\mathbb{P}^n$ and leaves the form ω invariant, so it suffices to prove positive definiteness at only one point. Now let $w_i := Z_i/Z_0$. be coordinates on the open set $U_0 = \{_0 \neq 0\}$ in $\mathbb{P}^n$ and use the lifting $Z := (1, w_1, \ldots, w_n)$ on U_0; we have, for $p_0 = (1 : 0 : \ldots : 0)$,

$$\omega_{p_0} = \frac{\sqrt{-1}}{2\pi}\sum dw_i \wedge d\overline{w}_i > 0$$

Thus ω defines a hermitian metric on $\mathbb{P}^n$. We check directly that this form is closed (although, as stated before, a $U(n+1)$-invariance argument is also valid). Indeed, take a local lifting Z on $U \subset \mathbb{P}^n$. Since $\partial\overline{\partial} = -\overline{\partial}\partial$, we can write

$$\begin{aligned} \omega &= \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}log(||Z||^2) \\ &= \frac{\sqrt{-1}}{4\pi}(\partial + \overline{\partial})(\overline{\partial} - \partial)log(||Z||^2) \\ &\quad \frac{\sqrt{-1}}{4\pi}d(\overline{\partial} - \partial)log(||Z||^2) \end{aligned}$$

which clearly shows that ω is closed, and therefore the Fubini-Study metric is Kähler. The above examined differential form arises naturally as the curvature form of the holomorphic tangent bundle of $\mathbb{P}^n$. For general computations on the first Chern class of a holomorphic vector bundle, and the derivation of the former differential form, see Wells [73] Ex.III.2.4, Prop. III.4.3, Ex. V.3.5.

More can be said about ω:

Lemma 3.9. *(Griffiths-Harris [20] p. 122) The above defined Kähler form ω in $\mathbb{P}^n$ is Hodge, and it generates $H^2(\mathbb{P}^n, \mathbb{Z})$. More precisely, the de Rham class of ω corresponds to the cycle class of a hyperplane in $\mathbb{P}^n$.*

Proof: Let l be a projective line in $\mathbb{P}^n$ and let H be a hyperplane. The following calculation is left to the reader:

$$\int_l \omega = 1 = \deg(H \cdot l).$$

It follows that ω is not exact by Stokes' theorem, and also that ω is Poincaré dual to H, thereby establishing the lemma.

As a consequence of last lemma, we obtain the following.

Proposition 3.10. *(Green [1] p.8) A smooth projective variety $M \subset \mathbb{P}^n$ has a Hodge metric obtained by restricting the Fubini-Study metric. Therefore, every projective manifold is compact Kähler. Moreover, the above obtained Hodge metric on M by restriction corresponds to the cycle class of a hyperplane section (of the mentioned embedding).*

The proof of last proposition is elementary. Its converse is a non-trivial result, and is due to Kodaira.

Theorem 3.11. (Kodaira embedding theorem) *(Griffiths-Harris [20]p.191) If a compact hermitian complex manifold admits a Hodge metric, then there exists an embedding of M in some $\mathbb{P}^N$ such that the Hodge metric on M is $\frac{1}{k}$ times the restriction of the Fubini-Study metric, for some positive integer k.*

Let us consider the following situation in Riemannian geometry. It is easy to see that at every point p of a Riemannian manifold M, there are local coordinates $x_1, \ldots, x_n$ for M centered at p such that

$$(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \delta_{ij} + O(||x||^2).$$

However, it is not true that at every point p of a hermitian complex manifold, there are local holomorphic coordinates $z_1, \ldots, z_n$ centered at p such that

$$(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}) = \delta_{ij} + O(||z||^2).$$

One may check that this is the case for $M = \mathbb{P}^n$ – it suffices to check it for $p := (1 : 0 : \ldots : 0)$ and the coordinates w_i defined above; it is then true for every point $p \in \mathbb{P}^n$ by using a suitable projective transformation. The following proposition makes the Kähler condition more natural in this context.

Proposition 3.12. *(Wells [73] Cor.V.3.11; Green [1] p.9) Let M be a complex hermitian manifold. The following are equivalent:*

1. *The metric is Kähler.*

2. *At every point p of M, there are local holomorphic coordinates z_i centered at p such that*
$$\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) = \delta_{ij} + O(||z||^2).$$

In other words, M is Kähler if and only if the metric on M osculates to order 2 to the Euclidean metric everywhere.

Proof: If z_i are local holomorphic coordinates on M, let $h_{ij}(z) := (\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j})$. Then
$$\omega = i \sum_{i,j} h_{ij} dz_i \wedge d\bar{z}_j.$$
If *2.* holds, then all first partials of the h_{ij} vanish at the origin, and hence $d\omega = 0$ at p; since the point was arbitrary, $d\omega = 0$ and the metric is Kähler. Conversely, if $d\omega = 0$, and we choose holomorphic coordinates z_i so that $h_{ij} = \delta_{ij}$ (this is possible), one has
$$h_{ij} = \delta_{ij} + \sum a_{ij}^k z_k + \sum \bar{a}_{ij}^k \bar{z}_k + O(||z||^2).$$
Then the change of variables $z_i = w_i + q_i(w, w)$, where the q_i are homogeneous and quadratic in the w's, changes the linear term of h_{ij} by $\frac{\partial q_i}{\partial z_j} + \frac{\partial q_j}{\partial z_i}$. Thus a_{ij}^k is changed by $\frac{\partial^2 q_i}{\partial z_j \partial z_k}$. The condition $d\omega = 0$ is equivalent to $a_{ij}^k = a_{kj}^i$ for all i, j, k, and thus if we take $q_j = -\frac{1}{2}\sum_{ik} a_{ij}^k z_i z_j$, the coordinates w_i satisfy *2.*

The following proposition contains an immediate consequence of the Kähler condition. For a more general statement, we refer the reader to Ueno [70] Cor.9.5.

Proposition 3.13. *([20] pp.109,110) For M a compact Kähler manifold, the holomorphic forms $H^0(M, \Omega_M^p)$ inject into the cohomology $H_{DR}^q(M, \mathbb{C})$, i.e. every such nonzero η is closed and never exact.*

Proof: Let $\varphi_1, \ldots, \varphi_n$ be a local unitary coframe; if

$$\eta = \sum_I \eta_I \varphi_I$$

then

$$\eta \wedge \overline{\eta} = \sum_{I,J} \eta_I \overline{\eta}_J \wedge \varphi_I \overline{\varphi}_I.$$

Now

$$\omega = \frac{\sqrt{-1}}{2} \sum \varphi_i \wedge \overline{\varphi}_i,$$

so

$$\omega^{n-q} = C_q (n-q)! \sum_K \varphi_K \wedge \overline{\varphi}_K;$$

for $C_q = \left(\frac{\sqrt{-1}}{2}\right)^{n-q}$; thus, for D_q a suitable nonzero constant,

$$\eta \wedge \overline{\eta} \wedge \omega^{n-q} = D_q \sum |\eta_I|^2 \cdot \Phi$$

where Φ is the volume form. Consequently,

$$\int_M \eta \wedge \overline{\eta} \wedge \omega^{n-q} \neq 0 \text{ if } \eta \neq 0.$$

Now suppose that $\eta = d\psi$ is exact. Then, since $\overline{\eta}, \omega$ are closed, Stokes' theorem yields

$$\int_M \eta \wedge \overline{\eta} \wedge \omega^{n-q} = \int_M d\left(\psi \wedge \overline{\eta} \wedge \omega^{n-q}\right) = 0.$$

Thus $\eta = d\psi$ implies $\eta \equiv 0$. Finally, since $d\eta = \partial\eta$ is a holomorphic $(q+1)$-form and is exact, it follows that $d\eta = 0$.

Let us return to the situation of a hermitian vector space $V, J, (,)$. We will explore the geometry existing in $\bigwedge^\bullet V^*$. Recall that a complex vector space has a natural orientation when viewed as a real vector space of twice the dimension, because $GL(n, \mathbb{C}) \subset GL(2n, \mathbb{R})$ is connected. We adopt the convention set by the ordering $x_1, y_1, \ldots x_n, y_n$, where $y_k = Jx_k$.

Definition Let V, J be a real vector space with an almost complex structure. The *natural orientation* of V is the pullback of the natural orientation of $V^{1,0}$ under the natural isomorphism of real vector spaces $V \to V^{1,0}$ given by $v \mapsto i.v + Jv$.

If $e_1, \ldots, e_n \in V$ are such that e_i, Je_i form a basis of V, then $e_1, Je_1, \ldots e_n, Je_n$ is a properly oriented basis for V. Once chosen the natural orientation for V, a natural positive-definite inner product is obtained on $\bigwedge^k V$ and $\bigwedge^k V^*$, by the standard construction on Euclidean spaces; this will also be denoted $(,)$. This may be extended to a positive definite (complex-valued) Hermitian inner product on $\bigwedge^k V_{\mathbb{C}}$ and $\bigwedge^k V_{\mathbb{C}}^*$, where $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of V. One may extend the *-operator to $\bigwedge^k V_{\mathbb{C}}$ and $\bigwedge^k V_{\mathbb{C}}^*$ so as to be complex linear, and then $(\alpha, \beta)\mathrm{Vol} = \alpha \wedge *\overline{\beta}$. The induced Hermitian inner product on $\bigwedge^k V_{\mathbb{C}}$ and $\bigwedge^k V_{\mathbb{C}}^*$ is usually called the *Hodge inner product.* Another frequently used operator is the complex conjugate version of $*$, namely $\overline{*}(\alpha) =^* \overline{\alpha}$. Thus the (complex Hermitian) Hodge inner product can be obtained by the formula $(\alpha, \beta)\mathrm{Vol} = \alpha \wedge^{\overline{*}} (\beta)$.

Remark Recall that the Hodge-*-operator needed a volume element only. In our case the volume element is $e_1 \wedge Je_1 \wedge \ldots e_n \wedge Je_n$, which is real and of type (n, n). Thus the subspaces $V^{p,q} = \bigwedge^p V^{1,0} \otimes \bigwedge^q V^{0,1}$ satisfy $\overline{*}(V^{p,q}) = V^{n-p,n-q}$.

Proposition 3.14. *(Green [1] p.10) Let $V, J, (,)$ be as in the above definition, and let $2n = dim_{\mathbb{R}} V$. Let ω denote the alternating form. Then*

$$\omega^{\otimes n} = n! \ Vol.$$

Definition (Operators L, Λ, H) Let $V, J, (,)$ be a real vector space of dimension $2n$ with almost-complex structure J and hermitian metric $(,)$. Let $L : \bigwedge^* V^* \to \bigwedge^* V^*$ be given by $L(\alpha) = \omega \wedge \alpha = \alpha \wedge \omega$, where ω is the alternating form associated to the metric. Since the metric of V induced metrics on $\bigwedge^k V^*$ for all k, the adjoint of L will be denoted by Λ, so that

$$(L\alpha, \beta) = (\alpha, \Lambda\beta)$$

for all $\alpha \in \bigwedge^k V^*$, $\beta \in \bigwedge^{k+2} V^*$. Finally, let $H : \bigwedge^* V^* \to \bigwedge^* V^*$ be the linear map whose restriction to $\bigwedge^k V^*$ is $(n-k)id_{\bigwedge^k V^*}$.

Recall: As already done in Lemma 3.1 for the Euclidean case, we give an explicit expression for Λ in terms of ω and the Hodge *-operator (for the second equality we use that L is a real operator):

$$\Lambda = \overline{*}^{-1} L \overline{*} = *^{-1} L * = w * L * .$$

The above defined operators satisfy non trivial commutation relations among themselves. The following proposition will take its full meaning in the next paragraph. For a proof, see Wells [73] pp. 161-165, especially Prop.V.1.1.(c).

Proposition 3.15. *(Wells [73] Th.V.1.6) The following relations hold*

$$[\Lambda, L] = H, \qquad [H, L] = -2L, \qquad [H, \Lambda] = 2\Lambda.$$

These relations are exactly those which hold among the standard generators of $\mathfrak{sl}_2$. This justifies an interlude on representations of the Lie algebra $\mathfrak{sl}_2$; see Subsection 3.2.3.

Definition (Primitive forms) The *primitive k-forms* are

$$\begin{aligned} P^k &= \{\eta \in \bigwedge^k V^* | \Lambda\eta = 0\} \\ &= \{\eta \in \bigwedge^k V^* | \omega \wedge *\eta = 0\} \end{aligned}$$

The meaning of last definition, as well as the relations in the former proposition, will be made clear in next paragraph on representation theory of the Lie algebra $\mathfrak{sl}_2$. We state now the following proposition, whose proof can be obtained as a straightforward consequence of the representation theory of $\mathfrak{sl}_2$. For an elementary but lengthy proof, see Wells [73] Th.V.1.8.

Proposition 3.16. (Lefschetz decomposition for forms).*([1] p.11)*

(1) $L^{n-k+1} : P^k \to \bigwedge^{2n-k+2} V^*$ *is zero;*

(2) $L^{n-k} : P^k \to \bigwedge^{2n-k} V^*$ *is injective;*

(3) $\bigwedge^k V^* = \oplus_j L^j P^{i-2j}$.

The final important bit of multilinear algebra is the following – on P^k, both $*$ and L^{n-k} land in $\bigwedge^{2n-k} V^*$, and we might hope for them to be multiples of each other, or even more generally, on $L^j P^k$. The following proposition gives a precise relation between both operators. The proof found in the usual literature is lengthy and shall be omitted, a reference being given.

Proposition 3.17. *(Green [1]p.11, Wells [73] Th.V.1.6) Suppose that* $\alpha = L^j\beta$, *where* $\beta \in P^{p,k-p}$ *and* $j \leq n-k$; *then*

$$*L^j\beta = (-1)^{\frac{k(k+1)}{2}} \frac{j!}{(n-k-j)!} L^{n-k-j} C\beta;$$

in other words,

$$\overline{*}\alpha = (-1)^{\frac{k(k+1)}{2}} i^{p-q} \frac{j!}{(n-k-j)!} L^{n-k-2j}\overline{\alpha}.$$

Definition On $\bigwedge^k V^*$, we define the hermitian inner product $\langle , \rangle$ by

$$\langle \alpha, \beta \rangle \mathrm{Vol} = i^{k^2} L^{n-k}(\alpha \wedge \overline{\beta}).$$

The power of i is necessary to make the form Hermitian.

Corollary 3.18. *(Green [1] p.12) On* $L^jP^{p,k-p}$, *the following identity holds*

$$(-1)^{k+p}\langle , \rangle = \frac{j!}{(n-j-k)!}(,),$$

and hence $(-1)^{k+p}\langle , \rangle$ *is a positive definite Hermitian form on* $L^jP^{p,k-p}$ *(note that* $(-1)^{k+p} = (-1)^{k-p}$.*)*

Proof: If $\alpha \in L^jP^k$, then

$$\begin{aligned} \langle \alpha, \alpha \rangle &= i^{k^2} L^{n-2j-k}(\alpha \wedge \overline{\alpha}) \\ = (-1)^{\frac{k(k+1)}{2}} i^{k^2-k+2p} &\frac{j!}{(n-j-k)!} \alpha \wedge *\overline{\alpha} \\ (-1)^{k+p} &\frac{j!}{(n-j-k)!} (\alpha, \alpha)\mathrm{Vol}. \end{aligned}$$

3.2.2 Representations of $\mathfrak{sl}_2$

In this paragraph we mainly follow Griffiths and Harris [20] pp.118 -122. Let $\mathfrak{sl}_2$ denote the Lie algebra of the Lie group $SL_2(\mathbb{C})$; it is realised as the vector space of 2×2 complex matrices with trace 0, and with the bracket

$$[A, B] = AB - BA.$$

We take as standard generators

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with the relations

$$[X,Y] = H, \qquad [H,X] = 2X, \qquad [H,Y] = -2Y.$$

Now, let V be a finite-dimensional complex vector space, $\mathfrak{gl}(V)$ its algebra of endomorphisms (seen as a Lie algebra with the usual commutator as Lie bracket). We want to study Lie algebra maps

$$\rho : \mathfrak{sl}_2 \to \mathfrak{gl}(V),$$

i.e., linear maps ρ such that

$$\rho([A,B]) = \rho(A)\rho(B) - \rho(B)\rho(A).$$

Such a map is called a *representation of* $\mathfrak{sl}_2$ *in* V; V is called an $\mathfrak{sl}_2$-module. A subspace of V fixed under $\rho(\mathfrak{sl}_2)$ is called a submodule; V (or ρ) is called *irreducible* if V has no nontrivial submodules. It is known that $\mathfrak{sl}_2$ is a semisimple Lie algebra, and thus every $\mathfrak{sl}_2$-module V decomposes into a direct sum of irreducible $\mathfrak{sl}_2$-modules (see Dixmier [13] Cor. 1.8.5 for the case of $\mathfrak{sl}_2$). More generally, for every semisimple Lie algebra $\mathfrak{g}$, every $\mathfrak{g}$-submodule W of a $\mathfrak{g}$-module V has a complementary submodule $W^\perp$, and thus V is the direct sum of irreducible $\mathfrak{g}$-modules – for this more general result we refer the reader to Dixmier [13] Th. 1.6.3. This means that to study $\mathfrak{sl}_2$-modules we need only look at irreducible representations of $\mathfrak{sl}_2$. Suppose then that V is an irreducible $\mathfrak{sl}_2$-module. The key to analysing the structure of V is to look at the eigenspaces for $\rho(H)$ – from now on, we omit the ρ's. These are called the *weight spaces.* First of all, note that if $v \in V$ is an eigenvector of H with eigenvalue λ, then Xv and Yv are also eigenvectors of H, with eigenvalues $\lambda + 2$ and $\lambda - 2$ respectively; this follows from

$$\begin{aligned} H(Xv) &= XH(v) + [H,X]v \\ &= X\lambda v + 2Xv \\ &= (\lambda+2)Xv, \end{aligned}$$

and similarly for Yv. Since H can have only a finite number of eigenvalues, we see from this that X and Y are nilpotent. We say that $v \in V$ is *primitive* if v is an eigenvector for H and $Xv = 0$; clearly primitive elements exist.

Proposition 3.19. *If $v \in V$ is primitive, then V is generated as a vector space by*

$$v, Yv, Y^2v, \ldots$$

Proof: Since V is irreducible, we need only show that the linear span V' of $\{Y^i v\}$ is fixed under $\mathfrak{sl}_2$. Clearly $HV' \subset V'$ and $YV' \subset V'$. We show $XV' \subset V'$ by induction: $Xv = 0$ trivially lies in V', and in general

$$XY^n v = YXY^{n-1}v + HY^{n-1}v;$$

so

$$XY^{n-1}v \in V' \Rightarrow XY^n v \in V'.$$

This completes the proof.

Note that the elements $\{Y^n v\}$ that are nonzero are linearly independent, since they are all eigenvectors for H with different eigenvalues. Therefore $V = \oplus V_\lambda$, where each V_λ is one-dimensional, and

$$HV_\lambda = V_\lambda, \qquad XV_\lambda = V_{\lambda+2}, \qquad YV_\lambda = V_{\lambda-2}.$$

Proposition 3.20. *All eigenvalues for H are integers, and we can write*

$$V = V_n \oplus V_{n-2} \oplus \cdots \oplus V_{-n+2} \oplus V_n.$$

Proof: Let v be primitive, and suppose

$$Y^n v \neq 0 = Y^{n+1}v,$$

and $Hv = \lambda v$. Then

$$\begin{aligned} Xv &= 0, \\ XYv &= YXv + Hv = \lambda v, \\ XY^2 v &= YXYv + HYv \\ &= Y\lambda v + (\lambda - 2)Yv = (\lambda + (\lambda - 2))Yv, \end{aligned}$$

and since $Y^n v \neq 0, Y^{n+1} v = 0$,

$$(n+1)\lambda - (n+1)^2 + n + 1 = 0 \Rightarrow \lambda = n.$$

In summary, the irreducible $\mathfrak{sl}_2$-modules are indexed by nonnegative integers n; for each such n the corresponding $\mathfrak{sl}_2$-module $V(n)$ has dimension $n+1$. Explicitly,

$$V(n) \cong Sym^n(\mathbb{C}^2)$$

is the n-th symmetric power of the vector space $\mathbb{C}^2$ (which is a standard sl_2-module via the obvious inclusion $SL_2(\mathbb{C}) \subset GL(\mathbb{C}^2)$). The eigenvalues of H acting on $V(n)$ are $-n, -n+2, \ldots, n-2, n$, each appearing with multiplicity 1 – for a list of examples on decompositions of $\mathfrak{sl}_2$-modules into irreducible submodules, we recommend Fulton and Harris [16] Sections 11.2 and 11.3. For any $\mathfrak{sl}_2$-module V, not necessarily irreducible, we define the *Lefschetz decomposition* of V as follows: let $PV := \text{Ker}X$; then

$$V = PV \oplus YPV \oplus \ldots,$$

and this decomposition is compatible with the decomposition of V into eigenspaces V_m for H. We also see that the maps

$$Y^m : V_m \to V_{-m}, X^m : V_{-m} \to V_m$$

are isomorphisms. Finally, in general,

$$(\text{Ker } X) \cap V_k = \text{Ker}\left(Y^{k+1} : V_k \to V_{-k-2}\right).$$

The above defined decomposition is the one mentioned in Proposition 3.16, now established.

3.2.3 The Hodge and Lefschetz decompositions on a compact Kähler manifold

We follow Green [1], and also Griffiths and Harris [20] pp.122-126. Let us return to our global framework, that is, a compact complex Hermitian manifold $M, J, (,)$. We will denote dim $M = n$. The operations L, Λ, H are defined pointwise, and hence may be globalised on M.

Definition Let M be a compact Hermitian complex manifold. The maps

$$L : A^k(M) \to A^{k+2}(M)$$
$$\Lambda : A^k(M) \to A^{k+2}(M)$$
$$H : A^k(M) \to A^k(M)$$

are the global extensions of the pointwise maps already defined on each $T_p(M)$ with the induced Hermitian metric. They satisfy the same commutation relations.

The difficulty that arises when working on an arbitrary Hermitian complex manifold is that neither the decomposition of differential forms

$$A^k(M) = \oplus_{p+q=k} A^{p,q}(M)$$

nor the operators L, Λ, H descend to cohomology. However, at the price of assuming that the metric is Kähler, both the (p,q)-decomposition and the above mentioned $\mathfrak{sl}_2$-representation do descend, and we have a situation incredibly rich in geometric structure.

Definition On a complex manifold M, the operators

$$\partial : A^{p,q}(M) \to A^{p+1,q}(M)$$
$$\overline{\partial} : A^{p,q}(M) \to A^{p,q+1}(M)$$

are uniquely defined by the equation

$$d = \partial + \overline{\partial}.$$

Proposition 3.21. *(Green [1] p.12) We have*

$$\partial^2 = 0$$
$$\partial\overline{\partial} + \overline{\partial}\partial = 0$$
$$\overline{\partial}^2 = 0$$

Proof: Decompose $d^2 = 0$ by type. Its $(p+2,q)$-type part is ∂^2, and its $(p+1,q+1)$ and $(p,q+2)$-type parts are $\partial\overline{\partial} + \overline{\partial}\partial$ and $\overline{\partial}^2$, respectively. The proposition follows.

Next proposition can be found in the reference cited below. We point out that there is a misprint in the cited reference, and we provide a correct statement.

Proposition 3.22. *(Green [1]p.13) ∂ and $\overline{\partial}$ have adjoints ∂^* and $\overline{\partial}^*$ given by*

$$\partial^* = -\overline{*}^{-1}\partial\overline{*} = - *^{-1}\overline{\partial} * = -w * \overline{\partial} *$$
$$\overline{\partial}^* = -\overline{*}^{-1}\overline{\partial}\overline{*} = - *^{-1}\partial * = -w * \partial * .$$

Proof: The proof is a direct consequence of Proposition 3.3 (it follows by decomposing d^* by type).

Definition The *p-th Dolbeault complex* on an arbitrary complex manifold M is $A^{p,\bullet}(M), \overline{\partial}$.

Theorem 3.23. (Dolbeault theorem).*(Griffiths and Harris [20] pp.45-46) For M a complex manifold,*

$$H^q(\Omega^p_M) \cong H^q_{\overline{\partial}}(A^{p,\bullet}(M)).$$

Proof: The complex $(A^{p,\bullet}(M), \overline{\partial})$ is a fine resolution of the sheaf Ω^p_M, by Poincaré 's $\overline{\partial}$-lemma (see [20] p. 38 for a proof of this lemma). See *loc. cit.* for further sheaf-theoretic details on the proof of this theorem.

Definition The *harmonic* (p,q) forms are the following subspace of $A^k(M)$, where $p+q=k$ and M is a (compact) complex manifold.

$$\mathcal{H}^{p,q}(M) = \{\eta \in A^{p,q}(M) | \Delta_{\overline{\partial}}\eta = 0\}.$$

The proof of the following theorem is based on elliptic operator theory, and it is analogous to that of the Hodge theorem - it makes explicit use of the existence of a Green operator for $\Delta_{\overline{\partial}}$. A detailed account of the proof may be found in Griffiths and Harris [20].

Theorem 3.24. *(Green [1] p.13; [20] pp.100) On a compact complex Hermitian manifold, the canonical map*

$$\mathcal{H}^{p,q}(M) \to H^q_{\overline{\partial}}(A^{p,\bullet}(M))$$

is an isomorphism. As a result, the vector spaces $H^q(M, \Omega^p_M)$ are finite-dimensional.

Remark We mention here that the $\bar{*}$-operator gives a complex-conjugate isomorphism between $A^{p,q}(M)$ and $A^{n-p,n-q}(M)$, and therefore such isomorphism restricts to a complex-conjugate isomorphism between $\mathcal{H}^{p,q}(M)$ and $\mathcal{H}^{n-p,n-q}(M)$, and therefore an isomorphism of complex vector spaces exists between $\mathcal{H}^{n-p,n-q}(M)$ and $\mathcal{H}^{p,q}(M)^*$, given by the Hermitian metric $(,)$. A classical result, known as Kodaira-Serre duality, to be stated below, generalises this statement.

We remark that the harmonic theory done so far -including the Dolbeault theorem- can be extended to include vector-valued k-forms, i.e. sections of the bundles $\Omega^p_M \otimes E$, where E is a holomorphic Hermitian vector bundle on M. Thus we obtain

Proposition 3.25. *([20] p.152) Let M be a compact complex Hermitian manifold, and let E be a holomorphic vector bundle endowed with a Hermitian form. Then the Hodge-$*$-operator admits a complex antilinear extension $\bar{*} = \bar{*}_E : A^p(E) \to A^{n-p}(E^*)$. Thus we may define $\Delta_{\bar{\partial}}$ on E. The following isomorphisms hold (as in the case where E is trivial)*

$$\mathcal{H}^{p,q}(E) = H^q_{\bar{\partial}}(A^{p,\bullet}(M) \otimes E) \cong H^q(M, \Omega^p_M \otimes E).$$

Thus the vector spaces $H^q(M, \Omega^p_M \otimes E)$ are finite-dimensional.

As a consequence of the former proposition, we obtain:

Theorem 3.26. (Kodaira-Serre duality). *(Wells [73]) Under the above hypotheses, the $\bar{*}$-operator gives a (complex linear) isomorphism*

$$H^q(M, \Omega^p(E)) \cong H^{n-q}(M, \Omega^{n-p}(E^*))^*.$$

So far, every result obtained in this paragraph has been established in the general context of compact Hermitian complex manifolds. The following theorem reveals the richness of the geometric implications of a manifold being Kähler.

Theorem 3.27. (Hodge-Kähler identities)*(Green [1] p.13; Wells [73] V.3.10) Let M be a compact Kähler manifold. Then*

(1) $[\Lambda, \bar{\partial}] = -i \cdot \partial^*, \quad [\Lambda, \partial] = i \cdot \bar{\partial}^*;$

(2) $[L, \partial^*] = i \cdot \overline{\partial}, \quad [L, \overline{\partial}^*] = -i \cdot \partial;$

(3) $[L, \partial] = 0, \quad [L, \overline{\partial}] = 0;$

(4) $[\Lambda, \partial^*] = 0, \quad [\Lambda, \overline{\partial}^*] = 0,$

(5) $\frac{1}{2}\Delta = \Delta_\partial = \Delta_{\overline{\partial}},$

(6) $[L, \Delta] = 0, \quad [\Lambda, \Delta] = 0, \quad [H, \Delta] = 0.$

Proof: Note that the formulas (1)-(4) depend only on the 1-jet of the coefficients of the Kähler form, and since they may be verified pointwise, the special holomorphic coordinates guaranteed for a Kähler metric (see Proposition 3.12) reduce the theorem to verifying the formulas for $\mathbb{C}^n$ with the flat metric. This is now a lengthy calculation (see [20] pp.111-115). Using *(2)* to write $\partial = i[L, \overline{\partial}^*]$, we obtain the formula

$$\Delta_\partial = i(L\overline{\partial}^*\partial^* - \overline{\partial}^* L\partial + \partial L\overline{\partial}^* - \partial^*\overline{\partial}^* L).$$

Using *(2)* again to substitute for $\overline{\partial}$ gives a similar formula for $\Delta_{\overline{\partial}}$, and the two are seen to be equal using the fact that ∂^*, $\overline{\partial}^*$ anti-commute. So $\Delta_\partial = \Delta_{\overline{\partial}}$. Expanding $d = \partial + \overline{\partial}$ and similarly d^*, we see that

$$\Delta = \Delta_\partial + \Delta_{\overline{\partial}} + \overline{\partial}\partial^* + \partial\overline{\partial}^* + \partial^*\overline{\partial} + \overline{\partial}^*\partial.$$

Again, substituting for ∂ and $\overline{\partial}$ using *(2)*, the last four terms cancel. This proves *(5)*. Finally, writing out Δ in terms of ∂, ∂^* and using *(1)* - *(4)* yields *(6)* (for more details, see [20] pp.115-116).

Notation Let $\pi^{p,q} : A^{p+q}(M) \to A^{p,q}(M)$ denote the canonical projection.

Theorem 3.28. *Let M be a compact Kähler manifold. Then*

$$\mathcal{H}^k(M) = \oplus_{p+q=k}\mathcal{H}^{p,q}(M).$$

Proof: Since $\Delta_{\overline{\partial}}$ commutes with $\pi^{p,q}$ and since the various Laplacians coincide (up to a constant) on a Kähler manifold, this is essentially automatic.

Notation It is traditional to denote $H^q(M, \Omega_M^p)$ by $H^{p,q}(M)$.

Corollary 3.29. (The Hodge Decomposition).

$$H^k(M,\mathbb{C}) = \oplus_{p+q=k} H^{p,q}(M).$$

Proof: Using Dolbeault's theorem and the Hodge Theorem, the corollary follows immediately.

Remark The Hodge-$\bar{*}$-operator acts on the Hodge decomposition of $\mathcal{H}^k(M)$, giving complex conjugate isomorphisms

$$H^{p,q}(M) \to H^{n-p,n-q}(M).$$

This isomorphism coincides with the Kodaira-Serre duality seen in Theorem 3.26.

Proposition 3.30. *(Ueno [70] Cor.9.3) Let M be a compact Kähler manifold and let Y be a compact complex manifold of dimension $\dim Y = \dim M$. Let $f : M \to Y$ be a surjective holomorphic map. Then the Hodge decomposition also holds for Y; in other words,*

$$H^k(Y,\mathbb{C}) = \oplus_{p+q=k} H^q(Y,\Omega^p_Y).$$

Thus every proper algebraic variety X admits a Hodge decomposition on its complex cohomology groups.

Sketch of Proof: One must show that the Hodge-de Rham spectral sequence for Y degenerates at E_1 - for a proof, see Griffiths and Harris [20]. One uses that $f^* : H^\bullet(Y,\mathbb{C}) \to H^\bullet(M,\mathbb{C})$ is injective (see Proposition 4.2) and also that $f^* : H^q(Y,\Omega^p_Y) \to H^q(M,\Omega^p_M)$ is injective; indeed, let $\alpha \in H^q(Y,\Omega^p_Y)$; then $\bar{*}\alpha \in H^q(Y,\Omega^p_Y)$, and $\int_M f^*\alpha \wedge f^*(\bar{*}(\alpha)) = \deg f \quad \int_Y \alpha \wedge \bar{*}\alpha \neq 0$, hence $f^*\alpha \neq 0$. The rest follows from spectral sequence arguments; indeed, all the differentials in the Hodge-de Rham spectral sequence of M vanish, therefore they will also vanish in the Hodge-de Rham spectral sequence for Y. For proper algebraic varieties Y, the proposition follows from Chow's lemma (see Hartshorne [63] Ex.II.4.11.) and Hironaka's theorem on resolution of singularities (see Theorem 5.6).

The (Hodge-)Kähler identities have the wonderful consequence that the operators L, Λ, H commute with Δ and thus take harmonic forms into harmonic forms. We now use this to globalise the concept of "primitive form" exposed in Proposition 3.16, and the Lefschetz decomposition constructed in paragraph 3.2.3, which, in virtue of the Kähler identities and the above mentioned commutation, will descend to cohomology.

Definition (Primitive differential forms).

$$P^k(M) := \{\eta \in A^k(M) | \Lambda\eta = 0\}$$

$$P^{p,q}(M) := \{\eta \in A^{p,q}(M) | \Lambda\eta = 0\}$$

$$\mathcal{P}^k(M) := \{\eta \in \mathcal{H}^k(M) | \Lambda\eta = 0\}$$

As a consequence of paragraph on representation theory of $\mathfrak{sl}_2$, we have the following decomposition for differential k-forms on M

$$A^k(M) = \oplus_{2j \leq k} L^j P^{k-2j}(M)$$

Now, Λ commutes with Δ and, as a result, the Lefschetz decomposition of $A^k(M)$ yields a Lefschetz decomposition for the finite-dimensional subspace of harmonic k-forms

$$\mathcal{H}^k(M) = \oplus L^j \mathcal{P}^{k-2j}(M).$$

We are now ready to prove the following theorem, also known as the **Hard Lefschetz theorem**.

Theorem 3.31. (Lefschetz decomposition for compact Kähler manifolds). *Let M be a compact Kähler manifold. Then*

$$\mathcal{H}^k(M) = \oplus_j L^j \mathcal{P}^{k-2j}(M).$$

As a result, the morphisms

$$L^k : H^{n-k}(M) \to H^{n+k}(M)$$

are isomorphisms for $k \leq n$.

Proof: The theorem follows from the previous lines, by using Hodge theorem on harmonic forms.

Next we quote an amusing result that holds for compact Kähler manifolds, and so for projective manifolds – it turns out that the Betti numbers of a compact Kähler manifold cannot be arbitrary, but are somehow constrained by the Hodge and Lefschetz decompositions. Its proof is straightforward and shall be omitted but for a few comments.

Proposition 3.32. *(Wells [73] Cor. V.4.2, Prop. V.5.3) The Betti numbers b_i of a compact Kähler manifold satisfy the following relations.*

1. $b_k = \sum_{p+q=k} h^{p,q}$
2. $h^{p,q} = h^{q,p}$ *(Hodge symmetry)*
3. b_k *is even for* k *odd.*
4. $h^{1,0} = \frac{1}{2} b_1$ *is a topological invariant.*
5. $b_r - b_{r-2} = dim\ P^r$ *for* $2 \leq r \leq n$.
6. $h^{p,p} > 0$ *for all* $p \leq dim\ M$;
7. *The Betti numbers of* M *admit the following ordering*

$$1 = b_0 \leq b_2 \leq \ldots \leq b_r \text{ for } 2r \leq n;$$
$$b_1 \leq b_3 \leq \ldots \leq b_{2r-1} \text{ for } 2r - 1 \leq n.$$

We make a few remarks on the proof of last proposition. The reason why $h^{p,p} > 0$ is because $\omega^p \in H^{p,p}$; we know that ω^p is not exact by Proposition 3.14 (plus Stokes' theorem), therefore **3.** holds. Also the spaces $H^{p,q}$ and $H^{q,p}$ are known to be conjugate of each other, hence **2.** holds.

Application of Proposition 3.32: As an application of last proposition, we give some examples of compact complex manifolds which do not admit a Kähler structure. A classical example is that of Hopf manifolds of dimension $n > 1$. Let $\alpha_i \in \mathbb{C}, \quad i \leq n$ be complex numbers such that $|\alpha_i| > 1$. We define the group Γ of automorphisms of $\mathbb{C}^n$ as the infinite cyclic group generated by

$$g(z_1, \ldots, z_n) = (\alpha_1 z_1, \ldots, \alpha_n z_n)$$

It is clear that the action of Γ on $\mathbb{C}^n$ is properly discontinuous and free. We define M as the quotient $\mathbb{C}^n/\Gamma$. It can be proven that M is diffeomorphic to $S^1 \times S^{2n-1}$ (for a proof see Kodaira [30] Examples 2.9, 2.15. Thus b_1 is odd for $n > 1$, and thus M does not admit a Kähler structure, by Proposition 3.32.

Definition Let M be a compact Kähler manifold of dimension n with Kähler form ω. Define a hermitian inner product on $H^k_{DR}(M, \mathbb{C})$ by

$$\langle \alpha, \beta \rangle := i^{k^2} \int_M \omega^{n-k} \wedge \alpha \wedge \overline{\beta}.$$

Note that the decomposition $A^k(M) = \oplus_{p+q=k} A^{p,q}(M)$ is orthogonal with respect to $\langle , \rangle$.

The proof of the following proposition is immediate.

Proposition 3.33. *If ω is integral, then $i^{-k^2}\langle , \rangle$ takes integral values on the image of integral cohomology.*

The following result motivates the notion of "polarised Hodge structure", to be defined in Section 3.4.

We define the following bilinear form on $H^k(M, \mathbb{R})$. Let $\xi = \sum L^s \xi_s$ and $\eta = \sum L^s \eta_s$ be two elements of $H^k(M, \mathbb{R})$ (and their Lefschetz decompositions). We define

$$\begin{aligned} Q(\xi, \eta) &:= \sum_s (-1)^{\frac{k(k+1)}{2}+s} \int_M L^{n-k+2s}(\xi_s \wedge \eta_s) \\ &= \sum_s (-1)^{\frac{k(k+1)}{2}+s} \int_M L^{n-k}(L^s \xi_s \wedge L^s \eta_s). \end{aligned}$$

Note that the complex Hermitian forms $(\alpha, \beta) \mapsto \langle , \rangle$ and $(\alpha, \beta) \mapsto Q(\alpha, C\overline{\beta})$ are equal on $L^j P^{k-2j}_{\mathbb{C}}$ up to sign. We can now state the following theorem.

Theorem 3.34. (Hodge-Riemann bilinear relations) *(Green [1] p. 14, Wells [73] Th. V.5.3) Let M be a compact Kähler manifold of dimension n. Then $(-1)^q \langle , \rangle$ is positive definite on $L^j P^{p,q}(M)$, and these spaces are mutually orthogonal. In terms of Q, the above defined bilinear form Q on $H^k(M, \mathbb{R})$ satisfies the following properties:*

(a) $Q(H^{p,q}, H^{r,s}) = 0$ *if* $(p,q) \neq (s,r)$,

(b) $Q(\xi, \eta) = (-1)^k Q(\eta, \xi)$,

(c) $Q(C\xi, C\eta) = Q(\xi, \eta)$;

(d) $Q(\xi, C\eta) = Q(C\xi, \eta)$;

(e) $Q(\xi, C\overline{\xi}) > 0$ *for* $\xi \in H^k(M, \mathbb{C}) - \{0\}$.

Proof: The only non-trivial statement is **(e)**. By Corollary 3.18, it holds for $\xi \in H^{p,q}$. Using **(a)** yields a proof of **(e)** in general.

The following paragraph is based on Griffiths and Harris [20] p.126. We point out that their statement holds only in the projective algebraic setting, and give a detailed account in the following lines.

The topological invariance of the Lefschetz decomposition Note, finally, the distinction between the Hodge and Lefschetz theorems on this section: the Hodge decomposition reflects the analytic structure of the particular manifold M, while the Lefschetz theorems are essentially topological (we shall precise this statement later) . For instance, if we take a real manifold and give two different Kähler complex structures, the Hodge decomposition of H^* may vary – the rank of the groups $(H^{p,q} \oplus H^{q,p}) \cap H^{p+q}(M, \mathbb{Z})$ may even jump – but the Lefschetz isomorphism and decomposition will yield spaces $P^k(M_i)$ of the same dimension; indeed, dim $P^k = b_k - b_{k-2}$. If we consider two given Kähler forms ω_0, ω_1, we may construct a path of Kähler forms $\omega_t = (1-t)\omega_0 + t\omega_1$ (which underlies a path of Kähler metrics). We have Euclidean scalar products Q_t on $H^\bullet(M, \mathbb{R})$, varying differentiably with t, associated to the Kähler structures ω_t, and $P_t^1(M) = H^1(M, \mathbb{R})$,$P_t^k(M) = L(H^{k-2}(M))^\perp \cap H^k(M)$ for $k \geq 2$. Thus the Lefschetz decomposition varies continuously with t.

We now show a situation where the Lefschetz decomposition is forced to be constant. Let us suppose that we have a projective morphism $p : X \to \mathbb{D}$ onto the complex unit ball $\mathbb{D} \subset \mathbb{C}^m$, that is an analytic map factorising through a closed embedding $X \hookrightarrow \mathbb{P}^n \times T$. Then we have an integral 2-form $\Omega = \mathrm{proj}_{\mathbb{P}^n}^*(\omega)$ on $\mathbb{P}^n \times T$

(where ω is the usual Fubini-Study form on $\mathbb{P}^n$) that gives an integral Kähler form Ω_t on $X_t = p^{-1}(t)$. Identifying $H^2(X_t, \mathbb{Q})$ with $H^2(X_0, \mathbb{Q})$ we get that the continuous path Ω_t on $H^2(X_0, \mathbb{Q})$ is necessarily constant, and so is the Lefschetz decomposition. Thus, the Lefschetz decomposition is (locally) constant on deformations of projective manifolds.

3.2.4 The Lefschetz hyperplane theorem and some applications

The following theorem relates the cohomology of a projective variety X to that of a hyperplane section Y. It can be proven for rational coefficients by using the Hard Lefschetz theorem (see Kleiman [28] pp.368-369); another proof may be found in Griffiths and Harris [20] pp.156-7 using Kodaira vanishing. We state this theorem in a stronger form, due to R. Bott. We refer the reader to Bott [7] for an even stronger statement and its proof.

Theorem 3.35. *(Bott [7] Corollary 1) Let X be a projective manifold, and Y be a nonsingular hyperplane section. Then X is obtained from Y by successively attaching cells of (real) dimension $\leq n$. Thus, the homomorphism induced by $j : Y \hookrightarrow X$ in both homotopy and integral homology is onto in dimensions $< n$, and is one-to-one in dimension $< n-1$. As a result, $j^* : H^i(X, \mathbb{Z}) \to H^i(Y, \mathbb{Z})$ is an isomorphism for $i < n-1$ and a monomorphism for $i = n-1$.*

The following corollaries seem to be well known. We include a proof of the first one, for the lack of a suitable reference.

Corollary 3.36. *Every Abelian variety is the Albanese variety of an algebraic surface.*

Proof: Indeed, consider an Abelian variety X of dimension n and its intersection with generic hyperplanes $S = X \cap H_1 \cap \ldots \cap H_{n-2}$ in a fixed projective embedding of X. Then $H_1(X, \mathbb{Z}) = H_1(S, \mathbb{Z})$, thus the inclusion $j : S \hookrightarrow X$ induces an isomorphism on H_1 (and on H^1). Using the standard theory of Abelian varieties, the lemma follows.

Corollary 3.37. *For every projective manifold X, there is a smooth projective surface $S \subset X$ such that $\pi_1(S) = \pi_1(X)$.*

By last corollary, the study of fundamental groups of projective manifolds reduces to that of algebraic surfaces.

Now let us apply the Lefschetz theorems to gather information about the Hodge conjecture. Let $n = \dim X$; then $Hodge(n,p)$ will denote the statement "the Hodge conjecture holds for codimension p cycles of every projective, n-dimensional manifold".

Theorem 3.38. *(Lewis [35] p.727) The Hodge conjecture in every codimension for projective smooth manifolds of any dimension follows from $Hodge(2p,p)$ for every $p \in \mathbb{N}$.*

Proof(Lewis):

It suffices to assume $2p \leq n$. Suppose, for example, that the Hodge conjecture holds for codimension p on a variety X of dimension n, and also that $2p < n$. Then the Hard Lefschetz theorem implies that

$$L^{n-2p} : H^{2p}(X) \to H^{2(n-p)}(X)$$

sends $B^p(X)$ isomorphically to $L^{n-2p}B^p(X) = B^{n-p}(X)$. Thus the Hodge conjecture holds also in codimension $n-p$. Note that the converse does not hold, for the inverse to L^{n-2p} need not be algebraic (this is part of the Grothendieck's standard conjectures, see Kleiman [28] [29]).

It suffices to assume $2p = n$. Let us suppose $2p < n$. Let $Y \subset X$ be the $2p$-dimensional submanifold obtained by cutting out X with $n-2p$ hyperplanes in general position. We assume $Hodge(2p,p)$ holds. Thus $j^* : H^{2p}(X) \hookrightarrow H^{2p}(Y)$; now let $\mu \in B^p(X)$. We will prove that μ is algebraic. We know that $j^*(\mu)$ is algebraic on Y, for $Hodge(2p,p)$ holds. If we consider Y as a general member of an $n-2p$-dimensional family of complete intersection subvarieties covering X and apply Hilbert scheme arguments, then we see that there exists an algebraic class $\xi \in H^{2p}(X)$ such that $j^*(\mu) = j^*(\xi)$. Injectivity shows $\xi = \mu$, therefore μ is algebraic. We have thus established $Hodge(2p,p) \Rightarrow Hodge(n,p)$.

Elementary proof, by the author: We prove the theorem by means of the well-known computation of the cohomology of $X \times \mathbb{P}^r$. Let $\zeta \in B^p(X)$ be a Hodge cycle of codimension p on X. Suppose that $2p > \dim X$. Let r be such that $2p = \dim X + r$. Then the Hodge cycle $\zeta \times \mathbb{P}^r$ belongs to $B^p(X \times \mathbb{P}^r)$, and the dimension of the ambient space is $2p$. Because

$$H^i(X \times \mathbb{P}^r) = \oplus H^{i-2j}(X)(-j),$$

ζ is algebraic on X if and only if $\zeta \times \mathbb{P}^r$ is algebraic on $X \times \mathbb{P}^r$.

Now suppose $2p < \dim X$; let r be such that $2p + r = \dim X$. Define the Hodge cycle $\zeta \times \{\text{point}\}$ on $X \times \mathbb{P}^r$; arguing as in the former case, algebraicity of ζ in X follows from algebraicity of $\zeta \times \{\text{point}\}$ on $X \times \mathbb{P}^r$. Thus we are done.

We now state a general theorem about what is known on the Hodge conjecture for every X. Its proof follows from the proof of last theorem and next Section 3.3.

Theorem 3.39. *Hodge$(n, 1)$ holds for every n (see next Section 3.3); also the statement Hodge$(n, n-1)$ holds for every n; precisely speaking, every Hodge cycle of codimension $n-1$ on an n-dimensional projective variety is an intersection of divisors (see first Proof of Th. 3.38). As a result, every Hodge cycle on a projective manifold of dimension $n \leq 3$ can be obtained as an intersection of divisors. Therefore, Hodge(n, p) holds for $p \leq n \leq 3$, and Hodge$(4, p)$ holds for $p = 0, 1, 3, 4$.*

Thus the first unknown case on the Hodge conjecture is $\dim X = 4, p = 2$. Our case falls into this category.

3.3 The Lefschetz (1,1) theorem

Let X be a projective manifold of dimension n. We are going to prove a classical result by Lefschetz that yields the Hodge conjecture for the Hodge cycles of codimension 1. The main tools will be the description of the Picard group of a complex projective manifold (see [20] Section 1.1) and the (holomorphic) exponential exact sequence of sheaves on X. We follow Griffiths and Harris [20] pp. 163-164.

The theorem is the following.

Theorem 3.40. (Lefschetz theorem on $(1,1)$-classes) *For X a projective manifold, the image of the cycle map $c_1 : Pic(X) \to H^2(X, \mathbb{Z})$ is exactly*

$$Ker\left(H^2(X,\mathbb{Z}) \to H^2(X, \mathcal{O}_X)\right)$$

induced by the inclusion of sheaves $\mathbb{Z}_X \hookrightarrow \mathcal{O}_X$, i.e. "$H^{1,1} \cap H^2(X,\mathbb{Z})$".

We now include a result which will clarify the relationship between algebraic equivalence and topological equivalence of divisors (or equivalently, line bundles). We note, however, that this result no longer holds in general for cycles of codimension.

Theorem 3.41. *Let L_1 and L_2 be two holomorphic line bundles on a projective manifold X. Then L_1 and L_2 are algebraically equivalent if and only if they are topologically (homologically) equivalent, i.e. $c_1(L_1) = c_1(L_2)$. Also a divisor D is numerically equivalent to 0 if and only if an integral multiple mD is algebraically equivalent to 0.*

Proof: (The proof requires vocabulary from the next section). It is clear that algebraic equivalence implies topological equivalence, as seen in section 2.2. For the converse, consider $L = L_1 \otimes L_2^{-1}$. Then it suffices to see that L is algebraically equivalent to $\mathcal{O}_X$. This follows from the existence of a Poincaré line bundle on $X \times \mathrm{Pic}^0 X$, as mentioned in the Remark following Theorem 10.17 on the properties of the Picard variety. For the equivalence between algebraic and numerical equivalence on divisors, we proceed to prove it by using Hard Lefschetz and Theorem 3.40. We know by Theorem 3.40 that $B^1(X)$ consists entirely of divisor classes. By Hard Lefschetz, the choice of an ample class $[H]$ yields an isomorphism of Hodge structures

$$L^{n-1} : H^2(X) \to H^{2n-2}(X)(n-2),$$

and therefore $B^{n-1}(X) = L^{n-1}B^1(X)$. Now, Theorem 3.34 splits $H^2(X)$ into two mutually orthogonal Hodge substructures

$$H^2(X) = B^1(X) \oplus T(X).$$

The fact that $Q = Q_H$ is nondegenerate and $T(X)$ is the orthogonal subspace to $B^1(X)$ yields a nondegenerate pairing

$$Q|B^1(X) : B^1(X) \times B^1(X) \to \mathbb{Q},$$

equal to the following (thus nondegenerate) pairing up to sign (. denotes the cup-product):

$$(\zeta, \eta) \mapsto \zeta . \eta . H^{n-2}.$$

We are just about done. Now $\zeta \in B^1(X)$ is non-zero if and only if there exists $\eta \in B^1(X)$ such that $0 \neq Q(\zeta, \eta) = \zeta . \gamma$, where $\gamma \in B^{n-1}(X)$ is an algebraic class of codimension $n-1$. We have thus proven that $c_1(L)$ is non-torsion if and only if L is numerically non-trivial. The proof is now complete.

Now we proceed to the proof of Lefschetz's theorem on $(1,1)$-classes. Consider the following exact sequence of sheaves on X:

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

whose left arrow is induced by the inclusion $n \mapsto 2\pi\sqrt{-1}n$ and its right arrow is defined locally by $f \mapsto e^{2\pi\sqrt{-1}f}$. Applying the cohomology exact sequence to it, we get the exact sequence

$$H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_{X,hol}) \to H^1(X, \mathcal{O}_{X,hol}^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_{X,hol}).$$

By Serre's GAGA principle [68], the terms involving $\mathcal{O}_{X,hol}$ and $\mathcal{O}_{X,hol}^*$ are equal to their algebraic counterparts. Thus the former exact sequence is equivalent to the following

$$H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$$

Thus we have the following exact sequence

$$Pic^0(X) = \frac{H^1(X, \mathcal{O}_X)}{H^1(X, \mathbb{Z})} \to Pic(X) = H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$$

The second and third arrows have a known interpretation. As seen in detail in Griffiths-Harris [20] p.139 and also p.407, the Chern class map for line bundles coincides with the connection morphism of the exponential exact sequence, which corresponds to the second arrow of last exact sequence of abelian groups. As for the third arrow, we have the following result:

Claim: The arrow $i_* : H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)$ above coincides with the composition of $H^2(\mathbb{Z} \to \mathbb{C})$ and the morphism $H^2(X, \mathbb{C}) \to H^2(X, \mathcal{O}_X)$ arising from the Hodge decomposition theorem.

The following proof may be found in [20] pp.163-164. We want to prove that the diagram

$$\begin{array}{ccc} H^2(X,\mathbb{Z}) & \xrightarrow{i_*} & H^2(X,\mathcal{O}_X) \\ \downarrow & & \downarrow \approx\text{Dolbeault} \\ H^2_{DR}(X,\mathbb{C}) & \xrightarrow{\pi^{0,2}} & H^{2,0}_{\overline{\partial}}(X) \end{array} \tag{3.1}$$

commutes. (The projection $\pi^{0,2}$ is defined on the form level, since for $\omega = \omega^{2,0} + \omega^{1,1} + \omega^{0,2}$, we have $\overline{\partial}\omega^{0,2} = (d\omega)^{0,3} = 0$.) To see check commutativity, let $z = (z_{\alpha\beta\gamma}) \in Z^2(X, \mathbb{Z})$; to find the image of z under the de Rham isomorphism, we take $f_{\alpha\beta} \in A^0(U_\alpha \cap U_\beta)$ such that

$$z_{\alpha\beta\gamma} = f_{\alpha\beta} + f_{\beta\gamma} - f_{\alpha\gamma} \text{ in } U_\alpha \cap U_\beta \cap U_\gamma$$

Since $z_{\alpha\beta\gamma}$ is constant,on differentiating we get

$$0 = df_{\alpha\beta} + df_{\beta\gamma} - df_{\alpha\gamma},$$

so $(df_{\alpha\beta}) \in Z^2(X, \mathcal{A}^1(X))$ and we can find $\omega_\alpha \in A^1(U_\alpha \cap U_\beta)$ such that

$$df_{\alpha\beta} = \omega_\alpha - \omega_\beta \text{ in } U_\alpha \cap U_\beta.$$

The global 2-form $d\omega_\alpha = d\omega_\beta$ then represents the image of z in $H^2_{DR}(X, \mathbb{C})$. On the other hand, take the image of $i_*(z)$ under the Dolbeault isomorphism: we write

$$z_{\alpha\beta\gamma} = f_{\alpha\beta} + f_{\beta\gamma} - f_{\alpha\gamma},$$

$$\overline{\partial} f_{\alpha\beta} = \omega_\alpha^{0,1} - \omega_\beta^{0,1},$$

and we see that $\overline{\partial}\omega_\alpha = (d\omega_\alpha)^{0,2}$ represents z in $H^2_{\overline{\partial}}(X)$. Now we are just about done: given $\gamma \in H^{1,1}(X) \cap H^2(X, \mathbb{Z})$, we have $i_*(\gamma) = 0$, and hence $\gamma = c_1(L)$ is the

Chern class of some line bundle $L \in H^1(X, \mathcal{O}^*)$ (recall that c_1 was defined as the connection morphism). It suffices to recall the connection between divisor classes and equivalence classes of line bundles: writing $L = [D]$ for some divisor D in X, we finally get $\gamma = c_1([D])$.

Alternative proof of Claim: We notice that the inclusion of sheaves $\mathbb{C}_X \hookrightarrow \mathcal{O}_X$ yields surjective maps $H^q(X, \mathbb{C}) \to H^q(X, \mathcal{O}_X)$ – they actually come from the Hodge decomposition of $H^q(X, \mathbb{C})$. We prove this by recurring to the Hodge-de Rham spectral sequence; the reference we followed is the section *(3.5.)* on Griffiths and Harris' book [20] entitled "Spectral sequences and applications". The following complexes of sheaves on X are quasi-isomorphic (the arrow $\mathbb{C} \to \mathcal{O}_X$ is the usual inclusion)

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathbb{C}_X & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \Omega^1_X & \longrightarrow & \Omega^2_X & \longrightarrow & \Omega^3_X & \longrightarrow & \dots \end{array} \tag{3.2}$$

The second complex above is acyclic by Poincaré $\overline{\partial}$-lemma, and the first row complex is obviously acyclic. Hence, the vertical rows yield a quasiisomorphism of complexes, which in turn induce an isomorphism on hypercohomology (again, see Griffiths-Harris [20] p.448). We recall that a Dolbeault resolution by fine sheaves exists for the second row complex (the *Dolbeault complex*, formed by the sheaves $\mathcal{A}^{p,q}(M)$ of differential forms of type (p, q)). The degeneration of the Hodge-de Rham spectral sequence yields an isomorphism

$$H^k(X, \mathbb{C}) \simeq \mathbb{H}^k(\Omega^\bullet_X) \simeq \oplus_{p+q=k} H^q(X, \Omega^p_X)$$

and the projection onto the first factor $H^k(X, \mathcal{O}_X)$ does come from the inclusion $\mathbb{C} \hookrightarrow \mathcal{O}_X$.

3.4 Generalities on Hodge structures

In Section 3.2 we reviewed the *Hodge decomposition* of $H^i(M, \mathbb{Q})$ for M a compact Kähler manifold. We then view the rational cohomology groups, or even the integral cohomology groups modulo torsion (resp. real cohomology groups) of a projective manifold (resp. compact Kähler manifold) as objects being endowed with an extra structure which depends upon the complex structure of our manifold M. This reflects our algebro-geometric viewpoint more faithfully than merely considering topological invariants. For instance, an important case takes place when $M = A$ is an Abelian variety, for in this case the Hodge decomposition of $H^1(A, \mathbb{Z})$ determines the complex structure on A – this will be exposed in detail in section 10.1.

We can observe, however, that the mere description of an Abelian variety by means of a Hodge decomposition for H^1 does not guarantee the projectiveness of A, but describes A only as a complex torus. Every complex torus can be described up to biholomorphism by means of these data. The concept of *polarisation* of a Hodge structure will lead us to know precisely when a complex torus is projective, i.e. an Abelian variety; this property is a particular case of the Hodge-Riemann bilinear relations. With this notion, every projective manifold M will admit a polarisation on its rational (or integral, modulo torsion) cohomology groups. Thus the category of objects of our interest will not be the full category of (finite-dimensional) Hodge structures, but its subcategory consisting of polarisable (rational) Hodge structures. This subcategory will turn out to be semisimple Abelian, as we will see in Theorem 3.48.

Definition A *(rational) Hodge structure of weight* n is a finite dimensional vector space V over $\mathbb{Q}$ equipped with one of the following equivalent data:

i. a decomposition of complex vector spaces

$$V \otimes \mathbb{C} = \oplus_{p+q=n} U^{p,q}$$

such that $U^{p,q} = \overline{U^{q,p}}$.

ii. a representation of real algebraic groups

$$h : S = Res_{\mathbb{C}|\mathbb{R}} \mathbb{G}_m \to GL(V_{\mathbb{R}})$$

such that $(h|\mathbb{R})(t) = t^n$ – notice that $S(\mathbb{R}) = \mathbb{C}^*$.

Remark: For a subring $A \subset \mathbb{R}$, we can introduce the notion of A-Hodge structure, morphism, Tate structure and polarisation. V is then a free A-module of finite rank (for details, see Deligne [8]). In this thesis, all the Hodge structures are considered to be rational, unless otherwise stated.

Example: Let M be a compact Kähler manifold. Then the Hodge decomposition for $H^k(M)$ (see Corollary 3.29) yields a canonical Hodge structure for $H^k(M) = H^k(M, \mathbb{Q})$ of weight k. Also, the torsion-free part of H^k, $H^k(M, \mathbb{Z})/tors$, is thus equipped with an integral Hodge structure of weight k.

Definition The *Tate Hodge structure* $\mathbb{Q}(n)$ is the unique Hodge structure of weight $-2n$ and dimension 1. More precisely, $\mathbb{Q}(n)$ is uniquely determined by the representation $h : S \to GL_1(\mathbb{R})$ given by $h(z) = (z\bar{z})^{-n}$.

Remark: A slightly different definition of the Tate Hodge structure can be found in Deligne [8] - $\mathbb{Q}(n)_{\mathbb{Q}} = (2\pi i)^n\mathbb{Q} \subset \mathbb{C}$. Our notation agrees with that of Deligne et al. [9] p.42 and van Geemen [17].

Definition A *morphism of Hodge structures* V,W is a linear map $f : V \to W$ equivariant with respect to the decomposition given above, or shorter, a $\mathbb{Q}$-defined linear map f such that $f_{\mathbb{R}}$is S-equivariant. The vector space of *Hodge morphisms*, or morphisms of Hodge structures from V to W, shall be denoted by $Hom_{Hodge}(V, W)$.

Definition Let V be a Hodge structure of weight n. We define the subspace of *Hodge cycles* of V as

$$B(V) = V_{\mathbb{Q}} \cap V^{p,p} \text{ if } 2p = n.$$

If n is odd, we define $B(V) = 0$.

Remark

1. The definitions given correspond to pure Hodge structures. Throughout this thesis, only pure Hodge structures are considered. Also, the above defined morphisms of Hodge structures are sometimes called *strict*, and the term "morphism" (in the broad sense) stands for strict morphisms of the type $V \to W(n)$ for some $n \in \mathbb{Z}$.

2. We can give an alternative description of the space of morphisms of Hodge structures: $Hom_{Hodge}(V, W) = B(V^* \otimes W)$ if the weights of V and W coincide: if not, then $B(V^* \otimes W)$ will coincide with the space of (non-strict) Hodge morphisms, as defined above.

3. Suppose that V is an integral Hodge structure of even weight $n = 2m$. Then we can define the submodule *integral Hodge cycles* of V (*of codimension* m) as the primitive submodule $B^m_{\mathbb{Z}}(V) = V \cap V^{m,m}$ (the subscript will not always be written when the coefficient ring is known to be $\mathbb{Z}$). The equivalence $B_{\mathbb{Z}}(V^* \otimes W) = Hom_{\text{Hodge},\mathbb{Z}}(V, W)$ holds also for integral Hodge structures.

Example: Let X be a projective manifold, and let D be a fixed cohomology class of an ample divisor (i.e. a rational Kähler class in $H^2(X)$. Let L, Λ, H be the operations on cohomology defined in Section 3.2. Then L, Λ and H define morphisms of Hodge structures of respective bidegrees $(1,1), (-1,-1)$ and $(0,0)$, i.e. $LH^{p,q} \subset H^{p+1,q+1}$ and so on. This is equivalent to say that the operators L, Λ, H define Hodge morphisms (in the broad sense) which are strict when contemplated as $H^\bullet(X) \to H^\bullet(X)(1)$, and so on.

If M is a general compact Kähler manifold, the observations above hold when considering real coefficients on cohomology - if a Kähler form exists on $H^2(M, \mathbb{Q})$, then M is Kähler projective, as stated in Theorem 3.11.

Definition The *Weil operator* C on a Hodge structure V, h of weight n is defined by $C := h(i)$. In other words, C is the real operator $C : V_{\mathbb{R}} \to V_{\mathbb{R}}$ defined by $C|V^{p,q} = i^{p-q} id_{V^{p,q}}$.

Definition A *polarisation* of a Hodge structure V is a morphism of Hodge structures

$$\psi : V \otimes V \to \mathbb{Q}(-n)$$

such that the following bilinear form on $V_{\mathbb{R}}$ is symmetric and positive definite:

$$\psi(v, Cw) = \psi(v, h(i)w).$$

The precise statement for ψ to be a morphism of Hodge structures is the following: ψ is a $\mathbb{Q}$-bilinear morphism $V \times V \to \mathbb{Q}$ satisfying

$$\psi\left(h(z)v, h(z)w\right) = (z\bar{z})^n \, \psi(v, w)$$

Remark The following properties hold for a polarised Hodge structure (V, ψ):

1. ψ is symmetric if n is even, and alternating if n is odd.

2. $\psi_{\mathbb{C}}(V^{p,q}, V^{r,s}) = 0$ if $(p,q) \neq (s,r)$. In particular, we can decompose $V_{\mathbb{R}} = \oplus V_p$ where $V_p \otimes_{\mathbb{R}} \mathbb{C} = V^{p,q} \oplus V^{q,p}$, and the former is an orthogonal decomposition with respect to ψ.

3. If $n = 2l$ is even, then the induced quadratic form Q on $V_{\mathbb{R}}$ satisfies: $(-1)^{l-p}Q > 0$ on V_p.

Example: We now give one of the most important examples of polarised Hodge structures. Let X be a projective manifold, and L, Λ, H be the Lefschetz operations on $H^\bullet(X)$ after fixing an ample class. Then the Lefschetz decomposition

$$\mathcal{H}^k(X) = \oplus_j L^j \mathcal{P}^{k-2j}(X)$$

(see Theorem 3.31) is defined over $H^k(X) = H^k(X, \mathbb{Q})$ (with rational coefficients), for so are L, Λ, H. The subspaces $P^k(X) = P^k(X, \mathbb{Q}) \subset H^k(X)$ are also (rational) Hodge substructures - indeed, $P^k(X) = (\text{Ker } \Lambda) \cap H^k(X)$; the rest follows from this and the above examples. Also, the bilinear form Q appearing in Theorem 3.34 is $\mathbb{Q}$-defined and is a polarisation of $H^k(X, \mathbb{Q})$. Indeed, the quadratic form $Q(\,.\,, C.\,)$ on $H^k(X, \mathbb{R})$ is positive definite, as is the Hermitian form $(\alpha, \beta) \mapsto Q(\alpha, C\overline{\beta})$, as seen in Theorem 3.34.

3.5 The Hodge and Mumford-Tate groups

Let V be a rational polarisable Hodge structure. There are two algebraic groups, the *Hodge and Mumford-Tate groups* (defined over $\mathbb{Q}$) that we can naturally attach to V. These algebraic groups will turn out to be reductive, which -as mentioned earlier- will give us semisimplicity of any polarisable (rational) Hodge structure - see Theorem 3.48. The reductiveness of these groups is of paramount importance in the computation of Hodge cycles on Abelian varieties, for the groups $MT(H^1(A))$ and $Hg(H^1(A))$ act naturally on the cohomology ring of A and the Hodge cycles are the $Hg(A)$- invariants of $H^\bullet(A)$. This will yield a quite explicit computation of the ring of Hodge cycles of A - for instance, when the ring of invariants is generated by degree 2 invariants, we obtain that the ring of Hodge cycles is generated by divisor classes and therefore the Hodge conjecture holds in this case. In general, though, it is not always possible to determine whether a system of generators of $H^{\bullet}(A)^{\mathbb{Hg}(A)}$ consists of algebraic classes or not; therefore, the Hodge conjecture does not directly follow for every Abelian variety A. The main references in this subsection will be B.B. Gordon [35] Appendix B, van Geemen [17], and Deligne's lecture notes in [9], I.3.

3.5.1 An interlude on algebraic groups

For this paragraph, we follow mainly Humphreys [27] Chap. 8, and also Waterhouse [72] Chap. 3. The first propositions are general, and we will suppose k to be an algebraically closed field, although this hypothesis may be superfluous – for the more general results, proven in the language of Hopf algebras and comodules, we refer the reader to [72], Chapters 1 and 3, which will be cited as needed. Let G be an affine algebraic group whose ring of regular functions is $k[G]$. Suppose that X is an affine variety endowed with a G-action

$$\varphi : G \times X \to X.$$

This means that φ is a morphism of algebraic varieties such that, for every $g \in G$, $\varphi(g, .)$ is an automorphism of X and the map

$$g \mapsto \varphi(g, .)$$

is a group homomorphism (for a more general definition in terms of Hopf algebras, see Waterhouse [72], Chapter 1).

We have the following natural actions of G on functions. For $g \in G$, we define $(\tau_g f)(x) := f(g^{-1}x)$ for every $f \in k[X]$. This gives a G-action on $k[X]$. We have also the *left and right regular representations* on $k[G]$:

$$(L_g f)(x) := f(g^{-1}x), \qquad (R_g f)(x) := f(xg) \qquad \text{for every } f \in k[G].$$

We begin with the following observation on closed subgroups H of G.

Lemma 3.42. *(Humphreys [27] Lemma 8.5) Let H be a closed subgroup of G and I be the ideal of H in G. Then H has the following characterisation:*

$$H = \{g \in G | R_g(I) \subset I\}.$$

The statement holds also if one replaces R_g by L_g.

The two following propositions provide a full description of all the representations of G.

Proposition 3.43. *(Humphreys [27] Prop. 8.6) Let G act on X, in the above hypotheses, and let F be a finite-dimensional subspace of $k[X]$. The following assertions hold:*

(a) There exists a finite-dimensional subspace E such that

$$F \subset E \subset k[X]$$

and such that E is G-invariant under the representation τ.

(b) F is G-stable if and only if $\varphi^* F \subset k[G] \otimes F$.

Theorem 3.44. *(Humphreys [27] Th.8.6, Waterhouse [72] Th. 3.4) An affine algebraic group G over an arbitrary field k is linear, i.e., admits a closed group embedding*

$$G \hookrightarrow GL_{n,k}.$$

Proof: We prove the theorem for algebraically closed fields only – for the general case, see [72].

The proof goes as follows. We will find such a linear embedding as a suitable subrepresentation of the regular representation $k[G]$. Indeed, choose a system of generators s_i of the k-algebra $k[G]$, and choose $F = \langle s_1, \dots, s_p \rangle$. Then, by Proposition 3.43(a), we can take a finite-dimensional representation E containing F which is invariant by the right regular representation $g \mapsto R_g$. We now prove that E gives the desired representation $\rho : G \hookrightarrow GL(E)$. The explicit description of ρ goes as follows. Let $(f_i)_{i \leq n}$ be a basis of E. Thanks to Proposition 3.43(b), there are functions $m_{ij} \in k[G]$ such that

$$\varphi^* f_i = \sum_j m_{ij} f_j.$$

Thus $(R_x f)(y) = f(yx) = \sum m_{ij}(x) f_j(y)$, and then the matrix

$$\rho(x) := \begin{pmatrix} \ddots & & \\ & m_{ij}(x) & \\ & & \ddots \end{pmatrix}$$

gives a morphism of algebraic groups $G \hookrightarrow GL_n$. The corresponding morphism of regular function rings

$$\rho^* : k[\{X_{ij}\}, \frac{1}{det(X_{ij})}] \to k[G]$$

is seen to be surjective because $f_i(x) = \sum m_{ij}(x) f_j(1)$ generate the algebra $k[G]$ by construction, hence m_{ij} generate $k[G]$. This concludes the proof – in general, ρ^* is surjective if and only if the elements $(m_{ij})_{i,j}, \frac{1}{\det(m_{ij})}$ generate $k[G]$ over k. In our case, m_{ij} suffice.

We now proceed to describe all the linear representations of an affine algebraic group as submodules of tensor representations of a given faithful G-module V.

Lemma 3.45. *(Waterhouse [72] Lemma 3.4) Let G be an affine group scheme over a field k. Every finite-dimensional linear representation of G embeds in a finite sum of copies of the regular representation.*

Proof: Let V be a (finite-dimensional) linear representation of G, and let F be the dual space of V, which has a structure of $k[G]$-"comodule" – in the language of [27], this is equivalent to say that it is a G-equivariant subspace of $k[X]$. The G-representation induces (or equivalently, the comodule structure is equivalent to) the following morphism of G-representations (or comodules)

$$\phi^* F \subset k[G] \otimes_k F.$$

Now, $k[G] \otimes_k F$ is isomorphic to $k[G] \otimes_k F_0 \cong k[G]^{\dim_k F}$ as a G-representation, F_0 being F with the trivial representation. This proves the lemma.

Theorem 3.46. *(Waterhouse [72] Th.3.4) Let k be a field, G a closed subgroup of* $\mathbf{GL_n}$*. Every finite-dimensional representation of G can be constructed from its original representation on k^n by the processes of forming tensor products, direct sums, subrepresentations, quotients, and duals.*

Proof: Let $A := k[G]$ and $B := k[GL_n]$. By the lemma it is enough to construct all the finite-dimensional V in A^m. Such a V is a "subcomodule" of the direct sum of its coordinate projections to A, so we may deal just with V in A. The original representation gives us a Hopf algebra surjection of $B = k[\{X_{ij}\}, 1/\det]$ onto A, and V is contained into the image of some subspace $(1/\det)^r\{f(X_{ij})|\deg(f) \leq s\}$ for some $r, s \in \mathbb{N}$. These subspaces are B-subcomodules of B, and thence also are A-subcomodules; it will be enough to construct them. Let v_j be the standard basis of k^n. The representation of GL_n has B-comodule structure $\rho(v_j) = \sum v_i \otimes X_{ij}$. For each i the map $v_j \mapsto X_{ij}$ is a comodule map to B. Thus the polynomials X_{ij} homogeneous of degree one are as a comodule the sum of n copies of the original representation. We can construct $\{f|$ homogeneous of degree $s\}$ as a quotient of the s-tensor product of $\{f|$ homogeneous of degree $1\}$ (more precisely, its s-fold symmetric power). For $s = n$ this space contains the one-dimensional representation $g \mapsto \det(g)$. From that we can construct its dual $g \mapsto 1/\det(g)$. Summing the homogeneous pieces we get $\{f|\deg(f) \leq s\}$, and tensoring r times with $1/\det(g)$ gives all we need. (We point out that dualisation was used here only to construct $1/\det(g)$.)

Reductive groups.- (Mumford [47]pp.24-27) Let G be an algebraic group of characteristic zero. We say that G is **reductive** if and only if its radical is equal to its centre. In this case -i.e. in characteristic zero- Nagata has shown that reductiveness is equivalent to the following condition: every linear representation V of G decomposes into a sum of irreducible representations.

Let G be a reductive group over a field k of characteristic zero, and let $(V_\alpha)_{\alpha\in A}$ be a faithful family of representations over k, so that the map $G \hookrightarrow \prod GL(V_\alpha)$ is injective. For any $m, n \in \mathbb{N}^{(A)}$ (i.e. only a finite number of indexes are nonzero) we can form the representation $T^{m,n} = \otimes V_\alpha^{m(\alpha)} \otimes (V_\alpha^*)^{n(\alpha)}$. For a closed subgroup H, we write H' for the subgroup of G fixing all tensors, occuring in some $T^{m,n}$, that are fixed by H. Clearly $H \subset H'$, and we shall need criteria guaranteeing their equality. The expression $X_k(G)$ will denote the group of characters $Hom(G, \mathbb{G}_m)$.

Proposition 3.47. *(Deligne et. al. [9] Part I, Prop.3.1) The notations are as above.*

(a) Any finite-dimensional representation of G is contained in a direct sum of representations $T^{m,n}$.

(b) (Chevalley's theorem). Any subgroup H of G is the stabiliser of a line L in some finite-dimensional representation of G.

(c) If H is reductive, or if $X_k(G) \to X_k(H)$ is surjective (or has finite cokernel), then $H = H'$.

Proof:

(a) G is noetherian, hence there exists a finite index subset $A' \subset A$ such that the representation $V := \oplus_{\alpha \in A'} V_\alpha$ is faithful. The rest follows from Theorem 3.46.

(b) Let I be the ideal of H in G. Then, as seen in Lemma 3.45, H is the stabiliser of I in the regular representation of G on $k[G]$. Choose a finite-dimensional subspace W of $k[G]$ that is G-stable and contains a generating set for the ideal I. Then H is the stabiliser of the subspace $I \cap W$ of W, and of $\bigwedge^d (I \cap W)$ in $\bigwedge^d W$, where d is the dimension of $I \cap W$ (see Borel [5] 5.1).

(c) According to (b), H is the stabiliser of a line L in some representation V of G and it follows from (a) that V can be taken to be a direct sum of $T^{m,n}$ s. Assume that H is reductive. Then $V = V' \oplus L$ for some H-stable V' and $V^* = V'^* \oplus L^*$. Thus H is the group fixing a generator of $L \otimes L^*$ in the G-module $V \otimes V^*$. Assume that $X_k(G) \to X_k(H)$ is surjective, i.e. that any character of H extends to a character of G. The one-dimensional representation of H on D can be regarded as the restriction to H of a one-dimensional representation of G. Now H is the group fixing a generator of $L \otimes L^*$ in the G-module $V \otimes L^*$. An analogous argument works if $X_k(G) \to X_k(H)$ has finite cokernel - in that case, there is a natural number q such that the one-dimensional representation on $L^{\otimes q}$ extends to a character of G, so one should consider the G-module $\otimes^q V$ and proceed as in the former case. We remark that, in the proofs of (b) and (c), it was necessary to tensor V with V^* or L^* in order to have the group G acting trivially on a line $(L \otimes L^*)$.

Remark ([9]I.3.2(a)) It is clearly necessary to have some condition on H in order to have $H' = H$. We give a nice counterexample now: let B be a Borel subgroup (i.e. a maximal connected solvable subgroup) of a reductive group G and let $v \in V$ be fixed by B. Then $g \mapsto g \cdot v$ defines a map of algebraic varieties $k : G/B \to V$, for G/B is a projective algebraic manifold. Hence, k is constant and thus v is fixed by G, and $B' = G$ (for details see Borel [5], 10.4, 10.5, 11.1, 11.2). However, the above argument shows the following (under the above hypotheses):let H' be the group fixing all tensors occurring in subquotients of $T^{m,n}$ s that are fixed by H; then $H = H'$.

3.6 The groups $MT(V)$ and $Hg(V)$ for general V

Definition Let V be a Hodge structure of weight k, and h as in 3.4. The following algebraic groups over $\mathbb{Q}$ are defined:

1. The *Mumford-Tate group* of V, $MT(V)$ is the least algebraic subgroup defined over $\mathbb{Q}$ such that $h(\mathbb{C}^*) \subset MT(V)_{\mathbb{R}}$

2. The *Hodge group* of V, $Hg(V)$ is the least algebraic subgroup defined over $\mathbb{Q}$ such that $h(U(1)) \subset Hg(V)_{\mathbb{R}}$

Remark Our definition follows van Geemen [17] and B.B. Gordon [35], and is slightly different from the one in Deligne, Milne et. al [9] p.43. However, they turn out to be equivalent (see Deligne et al. [9] I. Prop. 3.4).

Remark 1. It is clear that $MT(V)$ and $Hg(V)$ are connected algebraic groups, and that $MT(V) = \mathbb{G}_m Hg(V)$.

2. It is clear that if there exists a polarisation ψ for V, then: $MT(V) \subset GSp(V, \psi)$, $Hg(V) \subset Sp(V, \psi)$ in case k is odd. The following inclusions hold, $MT(V) \subset O(V, \psi)$ and $Hg(V) \subset SO(V, \psi)$ in case k is even.

Theorem 3.48. *Given a polarisable Hodge structure V of weight n, the groups $MT(V)$ and $Hg(V)$ are reductive. $Hg(V)$ can then be defined as the biggest algebraic subgroup of $GL(V)$ that leaves $B\left(V^{\otimes m} \otimes (V^*)^{\otimes s}\right)$ invariant.*

Proof: Write $C = h(i)$ as usual. For $v^{p,q} \in V^{p,q}, Cv^{p,q} = i^{p-q}v^{p,q}$, and so C^2 acts as $(-1)^n$.

We choose a polarisation ψ for V. Recall that ψ is a morphism $\psi : V \otimes V \to \mathbb{Q}(-n)$ such that the real-valued form $\psi(x, Cy)$ on $V_{\mathbb{R}}$ is symmetric a positive definite. Under the canonical isomorphism $\mathrm{Hom}(V \otimes V, \mathbb{Q}(-n)) \cong V^* \otimes V^*(-n)$, ψ corresponds to a (rational) tensor of bidegree $(0,0)$, which is fixed by $h(\mathbb{C}^*)$ and therefore fixed by $MT(V)$.

Recall that if H is a real algebraic group and σ is an involution of $H_{\mathbb{C}}$, then the *real form* of H defined by σ is a real algebraic group H_σ together with an isomorphism $H_{\mathbb{C}} \cong (H_\sigma)_{\mathbb{C}}$ under which complex conjugation on $H(\mathbb{C})$ corresponds to $\sigma \circ$ (complex conjugation) on $H_\sigma(\mathbb{C})$. We are going to use the following criterion (see Borel and Wallach [6] Chapter II): a connected algebraic group H over $\mathbb{R}$ is reductive if it has a compact real form H_σ. To prove the criterion it suffices to show that H_σ is reductive. On any finite-dimensional representation V of H there is an H_σ-invariant positive-definite symmetric form, namely $\langle u, v\rangle_0 = \int_H \langle u, v\rangle dh$, where $\langle , \rangle$ is any positive-definite symmetric form on V. If W is an H_σ-stable subspace of V, then its orthogonal complement is also H_σ-stable. Thus every finite-dimensional representation of H_σ is semisimple, and this implies that H_σ is reductive (see above note on reductive groups).

We apply the criterion to the Hodge group $Hg(V)$, which is a connected algebraic group. Since $C = h(i)$ acts as 1 on $\mathbb{Q}(-1)$, $C \in Hg(V)(\mathbb{R})$. Its square C^2 acs as $(-1)^n$ on V and therefore lies in the centre of $Hg(V)(\mathbb{R})$. The inner automorphism ad C of $MT(V)_{\mathbb{R}}$ defined by C is therefore an involution. For $u, v \in V_{\mathbb{C}}$ and $g \in Hg(V)(\mathbb{C})$ we have

$$\psi(u, C\overline{v}) = \psi(gu, gC\overline{v}) = \psi(gu, CC^{-1}gC\overline{v}) = \psi(gu, C\overline{g^*v})$$

where $g^* = C^{-1}\overline{g}C = (\text{ad } C)(\overline{g})$. Thus the positive-definite form $\phi(u, v) := \psi(u, Cv)$ on $V_{\mathbb{R}}$ is invariant under the real form of $Hg(V)$ defined by ad C, and so the real form is compact.

As a result, the Mumford-Tate group $MT(V) = \mathbb{G}_m Hg(V)$ is also reductive. We can now give a necessary and sufficient condition for the semisimplicity of $Hg(V)$.

Corollary 3.49. *If V is a polarisable nontrivial rational Hodge structure, $Hg(V)$ is semisimple if and only if the center of $MT(V)$ is $\mathbb{G}_m$, i.e. consists only of scalars.*

Proof: It follows immediately from the definition of both groups that $Z(MT(V)) = \mathbb{G}_m Z(Hg(V))$. Because both are reductive (i.e. their radical coincides with their centre), $Hg(V)$ will be semisimple if and only if its centre (i.e. its radical) is zero.

Corollary 3.50. *The category of polarisable Hodge structures is semisimple Abelian, i.e. for every such Hodge structure the algebra $End_{Hodge}(W)$ is a semisimple $\mathbb{Q}$-algebra.*

Proof: We make an essential use of the reductiveness of $MT(V)$ and $Hg(V)$. Let V be a polarisable Hodge structure of weight n, and let $G := Hg(V)$ be its Hodge group. Then, by Proposition 3.51, the Hodge substructures of V correspond to the subrepresentations of G. Now, G is a reductive algebraic group over $\mathbb{Q}$, which means that every finite-dimensional representation of G splits into a sum of irreducible subrepresentations.

Proposition 3.51. *Let V be a polarisable rational Hodge structure of weight n. Then:*

1. *the Hodge substructures of $V^{\otimes r} \otimes (V^*)^{\otimes s}$ are precisely the rational subrepresentations of $V^{\otimes r} \otimes (V^*)^{\otimes s}$ by $Hg(V)$ (or equivalently, by $MT(V)$).*

2. *for any r, s such that $(r - s)n$ is even,*

$$B\left(V^{\otimes r} \otimes (V^*)^{\otimes s}\right) = \left(V^{\otimes r} \otimes (V^*)^{\otimes s}\right)^{Hg(V)}$$

Proof: The proof is as follows. A reductive group $G \subset GL(V)$ for V a finite-dimensional vector space is characterised by its tensor invariants. Thus, $Hg(V)$ is characterised by the tensor invariants of V. For the corresponding proof in the case of $MT(V)$, we should view it as a subgroup of $GL(V) \times \mathbb{G}_m$ (see Deligne et al. [9] pp.43-45), so that the semiinvariants of $MT(V)$ will be exactly the invariants of its homomorphic image in $GL(V) \times \mathbb{G}_m$.

Chapter 4

Algebraic and Homological correspondences

The main references in this section are Kleiman [28] [29], Fulton [15], Scholl [54]. Note that, in some of the references, the varieties involved are (for instance, in Scholl [54]) allowed to be non-irreducible (for instance, see Scholl [54]) but, for the sake of simplicity, equidimensional. The notations used are common in the literature; when working on $X \times Y$, p_X will denote the projection onto the factor X, and so on. Also, when we work over $k = \mathbb{C}$ we consider the usual rational singular cohomology groups, understood as rational Hodge structures.

Let X, Y be projective smooth varieties over a field k. Let $H^\bullet$ denote any Weil cohomology theory on the category of projective smooth varieties over k; among other properties, the cohomology groups $H^i(M)$ are finite-dimensional vector spaces over a field K of characteristic zero, satisfy the usual functoriality properties and the Künneth formula, and admit a cycle map $c_{M,i} : CH^i(M) \to H^{2i}(M)$ with its usual features. There are morphisms $f^* : H^\bullet(Y) \to H^\bullet(X)$ and $f_* : H^\bullet(X) \to H^\bullet(Y)$ attached to a morphism $f : X \to Y$. We point out that f_* is deduced from f^* by Poincaré duality. Analogous morphisms f^* and f_* are also defined on Chow groups; the above mentioned morphisms are expected to satisfy concomitance with the cycle maps c_X, c_Y

$$\begin{array}{ccc} CH^i(X) & \xrightarrow{c_X} & H^{2i}(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ CH^{i+\dim X-\dim Y}(Y) & \xrightarrow{c_Y} & H^{2(i+\dim X-\dim Y)}(Y) \end{array} \tag{4.1}$$

In the case when $k = \mathbb{C}$, $H^\bullet = H^\bullet_{DR}$ there is no general way of conceiving the functor $f \to f_*$ in terms of f by means of a morphism $f_* : A^i(X) \to A^j(Y)$ on differential forms; only in special cases such as finite Galois covering spaces $X \to X/G$ (in which $f_* = \sum_{g\in G} g^*$), or in smooth fibrations, where it comes through "integration along the fibres". However, the operator f_* is well-behaved with respect to the underlying Hodge structures, for it is simply -up to Tate twists- the dual map of the Hodge morphism f^*. We are going to generalise the notions f^*, f_* and to prove the commutativity of last diagram, which will come as an easy consequence of elementary properties of correspondences. We are going to generalise the former notions with the definition of **correspondence**.

First we recall what a Weil cohomology is.

4.1 Weil cohomology

The main reference is Kleiman [28], 1.2. Fix a field K of characteristic zero, to be called the *coefficient field*. A contravariant functor $X \mapsto H^\bullet(X)$ from varieties to augmented, finite-dimensional, graded, anticommutative K-algebras is said to be a *Weil cohomology* if it satisfies the following properties:

A. *Poincaré duality.*

(i) The groups $H^i(X)$ are zero unless $i \in [0, 2n]$.

(ii) There is given an "orientation" isomorphism $H^{2n}(X) \simeq K$, where $n = \dim X$.

(iii) The canonical pairings

$$H^i(X) \otimes H^{2n-i}(X) \to H^{2n}(X)$$

are non-singular.

Define a *degree map* $\langle\rangle : H^*(X) \to K$ as zero on $H^i(X)$ for $i < 2n$ and as the orientation isomorphism on $H^{2n}(X)$. Let $H_i(X)$ denote the dual space to $H^i(X)$. Then Poincaré duality states that the map $a \mapsto \langle \cdot \, . \, a\rangle$ induces isomorphisms $H^{2n-i} \simeq H_i(X)$, which will be viewed as identifications.

Let $f : X \to Y$ be a morphism and $f^* = H^*(f) : H^*(Y) \to H^*(X)$. Then define a K-linear map $f_* : H^*(X) \to H^*(Y)$ as the transpose of f^*. Since f^* is a ring homomorphisms, it follows that f^* and f_* are related through the **projection formula**,

$$f_*((f^*a).\, b) = a.\, f_*b$$

which simply expresses that $H_*(X)$ is a left $H^*(X)$ module functorially.

B. *Künneth formula.*- Let p_X and p_Y denote the corresponding projections from $X \times Y$ onto X and Y, respectively. Then the canonical map $a \otimes b \mapsto p_X^* a \cdot p_Y^* b$ is an isomorphism (of graded-commutative algebras)

$$H^*(X) \otimes H^*(Y) \to H^*(X \times Y).$$

C. *Cycle map.*- Let $Z^p(X)$ denote, as usual, the (free) group of algebraic cycles of codimension p on X. There exist group homomorphisms

$$\gamma_X : Z^p(X) \to H^{2p}(X)$$

satisfying

(i) (functoriality).- If $f : X \to Y$ is a morphism, then

$$f^*\gamma_Y = \gamma_X f^* \text{ and } f_*\gamma_X = \gamma_Y f_*.$$

(ii) (multiplicativity) $\gamma_{X\times Y}(Z \times W) = \gamma_X(Z) \otimes \gamma_Y(W)$.

(iii) (non-triviality).- If P is a point, then $\gamma_P : C^*(P) = \mathbb{Z} \to H^*(P) = K$ is the canonical inclusion.

Proposition 4.1. *The cycle map γ_X induces a K-algebra homomorphism*

$$c_X : Z_\tau^*(X) \otimes_{\mathbb{Z}} K \to H^{\text{even}}(X)$$

where Z_τ^p denotes the quotient of Z^p by the τ-trivial cycles. We recall that τ-equivalence means algebraic equivalence modulo torsion.

Proof: A very similar argument is written in section 2.2. See also Kleiman [28] Propositions 1.2.1 and 1.2.2.

Proposition 4.2. *(Kleiman [28] Prop. 1.2.4) Let $f : X \to Y$ be a surjective morphism. Then $f^* : H^*(Y) \to H^*(X)$ is injective.*

Proof: Indeed, let $z = c_X(X)$. Then $f_* z \neq 0$. Let $a \in H^*(Y)$ such that $f^* a = 0$. Then, for any $b \in H^*(Y)$, $0 = f_*(f^*a.\, f^*b.\, z) = a.\, b.\, f_*z$. Hence, by Poincaré duality, $a = 0$.

4.2 Correspondences

Definition Let X and Y be two projective smooth varieties over a field k. Then an *algebraic* (resp. *homological*) *correspondence of degree* d is an element $\alpha \in CH^{dim(X)+d}(X \times Y)_{\mathbb{Q}}$ (resp $\alpha \in H^{2(dim(X)+d)}(X \times Y)$).

Example: The graph Γ_f of a morphism $f : X \to Y$ yields a fundamental example of algebraic correspondence (of degree 0). Note that the algebraic subset $\{(x, f(x)) : x \in X\}$ is of codimension dim Y in $X \times Y$, therefore one should consider Γ_f as the reduced subscheme associated to $\{(f(x), x) | x \in X\}$, $\Gamma_f \in CH^{\dim(Y)}(Y \times X)$. Its fundamental class (i.e., image under the cycle map) yields a homological correspondence $[\Gamma_f] \in H^{2\dim(Y)}(Y \times X)$.

Definition-Remark A correspondence α of degree d defines two morphisms on Chow groups, namely

$$\alpha_*(\zeta) = p_{Y*}(\alpha \cdot p_X^*(\zeta)), \qquad \alpha_* : CH^{\dim(X)-i}(X) \to CH^{\dim(Y)-i+d}(Y)$$

and

$$\alpha^*(\eta) = p_{X*}(\alpha \cdot p_Y^*(\eta)), \qquad \alpha^* : CH^i(Y) \to CH^{i+d}(X).$$

An analogous definition can be stated in the homological case.

Definition Let α be a correspondence of degree d in $X \times Y$, and let β be a correspondence of degree e in $Y \times Z$. We define the composition of α and β, $\alpha \circ \beta$ to be

$$\alpha \circ \beta = p_{XZ*}(p_{XY}^* \alpha . p_{YZ}^* \beta).$$

Then $\alpha \circ \beta$ is a correspondence of degree $d+e$ on $X \times Z$. Moreover, this composition law is associative (see Fulton [15] Prop. 16.1.1.)

Let us work out the case of homological correspondences in general. We first recall the existence of the *"orientation" isomorphism* $H^{2\dim(X)}(X) \simeq K$.

Suppose $\dim X = n, \dim Y = m$. Künneth formula yields the graded product

$$H^\bullet(X \times Y) = H^\bullet(X) \otimes H^\bullet(Y)$$

and Poincaré duality on X yields

$$H^\bullet(X) \otimes H^\bullet(Y) = H^\bullet(X)^* \otimes H^\bullet(Y) = Hom_K(H^\bullet(X), H^\bullet(Y)).$$

Under this identification, a correspondence α becomes identified to the operator $\alpha_* : H^*(X) \to H^*(Y)$ defined above. It suffices to prove this for an element of the form $a \otimes b$, which yields the linear operator $\langle \cdot . a \rangle b$ – we won't do this here.

One can check that the degree-0 algebraic correspondence Γ_f produces $f^* = \Gamma_{f*}$. Analogously, and justified by the formula $\tau(Z)_* = Z^*$ for an algebraic cycle on $X \times Y$ (where $\tau(x, y) = (y, x)$ is the transposition of coordinates), we obtain $f_* = \Gamma_f^*$. Thus the commutation of the diagram at the beginning of this chapter is proved.

Chapter 5

Blow-ups and resolution of singularities

In this section we compute the (Hodge) cohomology of a blowup; in the language of motives, we describe the (Hodge theoretical realisation of) the motive of a projective bundle and a blowup (modulo homological equivalence). For more precise results on this, and also for some applications, see Manin [37]. We also include Hironaka's singularity theorem. The results in this section will be used to prove that the Hodge conjecture for $S_1 \times S_2$ depends only on the birational type of S_i, and also to derive the important Lemma 9.3, which is a cornerstone of the proof of Theorem 1.1. We will use the Leray spectral sequence throughout the chapter. A good reference is Griffiths and Harris [20] pp.438-468 (especially pp.462-468). See also McCleary [36] pp.515,516.

5.1 The cohomology of a projective bundle

Let $E \to X$ be a vector bundle of rank r over a projective smooth variety X, and let $\pi : \mathbb{P}(E) \to X$ denote the projectivised bundle of E. The results we give in this section for the Chow groups hold for arbitrary fields.

Proposition 5.1. *(see Fulton [15]Th.3.3) Let* $\xi := c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$ *and let* $p_E(T) = T^r + c_1(E)T^{r-1} + c_2(E)T^{r-2} + \ldots + c_r(E) \in CH^\bullet(X)[T]$ *be the Chern polynomial of*

E. Then we can write

$$CH^\bullet(\mathbb{P}(E)) \simeq \frac{CH^\bullet(X)[T]}{(p(T))}$$

where the variable T goes to $c_1(\mathcal{O}_{\mathbb{P}}(E)$ and $CH^\bullet(\mathbb{P}(E))$ is made a $CH^\bullet(X)$-algebra through π^*. Thus the ring $CH^\bullet(\mathbb{P}(E))$ is free of rank r over $CH^\bullet(X)$

An analogous result holds for singular cohomology, or etale cohomology (over arbitrary fields). One possible proof uses Leray spectral sequence, which preserves Galois module and Hodge structures. This last sentence implies that the isomorphisms resulting from the degeneration of Leray spectral sequence will hold with underlying structures – the statement we use was first proved by Deligne in the case of projective smooth morphisms (see Deligne [10]) . For a good reference, see Griffiths and Harris [20] pp. 606-611. See also Manin [37]. Lecture notes by Deligne and Milne are a very nice introduction to the subject of motives, Hodge cycles and Tannakian categories [9] pp.101-228.

Proposition 5.2. *([20]pp. 606-607) For X any compact oriented C^∞ manifold, $\pi : \mathbb{P}(E) \to X$ any complex vector bundle of rank r, the cohomology ring $H^\bullet(\mathbb{P}(E), \mathbb{Q})$ is generated, as an $H^\bullet(X, \mathbb{Q})$-algebra, by the Chern class $\zeta = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$, with the single relation*

$$\zeta^r + c_1(E)\zeta^{r-1} + c_2(E)\zeta^{r-2} + \ldots + c_r(E) = 0.$$

(*)

Proof: We first establish the basic relation **(*)** – in fact, we will use $-\zeta = \xi = c_1(\mathcal{O}_{\mathbb{P}(E)}(-1))$, which is the Chern class of the tautological bundle. Let S be the quotient of the pullback π^*E by the tautological subbundle, and set $\eta_i := c_i(E)$. By Whitney product formula, we get

$$(1+\xi)(1+\eta_1+\ldots\eta_{r-1}) = \pi^*c(E)$$

and solving successively, we have

$$\eta_1 = c_1(E) - \xi$$

$$\eta_2 = c_2(E) - \xi.c_1(E) + \xi^2$$

$\vdots$

$$\eta_{r-1} = c_{r-1}(E) - \xi.c_{r-2}(E) + \ldots + (-1)^{r-1}\xi^{r-1}.$$

The final equation

$$c_r(E) = \xi.\eta_{r-1}$$

is then our basic relation (*). Now let $\{\psi_{i,\alpha}\}_\alpha$ be a basis for $H^i(X, \mathbb{Q})$, with $\psi_{i,\alpha}$ and $\psi_{n-i,\alpha}$ orthogonal – i.e., such that

$$\psi_{i,\alpha} \cup \psi_{n-i,\beta} = \pm\delta_{\alpha,\beta}.$$

We claim that the classes

$$\left(\pi^*\psi_{i,\alpha} \cup \xi^i\right)_{1\le i\le n, 1\le j\le r-1, \alpha}$$

are linearly independent in $H^\bullet(\mathbb{P}(E))$. First, for any pair of classes $\psi_{i,\alpha}$ and $\psi_{n-i,\alpha}$, the cup-product will be Poincaré dual to plus or minus the class of a fibre $\mathbb{P}(E)_p$ of $\mathbb{P}(E)$. But the restriction of ξ to $\mathbb{P}(E)$ is minus the class of a hyperplane in $\mathbb{P}(E)_p$, and consequently

$$\pi^*\psi_{i,\alpha} \cup \pi^*\psi_{n-i,\alpha} \cup \xi^{r-1} = \pm 1,$$

or, in other words, for any j,

$$\left(\pi^*\psi_{i,\alpha} \cup \xi^j\right) \cup \left(\pi^*\psi_{n-i,\alpha} \cup \xi^{r-j-1}\right) = \pm 1.$$

On the other hand, for $\alpha \neq \beta$,

$$\pi^*\psi_{i,\alpha} \cup \pi^*\psi_{n-i,\beta} = 0;$$

likewise, for $i < k$ and any α, β, j,

$$\left(\pi^*\psi_{i,\alpha} \cup \xi^j\right) \cup \left(\pi^*\psi_{n-k,\beta} \cup \xi^{r-j+k-1}\right) = 0.$$

Therefore the intersection matrix for the classes $(\psi_{i,\alpha} \cup \xi^j)_{i,j,\alpha}$ may be made upper triangular with ± 1's along the diagonal; in particular, we see that it is nonsingular, and so these elements are all linearly independent in $H^\bullet(\mathbb{P}(E), \mathbb{Q})$.

Finally, consider the Leray spectral sequence $(E_r^{p,q}, d_r)$ of the bundle $\pi : \mathbb{P}(E) \to X$ (see [20] pp.462-468; see also McCleary [36] pp. 515-516). We claim that the monodromy action of $\pi_1(X)$ on the fibres of π is as follows:

Claim: The monodromy action

$$\pi_1(X, x_0) \to \pi_0 Diff^+\mathbb{P}(E)$$

is trivial on cohomology.

Let us take the Claim for granted. Then the local systems $R^i\pi_*\mathbb{Q}_X$ are trivial, hence

$$E_2^{p,q} = H^p(X, R^q\pi_*\mathbb{Q}_X) = H^p(X, \mathbb{Q}) \times H^q(\mathbb{P}^{r-1}, \mathbb{Q}).$$

But since the classes $(\pi^*\psi_{i,\alpha} \cup \xi^i)$ are all independent in $H^\bullet(\mathbb{P}(E), \mathbb{Q})$,

$$\begin{aligned} r.\text{dim}H^\bullet(X) &\le \text{dim}H^\bullet(\mathbb{P}(E)) \\ &= \text{dim}E_\infty \\ &\le \text{dim}E_2 \\ &= r.\text{dim}H^\bullet(X) \end{aligned}$$

Equality must therefore hold everywhere, i.e., the classes $(\pi^*\psi_{i,\alpha} \cup \xi^i)$ span $H^\bullet(\mathbb{P}(E))$ so that ζ generates $H^\bullet(\mathbb{P}(E))$ as an $H^\bullet(X)$- algebra, and there can be other relations on ζ than (*) above.

Proof of the Claim. Let $m = r - 1$. The action ρ_i of $\pi_1(X)$ on cohomology of the fibres $\mathbb{P}^m$ is by means of graded ring automorphisms, and since the diffeomorphisms involved are positive, i.e., respect the orientation of the fibre, they must be *id* on $H^{2m}(\mathbb{P}^m)$. Let λ_i denote the character acting on $H^{2i}(\mathbb{P}^m)$; it is plain that λ_m is constant and equal to 1, and also that $\lambda_i = \lambda_1^i$, therefore $\lambda_1^m \equiv 1$. The claim is thus established in the case when m is odd. Suppose m is even. In that case, we can embed $\mathbb{P}(E) \hookrightarrow \mathbb{P}(E \oplus 1)$ as a relative hyperplane section. Then the former applies to $\mathbb{P}(E \oplus 1)$. It is not difficult to see that the monodromy action will be forced to be trivial on the fibres of $\mathbb{P}(E)$, thereby proving the claim.

The following proposition gives the Hodge structure of a projective vector bundle on a projective manifold X as a function of that of X. See Kleiman [29] p. 77 for an analogous computation on the category of motives modulo homological equivalence with respect to a given Weil cohomology..

Proposition 5.3. *Let $\pi : \mathbb{P}(E) \to X$ be a projective bundle over a projective smooth manifold X (we state the case $k = \mathbb{C}$). Then the following result holds (we recall our convention that $H^k(X)$ denotes the Hodge structure associated to $H^k(X, \mathbb{Q})$):*

$$H^i(\mathbb{P}(E)) = \bigoplus_{k \leq min\{i/2, r-1\}} H^{i-2k}(X)(-k)$$

In other words, the following rings are isomorphic

$$H^\bullet(\mathbb{P}(E)) \simeq H^\bullet(X) \otimes H^\bullet(\mathbb{P}^{r-1}) = H^\bullet(X \times \mathbb{P}^{r-1}).$$

Furthermore, the element $\zeta := c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$ generates $H^\bullet(\mathbb{P}(E))$ as a $H^\bullet(X)$-algebra with the single relation

$$\zeta^r + c_1(E)\zeta^{r-1} + c_2(E)\zeta^{r-2} + \ldots + c_r(E) = 0$$

5.2 The cohomology of a blowup

As in last paragraph, we follow mainly [20] pp. 606-611 and also Hartshorne [63]II.8. See also lecture notes by Deligne and Milne [9] pp.206-209 for the results involving "motives" of blowups (also [37] and Kleiman [29] for a detailed motivic version). Let X be a projective complex manifold of dimension n, and $Y \subset X$ a (smooth) non necessarily irreducible submanifold. For simplicity in exposition we suppose that Y is of pure dimension m. $\mathcal{I}$ will denote the ideal sheaf of Y in X (commonly represented by $\mathcal{I}_{Y/X}$). i will denote the natural inclusion $i : Y \hookrightarrow X$.

We quickly recall the construction of the blowup of X along the submanifold Y. For every point $y \in Y$, we have a decomposition of $T_X(y) = T_Y(y) \oplus N_{Y/X}(y)$. This decomposition would a priori depend on the metric chosen on X, but in fact we may view the normal vector space to Y in X as the cokernel of the natural inclusion $T_Y(y) \hookrightarrow T_X(y)$. We then present the normal bundle of X along Y in the following canonical and global fashion (see [63]p.182)

$$0 \to T_Y \to T_X|Y = i_* T_X \to N_{Y/X} \to 0$$

This sequence is the dual to the following sequence:

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega^1_X|Y \to \Omega^1_Y \to 0.$$

Recall that we may view a vector space as an algebraic variety in the following (canonical) fashion $V = SpecS(V^*)$ where $S(.)$ is the symmetric algebra of a vector space. Along these lines we can construct the normal bundle to Y as an algebraic vector bundle (see [63]II.Ex.5.18)

$$Vect(N_{Y/X}) := Spec_{\mathcal{O}_Y} S(\mathcal{I}/\mathcal{I}^2) \to Y$$

and the projectivised normal bundle (see [63]p.160 for an algebraic construction)

$$\mathbb{P}(N_{Y/X}) := Vect(N_{Y/X} - 0)/\mathcal{O}_Y^* \to Y$$

$\mathbb{P}(N_{Y/X})$ is also a projective variety (see [63]II.7.10), and its (closed) points denote "normal" directions to points $y \in Y$. We also observe that $\mathbb{P}(N_{Y/X})$ is smooth, and of dimension equal to $\dim X - 1$. For the description of blowing up along a submanifold, we follow [20] 602-603. The *blowing-up* of X along Y is defined as the morphism $\pi : \hat{X} \to X$ such that π is isomorphic outside Y, and is obtained by patching the following local data.

Choose, for every point $y \in Y$, a holomorphic chart $U = U_y$ biholomorphic to the complex unit ball and such that $U \cap Y$ becomes the coordinate plane $z_{k+1} = \dots z_n = 0$ in U. Then $W_y \pi^{-1}(U_y) \subset U_y \times \mathbb{P}^{\dim(Y)-1}$ is defined as $W_y := \{(z, \lambda) : z_i\lambda_j - z_j\lambda_i = 0 \text{ for } 1 \leq i < j \leq n\}$.

By construction, π is isomorphic outside Y, and $E := \pi^{-1}(Y) \cong \mathbb{P}(N_{Y/X})$ over Y. E is called the *exceptional divisor.* $\pi : \hat{X} \to X$ is then a birational morphism. We can now calculate the cohomology of $\hat{X}$ as a Hodge structure.

Proposition 5.4. *(Manin [37]; also Kleiman [29] p. 78) With the preceeding notations, the cohomology of $\hat{X}$ as a function of $H^\bullet(X)$ and $H^\bullet(Y)$ is the following:*

$$H^\bullet(\hat{X}) = H^\bullet(X) \oplus H^\bullet(E)/(\pi|Y)^* H^\bullet(Y)$$

Thus, the following formula holds for $H^i(\hat{X})$:

$$H^i(\hat{X}) \cong H^i(X) \oplus \bigoplus_{l=1}^{\min(r,[i/2])} H^{i-2l}(Y)(-l),$$

where [.] denotes the usual 'floor' function.

Proof: For the proof on singular cohomology, using Mayer-Vietoris exact sequence, we refer the reader to [20] p. 6. Note that, by Prop. 4.2, $H^\bullet(Y)$ injects into $H^\bullet(E)$. Now, using the fact that the maps π^*, π_* and the inclusion $Y \hookrightarrow X$ are (in this case, strict) Hodge morphisms, the computation holds also on the underlying Hodge structures. The proposition now follows by Proposition 5.3.

5.2.1 Birational morphisms and resolution of singularities

We now give a brief account on (bi)meromorphic maps, and finally state two classical results by Hironaka on resolution of singularities. We refer the reader to Ueno [70] I.2.

Let X, Y be proper algebraic varieties. Let $\varphi : X \to Y$ be a rational map. Then the set of points of indeterminacy of φ is known to be a proper closed subset S of X. We can define the graph of φ, $\Gamma_\varphi \subset X \times Y$, as the graph of the analytic map $X - S \to Y$. Now let us consider $G = \overline{\Gamma}_\varphi$. Then the projection onto X induces $f : G \hookrightarrow X \times Y \to Y$ is a birational morphism and the second projection $g : G \to Y$ coincides with φ outside S, i.e. $\varphi \circ f^{-1} = g$. Now recall that, for a complete algebraic variety X, a *resolution of singularities of* X is a smooth projective variety $\tilde{X}$ together with a birational morphism $f : \tilde{X} \to X$. It turns out that the resolutions of singularities of X form a filtered set.

Proposition 5.5. *Suppose X_1 and X_2 are resolutions of singularities of X. Then there exists a resolution X_3 of singularities of X dominating X_1, X_2. In other words, there exists a smooth projective variety X_3 together with birational morphisms onto X_1 and X_2.*

Proof: There exists a birational map $\varphi : X_1 \to X_2$. Consider the above defined irreducible subset $G \subset X_1 \times X_2$, and define X_3 as a resolution of singularities of G. The rest follows.

Theorem 5.6. Hironaka's Theorem*(Ueno [70] Th.I.2.12) Let X be an algebraic variety over $\mathbb{C}$. Then there exists a sequence of monoidal transformations (i.e. blowups)*

$$f_i : X_i \to X_{i-1}, \qquad i = 1, 2, \dots n,$$

satisfying the following conditions:

1. $X_0 = X$ *and* X_n *is non-singular;*

2. *the centre of the monoidal transformation* f_i *is non-singular and contained in the singular locus of* X_{i-1}.

Moreover if a finite group G *acts on* X, *then the above sequence of monoidal transformations can be chosen in such a way that the group* G *can be lifted to a group of analytic automorphisms of* X_n.

We recall the following result on birational morphisms of projective (proper) smooth varieties. Its proof follows from the valuative criterion of properness – see reference below.

Proposition 5.7. *(Hartshorne [63] Th. II.8.19, Ex.II.8.8) Let* $\pi : \tilde{X} \to X$ *be a birational morphism of proper smooth varieties. Then* $H^0(\tilde{X}, \Omega^p_{\tilde{X}}) = H^0(X, \Omega^p_X)$ *for* $0 \leq p \leq dim\ X$ *and* $P_n(\tilde{X}) = P_n(X)$ *for all* $n \in \mathbb{N}$.

The following theorem by Hironaka is somehow related to the former proposition.

Theorem 5.8. *(Freitag [14] Satz 1) Let* $f : \tilde{X} \to X$ *be a desingularisation of a normal analytic space* X, *with at most finite quotient singularities (a finite quotient singularity is a point that admits a neighbourhood* U *and a finite group quotient map* $V \to V/G = U$ *with* V *smooth). Then, if* X_0 *is the smooth locus of* X, *we have the following property. The morphism* f^* *on holomorphic differential forms is an isomorphism*

$$f^* : H^0(X_0, \Omega^p_{X_0}) \to H^0(\tilde{X}, \Omega^p_{\tilde{X}}),$$

i.e. every holomorphic differential form defined on X_0 *admits a unique extension to* $\tilde{X}$.

Another problem that Hironaka's resolution of singularities does fix is the resolution of indeterminacies of a morphism in characteristic 0.

Theorem 5.9. *(Ueno [70] Prop. 2.13) Let* φ *be a rational map from a proper algebraic variety* X *to another proper algebraic variety* Y. *Then there exists a birational morphism* $q : \tilde{X} \to X$ *such that the composition* $\varphi \circ q : \tilde{X} \to Y$ *is regular everywhere.* q *is constructed by the procedure of Theorem 5.6.*

We point out some consequences of last theorem that are relevant to us. Suppose that X and Y are two projective manifolds having the same birational type. Then, by Theorem 5.9, there exist morphism $f : \tilde{X} \to Y$ and $g : \tilde{Y} \to X$ where $\tilde{X}$ and $\tilde{Y}$ are constructed by a sequence of blowups on smooth centres from X, Y respectively. As a result, we obtain the following.

Proposition 5.10. *Let X and Y be two projective manifolds having the same birational type. Then $H^1(X) = H^1(Y)$ and $T^2(X) = T^2(Y)$, where T^2 stands for the complement of $NS_{\mathbb{Q}}$ in H^2. Also $H^0(X, \Omega^p_X) = H^0(Y, \Omega^p_Y)$ for $p = 0, \ldots, \dim X = \dim Y$ and their plurigenera $P_m(X) = P_m(Y)$.*

Proof: By using f and g above (and also Proposition 4.2), $H^\bullet(Y) \subset H^\bullet(\tilde{X})$ and $H^\bullet(X) \subset H^\bullet(\tilde{Y})$. The proposition now follows from Proposition 5.4.

Remark Compare the proof of last Proposition and that of Lemma 5.2.1.*(3)*, where another method is used which works also in this case.

Chapter 6

Generalities on surfaces and fibrations on curves

In this section we follow [4] [57] [58]. We put together results on algebraic surfaces and fibrations of an algebraic surface onto a curve that yield a criterion for a surface S to be a *quasi-bundle* over a curve. $e(X)$ will denote the topological Euler characteristic of a variety X. We also recall that $q(S) = h^{0,1}(S)$ (called *irregularity* of S) and $p_g(S) = h^{0,2}(S)$ (called *geometric genus* of S).

For the sake of completeness, we include a few basic results on the theory of algebraic surfaces.

Theorem 6.1. (Riemann-Roch for surfaces) *(see Hartshorne [63] Th. V.1.6) Let D be a divisor on a projective smooth surface S. Then the following formula holds*

$$\chi(\mathcal{O}_S(D)) - \chi(\mathcal{O}_S) = \frac{D(D - K_S)}{2}$$

Proposition 6.2. (Adjunction formula) *(see [4] I.15) Let C be a complete curve in S, S being as above. Then*

$$2p_a(C) - 2 = C(C + K)$$

Proof: The result appearing in [63],Chapter IV only applies to smooth curves – it is stated in its full generality in [4]I.15, and its proof applies in arbitrary characteristic. Consider the following exact sequence for an effective divisor D

$$0 \to \mathcal{O}_S(-D) \to \mathcal{O}_S \to \mathcal{O}_D \to 0$$

The former yields

$$\chi(\mathcal{O}_D) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-D))$$

Take $D = C$. The rest follows from Riemann-Roch applied to $D = C$ and the former equality.

Proposition 6.3. (Noether's formula) *(see Hartshorne [63]Appendix A, Example 4.1.2) For a smooth projective surface S, the following formula holds*

$$12\chi(\mathcal{O}_S) = K_S^2 + e(S)$$

where $e(S)$ denotes the topological Euler-Poincaré characteristic of S: $e(S) = \sum(-1)^i b_i$, with $b_i = dimH^i(S, \mathbb{Q})$. Over arbitrary characteristic, the formula holds for S provided we take $b_i = rk_{\mathbb{Q}_l} H^i_{et}(S, \mathbb{Q}_l)$ with $l \neq char(k)$.

The following result will be frequently used in the sequel.

Theorem 6.4 (Castelnuovo-de Franchis' inequality). *(Griffiths-Harris [20] p. 554) Let S be a minimal surface over $\mathbb{C}$, such that $e(S) < 0$. Then S is irrational ruled.*

We briefly recall the notion of rational map ϕ_L associated to a line bundle L and define the Kodaira dimension of a complete algebraic manifold. For a subspace $V = \text{span}(s_0, \ldots s_n)$ of $H^0(X, L)$ we can define the rational map $\phi_L : X \to \mathbb{P}^m$ by $P \mapsto (s_{0,P} : \ldots : s_{m,P})$, which will be defined for the points P which are not common zeroes of every section in $H^0(X, L)$. We now define the *D-dimension* of a line bundle (or equivalently, of a divisor) as follows:

$$\begin{aligned}\kappa(D, X) &= -\infty \text{ if } H^0(X, mD) = 0 \text{ for all } m \in \mathbb{N}, \\ &= max_{m \geq 1}(\dim \phi_{mD}(X)), \text{otherwise.}\end{aligned}$$

The Kodaira dimension.- The *Kodaira dimension* of a projective manifold is defined as $\kappa = \kappa(K_X)$; κ may take the values $-\infty$, $0, 1, \ldots, \dim X$. The Kodaira dimension is a birational invariant, and $P_m(X) = P_m(X')$ if X, X' are birationally isomorphic proper manifolds (see Hartshorne [63]Th. II.8.19, Ex. II.8.8, Sect. V.6; for a complete account, see also Ueno [70] Chap. 2 - Sects. 5,6). We say also that X is *of general type* if and only if $\kappa(X) = \dim X$.

This result will be used in the proof of Theorem 1.1.

Theorem 6.5. *(Beauville [4] Th. X.4) Let S be a surface of general type. Then $\chi(\mathcal{O}_X) > 0$.*

6.1 Elliptic surfaces

In this paragraph we give necessary and sufficient conditions for a surface to be elliptic. We follow Beauville [4] Chapter IX.

Lemma 6.6. *([4] Lemma IX.1) Let S be a non-ruled minimal surface. As usual, $K = K_S$ denotes the canonical divisor of S, $p_g = h^{2,0}$ and $P_r = h^0(rK)$.*

(a) If $K^2 > 0$, there exists an integer n_0 such that ϕ_{nK} maps S birationally onto its image for all $n \geq n_0$.

(b) If $K^2 = 0$ and $P_r \geq 2$, write $rK \sim Z + M$, where Z is the fixed part of the system $|rK|$ and M is the mobile part. Then

$$K.Z = K.M = Z^2 = Z.M = M^2 = 0.$$

Proposition 6.7. *([4] Prop. IX.2) Let S be a minimal surface with Kodaira dimension $\kappa = 1$.*

(a) We have $K^2 = 0$.

(b) There is a smooth curve B and a surjective morphism $p : S \to B$ whose generic fibre is an elliptic curve.

Proof of Proposition 6.7: By part **(a)** from last Lemma, we have $K^2 \leq 0$; so $K^2 = 0$ since otherwise S would be ruled ([4] VI.2). Let r be an integer such that $P_r \geq 2$. Write Z for the fixed part of the system $|rK|$, M for the mobile part, so that $rK \sim Z + M$. Part **(b)** from last Lemma gives $M^2 = K.M = 0$. It follows that $|M|$ defines a morphism from S to $\mathbb{P}^N$ whose image is a curve C. Consider the Stein factorisation $S \to B \to C$, where p has connected fibres (see Section 6.2). Let F be a fibre of p; since M is a sum of fibres of p and $K.M = 0$, we must have $K.F = 0$. It follows that $g(F) = 1$, so that the smooth fibres of p are elliptic curves.

A surface satisfying **(b)** in last Proposition is called an *elliptic surface*; the Proposition says that all surfaces with $\kappa = 1$ are elliptic. The converse is false, but we can say:

Proposition 6.8. *([4] IX.3) Let S be a minimal elliptic surface, $p : S \to B$ the elliptic fibration; for $b \in B$, put $F_b = p^*[b]$.*

(a) We have $K^2 = 0$.

(b) S is either ruled over an elliptic curve, or a surface with $\kappa = 0$, or a surface with $\kappa = 1$.

The surfaces such that $\kappa = 1$ are generally called *(honestly) elliptic*, this terminology being justified by last two Propositions.

Lemma 6.9. *([4] VI.4) Let $p : S \to B$ be a fibration of a surface S onto a smooth curve B. Let $\Sigma \subset B$ be the (finite) set of points over which p is not smooth, and let $\eta \in B - \Sigma$, $F = F_\eta$. Then*

$$e(S) = e(B)e(F) + \sum_{s \in B} (e(F_s) - e(F))$$

where the only nonzero terms in the sum correspond to points $s \in \Sigma$.

The following lemma is a classical result on algebraic curves, which we encode here in sheaf-theoretic language.

Lemma 6.10. *([4] VI.5) Let C be a reduced (but possibly reducible) complete curve. Then $e(C) \geq 2\chi(\mathcal{O}_C)$; equality holds if and only if C is smooth.*

Proof: Let $n : N \to C$ be the normalisation of C.

Topological remark: Let $n : N \to C$ be the normalisation of a reduced (but possibly reducible) complete curve C. Then n is a continuous and surjective map between compact (Hausdorff) topological spaces, and it is bijective up to a finite number of points in N. Thus, n is topologically equivalent to N/R where R is the following equivalence relation on N:

xRy if and only if $n(x) = n(y)$

Indeed, n induces a bijective map $N/R \to C$ between compact spaces, hence a homeomorphism between N/R and C.

Consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C}_C & \longrightarrow & n_*\mathbb{C}_N & \longrightarrow & \epsilon & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \varphi & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & n_*\mathcal{O}_N & \longrightarrow & \delta & \longrightarrow & 0 \end{array} \tag{6.1}$$

where $\mathbb{C}_X$ denotes the constant sheaf $\mathbb{C}$ on the variety X, and ϵ, δ are defined so as to make the rows exact. To say that φ is injective is the same (diagram chasing) as saying that a local section of $n_*\mathcal{O}_N$ that comes both from $n_*\mathbb{C}_N$ and $\mathcal{O}_C$ in fact comes from $\mathbb{C}_C$; this, however, is obvious.

Thus $h^0(\delta) \geq h^0(\epsilon)$, and it follows from the diagram that

$$e(N) = e(C) + h^0(\epsilon)$$

$$\chi(\mathcal{O}_N) = \chi(\mathcal{O}_C) + h^0(\delta),$$

and so $e(C) = 2\chi(\mathcal{O}_C) + h^0(\delta) + (h^0(\delta) - h^0(\epsilon))$, since $e(N) = 2\chi(\mathcal{O}_N)$. Hence

$$e(C) \geq 2\chi(\mathcal{O}_C)$$

and equality implies $h^0(\delta) = 0$, so that $\delta = 0$ and $C = N$.

We note that the property $H^i(C, n_*F) = H^i(N, F)$ for a sheaf on N was used. This fact is a consequence of Leray-Hirsch theorem, for n is a proper map with discrete fibres (i.e. 0-dimensional fibres), so $R^i f_*$ clearly vanish for $i > 0$, hence the equality. The reader may find the cited result in Godement [18] 4.17.1.

Another version of Lemma 6.10: The formula $\chi(\mathcal{O}_N) = \chi(\mathcal{O}_C) + h^0(\delta)$ can be read as follows:

$$p_a(C) = g(C) + \sum_{x \in C} (m_x - 1),$$

where the number $g(C)$ (called the *geometric genus* of C, and identical to $p_a(N) = h^0(\Omega_N)$ is a birational invariant. The number $m_x = \mathit{length}(\mathcal{O}_{N,x}/\mathcal{O}_{C,x}) = \dim\ \delta_x - 1$.

6.2 Some general facts about fibrations on algebraic varieties

We recall the fact that a **fibration** is a (proper) morphism with connected fibres. In the following we shall see some properties that characterise fibrations, as well as factorisation theorems for projective morphisms. Some of the results are cited following Hartshorne [63] III.11 - for projective morphisms. The reader may find the same results in EGA III.4. [23] for every proper morphism of Noetherian schemes.

To start with, let us consider a proper morphism of noetherian schemes $f : X \to Y$. Then $f_*\mathcal{O}_X$ is a coherent sheaf of $\mathcal{O}_Y$-algebras, and thus it determines a finite morphism $Y' \to Y$ given by $Y' := \underline{Spec}_{\mathcal{O}_Y} \to Y$. We observe that if f factors through a finite map $h : Z \to Y$, giving such h is equivalent to giving a coherent algebra $\mathcal{A} = h_*\mathcal{O}_Z$ over Y such that the following factorisation holds:

$$\mathcal{O}_Y \to \mathcal{A} \to f_*\mathcal{O}_X.$$

Thus we have a universal property for $Y' \to Y$ with respect to these morphisms: for every finite morphism $Z \to Y$ such that f factors through Z, necessarily Z factors through Y'. The morphism $f' : X \to Y'$ satisfies $f'_*\mathcal{O}_X = \mathcal{O}_{Y'}$ by construction.

The following proposition explains this condition.

Proposition 6.11. *(see Hartshorne [63] Cor.III.11.3) Let $f : X \to Y$ be a projective morphism of noetherian schemes, and suppose that $f_*\mathcal{O}_X = \mathcal{O}_Y$. Then $f^{-1}(y)$ is connected for every y.*

Last proposition and the above discussion yield the proof of next theorem.

Theorem 6.12. (Stein factorisation) *Let $f : X \to Y$ be a projective morphism of Noetherian schemes. Then f admits a factorisation $X \to Y' \to Y$ where $f' : X \to Y'$ is a projective fibration and $q : Y' \to Y$ is finite.*

Let X and Y be normal (irreducible) algebraic varieties defined over a base field k, and $f : X \to Y$ be a proper (not necessarily projective) morphism. We denote the generic point of Y by $\xi = \mathrm{Spec} k(Y)$. Let us find an algebraic interpretation of $L = f_*\mathcal{O}_X)_\xi = \bigcup_{U \subset Y} \mathcal{O}_X(f^{-1}(U))$; precisely speaking, we will describe the tower $k(Y) \subset L \subset k(X)$. Suppose $\varphi \in L$; then φ is regular over some open set $f^{-1}(U)$ for an affine open set $U \subset Y$; hence φ is algebraic over $k(Y)$. Conversely, let $\varphi \in k(X)$ be algebraic over $k(X)$, thus satisfying a monic equation

$$\varphi^n + a_1\varphi^{n-1} + \ldots + a_n.$$

Take an affine open set $U \subset Y$ where all the a_i are regular. Therefore $\varphi \in L$, as X is normal.

Thus we have proven the following; L is the algebraic closure of $f^*k(Y)$ in $k(X)$. Using this tower of function fields, it can be checked that the generic number of connected components of f equals the separability degree of $L|k(Y)$, and also that the generic fibre is non-reduced if and only if the extension $L|k(Y)$ is separable. We thus obtain a reciprocal statement to Proposition 6.11.

Proposition 6.13. *("Liouville's Theorem") Let $f : X \to Y$ be a morphism of normal varieties over an algebraically closed field k. Suppose that the general fibre of f is connected and reduced. Then $f_*\mathcal{O}_X = \mathcal{O}_Y$ (and thus every fibre of f is connected).*

We now give specific results in the case where $Y = B$ is a smooth curve.

Proposition 6.14. *([63] III.9.7.) Suppose we have a nonconstant morphism of algebraic varieties $p : X \to B$, where B is a smooth curve and X is normal. Then p is flat, for the affine rings of B are all Dedekind and every torsion-free R-module over a Dedekind ring R is flat. This implies in particular that the dimension of the fibres $p^{-1}(b)$ is constant and thus equal to dim $X - 1$ for every closed point $b \in B$.*

The proof of next result is straightforward in the case when dim $Y = 1$. In the general case, the proof follows from Zariski Main Theorem (see [63] Cor.III.11.4).

Proposition 6.15. *Suppose we have a surjective morphism of normal algebraic varieties $f : X \to Y$. Then the Stein factorisation $X \to Y' \to B$ of f is such that Y' is normal. In particular, if $Y = B$ is a smooth curve, then $Y' = C$ is also a smooth curve.*

6.3 The topological Euler formula and elliptic quasibundles

In this section we give necessary and sufficient conditions for a relatively minimal fibration (i.e. such that no fibre contains any exceptional divisor) to be an elliptic quasibundle. Our main tool will be a refinement of topological Euler formula given in Lemma 6.9. The main source will be Beauville's book [4], Chapters VIII and X; we also use [57] and [58].

Theorem 6.16. *([4] VI.6) Let $p : S \to B$ be a relatively minimal fibration from a smooth projective surface onto a smooth complete curve, and let F denote a general fibre. For a singular fibre F_b the following holds*

$$e(F_b) \geq e(F)$$

and equality holds if and only if $F_b = m_b E$, where E is a smooth elliptic curve in S. Thus the following formula for $e(S)$ holds

$$e(S) = e(B)e(F) + \sum_{b \in B} (e(F_b) - e(F))$$

where all the terms in the sum are non-negative and, for a branch point $b \in B$ $0 < e(F) - e(F_b)$ unless p is an elliptic fibration and $(F_b)_{red}$ is a smooth elliptic curve.

As a result, we obtain the following.

Corollary 6.17. *Under the above hypotheses, the following inequality holds:*

$$e(S) \geq e(F)e(B),$$

and equality holds only if p is smooth in case $g(F) \neq 1$, or p is an elliptic fibration whose reduced fibres are all smooth (in the latter case, p will be called elliptic quasi-bundle later on).

Before we prove last theorem, we will need a number of lemmas.

Lemma 6.18. *(Beauville [4] VIII.3) Let S be a surface, C_i irreducible curves on S, and $m_i > 0$ integers. Set $F = \sum m_i C_i$, and suppose that for each i, $F.C_i \leq 0$. Let $D = \sum r_i C_i$ with $r_i \in \mathbb{Q}$, and $D \neq 0$. Then:*

(a) $D^2 \leq 0$;

(b) if F is connected and $D^2 = 0$, then $D = r.F$ for some $r \in \mathbb{Q}$, and $F.C_i = 0$ for all i.

This means that the intersection matrix $(C_i.C_j)$ is negative semi-definite, and that its kernel has dimension at most 1 if F is connected.

Proof: Set $G_i = m_iC_i$, and $s_i = r_i/m_i$ so that $F = \sum G_i$ and $D = \sum s_iG_i$. Then

$$D^2 = \sum s_i^2 G_i^2 + 2\sum_{i<j} s_is_jG_i.G_j.$$

Write $G_i^2 = G_i.(F - \sum_{j\neq i} G_j)$:

$$\begin{aligned} D^2 &= \sum s_i^2 G_i.F - \sum_{i<j}(s_i^2 + s_j^2 - 2s_is_j)G_i.G_j \\ &= \sum s_i^2 G_i.F - \sum_{i<j}(s_i - s_j)^2 G_i.G_j. \end{aligned}$$

(a) is now clear. If $D^2 = 0$, we must have $s_i = s_j$ every time that $C_i \cap C_j$ is non-empty. Suppose F is connected; then any two components C_i and C_j can always be joined by a chain of C_k; it follows that all the s_i are equal (and hence non-zero). Moreover, we must also have $s_i^2G_i.F = 0$ for each i, and hence $C_i.F = 0$.

The following lemma is important in the proof of Theorem 6.16. Its proof is a direct consequence of last Lemma 6.18.

Lemma 6.19. *(Beauville [4] VIII.4) Let S be a surface, B a smooth curve and $p: S \to B$ a fibration. Suppose that $F_b = p^*b = \sum m_iF_i$ for some $b \in B$. Then for $D = \sum r_iF_i$, with $r_i \in \mathbb{Z}$, we have $D^2 \leq 0$, with equality if and only if $D = r.F_b$ for some $r \in \mathbb{Q}$.*

Proof of Theorem 6.16:

Step 1: Suppose $F_{b,red}$ is irreducible, i.e. $F_b = m.C$; then $C^2 = 0$ and

$$e(F_b) = e(C) \geq 2\chi(\mathcal{O}_C) \geq -C.K = -\frac{F.K}{n} = \frac{e(F)}{n}$$

Suppose $g(F) = 0$. Then $n|2$, and $C.K = -C^2 + 2p_a(C) - 2 = 2p_a(C) - 2$ is negative, so $0 = p_a(C) \geq g(C)$, hence $C \simeq \mathbb{P}^1$ and thus C is smooth and $n = 1$. Also if $g(F) \geq 2$ then $e(F) < 0$ and the former yields $n = 1$ and $e(C) = 2\chi(\mathcal{O}_C)$, hence C is smooth by Lemma 6.10, $F_b = C$ is a general fibre. The remaining case is then $g(F) = 1$; we see then that $e(C) = e(F)$ forces $e(C) = 2\chi(\mathcal{O}_C)$ and thus, again by Lemma 6.10 we get C to be smooth; therefore, $(F_b)_{red} = C$ is a (smooth) elliptic curve.

Step 2: Suppose now $F_b = \sum_{i=1}^{t} n_i C_i$, where t denotes the number of irreducible components of F_b. Then if we write $(F_b)_{red} = C = \sum C_i$, Lemma 6.10 yields once more the inequality

$$e(F_b) = e(C) \geq 2\chi(\mathcal{O}_C) = -(\sum C_i)^2 - (\sum C_i).K$$

Corollary 6.19 shows that $(\sum C_i)^2 \leq 0$ and $C_i^2 < 0$ for each i; since C_i is not an exceptional divisor, $C_i.K \geq 0$ – indeed, since $C_i^2 < 0$, $C_i K < 0$ would imply (adjunction formula) $2p_a(C_i) - 2 \leq -2$, C_i thus being isomorphic $\mathbb{P}^1$ and hence an exceptional divisor.

$$e(F_b) \geq -\sum n_i C_i.K = -F_b.K = -F.K = e(F),$$

thus establishing the result.

Let us analyse the case when equality $e(F_b) = e(F)$ holds. The divisor C turns out to be smooth by Lemma 6.10; therefore, C_i are pairwise disjoint, therefore C is disconnected unless irreducible. Suppose C is not irreducible; this leads to a contradiction with Proposition 6.13. Therefore C is smooth irreducible, hence elliptic by Step 1. This concludes the proof.

We now turn our attention to a special type of fibrations, which will be closely related to smooth fibrations.

Definition Let $p : S \to B$ be a fibration of a smooth complete surface onto a smooth projective curve. Then p is called *quasi-bundle* if and only if every fibre of p is an integral multiple of a smooth curve in S.

Remark We recall that a smooth fibration is differentiably a fibre bundle. One should note, however, that a smooth fibration is not necessarily a holomorphic fibre bundle, for its fibres need not be biholomorphically equivalent.

Definition A fibration p is called *isotrivial* if all smooth fibres of p are pairwise isomorphic.

Remark 1. Let $p : S \to B$ be a smooth elliptic fibration, and B be smooth projective. Then p is isotrivial, for the j-invariant of the family

$$j : B \to \mathbb{A}^1$$

must be constant. The same holds for an elliptic quasibundle, thanks to Lemma 6.20 below.

2. Suppose $p : S \to B$ is a smooth fibration on curves of genus 2 (over a smooth projective basis). By a result of S. Diaz [12] giving upper bounds for complete subvarieties of moduli spaces of curves $\mathcal{M}_g$, the moduli map of the family $p : B \to \mathcal{M}_2$ is zero-dimensional, and thus p is isotrivial. The same type of result would hold for p a quasibundle whose fibres have genus 2.

Lemma 6.20. *([4] VI.7) Let $p : S \to B$ be a quasibundle. Then there exists a ramified Galois cover $q : B' \to B$ with Galois group G, say, a surface S' and a commutative diagram*

$$\begin{array}{ccc} S' & \xrightarrow{q'} & S \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{q} & B \end{array} \tag{6.2}$$

such that the action of G on B' lifts to S', q' induces an isomorphism $S'/G = S$ and p' is smooth.

Proof: It is enough to eliminate each multiple fibre by taking successive branched covers, and so the lemma follows from the following local version:

Lemma 6.21. *([4] VI.7') Let $\Delta \subset \mathbb{C}$ be the unit disk, U a non-compact smooth analytic surface and $p : U \to \Delta$ a proper fibration that is smooth outside* 0*, such that $p^*(0) = nC$ for some smooth curve $C \subset U$. Let $q : \Delta \to \Delta$ be the morphism defined by $z \mapsto z^n$, $V = U \times_{\Delta,q} \Delta$, U' the normalisation of V and p', q' the projections of U' onto Δ, U. The group μ_n of nth roots of unity acts on Δ (by $z \mapsto \zeta z$) and so on V (via the second factor), and so on U'; q' induces an isomorphism $U'/\mu_n = U$. The fibration $q' : U' \to \Delta$ is smooth.*

Note that the local construction described above does not tell us that the cover $q : B' \to B$ is ramified only over those points of B that correspond to multiple fibres, nor that $q' : S' \to S$ is etale. We now obtained a smooth fibration (*i.e.* a fibre bundle) from a quasibundle by means of the construction above. We will now obtain every smooth elliptic fibration as a quotient of a natural projection $C \times F \to C$ by the (diagonal) action of a finite group G; we will be able to do so thanks to the structure of the moduli space of elliptic curves.

We can actually give a more explicit construction for the obtention of a fibre bundle (i.e. smooth fibration) from a quasibundle. The following Proposition plays this rôle. The first part is simply a global statement of last local lemma 6.21. For a proof of its last part, see reference given below.

Proposition 6.22. *(Serrano [58], Prop.3.3) Let $p : S \to B$ be a quasibundle. Let $m_1D_1, \cdots m_tD_t$ be the singular (and therefore multiple) fibres of p, with D_i smooth. Let μ be a common multiple of $m_1, \cdots, m_t$, and $e \geq 0$ any integer such that $\mu | t + e$. Let $p(D_i) = P_i \in B, 1 \leq i \leq t$, and choose points $P_{t+1}, \cdots, P_{t+e} \in B$ whose fibres by p are smooth. Furthermore, let $\mathcal{L}$ be any line bundle on B satisfying*

$$\mathcal{L}^\mu \cong \mathcal{O}_B(P_1 + \cdots P_{t+e})$$

$\mathcal{L}$ defines a cyclic covering $\varepsilon : C \to B$ of degree μ, totally ramified at $P_1, \cdots, P_{t+e}$. Finally, denote by R the normalisation of $S \times_B C$. Then, with these conditions, R is a smooth surface and $q(R) - g(C) = q(S) - g(B)$.

Theorem 6.23. *(Beauville [4] Prop. VI.8) Let $p : S \to B$ be a smooth fibration from a surface to a (smooth) curve, and let F be a fibre of p. Assume either that $g(B) = 1$ and $g(F) \geq 1$ or $g(F) = 1$. Then there exists an etale cover $q : B' \to B$ such that the fibration $p' : S' = S \times_B B' \to B'$ is trivial, i.e. $S' \cong B' \times F$. Furthermore, we can take the cover $B' \to B$ to be Galois with group G, say, so that $S \cong (B' \times F)/G$.*

Proof: This depends upon a series of facts from the theory of moduli spaces for curves; a good reference is Grothendieck [22].

Let T be a variety. A curve of genus g over T is a smooth morphism $f : X \to T$ whose fibres are curves of genus g. f is a topological fibre bundle, and so the sheaf $R^1 f_*(\mathbb{Z}/n\mathbb{Z})$ is locally constant for all n. There is a symplectic form on it given by the cup product:

$$R^1 f_*(\mathbb{Z}/n\mathbb{Z}) \otimes R^1 f_*(\mathbb{Z}/n\mathbb{Z}) \to (\mathbb{Z}/n\mathbb{Z})_T.$$

Having such a sheaf is equivalent to knowing its fibre at a point $t \in T$ (i.e. $H^1(X_t, \mathbb{Z}/n\mathbb{Z})$) together with the action of the fundamental group $\pi_1(T,t)$: this action preserves the symplectic form on $H^1(X_t, \mathbb{Z})$ and therefore that on $H^1(X_t, \mathbb{Z}/n\mathbb{Z})$ (remember that the monodromy can be thought of a map $\pi_1(T,t)\pi_0 Diff^+(F_t)$, hence the action is induces a diffeomorphism action on homotopy and (co)homology groups). Let us endow the constant sheaf $(\mathbb{Z}/n\mathbb{Z})_T^{2g}$ with its standard symplectic form. A J_n-rigidified curve of genus g over T is a curve of genus g over T together with a symplectic isomorphism $(\mathbb{Z}/n\mathbb{Z})_T^{2g} \to R^1 f_*(\mathbb{Z}/n\mathbb{Z})$. A curve of genus g over T can be J_n-rigidified if T acts trivially on $H^1(X_t, \mathbb{Z}/n\mathbb{Z})$. Since $\mathrm{Aut}(H^1(X_t, \mathbb{Z}/n\mathbb{Z}))$ is finite, $\pi_1(T,t)$ has a subgroup of finite index which acts trivially on $H^1(X_t, \mathbb{Z}/n\mathbb{Z})$, and so there is an étale (Galois) cover $T' \to T$ such that the pullback of the given curve is J_n-rigidifiable over T'.

Let $n \geq 3$. One then shows that J_n-rigidification eliminates automorphisms (see Corollary 10.7), and this implies that there is a universal J_n-rigidified curve of genus g, denoted by $U_{g,n} \to T_{g,n}$. The spaces $R_{g,n}$ are in fact quasi-projective varieties, but it is enough for our purposes that they exist as analytic spaces. We shall need also the following properties:

(1) For $g \geq 2$, there is non non-constant analytic morphism $h : \mathbb{C} \to T_{g,n}$.

(2) For $g = 1$, there is no non-constant analytic morphism from a connected compact analytic variety X to $T_{1,n}$.

(2) is elementary, since the j-invariant defines a holomorphic function on X, which must be a constant. (1) is more subtle. One can use the fact that the universal cover T_g (Teichmüller space) of $T_{g,n}$ is a bounded domain, and so cannot be the target of any non-trivial morphism from $\mathbb{C}$. One can also conside the space $A_{g,n}$, which classifies J_n-rigidified principally polarised Abelian varieties of dimension g; by construction, its universal cover is the Siegel upper half-space $\mathcal{H}_g$, which is a bounded domain. Finally one applies the Torelli theorem, which shows that the map $T_{g,n} \to A_{g,n}$ obtained by sending a curve to its Jacobian is finite.

We shall show how the theorem follows from properties **(1)** and **(2)**. Let $p : S \to B$ be a smooth morphism, with fibres of genus g; then for fixed $n \leq 3$, there is an étale cover $B' \to B$ such that the curve $S' = S \times_B B' \to B'$ is J_n-rigidifiable. Choose some J_n-rigidification: we get a morphism $h : B' \to T_{g,n}$ such that $S' = U_{g,n} \times_{T_{g,n}} B'$. If $g(B) = 1$, then $g(B') = 1$; if $g \geq 2$, then h is trivial by **(1)**, since the universal cover of B' is $\mathbb{C}$. If $g = 1$, then h is trivial by **(2)**. The theorem follows.

The following proposition characterises elliptic quasibundles among the (relatively minimal) elliptic fibrations. The numerical criterion below makes use of Lemma 6.20 and Theorem 6.16.

Proposition 6.24. *(Serrano [58], Prop. 1.5.) Let $p : S \to B$ be an elliptic fibration of a (smooth projective) surface onto a smooth projective curve. Suppose p is relatively minimal. Then the following are equivalent:*

a. $\chi(\mathcal{O}_S) = 0$

b. p is an elliptic quasibundle

c. $e(S) = 0$

Under (any of) the former hypotheses, p can be constructed in the following fashion. Denoting E for a general fibre of p, there exists a Galois covering $B' \to B$ of Galois group G, say, and an action of G on E such that $S = (B' \times E)/G$ and p is equivalent to the natural projection

$$S = (B' \times E)/G \to B'/G = B$$

Proof: The fact that p is a relatively minimal elliptic fibration yields $K_S^2 = 0$ and thus the result follows from Noether's formula and Lemma 6.9

$$12\chi(\mathcal{O}_S) = K^2 + e(S) = e(S).$$

Now, $e(S) = 0$ if and only if p is an elliptic quasibundle, by Theorem 6.16. The proposition is thus established.

Chapter 7

The Hodge Conjecture for $S_1 \times S_2$

Let S_1 and S_2 be smooth projective surfaces over $\mathbb{C}$. Then the Hodge conjecture for $S_1 \times S_2$ needs to be proven in codimension 2 only (see Proposition 3.38). For this reason we want to find the Hodge cycles in H^4 and determine their geometric origin when possible.

$$\begin{aligned} H^4(S_1 \times S_2) = H^0(S_1) \otimes H^4(S_2) \oplus H^4(S_1) \otimes H^0(S_2) \oplus \\ \oplus H^1(S_1) \otimes H^3(S_2) \oplus H^3(S_1) \otimes H^1(S_2) \oplus \\ \oplus H^2(S_1) \otimes H^2(S_2) \end{aligned}$$

Claim: Let X and Y be two smooth projective varieties of respective dimensions m and n. Then the Hodge cycles in the Hodge structures $H^1(X) \otimes H^{2n-1}(Y)$ and $H^{2m-1}(X) \otimes H^1(Y)$ are spanned by intersections of divisors, hence algebraic.

Let us take the Claim for granted. Let us consider the case $X = S_1$, $Y = S_2$. The Künneth formula for $H^4(S_1 \times S_2)$ above yields a decomposition of this Hodge structure into Hodge substructures, and thus the Hodge conjecture for $S_1 \times S_2$ would follow from the statement that every Hodge cycle in every summand of the Künneth decomposition of $H^4(S_1 \times S_2)$ is algebraic. The Hodge cycles in each of the summands but the last are known to be algebraic – this follows by Claim above. Therefore, the Hodge conjecture for $S_1 \times S_2$ holds in codimension two if and only if every Hodge cycle in $H^2(S_1) \otimes H^2(S_2)$ is algebraic.

Ad-hoc proof of the Claim: (compare Proposition 10.19) Let us first prove that every Hodge class in the Hodge structure $H^1(X) \otimes H^1(Y)$ is a linear combination of intersections of divisors. A quick proof of this fact stems from the observation that $X \times Y$ satisfies the Lefschetz (1,1)-theorem and hence for every summand in the following formula

$$H^2(X) \oplus H^2(Y) \oplus H^1(X) \otimes H^1(Y) = H^2(X \times Y)$$

the subspaces of Hodge cycles and of algebraic classes coincide.

The rest follows from the Hard Lefschetz theorem: indeed, an ample class in Y yields a *Lefschetz isomorphism* of Hodge structures (in the broad sense)

$$id_X^* \otimes L_Y^{n-1} : H^1(X) \otimes H^1(Y) \to H^1(X) \otimes H^{2n-1}(Y)$$

inducing isomorphisms of Hodge and algebraic classes, which will then coincide in the target space. The same proof applies to $H^{2m-1}(X) \otimes H^1(Y)$, by switching the variables.

Let us come back to the Künneth decomposition of $H^4(S_1 \times S_2)$. As shown above, the Hodge conjecture for $S_1 \times S_2$ is equivalent to proving that every Hodge cycle in $H^2(S_1) \otimes H^2(S_2)$ is algebraic. We could further decompose this Hodge structure, as will be seen below. For a surface S, we have the following decomposition for H^2

$$H^2(S) = T(S) \oplus NS(S)_{\mathbb{Q}} \tag{7.1}$$

where $NS(S)$ denotes the Néron-Severi group of S and $T(S) = NS(S)^{\perp}$ is the (rational) *transcendental lattice* of S. Notice that $T(S)$ has no non-zero Hodge cycles; also, the non-degeneracy of cup-product on $H^2(S)$ yields the so-called *self-duality* property for $T(S)$: $T(S)(2) \simeq T(S)^*$. Now, with this decomposition for $H^2(S)$, we can write Equation 7.1 in a more explicit form

$$H^2(S_1) \otimes H^2(S_2) = NS(S_1)_{\mathbb{Q}} \otimes NS(S_2)_{\mathbb{Q}} \oplus T(S_1) \otimes T(S_2) \tag{7.2}$$

$$\oplus NS(S_1)_{\mathbb{Q}} \otimes T(S_2) \oplus T(S_1) \otimes NS(S_2)_{\mathbb{Q}} \tag{7.3}$$

Then the Hodge conjecture holds if and only if every Hodge cycle in $T(S_1) \otimes T(S_2)$ is algebraic. We recall that the cup-product is a morphism of Hodge structures, hence for a surface S the natural morphism $H^2(S) = H^4(S) \otimes H^2(S)^* = H^2(S)^*(-2)$ sends Hodge cycles isomorphically to Hodge cycles. As a result, $T(S)^* = T(S)(2)$ ("self duality"), therefore the algebraicity of every Hodge cycle in $T(S_1) \otimes T(S_2)$ is equivalent to $Hom_{Hodge-str}(T(S_1), T(S_2))$ consisting of morphisms induced by algebraic cycles.

Remark on birational invariance We notice the birational invariance of the Hodge conjecture for products of two smooth projective surfaces – see Equations 7.1,7.2, and Proposition 5.10. Thus, it suffices to prove the Hodge conjecture for a product $S_1 \times S_2$ and then it automatically holds for $T_1 \times T_2$, with T_i birationally isomorphic to S_i, $i = 1, 2$.

Chapter 8

Hodge structures of weight 2, $dim(W^{2,0}) = 1$

Let W be a (polarisable) rational Hodge structure of weight 2 such that dim $W^{2,0} = 1$. Let $\mathcal{R}$ be the ring of endomorphisms of W as a Hodge structure. It turns out that the condition on dim $W^{2,0}$ gives W a special property, namely, W can be viewed as a Hodge substructure of a Hodge structure generated by cohomology groups of Abelian varieties – this was done by M.Kuga and I. Satake in [31], and used by P.Deligne in his direct proof of the Weil conjecture for K3 surfaces [11]. We give a precise statement of this fact in this paragraph, which we use in the proof of Theorem 1.1.

Proposition 8.1. *Under the above hypotheses, if W has no non-trivial Hodge cycles, then W is an irreducible Hodge structure, and $\mathcal{R}$ is a number field of degree $d|dim_{\mathbb{Q}}W$ over $\mathbb{Q}$.*

Proof: We first prove that W is irreducible. Indeed, suppose W is not irreducible. As the category of polarisable Hodge structures is semisimple Abelian, W admits a decomposition

$$W = M \oplus N$$

Suppose that $M^{2,0}$ is one-dimensional. In that case, $N^{2,0} = 0 = N^{0,2}$, hence N must consist only of Hodge cycles, hence $N = 0$ and therefore W is irreducible. Since W is irreducible, it suffices to see the action of a Hodge-endomorphism on its

(2, 0)-component, which is one-dimensional. This yields automatically an injection of rings

$$\mathcal{R} \hookrightarrow End_{\mathbb{C}} W^{2,0} = \mathbb{C}.$$

We have seen that $\mathcal{R}$ is a number field. Let α be a primitive element of the extension $\mathcal{R}|\mathbb{Q}$. Then the minimal polynomial of α over W is irreducible, and therefore the characteristic polynomial is a power of the minimal polynomial, thus establishing the proposition.

Examples: 1. Let A be an Abelian surface, and let $W = NS(A)^{\perp}_{\mathbb{Q}}$ under the polarisation Q given in Theorem 3.34. By Mumford [45] Chapter 1, $H^{2,0}(A)$ is a complex vector space of dimension $\binom{\dim A}{2} = 1$ in our case. Hence W is of weight 2, irreducible, and $W^{2,0} = H^{2,0}(A)$ is one-dimensional.

2. Let X be a K3 surface. Consider the (rational) trascendental lattice $T(X)$, as defined in (7). Then $h^{2,0}(X) = 1$, hence $T(X)$ satisfies the hypotheses of this paragraph as well.

We recall also that every such polarised Hodge structure W can be assigned a Kuga-Satake variety Abelian variety A (see [17]) – A is constructed by means of the theory of Clifford algebras applied to the quadratic space (W, ψ) given by the polarisation. $C(E, \beta)$ will denote the Clifford algebra associated to a quadratic space (E, β) where β is a symmetric bilinear form on E. $C^+(E, \beta)$ will denote the even Clifford algebra, which is the subalgebra of $C(E, \beta)$ spanned by the products of an even number of elements of E.

Theorem 8.2. *[Existence of a Kuga-Satake variety] (Kuga-Satake [31]; van Geemen [17] sect. 5) Let (W, ψ) be a polarised Hodge structure of weight 2, such that dim $W^{2,0}_{\mathbb{C}} = 1$. There exists a polarised weight-one Hodge structure on the even Clifford algebra $V = C^+(W, \psi)$ together with a polarised embedding*

$$W \hookrightarrow V \otimes V.$$

Any polarised Abelian variety A such that $H^1(A) \equiv V$ will be called a Kuga-Satake variety for W.

Remark: (see van Geemen [17] 8.3.)

1. We point out that Theorem 8.2 holds for a Hodge structure over a subfield F of $\mathbb{R}$; the construction in van Geemen [17], which makes use of Clifford algebra, remains unchanged, and we thus obtain an F- Hodge structure of weight 1.

2. Note that the former theorem yields a geometric realisation of every Hodge structure of weight two with one-dimensional $(2,0)$-part. This applies in particular to $H^2(S)$ and to $T(S)$ for any surface such that $p_g = 1$. The above construction, however, is not guaranteed to have a geometric origin.

3. The hypothesis on W, namely, $dim\ W_{\mathbb{C}}^{2,0} = 1$, turns out not to be casual for a weight two Hodge structure; the theorem asserts that every polarised weight two Hodge structure W with $dim\ W_{\mathbb{C}}^{2,0} = 1$ is a Hodge substructure of the cohomology of an algebraic variety (namely, $A \times A$, where A is a Kuga-Satake variety for W). This does not hold in general for $dim\ W_{\mathbb{C}}^{2,0} > 1$. Griffiths' work on variations of Hodge structures ('Griffiths transversality') implies that the general polarised Hodge structure W of weight two with $dim\ W_{\mathbb{C}}^{2,0} > 1$ is not a Hodge substructure of the cohomology of any algebraic variety – see [38] Theorem 1.1 for details.

The following theorem is the main result of Morrison [43].

Theorem 8.3. *[43] Let A be an Abelian surface. Then the Kuga-Satake Abelian variety assigned to the polarised Hodge structure $T(A)$ of weight 2 (polarised by the cup-product, see Theorem 3.34) is isogenous to a power of A.*

Chapter 9

The case $p_g = 1$, $q = 2$

Let S be a minimal smooth projective surface over $\mathbb{C}$ and let A be an Abelian surface. Suppose $p_g(S) = 1$. In this case the rational Hodge structure $T(S)$ is irreducible, as seen in Proposition 8.1; this applies to an Abelian surface A also. Without loss of generality we suppose S to be minimal – see Section 7. Let A be an Abelian surface; then either of the following holds:

i. $Hom(T(S), T(A)) = 0$, in which case the Hodge structure $T(S) \otimes T(A)$ has, by self-duality of the lattice $T(S)$ (up to Tate twists), the following subspace of Hodge cycles

$$B(T(S) \otimes T(A)) = Hom_{Hodge}(T(S), T(A)) \otimes \mathbb{Q}(-1) = 0$$

and thus the Hodge conjecture plainly holds; or

ii. $T(S)$ and $T(A)$ are isomorphic

No other case may hold, for they both are irreducible Hodge structures.

Proposition 9.1. *Provided that S satisfies the hypotheses of Theorem 1.1, the Hodge conjecture for $S \times A$:*

(a) holds trivially, in case $T(S)$ and $T(A)$ are not isomorphic.

(b) in case $T(S) \cong T(A)$, it follows from the existence of an algebraic class in $T(S) \otimes T(A)$ and the Hodge conjecture for $A \times A$; in other words, suppose that $Hom_{Hodge}(T(S), T(A))$ contains a nonzero, algebraically induced morphism. Then the Hodge conjecture for $A \times A$ implies the Hodge conjecture for $S \times A$.

Proof: $Hom_{Hodge}(T(S), T(A))$ is either 0 or a free rank-1 module over $End_{Hodge}T(A)$, where the ring action is defined by the usual composition of algebraic correspondences. The proposition follows.

Assumption: Henceforth we will assume, as stated before, that $p_g(S) = 1$; we also assume that $q = 2$, S is minimal and also that S is not an Abelian surface. Without loss of generality we suppose also that $T(S)$ is isomorphic to $T(A)$ for A an Abelian surface - see last Proposition 9.1.

A first consequence of the assumptions on S is that S must be nonruled, as H^2 is algebraic (i.e. $p_g = 0$) for every ruled surface. Now S cannot be a K3 or Enriques surface (for which $q = 0$), or a bielliptic surface (for which $p_g = 0$); by hypothesis, S is not Abelian either, hence (Beauville [4] Th. VIII.2) S must be either (honestly) elliptic or of general type.

We now recall the fact that the arithmetic genus (and therefore $\chi(\mathcal{O}_{\mathcal{X}})$) is a birational invariant for a projective smooth surface – see Proposition 5.7; see also [63] Cor. V.5.6. The hypotheses on S yield $\chi(\mathcal{O}_S) = 0$, hence S is not of general type by Theorem 6.5 and is therefore an honestly elliptic surface.

By Proposition 6.7, we have $K_S^2 = 0$, and Proposition 6.24 implies that S is an elliptic quasibundle $S \to B$. S then satisfies $e(S) = 0 = b_2 - 6$, which yields $b_2(S) = 6$. Hence the Betti (and also the Hodge) numbers of S are those of an Abelian surface.

We will concentrate on producing an algebraic cycle linking S and A in this section, and we include a detailed proof of the Hodge conjecture in detail for $A \times A$ later on in this thesis. As A will turn out to be non-simple, we only need this explicit result for A consisting of a product of two elliptic curves. This last result is stated and proved in Proposition 10.29.

We now have $S = (C' \times E')/G$, where the finite group G acts faithfully on each component and freely on their product.

Now consider the elliptic fibration p in this guise:

$$S = (C' \times E')/G \to C'/G = B.$$

Remark As G acts freely on the product, for any $c \in C'$ we get that G_c acts on E' as a group of translations (which is equivalent to saying that no element of G_c has fixed points in E'). Thus E'/G_c is an elliptic curve, and in fact for $b \in B$ the image of c we have

$$p^*(b) = \frac{card(G)}{card(G_c)}(E/G_c).$$

We have thus described every singular fibre of p in terms of the G-action on E and on C'. For more details, see Fulton [15] Proposition 1.7.

Remark on the Picard number of S Notice that the Picard number $\rho(A) \geq 2$. This follows from the fact that $H^2(S)$ and $H^2(A)$ are isomorphic as Hodge structures – indeed, their transcendant lattices are isomorphic and they have equal dimension, hence their Picard numbers must coincide $\rho(S) = \rho(A)$. This observation will play an important rôle in showing that A is non-simple, in the last chapter of this thesis.

By Lemma 9.3, we obtain the following equality $2 = q(S) = g(C'/G) + g(E'/G) = g(B) + g(E'/G)$ (for a more general inequality, see Proposition 10.16). The number $g(E'/G)$ is either 0 or 1; if it is 0 (i.e. $E'/G \cong \mathbb{P}^1$), then $g(B) = 2$ and thus p is the Albanese fibration of S by Proposition 10.15; otherwise, $B = C'/G$ is an elliptic curve.

We are now going to study the case $g(E'/G) = 1$. The group G acts then by translations on E', which is equivalent to saying that G acts trivially on the cohomology of E', $H^\bullet(E', \mathbb{Z})$. It follows from last observation and Lemma 9.3 that

$$\begin{aligned} H^\bullet(S) &= H^\bullet(C' \times E')^G = (H^\bullet(C') \otimes H^\bullet(E'))^G \\ &= H^\bullet(C')^G \otimes H^\bullet(E'/G) = H^\bullet(B) \otimes H^\bullet(E'/G). \end{aligned}$$

The above lines show precisely the following: if E'/G is an elliptic curve, then B is also elliptic, and the morphism

$$\varphi : S = (C' \times E')/G \to C'/G \times E'/G = B \times E'/G$$

yields an isomorphism φ^* of the cohomology ring of S with the cohomology ring of the Abelian variety $A_1 := B \times E'/G$ (which is a Hodge isomorphism, since φ^* is a

Hodge morphism). Thus A_1 is isogenous to Alb S, by looking at the H^1's, and also $T(A_1) \cong T(S)$ via φ^* (we can say that the motive of S, $h(S)$ is then isomorphic to the motive of A_1, $h(A_1)$ modulo homological equivalence; see Scholl [54] for an explanation of this vocabulary). Thus we see that proving the Hodge conjecture for $S \times A$ in this case reduces to proving it for $A \times A_1$, where A is an arbitrary Abelian surface and A_1 is a non-simple Abelian surface. Chapter 7 and our Assumptions on S show that the last statement reduces to that of Lemma 10.34, thereby proving Theorem 1.1 in case of existence of an elliptic fibration of S onto an elliptic curve..

In the case when $q(S) = g(B)$ and thus $g(B) = 2$, we are going to need further reductions to reduce our problem to one of Hodge cycles on Abelian varieties.

9.1 Reduction to the cyclic monodromy case

Given an elliptic quasibundle $S = (C' \times E')/G$ as above, we want to draw conclusions about the nature of the motive of S. The case when E'/G is elliptic is worked out above; thus we concentrate on the case where G acts nontrivially on $H^1(E')$. In this case, an important simplification in our computation will be the reduction to the case in which G is cyclic.

We first need a result to compute the spaces of holomorphic differentials on finite quotients. After that, we will be able to state and prove a lemma giving the Hodge structure of $H^\bullet(Y)$ fairly explicitly, for Y a desingularisation of a finite quotient X/G of a smooth projective surface X.

Proposition 9.2. *(Freitag [14] Satz 6) Let X be a projective manifold and G be a finite group of analytic automorphisms on X. Then, for Z a desingularisation of X/G, we have a natural isomorphism*

$$H^0(Z, \Omega_Z^p) = H^0(X, \Omega_X^p)^G$$

for $p = 0, \ldots, dim\ X$.

Proof: Let $F \subset X$ be defined by $\{x \in X | \text{Card}(G.x) < \text{Card}(G)\}$. F is exactly the ramification locus of the canonical projection map $p : X \to X/G$. Let $\tilde{F} := p(F)$ be the branch locus of p. We know that X/G is normal, and therefore its singular locus is of codimension ≥ 2 (see Hartshorne [63] Th.II.8.22A). We denote by $V \to U$ the étale part of p, V thus denoting X/F and $U = (X/G) - \tilde{F}$.

By Theorem 5.8, we know that every holomorphic differential form on $\tilde{X} := (X/G)_{\text{reg}}$ extends uniquely to a holomorphic differential form on Y. Therefore it suffices to prove

$$H^0(\tilde{X}, \Omega^p_{\tilde{X}}) \cong H^0(X, \Omega^p_X)^G \quad \text{via } p^*.$$

Let us check that the morphism p^* above is well defined on differential forms. Indeed, if ω is a differential form that is regular on $\tilde{X}$, then $p^*\omega$ is regular outside a subset of codimension ≥ 2 on X, hence regular on all X (see proof of Proposition 5.7, ref. given). Also the fact that $p \circ g = p$ for all $g \in G$ yields $p^* H^0(\tilde{X}, \Omega^p_{\tilde{X}}) \subset H^0(X, \Omega^p_X)^G$. It remains to prove the converse: let ω be a G-invariant regular differential on X, and let η be the meromorphic differential on X/G, regular on $U \subset (X/G)_{\text{reg}}$, such that $p^*\eta = \omega$. We are going to prove that η can be extended to a regular differential outside a closed subscheme of codimension ≥ 2.

Let Y be a codimension 1 component of F, and let $\tilde{Y} = p(Y) \subset \tilde{F}$ be its image, also of codimension 1. We recall that the rings $\mathcal{O} = \mathcal{O}_{X,Y}$ and $\mathcal{O}' = \mathcal{O}_{\tilde{X},\tilde{Y}}$ are local, normal, one-dimensional (Noetherian) and hence discrete valuation rings. Thus we have a Galois finite morphism of local discrete valuation rings $p^* : \mathcal{O}' \to \mathcal{O}$ of Galois group $G_Y = \{g \in G | g(Y) = Y\}$; G_Y is cyclic (see Serre [69] IV. (1.1),(2.6),(2.7)), hence $\mathcal{O} = \mathcal{O}'[Z]/(Z^e - t)$ for t a local coordinate of $\mathcal{O}'$. This implies that, for a generic point $y \in Y$ (i.e. outside a subset of codimension ≥ 2 in X), the map p can be locally described as

$$(z_1, \ldots, z_n) \mapsto (w_1, \ldots, w_n) = (z_1, \ldots, z_{n-1}, z_n^e).$$

where z_i are coordinates on X around y and w_i are coordinates of y around $p(y)$ (remember that this is for generic y, more precisely, for y a smooth point of Y such that the quotient map $Y \to Y/G_Y$ is étale, which is clearly generic on Y. Take then $z_1, \ldots, z_{n-1}$ to be local coordinates on Y around y, and do the same with w_i around $p(y)$).

Now let us write $\eta = \sum_{|I|=k} \eta_I \quad dw_I$, where η_I are a priori meromorphic functions around $p(y)$. The equality $p^*\eta = \omega = \sum_{|I|=k} \omega_I \quad dz_I$ leads to

$$\eta_I(z_1, \dots, z_{n-1}, z_n^e)\cdot e\cdot z_n^{e-1} = \omega_I \text{ if } n \in I;$$

$$\eta_I(z_1, \dots, z_{n-1}, z_n^e) = \omega_I \text{ if } n \in \{1, \dots, n\} - I.$$

It is then easy to see that in either case η_I must be regular on $p(y)$, as

$$z_n^{e-1}\eta_I(z_1, \dots, z_{n-1}, z_n^e)$$

is regular on y and thus η_I cannot have negative integral powers of w_n. Therefore, by looking at the analytic expansion of ω_I, $\eta_I(z_1, \dots, z_{n-1}, z_n^e)$ is regular, which clearly implies that η_I is regular on $p(y)$. This completes the proof.

The following lemma yields a tool for studying the Hodge structures on the cohomology of desingularisations of finite quotients, specially in the case of surfaces. It seems a well-known result, although we provide a proof for the lack of a suitable reference.

Lemma 9.3. *1. Let X be a differentiable manifold, and G a finite group of diffeomorphisms acting freely on X. Then $H^i(X/G, \mathbb{Q}) = H^i(X, \mathbb{Q})^G$*

2. Let X be a compact Kähler manifold, and G a finite group of biholomorphisms acting freely on X. Then $H^{p,q}(X/G) = H^{p,q}(X)^G$

3. Let X be a projective manifold, and G a finite group of biholomorphisms of X. Then $H^1(X)^G = H^1(Y)$, for Y a desingularisation of X/G, and also $T^2(Y) = T^2(X)^G$, where T^2 denotes a complementary of $NS_{\mathbb{Q}}$ in H^2.

Proof: Parts *(1)* and *(2)* have a simple proof, by taking holomorphic coordinates on points of the same G-orbit which are G-translates of one another (the result over $\mathbb{Q}$ can be proven by using triangulations, but it cannot be improved to $\mathbb{Z}$ because the projection operator $\frac{1}{\text{card}(G)} \sum_{g\in G} g$ is not $\mathbb{Z}$-defined).

Let us prove *(3)*. First of all, $H^0(Y, \Omega^1_Y) = H^0(X, \Omega^1_X)^G$ by Proposition 9.2. Thus we can think of a G-equivariant birational morphism $f : X' \to X$ such that the following diagram is commutative:

$$\begin{array}{ccc} X' & \xrightarrow{q} & Y \\ f\downarrow & & \downarrow h \\ X & \xrightarrow{p} & X/G. \end{array}$$

We know by Theorem 5.8 and Proposition 9.2 that q^* yields an isomorphism $H^0(Y,\Omega^1_Y) \cong H^0(X,\Omega^1_X)^G$, which is equivalent to saying that q^* yields an isomorphism of Hodge structures $H^1(Y) \to H^1(X')^G = H^1(X)^G$.

As for the second statement of *(3)*, we can consider the map $q^* : T^2(Y) \to T^2(X')$. We know that p^* yields an isomorphism $H^0(Y,\Omega^2_Y) \to H^0(X',\Omega^2_{X'})^G$ by Proposition 9.2. Let W be the minimal (rational) Hodge structure of $H^2(Y)$ containing $H^{2,0}_Y = H^0(Y,\Omega^2_Y)$; it is clear that it is $T^2(Y)$; indeed, $H^2(Y) = W \oplus V$; then W contains $H^{2,0}$, hence V consists of Hodge cycles and therefore $T^2(Y) = W$. Similarly, we obtain $T(X')^G$ on X', and therefore q^* induces an isomorphism $T(Y) \cong T(X')^G \cong T(X)^G$ (see Proposition 5.4 and Proposition 5.10 for last isomorphism). Thus we are done.

We need the following two well-known elementary lemmas on elliptic curves; for the background we refer to Silverman [59] Chapter IV.

Lemma 9.4. *Let E be an elliptic curve over $\mathbb{C}$, and let O denote its origin. Let $f : E \to E$ be a holomorphic map such that $f \neq Id_E$. Then the following are equivalent:*

1. *f is a translation;*
2. *$f_*|H_1(E,\mathbb{Z}) = Id_{H_1(E,\mathbb{Z})}$;*
3. *f has no fixed points.*

Proof: Let g be the isogeny such that $f(x) = g(x) + a$ $(a = f(O))$ (i.e. $g =$ alb f). Then, if f has no fixed points, the isogeny $x \mapsto g(x) - x$ is non-surjective, hence constant equal to zero, which means that f is a translation. The rest follows from the fact that, if $E = \frac{\mathbb{C}}{\Lambda}$, then $\Lambda = H_1(E,\mathbb{Z})$.

Remark We point out that Lemma 9.4 holds as well if we change E for a simple Abelian variety A. A counterexample can be easily constructed if $A = A_1 \times A_2$; choose $f = (f_1, f_2)$ where f_1 is a non-trivial translation on A_1 and f_2 is any endomorphism of A_2. Then it is clear that f is fixed-point free but is not a translation.

Lemma 9.5. *Let E be an elliptic curve defined over $\mathbb{C}$. The following statements hold.*

1. *$Aut_0(E)$ (group of invertible isogenies, i.e. those which fix the origin O) is a cyclic group of cardinal* 2, *or* 4 *(in case $E = \frac{\mathbb{C}}{\mathbb{Z}[i]}$), or* 6 *(in case $E = \frac{\mathbb{C}}{\mathbb{Z}[\omega]}$ where $\omega^3 = 1 \neq \omega$).*

2. *Let G be a finite group of biholomorphisms of E, and H be the subgroup of translations of G. Then $G/H = \mathbb{Z}_n$, where n belongs to the set* 2, 3, 4 *or* 6. *More precisely, G is a semidirect product of $\mathbb{Z}_n$ by H; i.e. the exact sequence*

$$1 \to H \to G \to G/H = \mathbb{Z}_n \to 1$$

splits.

Proof:

1. We recall (see Silverman [59]) that the ring of isogenies of E (denoted $End(E)$) has the following description:

$$End(E) = \{\alpha \in \mathbb{C} : \alpha\Lambda \subset \Lambda\}.$$

Therefore $End(E)$ is either $\mathbb{Z}$ or an order in an imaginary quadratic field. The group of invertible isogenies of E is then forced to be finite (the equation $N_{K|\mathbb{Q}}(x) = 1$ has only a finite number of integer solutions $x \in \mathcal{O}_K$ for K a field with no real embeddings). The following Claim proves 1; we leave the details to the reader.

Claim: Let α be a root of unity, and let m be its exact order in $\mathbb{C}^*$. Suppose that α is quadratic over $\mathbb{Q}$. Then m belongs to the set $\{3, 4, 6\}$.

Proof(s) of the Claim: We know that α generates a field extension of degree $\phi(m)$ over $\mathbb{Q}$. The hypothesis leads to $\phi(m) = 2$, which only the above listed numbers satisfy. This concludes the proof.

Another different proof comes from the following observation. We may write $\alpha = e^{2\pi i r}$, for some $r \in \mathbb{Q}$. Therefore, the number $\alpha + \overline{\alpha} = 2\cos(2\pi r)$ is rational (for α is quadratic) and simultaneously an algebraic integer, hence an integer. This forces $2\cos(2\pi r)$ to be one of the following numbers $-2, -1, 0, 1, 2$. The Claim follows.

2. By Lemma 9.4, we know that H is the kernel of the representation $\rho : G \to GL(H_1(E,\mathbb{Z}))$ given by $g \mapsto H_1(g)$ (we point out that the H^1 version is the contragredient representation of ρ). Also, again by Lemma 9.4, ρ factors through the group monomorphism

$$\mathrm{Aut}_0(E) \hookrightarrow GL(H_1(E,\mathbb{Z}))$$

by sending $g \in G$ to the isogeny $g - g(O)(=$ alb $g)$ in $\mathrm{Aut}_0(E)$. Therefore, the quotient G/H is injected as a subgroup of $\mathrm{Aut}_0(E)$, hence cyclic, $G/H \cong \mathbb{Z}_n$, and n divides 2, or 4 or 6; the case $n = 3$ or 6 implies $E = \frac{\mathbb{C}}{\mathbb{Z}[\omega]}$, and the cases 4 and 6 are already mentioned in Statement 1.

Let p be the canonical projection $G \to G/H$. Let us construct a group homomorphism $s : G/H(\cong \mathbb{Z}_n) \hookrightarrow G$ such that $p \circ s = id_{\mathbb{Z}_n}$. Choose an element $g \in G$ such that $p(g)$ generates G/H, i.e. such that n is the least positive integer satisfying $g^n \in H$. Then, by Lemma 9.4, g has a fixed point, hence g^n is a translation with a fixed point, therefore $g^n = \mathrm{Id}$ again by Lemma 9.4. This proves s to be a section, hence G is a semidirect product of H and G/H by taking the action of G/H on H by inner automorphisms (here we use that H is abelian), thereby establishing the statement.

Proposition 9.6. *Let $S = (C' \times E')/G$ satisfy the hypotheses of this Chapter 9 ($p_g = 1, q = 2$); suppose also that G does not act trivially on $H^1(E')$. Let $H = Ker\,(G \to Aut\ H^1(E,\mathbb{Z}))$. Then $G/H = \mathbb{Z}_n$ with $n = 2,3,4,6$. Also, there exists another elliptic isotrivial fibration S' such that the monodromy group of S' is cyclic and $T(S) = T(S')$ are isomorphic Hodge structures. Furthermore, $\mathbb{Z}_n$ acts faithfully on H^1 of a fibre.*

More precisely, if S' is a desingularisation of the surface

$$(C'/H \times E'/H)/\mathbb{Z}_n = (C \times E)/\mathbb{Z}_n,$$

where $C'/H = C$ and $E'/H = E$ (E is an elliptic curve), then S' satisfies the above properties.

Proof: Take S' to be a minimal desingularisation of the following surface: $S_1 = (C'/H \times E'/H)/\mathbb{Z}_n$. Henceforth we write $C := C'/H$, $E = E'/H$. Note that H acts on E' by translations, therefore E is also elliptic. We define φ to be the natural morphism $S \to S_1$ coming from the G-action. Then:

- φ yields an isogeny Alb $S \to$ Alb $S_1 =$ Alb S'. Indeed, by Proposition 9.2

$$q(S') = \dim H^1(C \times E, \mathcal{O})^{\mathbb{Z}_n} = g(C/\mathbb{Z}_n) + g(E/\mathbb{Z}_n)$$
$$= g(C'/G) + g(E'/G) = q(S).$$

The second equality above holds, for instance, because $H^1(C \times E)^{\mathbb{Z}_n} = H^1(C)^{\mathbb{Z}_n} \oplus H^1(E)^{\mathbb{Z}_n}$ for a diagonal action of $\mathbb{Z}_n$ on each component.

- φ induces an isomorphism of transcendant lattices $T(S) = T(S')$. Indeed, let us write

$$H^2(S) = H^2(C' \times E')^G = \left(H^2(C' \times E')^H\right)^{\mathbb{Z}_n}$$
$$= H^2(C') \oplus H^2(E') \oplus \left[\left(H^1(C') \otimes H^1(E')\right)^H\right]^{\mathbb{Z}_n}$$

Now, as H acts trivially on the homology of E', the natural morphism

$$C' \times E'/H \to C \times E$$

induces an isomorphism on cohomology, as can be seen from the isomorphism

$$\left(H^1(C') \otimes H^1(E')\right)^H = H^1(C) \otimes H^1(E).$$

Both surfaces above are acted on by $G/H = \mathbb{Z}_n$, which in turn gives, by Lemma 9.3, the following isomorphism

$$T(S) = T(C \times E)^{\mathbb{Z}_n} = T(S').$$

This completes the proof.

Proposition 9.7. *Let S satisfy the hypotheses of Proposition 9.6. Then there exists a birational morphism $h : R \to S$ such that h is a composition of blow-ups of S centred at finite subsets, and there exists a morphism $t : R \to S'$ above the canonical projection*

$$p : (C' \times E')/G \to (C \times E)/H.$$

The induced morphism h_* *on cohomology is an epimorphism, and the following map* $t_* \circ h^*$ *induces a Hodge isomorphism between* $H^1(S)$ *and* $H^1(S')$ *and also between* $T(S)$ *and* $T(S')$. *These isomorphisms are induced by an algebraic cycle on* $S \times S'$, *by Subsection 4.2.*

Proof: The proof follows from Lemma 9.3, Proposition 9.6 and also Subsection 4.2.

Chapter 10

Hodge structures coming from Abelian surfaces

The idea of the proof of Theorem 1.1 is the following. We notice that the Hodge structure $T(S)$ of weight two is isomorphic to $T(A')$, where A' is an Abelian surface, possibly different from A; this has been proven in some of the cases for S, at the beginning of Chapter 9, and will be seen to hold for every S in Section 11. This basically means that we will end up by working with Hodge structures associated to Abelian varieties. This section is devoted to a thorough understanding of the Hodge structure $T(A)$ and the tools needed for the proof of Theorem 1. An excellent reference providing many of the results we need in this thesis is Moonen and Zarhin [66].

It is well known that the ring of endomorphisms of an Abelian variety with coefficients over $\mathbb{Q}$ is a semisimple algebra with a positive antiinvolution. This gives a complete classification of the endomorphism algebra of an Abelian variety $End^0(A) = End(A) \otimes \mathbb{Q}$.

The following results will be used throughout the section.

Theorem 10.1 (Albert classification for Abelian surfaces). *[51] Let A be a simple complex Abelian surface. Let ρ denote its Picard number. Then the following cases occur for $End^0(A)$:*

1. *$End^0(A) = \mathbb{Q}$ (Type $I(1)$). We have $\rho(A) = 1, dim\ T(A) = 5$.*

2. *$End^0(A) = F_0$ a real quadratic field (Type $I(2)$). Also $\rho(A) = 2$, $dim\, T(A) = 4$.*

3. *$End^0(A) = D$ where D is an indefinite quaternion algebra over $\mathbb{Q}$ (Type $II(1)$). In this case, $\rho(A) = 3$, $dim\, T(A) = 3$. We say that A is of QM (quaternionic multiplication) type.*

4. *$End^0(A) = F$ where F is a CM-field of degree 4 and F_0 is its (quadratic) totally real subfield. (Type $IV(2,1)$). $\rho(A) = 2$, $dim\, T(A) = 4$.*

Theorem 10.2. *Let A be a non-simple Abelian surface. The values of ρ according to its decomposition are:*

1. *$\rho = 2$ (and then $dim\, T(A) = 4$) if $A \sim E_1 \times E_2$, E_1 nonisogenous.*

2. *$\rho = 3$ (and then $dim\, T(A) = 3$) if $A \sim E \times E$, E being non-CM.*

3. *$\rho = 4$ (and then $dim\, T(A) = 2$) if $A \sim E \times E$, E a CM-elliptic curve.*

10.1 Abelian varieties over $\mathbb{C}$

For this section we follow van Geemen [1] Sect. 3. See also Mumford [45], Lange-Birkenhake [34]. Let X be an Abelian variety over the complex numbers, that is $X \simeq V/\Lambda$ for some lattice Λ in a complex vector space V of dimension $dim_{\mathbb{C}} V = g$, and X is a projective variety. Note that $\Lambda = \pi_1(X, 0) = H_1(X, \mathbb{Z})$ and $V = H_1(X, \mathbb{R})$ is the universal cover of X; also $V = T_0(X)$ and multiplication by $i \in \mathbb{C}$ on V corresponds to an $\mathbb{R}$-linear map:

$$J : H_1(X, \mathbb{R}) \to H_1(X, \mathbb{R}) \text{ with } J^2 = -I.$$

The map J allows us to recover this structure of complex vector space on $H_1(X, \mathbb{R})$ - it actually corresponds to the complex structure on X, i.e. on the (trivial) real-valued C^∞ tangent bundle of X. Thus

$$T_0(X) \simeq (H_1(X, \mathbb{R}), J).$$

Any embedding $\theta : X \hookrightarrow \mathbb{P}^n$ defines a polarisation $E := c_1(\mathcal{O}(1)) \in B^1(X) \subset H^2(X, \mathbb{Q})$. By duality, E defines a map, which we denote by the same name

$$E : \bigwedge^2 H_1(X, \mathbb{Q}) \to \mathbb{Q}$$

and this map satisfies the Riemann relations (here we extend E $\mathbb{R}$-linearly)

$$E(Jx, Jy) = E(x, y),\ E(x, Jx) > 0$$

for all $x, y \in H_1(X, \mathbb{R})$, with $x \neq 0$ for the last condition. This last condition also shows E to be non-degenerate.

Conversely, $X = V/\Lambda$ is an Abelian variety if and only if there exists $E : \bigwedge^2 \Lambda \to \mathbb{Q}$ satisfying the Riemann relations. For more details, see Paragraph 10.1.1.

The cohomology of X and its Hodge structure are completely determined by $H^1(X, \mathbb{Q})$ and its Hodge structure:

$$H^k(X, \mathbb{Q}) = \bigwedge^k H^1(X, \mathbb{Q}),\ H^{p,q} = \left(\bigwedge^p H^{1,0}(X)\right) \otimes \left(\bigwedge^q H^{0,1}(X)\right)$$

One may ask whether the biholomorphism class of X can be retrieved from the Hodge decomposition on $H^1(X)$. The question has a positive answer if we take integer coefficients; if not, we will almost recover X, i.e. X will be obtained modulo isogeny. This will be proven in the following lines.

The Hodge decomposition on $H^1(X, \mathbb{C})$ yields

$$H^1(X, \mathbb{R}) \hookrightarrow H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$$

and the image of $H^1(X, \mathbb{R})$ is well known to be $\{(v, \bar{v}) : v \in H^2(X, \mathbb{R})\}$. Thus one can define a linear map $J_{\mathbb{C}}$ on $H^1(X, \mathbb{C})$ by setting $J'_{\mathbb{C}}|H^{1,0}(X) = i\ id_{H^{1,0}(X)}$ and $J'_{\mathbb{C}}|H^{0,1}(X) = -i\ id_{H^{0,1}(X)}$ that restricts to a $\mathbb{R}$-linear operator J' $H^1(X, \mathbb{R})$:

$$J' : H^1(X, \mathbb{R}) \to H^1(X, \mathbb{R}),\ \text{with } (J')^2 = -I.$$

By the well-known duality between H^i and H_i we define

$$J := (J')^* : H^1(X, \mathbb{R}) \to H^1(X, \mathbb{R}),\ \text{with } J^2 = -I.$$

To obtain X, we have to take the quotient of the vector space $(H_1(X, \mathbb{R}), J)$ by a lattice $\Lambda \subset H_1(X, \mathbb{R})$. This is where we need integer coefficients: we know $\Lambda \subset$

$H_1(X,\mathbb{Q}) \subset H_1(X,\mathbb{R})$, and $H_1(X,\mathbb{Q})$ is obtained as the dual of $H^1(X,\mathbb{Q})$, but we cannot reconstruct $H_1(X,\mathbb{Z})$ unless we are given $H^1(X,\mathbb{Z})$ from the very beginning. However, the inclusion $\Lambda \subset H_1(X,\mathbb{Q})$ allows us to know Λ up to commensurability, i.e. any other lattice Λ' in $H_1(X,\mathbb{Q})$ satisfies $\Lambda : \Lambda \cap \Lambda'$ and $\Lambda' : \Lambda \cap \Lambda'$ are finite abelian groups. This leads to the following definitions.

Definition Two Abelian varieties X and Y are said to be **isogenous**, $X \sim Y$ (also denoted $X \approx Y$) if and only if there is a finite, surjective map (an *isogeny*) $\varphi : Y \to X$. (If $X \sim Y$ then it is not difficult to see that there exists also a finite, surjective map $X \to Y$; see below.)

Remark We note also that the existence of such maps implies the existence of homomorphisms with the same properties: indeed, any holomorphic map between two Abelian varieties is the translate of a homomorphism of group schemes (see Mumford [45] .)

Given an isogeny $\varphi : Y \to X$, the group $\varphi_*(\pi(Y))$ is a subgroup of finite index of $\pi_1(X)$ and thus $\pi_1(X) \simeq N\pi_1(X) \subset \varphi_*(\pi_1(X))$ for some integer N. Therefore one has a finite, surjective map $X \to Y$. Also, φ induces φ^* on cohomology

$$\varphi^* : H^1(X,\mathbb{Q}) \to H^1(Y,\mathbb{Q}), \text{ and } \varphi^*(H^{1,0}(Y)) \subset H^{1,0}(X).$$

The line above is equivalent to saying that φ^* induces an isomorphism of Hodge structures. Conversely, the existence of such map φ^* implies that the Abelian varieties X and Y are isogenous. An isogeny $\varphi : Y \to X$ thus induces isomorphisms $B^p(X) \to B^p(Y)$. Moreover, using pullback of cycles by φ and $\hat{\varphi}$ we obtain the following.

Lemma 10.3. *Let X and Y be two isogenous Abelian varieties. Then the Hodge (p,p)-conjecture holds for X if and only if it holds for Y.*

Now we will describe all the homomorphisms from an Abelian variety A into another Abelian variety B in terms of their integral Hodge structures.

Proposition 10.4. *Let A, B be two complex Abelian varieties. Then the canonical map*

$$B_{\mathbb{Z}}(H^1(B,\mathbb{Z})^* \otimes H^1(A,\mathbb{Z})) \leftarrow Hom(A,B)$$

is an isomorphism. Here * *denotes the usual dual group* $Hom(G, \mathbb{Z})$ *of a free Abelian group of finite type.*

Proof: The left hand side of the isomorphism consists of group homomorphisms $\phi : H^1(B, \mathbb{Z}) \to H^1(A, \mathbb{Z})$ that preserve the Hodge structures, i.e. such that they induce isomorphisms of Hodge structures $\phi(H^{1,0}(B)) \subset H^{1,0}(A)$. Let $\varphi := \phi^*$; then φ induces a homomorphism $H_1(A, \mathbb{R}) \to H_1(B, \mathbb{R})$ such that both the complex structure and the integral lattices are preserved - $\varphi(H_1(A, \mathbb{Z})) \subset H_1(B, \mathbb{Z})$ by duality; the assertion on the complex structures is already checked above in this paragraph.

10.1.1 Polarisations of Abelian varieties

The main references in this paragraph are [34] Sect.5.2. and van Geemen [1] Sect.3.

Definition Let $X = V/\Lambda$ be a complex torus. Then a *polarisation* on X is a skew-symmetric bilinear form

$$E : \Lambda \times \Lambda \to \mathbb{Z}$$

such that E gives a polarisation of the integral Hodge structure $H_1(X, \mathbb{Z}) = (H^1(X, \mathbb{Z}))^*$.

Definition (*Equivalent definition*) Let $X = V/\Lambda$ be a complex torus. and let E be a skew-symmetric bilinear form

$$\Lambda \times \Lambda \to \mathbb{Z}$$

with integral values. Then E naturally defines an element in $H^2(X, \mathbb{Z}) = \bigwedge^2 H^1(X, \mathbb{Z})$. E will be called a *polarisation* if and only if the following conditions are fulfilled:

(i) E corresponds to an element in $B^1_{\mathbb{Z}}(X) = H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$.

(ii) For every element $x \in V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, $E(x, Jx) > 0$; i.e. the bilinear form $E(., J.)$ defined on V is positive definite.

Remark There are two separate conditions defining a polarisation on X. The property (i) alone is equivalent to the assertion that $E = c_1(L)$ for some holomorphic line bundle on X. Both properties (i) and (ii) holding simultaneously are equivalent to saying that such L is a positive definite line bundle and therefore ample (see Theorem 3.11; also Griffiths-Harris [20] p.148).

Proposition 10.5. *A complex torus X admits a polarisation if and only if X is an Abelian variety. In that case, positive definite line bundles are in one-to-one correspondence with ample line bundles on X, and the polarisations are in bijection with ample classes in $NS(X)$.*

Proof: It is an immediate consequence of the Kodaira embedding Theorem 3.11 – see last Remark.

Remark Let A be an Abelian variety. Any 2-form $E = c_1(L)$ for L a line bundle on X defines a morphism

$$\phi_L : A \to \hat{A}$$

which depends only on $E = c_1(L)$. A polarisation defines an isogeny $A \to \hat{A}$ (see Lange-Birkenhakke [34] Sect.2.4. for more details). Also *op. cit.* Theorem 2.5.5. yields a complete description of the isogenies $A \to \hat{A}$ induced by polarisations.

Definition A *polarised morphism* (or *morphism of polarised Abelian varieties* $f : (A, L) \to (B, M)$ is a homomorphism $f : A \to B$ such that $f^*c_1(M) = c_1(L)$.

Proposition 10.6. *(Lange-Birkenhake [34] Cor. 5.1.9) The group of automorphisms of the polarised Abelian variety (X, L) is finite.*

Proof: Suppose f is an automorphism of (X, L). Then $f^*L \otimes L^{-1} \in Pic^0(X)$ so that $\phi_L = \phi_{f^*L} = \hat{f}\phi_L f$, hence $f'f = 1$. Thus $Aut(X, L) \subset \{f \in End(X) | \mathrm{Tr}(f'f) \leq M\}$, i.e. $Aut(X, L)$ is the intersection of a compact set of $End(H_1(X, \mathbb{R}))$ with the discrete subset $End(X)$, hence finite. The positive definiteness of the Rosati involution $f \mapsto f'$ is established in Proposition 10.8.

Corollary 10.7. *(Serre; see Lange-Birkenhakke [34] Cor. 5.1.10) Let f be an automorphism of a polarised Abelian variety (X, L) and $n \geq 3$ an integer. Let X_n denote the subgroup of points of n-torsion of X. If $f|X_n = id_{X_n}$, then $f = 1_X$. As a result, we get an injection $Aut(X, L) \hookrightarrow Aut(X_n) = GL_{2g}(\mathbb{Z}/n\mathbb{Z})$ for any $n \geq 3$, which yields an easy bound for the cardinal of $Aut(X, L)$.*

Proof: Note that the only unipotent automorphism of (X, L) is the identity, as $End^0(X) = End(X) \otimes \mathbb{Q}$ is a semisimple algebra. Without loss of generality, we may suppose that f is of prime order p; let ξ be a primitive p–th root of unity which is an eigenvalue of $f|T_0X$. By assumption, $X_n \subset \mathrm{Ker}(1_X - f)$. Hence there is a $g \in End(X)$ such that $ng = 1_X - f$. This implies that there is an algebraic integer η, namely an eigenvalue of g, such that

$$n\eta = 1 - \xi.$$

Applying the norm of the field extension $\mathbb{Q}(\xi)|\mathbb{Q}$ we get

$$n^{p-1}N_{\mathbb{Q}(\xi)|\mathbb{Q}}(\eta) = N_{\mathbb{Q}(\xi)|\mathbb{Q}}(1-\xi) = (1-\xi)\dots(1-\xi^{p-1}) = p.$$

This is impossible, since p is a prime and $n \geq 3$.

10.1.2 Endomorphisms of Abelian varieties

Let A be a complex Abelian variety. Then the *endomorphism ring* of A, $End(A)$, is defined to be the group of morphisms $f : A \to A$ such that $f(0) = 0$ – by a rigidity argument, these are precisely the endomorphisms of A as an algebraic group (Mumford [45] p. 43). We call $End^0(A) = End(A)_{\mathbb{Q}}$.

Let A, L be a polarised Abelian variety. Then we may define the *Rosati involution* (or more precisely, antiinvolution) $\iota = \iota_L$ on $End^0(A)$ as $\iota(\varphi) := \phi_L^{-1} \circ \hat{\varphi} \circ \phi_L$ (this involution is not always defined over $\mathbb{Z}$).

Proposition 10.8. *(B.B. Gordon [35], Lemma 1.12.1; Mumford [45] Th. IV.21.1) $End^0 A$ together with ι_L is a semisimple algebra endowed with a positive antiinvolution.*

Proof: The Poincaré reducibility theorem yields $A \sim B_1^{n_1} \times \dots \times B_r^{n_r}$ for B_i pairwise nonisogenous Abelian varieties; thus

$$End^0(A) = \prod_{i=1}^{r} M_{n_i}(End^0(B_i)),$$

where $End^0(B_i)$ are division algebras and $M_n(D)$ stands for the ring of $n \times n$ matrices over D. We now prove that ι is positive. There exists a positive definite Hermitian form H on $T_0A = H_1(A, \mathbb{R})$ defined by $H(u,v) = E(u, iv) + iE(u,v)$, where E is given by L. Then $\iota(\varphi)$ is exactly the adjoint of φ with respect to the Hermitian metric H, and therefore $Tr(\varphi\iota(\varphi))$ is positive for every nonzero $\varphi \in End^0(A)$.

Let us suppose A to be simple. Last proposition reduces the structure of $End^0(A)$ to a certain number of possibilities, to be listed below.

Theorem 10.9 (Albert classification). *(B.B. Gordon [35] Th. 1.12.2; also Mumford [45] pp. 201-203) Let A be a simple complex Abelian variety. Let F be the centre of $End^0(A)$ and let F_0 be the subfield of elements of F fixed by the Rosati involution. Then F_0 is a totally real number field of degree e_0 over $\mathbb{Q}$ and F (whose degree $[F : \mathbb{Q}] =: e$) is either equal to F_0 or a totally imaginary extension of F_0, hence a CM-field. We will denote $d^2 = [End^0(A) : F]$. $End^0(A)$ is one of the following types:*

1. *$End^0(A) = F = F_0$ is a totally real algebraic number field, and the Rosati involution acts as the identity. The relation $e|g$ holds. (We say then that A is of type $I(e_0)$).*

2. *$F = F_0$ is a totally real number field, and $End^0(A)$ is a division quaternion algebra over F such that every simple component of $End^0(A) \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to $M_2(\mathbb{R})$; there is an element $\beta \in End^0(A)$ such that ${}^t\beta = -\beta$, and $\beta^2 \in F$ is totally negative; and the Rosati involution is given by $\iota(\alpha) = \beta^{-1} \cdot {}^t\alpha \cdot \beta$. The relation $2e|g$ holds. (A is said to be of type $II(e_0)$).*

3. *$F = F_0$ is a totally real number field, and $End^0(A)$ is a division quaternion algebra over F such that every simple component of $End^0(A) \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to the Hamiltonian quaternion algebra $\mathbb{H}$ over $\mathbb{R}$; and $\iota(\alpha) =^t \alpha$. The relation $2e|g$ holds. (A is of type $III(e_0)$).*

4. *F_0 is a totally real number field, and F is a totally imaginary quadratic extension of F_0 (i.e. $e = 2e_0$), and $End^0(A)$ is a division algebra with centre F; the restriction of the Rosati involution to K acts as the restriction of complex conjugation to K. The relation $e_0 \cdot d^2|g$ holds. (A is of type $IV(d, e_0)$).*

10.2 The Albanese map

To every algebraic variety X we can associate an Abelian variety – the *Albanese variety* Alb X – satisfying a certain universal property. In the case when X is smooth and projective, Alb X will turn out to be connected to another Abelian variety which is canonically associated to every projective smooth variety: the Picard variety.

Definition Let X be an algebraic variety defined over an algebraically closed field k. An *Albanese variety* (or *map*) is a couple (A, f) consisting of an Abelian variety A and a morphism $f : X \to A$ such that the following properties hold:

i. (X, f) generates A as a group;

ii. for every morphism $g : X \to B$ into an Abelian variety B, there exists a homomorphism $g_0 : A \to B$ and a constant $c \in B$ such that $g = g_0 f + c$.

We observe immediately that the homomorphism g_0 in *ii.* is unique. If we have $g = g'f + c'$, with a homomorphism g', then $g' = g_0$. We shall say that g_0 is the homomorphism *induced* by g.

Remark: We observe that properties *i.* and *ii.* above are equivalent to the property **(P)** For every morphism $g : X \to B$ into an Abelian variety B, there exists a unique morphism of Abelian varieties $h : A \to B$ such that $g = h \circ f$ up to translation.

In case **(P)** is adopted as a definition, it should then be proven that (X, f) generates A as a group, and thus the equivalence between this new definition and the one above will be established. This is proven in next proposition below – the reader may find a wider statement in the given source.

Proposition 10.10. *(see Ueno [70] Lemma 9.14) For a complete algebraic variety$V$, the Albanese map (V, f) generates $A =$ Alb$X$ as a group. More precisely, there exists a natural number n such that the following map is surjective*

$$f^n : V \times V \times \ldots \times V \to A$$

$$f^n(p_1, \ldots, p_n) = f(p_1) + \ldots + f(p_n)$$

Proof: We observe that, if $Z_k = Im(f^k)$ and $0 \in Im(f)$, then the $_k$ form an ascending chain of irreducible closed subsets of A

$$0 \subset Z_1 \subset \ldots \subset Z_k \subset \ldots = Z,$$

where Z is the union of all the Z_k, hence closed and irreducible, for the chain Z_k stops at some k_0. Also, the fact that $Z_k + Z_l \subset Z_{k+l}$ implies that Z is a subgroup of A. Indeed, translations by elements of Z act as automorphisms of Z by restriction, which also means that their inverses take Z into Z. This is exactly the same as saying that image of the morphism of varieties $Z \times Z \to Z$ given by $(x, y) \mapsto x - y$ falls into Z, thereby proving that Z is a subgroup.

In the case when $Im(f)$ does not contain 0 follows from the observation that for the map $g(x) = f(x) - f(x_0)$, the sets $Z_k(f)$ correspond to translates of $Z_k(g)$, $Z_k(f) = Z_k(g) + k.f(x_0)$.

As said in the beginning of this section, the Albanese map can be constructed for general algebraic varieties. The most general result to be found in the literature is the following:

Theorem 10.11. *(see Lang [65] Thm.II.3.11) Let V be a (non necessarily projective or smooth) variety over an algebraically closed field k. Then there exists an Albanese variety (A, f) of V, f being a rational map. The Abelian variety is uniquely determined up to isomorphism, and f is determined up to a translation. Also, it is seen from the construction of (A, f) that the Albanese variety (or map) depends only on the birational isomorphism class of V. f is regular at smooth points of V.*

A special case of the Albanese variety is when $X = C$ is a smooth projective curve. In this case we call $Alb(C) =: JC$ the *Jacobian* of C. The Jacobian variety of C is isomorphic to its Picard variety, and -after choosing a point $O \in C$ - the Albanese map reads

$$C \to Pic^0(C), \; P \mapsto (P) - (O).$$

One may ask whether this map is an embedding. Indeed, this is the case when $g(C)$ is positive. When C is a curve of genus one, then C is an elliptic curve and thus $C = JC$. If $g(C) \geq 2$, then it is a strict embedding. A proof for this may be found in Lang [65] Prop. II.2.4.

The Albanese variety is closely related to the Picard variety, as the next Proposition shows.

Proposition 10.12. *(Matsusaka; see Lang [65] IV.4) The Picard variety and the*

Albanese variety of a projective smooth variety over a field k are a dual pair of Abelian varieties.

We include the following theorem, frequently used throughout this thesis.

Theorem 10.13. *(Ueno [70] Th. 9.7) Let X be a complex projective variety. Then the Picard variety of X has the following structure as a complex torus:*

$$Pic^0(X) = \frac{H^1(X,\mathcal{O}_X)}{H^1(X,\mathbb{Z})/\mathrm{tors}},$$

where the inclusion of $H^1(X,\mathbb{Z})/\mathrm{tors}$ into $H^1(X,\mathcal{O}_X)$ comes from the Hodge decomposition of $H^1(X,\mathbb{C})$ (or equivalently, from the exponential exact sequence, see Section 3.3). The Albanese variety of X is the dual complex torus of the Picard variety.

10.2.1 The Albanese fibration

Let us consider a smooth projective variety X defined over an algebraically closed field k, and its Albanese map $a = alb_X$, $A =$ Alb X. Then a factorises as

$$X \to a(X) \subset A$$

. The resulting morphism $X \to a(X)$ is proper and surjective. We have the following definition.

Definition For the Albanese mapping $a : X \to a(X)$, let $X \to W \to a(X)$ be its the Stein factorisation. We call $p : X \to W$ the *Albanese fibration of* X.

Proposition 10.14. *(Ueno [70] Prop. 9.19) Assume that the image of the Albanese map $a : X \to \mathrm{Alb}X$ is a curve C. Then C is a non-singular curve of genus $g = dim(Alb\,X)$ and the fibres of $a : X \to C$ are connected.*

Proof: Let $\nu : \hat{C} \to C$ be the normalisation of the curve C and let JC be the Jacobian variety of C. Then a factorises through ν, since X is normal (see Proposition 6.15); we call $p : X \to C$ and $q := \mathrm{inc} \circ \nu : C \to \mathrm{Alb}\ X$. Thus $a = q \circ p$. Let $g = \mathrm{alb}_C : C \to JC$. Then $\mathrm{alb}(p) : \mathrm{Alb}\ X \to JC$ satisfies $Id_{\mathrm{Alb}(X)} = \mathrm{alb}(a) =$

alb$(q) \circ$ alb(p). Therefore alb(p) is a monomorphism of Abelian varieties, and is also surjective as p is, hence an isomorphism. An immediate consequence of this fact is that $C \hookrightarrow JC \cong \text{Alb } X$, and therefore $C \cong a(X)$, thereby establishing the proposition.

Proposition 10.15. *Let S be a projective smooth surface and B be a smooth curve. Let $p : S \to B$ be a fibration. Then p is the Albanese fibration if and only if $q(S) = g(B)$.*

Proof: One implication is already solved in Proposition 10.14. Suppose $q(S) = g(B)$. Then alb(p) is surjective and, by a dimension argument, of finite kernel, hence an isogeny. This implies directly that alb_S has a one-dimensional image C, and thus p factors through C. Hence $C = B$ and p is the Albanese fibration, thus concluding the proof.

The following Proposition seems to be well-known, and is a generalisation of an Exercise proposed in Beauville [4]. We include a proof for the lack of a suitable reference.

Proposition 10.16. *Let $p : X \to B$ be a fibration of complex projective smooth varieties. Then $q(X) \leq b + f$, where $b = q(B)$ and $f = q(F_b)$ for $b \in B$ in the smooth locus of p and $F_b = p^{-1}(b)$. Also, if equality holds, then Alb(F_b) is a constant family over the smooth locus of p.*

Proof: Consider the map $p^* : \text{Pic}(B) \to \text{Pic}(X)$. For $\mathcal{M}$ a line bundle over B we have $p_* p^* \mathcal{M} = \mathcal{M}$, hence p^* is injective on Picard groups. We also note the following. Let $\tilde{B}$ denote the smooth locus of p, and let $b \in \tilde{B}$. Then the inclusion $F_b \subset X$ yields a family of maps

$$\alpha_b : \text{Alb } F_b \to \text{Alb } X.$$

The images $\alpha_b(\text{Alb } F_b)$ $(b \in \tilde{B})$ form a continuous family of subvarieties of Alb X, which is constant. Indeed, if $A = V/\Lambda$ is an Abelian variety over $\mathbb{C}$, then Abelian subvarieties are in one-to-one correspondence with primitive sublattices $\Lambda' \subset \Lambda$ such that $\mathbb{R}\Lambda'$ forms a complex subspace of V; this is clearly a discrete subset of A, which completes the argument. We denote this common immage by A.

Consider the sequence

$$0 \to \mathrm{Pic}^0(B) \to \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(F_b),$$

for $b \in \tilde{B}$. The first map is p^* and the second is the usual restriction $r(\mathcal{L}) = \mathcal{L}|F_b$. Let us prove that the kernel of the restriction map r is a finite union of translates of $p^*\mathrm{Pic}^0(B)$.

It is clear that $r \circ p^* = 0$ - apply Lemma 10.18 (or either use moving lemma to have the support of a divisor D away from b). To prove the second part of the assertion, we use the well-known triviality criterion for a line bundle L: if L is a line bundle over a proper variety X, then L is trivial if and only if $h^0(L) \neq 0 \neq h^0(L^{-1})$.

Let $\mathcal{L} \in \mathrm{Pic}^0(X)$ be such that $\mathcal{L}|F_b$ is trivial. This means that $h^0(F_b, \mathcal{L}|F_b)$ and $h^0(F_b, \mathcal{L}^*|F_b)$ are non-zero. As these numbers are constant for every $b \in B$ (not necessarily smooth), this implies that $\mathcal{L} = p^*(\mathcal{M})$ for some $\mathcal{M} \in \mathrm{Pic}^(B)$ (see Lemma 10.18). It is not hard to see, by means of a divisor argument, that $\mathcal{M}(= p_*\mathcal{L})$ is numerically equivalent to 0 in B, which in turn yields $\mathcal{M} \in \mathrm{Pic}_\tau(B)$, containing $\mathrm{Pic}^0(B)$ as a subgroup of finite index.

The sequence

$$\mathrm{Alb}\ F_b \to A \subset \mathrm{Alb}\ X \to \mathrm{Alb}\ B$$

has the following dual sequence

$$\mathrm{Pic}^0(B) \to \mathrm{Pic}^0(X) \to \hat{A} \to \mathrm{Pic}^0(F_b),$$

where the map $\mathrm{Pic}^0(X) \to \hat{A}$ is surjective and the map $\hat{A} \to \mathrm{Pic}^0(F_b)$ has a finite kernel. Two conclusions may be drawn from these facts:

1. $q(X) - q(B) = \dim \mathrm{A} \leq q(F_b)$; and

2. in case $q(X) = q(B) + q(F_b)$, we have a continuous family

   $$\mathrm{Alb}\ F_b \to \mathrm{A}$$

   of isogenies, which must then be constant, and therefore Alb F_b is a constant family, isogenous to A.

This concludes the proof.

10.2.2 Codimension-one cycles and the Picard functor

We include a summary of the basic properties of the Picard functor, following Grothendieck [21]; another very useful reference is Mumford [47] pp.22-24. For the preliminaries needed, the most suitable reference is Hartshorne [63], Sect. III.9-III.12. We include an application to the computation of $CH^1(X \times Y)$ in terms of X and Y, which we use several times along this thesis. In this section we will use $\mathcal{S}ch/k$ to denote the category of k-schemes of finite type. $\mathcal{V}ar$ will denote the category of varieties over k.

Definition Let X be a projective smooth variety over ar arbitrary field k. The *Picard functor* of X is the (contravariant) functor

$$(\mathcal{S}ch/k)^{opp} \to \{\text{Abelian groups}\}$$

defined by $Pic_{X/k}(S) := Pic(X \times S)/pr_S^* Pic(S)$.

Theorem 10.17. *([47] p.22) Let us suppose that X is a projective smooth variety over a field k, and also that X has at least one k-rational (closed) point. Then there exists a group-scheme $\underline{Pic}_{X/k}$ over k, proper and locally of finite type over k, representing the Picard functor. More precisely, $\underline{Pic}_{X/k}$ satisfies the following property*

$$\frac{Pic(X \times T)}{pr_T^* Pic(T)} = Hom_k(T, \underline{Pic}_{X/k}).$$

The identity component of $\underline{Pic}_{X/k}$ represents the functor $Pic^0_{X/k}$ defined analogously for Pic^0.

Basic properties of the Picard functor: Let us draw some immediate consequences from last theorem. From now on suppose for simplicity that k is an algebraically closed field of characteristic zero. Let $S = \text{spec } k$. Then the theorem yields an isomorphism of abstract groups $Pic(X) = \underline{Pic}_{X/k}$. It can be checked that this isomorphism restricts to $\underline{Pic}^0_{X/k} = Pic^0 X$. Henceforth we work with the Picard functor $Pic^0_{X/k}$, which is reduced (since char $k = 0$) and is therefore an Abelian variety (see Mumford [45] p. 131, p. 150, p.163).

Let us take now $S = \underline{Pic}^0_{X/k}$. In this case, the identity morphism

$$I \in Hom_k(\underline{Pic}^0_{X/k}, \underline{Pic}^0_{X/k})$$

corresponds to $\mathcal{P} \in Pic^0_{X/k}(\underline{Pic}^0_{X/k}) = \frac{Pic^0(X \times \underline{Pic}^0_{X/k})}{pr_2^* Pic^0(\underline{Pic}^0_{X/k})}$. We consider $\mathcal{P}$ as an invertible sheaf on $X \times \underline{Pic}^0_{X/k}$, unique up to tensor product with elements of $pr_2^* Pic(\underline{Pic}^0_{X/k})$. Such a $\mathcal{P}$ will be called a *Poincaré sheaf.* From now on we will not underline the Picard scheme, the connotations being understood from the context.

Let $L \in Pic^0(X \times S)$ correspond to a map $S \to Pic^0_{X/k}$. The natural composition $Hom_k(S, Pic^0_X) \times Hom_k(Pic^0_X, Pic^0_X) \to Hom_k(S, Pic^0_X)$ given by $(f, I) \mapsto I \circ f = f$ corresponds to the following in terms of the Picard functor:

$$(1 \times f)^* \mathcal{P} = L \quad \text{up to tensoring with elements of } pr_S^* Pic^0 S.$$

As a result, taking an invertible sheaf $L \in Pic^0 X$ and $S := \xi \in Pic^0_{X/k}$ its corresponding point, we get $L = \mathcal{P}|X \times \xi$, or $\xi = \mathcal{P}|X \times \xi$ by *abus de langage.* Thus the Poincaré line bundle is a universal line bundle, though unique only up to line bundles on $Pic^0_{X/k}$ – compare Lemma 10.18 below regarding this "uniqueness" condition.

The Picard group of a product: Let X, Y be two projective smooth varieties defined over an algebraically closed field k. We are going to describe the divisor group $CH^1(X \times Y)$ in terms of $CH^1(X)$ and $CH^1(Y)$. This property will turn out to be very useful in proving (for $k = \mathbb{C}$) that certain Hodge classes on $X \times Y$ are algebraic. More precisely, we go along the lines of Tate [61] in his proof that the Tate conjecture in codimension 1 for X and Y implies the Tate conjecture in codimension 1 for $X \times Y$ when the ground field is $k = \mathbb{F}_q$ finite. Although there is an ad-hoc proof in Section 7, this fact is used in other computations in this thesis. First we need the following basic

Lemma 10.18. *([63]III.12.Ex.5) Let Y be an integral scheme of finite type over an algebraically closed field k. Let $f : T \to Y$ be a flat projective morphism whose fibres are all integral schemes. Let $\mathcal{L}$, $\mathcal{M}$ be invertible sheaves on T, and assume for each $y \in Y$ that $\mathcal{L}_y \simeq \mathcal{M}_y$ on the fibre X_y. Then there is an invertible sheaf $\mathcal{N}$ on Y such that $\mathcal{L} \simeq \mathcal{M} \otimes f^*\mathcal{N}$.*

Proof: We will first prove uniqueness of $\mathcal{N}$. Suppose there exists $\mathcal{N} \in Pic(Y)$ such that $\mathcal{L} \simeq \mathcal{M} \otimes f^*(\mathcal{N})$. Then $f^*(\mathcal{N}) \simeq \mathcal{L} \otimes \mathcal{M}^{-1}$. Projection formula

([63]III.Cor.12.9) yields

$$f_*(\mathcal{L} \otimes \mathcal{M}^{-1}) \simeq f_*f^*(\mathcal{N}) = f_*\mathcal{O}_T \otimes \mathcal{N} = \mathcal{N}$$

Last equality holds by "Liouville's theorem" for proper maps with connected fibres, namely $f_*(\mathcal{O}_T) = \mathcal{O}_Y$. We consider the sheaf $\mathcal{F} = f_*(\mathcal{L} \otimes \mathcal{M}^{-1})$ on Y; the lemma follows if we prove that $\mathcal{F}$ is invertible on Y. Indeed, the function $\varphi(y) = \dim_{k(y)} \mathcal{F}_y \otimes k(y)$ is constant and equal to 1 on Y. Hence, by [63]III.Cor.12.9, $\mathcal{F}$ is locally free of rank 1, i.e. is an invertible sheaf on Y, thereby proving the lemma.

Proposition 10.19. *(see Tate [61] p.144) Within the above hypotheses, the following formula holds:*

$$\frac{CH^1(X \times Y)}{p_X^*CH^1(X) + p_Y^*CH^1(Y)} = Hom(Alb(Y), Pic^0(X)) = Hom(Alb(X), Pic^0(Y))$$

where p_X, p_Y denote the natural projections from $X \times Y$ onto its factors X, Y respectively.

Proof: We are going to use the the above listed properties of the Picard functor. The symbol $\mathcal{V}ar$ denotes the category of varieties over k. Here Pic$X$ (resp.Pic^0_X) stand for the reduced Picard scheme (resp. the Picard variety), therefore one should imagine the subscript *red* whenever reading a scheme statement.

The Picard functor Pic_X is defined by

$$Pic_X(Y) = Pic(X \times Y)/p_Y^*(Pic(Y))$$

The representability of this functor yields

$$Hom_{\mathcal{V}ar}(Y, \underline{Pic}(X)) = \frac{Pic(X \times Y)}{p_Y^*Pic(Y)}$$

Note that, for a connected variety T, any morphism $f : T \to \underline{Pic}(X)$ sends its image to one connected component of the Picard scheme of X. In our case, given a line bundle $\mathcal{L}$ on $X \times Y$ we can choose a closed point $y_0 \in Y$ and consider the new line bundle $\mathcal{L} \otimes \mathcal{L}_{(.,y_0)}$ on $X \times Y$. On the other side of the isomorphism, this reflects as choosing $f - f(y_0)$ as a new morphism. What we are doing on the left hand side is actually quotienting out by constant morphisms $Y \to \underline{Pic}(X)$. By Lemma

10.18 above, the line bundles L on $X \times Y$ that correspond to a constant function $M \in Pic(X)$ are exactly the elements of the form $p_X^*(M) \otimes p_Y^* N$; indeed, consider the line bundle $L \otimes p_X^*(M)^{-1}$. This line bundle is trivial on every subvariety $X \times y$, and hence it is isomorphic to an element of $p_Y^* Pic(Y)$. This implies that

$$Hom_{\mathcal{V}ar}(Y, \underline{Pic}(X))/\text{constant maps} = \frac{Pic(X \times Y)}{p_X^* Pic(X) + p_Y^* Pic(Y)}$$

It is clear, though, that

$$\frac{Hom_{\mathcal{V}ar}(Y, \underline{Pic}(X))}{\text{constant maps}} = \frac{Hom_{\mathcal{V}ar}(Y, \underline{Pic}(X))}{\underline{Pic}(X)} = \frac{Hom_{\mathcal{V}ar}(Y, \underline{Pic}^0(X))}{\underline{Pic}^0(X)}$$

We now can use the property of the Albanese map:

$$\frac{Hom_{\mathcal{V}ar}(Y, \underline{Pic}^0(X))}{\text{constant maps}} = \frac{Hom_{\mathcal{V}ar}(Y, \underline{Pic}^0(X))}{\text{translation on } Pic^0(X)} = Hom(Alb(Y), \underline{Pic}^0(X))$$

thus establishing the proposition.

10.3 Hodge theory of Abelian varieties

The main references used in this section are B.B. Gordon [35], and occasionally Ribet [53], Mumford [46].

Let A be an Abelian variety. We define $MT(A)$ and $Hg(A)$ to be $MT(H_1(A))$ and $Hg(H_1(A))$. We will also denote by $\mathcal{H}(A) := B(H^\circ(A))$, which will be called the *Hodge algebra* of A, and $\mathcal{D}(A)$ to be the subalgebra of $\mathcal{H}(A)$ generated by classes of divisors.

We include the following group-theoretic lemma, which will be used in the sequel.

Proposition 10.20 (Goursat's Lemma). *(Gordon [35] Prop. 2.16)*

1. *Let G and G' be groups and suppose H is a subgroup of $G \times G'$ for which the projections $p : H \to G$ and $p' : H \to G'$ are surjective. Let N be the kernel of p' and let N' be the kernel of p. Then N is a normal subgroup of G and N' is a normal subgroup of G', and the image of H in $G/N \times G'/N'$ is the graph of an isomorphism $G/N \cong G'/N'$.*

2. *Let V_1, V_2 be two finite-dimensional complex vector spaces. Let $\mathfrak{s}_1, \mathfrak{s}_2$ be simple complex subalgebras of $\mathfrak{gl}(V_1)$, $\mathfrak{gl}(V_2)$ respectively, of type A, B or C. Let $\mathfrak{s}$ be a Lie subalgebra of $\mathfrak{s}_1 \times \mathfrak{s}_2$ whose projection to each other is surjective. Then either $\mathfrak{s} = \mathfrak{s}_1 \times \mathfrak{s}_2$ or $\mathfrak{s}$ is the graph of an isomorphism induced by an $\mathfrak{s}_1$-isomorphism $V_2 \simeq V_1$ or $V_2 \simeq V_1^*$.*

3. *Let $\mathfrak{s}_1, \mathfrak{s}_2, \cdots, \mathfrak{s}_d$ be simple finite-dimensional Lie algebras and let $\mathfrak{g}$ be a subalgebra of $\mathfrak{s}_1 \times \cdots \times \mathfrak{s}_d$. Assume that the projection $\mathfrak{g} \to \mathfrak{s}_i$ is surjective for all i, and that whenever $i < j$ the projection of $\mathfrak{g}$ onto $\mathfrak{s}_i \times \mathfrak{s}_j$ is surjective. Then $\mathfrak{g} = \mathfrak{s}_1 \times \cdots \times \mathfrak{s}_d$.*

4. *Let I be a finite set, and for each $\sigma \in I$, let $\mathfrak{s}_\sigma$ be a finite-dimensional complex simple Lie algebra. Let $\mathfrak{g}, \mathfrak{h}$ be two algebras such that*

   (a) $\mathfrak{g} \subset \mathfrak{h}$.

   (b) $\mathfrak{h}$ is a subalgebra of $\prod_{\sigma \in I} \mathfrak{s}_\sigma$ such that the projection to each $\mathfrak{s}_\sigma$ is surjective.

(c) $\mathfrak{g}, \mathfrak{h}$ *have equal images on* $\mathfrak{s}_\sigma \times \mathfrak{s}_\tau$ *for all pairs* $(\sigma, \tau) \in I \times I$, $\sigma \neq \tau$.

Then $\mathfrak{g} = \mathfrak{h} = \prod_{\sigma \in J} \mathfrak{s}_\sigma$ *for some subset* $J \in I$.

The following corollary is an immediate consequence of last Proposition.

Corollary 10.21. *(Gordon [35], Proof of Th.3, p.329) Let* A *be an Abelian variety isogenous to* $B \times C$, *with* $Hg(B)$ *a torus and* $Hg(C)$ *semisimple (we exclude one-dimensional abelian groups in the definition of semisimple algebraic group). Then* $Hg(A) = Hg(B) \times Hg(C)$.

Proof: We make use of Proposition 10.25 below. Indeed, suppose that the inclusion $Hg(B \times C) \subset Hg(B) \times Hg(C)$ is not an equality. Then, as projections onto both factors are surjective, we can apply Proposition 10.20 below: there are normal algebraic subgroups N, N' such that

$$Hg(B)/N \cong Hg(C)/N',$$

which leads to a contradiction with the hypotheses.

We now include a group-theoretical lemma which will be used later on.

Proposition 10.22. *Let* F *be a field of characteristic zero, and*

$$D = \left(\frac{a, b}{F}\right)$$

be a quaternion algebra (which may be split). Let $D_1^\times$ *be the algebraic group over* F *of norm 1 quaternions, and let* $H_{a,b}$ *be its Lie algebra. Then, for* $a_2, b_2 \in F$, *we have* $H_{a,b} \cong H_{a_2,b_2}$ *as* F*-Lie algebras if and only if the corresponding quaternion algebras are isomorphic over* F.

Proof: Let us calculate the structure constants of the Lie algebra $H_{a,b}$ first. We choose i, j, k such that $i^2 = a, j^2 = b, k = ij = -ji$. Then:

$$[i, j] = ij - ji = 2k$$

$$[j, k] = jk - kj = -2j^2 i = -2bi$$

$$[i, k] = ik - ki = 2i^2 j = 2aj.$$

We denote by $\langle , \rangle$ the following quadratic form on $H_{a,b}$:

$$\langle x.\, i + y.\, j + z.\, k, x'.\, i + y'.\, j + z'.\, k\rangle = axx' + byy' - abzz'.$$

We have the following relation. For every $u, v \in H_{a,b}$,

$$u.\, v + v.\, u = 2\langle u, v\rangle.$$

Suppose that the Lie algebras $H_{a,b}$ and H_{a_2,b_2} are isomorphic. This means that there exist $i_2, j_2, k_2 \in H_{a,b}$ such that

$$[i_2, j_2] = i_2 j_2 - j_2 i_2 = 2k_2$$

$$[j_2, k_2] = j_2 k_2 - k_2 j_2 = -2j_2^2 i_2 = -2b_2 i_2$$

$$[i_2, k_2] = i_2 k_2 - k_2 i_2 = 2i_2^2 j_2 = 2a_2 j_2.$$

We will be done by showing that

$$i_2^2 = a_2, j_2^2 = b_2, k_2 = i_2 j_2 = -j_2 i_2.$$

Indeed, the hypotheses on the Lie brackets jointly with tha formula for $uv + vu$ above yield, for instance

$$i_2 j_2 = \langle i_2, j_2\rangle + k_2$$

and

$$j_2 k_2 = \langle j_2, k_2\rangle - b_2 i_2.$$

The vector $i_2 j_2 k_2$ may then be written in two ways:

$$(i_2 j_2)k_2 = i_2(j_2 k_2).$$

Substituting the parentheses by the expressions above we get

$$k_2^2 + \langle i_2, j_2\rangle k_2 = \langle j_2, k_2\rangle i_2 - b_2 i_2^2.$$

Note that, for $u \in H_{a,b}$, $u^2 = \langle u, u\rangle \in F.\, 1$, hence the former equation yields the vectors i_2, j_2, k_2 to be $\langle , \rangle$-orthogonal. It remains to check the identities $i_2^2 = a_2, j_2^2 = b_2$.We sketch the argument only. It suffices to develop the expression $\langle [u, v], w\rangle$ and to evaluate on suitable permutations of i_2, j_2, k_2, thereby completing the proof.

Definition The *Lefschetz group* of a polarised Abelian variety (A, L) is the rational algebraic group defined by

$$Lf(A, E) := \{g \in \mathrm{Sp}(H_1(A), E) \mid \phi g = g\phi \ \forall \phi \in End(A)_{\mathbb{Q}}\}^0,$$

where E is the Riemann form associated to the ample line bundle L. In other words, $Lf(A, E)$ is the (connected component of the identity in the) centraliser of $End^0(A)$ in $\mathrm{Sp}(H_1(A), E)$.

It is clear that $Lf(A, E)_{\mathbb{R}}$ contains $h(U(1))$, therefore $Hg(A) \subset Lf(A, E)$ for every polarisation E.

Remark 1. Suppose that E, E' are two Riemann forms on $H_1(A)$. Then the groups $Lf(A, E)$ and $Lf(A, E')$ are conjugate subgroups in $GL(H_1(A))$. Indeed, if ϕ, ϕ' are the isogenies $A \to \hat{A}$ induced by E, E' then $\phi \circ (\phi')^{-1}$ yields an element $M \in End^0(A)$ such that $MLf(A, E)M^{-1} = Lf(A, E')$. Therefore we will often refer to the Lefschetz group of an Abelian variety A as the Lefschetz group of A with respect to an arbitrary fixed polarisation.

2. The Hodge structures $H_1(A)$ and $H^1(A)$ differ by a twist; i.e. $H^1(A)(1) \cong H_1(A)$ by means of a polarisation on A. Thus we can either regard $MT(A)$ and $Hg(A)$ as groups acting on $H^1(A)$ or on $H_1(A)$; the former case entails contravariance with respect to morphisms; for instance, the Lefschetz group appears as a subgroup of $MT(H^1(A))$ under the form

$$Lf(A) := \{g \in Sp(H^1(A)) \mid \phi^* g = g\phi^* \ \forall \phi \in End(A)_{\mathbb{Q}}\}^0.$$

Henceforth on the terms $Hg(A)$ and $MT(A)$ will denote $Hg(H^1(A))$ and $MT(H^1(A))$, or $Hg(H_1(A))$ and $MT(H_1(A))$ without distinction, and the appropriate connotations are to be inferred from their context.

The following proposition gives a sharper upper bound for the Hodge and Mumford-Tate groups, and is used widely in the literature, particularly in [53] and [50].

Proposition 10.23. *Let A be a simple Abelian variety, and let ψ be a polarisation on $V = H_1(A)$. Let W denote V as a $Z(End^0(A))$-vector space. The following inclusions hold:*

1. *Suppose $E = End^0(A)$ is a totally real field. Then there exists an E-bilinear alternate form $\phi : W \times W \to E$ such that*

$$Hg(A) \subset Res_{E|\mathbb{Q}} Sp(W, \phi).$$

2. *Suppose $F = End^0(A)$ is a CM-field and let $E = F \cap \mathbb{R}$. Then there exists a Hermitian form H on W such that*

$$Hg(A) \subset Res_{E|\mathbb{Q}} U(W, H).$$

To sum up, suppose that $End^0(A)$ is a number field. Then the expression on the right is equal to $Lf(A)$ (or more precisely, to $Lf(A, \psi)$).

Proof: The proof relies on the following observations.

1. In both cases, the centraliser of $Z(End^0(A))$ in $\mathrm{Sp}(V, \psi)$ is

$$\mathrm{Res}_{E|\mathbb{Q}}\mathrm{Sp}(W, \phi)$$

and

$$\mathrm{Res}_{E|\mathbb{Q}}\mathrm{U}(W, H)$$

respectively - this is proven in the next two points. (The above mentioned groups are connected and hence equal to $Lf(A, \psi)$).

2. To prove the last assertion, we note the following. Let ψ be a polarisation on V. Then ψ satisfies

$$\psi(e.\, u, v) = \psi(u, e.\, v) \text{ for } e \in E$$

and

$$\psi(a.\, u, v) = \psi(u, \bar{a}.\, v) \text{ for } a \in F.$$

3. In the first case, the former observation yields the existence of an alternate E-bilinear form ϕ on W such that

$$\psi = \mathrm{Tr}_{E|\mathbb{Q}}(\phi).$$

In the second case of the proposition, ψ has a kind of F-antiequivariance, which can be interpreted as follows. Due to the isomorphism of the E-subspaces of Hermitian and skew-Hermitian bilinear forms on W (with respect to the imaginary extension $F|E$), we can find a Hermitian form H on W and a purely imaginary element $\theta \in F, \overline{\theta} = -\theta$ such that

$$\psi = \mathrm{Tr}_{F|\mathbb{Q}}(\theta H(u, v)).$$

The proposition now follows.

Corollary 10.24. *Let A be an Abelian variety of CM type. Let $F = End^0(A)$ and let $F_0 = F \cap \mathbb{R}$. The Lefschetz group of A is given by the following expression:*

$$Lf(A) = Res_{F_0|\mathbb{Q}} U_F(1),$$

where

$$U_F(1) := Ker\ N_{F|F_0} : \mathbb{G}_{m,F} \to \mathbb{G}_{m,F_0}.$$

Proof: The basic fact in this case is that $\dim_F W = 1$. The rest follows from Proposition 10.23..

Proposition 10.25. *(Gordon [35]; Hazama [24] Props. 1.9, 1.10) Let A, B be two Abelian varieties. The following statements hold.*

1. *$Hg(A \times B) \subset Hg(A) \times Hg(B)$. Also if we denote the former natural inclusion by ι and we define p_1, p_2 to be the first and second projections of $Hg(A) \times Hg(B)$ onto its factors, we have that*

$$p_1 \circ \iota : Hg(A \times B) \to Hg(A) \quad and \quad p_2 \circ \iota : Hg(A \times B) \to Hg(B)$$

   are surjective.

2. *For any $n \in \mathbb{N}$, we have $Hg(A^n \times B) = Hg(A \times B)$. In particular, for $B = 0$ we have $Hg(A^n) = Hg(A)$ and, as a result, $MT(A^n) = MT(A)$.*

Proof:

1. The inclusion $Hg(A \times B) \subset Hg(A) \times Hg(B)$ is clear. To prove the surjectivity of both projections, we take into account the following fact. If $T(E)$ symbolises the tensor algebra of a vector space E over $\mathbb{Q}$, then, for a rational Hodge structure of weight w, $Hg(V)$ is the stabiliser of the Hodge cycles in the ring

$$T(V \oplus V^*) = \bigoplus_{r \geq 0} V^{\otimes r} \otimes \bigoplus_{s \geq 0} V^{*\otimes s}.$$

Let $V_1 = H^1(A)$ and $V_2 = H^1(B)$. Then $Hg(A \times B)$ acts on the tensor algebra $T(V \oplus V^*)$ where $V = V_1 \oplus V_2$. The action of $Hg(A \times B)$ induced on the Hodge substructures $T(V_i \oplus V_i^*)$ is through the projections p_i. More precisely, the subspace of Hodge cycles of $T(V_i \oplus V_i^*)$ is the subspace of $Hg(A \times B)$-invariants inside $T(V_i \oplus V_i^*)$, and is also the subspace of $Hg(V_i)$-invariants of $T(V_i \oplus V_i^*)$:

$$T(V_i \oplus V_i^*)^{p_i(Hg(A\times B))} = T(V_i \oplus V_i^*)^{Hg(V_i)}.$$

Therefore, by Proposition 3.47(**c**), $p_i(Hg(A \times B)) = Hg(V_i)$, thus concluding the proof of 1. Note that $p_i(Hg(A \times B)) \subset Hg(V_i)$ is a reductive closed subgroup (Humphreys [27] 7.4.B), so Proposition 3.47 applies to our case.

2. Just as in 1. above, we have the natural inclusion

$$\iota_n = (\alpha_1, \cdots \alpha_n, \beta) : Hg(A^n \times B) \hookrightarrow Hg(A) \times \overset{n}{\cdots} \times Hg(A) \times Hg(B).$$

Let us take real coefficients in the above morphism. The fact that the embedding

$$(\alpha_1 \cdots \alpha_n)|U(1) : U(1) \hookrightarrow Hg(A) \times \overset{n}{\cdots} \times Hg(A)$$

(giving the complex structure) is diagonal implies $\alpha_1 = \alpha_2 = \cdots = \alpha_n$. Therefore, ι_n may be described as $(\Delta_n, \mathrm{Id}) \circ \iota$

$$Hg(A \times B) \hookrightarrow Hg(A) \times Hg(B) \overset{(\Delta,1)}{\to} Hg(A)^n \times Hg(B),$$

where $\iota = \iota_1$ as in 1. The image of the above defined morphism in $Hg(A)^n \times Hg(B)$ turns out to be $Hg(A^n \times B)$, which is isomorphic to $Hg(A \times B)$.

A different proof of 2. may be obtained by using Theorem 3.48, Proposition 3.47(**c**).

Remark: We have already argued in the Definition of Lefschetz group that the inclusion

$$Hg(A) \subset Lf(A, E)$$

holds. The equality may be strict, however (see examples of this phenomenon in [46]). For instance, the Hodge and Lefschetz groups are found to have rather different behaviour with respect to products. The following proposition shows the good behaviour of Lf with respect to products.

Proposition 10.26. *(Gordon [35] Lemma 2.15) $Lf\left(B_1^{n_1} \times \cdots \times B_r^{n_r}\right) = Lf(B_1) \times \cdots \times Lf(B_r)$, where B_i are pairwise non-isogenous, simple Abelian varieties.*

The following proposition characterises the Abelian varieties A of CM type by means of their Hodge group.

Proposition 10.27. *(Gordon [35] Prop. 2.12) A complex Abelian variety is of CM-type if and only if $Hg(A)$ is an algebraic torus.*

Proof: Suppose first that A is of CM-type. Then $End^0(A)$ contains a commutative semisimple algebra of dimension 2 dim A over $\mathbb{Q}$. As

$$End^0(A) = End(H_1(A))^{Hg(A)},$$

it follows that $Hg(A)$ commutes with a maximal comutative semisimple subalgebra $R' \subset End(H_1(A))$. Therefore $Hg(A)$ is contained in the units of R' and thus must be an algebraic torus. Conversely, if $Hg(A)$ is an algebraic torus, then it is diagonalisable over $\mathbb{C}$. Therefore its centraliser in $End(H_1(A))$ contains a maximal commutative semisimple subalgebra $R' \subset End(H_1(A))$. But then $[R' : \mathbb{Q}] = 2$ dim A and $R' \subset End^0(A)$, so A is of CM-type.

The following proposition is included as Proposition 2.7. in Gordon [35]. For its proof, we follow Zarhin [75].

Proposition 10.28. *(Gordon [35] Proposition 2.7, Ribet [53] p.533)*

1. *Let A be an Abelian variety. Then the centre $Z(MT(A))$ is an algebraic subgroup of $Z(End^0(A))$.*

2. *Let A be an Abelian variety such that the centre of $End^0(A)$ is equal to a central simple algebra over $\mathbb{Q}$. Then $Hg(A)$ is simple (i.e. simple as an algebraic group over $\mathbb{Q}$).*

Proof: We know that $End^0(A)$ is precisely the centraliser of $MT(A)$ in $\mathrm{End}(H_1(A,\mathbb{Q}))$; this happens even after extending scalars over $\mathbb{C}$, as seen above. As usual, $Z(.)$ will denote the centre of an (associative or Lie) algebra or of an algebraic group.

To avoid rationality questions, we work directly with Lie algebras. Let $\mathfrak{z}$ be the centre of $\mathfrak{mt}(A)$; then $\mathfrak{z}$ is contained in $End^0(A)$ (now regarded as a Lie algebra). Indeed, $End^0(A)$ is the centraliser of $\mathfrak{mt}(A)$ in $End^0(H_1(A,\mathbb{Q}))$, hence $\mathfrak{z} \subset End^0(A)$. Even more, since $End^0(A)$ commutes with $\mathfrak{z} \subset End^0(A)$, we have $\mathfrak{z} \subset Z(End^0(A))$.

The second statement follows easily from this fact, since in our case

$$\mathbb{Q}.\mathrm{Id} \subset \mathfrak{z} \subset Z(End^0(A)) = E,$$

hence $Z(Hg(A))$ has a finite centre and is therefore semisimple. The fact that $End^0(A)$ is a division algebra yields A to be simple. This together with the semisimplicity of $Hg(A)$ leads to the simplicity of $Hg(A)$ as an algebraic group over $\mathbb{Q}$, thereby concluding the proof.

Proposition 10.29. *(Gordon [35]Th. 3) Let E, E_i be elliptic curves.*

1. *Suppose E is non-CM. Then $MT(E) = GL(H_1(E)), Hg(E) = SL(H_1(E))$. If $End(E)_{\mathbb{Q}} = K$ imaginary quadratic field, then $MT(E) = Res_{K|\mathbb{Q}}(\mathcal{G}_m)$ and $Hg(E) = Ker\left(Nm_{K|\mathbb{Q}} : MT(E) \to \mathbb{G}_{m,\mathbb{Q}}\right)$.*

2. *$S^2H^1(E)$ is an irreducible Hodge structure if and only if E is not of CM type.*

3. *$Hg\left(E_1^{n_1} \times \cdots \times E_r^{n_r}\right) = Hg(E_1) \times \cdots \times Hg(E_r)$*

Proof:

Proof of 1.- Suppose $End(E) = \mathbb{Z}$. Then $Hg(E) \subset SL(H^1(E))$, and $\mathfrak{hg}(E)$ is a subalgebra of the simple algebra $\mathfrak{sl}(H^1(E))$, hence $Hg(E) = SL(H^1(E))$ (use the fact that the $Hg(E)$-invariants in $H^2(E)$ coincide with their $SL(H^1(E))$-invariants). Ttherefore $MT(E) = GL(H^1(E))$.

Proof of 2.- In case $End^0(E) = K$ imaginary quadratic field, then the proof of Proposition 10.27 shows that $MT(E, \mathbb{Q}) \subset K^\times$ as algebraic groups over $\mathbb{Q}$; taking real points ($\mathbb{C}^\times \subset MT(E)(\mathbb{R})$) we see that $MT(E) = \mathrm{Res}_{K|\mathbb{Q}}\mathbb{G}_{m/K}$.

Proof of 3.- The first step is to consider $A = E^n$. Then

$$\begin{aligned}\mathcal{H}(A) &= H^*(E^n, \mathbb{Q})^{Hg(A)} \\ &= \left(\bigwedge^{*}(H^1(E,\mathbb{Q})^{\oplus n})\right)^{Hg(E)} \\ &= \bigoplus (H^1(E,\mathbb{Q}) \otimes \cdots \otimes H^1(E,\mathbb{Q}))^{Hg(E)}.\end{aligned}$$

Now if E has complex multiplication by K, then $\alpha \in K^\times = MT(E)$ acts on $H^1(E,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathbb{C} \oplus \mathbb{C}$ by $\alpha(z,w) = (\alpha z, \overline{\alpha} w)$. Let $K_1^\times$ denote the elements of $K^\times$ of norm 1 (in the language of algebraic groups, i.e. we are referring to an algebraic subgroup). Then

$$\left(\otimes^r H^1(E,\mathbb{Q})\right)^{Hg(E)} \otimes_{\mathbb{Q}} \mathbb{C} = ((H^1(E,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}) \otimes \cdots \otimes (H^1(E,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}))^{K_1^\times},$$

in which any invariant class arises as a combination of products of elements of

$$((H^1(E,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}) \otimes (H^1(E,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}))^{K_1^\times} \subset (H^2(E \times E, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C})^{K_1^\times}.$$

Therefore the invariants are generated by those in $H^2(A,\mathbb{Q})$, which means that $\mathcal{H}(A) = \mathcal{D}(A)$ in this case.

Next suppose that E does not have complex multiplication. Then $Hg(E) = SL(2)$ acts on $H^1(E,\mathbb{Q})$ by the standard representation. Now we invoke the well-known fact that the tensor invariants of $SL(2)$ are generated by the determinant; see Weyl [74]. Since the determinant is a representation of degree 2 lying in $H^1(E,\mathbb{Q})^{\otimes 2} \subset H^2(A,\mathbb{Q})$, again we find that the Hodge cycles of $A = E^n$ are generated by divisor classes.

Finally, let $A = E_1^{n_1} \times \cdots \times E_r^{n_r}$, where the E_i are pairwise non-isogenous elliptic curves. First suppose that all the E_i have complex multiplication by (pairwise distinct) fields K_i. Then

$$Hg(A) \subset K_{1,1}^\times \times \cdots \times K_{r,1}^\times,$$

and moreover from the definition, $Hg(A)$ surjects onto each factor. Therefore we have a surjection of character groups

$$\lambda : M \to X(Hg(A)),$$

where

$$M := X(K_{1,1}^{\times}) \oplus \cdots \oplus X(K_{r,1}^{\times}).$$

We observe that for each i the composition

$$X(K_{i,1}^{\times}) \subset M \to X(Hg(A))$$

of $m \mapsto (0, \cdots, m, \cdots, 0)$ with λ is injective. In addition, all of these character groups are $\mathcal{G} = Gal(\overline{\mathbb{Q}}|\mathbb{Q})$-modules and the maps $\mathcal{G}$-equivariant. It can be shown by induction, by using that the fields K_i are linearly disjoint, that λ is injective. Indeed, there is some $\sigma \in \mathcal{G}$ that acts as $+1$ on $X(K_{1,1}^{\times})$ and -1 on the other components. Thus if $m = (m_1, \ldots, m_r)$ is in the kernel of λ then $\sigma m + m = (2m_1, 0, \ldots, 0)$ must be as well. Then the injectivity of the composition above forces $m_i = 0$, and by induction the kernel of λ is zero.

Next suppose that none of the E_i has complex multiplication. Then we have

$$\mathfrak{hg}(A) \subset \mathfrak{hg}(E_1) \times \cdots \times \mathfrak{hg}(E_r),$$

and mapping surjectively onto each factor. Then by Proposition 10.20.4, it follows that if it also maps surjectively onto each pair of factors, then $\mathfrak{hg}(A)$ is the entire product. But by Proposition 10.20.2, if $\mathfrak{hg}(A)$ does not project onto $\mathfrak{hg}(E_i) \times \mathfrak{hg}(E_j)$ for a pair $i < j$, then it projects onto the graph of an isomorphism between $\mathfrak{hg}(E_i)$ and $\mathfrak{hg}(E_j)$, contrary to our assumptions.

Finally it remains to see that if A is an Abelian variety isogenous to a product $B \times C$ with B a product of non-CM elliptic curves and C a product of CM-elliptic curves, then $Hg(A) = Hg(B) \times Hg(C)$. However, this is a consequence of Corollary 10.21, for B is CM, hence $Hg(B)$ is an algebraic torus by Proposition 10.27 and $Hg(C)$ is isomorphic to a product of copies of $SL_{2,\mathbb{Q}}$ (and hence semisimple) as seen above. This completes the proof of the present Proposition.

Proposition 10.30. *(Gordon [35] Sect.6; see also Ribet [53]) Let A be an Abelian variety of dimension d, and suppose that one of the following holds:*

1. *$End(A)_{\mathbb{Q}} = F_0$ totally real field, $e = [F_0 : \mathbb{Q}]$ and d/e is odd, or*

2. *d is a prime number and A is simple, or*

3. *$End(A)_{\mathbb{Q}}$ is an imaginary quadratic field K and the multiplicities n' and n'' with which $\alpha \in K$ acts are relatively prime.*

Then $Hg(A) = Lf(A)$ and $\mathcal{H}(A^n) = \mathcal{D}(A^n)$ for $n \geq 1$. The Hodge conjecture will then hold trivially for A^n.

10.4 The Mumford-Tate and Hodge groups of an Abelian surface and $T(A)$

Let A be an Abelian surface. Then A satisfies the hypotheses of Proposition 10.30, and therefore $Lf(A) = Hg(A)$. This means that we may obtain an explicit expression for the Hodge group of A. Actually (see Mumford [46]), for Abelian varieties of dimension ≤ 3 the equality $Lf(A) = Hg(A)$ holds.

The following proposition calculates $Hg(A)$ in terms of $End(A)$.

Proposition 10.31. *(Mumford [46], Gordon [35]) Let A be a simple Abelian surface.*

1. *(Type I(1)) If $End(A)_{\mathbb{Q}} = \mathbb{Q}$ then $Hg(A) = Sp(H_1(A), \psi)$ where ψ is the only polarisation for A - up to a constant.*

2. *(Type I(2)) If $End(A)_{\mathbb{Q}} = F_0$ real quadratic, then $Hg(A) = Res_{F_0|\mathbb{Q}}(SL_{2,F_0})$.*

3. *(Type II(1)) If $End(A)_{\mathbb{Q}} = D$ indefinite quaternion algebra, then $Hg(A) \simeq D_1^{\times}$, where $D_1^{\times} = Ker\ (Nrd : D^{\times} \to \mathbb{G}_m)$. Note that $(D_1^{\times})_{\mathbb{R}} \simeq SL_{2,\mathbb{R}}$.*

4. *(Type II(2,1)) If $End^0(A) = F$ is a CM-field of degree 4 over $\mathbb{Q}$ and $F_0 = F \cap \mathbb{R}$, then*

$$Hg(A) = Res_{F_0|\mathbb{Q}} U_F(1),$$

where

$$U_F(1) := Ker\ N_{F|F_0} : \mathbb{G}_{m,F} \to \mathbb{G}_{m,F_0}.$$

(Note that $Hg(A)$ is in this case a 2-dimensional algebraic group over $\mathbb{Q}$.)

If A is non-simple, the following cases hold (see Proposition 10.29)

1. $A \sim E_1 \times E_2$, E_i *non-isogenous. Then* $Hg(A) = Hg(E_1) \times Hg(E_2)$.
2. $A \sim E \times E$. *Then* $Hg(A) = Hg(E)$.

Proof: V will denote the Hodge structure associated to $H_1(A, \mathbb{Q})$ as usual, and ψ will be a polarisation for V.

The proposition follows from Proposition 10.29 for A non-simple. From now on we assume that A is simple, and proceed to compute $Lf(A)$ (which in our case coincides with $Hg(A)$). We proceed case by case as stated in the lemma:

- In the cases where A is of Type I, i.e. A is of type I(1) or I(2), we can apply Proposition 10.23 and we get

  $$Lf(A) = \mathrm{Sp}(V, \psi) = \mathrm{Sp}(4)$$

  when A is of type I(1), and

  $$Lf(A) = \mathrm{Res}_{F_0|\mathbb{Q}}\mathrm{Sp}(W, \phi) \simeq \mathrm{Res}_{F_0|\mathbb{Q}}\mathrm{SL}_{2,F_0}$$

  when A is of Type I(2) (F_0 is a real quadratic field).

- Suppose that A is of type II(1), i.e. there exists an indefinite quaternionic algebra D over $\mathbb{Q}$ endowed with a positive antiinvolution (which we shall denote by) such that $D = End^0(A)$: in this case, V is a right D-module of rank 1. The centraliser of D in $End(V)$ is easily calculated from the following paragraph.

  The morphism of algebras

  $$D \otimes D^{opp} \to End_{\mathbb{Q}}(D)$$

  given by

  $$a \otimes b \mapsto (x \mapsto a.\, x.\, b)$$

  is an isomorphism of $\mathbb{Q}$ algebras (this is valid for semisimple algebras in general). Therefore the centraliser of D in $End_{\mathbb{Q}}(D)$ is equal to $1 \otimes D^{opp} \simeq D^{opp}$, which is isomorphic to D in our case.

It now remains to impose the preservation of a fixed polarisation on V. The right D-module V is free of rank 1, i.e. V is isomorphic to D as a right D-module. A polarisation on V is given through this isomorphism as a bilinear form

$$D \times D \to \mathbb{Q}$$

by the formula

$$\psi(\alpha, \beta) = \mathrm{Tr}_{D|\mathbb{Q}}(\theta\overline{\alpha}\beta),$$

where is the involution on D and θ is a traceless element (i.e. such that $\overline{\theta} = -\theta$) such that $\theta^2 < 0$ in $\mathbb{Q}$ (see Gordon [35] 1.13.7). The endomorphisms of A act by right multiplication on D, but then the centraliser corresponds to the left multiplication operators L_γ. Let us compute the action of such an operator on a fixed polarisation:

$$\psi(\gamma.\alpha, \gamma.\beta) = \mathrm{Tr}_{D|\mathbb{Q}}(\theta\overline{\alpha}.\overline{\gamma}\gamma\beta) = (\overline{\gamma}\gamma).\psi(\alpha, \beta).$$

This means that the algebraic subgroup

$$Lf(A, \psi) = \mathrm{Sp}(V, \psi) \cap \mathrm{Centraliser}_{End(V)}(D)$$

coincides with the kernel of the "norm map"

$$D_1^\times := \mathrm{Ker}\ \mathrm{Nrd}_{D|\mathbb{Q}} : GL_1(D) \to \mathbb{G}_{m,\mathbb{Q}}$$

thus proving the statement.

- In the case when A has complex multiplication by a CM-field F, we have $Hg(A) = Lf(A)$ (for dim A $= 2$). Now $Lf(A)$ is calculated in Corollary 10.24, and thus the statement follows.

We now proceed to calculate the number of polarisations of $T(A)$ and then use this to prove our key lemma.

Proposition 10.32. *Let A be an Abelian surface. Then:*

i. If A is simple and not of CM-type, then dim $B(S^2T(A)) = 1$, and thus $T(A)$ admits only one polarisation.

ii. *If* $A \sim E_1 \times E_2$ *where* E_1 *and* E_2 *are not simultaneously of CM-type, then* $T(A)$ *admits only one polarisation*

iii. *If* $A \sim E_1 \times E_2$ *where* E_i *are nonisogenous CM elliptic curves and* K_i *are the associated fields, then*

$$End_{Hodge}T(A) = K_1K_2$$

where K_1K_2 *denotes the compositum of both fields in* $\mathbb{C}$. *In this case* $T(A)$ *will admit exactly two (linearly independent) polarisations.*

iv. *If* A *is simple of CM type and* $End^0(A) = F$ *is Galois over* $\mathbb{Q}$ *then*

$$End_{Hodge}T(A) = F.$$

v. *If* A *is of CM type and* A *is not isogenous to a product* $E \times E$, *then dim* $B(S^2T(A))$ *is equal to* 2, *i.e.* $T(A)$ *admits exactly two polarisations.*

Proof:

i. If A is of type *I(1)*, it follows from [16] that $End_{Hodge}(T(A)) = End(T(A))^{Hg(A)}$ equals $\mathbb{Q}$. The *I(2)* and *II(1)* cases are similar (proceed tensoring with $\mathbb{C}$.)

ii. We will prove the case when E_i are non-isogenous. The case $E_1 = E_2$ is similar. Let $H^1(E_i) = W_i$. Then

$$T(A) = W_1 \otimes W_2$$

The fact that dim $W_i = 2$ yields

$$S^2 (W_1 \otimes W_2) = S^2W_1 \otimes S^2W_2 \oplus \bigwedge^2 W_1 \otimes \bigwedge^2 W_2 \qquad (10.1)$$

It is now easy to see that if either of E_i is not CM, then the Hodge cycles restrict to $\bigwedge^2 W_1 \otimes \bigwedge^2 W_2$. The lemma follows.

iii. Let $K_i = \mathbb{Q}(\sqrt{-d_i})$. Let us fix the product polarisation on $E_1 \times E_2$. We know that $End_{Hodge}T(A)$=K is a number field whose degree divides dim $T(A) = 4$. The endomorphisms $\alpha_1 = \left(\sqrt{-d_1}, id_{E_2}\right)$ and $\alpha_2 = \left(id_{E_1}, \sqrt{-d_2}\right)$ yield subfields K_i of F. Also, from the formula (9) we derive the fact that there are two polarisations on $T(A)$, corresponding to the two endomorphisms coming from the following endomorphisms of A: $id_A, \alpha_1\alpha_2$ via the standard polarisation of a product.

iv. It follows from the observation that there exists a natural morphism of algebraic groups

$$\rho : F^{\times} \to Aut_{Hodge}T(A)$$

coming from the group structure of A. Let $\sigma_i : F \hookrightarrow \mathbb{C}$ bethe immersions corresponding to the CM-type of A. Then $\rho|H^{2,0} = \sigma_1\sigma_2$. Let L be the field generated by Im ρ; it is not difficult to see that it includes $(\sigma_1 + \sigma_2)(F)$. Now $\dim_{\mathbb{Q}} (\sigma_1 + \sigma_2)(F) = 3$. Since F is Galois, this establishes the lemma.

v. The proof is analogous to that of *iii.*, as the Hodge group of A is a two-dimensional compact torus and the representation $(Hg(A)_{\mathbb{R}}, H^1(A, \mathbb{R}))$ is equivalent to that of *iii.*, therefore having the same invariants. This completes the proof of *v.*

There are other ways to obtain information about the Hodge groups of Abelian surfaces. The following proposition derives the Hodge group of $T(A)$ from that of A itself.

Proposition 10.33. *($Hg(T(A))$ as a function of $Hg(A)$) Let V be a polarisable Hodge structure (of pure weight k) and let W be a Hodge substructure of $\otimes^r V$. Then each one of the following maps is surjective:*

$$Hg(V) \to Hg(\otimes^r V) \to Hg(W).$$

As a corollary, $Hg(T(A)) = Hg(A)/\mu_2$. The assertion is also valid for A an Abelian variety of higher dimension, replacing $T(A)$ by $T^2(A) \subset H^2(A)$.

The first assertion is clear from Proposition 3.47; see Proposition 10.25 for an analogous argument. As for the second statement, consider the following

Fact: Let V be a vector space and let $r \in \mathbb{N}$ satisfy $2r \leq \dim V$. Then, for any two endomorphisms f, g of V, the equality $\bigwedge^r f = \bigwedge^r g$ implies $f = g$.

Let us apply this fact to the transcendental lattices $T(A)$. Since $\bigwedge^2 H^1(A) = T(A) \oplus H^2(A)^{Hg(A)}$, the kernel of the surjection $Hg(A) \to Hg(T(A)) = Hg(H^2(A))$ is precisely μ_2.

Lemma 10.34 (Key Lemma). *Let A be an Abelian surface and E_i elliptic curves such that the rational Hodge structures $T(A)$ and $T(E_1 \times E_2)$ are isomorphic. Then*

$$A \sim E_1 \times E_2.$$

Proof: First, let us suppose that E_i are not both of CM type. We know that by Theorem 8.3 both Abelian surfaces must be polarised isogenous if there exists a polarised isomorphism (i.e. Hodge isometry) on their trascendental lattices. In our situation, we do not necessarily have a Hodge isometry, but a Hodge conformal isomorphism. Still, minor modifications of the Clifford algebra construction in [17] yield a Hodge isomorphism between the corresponding Kuga-Satake Hodge structures; we proceed to elaborate this point below.

Suppose that (W_i, ψ_i) are two rational polarised weight-two Hodge structures such that dim $W_i^{2,0} = 1$. Assume that (W_i, ψ_i) are conformally equivalent, i.e. there exist an isomorphism $f : W_1 \to W_2$ and a constant $a \in \mathbb{Q}_{>0}$ such that

$$f^* \psi_2 = a.\, \psi_1.$$

Let $F = \mathbb{Q}(\sqrt{a})$. Then the morphism

$$\frac{1}{\sqrt{a}} f : W_1 \otimes_{\mathbb{Q}} F \to W_2 \otimes_{\mathbb{Q}} F$$

is an isometry of polarised F-Hodge structures, and therefore by Theorem 8.2 the resulting Kuga-Satake F-Hodge structures of $W_i \otimes F$ are isometric. The point here is that they are actually $\mathbb{Q}$-isometric. Indeed, the linear morphism $\frac{1}{\sqrt{a}} f$ becomes

$$(\frac{1}{\sqrt{a}})^{2i} \Lambda^{2i}(f)$$

on even Clifford algebras and is thus $\mathbb{Q}$-defined.

Applying the former observation to our situation, let A_i be two Abelian surfaces such that $T(A_i)$ are conformally equivalent for the standard polarisations ψ_i on

$T(A_i)$ induced by the cup-product. By the last remark, we have that the Kuga-Satake varieties $KS(A_i, \psi_i)$ are polarised isogenous, hence $A_1 \sim A_2$ by Theorem 8.3. In the case where $E_1 \times E_2$ is not of CM-type, there is only one polarisation on $T(E_1 \times E_2)$ (up to constant multiples), so it must be constructed out ot the cup-product (see Proposition 10.32); since $T(A)$ and $T(E_1 \times E_2)$ are isomorphic Hodge structures, so it is for $T(A)$ and therefore the Lemma follows in this case from the last argument.

Again, by Proposition 10.32 it only remains to check the following case. Let A and $E_1 \times E_2$ be of CM type, and let $K_i = End^0(E_i)$. We consider

$$End_{Hodge}T(E_1 \times E_2) = End_{Hodge}(T(A)),$$

both then subfields of $L = End^0(A)$ if A is simple (see Proposition 10.32). This forces L to coincide with K_1K_2, which leads to contradiction by Lemma 11.6 below. This will mean that A is also non-simple CM, and then $A \sim E_1 \times E_2$ as can be read off from Proposition 10.32.iii.

Remark: Let A, A' be two Abelian surfaces. A more general statement than Lemma 10.34 holds. Namely, suppose that $T(A) \cong T(A')$. Then $A \sim A'$. This follows from Moonen-Zarhin [62], where an important result of Hazama [25] is used. Also an alternative proof to this may be given by using Proposition 10.33 - we leave this to the reader.

Chapter 11

Proof of Theorem 1.1

By Proposition 9.7, we only need to prove Theorem 1.1 in the case when S is an isotrivial elliptic surface birational to $(C \times E)/G$ with cyclic monodromy group $G = \mathbb{Z}_n$ such that $\mathbb{Z}_n$ acts faithfully on $H^1(E)$ (and thus $E/\mathbb{Z}_n = \mathbb{P}^1$). By Proposition 9.1, we will be done by constructing a Hodge isomorphism between $T(S)$ and $T(A)$ coming from an algebraic cycle – first some geometric restrictions on both S and A will be drawn from our assumptions, and, these restrictions being imposed, one such algebraic cycle will be constructed at the end of the proof.

Suppose that S is minimal and that $T(S) \cong T(A)$. We know that S will be isomorphic to an etale quotient $(C' \times E')/G$ where the action of G is diagonal and faithful on each component. In the case when $g(E'/G) = 1$, we costructed a finite morphism of S onto a product of two elliptic curves in Chapter 9 and proved that it yields an isomorphism on cohomology groups. The remaining case occurs when $g(E'/G) = 0$. By Proposition 9.7, we found that S admits a map onto a (possibly non-smooth) quotient $S_1 = (C \times E)/\mathbb{Z}_n$, where $C'/H = C$, $E'/H =: E$ is an elliptic curve and $E/\mathbb{Z}_n = \mathbb{P}^1$. We have the following expression for $H^2(S)$:

$$H^2(S) = H^2(C) \oplus H^2(E) \oplus \left(H^1(C) \otimes H^1(E)\right)^{\mathbb{Z}_n}$$

Now, let S' be a desingularisation of $S_1 = (C \times E)/\mathbb{Z}_n$. It is clear (see Propositions 9.6 and 9.7) that the transcendental lattices of S and S' are isomorphic; hence, as $H^2(C)$ and $H^2(E)$ are Hodge (even algebraic) classes, we have

$$T(S) = T((H^1(C) \otimes H^1(E))^{\mathbb{Z}_n}) = T(H^1(C) \otimes H^1(E))^{\mathbb{Z}_n} = T(S').$$

Notation for this Chapter: 1. Let A be an Abelian variety, and let B be a closed subgroup. We denote by B^0 the identity component of B. B^0 is then an Abelian subvariety of A.

2. We denote the group of n-th roots of unity by $\mu_n \subset \mathbb{C}^\times$.

3. Let $n \in \mathbb{N}$. We denote the cyclotomic polynomial of order n by $q_n(x)$, i.e. if $\zeta = e^{2\pi i/n}$, then
$$q_n(x) = \prod_{k \in \mathbb{Z}_n^\times} (x - \zeta^k).$$
It is well-known that q_n is a monic irreducible polynomial with integer coefficients and its degree is equal to $\phi(n)$ (here $\phi(\cdot)$ stands for the Euler function).

4. During the whole chapter, $\zeta = e^{2\pi i/n}$.

5. ϕ will denote the element of $G \cong \mathbb{Z}_n$ such that $\phi_E^*|H^{0,1}(E) = \zeta \mathrm{Id}$.

The following lemma is well-known, and it is proven by a slight variation of the proof of Poincare's reducibility theorem for Abelian varieties and the comments prior to Lemma 10.3.

Lemma 11.1. *Let A be an Abelian variety, and $f : A \to A$ be an endomorphism of A. Let A_1, A_2 be sub-Abelian varieties such that $f(A_i) \subset A_i$, $i = 1, 2$ and also the canonical map $u : A_1 \times A_2 \to A$ sending $(x, y) \mapsto x + y$ is an isogeny (we remark it is also f-equivariant). Then there exists an f-equivariant isogeny*

$$v : A \to A_1 \times A_2$$

such that $u \circ v = N \cdot Id_A$ and $v \circ u = N \cdot Id_{A_1 \times A_2}$ for some natural number $N > 0$.

11.1 Splitting $H^1(C) = H^1(JC)$

Let φ be a generator of the group $\mathbb{Z}_n$ chosen according to the notation above. Let ϕ denote the morphism induced on JC by $\varphi_{C,*}$. Then Lemma 11.1 yields the following ϕ-equivariant decomposition for JC:

$$JC \sim \prod_{d|n} [Ker(q_d(\phi))]^0.$$

Lemma 11.2. *Let $P := [Ker\ q_n(\phi)]^0$. The following identities hold:*

$$\left[H^1(C) \otimes H^1(E)\right]^{\mathbb{Z}_n} = \left[H^1(P) \otimes H^1(E)\right]^{\mathbb{Z}_n}, \text{ and}$$

$$T\left(\left(H^1(P) \otimes H^1(E)\right)^{\mathbb{Z}_n}\right) \cong T(A).$$

Proof: The proof relies on the following fact of elementary group theory: the product of two characters $\chi_1\chi_2$, with χ_1 of exact order n and χ_2 of order $d|n$, $d < n$ is never the trivial character (apply this to the tensor product $H^1(C) \otimes H^1(E)$ after taking $\mathbb{Q}(\mu_n)$-coefficients so as to have both components as a sum of characters over $\mathbb{Q}(\mu_n)$, and use that $\mathbb{Z}_n$ acts faithfully on $H^1(E)$). Hence, a similar identity holds between transcendental parts (use $T(V)^G = T(V^G) = V^G \cap T(V)$) for a Hodge action of G on a Hodge structure V); namely

$$T\left(\left(H^1(P) \otimes H^1(E)\right)^{\mathbb{Z}_n}\right) \cong T(A).$$

Remark Since the automorphism $\phi|P$ acts with exact order n, the ϕ-action induces an injection $\mathbb{Q}(\mu_n) \subset End^0(P)$. This will be used in the sequel, together with Lemma 11.6, restricting the arbitrariness of the simple sub-Abelian varieties of P.

A further decomposition of P can be found. This the content of the next proposition.

Remark In the sequel, the reader will note a separation of cases, $n = 2$ and $n = 3, 4, 6$, which is mainly due to the fact that the non-trivial irreducible representations of $\mathbb{Z}_n$ are two-dimensional for $n = 3, 4, 6$ and the only nontrivial irrep of $\mathbb{Z}_2$ is one-dimensional. Also the fact that $\mathbb{Q}(\mu_n)$ is quadratic if and only if $n = 3, 4, 6$ yields certain constraints on the Jacobian JC and on E.

Proposition 11.3. *Let P and E be as above. Let Q be the maximal ϕ-invariant Abelian subvariety of P such that Q is isogenous to a power of E. Then there exist a ϕ-invariant Abelian subvariety P' of P and ϕ-equivariant splittings*

$$P \sim P' \times Q.$$

Also the following statements hold.

1. *$Hom(P', E) = 0$ and, if $P' \neq 0$, then $\mathbb{Q}(\mu_n) \subset End^0(P')$. Therefore, if $n \geq 3$ and $P' \neq 0$, then dim $P' \geq 2$. Precisely speaking, if $n \geq 3$ then P' contains no ϕ-stable elliptic curves.*

2. *$H^1(P') \otimes H^1(E)$ is a transcendental Hodge structure, i.e. has no non-zero Hodge cycles.*

3. *If $n = 2$ then ϕ acts trivially on $H^1(P) \otimes H^1(E)$.*

4. *There exists a ϕ-equivariant isogeny*

$$f : Q \to E^k$$

such that the action of ϕ on E^k is diagonal, i.e. every component of the product is ϕ-invariant. Therefore, ϕ acts on Q (up to ϕ-equivariant isogeny) as

$$\phi|Q \sim diag(\zeta, \overset{m}{\ldots}, \zeta, \overline{\zeta}, \overset{k-m}{\ldots}, \overline{\zeta}).$$

The expression above should be understood as that of an element of $End^0(E^k) = M_k(End^0(E))$, and we choose ζ to map to φ_E; for $n = 2$ it is still valid, though a more precise statement is given in 3. above.

Proof:

1. is immediate by Lemma 11.2; if E_0 is a ϕ-stable elliptic curve contained in P', then the condition $\mathcal{O}_{\mathbb{Q}(\mu_n)} \subset \mathrm{End}(P')$ derived from ϕ-invariance determines not only the isogeny type of E_0, but also its isomorphism type, which would be that of E. This contradicts the assumption that $\mathrm{Hom}(P', E) = 0$.

2. is an immediate consequence of Section 10.1.

3. The fact that ϕ has exact order $n = 2$ in this case forces $\phi|P$ to be $-\mathrm{Id}_P$; thus the action of ϕ on both $H^1(P)$ and $H^1(E)$ corresponds to the homothety of ratio -1, yielding a trivial action on their tensor product. The assertion is thus established.

4. It follows from the elementary fact that ϕ yields a semisimple matrix in $\mathrm{End}^0(Q) \cong M_k(\mathrm{End}^0(E))$; the zeroes of its characteristic polynomial are defined over $\mathrm{End}^0(E) = \mathbb{Q}(\mu_n)$, hence rational (over $\mathrm{End}^0(E)$). As a result, ϕ is conjugate to a diagonal element

$$\delta = \mathrm{diag}(\zeta, \overset{m}{\dots}, \zeta, \overline{\zeta}, \overset{k-m}{\dots}, \overline{\zeta})$$

by an isogeny u, i.e. $\phi \circ u = u \circ \delta$. We remark that $\zeta, \overline{\zeta}$ are the only primitive n-th roots of unity for $n = 3, 4, 6$. Another way to prove this is to use the canonical form of the Hodge automorphism ϕ_* in $H_1(Q) = H_1(E)^k$, which yields k ϕ-stable elliptic curves E_i necessarily isomorphic to E (for E_i admit an automorphism $\phi|E_i$ of order $n \geq 3$); the proof follows from Lemma 11.1 above.

Proposition 11.4. *Under the above hypotheses and notations, the following statements hold.*

1. *Let n be $3, 4$ or 6 and let $\rho = (\rho_1, \rho_2)$ be a diagonal action of $\mathbb{Z}_n$ on $E \times E$ faithful in both variables (therefore n satisfies the usual restrictions and the biholomorphisms involved have a fixed point, which we suppose to be the origin of E). Then two cases are possible: either $\rho_1 = \rho_2$ or $\rho_1(g) = \rho_2(g)^{-1}$ for every $g \in \mathbb{Z}_n$ (the reason why is that $2 = |\mathbb{Z}_n^\times|$.)*

2. *Under the above hypotheses, the Hodge substructure of invariants of $H^2(E \times E)$ by ρ is:*

$$H^2(E) \oplus H^2(E) \oplus NS^0(E \times E) \text{ in the first case;}$$

and

$$H^2(E) \oplus H^2(E) \oplus T(E \times E) \text{ in the second case.}$$

As a result, the Hodge substructure of ρ-invariants $T(E \times E)^\rho = T(E \times E)$ in case $\rho_1 = \rho_2$ and $T(E \times E)^\rho = 0$ in the case when $\rho_1 = \rho_2^{-1}$.

Proof: The proof of 1. follows from the fact that, in our case, there are only two ($= \phi(n)$) characters of $\mathbb{Z}_n$ for $n = 3, 4, 6$. For the proof of 2., take a basis of $H^1(E)$ consisting of the only holomorphic differential form ω on E (up to a constant) and its conjugate $\overline{\omega}$. It is then easy to see that, for a suitable generator g of $\mathbb{Z}_n$,

$$g^*(\omega \otimes \omega) = \zeta^2 \omega \otimes \omega$$

for the first action in 1. and

$$g^*(\omega \otimes \omega) = \omega \otimes \omega$$

for the second; so far, we have proven that $T(E \times E)$ has no invariants in the first case and is invariant in the second case (indeed, so we have seen for $H^{2,0}$). Proceeding as shown with the other wedge products, the whole proposition follows.

Remark Last lemma will be used to show how ϕ will act on $H^1(Q) \otimes H^1(E)$, in order to obtain a clearer expression for $T\left(H^1(P) \otimes H^1(E)\right)^{\mathbb{Z}_n}$.

Theorem 11.5. *Under the above hypotheses, the following statements hold.*

1. *For* $n = 2$, $\left(H^1(C) \otimes H^1(E)\right)^{\mathbb{Z}_2} = H^1(P) \otimes H^1(E)$, *which implies that* P *is an elliptic curve. More precisely,* $C \to C/\mathbb{Z}_2 = B$ *is an étale double covering and* P *is (isogenous to) the Prym variety of such covering.*

2. *For* $n = 3, 4$ *or* 6, $\left(H^1(P') \otimes H^1(E)\right)^{\mathbb{Z}_n}$ *is a Hodge structure of transcendental cycles whose rank over* $\mathbb{Q}$ *is equal to* 2 *dim* P'.

3. *Suppose that the following occurs: for every* ϕ*-stable non-trivial homomorphism*

$$w : E \to Q$$

, the endomorphism induced by ϕ *on* E *via* w *and the one induced by* φ_E *on* E *coincide (i.e.* $(w, Id_E) : E \times E \to Q \times E$ *admits a* $\mathbb{Z}_n$*-action as in the first case of Proposition 11.4). Then* P' *is an Abelian surface admitting a ring homomorphism* $\mathbb{Q}(\mu_n) \subset End^0(P')$, *and* $\left(H^1(P') \otimes H^1(E)\right)^{\mathbb{Z}_n} = T(A)$. *Therefore dim* $T(A) = 4$ *(or equivalently,* $\rho(A) = 2$*). (This case occurs precisely when the contribution of* $H^1(Q) \otimes H^1(E)$ *to* $T(S)$ *is zero.)*

4. *Suppose that there exists a non-constant morphism $E \to Q$ onto a ϕ-stable curve $\tilde{E}$ such that $\phi|\tilde{E}$ coincides with φ_E^{-1}. Then $\tilde{E} \cong E$. Also $T(A) = T(E \times E)$, and so $P' = 0$. In this case dim $T(A) = 2$. This case occurs precisely when the contribution of $H^1(Q) \otimes H^1(E)$ to $T(S)$ is non-zero; by 2. above, $P' = 0$ in this case.*

5. *Suppose $n \geq 3$. Then either $T(A) = T(E \times E)$ and $P' = 0$ (3. above)or $T(A)$ is 4-dimensional and P' is an Abelian surface (4. above).*

Proof:

1. By Proposition 11.3, we have the following equality:

$$T(S) = T(H^1(P) \otimes H^1(E)).$$

Therefore, by equating $(2,0)$-pieces of the Hodge decomposition on both sides we get $1 = h^{1,0}(P)h^{1,0}(E) = h^{1,0}(P) =$ dim P. As $H^1(C/G) = H^1(C)^G$ by Lemma 9.3, this means that

$$\dim P = \dim JC - \dim JB = g(C) - g(B) = g(C) - 2 = 1,$$

hence $g(C) = 3$. By the Riemann-Hurwitz formula (see, for instance, Miranda [41]), the degree-2 morphism $f : C \to C/\mathbb{Z}_2 = B$ has no ramification. Indeed, let β denote the branching number, which is 0 precisely when the morphism is étale. Then

$$2 = g(C) - 1 = 2g(B) - 2 + \frac{1}{2}\beta = 2 + \frac{1}{2}\beta \Rightarrow \beta = 0.$$

Therefore the action of $\mathbb{Z}_2$ has no fixed points. P turns out to be isogenous to the *Prym variety* of the étale double covering $C \to B$ (for precise definitions and properties see Mumford [48] pp.283-288).

2. By Proposition 11.3, the Hodge structure in question is transcendental, i.e. contains no non-zero Hodge classes. Also, using the mean value trick for group representations shows that, for a linear representation of a finite group G on a vector space V over a field K of characteristic zero, dim $V^G = \dim_L(V \otimes_K L)^G$. Indeed, if $p = p_G = \frac{1}{|G|}\sum \rho(g)$, we have $V^G = \text{Im}\, p$ and analogously for

$V_L^G = \mathrm{Im}(p \otimes \mathrm{Id}_L) = V^G \otimes_K L$. Therefore in our case it suffices to compute the dimension after extension of scalars by $L = \mathbb{Q}(\mu_n)$.

In our case there are precisely two primitive characters, χ and $\overline{\chi} = \chi^{-1} \neq \chi$. Therefore $H^1(Q) \otimes_{\mathbb{Q}} L = a\chi + a\chi^{-1}$ ($a =$ dim P') and $H^1(E) \otimes_{\mathbb{Q}} L = \chi + \chi^{-1}$. Tensoring yields

$$\dim \left(H^1(P') \otimes H^1(E)\right)^{\mathbb{Z}_n} = 2a = 2 \dim P',$$

thereby establishing 2.

3. Proposition 11.3 implies that the contribution of $H^1(Q) \otimes H^1(E)$ to $T(S)$ is zero, hence $T(S) = \left(H^1(P') \otimes H^1(E)\right)^{\mathbb{Z}_n}$. Part 2. of this Theorem yields

$$2 \dim P' = \dim T(S) = \dim T(A) \leq 5,$$

where last inequality holds by Theorem 10.1; we also know that P' is simple of dimension $\dim P' \geq 2$ by Proposition 11.3, hence by last inequality P' is an Abelian surface endowed with multiplication by the quadratic imaginary field $\mathbb{Q}(\mu_n)$. The equality above yields $\dim T(A) = 4$ in this case.

4. The hypothesis on the ϕ-action implies that the contribution of $H^1(Q) \otimes H^1(E)$ to $T(S)$ is non-trivial by Proposition 11.4. Therefore, as $T(S)$ is an irreducible Hodge structure by Proposition 8.1, Part 2. of the present theorem shows $P' = 0$.

5. Follows from the statements 1-4 above.

11.2 Construction of an algebraic cycle on $S \times A$

Under the hypotheses and notations of this Chapter and of last Section, we obtain more results on the structure of P and A and finally use them to construct an algebraic cycle on $S \times A$ satisfying the desired properties. In particular, we arrive to the conclusion that, for $n \geq 3$, $P' = 0$ unconditionally by using the methods of Mumford-Tate groups given in Sections 10.3 and 10.4.

The next lemma is a cornerstone of the proof of Theorem 1.1.

Lemma 11.6. *Let B be an Abelian surface not of QM (quaternionic multiplication) type, and let K be an imaginary quadratic field such that $K \hookrightarrow End(B)_{\mathbb{Q}}$. Then B is non simple. Precisely speaking, B is isogenous to $E_1 \times E_1$, where E_1 is an elliptic curve.*

Proof: Suppose that B is simple. Then B must be of CM type, by Theorem 10.1. The following lines contain an elementary proof of the Lemma; for more general results on Abelian varieties of CM-type, see Lang [64] Chapter 1.

Let $L = End^0(B)$; then $L = K'K$, where K is another imaginary quadratic field. Now, as shown in [64] Theorem 1.4.2, the isogeny type of B is determined by $L = End^0(B)$ and a CM type, which in our case is the class modulo complex conjugacy of two (nonconjugate) immersions $\sigma_i : L \hookrightarrow \mathbb{C}$: the isogeny class of B is then determined by the variety $B' := \mathbb{C}^2/\Psi(\mathcal{O}_L)$, where $\Psi = (\sigma_1, \sigma_2)$ as defined above. As σ_i do not form a conjugate pair, $\sigma_1 = \sigma_2$ on, say, K. Define the elliptic curve $E_K := \mathbb{C}/\mathcal{O}_K$ to be the only elliptic curve (up to isogeny) such that $\mathrm{End}^0(E_K) = K$. We now construct a non-constant morphism from E_K to B. The holomorphic map

$$\mathbb{C} \to B$$

given by the linear map

$$h : \mathbb{C} \to \mathbb{C}^2 \qquad \text{defined by } z \mapsto (z, z)$$

is such that $h(\mathcal{O}_K) \subset \Psi(\mathcal{O}_L)$, as $(\sigma_1, \sigma_2)|\mathcal{O}_K$ coincides with the diagonal morphism $h|\mathcal{O}_K$. Thus h yields a non-constant morphism $E \to B$, thereby showing that B is necessarily non-simple (and therefore not of CM type). For general theorems on CM-types and decomposition of Abelian varieties of CM-type, we refer the reader to Lang [64] Theorems 1.3.4-1.3.6.

It remains to check that B is isogenous to a power of an elliptic curve E_1. Indeed, B is already non-simple and therefore isogenous to $E_1 \times E_2$, by the above argument. Suppose that E_2 is not isogenous to E_1; we have a unital ring homomorphism

$$K \hookrightarrow \mathrm{End}^0(E_1) \times \mathrm{End}^0(E_2),$$

which means that E_1, E_2 have both CM-type by the field K, hence $E1$ and E_2 are isogenous, thus leading to a contradiction. We point out that the last bit could be checked directly; for general constructions, see Lang [64] Thms. 1.3.4-1.3.6.

Remark Note that the isogeny type of E_1 in last lemma is not at all determined. Indeed, fix an arbitrary elliptic curve E_1. Now choose the representation $K \to M_2(\mathbb{Q})$ given by the choice of a $\mathbb{Q}$-basis in K. The embedding $\mathbb{Q} \to \mathrm{End}^0(E_1)$ yields an injection

$$K \hookrightarrow \mathrm{End}^0(E_1 \times E_1)$$

thus satisfying the hypotheses of Lemma 11.6. This example is also valid for $K = \mathbb{Q}(\mu_n)$, for $n = 3, 4, 6$.

Our last auxiliary result is the following lemma, which generalises Proposition 11.4.

Lemma 11.7. *Let E_1 be an arbitrary elliptic curve and $n = 2, 3, 4$ or 6. Let $E = \frac{\mathbb{C}}{\mathbb{Z}[\mu_n]}$ if $n > 2$ and of arbitrary moduli if $n = 2$. Suppose that $\rho : \mathbb{Z}_n \to Aut(E_1 \times E_1)$ is a group monomorphism given by isogenies, such that every root of $\rho(1)|H^1(E_1 \times E_1)$ is primitive and consider the usual representation on $H^1(E)$ given by $\mathbb{Z}_n \subset End(E)^\times$. Then the following equality holds on the tensor product representation $H^1(E_1 \times E_1) \otimes H^1(E)$:*

$$T(E_1 \times E) = [T(E_1 \times E) \oplus T(E_1 \times E)]^{\mathbb{Z}_n}.$$

(We note that $T(E_1 \times E) = H^1(E_1) \otimes H^1(E)$ if E_1 is not isogenous to E).

Proof: We prove the lemma for the cases $n = 3, 4, 6$ in detail. The case $n = 2$ is analogous and will be left to the reader. $T(E_1 \times E)$ is known to be irreducible because its $(2, 0)$-part is one-dimensional (see Proposition 8.1), therefore it suffices to show that the $(2, 0)$-invariants by the $\mathbb{Z}_n$-action form a one-dimensional subspace. As in the proof of Theorem 11.5 (2.), this can be done by extending scalars to $L = \mathbb{Q}(\mu_n)$. Therefore, the vector space $H^1(E_1 \times E_1) \otimes_{\mathbb{Q}} L$ has two $\mathbb{Z}_n$-invariant one-dimensional subspaces, and thus the representation ρ decomposes over L as a sum $\chi^{\oplus 2} \oplus \overline{\chi}^{\oplus 2}$. Now $H^1(E) \otimes_{\mathbb{Q}} L = \chi \oplus \overline{\chi}$, where χ is supposed to be the character on $H^{1,0}(E)$. Because the decomposition into characters is equivariant with respect to the corresponding Hodge structures, one needs only to consider two cases: either $H^{1,0}(E_1 \times E_1)$ is isotypic or its representation contains both χ and $\overline{\chi}$ once. By the standard procedure, the Lemma follows in either case.

We give the last couple of preparatory results before proving Theorem 1.1.

Theorem 11.8. *In the cases $n = 3, 4, 6$, P' cannot be an simple Abelian surface. As a result, $T(A) \cong T(E' \times E)$ for a certain elliptic curve E' and thus $A \sim E' \times E$. In the case $n = 2$, $T(A)$ is also Hodge-isomorphic to a transcendental lattice $T(E' \times E)$ and therefore $A \sim E' \times E$ as well.*

Proof: Let us examine the case $n = 2$ first. In this case, Theorem 11.5.1 shows $T(E' \times E)$, for $E' = P$. Thus we are done in this case.

Let us assume $n \geq 3$. We know that $\rho(A) \geq 2$ by the Remark on Picard numbers in the beginning of Chapter 9 (see also Proof of Theorem 11.5). We know by Theorem 11.5(5.) that P' is either 0 or an Abelian surface having multiplication by the field $\mathbb{Q}(\mu_n)$, which is clearly imaginary quadratic in our case. The case $P' = 0$ is clear, since $T(A) = T(E \times E)$ by Theorem 1.1. It remains to prove the Theorem in the case when P' is an Abelian surface.

Suppose dim $P' = 2$. We prove that P' is non-simple by contradiction. Suppose that P' is a simple abelian surface; as P' admits complex multiplication by an imaginary quadratic field, P' is either of CM type or of QM type. Then P' must be of QM type by Lemma 11.6, and also the Picard number $\rho(A) = 2$ by Theorem 11.5.4. Lemme 11.9 below shows that this case cannot hold. Let us take Lemma 11.9 below for granted. Then P' must be isogenous to a product $E_1 \times E_1$. Lemma 11.7 yields $[H^1(P') \otimes H^1(E)]^{\mathbb{Z}_n} = H^1(E_1) \otimes H^1(E) = T(E_1 \times E)$ (recall that P' does not contain factors isogenous to E, and therefore E_1 and E are non-isogenous); we state and prove Lemma 11.9, and subsequently prove the non-simplicity of P' by using Lemma 11.9.

Lemma 11.9. *Let B be an Abelian surface of QM type and X be an Abelian variety of CM type. Then $Hg(B \times X) = Hg(B) \times Hg(X)$. Therefore, the following identity holds for the Hodge ring of $B \times X$:*

$$\mathcal{H}^\bullet(B \times X) = \mathcal{H}^\bullet(B) \otimes \mathcal{H}^\bullet(X).$$

Likewise, if A is an Abelian surface of type I(2) (i.e. $End^0(A)$ is a real quadratic field F_0) and B is an Abelian surface of QM type, then

$$Hg(A \times B) = Hg(A) \times Hg(B).$$

Proof of the Lemma: By Proposition 10.29, $Hg(B)$ is a semisimple algebraic group over $\mathbb{Q}$ such that $Hg(B)_{\mathbb{R}} \cong SL_{2,\mathbb{R}}$, and therefore any normal subgroup of $Hg(B)$ must be discrete. Also $Hg(X)$ is an algebraic torus by Proposition 10.27. The first statement is thus established by Corollary 10.21. As for the second statement, the general fact that $Hg(A \times B)$ admits canonical projections onto $Hg(A)$ and $Hg(B)$ respectively implies by Proposition 10.20 that if $Hg(A)$ and $Hg(B)$ have simple Lie algebras over $\mathbb{Q}$, then either $Hg(A \times B)$ equals the product or maps surjectively onto the graph of an isomorphism $Hg(A)/N \cong Hg(B)/N'$ where N, N' are finite; last case is impossible, since the algebraic groups $D_1^{\times}$ and $\mathrm{Res}_{F_0|\mathbb{Q}} SL_{2,F_0}$ are not isogenous ($D = \mathrm{End}^0(B)$). One way to see this is by noticing that their dimensions are different. This concludes the proof.

Proof that Lemma 11.9 yields P' non-simple.- We know that $\rho(A) = 2$, hence A is of type I(2) or of CM-type (or non-simple).

1. Suppose that A is of CM-type. Given the fact that P' is of QM type and $A \times E$ is also of CM type (since $n \geq 3$), we have that $Hg(P')$ is semisimple and $Hg(A \times E)$ is a torus. By Lemma 11.9 we obtain that $Hg(P' \times A \times E) = Hg(P') \times Hg(A \times E)$ and therefore

$$\mathcal{H}(P' \times A \times E) = \mathcal{H}(P') \otimes \mathcal{H}(A \times E).$$

It is then clear that

$$H^1(P') \otimes H^1(E) \otimes H^2(A)$$

has no non-trivial Hodge cycles, which leads to a contradiction with the fact that $T(S) = T(A) \hookrightarrow H^1(P') \otimes H^1(E)$. Hence A cannot be of CM-type.

2. In the case when $\mathrm{End}^0(A) = F_0$ is a real quadratic field, a similar argument is valid. By Lemma 11.9, $Hg(A \times P') = Hg(A) \times Hg(P')$, hence $Hg(A \times P')$ is semisimple (and non-abelian). As E is of CM type, we can apply Lemma 11.9 again, finally obtaining

$$Hg(P' \times A \times E) = Hg(P') \times Hg(A) \times Hg(E).$$

This leads to a contradiction in the same way as in the last case where A was of CM type.

3. Suppose that A is non-simple. Then it can be checked that there are no nontrivial group homomorphisms between quotients of $Hg(A)$ and quotients of $Hg(P')$, and therefore $Hg(P' \times A \times E) = Hg(P') \times Hg(A \times E)$. Indeed, in this case $A \sim E_1 \times E_2$, therefore $Hg(A \times E)$ is a product of groups which are either SL_2's or one-dimensional compact tori. If the former identity does not hold, then again by Goursat's lemma (Proposition 10.20) there exist normal subgroups N, N' of $Hg(P')$ and $Hg(A \times E)$ respectively such that

$$Hg(P')/N \cong Hg(A \times E)/N',$$

hence one of the curves E_i is not of CM type and

$$Hg(P')/\mu_2 \cong Hg(E_i)/\mu_2,$$

in which case the Lie algebras of both groups are isomorphic, thus leading to a contradiction with Proposition 10.22. Finally, arguing as in the former case, we arrive to a contradiction, thereby concluding that the case when P' is QM and A is non-simple cannot hold either.

We have just proven that P' cannot be of QM type (or of CM-type). As a result, $P' \sim E' \times E'$ and therefore $T(A) = T(E' \times E)$ as shown above (just before the statement of Lemma 11.9).

Let us finally construct the desired algebraic cycle. As a consequence of the former claim and Lemma 10.34, we know that $A \sim E' \times E$ in all cases. It therefore suffices to find an algebraic cycle on $S \times E' \times E$ inducing an isomorphism between $T(S)$ and $T(E' \times E)$.

Construction in the case $n = 2$.- Let $n = 2$. Then $S = (C \times E)/\mathbb{Z}_2$ (here the moduli of E is unrestricted). We have that $JC \sim JB \times P$, where $B = C/\mathbb{Z}_2$ and $P = E'$ is an elliptic curve and $JB \sim$ by Theorems 11.5 and 11.8. We have the isomorphism

$$H^2(S) = \{H^1(C) \otimes H^1(E)\}^{\mathbb{Z}_2} \oplus H^2(C) \oplus H^2(E) \cong H^2(E' \times E)$$

given by

$$\Psi : C \times E \to JC \times E \to E' \times E,$$

where the maps are $(alb_C, Id_E), (proj_{E'}, Id_E)$ respectively. More precisely, we have the inclusion $H^2(S) = H^2(C \times E)^{\mathbb{Z}_2}$ given by Lemma 9.3 via $p*$, where

$$p : C \times E \to S = (C \times E)/\mathbb{Z}_2$$

is the canonical projection. The correspondence between $H^2(S)$ and $H^2(E' \times E)$ given by $\Psi_* \circ p^*$ is clearly algebraic and yields the above described isomorphism. We are just about done, since the Abelian surfaces $E' \times E$ and A are isogenous and hence have isomorphic H^i's.

Let us describe the isomorphism of $H^i(A)$ and $H^i(A')$ isogenous Abelian varieties. Suppose that $f : A \to A'$ and $g : A' \to A$ are two morphisms such that $fg = N.1_{A'}$ (and thus $gf = N.1_A$). Then the correspondences f^*, g^* viewed as morphisms $H^i(A') \to H^i(A)$(resp. swapping A' and A) yield

$$f^* \circ g^* = N^i 1_{H^i(A)}$$

(resp. swapping g and f on A').

Remark We have thus constructed the desired algebraic cycle. Note that we only needed to construct an algebraic isomorphism of transcendental lattices and found one on the whole of H^2's instead. This will not occur in the remaining cases, as the desingularisation process forces us to restrict to transcendental lattices.

In the cases $n = 3, 4, 6$ the construction will be taken up in four steps:

Step 1.- We consider the equality

$$T(S) = T(H^1(C) \otimes H^1(E))^{\mathbb{Z}_n}.$$

There exists an algebraic cycle α of codimension 2 on $C \times E \times S$ such that α^* induces an isomorphism between $T(S)$ and $T(C \times E)^{\mathbb{Z}_n} = T(H^1(C) \otimes H^1(E))^{\mathbb{Z}_n}$. If the $\mathbb{Z}_n$-action on $C \times E$ has no fixed points, then we may choose the graph of the canonical projection; if not, then there exists a smooth projective surface S'' and morphisms a, b such that the following diagram commutes:

$$\begin{array}{ccc} S'' & \xrightarrow{a} & S \\ \downarrow{\scriptstyle b} & & \downarrow \\ C \times E & \xrightarrow{p} & (C \times E)/\mathbb{Z}_n. \end{array}$$

The (iso)morphism $T(S) \to T(C \times E)^{\mathbb{Z}_n}$ is thus given by $b_* a^*$, or equivalently by the algebraic correspondence ${}^t[\Gamma_b] \circ [\Gamma_a]$ (see Section 4.2). Our initial statement is therefore reduced to finding an algebraic cycle of codimension 2 in $C \times E \times A$ mapping $T(C \times E)^{\mathbb{Z}_n}$ isomorphically onto $T(A)$.

Step 2.- A is isogenous to $E' \times E$ by Theorem 11.8, therefore Theorem 1.1 follows from the existence of an algebraic cycle of codimension 2 in $C \times E \times E' \times E$ mapping $T(C \times E)^{\mathbb{Z}_n}$ isomorphically onto $T(E' \times E)$.

Step 3.- We know by Lemma 11.2 that

$$T(C \times E)^{\mathbb{Z}_n} = T(H^1(C) \otimes H^1(E))^{\mathbb{Z}_n} = T(H^1(P) \otimes H^1(E))^{\mathbb{Z}_n}.$$

Therefore, if $q : JC \to P$ is a $\mathbb{Z}_n$-equivariant (i.e. ϕ-equivariant) projection (see Lemma 11.1) and $h = (q \circ \mathrm{alb}_C, \mathrm{id}_E)$ then h^* yields an isomorphism of $T(C \times E)^{\mathbb{Z}_n}$ onto $[H^1(P) \otimes H^1(E)]^{\mathbb{Z}_n}$. Therefore Theorem 1.1 follows from the existence of an algebraic cycle of codimension 2 in $P \times E \times E' \times E$ inducing an isomorphism between $T^2(P \times E)^{\mathbb{Z}_n}$ and $T(E' \times E)$.

Step 4 (Case 1).- In Part 4. of Theorem 11.5 (i.e. $P' = 0$), there exists a ϕ-stable elliptic curve $E_2 \subset P$ isomorphic to E such that $T(E_2 \times E) = T(P \times E)^{\mathbb{Z}_n}$ (indeed, counting Aut E_2 shows $E_2 \cong E$); also, the fact that E' is isogenous to E follows from the Hodge isomorphisms

$$T(E_2 \times E) = T(E \times E) \cong T(A) = T(E' \times E)$$

(one may simply argue that dim $T(E' \times E) \leq 3$, therefore $E' \sim E$; we may also cite Lemma 10.34). Thus, for w a ϕ-invariant projection $w : P \to E$ (see Lemma 11.1) mapping E_2 onto E, the morphism $\Psi = (w, \mathrm{id}_E)$ yields an isomorphism

$$\Psi^* : T(E \times E) \to T(P \times E)^{\mathbb{Z}_n} = T(E_2 \times E).$$

Theorem 1.1 follows from the last line and from the fact that $E' \sim E$.

Step 4 (Case 2).- We suppose that E' and E are not isogenous; then $T(P \times E)^{\mathbb{Z}_n} = T(P' \times E)^{\mathbb{Z}_n}$, where $P' \sim E' \times E'$ is an isogeny of $\mathbb{Z}_n$-varieties, considering the appropriate action on the right hand side. We are going to prove the existence of an algebraic cycle δ of codimension 2 on $\{(E' \times E') \times E\} \times E' \times E$ inducing an isomorphism

$$[T^2(E' \times E' \times E)]^{\mathbb{Z}_n} \to T(E' \times E)$$

or equivalently

$$[(H^1(E') \oplus H^1(E')) \otimes H^1(E)]^{\mathbb{Z}_n} \to H^1(E') \otimes H^1(E).$$

A Hodge isomorphism of the above Hodge structures exists by Lemma 11.7; it suffices to show that it is induced by an algebraic cycle. Indeed, the above isomorphism yields a Hodge class in the cohomology ring of the Abelian variety

$$E' \times E' \times E \times E' \times E.$$

Now, by Proposition 10.29 every Hodge class on a product of elliptic curves is algebraic, hence Theorem 1.1 holds also in this case.

The above steps conclude the proof of Theorem 1.1 in every case.

Bibliography

[1] A. Albano, F. Bardelli (eds.) *Algebraic cycles and Hodge Theory*, LNM 1594, Springer-Verlag, Berlin-Heidelberg 1994.

[2] E. Ballico, F. Catanese, C. Ciliberto, eds. *Classification of Irregular Varieties, Minimal Models and Abelian Varieties. Proceedings of the Trento Conference 1990.* LNM 1515, Springer-Verlag, Berlin-Heidelberg 1992.

[3] W. Barth, C. Peters, A. van de Ven, *Compact Complex Surfaces*, Springer-Verlag. Berlin-Heidelberg-New York, 1984.

[4] A. Beauville, *Complex Algebraic Surfaces*, second English edition, Cambridge Univ. Press, 1996.

[5] A. Borel, *Linear Algebraic Groups.* 2nd. enlarged edition. Springer GTM 126. Springer-Verlag, New York 1991.

[6] A. Borel, N. Wallach, *Continuous Cohomology, Discrete Subgroups and Representations of Reductive Groups.* 2nd Edition. AMS, Providence RI, 2000.

[7] R. Bott, *On a theorem of Lefschetz. Collected Works, Volume 1*, pp. 396-401. Birkhäuser, Boston 1994.

[8] P. Deligne, *Travaux de Griffiths.* Séminaire Bourbaki 1969/70, Exposé 376. LNM 180, Springer-Verlag, Heidelberg 1971.

[9] P. Deligne, J.S. Milne, A. Ogus, K.Y. Shih, *Hodge cycles, motives and Shimura varieties*, LNM 900. Springer-Verlag, Berlin-Heidelberg-New York, 1982.

[10] P. Deligne, *Theoreme de Lefschetz et criteres de degenerescence de suites spectrales*, Publ. Math. IHES 35 (1968) 107-126.

[11] P. Deligne, *La Conjecture de Weil pour les surfaces K3*, Inv. Math 15 (1972), 206-226.

[12] S. Diaz, *A bound on the dimensions of complete subvarieties of $\mathcal{M}_g$*. Duke Math. J. 51 (1984), pp. 405-408.

[13] J. Dixmier, *Enveloping Algebras*, North-Holland, Amsterdam 1974.

[14] E. Freitag, *Über die Struktur der Funktionenkörper zu hyperabelschen Gruppen.I.* J.Reine und Angew. Math 247(1971), 97-117.

[15] W. Fulton, *Intersection Theory*, 2nd. Edition. Springer-Verlag, NY 1998.

[16] W. Fulton, J. Harris, *Representation Theory - a first course.* Springer GTM 129, Springer-Verlag, New York, 1991.

[17] B.L. van Geemen, *Kuga-Satake varieties and the Hodge conjecture.* To be found in *The Geometry and Arithmetic of Algebraic Cycles*, NATO Science Series 548, World Scientific 2001. See also arXiv math.AG/9903146.

[18] R. Godement, *Topologie Algébrique et Théorie des Faisceaux.* Hermann, Paris 1958.

[19] P. Griffiths, J. Adams, *Topics in Algebraic and Analytic Geometry*, Mathematical Notes 13, Princeton U. Press, Princeton 1974.

[20] P. Griffiths, J. Harris, *Principles of Algebraic Geometry.* Wiley Interscience, NY 1978.

[21] A. Grothendieck, *Techniques de descente et theoremes d'existence en Geometrie Algebrique. V. Les Schemas de Picard. Seminaire Bourbaki 1961/1962, no. 232.*

[22] A. Grothendieck, *Familles d'espaces complexes et fondements de la géométrie analytique.* Exposés 7-8, 16-17 in Séminaire Cartan 13 (1960-61).

[23] A. Grothendieck, *Elements de Géométrie Algébrique.* III. Étude cohomologique des faisceaux cohérents, Publ. Math. IHES 11 (1961) and 17 (1963).

[24] F. Hazama, *Algebraic cycles on certain Abelian varieties and powers of special surfaces.* J.Fac.Sci. Univ.Tokyo Sect.IA Math. 31 (1984), 487-520.

[25] F. Hazama, *Algebraic Cycles on nonsimple Abelian varieties,* Duke Math. J. Vol 58 (1989), no. 1, pp. 31-37.

[26] F. Hirzebruch, *Topological Methods in Algebraic Geometry,* 3rd Edition. Springer-Verlag, Berlin-Heidelberg 1966.

[27] J. Humphreys, *Linear Algebraic Groups*, Springer-Verlag, New York 1975.

[28] S. Kleiman, *Algebraic Cycles and the Weil Conjectures, Dix exposés sur la cohomologie des schemas*, pp. 359-386. North-Holland, Amsterdam, 1968.

[29] S. Kleiman, *Motives*, Algebraic Geometry Oslo 1970, North-Holland.

[30] K. Kodaira, *Complex Manifolds and Deformations of Complex Structures*, Springer-Verlag, New York 1986.

[31] M. Kuga, I. Satake, *Abelian varieties attached to polarized K3-surfaces*, Math. Annalen 169 (1967), 239-242.

[32] K. Künnemann, *On the Chow motive of an Abelian scheme*, Motives (Seattle, WA 1991). Proc. Symp. Pure Math. 55, Part 1, pp. 189-205. AMS, Providence RI, 1995.

[33] S. Lang, A. Néron, *Rational points of Abelian varieties over function fields*, Amer. J. Math 81(1959), 95-118.

[34] H.Lange, C.Birkenhakke, *Complex Abelian Varieties*, Springer-Verlag, Heidelberg 1992.

[35] J.D. Lewis, *A survey on the Hodge Conjecture*, 2nd Edition, AMS Providence, RI 1998 – with an appendix by B. Brent Gordon, "The Hodge conjecture for Abelian varieties", also alg-geom/9709030

[36] J. McCleary, *A user's guide to spectral sequences*, 2nd. Edition. Cambridge Univ. Press, Cambridge (UK) 2001.

[37] Yu I. Manin, *Correspondences, motifs and monoidal transformations*, Math. USSR Sbornik 6 (1968), 439-470.

[38] R. Mayer, *Coupled contact systems and rigidity of maximal dimensional variations of Hodge structure*, Trans. Amer. Math. Soc 352 (2000) 2121-2144.

[39] J.S. Milne, *Lefschetz Motives and the Tate Conjecture*, Compositio Math. 117 (1999), pp. 47-81.

[40] J.S. Milne, *Etale cohomology*, Princeton University Press, Princeton NJ, 1980.

[41] Rick Miranda, *Algebraic Curves and Riemann Surfaces,*, Graduate Studies in Math. 5, AMS, Providence RI 1995.

[42] D. R. Morrison, *On K3 surfaces with large Picard number*, Inv Math. 75 (1984) 105-121.

[43] D. R. Morrison, *The Kuga-Satake variety of an abelian surface*, J. Algebra 92 no. 2(1985) 454-476.

[44] S.Mukai, *On the moduli space of bundles of K3 surfaces, I.* Vector Bundles on Algebraic Varieties, Bombay Colloquium 1984, Oxford 1987.

[45] D. Mumford, Abelian Varieties, Oxford University Press, Oxford 1970.

[46] D. Mumford, *On Shimura's paper "Discontinuous groups and abelian varieties.* Math. Ann. 181 (1969), 345-351

[47] D. Mumford, J. Fogarty, F. Kirwan, *Geometric Invariant Theory*, 3rd enlarged edition. Springer-Verlag, Berlin-Heidelberg-New York 1994.

[48] D. Mumford, *The Red Book of Varieties and Schemes*, 2nd. expanded edition – includes *Curves and Their Jacobians.* LNM 1358, Springer-Verlag, New York 1999.

[49] J.P. Murre, *On the motive of an algebraic surface*, J.Reine und Angewandte Math. 409(1990), 190-204.

[50] V.K. Murty, *Exceptional Hodge cycles on certain Abelian varieties*, Math. Ann. 268 (1984), 197-206.

[51] F. Oort, *Endomorphism algebras of Abelian varieties.* Algebraic Geometry and Commutative Algebra in honor of Masayoshi Nagata, Part II, Kinokuniya Company Ltd., Tokyo 1988, pp. 469-502

[52] Miles Reid, *"Chapters on algebraic surfaces". Complex Algebraic Geometry,* J. Kollar ed. Park City Series, AMS, Providence RI 1997.

[53] K. Ribet, *Hodge classes of certain Abelian varieties*, Amer. J. Math 115 (1983), 523-538.

[54] A.J. Scholl, *Classical Motives*, Proc. Symp. Pure Math. 55 (Seattle, WA 1991), Part 1, pp. 163-187. AMS, Providence, RI 1995.

[55] C. Schoen, *Hodge classes on self-products of a variety with an automorphism*, Comp. Math. 65(1988) 3-32.

[56] C. Schoen, *Addendum to: Hodge classes on self-products of a variety with an automorphism*, Comp. Math. 114 (1998) 329-336.

[57] F. Serrano, *On isotrivial fibered surfaces*, preprint, Universitat de Barcelona 1992

[58] F. Serrano, *The Picard group of a quasi-bundle*, Manuscripta Mathematica 73 (1991), 63-82.

[59] J.H. Silverman, *The Arithmetic of Elliptic Curves*, Springer-Verlag, New York, 1986.

[60] S.G.Tankeev, *Abelian Varieties and Lie Algebras*, Selecta Math. Vol. 13, No.1 (1994).

[61] J.T. Tate, *Endomorphism algebras of abelian varieties over finite fields.* Inv. Math. 2 (1966), 134-144.

[62] B.J.J. Moonen, Yu G. Zarhin, *Hodge classes and Tate classes on simple Abelian fourfolds*, Duke Math J. 77(1995), 553-581.

[63] R. Hartshorne, *Algebraic Geometry*, Springer GTM 52, Springer-Verlag, New York, 1977.

[64] S. Lang, *Complex Multiplication*, Springer-Verlag, New York 1983.

[65] S. Lang, *Abelian Varieties*, Interscience Publishers, NY, 1959.

[66] B.J.J. Moonen, Yu G. Zarhin, *Hodge cycles on abelian varieties of low dimension.* Math.Ann. 315(1999), No. 4, 711-733. Also available in math.AG/9901113.

[67] P. Roquette, *Abschatzüng der Automorphismenanzahl von Funktionenkörpern bei Primzahlcharakteristik*, Math. Zeitschrift 117 (1970) 157-163.

[68] J.P. Serre, Géométrie Algébrique et Géométrie Analytique. Ann. Institut Fourier 6 (1955), 1-42.

[69] J.P. Serre, *Local Fields,* Springer GTM 67, New york 1979 (English translation)

[70] K. Ueno, *Classification Theory of Algebraic Varieties and Compact Complex Spaces*, LNM 439, Springer-Verlag, Berlin-Heidelberg, 1975.

[71] F.Warner, *Foundations of Differential Manifolds and Lie groups.* Scott, Foresman and Co., Glenview IL 1971.

[72] W.C. Waterhouse, *Introduction to Affine Group Schemes,* Springer GTM 66, Springer-Verlag, NY 1979.

[73] R.O. Wells, *Differential Analysis on Complex Manifolds*, Prentice-Hall, Englewood Cliffs NJ, 1973.

[74] H. Weyl, *The Classical Groups*, Princeton Univ. Press, Princeton 1946.

[75] Yu. G. Zarhin, *Abelian Varieties and Lie Algebras*, Selecta Math. vol. 13 (1994), no. 1, pp.55-95.