Bundy, D. and Hart, S.

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The case of equality in the Livingstone-Wagner Theorem

David Bundy, Sarah Hart

Abstract

Let G be a permutation group acting on a set Ω of size $n \in \mathbb{N}$ and let $1 \leq k < (n-1)/2$. Livingstone and Wagner proved that the number of orbits of G on k-subsets of Ω is less than or equal to the number of orbits on $(k+1)$-subsets. We investigate the cases when equality occurs.

1 Introduction

Throughout this article we let G be a permutation group acting on a set Ω of size $n \in \mathbb{N}$ and let $1 \leq k < (n-1)/2$. In [6] Livingstone and Wagner proved the following theorem.

Theorem 1.1. *(Livingstone, Wagner) [6] The number of orbits of G on k-subsets of Ω is less than or equal to the number of orbits on $(k+1)$-subsets.*

Alternative proofs were subsequently given by Robinson [7] and Cameron [1] who extended the result to Ω infinite. An investigation of the cases when equality occurs for Ω infinite was then made by Cameron [1], [2] and Cameron and Thomas [5]. The case of equality also follows from a stronger "intersection property" examined by Cameron, Neumann and Saxl [4]. In this article, we will prove some similar results about the case of equality when Ω is finite.

In Section 2 we consider the case when G is intransitive. We show (see Lemma 2.1) that G must have one orbit of length at least $n-k$ and (see Proposition 2.2) that the action of G on this orbit satisfies a strong condition which in almost all cases forces G to be k-homogeneous on this orbit.

Transitive but imprimitive groups are then investigated in Section 3. In this case there are too many examples for a complete classification to be feasible, so we concentrate on finding a necessary condition for the sizes and number of blocks in a system of imprimitivity. This quickly reduces to a combinatorial problem of determining when the number of partitions of k into at most r parts of size at most s is the same as for $k+1$. This problem is also of independent interest in invariant theory, where such partitions can be used to count the number of linearly independent semi-invariants of degree r and weight k of a binary form of degree s. We are able to determine all the cases of equality for $r \leq 4$ (see Theorem 3.1) and conjecture that for $s \geq r \geq 5$, there are only finitely many cases of equality (see Conjecture 3.2 for details). Theorem 3.7 shows that for $s \geq r \geq 5$, equality can only occur when $2k \geq r(s-1)-1$, that is k is close to half n. We have strong experimental evidence for believing Conjecture 3.2 to be true. We observe that for large enough fixed r and s the number of partitions of k into at most r parts of size at most s approximates to a Gaussian distribution whose peak becomes sharper for larger r and s.

In the final section we make some observations about the case when G is primitive. Aside from $(k+1)$-homogeneous groups the only examples we know are the affine general linear groups over a field of size 2 (see Proposition 4.2) and a list of 19 further examples of degree at most 24, many of which are subgroups of M_{24}. The absence in [4] of any examples of degree greater than 24 suggests that such examples may also be rare or non-existent in our situation.

Notation and preliminary results

For each $0 \leq l \leq n$, let $\sigma_l(G)$ be the number of orbits of G on the set of l-subsets of Ω. A permutation group is said to be *l-homogeneous* if it is transitive in its action on l-subsets, that is $\sigma_l(G) = 1$. Let Δ be a G-invariant subset of Ω. Then G^Δ will denote the permutation group induced by G in its action on Δ.

Let H be a subgroup of a group G, χ be a character of G and ψ a character of H. Then $\chi \downarrow H$ will denote the restriction of χ to H and $\psi \uparrow G$ will denote the character induced by ψ on G. Furthermore 1_G will denote the trivial character on G.

Lemma 1.2. *Let $G \leq \mathrm{Sym}(n)$, $0 \leq l \leq n$ and ψ_l be the character of $\mathrm{Sym}(n)$ induced by the trivial character on $\mathrm{Sym}(l) \times \mathrm{Sym}(n-l)$. Then $\langle \psi_l \downarrow G, 1_G \rangle$ is the number of orbits of G on l-subsets of $\{1, \ldots, n\}$ and if $0 \leq l < (n-1)/2$, then $\psi_{l+1} - \psi_l$ is an irreducible character of $\mathrm{Sym}(n)$.*

Proof. See [7]. □

Lemma 1.3. *Let $H \leq G \leq \mathrm{Sym}(n)$ and $1 \leq k < (n-1)/2$. Then $\sigma_{k+1}(G) - \sigma_k(G) \leq \sigma_{k+1}(H) - \sigma_k(H)$. In particular, if $\sigma_{k+1}(H) = \sigma_k(H)$, then $\sigma_{k+1}(G) = \sigma_k(G)$.*

Proof. Let $\chi := \psi_{k+1} - \psi_k$ be the irreducible character in the conclusion of Lemma 1.2. Then

$$\sigma_{k+1}(G) - \sigma_k(G) = \langle \chi \downarrow G, 1_G \rangle \leq \langle \chi \downarrow H, 1_H \rangle = \sigma_{k+1}(H) - \sigma_k(H).$$

In particular, if $\sigma_{k+1}(H) = \sigma_k(H)$, then the right-hand side is zero and by Theorem 1.1 the left-hand side is non-negative, so must also be zero. □

2 Intransitive groups with equality

In this section we investigate intransitive permutation groups which achieve equality in the Livingstone-Wagner Theorem.

Lemma 2.1. *Let $G \leq \mathrm{Sym}(n)$ and suppose $\sigma_k(G) = \sigma_{k+1}(G)$ for some $1 \leq k < (n-1)/2$. Then G has an orbit of length at least $n-k$.*

Proof. Suppose G has no orbit of length at least $n-k$. Then $G \leq \mathrm{Sym}(n-l)\times\mathrm{Sym}(l) =: M$, for some $l \geq k+1$. But $\sigma_{k+1}(M) = k+2 > k+1 = \sigma_k(M)$, which contradicts Lemma 1.3. □

Proposition 2.2. *Let $G \leq \mathrm{Sym}(n)$ and $1 \leq k < (n-1)/2$ with $\sigma_k(G) = \sigma_{k+1}(G)$. Let Δ be an orbit of G of length at least $n-k$. Then $\sigma_l(G^\Delta) = \sigma_{l+1}(G^\Delta)$, for all $k-(n-|\Delta|) \leq l \leq \min(k, |\Delta|-k-2)$.*

Proof. Note that an orbit of length at least $n-k$ exists by Lemma 2.1. Let $M := G^\Delta \times \mathrm{Sym}(\Omega \setminus \Delta) \geq G$ and let $m := |\Delta|$. For $t \in \mathbb{N}$, two t-subsets of Ω are in the same M-orbit if and only if their intersections with Δ are in the same G^Δ-orbit. In particular, these intersections must be of the same size. Hence

$$\sigma_t(M) = \sum_{l=\max(0,t-(n-m))}^{min(t,m)} \sigma_l(G^\Delta).$$

Now $m \geq n-k \geq (2k+1)-k = k+1$. Also $k-(n-m) \geq k+(n-k)-n = 0$. Therefore

$$\begin{aligned} 0 &= \sigma_{k+1}(M) - \sigma_k(M) = \sum_{l=k+1-(n-m)}^{k+1} \sigma_l(G^\Delta) - \sum_{l=k-(n-m)}^{k} \sigma_l(G^\Delta) \\ &= \sigma_{k+1}(G^\Delta) - \sigma_{k-(n-m)}(G^\Delta). \end{aligned}$$

That is, $\sigma_{k+1}(G^\Delta) = \sigma_{k-(n-m)}(G^\Delta)$. If $2k < m-1$ then the Livingstone-Wagner Theorem forces $\sigma_l(G^\Delta) = \sigma_{l+1}(G^\Delta)$, for each $k-(n-m) \leq l \leq k$.
On the other hand, suppose $2k \geq m-1$. Then $\sigma_{k+1}(G^\Delta) = \sigma_{m-(k+1)}(G^\Delta)$ and $m-(k+1)$ is within the range to which the Livingstone-Wagner Theorem applies. We also have that

$$(m-(k+1)) - (k-(n-m)) = (n-1) - 2k > 0.$$

Hence, by the Livingstone-Wagner Theorem, $\sigma_l(G^\Delta) = \sigma_{l+1}(G^\Delta)$, for each $k-(n-m) \leq l \leq m-k-2$. Note that $\min(k, m-k-2)$ is k precisely when $2k < m-1$ and $m-k-2$ otherwise, so the proof is complete. □

Proposition 2.2 provides the means to reduce the case of equality for an intransitive group to that of equality for a transitive group. Indeed if G is intransitive with an orbit Δ satisfying the condition of Proposition 2.2, then we nearly always have equality $\sigma_l(G^\Delta) = \sigma_{l+1}(G^\Delta)$ for several consecutive values of l. (If there is just one value of l then either G is already transitive or $n = 2k + 2$.) This almost forces G^Δ to be k-homogeneous. The only known exceptions with $k < (n-1)/2$ are where $G^\Delta \cong M_{24}$ or M_{23}.

3 Imprimitive groups with equality

There is an abundance of imprimitive groups which achieve equality in the Livingstone-Wagner Theorem and a complete classification of them seems intractable. Nevertheless, we are able to give a condition on the block sizes which is necessary if equality in the Livingstone-Wagner Theorem holds. Observe that by Lemma 1.3, if $\sigma_k(H) = \sigma_{k+1}(H)$ holds for an imprimitive group H with r blocks of size s, then $\sigma_k(G) = \sigma_{k+1}(G)$, where $G \cong \mathrm{Sym}(s) \wr \mathrm{Sym}(r)$ is the full stabiliser in $\mathrm{Sym}(rs)$ of the blocks of H. Note also that the number of orbits of G on k-subsets is equal to the number of ways, $P(r, s, k)$, to partition k into at most r parts of size at most s. We require $P(r, s, k) = P(r, s, k + 1)$. The following result is established by Lemma 3.5, Proposition 3.6 and Proposition 3.9.

Theorem 3.1. *Let $r \in \{2, 3, 4\}$ with $r \le s$ and $1 \le k < (rs-1)/2$. Then $P(r, s, k) = P(r, s, k + 1)$ if and only if one of the following holds.*

(a) $r = 2$ and k is even.

(b) $r = 3$ and

$$k = \begin{cases} \frac{3s-3}{2}, & \text{if } s \text{ is odd}, \\ \frac{3s-4}{2}, & \text{if } s \equiv 0 \mod 4, \\ \frac{3s-2}{2} \text{ or } \frac{3s-6}{2}, & \text{if } s \equiv 2 \mod 4. \end{cases}$$

(c) $r = 4$ and $k = 2s - 2$ or $r = s = k = 4$.

We also make the following conjecture.

Conjecture 3.2. *Let $1 < r \le s$, $1 \le k < (rs-1)/2$ and suppose $P(r, s, k) = P(r, s, k+1)$. Then one of the following holds:*

(a) $r \in \{2, 3, 4\}$ and the possibilities for s and k are as in Theorem 3.1; or

(b) r, s and k have the values given by a column of the following table

r	5	5	5	6	6	6	6	6	7
s	6	10	14	6	7	9	11	12	10
k	14	24	34	16	20	26	32	38	34

Remark 3.3. *The quantity* $P(r,s,k) - P(r,s,k-1)$ *is of interest in invariant theory. By a theorem of Cayley and Sylvester (see Satz 2.21 of [8]) it is equal to the number of linearly independent semi-invariants of degree* r *and weight* k *of a binary form of degree* s*. Conjecture 3.2, if proven, would then give the values of* r, s *and* k *for which no such semi-invariant exists.*

We now define some more notation which we will use in this section. Let $\mathcal{P}(r,s,k)$ be the set of partitions of k into at most r parts of size at most s, so $P(r,s,k) = |\mathcal{P}(r,s,k)|$. We will use the convention that $P(r,s,k) = 0$ if $k < 0$ or $k > rs$. By considering dual partitions we observe that $P(r,s,k) = P(s,r,k)$, so without loss we will assume that $r \le s$. Elements of $\mathcal{P}(r,s,k)$ will be written $(a_1, a_2, \dots, a_r)$ where $\sum_{i=1}^{r} a_i = k$ and $s \ge a_1 \ge \cdots \ge a_r \ge 0$. Let $\mathcal{A}(r,s,k)$ be the subset of $\mathcal{P}(r,s,k)$ consisting of all partitions of the form $(s, a_2, \dots, a_r)$ and let $\mathcal{B}(r,s,k+1)$ be the subset of $\mathcal{P}(r,s,k+1)$ consisting of all partitions of the form $(x, x, a_3, \dots, a_r)$, for some $x \le s$. Furthermore, let $A(r,s,k) = |\mathcal{A}(r,s,k)|$ and $B(r,s,k) = |\mathcal{B}(r,s,k)|$. Note that $A(r,s,k) = P(r-1,s,k-s)$. We will define a bijection from a subset of $\mathcal{P}(r,s,k)$ to a subset of $\mathcal{P}(r,s,k+1)$. Let $(a_1, a_2, \dots, a_r) \in \mathcal{P}(r,s,k)$ with $s > a_1 \ge a_2 \ge \dots \ge a_r \ge 0$, and define

$$f(a_1, a_2, \dots, a_r) = (a_1 + 1, a_2, \dots, a_r).$$

Then f is a bijection from $\mathcal{P}(r,s,k) \setminus \mathcal{A}(r,s,k)$ to $\mathcal{P}(r,s,k+1) \setminus \mathcal{B}(r,s,k+1)$. In particular we have the following result.

Lemma 3.4. *Let* $r, s, k \ge 1$*. Then*

$$P(r,s,k+1) - P(r,s,k) = B(r,s,k+1) - A(r,s,k).$$

So the problem of determining when $P(r,s,k) = P(r,s,k+1)$ reduces to that of determining when $B(r,s,k+1) = A(r,s,k)$. We now consider in turn the cases when $r = 2, 3$ and 4.

Lemma 3.5. *Let* $s \ge 0$*. Then*

$$P(2,s,k) = \begin{cases} 0, & \text{if } k > 2s, \text{ or } k < 0, \\ s - \left\lceil \frac{k}{2} \right\rceil + 1, & \text{if } s \le k \le 2s, \\ \left\lfloor \frac{k}{2} \right\rfloor + 1, & \text{if } 0 \le k \le s. \end{cases}$$

In particular, if $1 \le k < s$*, then* $P(2,s,k) = P(2,s,k+1)$ *if and only if* k *is even.*

Proof. Elementary. □

Proposition 3.6. *Let $s \geq 3$ and $1 \leq k < (3s-1)/2$. Then $P(3,s,k) = P(3,s,k+1)$ if and only if one of the following holds:*

(a) s is odd and $k = (3s-3)/2$,

(b) $s \equiv 0 \mod 4$ and $k = (3s-4)/2$,

(c) $s \equiv 2 \mod 4$ and $k = (3s-2)/2$ or $(3s-6)/2$.

Proof. Let $d_k = P(3,s,k+1) - P(3,s,k) = B(3,s,k+1) - A(3,s,k)$. By Lemma 3.5,

$$A(3,s,k) = P(2,s,k-s) = \begin{cases} \lfloor \frac{k-s}{2} \rfloor + 1 & \text{if } s \leq k < (3s-1)/2, \\ 0 & \text{if } k \leq s. \end{cases}$$

Moreover,

$$B(3,s,k+1) = |\{(a,a,b) \mid s \geq a \geq b, 2a+b = k+1\}| = \left\lfloor \frac{k+1}{2} \right\rfloor - \left\lceil \frac{k+1}{3} \right\rceil + 1.$$

Hence

$$B(3,s,k+1) \geq \frac{k}{2} - \frac{k+3}{3} + 1 = \frac{k}{6} > 0.$$

So if $A(3,s,k) = 0$, then $d_k \geq k/6 > 0$. We may therefore assume that

$$s \leq k < (3s-1)/2 \text{ and } A(3,s,k) = \left\lfloor \frac{k-s}{2} \right\rfloor + 1.$$

Thus

$$d_k = \left\lfloor \frac{k+1}{2} \right\rfloor - \left\lceil \frac{k+1}{3} \right\rceil - \left\lfloor \frac{k-s}{2} \right\rfloor.$$

Suppose s is odd. Then $k+1 \equiv k-s \mod 2$. Hence

$$d_k = \frac{k+1-(k-s)}{2} - \left\lceil \frac{k+1}{3} \right\rceil = \frac{s+1}{2} - \left\lceil \frac{k+1}{3} \right\rceil.$$

Therefore

$$d_k = 0 \Leftrightarrow k \in \left\{ \frac{3s+1}{2}, \frac{3s-1}{2}, \frac{3s-3}{2} \right\}.$$

Since $k < (3s-1)/2$, this forces $k = \frac{3s-3}{2}$.
Suppose s is even. Then

$$d_k \geq \frac{k}{2} - \frac{k+3}{3} - \frac{k-s}{2} = \frac{1}{6}(3s-2k-6).$$

Assume $d_k = 0$. Then $2k \geq 3s - 6$. Thus $3s/2 - 3 \leq k \leq 3s/2 - 1$ and so $\lceil \frac{k+1}{3} \rceil = \frac{s}{2}$. Therefore

$$d_k = \begin{cases} \frac{k}{2} - \frac{s}{2} - \frac{k-s}{2} = 0, & \text{if } k \text{ is even} \\ \frac{k+1}{2} - \frac{s}{2} - \frac{k-s-1}{2} = 1, & \text{if } k \text{ is odd, a contradiction.} \end{cases}$$

Thus k is even, $\frac{3s-6}{2} \leq k \leq \frac{3s-2}{2}$ and hence

$$k = \begin{cases} \frac{3s-4}{2} & \text{if } s \equiv 0 \mod 4 \\ \frac{3s-2}{2} \text{ or } \frac{3s-6}{2} & \text{if } s \equiv 2 \mod 4. \end{cases}$$

□

Theorem 3.7. *Let $4 \leq r \leq s$ and $1 \leq k < (rs-1)/2$. If $P(r,s,k) = P(r,s,k+1)$, then $k \geq (r(s-1)-1)/2$ or $r = s = k = 4$.*

Proof. Suppose first that $k < s$. Then $A(r,s,k) = 0$ but $B(r,s,k) > 0$, since $r \geq 4$. Therefore by Lemma 3.4 $P(r,s,k) < P(r,s,k+1)$. Now suppose that $k = s \geq 5$. Then

$$P(r,s,k) = P(r,k,k) = 2 + P(r,k-2,k)$$

and

$$P(r,s,k+1) = P(r,k,k+1) = 3 + P(r,k-2,k+1).$$

Since $(r(k-2)-1)/2 \geq (4(k-2)-1)/2 = 2k - 9/2 > k$, applying Theorem 1.1 yields $P(r,k-2,k) \leq P(r,k-2,k+1)$ and so $P(r,s,k) < P(r,s,k+1)$ in this case.
It remains to show for $s < k < (r(s-1)-1)/2$ that $P(r,s,k) < P(r,s,k+1)$. So we assume for a contradiction that $P(r,s,k) = P(r,s,k+1)$ in this case. Observe that

$$P(r,s,k) = P(r,s-1,k) + P(r-1,s,k-s).$$

Since $k < (r(s-1)-1)/2$ and $k - s < (r(s-1)-1-2s)/2 < ((r-1)s-1)/2$, by Theorem 1.1, $P(r,s-1,k) \leq P(r,s-1,k+1)$ and $P(r-1,s,k-s) \leq P(r-1,s,k-s+1)$. So under our assumption we have $P(r-1,s,k-s) = P(r-1,s,k-s+1)$. We now proceed by induction on r.
Suppose first that $r = 4$. Then by Proposition 3.6, $P(3,s,k-s) = P(3,s,k-s+1)$ implies $3s/2 - 3 \leq k - s \leq 3s/2 - 1$. However $k < (4(s-1)-1)/2 = 2s - 5/2$, so $k - s \leq s - 3 < 3s/2 - 3$, a contradiction.
Now suppose $r > 4$ and the result holds for $r-1$ in place of r. Since $P(r-1,s,k-s) = P(r-1,s,k-s+1)$, we obtain by induction that

$$k - s \geq \tfrac{(r-1)(s-1)-1}{2} = \tfrac{rs-r-s}{2}.$$

Hence $k \geq (rs-r+s)/2 > (rs-1)/2$, a contradiction. Therefore by induction the result holds for all $r \geq 4$. □

Proposition 3.8. *Let $s \geq 4$ and $2s-2 \leq k \leq 2s-1$. Then $P(4,s,k) = P(4,s,k+1)$ if and only if $k = 2s-2$.*

Proof. Since $r = 4$ is fixed, for this proof we will abbreviate $A(r,s,k)$ by $A(s,k)$ and $B(r,s,k)$ by $B(s,k)$. We first show that for all $s \geq 4$, $P(4,s,2s-2) = P(4,s,2s-1)$. We need to evaluate $B(s,k)$ more precisely. Now

$$\mathcal{B}(s,k) = \{(a,a,b,c) : s \geq a \geq b \geq c \geq 0, 2a+b+c = k\}.$$

Now $0 \leq b+c \leq 2a$ implies $2a \leq k \leq 4a$. Hence $\lceil \frac{k}{4} \rceil \leq a \leq \lfloor \frac{k}{2} \rfloor$. Thus

$$B(s,k) = \sum_{a=\lceil \frac{k}{4} \rceil}^{\lfloor \frac{k}{2} \rfloor} P(2,a,k-2a).$$

By Lemma 3.5, the value of $P(2,a,k-2a)$ depends on whether $0 \leq k-2a \leq a$ or $a \leq k-2a \leq 2a$. Now $2a-(k-2a) = 4a-k \geq 0$. Also $k-2a \geq a$ whenever $a \leq \lfloor \frac{k}{3} \rfloor$. Therefore by Lemma 3.5

$$B(s,k) = \sum_{a=\lceil \frac{k}{4} \rceil}^{\lfloor \frac{k}{3} \rfloor} \left(a - \lceil \tfrac{k-2a}{2} \rceil + 1\right) + \sum_{a=\lfloor \frac{k}{3} \rfloor + 1}^{\lfloor \frac{k}{2} \rfloor} \left(\lfloor \tfrac{k-2a}{2} \rfloor + 1\right). \qquad (1)$$

It follows that

$$\begin{aligned}
B(s,2s-1) &= \sum_{a=\lceil \frac{2s-1}{4} \rceil}^{\lfloor \frac{2s-1}{3} \rfloor} \left(a - \lceil \tfrac{2s-1-2a}{2} \rceil + 1\right) + \sum_{a=\lfloor \frac{2s-1}{3} \rfloor + 1}^{\lfloor \frac{2s-1}{2} \rfloor} \left(\lfloor \tfrac{2s-1-2a}{2} \rfloor + 1\right) \\
&= \sum_{a=\lceil \frac{s}{2} \rceil}^{\lfloor \frac{2s-1}{3} \rfloor} (2a-s+1) + \sum_{a=\lfloor \frac{2s-1}{3} \rfloor + 1}^{s-1} (s-a) \\
&= (1-s)\left(\lfloor \tfrac{2s-1}{3} \rfloor - \lceil \tfrac{s}{2} \rceil + 1\right) + \lfloor \tfrac{2s-1}{3} \rfloor \left(\lfloor \tfrac{2s-1}{3} \rfloor + 1\right) - \lceil \tfrac{s}{2} \rceil \left(\lceil \tfrac{s}{2} \rceil - 1\right) \\
&\quad + s\left(s-1-\lfloor \tfrac{2s-1}{3} \rfloor\right) - \tfrac{1}{2}(s-1)s + \tfrac{1}{2}\lfloor \tfrac{2s-1}{3} \rfloor \left(\lfloor \tfrac{2s-1}{3} \rfloor + 1\right) \\
&= \lfloor \tfrac{2s-1}{3} \rfloor \left(\tfrac{3}{2}\lfloor \tfrac{2s-1}{3} \rfloor + 1 + 1 - s - s + \tfrac{1}{2}\right) + \lceil \tfrac{s}{2} \rceil \left(-\lceil \tfrac{s}{2} \rceil + s - 1 + 1\right) \\
&\quad + 1 - s + s^2 - s - \tfrac{1}{2}s^2 + \tfrac{1}{2}s \\
B(s,2s-1) &= \underbrace{\tfrac{1}{2}\lfloor \tfrac{2s-1}{3} \rfloor \left(3\lfloor \tfrac{2s-1}{3} \rfloor + 5 - 4s\right)}_{X_3(B)} + \underbrace{\lceil \tfrac{s}{2} \rceil \left(s - \lceil \tfrac{s}{2} \rceil\right) + \tfrac{1}{2}\left(s^2 - 3s + 2\right)}_{X_2(B)}
\end{aligned}$$

Note that $X_2(B)$ depends only on s modulo 2 and $X_3(B)$ depends only on s modulo 3.

We now work out $A(s, 2s-2)$ in a similar fashion. Firstly note that $A(s, 2s-2) = P(3, s, s-2) = P(3, s-2, s-2)$, and

$$P(3, s-2, s-2) = \#\{a, b, c : a \geq b \geq c \geq 0, a+b+c = s-2\}.$$

This implies that $\lceil \frac{s-2}{3} \rceil \leq a \leq s-2$. Thus $A(s, 2s-2) = \sum_{a=\lceil \frac{s-2}{3} \rceil}^{s-2} P(2, a, s-2-a)$. From Lemma 3.5, and noting that $s-2-a \geq a$ when $a \leq \lfloor \frac{s-2}{2} \rfloor$, we make the following calculation.

$$\begin{aligned}
A(s, 2s-2) &= \sum_{a=\lceil \frac{s-2}{3} \rceil}^{\lfloor \frac{s-2}{2} \rfloor} \left(a - \lceil \tfrac{s-2-a}{2} \rceil + 1\right) + \sum_{a=\lfloor \frac{s-2}{2} \rfloor + 1}^{s-2} \left(\lfloor \tfrac{s-2-a}{2} \rfloor + 1\right) \\
&= \sum_{a=\lceil \frac{s-2}{3} \rceil}^{\lfloor \frac{s}{2} \rfloor - 1} (a - (s-2-a)) + \sum_{a=\lceil \frac{s-2}{3} \rceil}^{s-2} \left(\lfloor \tfrac{s-a}{2} \rfloor\right) \\
&= \sum_{a=\lceil \frac{s-2}{3} \rceil}^{\lfloor \frac{s}{2} \rfloor - 1} (2a - s + 2) + \sum_{a=\lceil \frac{s-2}{3} \rceil}^{s-2} \left(\tfrac{s-a}{2}\right) - \tfrac{1}{2} \# \left\{ i \in \left\{2, \ldots, s - \lceil \tfrac{s-2}{3} \rceil \right\} : i \text{ odd} \right\}
\end{aligned}$$

Now the number of odd numbers in the range $\{2, \ldots, x\}$ is $\left\lfloor \frac{x-1}{2} \right\rfloor$, so the number of odd numbers in $\left\{2, \ldots, s - \lceil \frac{s-2}{3} \rceil \right\}$ is $\left\lfloor \frac{\lfloor \frac{2s+2}{3} \rfloor - 1}{2} \right\rfloor = \left\lfloor \frac{2s-1}{6} \right\rfloor = \left\lfloor \frac{s-1}{3} \right\rfloor$. Therefore

$$\begin{aligned}
A(s, 2s-2) &= (2-s)\left(\lfloor \tfrac{s}{2} \rfloor - \lceil \tfrac{s-2}{3} \rceil\right) + \lfloor \tfrac{s}{2} \rfloor \left(\lfloor \tfrac{s}{2} \rfloor - 1\right) - \lceil \tfrac{s-2}{3} \rceil \left(\lceil \tfrac{s-2}{3} \rceil - 1\right) \\
&\quad + \tfrac{s}{2}\left(s - 2 - \lceil \tfrac{s-2}{3} \rceil + 1\right) - \tfrac{1}{4}(s-2)(s-1) + \tfrac{1}{4} \lceil \tfrac{s-2}{3} \rceil \left(\lceil \tfrac{s-2}{3} \rceil - 1\right) - \tfrac{1}{2} \left\lfloor \tfrac{s-1}{3} \right\rfloor \\
&= \left\lfloor \tfrac{s}{2} \right\rfloor \left(2 - s + \lfloor \tfrac{s}{2} \rfloor - 1\right) + \tfrac{s}{2}(s-1) - \tfrac{1}{4}(s-2)(s-1) \\
&\quad + \left\lceil \tfrac{s-2}{3} \right\rceil \left(s - 2 - \tfrac{3}{4} \lceil \tfrac{s-2}{3} \rceil + \tfrac{3}{4} - \tfrac{s}{2}\right) - \tfrac{1}{2} \left\lfloor \tfrac{s-1}{3} \right\rfloor \\
A(s, 2s-2) &= \underbrace{\left\lfloor \tfrac{s}{2} \right\rfloor \left(\lfloor \tfrac{s}{2} \rfloor + 1 - s\right) + \tfrac{1}{4}(s-1)(s+2)}_{X_2(A)} + \underbrace{\tfrac{1}{4} \left\lceil \tfrac{s-2}{3} \right\rceil \left(2s - 5 - 3 \lceil \tfrac{s-2}{3} \rceil\right) - \tfrac{1}{2} \left\lfloor \tfrac{s-1}{3} \right\rfloor}_{X_3(A)}.
\end{aligned}$$

Again note that $X_2(A)$ depends only on s modulo 2 and $X_3(A)$ depends only on s modulo 3.

Now $P(4, s, 2s-2) = P(4, s, 2s-1)$ if and only if $B(s, 2s-1) = A(s, 2s-2)$, which is if and only if $X_2(B) - X_2(A) = X_3(A) - X_3(B)$. We have

$$X_2(B) - X_2(A) = \left(\lceil \tfrac{s}{2} \rceil \left(s - \lceil \tfrac{s}{2} \rceil\right) + \tfrac{1}{2}\left(s^2 - 3s + 2\right)\right) - \left(\lfloor \tfrac{s}{2} \rfloor \left(\lfloor \tfrac{s}{2} \rfloor + 1 - s\right) + \tfrac{1}{4}(s-1)(s+2)\right).$$

A simple calculation shows that regardless of whether s is odd or even, $X_2(B) - X_2(A) = \frac{1}{4}(3s^2 - 9s + 6)$.

$$X_3(A) - X_3(B) = \left(\tfrac{1}{4} \left\lceil \tfrac{s-2}{3} \right\rceil \left(2s - 5 - 3 \lceil \tfrac{s-2}{3} \rceil\right) - \tfrac{1}{2} \left\lfloor \tfrac{s-1}{3} \right\rfloor\right) - \left(\tfrac{1}{2} \lfloor \tfrac{2s-1}{3} \rfloor \left(3 \lfloor \tfrac{2s-1}{3} \rfloor + 5 - 4s\right)\right).$$

Calculating for each possible value of s modulo 3 shows that in each case, $X_3(A)-X_3(B)=\frac{1}{4}(3s^2-9s+6)=X_2(B)-X_2(A)$. Therefore, for all $s\geq 4$, $P(4,s,2s-2)=P(4,s,2s-1)$.
We now show that $P(4,s,2s-1)<P(4,s,2s)$ for all $s\geq 4$. Since $P(4,s,2s-2)=P(4,s,2s-1)$ for all $s\geq 4$, by substituting $s+1$ for s in Lemma 3.4 we have

$$A(s+1,2s)=B(s+1,2s+1). \tag{2}$$

Now $A(s,2s-1)=P(3,s,s-1)=P(3,s-1,s-1)$ as no part of a partition of $s-1$ can exceed $s-1$. Similarly $A(s+1,2s)=P(3,s+1,s-1)=P(3,s-1,s-1)$. Hence

$$A(s,2s-1)=A(s+1,2s). \tag{3}$$

Now we consider $B(s+1,2s+1)$ compared to $B(s,2s)$.
Setting $k+1=2s$ and $k+1=2s+1$, respectively, gives:

$$\begin{aligned}
B(s,2s) &= \sum_{a=\lceil \frac{s}{2}\rceil}^{\lfloor \frac{2s}{3}\rfloor}(a-(s-a)+1)+\sum_{a=\lfloor \frac{2s}{3}\rfloor+1}^{s}(s-a+1)\\
&= \sum_{a=\lceil \frac{s}{2}\rceil}^{\lfloor \frac{2s}{3}\rfloor}(2a-s+1)+\sum_{a=\lfloor \frac{2s}{3}\rfloor+1}^{s}(s-a+1)\,;\\
B(s+1,2s+1) &= \sum_{a=\lceil \frac{2s+1}{4}\rceil}^{\lfloor \frac{2s+1}{3}\rfloor}(a-(s-a+1)+1)+\sum_{a=\lfloor \frac{2s+1}{3}\rfloor+1}^{s}((s-a)+1)\\
&= \sum_{a=\lceil \frac{s+1}{2}\rceil}^{\lfloor \frac{2s+1}{3}\rfloor}(2a-s)+\sum_{a=\lfloor \frac{2s+1}{3}\rfloor+1}^{s}(s-a+1)\,.
\end{aligned}$$

If $\lfloor\frac{2s}{3}\rfloor=\lfloor\frac{2s+1}{3}\rfloor$, then $B(s,2s)-B(s+1,2s+1)\geq\sum_{a=\lceil s/2\rceil}^{\lfloor(2s+1)/3\rfloor}1\geq\frac{2s-1}{3}-\frac{s-1}{2}>0$. If $\lfloor\frac{2s}{3}\rfloor<\lfloor\frac{2s+1}{3}\rfloor$ then $\lfloor\frac{2s}{3}\rfloor=\frac{2s-2}{3}$, $\lfloor\frac{2s+1}{3}\rfloor=\frac{2s+1}{3}$ and

$$\begin{aligned}
B(s,2s)-B(s+1,2s+1) &\geq \left(\sum_{a=\lceil\frac{s}{2}\rceil}^{\lfloor\frac{2s}{3}\rfloor}1\right)-\big(2\lfloor\tfrac{2s+1}{3}\rfloor-s\big)+\big(s-\big(\lfloor\tfrac{2s}{3}\rfloor+1\big)+1\big)\\
&\geq \tfrac{2s-2}{3}-\tfrac{s-1}{2}-\tfrac{4s+2}{3}+2s-\tfrac{2s-2}{3}\\
&= \tfrac{1}{6}(-3s+3-8s-4+12s)=\tfrac{1}{6}(s-1)>0.
\end{aligned}$$

Thus in any case $B(s,2s)>B(s+1,2s+1)$. Therefore by equations (2) and (3),

$$B(s,2s)-A(s,2s-1)>B(s+1,2s+1)-A(s+1,2s)=0.$$

Hence by Lemma 3.4 $P(4,s,2s-1)<P(4,s,2s)$. □

Proposition 3.9. *Let $s \geq 4$ and $1 \leq k \leq 2s-1$. Then $P(4,s,k) = P(4,s,k+1)$ if and only if $k = 2s-2$ or $s = k = 4$.*

Proof. In the case $s = k = 4$ it be easily computed that $P(4,4,4) = P(4,4,5) = 5$. Otherwise, by Theorem 3.7, if $P(4,s,k) = P(4,s,k+1)$, then $4(s-1)-1 \leq 2k$ and, since k is an integer, $2s-2 \leq k$. We may now apply Proposition 3.8 to get the result. □

Theorem 3.1 now follows immediately from Lemma 3.5, Proposition 3.6 and Proposition 3.9.

4 Primitive groups with equality

Primitive groups which are not $(k+1)$-homogeneous but achieve equality in the Livingstone-Wagner Theorem for some $k < (n-1)/2$ are fairly rare.

Remark 4.1. *The known primitive but not $(k+1)$-homogeneous groups G such that $\sigma_k(G) = \sigma_{k+1}(G)$, for some $k < (n-1)/2$, are:*

(a) $AGL(m,2)$, *for* $m \geq 4, n = 2^m, k = 4$,

(b) $ASL(2,3)$ *or* $AGL(2,3)$, *for* $n = 9, k = 3$,

(c) $\mathrm{Sym}(5), \mathrm{Sym}(6), PGL(2,9)$ *or* $P\Gamma L(2,9)$, *for* $n = 10, k = 4$,

(d) $M_{11}, PSL(2,11), PGL(2,11)$, *for* $n = 12, k = 4$,

(e) $PSL(3,3)$, *for* $n = 13, k = 4$,

(f) $PGL(2,13)$, *for* $n = 14, k = 4$,

(g) $2^4 : \mathrm{Alt}(6), 2^4 : \mathrm{Sym}(6), 2^4 : \mathrm{Alt}(7)$, *for* $n = 16, k = 6$,

(h) $PGL(2,17)$, *for* $n = 18, k = 6$ *or* 8,

(i) M_{22} *or* $\mathrm{Aut}(M_{22})$, *for* $n = 22, k = 8$,

(j) M_{23}, *for* $n = 23, k = 8, 9$,

(k) M_{24}, *for* $n = 24, k = 6, 8, 9$ *or* 10.

Observe that many of these groups are subgroups of M_{24}.

Regarding case (a), we prove the following.

Proposition 4.2. *Let $G = AGL(m,2)$, for $m \geq 4$, acting naturally on an m-dimensional vector space V over $GF(2)$. Then $\sigma_4(G) = \sigma_5(G) = 2$.*

Proof. Observe that the stabiliser in G of any three points of V fixes the fourth point in the unique affine plane containing these three points and is transitive on the remaining points of V. It follows that $\sigma_4(G) = 2$ and also G has a single orbit on the set of 5-subsets which contain affine planes. Let Δ be any set of five distinct points in V which does not contain any affine plane. Then Δ is not contained in an affine 3-dimensional subspace of V. Furthermore the stabiliser in G of an affine 3-dimensional subspace W is transitive on pairs (α, Λ), where α is a point not in W and Λ is any set of four points in W which is not an affine plane. Therefore G has a single orbit on 5-subsets which do not contain any affine plane. Thus $\sigma_5(G) = 2$. □

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