Analysis of complex nonlinear reaction-diffusion equations

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Dedicated to

my Parents and my Wife

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Abstract

A mathematical analysis has been carried out for some nonlinear reaction-diffusion equations on open bounded convex domains $\Omega \subset \mathbb{R}^d (d \leq 3)$ with Robin boundary conditions. Existence, uniqueness and continuous dependence on initial data of weak and strong solutions are proved.

A numerical analysis has also been undertaken for these nonlinear reaction-diffusion equations on the above domains. A fully practical piecewise linear finite element approximation is proposed for which existence and uniqueness of the numerical solution are proved. Semi-discrete and fully discrete error estimates are given. A practical algorithm for computing the numerical solution is given and its convergence is proved. Finally, some numerical simulations in one-dimensional space are exhibited.

Declaration

The work presented in this thesis was carried out in the Numerical Analysis Group, the Department of Mathematical Sciences, University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

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Chapter 1

Introduction

Reaction diffusion systems, which are systems of nonlinear parabolic partial differential equations, have been the subject of active research for many years. These systems have numerous applications in physics, chemistry, ecology, biology, and other disciplines. For a review of the theory and applications of reaction-diffusion systems see Britton (1986), Fife (1979), Murray (1993), Smoller (1983), and Volpert et al. (1994).

For instance, a system of reaction-diffusion equations mapping a vector function $u(x,t) = (u_1(x,t), \ldots, u_m(x,t))$ from $\Omega_T := \Omega \times (0,T), T > 0$ into $\mathbb{R}^m$ can be considered in the following general form:

$$\frac{\partial u}{\partial t} - D\Delta u + g(u,x) = 0 \text{ in } \Omega_T\,, \tag{1.1.1}$$

supplemented by initial and boundary conditions, where $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, is a bounded domain with Lipschitz boundary $\partial\Omega$. Here, Δ is the Laplace operator, $D = \text{diag}(d_1, \cdots, d_m), d_i > 0$ is the diagonal matrix of diffusion coefficients, and $g = (g_1, \cdots, g_m)$ accounts for the reaction terms.

In this thesis, we consider the scalar reaction-diffusion equation:

$$\frac{\partial u}{\partial t} - d\Delta u + g(u, x) = 0 \text{ in } \Omega_T\,, \tag{1.1.2}$$

where $d > 0$ is given. The nonlinear reaction term g is a C^1 function and satisfies the following *growth condition*:

$$|g(u)| \leq \alpha |u|^p + C\,, \tag{1.1.3}$$

where $\alpha, p \in \mathbb{R}$ and $C > 0$. One example of such a function $g(u)$ is an odd degree polynomial (see Section 3.2). Together with the above form we include the following *initial* and *Robin* boundary conditions:

$$u(x, 0) = u_0(x)\,, \quad x \in \Omega, \tag{1.1.4}$$

$$\frac{\partial u}{\partial n} + \beta u = 0 \quad \text{on } \Sigma\,, \tag{1.1.5}$$

where $\Sigma := \partial\Omega \times (0, T)$. Here, n is the outer unit normal to the boundary $\partial\Omega$ of Ω and β is a positive constant (later we shall see that the positivity of β is required to guarantee coercivity of the spatial operator associated with the weak form). It is well known that *Dirichlet* and *Neumann* boundary conditions correspond to two extreme cases, namely "$\beta = \infty$" and "$\beta = 0$", respectively (see, e.g., Daners (2000), p.4207). Although the elliptic boundary value problem with Robin boundary conditions introduced some time ago by Maz'ya (1981), It was, at that time, not very well known that such a problem could be characterized as a variational problem in a Hilbert space. However, Showalter (1985) showed that elliptic boundary value problems with Robin conditions have interpretations as weak formulations in Hilbert spaces.

There are many reasons behind the importance of the reaction-diffusion systems with Robin conditions which are to be studied. Although reaction-diffusion systems with Dirichlet and Neumann boundary conditions have been extensively studied, very little work has been done for Robin boundary conditions. Sherratt (2004) considers the system of "oscillatory" reaction-diffusion equations can be interpreted in the context of ecological applications where the Dirichlet condition is often used as a simple approximation to a more realistic Robin condition[1]. An example of this system combining the properties of equations (1.1.1) and (1.1.3)-(1.1.5) is the standard predator-prey equations, which can be represented by the standard Hopf normal form:

$$\frac{\partial u}{\partial t} = \Delta u + (1 - r^2)u - (\omega_0 - \omega_1 r^2)v\,, \tag{1.1.6}$$

$$\frac{\partial v}{\partial t} = \Delta v + (\omega_0 - \omega_1 r^2)u + (1 - r^2)v\,, \tag{1.1.7}$$

[1]This more realistic boundary condition gives rise to periodic travelling moving away from the boundary (see Sherratt (2004), pp.2-4).

with $r^2 = u^2 + v^2, \omega_i > 0$, and the Robin boundary condition:

$$\frac{\partial u}{\partial n} + \mu u = 0 \qquad \text{and} \qquad \frac{\partial v}{\partial n} + \mu v = 0\,, \tag{1.1.8}$$

where μ is a positive parameter. This realistic boundary condition implies that the flux $\partial u/\partial n$ and $\partial v/\partial n$ out of the domain is proportional to the density of u and v rather than a zero flux condition.

Another example combining the properties of equations (1.1.2) and (1.1.3)-(1.1.5) is the Ginzburg-Landau equation, which has the form:

$$\frac{\partial u}{\partial t} - \gamma \Delta u + u(u^2 - 1) = 0\,, \tag{1.1.9}$$

where $u \in \mathbb{R}$ and $\gamma > 0$. This equation arises in the study of superconductivity of liquids (see Smoller (1983), p.210, and the references therein). There are also other examples of reaction-diffusion systems that satisfy the properties of equations (1.1.1)-(1.1.5) such as the Fitz-Hugh Nagumo equations. The system of the Fitz-Hugh Nagumo equations is as follows:

$$\frac{\partial u_1}{\partial t} = d_1 \frac{\partial^2 u_1}{\partial x^2} + f(u_1) - u_2\,, \tag{1.1.10}$$

$$\frac{\partial u_2}{\partial t} = d_2 \frac{\partial^2 u_2}{\partial x^2} + \delta u_1 - \gamma u_2\,, \tag{1.1.11}$$

where $u = (u_1, u_2) \in \mathbb{R} \times \mathbb{R}$. Here $d_i, \delta, \gamma >$ are positive constants and $f(u_1) = -u_1(u_1 - \sigma)(u_1 - 1)$ where $0 < \sigma < 1/2$. This system is intended to describe signal transmission across axons (see, e.g., Smoller (1983), p.209, and Temam (1997), p.99-100). Note that in the previous examples we have that $g(u)$ is an odd degree polynomial.

This thesis can be viewed as a first step towards developing a framework for analyzing parabolic problems with Robin boundary conditions. It focuses mainly on a classical mathematical and numerical analysis of the following system:

$$\frac{\partial u}{\partial t} - \Delta u + g(u) = 0 \qquad \text{in} \quad \Omega_T\,, \tag{1.1.12}$$

with initial and Robin boundary conditions

$$u(\cdot, 0) = u_0(\cdot)\,, \tag{1.1.13}$$

$$\frac{\partial u}{\partial n} + \beta u = 0 \qquad \text{on} \quad \Sigma\,, \tag{1.1.14}$$

which we repeat here for clarity. Moreover, some numerical experiments are presented. To our knowledge there have been no studies of the numerical analysis of the system (1.1.12)-(1.1.14). Furthermore, the thesis includes two important results that are related to our work on reaction diffusion problems with Robin boundary conditions. The first result is the spectral theory of Robin boundary value problems shown in Chapter 2, which to our knowledge has not been documented elsewhere. However, there are similar results of spectral theories that consider Dirichlet and Neumann boundary value problems (see, e.g., Thomée and Larsson (1999), p.71, Robinson (2001), p.163, Garvie (2003), p.138). The second result is the regularity of the scalar Robin boundary value problem shown in Chapter 4. Although elliptic eigenvalue problems with Robin conditions have been studied in Dancer and Daners (1994), (1997), Daners (2000), and Showalter (1985), the regularity of the solution has not appeared.

We now give a brief description for each chapter of this thesis. Each of these descriptions is followed by the methodology that has been used.

In Chapter 2 we discuss the spectral theory of Robin boundary value problems. We show that there is an orthonormal basis for $L^2(\Omega)$ and an orthogonal basis for $H^1(\Omega)$ consisting of eigenfunctions of the operator $A = -\Delta + I$ with Robin boundary conditions. This was achieved using the Hilbert-Schmidt theorem (see, e.g., Robinson (2001)).

In Chapter 3 we prove the existence and uniqueness of a weak solution for the system (1.1.12)-(1.1.14) using the Faedo-Galerkin method of Lions (1969) and the Alaoglu compactness theorem (see, e.g., Robinson (2001)). The basic idea is to reduce the infinite dimensional dynamical system to a finite dimensional one using a truncated eigenfunction expansion. Then we deduce from the finite weak form of the reaction-diffusion equation the *local existence* [2] (and uniqueness) of solutions using the Picard's existence theorem (see The-

[2]Picard's existence theorem gives local existence of the approximated solution u^k on the finite time interval $(0, t_k), t_k > 0$. This relies on the local Lipschitz of the nonlinear term in the system of ODEs.

orem A.0.13). We also deduce *global existence* [3], uniqueness, and continuous dependence of weak solutions on the initial data in $H = L^2(\Omega)$. These results can be obtained by using the Alaoglu compactness theorem and some energy estimates.

In Chapter 4, in the first section, we study a regularity result for the Robin boundary value problem using the methodology of Grisvard (1985). In the second section, we prove the existence, uniqueness, and continuous dependence of strong solutions on the initial data in $V = H^1(\Omega)$. These results can be obtained by regularity estimates (introduced in Chapter 3) and using low regularity of the initial data.

In Chapter 5, we describe some technical tools necessary for analysis in this chapter and Chapter 6. Then we discretise the system (1.1.12)-(1.1.14) in *space* using the finite element method to present the *semi-discrete finite element approximation.* Then we prove the existence and uniqueness of the semi-discrete approximations. Finally, an error bound between the semi-discrete and continuous solutions is given. This was achieved using the finite element method (see, e.g., Ciarlet (1978)) with piecewise linear basis functions and some assumptions on the partitioning of Ω.

In Chapter 6 we discretise the system (1.1.12)-(1.1.14) in *space* using the finite element method and discretise in *time* using backward Euler method

[3] If the solutions u^k are uniformly (independently of k) bounded w.r.t. some norm, then we have global existence of u^k.

to present the *fully discrete finite element approximation.* Then we prove the existence and uniqueness of the fully discrete approximations. Finally, an error bound between the fully discrete and continuous solutions is given. The basic idea is to discretise the system (1.1.12)-(1.1.14) in time using backward Euler method and to use the results of Chapter 5 in order to achieve the fully discrete finite element approximation and its error bound.

In Chapter 7 we describe an algorithm for computing the numerical solution. Some numerical experiments are performed and discussed in one space dimension.

Chapter 2

Spectral Theorem

In this chapter we first introduce some notation and definitions. Then we consider the Robin eigenvalue problem in n-dimensional space. We show that an infinite set of eigenfunctions of this problem can form a basis for some Hilbert spaces, namely, there is a basis for the space $H^1(\Omega)$ consisting of eigenfunctions of the operator $A = -\Delta + I$ with Robin boundary conditions.

Now, we spend some time presenting notation and definitions needed for this chapter:

Let V be a vector space with real scalars. If $(\cdot,\cdot)$ is a scalar product on V, then the space V is said to be *complete* with respect to the corresponding norm $\|\cdot\|_V = (\cdot,\cdot)^{\frac{1}{2}}$ if every Cauchy sequence $\{v_i\}$ in V converges to some $v \in V$. Let $(V,(\cdot,\cdot))$ be an inner product space. If the associated normed space $(V,\|\cdot\|)$ is complete, then $(V,(\cdot,\cdot))$ is called a *Hilbert space.*

Let v be a function $v : \mathbb{R}^n \to \mathbb{R}$. We define its partial derivatives of order $|\alpha|$ as follows:

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \quad |\alpha| = \sum_{i=1}^{n} \alpha_i \,,$$

where α is a multi-index, $\alpha = (\alpha_1, \cdots, \alpha_n)$, and α_i are non-negative integers.

We define the Hilbert spaces $H^k(\Omega)$, for a non-negative integer k, as follows:

$$H^k(\Omega) = \{v \in L^2(\Omega) : D^\alpha v \in L^2(\Omega) \text{ for } |\alpha| \leq k\} \,, \tag{2.2.1}$$

equipped with the scalar product and the corresponding norm:

$$(u, v)_{H^k(\Omega)} = \sum_{|\alpha| \leq k} \int_\Omega D^\alpha u \cdot D^\alpha v dx \,, \tag{2.2.2}$$

$$\|v\|_{H^k(\Omega)} = \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha v|^2 dx \right)^{\frac{1}{2}} \,, \tag{2.2.3}$$

respectively, where we sum over all multi-indices α with $|\alpha| \leq k$. Note that in the literature, $H^k(\Omega)$ is often denoted by $W^{k,2}(\Omega)$, and if additionally $k = 0, W^{0,2}(\Omega) \equiv H^0(\Omega) \equiv L^2(\Omega)$, which is a Hilbert space. The space $L^2(\Omega)$ consists of functions defined on Ω that are square integrable with respect to the Lebesgue measure, i.e.

$$\int_\Omega [v(x)]^2 dx < \infty \,,$$

see, e.g., Halmos (1950) for the concept of Lebesgue measure.

A *bilinear functional* $a(\cdot,\cdot)$ on V is a function $a : V \times V \to \mathbb{R}$ such that for all $u, v, w \in V$ and $\lambda, \mu \in \mathbb{R}$,

$$a(\lambda u + \mu v, w) = \lambda a(u, w) + \mu a(v, w)\,, \tag{2.2.4}$$

$$a(u, \lambda v + \mu w) = \lambda a(u, v) + \mu a(u, w)\,. \tag{2.2.5}$$

The bilinear functional $a(\cdot,\cdot)$ is said to be *symmetric* if

$$a(u, v) = a(v, u) \qquad \forall u, v \in V\,, \tag{2.2.6}$$

and $a(\cdot,\cdot)$ is, on a Hilbert space V with norm $\|\cdot\|_V$, said to be *coercive* if there is a positive constant α such that

$$a(v, v) \geq \alpha \|v\|_V^2 \qquad \forall v \in V\,. \tag{2.2.7}$$

We shall denote by $\langle\cdot,\cdot\rangle$ the duality pairing between a Banach space X and its dual X'. In this chapter, $\langle\cdot,\cdot\rangle$ will represent the duality pairing between $[H^1(\Omega)]'$ and $H^1(\Omega)$.

Finally, we recall the Cauchy-Schwarz inequality:

$$|(u, v)| \leq \|u\|_V\| \, \|v\|_V \qquad \text{for } u, v \in V\,. \tag{2.2.8}$$

Now, let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. We here try to show that, for Lipschitz boundary $\partial\Omega$, an infinite set of eigenfunctions $\{z_i\}_{i=1}^{\infty}$ of the Robin eigenvalue problem can form an orthogonal basis for the Hilbert space $V = H^1(\Omega)$ and an orthonormal basis for the Hilbert space $H = L^2(\Omega)$.

Consider the Robin eigenvalue problem:

$$-\Delta z_i + z_i = \mu_i z_i \qquad \text{a.e. in} \quad \Omega,\, z_i \neq 0\,, \tag{2.2.9}$$

$$\frac{\partial z_i}{\partial n} + \beta z_i = 0 \qquad \text{a.e. on} \quad \partial\Omega\,, \tag{2.2.10}$$

where $\beta > 0$. Equation (2.2.9) may be written in the form:

$$Az_i = \mu_i z_i \qquad \text{a.e. in} \quad \Omega,\, z_i \neq 0, \tag{2.2.11}$$

where $A = -\Delta + I$ is a linear operator. Now consider the elliptic boundary value problem with Robin boundary condition:

$$Au = f \qquad \text{a.e. in} \quad \Omega,\, f \in V'\,, \tag{2.2.12}$$

$$\frac{\partial u}{\partial n} + \beta u = 0 \qquad \text{a.e. on} \quad \partial\Omega, \beta > 0\,, \tag{2.2.13}$$

where $u \in V$. Note that considering problem (2.2.12)-(2.2.13) gives us some foundations for the full reaction-diffusion problem of the next chapter. In fact, we will use the orthogonal basis in this chapter to construct the Galerkin approximations for the full reaction-diffusion problem. This is known as the Faedo-Galerkin method (see Chapter 3).

Now, multiplying equation (2.2.12) by a function v (say $v \in H^1(\Omega)$) and using the application of Green's identity (Theorem A.0.5), recalling the homogeneous Robin boundary conditions, we rewrite (2.2.12)-(2.2.13) in weak form as follows:

Find $u \in V$ such that

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega uv dx + \beta \int_{\partial\Omega} uv d\sigma \tag{2.2.14}$$

$$= \langle f, v \rangle, \qquad v \in V.$$

Here, σ is the $(d-1)$-dimensional Hausdorff measure restricted to $\partial\Omega$ which coincides with the usual surface measure if $\partial\Omega$ is smooth.

We recall the well-known results of the Hilbert spaces

$$V \overset{c}{\hookrightarrow} H \equiv H' \hookrightarrow V',$$

where each space is dense in the following one; '$\hookrightarrow$' denotes continuous injection (see Temam (1997), p.55), '$\overset{c}{\hookrightarrow}$' denotes compact injection (where the possible compactness of the injections depend on the dimension of Ω (see Theorem A.0.8)), and the identity of H and its dual H' is due to the Riesz Representation Theorem (Theorem A.0.1). We observe that

$$\langle f, v\rangle = (f, v), \quad \forall f \in H, \forall v \in V ,$$

where $\langle f, v\rangle$ denotes the pairing between $f \in H$ and $v \in V$. Thus we re-express (2.2.14) as

$$\begin{gathered} \text{Given} \quad f \in V', \\ \text{find} \quad u \in V, \text{ such that} \\ a(u, v) = \langle Au, v\rangle = \langle f, v\rangle \text{ for all } v \in V. \end{gathered} \tag{2.2.15}$$

We will now show that equation (2.2.15) satisfies the conditions of the Lax-Milgram lemma (Theorem A.0.2) with $V = H^1(\Omega)$, so that this equation has a unique solution in V, thus the inverse of operator A is a linear operator and we define

$$u = A^{-1} f . \tag{2.2.16}$$

We first verify that the bilinear form:

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega uv dx + \beta \int_{\partial\Omega} uv d\sigma ,$$

is bounded. By the Cauchy-Schwarz inequality (2.2.8), it follows that

$$|a(u, v)| \leq C_1 \|u\|_V \|v\|_V + \beta \int_{\partial\Omega} |uv| d\sigma ,$$

is bounded since the first term on the right-hand side is obviously bounded with respect to the V-norm and the second term, which is the *boundary integral*, is also bounded. In fact, the boundedness of the boundary integral is assured by the following theorem:

Theorem 2.2.1 Suppose that Ω, has a Lipschitz boundary $\partial\Omega$, and that p is a real number $1 \leq p \leq \infty$. Then there is a constant, C, such that

$$\|v\|_{L^p(\partial\Omega)} \leq C\,\|v\|_{L^p(\Omega)}^{1-1/p}\,\|v\|_{W^{1,p}(\Omega)}^{1/p} \qquad \forall v \in W^{1,p}(\Omega)\,.$$

Proof. See Brenner and Scott, pp.36-39.

Note that this theorem is compatible with the trace embedding theorems (Theorem A.0.9) since we have that $p = 2$ and $\|v\|_{L^2(\Omega)} \leq C\|v\|_{W^{1,2}(\Omega)}$, so it gives

$$\begin{aligned}\|v\|_{L^2(\partial\Omega)} &\leq C\,\|v\|_{L^2(\Omega)}^{1/2}\,\|v\|_{W^{1,2}(\Omega)}^{1/2} \\ &\leq C\|v\|_{W^{1,2}(\Omega)} \quad \forall v \in W^{1,2}(\Omega)\,,\end{aligned}$$

where $W^{1,2}(\Omega) = H^1(\Omega) = V$ in our case. Again by Cauchy-Schwarz inequality we also have that

$$|\langle f, v\rangle| \leq \|f\|_{V'}\|v\|_V\,.$$

Finally, it remains to verify the coercivity condition of $a(\cdot,\cdot)$. Recall that $a(\cdot,\cdot)$ is a bilinear form and

$$a(v,v) = \int_\Omega |\nabla v|^2 dx + \int_\Omega v^2 dx + \beta \int_{\partial\Omega} v^2 d\sigma,\ \beta > 0\,.$$

$$= \|v\|_V^2 + \beta \int_{\partial\Omega} v^2 d\sigma\,.$$

Since $\beta \int_{\partial\Omega} v^2 d\sigma \geq 0$ we can omit this term to get

$$a(v,v) \geq \|v\|_V^2\,,$$

i.e. $a(v,v)$ is coercive. Thus equation (2.2.15) has a unique solution $u = A^{-1}f$.

Now, we want to show that A^{-1} is a self-adjoint bounded and compact operator from H to H so that we can apply the Hilbert-Schmidt theorem (Theorem A.0.3). Since the bilinear form $a(.,.)$ is continuous on V we can associate with $a(.,.)$ a linear continuous operator A from V into V', i.e. $A \in L(V, V')$, such that

$$a(u,v) = \langle Au, v\rangle \quad \forall u, v \in V\,, \tag{2.2.17}$$

where $Au = f \in V'$. Hence, by the Riesz Representation theorem (Theorem A.0.1) we have

$$\|f\|_{V'} = \|u\|_V \, . \tag{2.2.18}$$

Recall that $H \hookrightarrow V'$ so for $f \in H$ this leads to

$$\|f\|_{V'} \leq C\|f\|_H \, , \tag{2.2.19}$$

for some constant C. Thus for $f \in H$ from (2.2.18) and (2.2.19) we have

$$\|u\|_V \leq C\|f\|_H \text{ or } \|A^{-1}f\|_V \leq C\|f\|_H \; ,$$

i.e. A^{-1} is a bounded operator from H to V. Now, by the above assumption or by "Kondrasov embedding theorems" (Theorem A.0.8) we have the following compact injection map:

$$V \overset{c}{\hookrightarrow} H \, ,$$

and so A^{-1} is a bounded compact operator from H to H.

To show that A^{-1} is a self-adjoint operator, let

$$D(A) = \{u \in V | \, \frac{\partial u}{\partial n} + \beta u = 0 \text{ on } \partial\Omega\}$$

be the domain of the operator A. Note that A is symmetric due to

$$\langle Au, v\rangle = a(u,v) = a(v,u) = \langle Av, u\rangle \, . \tag{2.2.20}$$

Since $\langle Au, v\rangle = \langle f, v\rangle = (f, v)$ for all $v \in V$, $f \in H$, the symmetry condition for A becomes

$$(Au, v) = (u, Av) \quad \forall u, v \in V,\ Au, Av \in H\,, \tag{2.2.21}$$

where $(.,.)$ is the inner product on H. We have to be a little careful as A is an unbounded operator and the domain of an unbounded operator becomes an integral part of the definition of the operator. However, $D(A) = V$ is dense in H and hence equation (2.2.21) is valid (see Renardy and Rogers (1993), p.253). Now let $Au = x, Av = y$ for all $x, y \in H$, then

$$(x, A^{-1}y) = (A^{-1}x, y) \qquad \forall x, y \in H\,. \tag{2.2.22}$$

Thus A^{-1} is self-adjoint. We now apply the Hilbert-Schmidt theorem (Theorem A.0.3) with $L := A^{-1}$, noting that

$$Az_i = \mu_i z_i \quad \Leftrightarrow \quad A^{-1}z_i = \mu_i^{-1} z_i\,, \tag{2.2.23}$$

thus the μ_i^{-1} are real and we have the infinite sequence

$$\lim_{i\to\infty} \mu_i^{-1} = 0\,, \cdots \leq |\mu_{i+1}^{-1}| \leq |\mu_i^{-1}| \leq \cdots \leq |\mu_1^{-1}|\,, \tag{2.2.24}$$

where the eigenfunctions z_i form an orthonormal basis for the whole of H and an orthogonal basis for V. To show this we recall the result that if H

is a Hilbert space, then a subspace M of H is dense if and only if $M^{\perp} = \{0\}$, (see Renardy and Rogers (1993), Corollary 6.27, p. 186). Now take $M := \text{span}\{z_i\}_{i=1}^{\infty} \subset V \subset H$ and as V is dense in H we have $V^{\perp} = \{0\}$, which implies $M^{\perp} = \{0\}$ (with respect to H), which implies M is dense in H, i.e. $\{z_i\}_{i=1}^{\infty}$ is an orthonormal basis for the whole of H,

$$(z_i, z_j) = \delta_{ij}\,. \tag{2.2.25}$$

Also, from the weak form of the eigenvalue problem (2.2.9)-(2.2.10) we have

$$a(z_i, z_j) = \mu_i \delta_{ij}\,, \tag{2.2.26}$$

that is the z_i are an orthogonal basis for V. From equations (2.2.25) and (2.2.26) we obtain

$$\int_{\Omega} \nabla z_i \cdot \nabla z_j dx + \beta \int_{\partial\Omega} z_i z_j d\sigma = (\mu_i - 1)\delta_{ij}\,. \tag{2.2.27}$$

This result is useful and is to be used in next chapter.

We summarise this chapter in the following theorem:

Theorem 2.2.2. (Spectral theorem)

There is a basis of $V = H^1(\Omega)$ consisting of eigenfunctions of the operator $A = -\Delta + I$ with Robin boundary conditions. These eigenfunctions are linearly independent elements of V.

Chapter 3

Weak Solutions

This chapter is divided into four sections. In Section 3.1 we introduce basic notation of Sobolev spaces and time-dependent Sobolev spaces. We consider the reaction-diffusion system that was introduced in Chapter 1. At the end of Section 3.1 we will give a statement of the main theorem of this chapter; this statement shows that there exists a unique weak solution for the reaction-diffusion system considered. In Section 3.2 and Section 3.3 we prove local existence and global existence of the weak solutions for weak formulations, using passage to the limit of the Galerkin approximations. Finally, in Section 3.4, the uniqueness is proven and the continuity of a solution is showed.

Section 3.1: Notation and main result

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Throughout this Thesis we denote X' to be the dual space of a Banach space X. The Sobolev space notation $W^{m,p}(\Omega)(m \in \mathbb{N}, p \in [1,\infty])$ is adopted along with associated norms and semi-norms defined by

$$\|u\|_{m,p} := \left(\sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_{0,p}^p \right)^{1/p} ,$$

$$|u|_{m,p} := \left(\sum_{|\alpha| = m} \|D^\alpha u\|_{0,p}^p \right)^{1/p} ,$$

respectively. We recall some well-known results of Sobolev spaces (see Theorem A.0.6). For $p = 2$, $W^{m,p}(\Omega)$ will be denoted by $H^m(\Omega)$ with the associated norm and semi-norm written as $\|\cdot\|_m$ and $|\cdot|_m$, respectively.

We also define function spaces depending on space and time. Let X be a Banach space, then the space of continuous functions from $(0,T)$ into X, $C^0(0,T;X)$, consists of those $u(t) : (0,T) \to X$ such that $u(t) \to u(t_0)$ in X as $t \to t_0$. Let $L^p(0,T;X)$ be the Banach spaces that consist of all those functions $u(t) : (0,T) \to X$ such that $t \to \|u(t)\|_X$ is in $L^p(0,T)$, with norm

$$\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u(t)\|_X^p dt \right)^{1/p} \qquad \text{for} \quad 1 \leq p < \infty ,$$

$$\|u\|_{L^\infty(0,T;X)} := \operatorname*{ess\,sup}_{t \in (0,T)} \|u(t)\|_X \qquad \text{if} \quad p = \infty .$$

Note that $C^0(0,T;X)$ is dense in $L^p(0,T;X)$ with respect to the norm $\|\cdot\|_{L^p(0,T;X)}$. In addition if, for example $X = L^p(\Omega)$, then we can write $L^p(\Omega_T) \equiv L^p(0,T;L^p(\Omega))$. We recall some well-known results of these time-dependent Sobolev spaces (see Theorem A.0.11). We shall also need to use $C_0^\infty(0,T;X)$, the space of infinitely differentiable functions from $(0,T)$ into X, with compact support in $(0,T)$. This space is also dense in $L^p(0,T;X)$ with respect to the norm $\|\cdot\|_{L^p(0,T;X)}$.

We now consider the reaction-diffusion equation

$$\frac{\partial u}{\partial t} - \Delta u + g(u) = 0 \qquad \text{in} \quad \Omega_T\,, \tag{3.1.1}$$

where $\Omega_T := \Omega \times (0,T), T > 0$, with initial and Robin boundary conditions

$$u(\cdot,0) = u_0(\cdot)\,, \tag{3.1.2}$$

$$\frac{\partial u}{\partial n} + \beta u = 0 \qquad \text{on} \quad \Sigma\,, \tag{3.1.3}$$

where $\beta > 0$ and $\Sigma := \partial\Omega \times (0,T)$. For later use we recall Young's inequality in the form

$$ab \leq \varepsilon^{m/n}\,\frac{a^m}{m} + \varepsilon^{-1}\,\frac{b^n}{n}\,, \quad \frac{1}{m} + \frac{1}{n} = 1\,, \tag{3.1.4}$$

valid for any $\varepsilon > 0, a, b \geq 0$ and $m, n > 1$.

We define $H := L^2(\Omega)$ and $V := H^1(\Omega)$ so $V' = (H^1(\Omega))'$. Now, we state the main theorem of this chapter:

Theorem 3.1.1. Let $\Omega \subset \mathbb{R}^d$ be an open and bounded convex domain[4]. Let the function g satisfies the assumption (1.1.3), $u_0 \in H$, and $\beta > 0$, then the reaction-diffusion system (3.1.1)-(3.1.3) possesses at least one weak solution u satisfying

$$u \in L^2(0,T;V) \cap L^{2s}(\Omega_T) \cap L^\infty(0,T;H),$$

and the equation (3.1.1) holds as equality in $L^q(0,T;V')$, where $s > 1$ and q is the conjugate of $2s$ (i.e. $\frac{1}{2s} + \frac{1}{q} = 1$). Furthermore, the weak solution is unique and the map

$$u_0(\cdot) \longmapsto u(\cdot, t),$$

is continuous on H.

Proof. We will prove this theorem using the **Faedo-Galerkin** method of Lions (see Lions (1969)) and the **Alaoglu** compactness theorem (Theorem A.0.12). We separate the proof into three parts: local existence of the Galerkin approximations, global existence of the Galerkin approximations, and uniqueness and continuity of the weak solution u in H.

[4]Recall that an open and bounded convex domain has a Lipschitz continuous boundary (see Grisvard (1985), Corollary 1.2.2.3).

Section 3.2: Local existence of the Galerkin approximations

In this section we will seek the local existence of the Galerkin approximations for the reaction-diffusion system (3.1.1)-(3.1.3). We will require that $g(x)$ be an odd degree polynomial

$$g(x) = \sum_{j=0}^{2s-1} b_j x^j \qquad \forall x \in \mathbb{R}\,, \tag{3.2.1}$$

with a positive leading coefficient, i.e. $b_{2s-1} > 0$. For any function g given by (3.2.1), there exists a constant $C > 0$ such as

$$\left| \sum_{j=0}^{2s-2} b_j x^{j+1} \right| \leq \frac{1}{2} b_{2s-1} \left|x\right|^{2s} + C\,,$$

(see Temam (1997), pages 84-85, Robinson (2001), page 213).

We now multiply equation (3.1.1) by a function v and using the application of Green's identity (Theorem A.0.5) and recalling equation (3.1.3) we rewrite (3.1.1)-(3.1.3) in weak form as follows:

(P) Find $u \in V = H^1(\Omega)$ such that $u(\cdot, 0) = u_0(\cdot)$ and for almost every $t \in (0, T)$

$$\left(\frac{\partial u}{\partial t}, v\right) + (\nabla u, \nabla v) + \beta \int_{\partial\Omega} uv d\sigma + \int_{\Omega} g(u) v dx = 0 \quad \forall v \in V. \tag{3.2.2}$$

Now, set V^k :=span$\{z_i\}_{i=1}^k \subset V$. Note that the z_i is an infinite set of eigenfunctions of the eigenvalue problem (2.2.9)-(2.2.10) where

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_k \leq \cdots \text{ with } \lim_{i \to \infty} \mu_i = \infty$$

is an infinite set of corresponding eigenvalues (see the proof of Theorem 2.2). We seek a finite dimensional weak form corresponding to **(P)** :

($\mathbf{P}^k$) Find $u^k \in V^k$ such that $u^k(\cdot, 0) = u_0^k(\cdot)$ and for almost every $t \in (0, T)$

$$\left(\frac{\partial u^k}{\partial t}, v^k\right) + (\nabla u^k, \nabla v^k) + \beta \int_{\partial\Omega} u^k v^k d\sigma + \int_\Omega g(u^k) v^k dx = 0 \quad \forall v^k \in V^k, \tag{3.2.3}$$

where u_0^k is given, and defined by (3.2.9) later.

We introduce $P^k : H \longrightarrow V^k$, defined to be the orthogonal projection from H onto V^k, which satisfies

$$(P^k u, v) = (u, v), \quad \forall v \in V^k, \, u \in H\,. \tag{3.2.4}$$

This definition clearly makes sense for elements of $V \subset H$. We spend some time analyzing properties of P^k as these properties are repeatedly needed in this chapter and the next chapters.

Lemma 3.2.1. For any $u \in V$ we have

$$\|P^k u\|_0 \leq \|u\|_0 . \tag{3.2.5}$$

Proof. Take $v = P^k u \in V^k$ in (3.2.4), we obtain

$$\|P^k u\|_0^2 = (u, P^k u) \leq \|u\|_0 \|P^k u\|_0 ,$$

after applying the Cauchy-Schwarz inequality. Thus dividing both sides of this inequality by $\|P^k u\|_0$ gives the desired result. □

We know from the proof of Theorem 2.2 that $\{z_i\}_{i=1}^{\infty}$ is an orthonormal basis for H where

$$(z_i, z_j) = \delta_{ij} .$$

Thus we can write $u \in H$ as

$$u = \sum_{i=1}^{\infty} (u, z_i) z_i ,$$

see, e.g., Kreyszig (1978), Section 3.5. We also need the following lemma, which will be needed for proving the next lemma,

Lemma 3.2.2. For any $u \in V$ we have

$$\|\nabla(P^k u)\|_0 \leq \|\nabla u\|_0 . \tag{3.2.6}$$

Proof. As $u \in V \subset H$ we have

$$P^k u = \sum_{i=1}^{k} (u, z_i) z_i \,,$$

and hence

$$\nabla(P^k u) = \sum_{i=1}^{k} (u, z_i) \nabla z_i \,.$$

Thus

$$\begin{aligned}
\|\nabla(P^k u)\|_0^2 &= \big(\nabla(P^k u), \nabla(P^k u)\big) \\
&= \sum_{j=1}^{k} \sum_{i=1}^{k} (u, z_j)(u, z_i)(\nabla z_i, \nabla z_j) \\
&\leq \sum_{j=1}^{k} \sum_{i=1}^{k} (u, z_j)(u, z_i)(\mu_i - 1)\delta_{ij} \\
&= \sum_{j=1}^{k} (u, z_j)(u, z_j)(\mu_j - 1) \\
&= \sum_{j=1}^{k} (u, z_j)(\nabla u, \nabla z_j) \\
&= \big(\nabla(P^k u), \nabla u\big) \\
&\leq \|\nabla(P^k u)\|_0 \|\nabla u\|_0 \,,
\end{aligned}$$

after noting that

$$(\nabla u, \nabla z_i) = (\mu_i - 1)(u, z_i), \quad \forall u \in V \,,$$

which is deduced from equations (2.2.25) and (2.2.26) in the previous chapter, and applying the Cauchy-Schwarz inequality. Thus dividing both sides of the above inequality by $\|\nabla(P^k u)\|_0$ gives

$$\|\nabla(P^k u)\|_0 \leq \|\nabla u\|_0 \,,$$

as desired. $\square$

We now require the following lemma for the work that follows:

Lemma 3.2.3. Let $u \in V$. Then

$$\|P^k u\|_1 \leq \|u\|_1 \,. \tag{3.2.7}$$

Proof. Squaring and combining the inequalities (3.2.5) and (3.2.6) gives the desired result. $\square$

Now, we will show the existence and uniqueness of local solutions of problem $(\mathbf{P}^k)$. We write u^k as a Galerkin approximation:

$$u^k(\cdot,t) = \sum_{i=1}^{k} a_{ik}(t) z_i(\cdot) \,, \tag{3.2.8}$$

and set $v^k = z_j$ for $j = 1, \cdots, k$ in the finite dimensional weak form (3.2.3), where $a_{ik}(t) = (u^k, z_i)$. This will give a system of k ODEs for $a_{ik}(t)$. Then we deduce the existence and uniqueness of local solutions by using the local existence theorem (Theorem A.0.13).

For the initial approximations we take

$$u_0^k := P^k u_0(\cdot)\,. \tag{3.2.9}$$

Note that the argument for convergence, in Lemma 3.2.1, is the same for H. Thus we have the strong convergence in H of the initial approximations to the initial data, that is

$$u_0^k \longrightarrow u_0 \quad \text{in} \quad H \quad \text{as} \quad k \to \infty\,. \tag{3.2.10}$$

With the above setup the substitution of u^k into the finite dimensional weak form (3.2.3) leads to

$$\sum_{i=1}^{k} \frac{da_{ik}}{dt}(z_i, z_j) + \sum_{i=1}^{k} a_{ik}(\nabla z_i, \nabla z_j) + \beta \int_{\partial\Omega} a_{ik} z_i z_j d\sigma = - \int_{\Omega} z_j g(u^k) dx\,,$$

for $j = 1, \ldots, k$.

After recalling equations (2.2.25) and (2.2.27) from the previous chapter we obtain an initial value problem for a system of k ODEs in the components

a_{jk}:

$$\frac{da_{jk}}{dt} + a_{jk}(\mu_j - 1) = -\int_\Omega z_j g(u^k)dx\,, \tag{3.2.11}$$

where $a_{jk}(0) = (u_0, z_j)$, $\quad j = 1, \cdots, k$. We could also write this equation as

$$\frac{du^k}{dt} = \Delta u^k - P^k g(u^k), \quad u^k(\cdot, 0) = P^k u_0(\cdot)\,, \tag{3.2.12}$$

where $-\Delta u^k = \sum_{j=1}^{k}(\mu_j - 1)z_j a_{jk}$. Here u^k is the *composite function* given in terms of the components by equation (3.2.8). We can also write equation (3.2.12) in the form

$$\frac{du^k}{dt} = -Au^k - P^k g(u^k)\,, \tag{3.2.13}$$

where $A = -\Delta$ is a linear operator. In fact, the advantage of equation (3.2.12) or (3.2.13) is to derive the estimates that follow in the next chapter.

We now need to show that the nonlinearity on the right-hand side of the system of ODEs is, in fact, locally Lipschitz. If this holds, from the local existence theorem (Theorem A.0.13), it follows that the system of the k ODEs (3.2.11) has a unique solution u^k on some finite time interval $(0, t_k)$, $t_k > 0$. We recall that $g(x)$ is an odd degree polynomial and deal with it as follows

$$|g(u) - g(v)| \;=\; \left|\sum_{i=0}^{2s-1} b_i u^i - \sum_{i=0}^{2s-1} b_i v^i\right|$$

$$= \left| \sum_{i=1}^{2s-1} b_i(u^i - v^i) \right|$$

$$= \left| \sum_{i=1}^{2s-1} b_i(u-v) \sum_{k=0}^{i-1} u^{i-1-k} v^k \right| . \qquad (3.2.14)$$

We shall now apply the Cauchy's inequality to the term $\sum_{k=0}^{i-1} u^{i-1-k} v^k$ to get

$$\sum_{k=0}^{i-1} \left| u^{i-1-k} v^k \right| \leq \left(\sum_{k=0}^{i-1} (u^{i-1-k})^2 \right)^{1/2} \left(\sum_{k=0}^{i-1} (v^k)^2 \right)^{1/2}$$

$$= \left(\sum_{k=0}^{i-1} u^{2k} \right)^{1/2} \left(\sum_{k=0}^{i-1} v^{2k} \right)^{1/2}.$$

$$\leq \left(\sum_{k=0}^{i-1} u^k \right) \left(\sum_{k=0}^{i-1} v^k \right), \qquad (3.2.15)$$

since $u^{2k}, v^{2k} \geq 0$[5]. Substituting (3.2.15) in (3.2.14) yields

$$|g(u) - g(v)| \leq \left(\sum_{i=1}^{2s-1} |b_i| \left(\sum_{k=0}^{i-1} u^k \right) \left(\sum_{k=0}^{i-1} v^k \right) \right) |u - v|$$

$$\leq \left(\sum_{i=1}^{2s-1} |b_i| \left(\sum_{k=0}^{i-1} u^k \right)^2 \right)^{1/2} \left(\sum_{i=1}^{2s-1} |b_i| \left(\sum_{k=0}^{i-1} v^k \right)^2 \right)^{1/2} |u - v|$$

[5]Recall the elementary result, $(a+b)^p \leq a^p + b^p, a, b \geq 0, p < 1$.

$$\leq \max\{|b_i|\} \left(\sum_{i=1}^{2s-1} \sum_{k=0}^{i-1} u^k \right) \left(\sum_{i=1}^{2s-1} \sum_{k=0}^{i-1} v^k \right) |u - v|$$

$$\leq \max\{|b_i|\}\, C \left(1 + \|u\|_{2s-2}^{2s-2}\right) \left(1 + \|v\|_{2s-2}^{2s-2}\right) |u - v|$$

$$\leq C(u, v)\, |u - v| \;, \tag{3.2.16}$$

where $C(u, v)$ is the *Lipschitz constant* of the function g.

Section 3.3: Global existence of the Galerkin approximations

In this section we shall show that the solutions are bounded in time and that uniform bounds on u^k (independently of k) hold in various Banach spaces. We now set $v^k = z_j$, for $j = 1, \ldots, k$, in the finite dimensional weak form (3.2.3). Recall equation (3.2.8), multiply equation (3.2.3) by a_{jk} and sum from $j = 1, \cdots, k$. This is equivalent to taking $v^k = u^k$ in the finite dimensional weak form (3.2.3), yielding

$$\left(\frac{\partial u^k}{\partial t}, u^k \right) + (\nabla u^k, \nabla u^k) + \beta \int_{\partial\Omega} (u^k)^2 d\sigma + \int_{\Omega} g(u^k) u^k dx = 0, \tag{3.3.1}$$

and hence

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u^k|^2 dx + \int_{\Omega} |\nabla u^k|^2 dx + \beta \int_{\partial\Omega} (u^k)^2 d\sigma + \int_{\Omega} g(u^k) u^k dx = 0 \,. \tag{3.3.2}$$

Note that the third term on the left-hand side of this equation is the boundary integral which is bounded as shown in Theorem 2.1. Thus $0 \le \beta \int_{\partial\Omega} (u^k)^2 d\sigma < \infty$. Therefore we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u^k|^2 dx + \int_\Omega |\nabla u^k|^2 dx + \int_\Omega g(u^k) u^k dx \le 0\,. \tag{3.3.3}$$

For any function g given by (3.2.1), there exists a constant $C > 0$ such as

$$\left|\sum_{j=0}^{2s-2} b_j (u^k)^{j+1}\right| \le \frac{1}{2} b_{2s-1} |u^k|^{2s} + C_1, \quad C_1 > 0\,, \tag{3.3.4}$$

and this implies

$$\frac{1}{2}\, b_{2s-1}\, |u^k|^{2s} - C_1 \le g(u^k) u^k \le \frac{3}{2}\, b_{2s-1}\, |u^k|^{2s} + C_1\,. \tag{3.3.5}$$

With these inequalities (3.3.4) and (3.3.5) we can simplify equation (3.3.3) to

$$\frac{d}{dt}\int_\Omega |u^k|^2 dx + 2\int_\Omega |\nabla u^k|^2 dx + b_{2s-1}\int_\Omega |u^k|^{2s} dx \le C_1 |\Omega|\,. \tag{3.3.6}$$

We denote the measure of Ω by $|\Omega|$, i.e. $|\Omega| = \int_\Omega dx$. Since $V = H^1(\Omega)$ then a Poincare inequality is not available for the inequality (3.3.6). Therefore we have to seek an alternative way to make inequality (3.3.6) useful as we need this later! By Hölder's inequality for $s > 1$

$$\|u\|_0^2 \leq \left(\int_\Omega u^{2s} dx \right)^{1/s} (|\Omega|)^{1/s'} ,$$

where s and s' are conjugate. Now Young's inequality yields

$$\|u\|_0^2 \leq \frac{1}{2} b_{2s-1} \int_\Omega u^{2s} dx + C_2 \, b_{2s-1}^{-s'/s} |\Omega|, \quad C_2 > 0 . \tag{3.3.7}$$

From equations (3.3.6) and (3.3.7) we obtain

$$\frac{d}{dt} \int_\Omega |u^k|^2 dx + 2 \int_\Omega |\nabla u^k|^2 dx + \|u^k\|_0^2 + \frac{1}{2} b_{2s-1} \int_\Omega |u^k|^{2s} dx \leq C_0 , \tag{3.3.8}$$

where $C_0 = C_1 |\Omega| + C_2 b_{2s-1}^{-s'/s} \, |\Omega|$. Applying a Grönwall lemma (Theorem A.0.14) to inequality (3.3.8) yields

$$\begin{aligned} &\|u^k(T)\|_0^2 + \exp(-T) \int_0^T \left(2|u^k|_1^2 + \frac{1}{2} b_{2s-1} \, \|u^k\|_{0,2s}^{2s} \right) dt \\ &\qquad \leq \|u^k(0)\|_0^2 \exp(-T) + C_0(1 - \exp(-T)), \end{aligned} \tag{3.3.9}$$

where $t_k = T$ (independent of k). Recalling $u_0 \in H = L^2(\Omega)$ so $\|u^k(0)\|_0 \equiv \|P^k u_0\|_0 \leq \|u_0\|_0 \leq C$ we have

$$u^k \text{ is uniformly bounded in } L^\infty(0, T; H) \cap L^{2s}(\Omega_T) \quad .$$

By noting the injection $L^\infty \hookrightarrow L^2$, the semi-norm bound for V and the density of V^k in V we have

$$u^k \text{ is uniformly bounded in } L^2(0, T; V) \quad .$$

We can also write these as

$$u^k \in L^2(0,T;V) \cap L^{2s}(\Omega_T) \cap L^\infty(0,T;H) \quad \text{(uniformly)}\ .$$

Passage to the limit:

We will now show passage to the limit of the terms in (3.2.13). We first consider the term du^k/dt. We use the fact that A is bounded linear operator from V into V', so that for time-dependent problems if $u^k \in L^2(0,T;V)$ then $Au^k \in L^2(0,T;V')$. Furthermore, we claim that $P^k g(u^k) \in L^q(\Omega_T)$, where q is conjugate to $2s$. This result will be shown very soon. Thus it follows from the equation (3.2.13) that

$$\frac{\partial u^k}{\partial t} \text{ is uniformly bounded in } L^2(0,T;V') + L^q(\Omega_T)\ ,$$

where $L^2(0,T;V')+L^q(\Omega_T)$ is the dual space of $L^2(0,T;V)\cap L^{2s}(\Omega_T)$ (see (3.4.10), (3.4.11), and Lemma 3.4.1).

We now use the **Alaoglu** compactness theorem (Theorem A.0.12) to extract a subsequence such that du^k/dt converges weakly to some $\dot{v}$. We adapt an argument in (Robinson (2001), Subsection 7.4.3) to give $\dot{v} = du/dt$, i.e.

$$\frac{du^k}{dt} \rightharpoonup \frac{du}{dt} \quad \text{in} \quad L^2(0,T;V') + L^q(\Omega_T) \quad \text{as} \quad k \to \infty\,.$$

First, u^k is uniformly bounded in $L^2(0,T;V) \cap L^{2s}(\Omega_T)$, so since $L^2(0,T;V)$ and $L^{2s}(\Omega_T)$ are reflexive we can extract a subsequence that converges weakly

$$u^k \rightharpoonup u \quad \text{in} \quad L^2(0,T;V) \cap L^{2s}(\Omega_T) \quad \text{as} \quad k \to \infty\,,$$

with dual space $L^2(0,T;V') + L^q(\Omega_T)$. Furthermore, from the Sobolev embedding theorem (Theorem A.0.7) and the fact V is dense in H, we have the dense inclusion $V \hookrightarrow L^{2s}(\Omega)$, thus $L^q(\Omega) \hookrightarrow V'$ and so $L^2(0,T;V') + L^q(\Omega_T) \subset L^q(0,T;V')$. Now consider an arbitrary $\phi(t) \in C_0^\infty(0,T;V) \subset L^{2s}(0,T;V)$. Integrating by parts, noting that functions in $C_0^\infty(0,T;V)$ have compact support in $(0,T)$ and using the weak convergence of u^k to u in $L^2(0,T;V') + L^q(\Omega_T)$ and hence in $L^q(0,T;V')$ yields

$$\int_0^T \left(\frac{du^k}{dt}, \phi\right) dt = -\int_0^T \left(u^k, \frac{d\phi}{dt}\right) dt \to -\int_0^T \left(u, \frac{d\phi}{dt}\right) dt$$

$$= \int_0^T \left(\frac{du}{dt}, \phi\right) dt\,,$$

where we note that $d\phi/dt \in C_0^\infty(0,T;V)$, due to the smoothness of the functions in this space. From the weak convergence of du^k/dt to $\dot{v}$ in $L^q(0,T;V')$ we also have

$$\int_0^T \left(\frac{du^k}{dt}, \phi\right) dt \longrightarrow \int_0^T (\dot{v}, \phi)\, dt \quad \text{as} \quad k \to \infty$$

(see Theorem A.0.18), and so by the uniqueness of weak limits we have $\dot{v} = du/dt$ as required. Due to the density of $C_0^\infty(0,T;V)$ in $L^{2s}(0,T;V)$ the convergence results that hold for functions in $C_0^\infty(0,T;V)$ also hold by extension for functions in $L^{2s}(0,T;V)$.

To obtain the same convergence of Au^k, we use the fact that A is a bounded linear operator from V into V', so that the weak convergence

$$u^k \rightharpoonup u \quad \text{in} \quad L^2(0,T;V) \quad \text{as} \quad k \to \infty,$$

implies the following weak convergence

$$Au^k \rightharpoonup Au \quad \text{in} \quad L^2(0,T;V') \quad \text{as} \quad k \to \infty\,.$$

Note that, from the previous chapter, we have the symmetry condition for the operator A as follows

$$(Au, v) = (u, Av) \quad \forall u, v \in V,\ Au, Av \in H.$$

Thus, considering $\psi \in L^2(0,T;V)$

$$\int_0^T \left(Au^k, \psi\right) dt = \int_0^T \left(u^k, A\psi\right) dt \to \int_0^T (u, A\psi)\, dt$$

$$= \int_0^T (Au, \psi)\, dt\,,$$

where $A\psi \in L^2(0,T;V')$. Since $L^2(0,T;V') \subset L^q(0,T;V')$ (we have $q < 2$ since $2s > 2$ as $s > 1$), we have

$$Au^k \rightharpoonup Au \quad \text{in} \quad L^q(0,T;V') \quad \text{as} \quad k \to \infty.$$

This completes the weak convergence of Au^k.

Finally, we consider the last term $P^k g(u^k)$. We use the bound on u^k in $L^{2s}(\Omega_T)$ to obtain bounds on the nonlinear term $g(u^k)$ in $L^q(\Omega_T)$. From the polynomial (3.2.1) we have that there exists a constant $C > 0$ such that

$$\left| \sum_{j=0}^{2s-2} b_j \, (u^k)^j \right| \le \frac{1}{2} \, b_{2s-1} |u^k|^{2s-1} + C \, . \tag{3.3.10}$$

Therefore

$$\left| g(u^k) \right| \le \frac{3}{2} \, b_{2s-1} |u^k|^{2s-1} + C \, . \tag{3.3.11}$$

It follows from inequality (3.3.11) that

$$\begin{aligned} \int_0^T \int_\Omega |g(u^k)|^q \, dxdt &\le \int_0^T \int_\Omega \left(\frac{3}{2} b_{2s-1} |u^k|^{2s-1} + C \right)^q \, dxdt \\ &\le 2^{q-2} (3 b_{2s-1})^q \int_0^T \int_\Omega |u^k|^{q(2s-1)} dxdt + 2^{q-1} C^q |\Omega|^q T \, , \end{aligned}$$

where $q = 2s/(2s-1)$. Since $q(2s-1) = 2s$ it follows that

$$\|g(u^k)\|^q_{L^q(\Omega_T)} \le 2^{q-2} (3 b_{2s-1})^q \|u^k\|^{2s}_{L^{2s}(\Omega_T)} + C < \infty \, , \tag{3.3.12}$$

where C depends on q, $|\Omega|$ and T. Recalling that $u^k \in L^{2s}(\Omega_T)$ and $L^{2s} \hookrightarrow L^q$ we have

$$g(u^k) \text{ is uniformly bounded in } L^q(\Omega_T),$$

and so from weak compactness arguments there exists some $\chi \in L^q(\Omega_T)$ such that

$$g(u^k) \rightharpoonup \chi \quad \text{in} \quad L^q(\Omega_T) \quad \text{as} \quad k \to \infty \,.$$

We now want to show that

$$P^k g(u^k) \rightharpoonup \chi \quad \text{in} \quad L^q(\Omega_T) \quad \text{as} \quad k \to \infty \,.$$

Define $Q^k := I - P^k$, the projection orthogonal to P^k. Therefore, for all $\phi \in L^{2s}(\Omega_T)$ we can write

$$\begin{aligned} \left| \int_0^T (P^k g(u^k) - \chi, \phi) dt \right| &= \left| \int_0^T (g(u^k) - \chi, \phi) dt - \int_0^T (g(u^k), Q^k \phi) dt \right| \\ &\leq \left| \int_0^T (g(u^k) - \chi, \phi) dt \right| + \left| \int_0^T (g(u^k), Q^k \phi) dt \right| . \end{aligned}$$

The first term on the right-hand side of this inequality tends to zero due to the weak convergence of $g(u^k)$ to χ in $L^q(\Omega_T)$. For the second term, we know from the Sobolev embedding theorems and the fact V is dense in H that we have the dense inclusion $V \hookrightarrow L^{2s}(\Omega)$, thus using Lemma 3.2.3 we deduce that $P^k \phi \to \phi$ in $L^{2s}(\Omega)$, $\forall \phi \in L^{2s}(\Omega)$, i.e. $Q^k \phi \to 0$ in $L^{2s}(\Omega)$. This shows

$$\left| \int_0^T (P^k g(u^k) - \chi, \phi) dt \right| \longrightarrow 0 \qquad \text{in} \quad L^q(\Omega_T) \,,$$

and hence the desired result.

It remains to show that $\chi \equiv g(u)$. We will first require the following theorem:

Theorem 3.3.1 (Lions-Aubin compactness theorem)
Let X_0, X, X_1 be three Banach spaces such that

$$X_0 \overset{c}{\hookrightarrow} X \hookrightarrow X_1 \,,$$

and X_i is reflexive, $i = 0, 1$. Let $T > 0$ be a fixed finite number and $1 < p_i < \infty, i = 0, 1$, then the space

$$\begin{aligned} Y &\equiv Y(0, T; p_0, p_1; X_0, X_1) \\ &= \left\{ v | v \in L^{p_0}(0, T; X_0), \quad \frac{dv}{dt} \in L^{p_1}(0, T; X_1) \right\} , \end{aligned}$$

is a Banach space for the norm

$$\|v\|_Y = \|v\|_{L^{p_0}(0,T;X_0)} + \left\| \frac{dv}{dt} \right\|_{L^{p_1}(0,T;X_1)} .$$

Moreover, the injection of Y into $L^{p_0}(0, T; X)$ is compact.

Proof. R. Temam (1984), pp. 271-273.

We have that $u^k \in L^2(0, T; V)$, $\dfrac{du^k}{dt} \in L^q(0, T; V')$, so the above Theorem guarantees that $Y \equiv Y(0, T; 2, q; V, V') \overset{c}{\hookrightarrow} L^2(0, T; H)$ and we can extract a further subsequence such that

$$u^k \longrightarrow u \qquad \text{(strongly)} \quad \text{in} \quad L^2(0,T;H).$$

We also need the following lemma:

Lemma 3.3.2

Let O be an open set in $\mathbb{R}^m \times \mathbb{R}$.

1. If $u^k \to u$ in $L^p(O) (1 \leq p < \infty)$, then there is a subsequence, u^k, that converges to u a.e. in O (see Robinson (2001), p.27, Rodrigues (1987), p.59).

2. If, in addition, O is bounded, $g \in L^q(O)$, and $\{g^k\}$ is a sequence of functions such that

$$\|g^k\|_{L^q(O)} \leq C \quad \text{and} \quad g^k \to g \text{ a.e. in } O,$$

   then $g^k \rightharpoonup g$ in $L^q(O)$ (see Robinson (2001), p.218).

According to the first part of this lemma, there is a subsequence u^k such that $u^k(x,t) \to u(x,t)$ a.e $(x,t) \in \Omega_T$. As g is locally Lipschitz in Ω_T (see Section 3.2), it follows that $g(u^k(x,t)) \to g(u(x,t))$, $(x,t) \in \Omega_T$. Now, the second part of the above lemma gives

$$g(u^k) \rightharpoonup g(u) \quad \text{in} \quad L^q(\Omega_T) \quad \text{as} \quad k \to \infty\,.$$

By the uniqueness of weak limits we deduce $\chi \equiv g(u)$.

Finally, we show that $u(0) = u_0$ by adapting the technique used in Robinson (2001), p.205-206 . Let $\phi \in C^1([0,T];V)$ be an arbitrary function of the form

$$\phi(\cdot,t) \equiv \phi^k(\cdot,t) = \sum_{i=1}^{k} b_i(t) z_i(\cdot)\,,$$

with properties that $\phi(T) = 0$, $\phi^k(0) = P^k\phi_0$, and $b_i(t) \in C^1([0,T])$ are arbitrary. Taking $v^k = \phi^k = \phi$ and integrating the finite dimensional weak form (3.2.3) from 0 to T gives

$$\int_0^T (\frac{\partial u^k}{\partial t},\phi)dt + \int_0^T (\nabla u^k, \nabla\phi)dt + \beta\int_0^T \langle u^k,\phi\rangle dt + \int_0^T (g(u^k),\phi)dt = 0, \tag{3.3.13}$$

where $\langle u, v\rangle = \int_{\partial\Omega} uv\,d\sigma$ representing the boundary integral from now on. Note that this equation holds for all $\phi \in L^{2s}(0,T;V)$ and this is due to the density of $V^k \subset V$ in H and the dense inclusion $L^{2s}(0,T;V) \hookrightarrow L^2(0,T;V) \cap L^{2s}(\Omega_T)$. Similarly, we take $v = \phi$ and integrate the weak form (3.2.2) from 0 to T to obtain

$$\int_0^T (\frac{\partial u}{\partial t},\phi)dt + \int_0^T (\nabla u, \nabla\phi)dt + \beta\int_0^T \langle u,\phi\rangle dt + \int_0^T (g(u),\phi)dt = 0, \tag{3.3.14}$$

for all $\phi \in L^{2s}(0,T;V)$. We now integrate the first term of (3.3.13) and (3.3.14) by parts with respect to t to get

$$\int_0^T (\nabla u^k, \nabla\phi)dt + \beta\int_0^T \langle u^k, \phi\rangle dt + \int_0^T (g(u^k), \phi)dt$$

$$= \int_0^T (u^k, \frac{\partial\phi}{\partial t})dt + (u^k(0), \phi(0))\,, \qquad (3.3.15)$$

and

$$\int_0^T (\nabla u, \nabla\phi)dt + \beta\int_0^T \langle u, \phi\rangle dt + \int_0^T (g(u), \phi)dt$$

$$= \int_0^T (u, \frac{\partial\phi}{\partial t})dt + (u(0), \phi(0))\,. \qquad (3.3.16)$$

From (3.2.10) we have that $P^k u_0 \equiv u^k(0) \to u_0$ in H. Thus taking limits in all terms of equation (3.3.15) we obtain

$$\int_0^T (\nabla u, \nabla\phi)dt + \beta\int_0^T \langle u, \phi\rangle dt + \int_0^T (g(u), \phi)dt$$

$$= \int_0^T (u, \frac{\partial\phi}{\partial t})dt + (u_0, \phi(0))\,. \qquad (3.3.17)$$

Since $\phi(0) \in V \hookrightarrow H$ is arbitrary, a comparison of (3.3.16) and (3.3.17) shows that $u(0) = u_0$, as required.

Section 3.4: Uniqueness and Continuity

To prove the unique dependence of a solution of problem **(P)** on the initial data in H we suppose that there are two solutions u_1 and u_2 of the weak form (3.2.2) with initial conditions $u_1(0), u_2(0) \in H$, respectively. Then, letting $v = w := u_1 - u_2$, we obtain

$$\left(\frac{\partial w}{\partial t}, w\right) + (\nabla w, \nabla w) + \beta \int_{\partial\Omega} |w|^2 d\sigma$$

$$= - \int_{\Omega} [g(u_1) - g(u_2)]\,(u_1 - u_2) dx\,, \tag{3.4.1}$$

and hence

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} |w|^2 dx + \int_{\Omega} |\nabla w|^2 dx + \beta \int_{\partial\Omega} |w|^2 d\sigma = \int_{\Omega} [g(u_2) - g(u_1)]\,(u_1 - u_2) dx\,. \tag{3.4.2}$$

Note that the second and third terms on the left-hand side of this equation are bounded and non-negative. Thus

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}|w|^2dx \leq \int_{\Omega}[g(u_2)-g(u_1)]\,(u_1-u_2)dx\,. \tag{3.4.3}$$

Now, recalling inequality (3.2.16) we have

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}|w|^2dx \leq C(t)\int_{\Omega}|u_1-u_2|^2dx\,, \tag{3.4.4}$$

where

$$C(t) \equiv C(u_2,u_1) = \max\{|b_i|\}\,C\left(1+\|u_2\|_{0,2s-2}^{2s-2}\right)\left(1+\|u_1\|_{0,2s-2}^{2s-2}\right) \tag{3.4.5}$$

is a positive Lipschitz constant of the function g. We therefore obtain

$$\frac{d}{dt}\int_{\Omega}|w|^2dx \leq 2C(t)\|w\|_0^2. \tag{3.4.6}$$

Applying the usual Grönwall lemma (Theorem A.0.14) to this inequality yields

$$\|w(t)\|_0^2 \leq \exp\left(2\int_0^t C(s)\,ds\right)\|w(0)\|_0^2. \tag{3.4.7}$$

Thus we have

$$\|u_1(t)-u_2(t)\|_0^2 \leq \exp\left(2\int_0^t C(s)\,ds\right)\|u_1(0)-u_2(0)\|_0^2. \tag{3.4.8}$$

So the uniqueness follows if $u_1(0) = u_2(0)$. However, if $u_1(0) \neq u_2(0)$, then we have continuous dependence in $H = L^2(\Omega)$.

Finally, it remains to show that the map

$$u_0(\cdot) \longrightarrow u(\cdot, t)\,,$$

is continuous on H, i.e. $u \in C([0,T]; H)$. First let

$$W := L^2(0,T;V) \cap L^{2s}(0,T;L^{2s}(\Omega)) \tag{3.4.9}$$

and

$$W' := L^2(0,T;V') + L^q(0,T;L^q(\Omega))\,, \tag{3.4.10}$$

where $q = 2s/(2s-1), s > 1$ and W' is the dual space of W.

We note the following lemma:

Lemma 3.4.1. Let V, H, V' be three Hilbert spaces, each space included and dense in the following one, V' being the dual space of V. If

$$u \in W \qquad \text{and} \qquad \frac{du}{dt} \in W'\,,$$

where W and W' are as given in (3.4.9) and (3.4.10), then u is almost everywhere equal to a function continuous from $[0,T]$ into H, i.e.

$$u \in C([0,T];H) \qquad \text{a.e.}\,.$$

From Section 3.3 we have already noted that $u \in W$ and $\dfrac{du}{dt} \in W'$. Thus, by applying this lemma, we obtain that $u \in C([0,T];H)$ as required. This completes the proof of Theorem 3.1.1. $\square$

Note that the proof of Lemma 3.4.1 is an adapted version of the proof of Robinson (2001), pp.191-193, but for completeness we include it here.

Proof. Since $V \overset{c}{\hookrightarrow} H \hookrightarrow V'$ and considering the associated Banach norms, it is easy to see the following continuous injections:

$$W \hookrightarrow L^2(0,T;H) \hookrightarrow W'\,. \tag{3.4.11}$$

As a consequence of (3.4.11), the scalar product in $L^2(0,T;H)$ of $f \in L^2(0,T;H)$ and $u \in W$ is the same as the scalar product of f and u in W' and W, respectively. Thus we have

$$\langle f,u\rangle_{W',W} = (f,u) = \int_0^T \langle f,u\rangle \; dt\,, \tag{3.4.12}$$

for all $f \in L^2(0,T;H)$ and $u \in W$.

Now, we adapt a proof that is given in Robinson (2001) (Theorem 7.2, Chapter 7). We shall regularize the function $\widehat{u}$, from $\mathbb{R}$ into V, which is equal to u on $[0,T]$ and to zero outside this interval. We shall also use the technique of *mollification* which allows us to approximate less regular functions by smooth functions. A mollification u_k of u is

$$u_k(t) = k^{-1}\int_0^T \rho\left(\frac{t-r}{k}\right) u(r)dr\,,$$

and the mollification $u_{1/k}$ of u is

$$(u(t))_{1/k} = k\int_0^T \rho\left(k(t-r)\right) u(r)dr\,,$$

where $\rho(t) \in C_c^\infty(\mathbb{R})$ (for the definitions of $\rho(t)$ and general mollified functions the reader may refer to Robinson (2001), p.19). From the definition of ρ and the fact that

$$\int_{\mathbb{R}} \rho(t)dt = 1\,,$$

we deduce that

$$k^{-1}\int_0^T \rho\left(\frac{t-r}{k}\right) dr = k\int_0^T \rho\left(k(t-r)\right) dr = 1\,,$$

which leads to a mollified version $u_k(t) = (u(t))_{1/k}$ of u with respect to the variable t . We hence obtain a sequence of functions $u_k \in C^1([0,T];V)$, which converges to u such that

$$u_k \to u \quad \text{in} \quad W \qquad \text{and} \qquad \frac{du_k}{dt} \to \frac{du}{dt} \quad \text{in} \quad W' . \tag{3.4.13}$$

Then, for any $t_0 \in [0, T]$,

$$\|u_k(t)\|_0^2 = \|u_k(t_0)\|_0^2 + 2\int_{t_0}^{t} \langle \frac{du_k(s)}{ds}, u_k(s)\rangle \, ds \, ,$$

where $\langle \frac{du_k(s)}{ds}, u_k(s)\rangle = \frac{1}{2}\frac{d}{ds}\|u_k(s)\|_0^2$. This equation is equivalent to

$$\|u_k(t)\|_0^2 = \|u_k(t_0)\|_0^2 + 2\langle \frac{du_k}{dt}, u_k\rangle_{W',W} . \tag{3.4.14}$$

Choose t_0 such that

$$\|u_k(t_0)\|_0^2 = \frac{1}{T}\int_0^T \|u_k(t)\|_0^2 \, dt \, , \tag{3.4.15}$$

this follows since $u_k \in C([0,T]; V)$, so that

$$T \min_{t\in[0,T]} \|u_k(t)\|_0^2 \leq \int_0^T \|u_k\|_0^2 \, dt \leq T \max_{t\in[0,T]} \|u_k(t)\|_0^2 \, ,$$

and the intermediate value theorem applies. Substituting equation (3.3.15) in equation (3.4.14) yields

$$\|u_k(t)\|_0^2 = \frac{1}{T}\int_0^T \|u_k(t)\|_0^2 \, dt + 2\langle \frac{du_k}{dt}, u_k\rangle_{W',W} . \tag{3.4.16}$$

By Cauchy-Schwarz inequality (2.2.8) and the Young's inequality (3.1.4) (with $n = m = 2$ and $\varepsilon = 1$), we have

$$\|u_k(t)\|_0^2 \leq \frac{1}{T}\|u_k\|_{L^2(0,T;H)}^2 + 2\|\frac{du_k}{dt}\|_{W'}\|u_k\|_W$$

$$\leq C\|u_k\|_W^2 + \|\frac{du_k}{dt}\|_{W'}^2 + \|u_k\|_W^2 \,.$$

Thus

$$\sup_{t\in[0,T]} \|u_k(t)\| \leq C(\|u_k\|_W + \|\frac{du_k}{dt}\|_{W'})\,. \tag{3.4.17}$$

Since u_k is a Cauchy sequence in W and du_k/dt is a Cauchy sequence in W', it follows that u_k is a Cauchy sequence in $C([0,T];H)$ and hence $u \in C([0,T];H)$ as desired. $\square$

Chapter 4

Strong Solutions

In this chapter we deduce further regularity results to the weak form (3.2.2) from additional estimates, which lead to results for *strong solutions*[6]. Moreover, such these estimates will be useful for the following chapter. In Section 4.1 we present a regularity result for the elliptic boundary value problem with Robin boundary conditions. In Section 4.2 we improve the results for weak solutions in Chapter 3 by increasing the regularity of problem (**P**) and the initial data.

Section 4.1: Regularity Result

In this section we consider the elliptic boundary value problem with Robin boundary conditions. We show that the existence of the strong solution of the Robin problem

$$-\Delta u + u = f \qquad \text{in} \quad \Omega \tag{4.1.1}$$

[6]Here, e.g., a strong solution is a solution of the problem (4.1.1)-(4.1.2) which is $u \in H^2(\Omega) \cap H^1(\Omega)$, where $f \in L^2(\Omega)$ is given. For the definition of a strong solution in the context of second order elliptic PDEs see, e.g., Renardy and Rogers (1993),pp.287-288, Robinson (2001),p.160.

$$\frac{\partial u}{\partial n} + \beta u = 0 \qquad \text{on} \quad \partial\Omega \tag{4.1.2}$$

can be obtained with the help of an *a priori* estimate. Throughout this section we will treat this system (4.1.1)-(4.1.2) on a bounded convex domain $\Omega \subset \mathbb{R}^n$ and $\beta > 0$. The uniqueness of a strong solution follows from the fact that a strong solution is also a weak solution (the uniqueness of weak solutions has been shown in Section 3.4).

We shall also need the so-called Young's inequality

$$ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2},$$

valid for any $a, b\mathbb{R}$ and $\varepsilon > 0$.

We now state the main theorem of this section:

Theorem 4.1.1. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded convex domain, and let β be a positive constant. Then for every $f \in L^2(\Omega)$, there exists a unique $u \in H^2(\Omega)$ which is the solution of the system (4.1.1)-(4.1.2).

Proof. The proof follows the methodology of Grisvard (1985) where the zero Neumann boundary condition (see Theorem 3.2.1.3) is considered. We shall first multiply equation (4.1.1) by a function $v \in H^1(\Omega)$ and using the application of Green's identity (Theorem A.0.5) and recalling equation (4.1.2) we find the system (4.1.1)-(4.1.2) can be written in weak form as follows:

$$\int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega uv dx + \beta \int_{\partial\Omega} uv d\sigma = \int_\Omega fv dx\,. \tag{4.1.3}$$

We separate the proof into three subsections. Subsection 4.1.1 shows the existence and uniqueness of u_k for a solution in $H^2(\Omega_k)$ provided that each Ω_k is a bounded convex open set with a C^2 boundary and $\Omega \subset \Omega_k$. In Subsection 4.1.2 we deduce an *a priori* estimate which leads to a bounded sequence $\{u_k\}$ in $H^2(\Omega_k)$, and hence in $H^2(\Omega)$ by restricting the u_k to Ω (see Lemma 4.1.2). We will also see that the constant of the *a priori* estimate does not depend on the curvature of $\partial\Omega_k$, i.e. on the fact that $\partial\Omega_k$ is $\{C^{1,1}\}$[7] which allows us to take the limit in k with respect to a general bounded convex domain Ω. In the final subsection, Subsection 4.1.3, we will show that the strong solution u of equation (4.1.1) and (4.1.2) is achieved by taking the limit in k with the help of an *a priori* estimate, i.e. inequality (4.1.6).

Subsection 4.1.1: Existence and Uniqueness

We first need the following lemma for the work that follows:

Lemma 4.1.2. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded convex domain. Then for every $\varepsilon > 0$, there exist two convex open subsets Ω_1 and Ω_2 in $\mathbb{R}^n$ such

[7] $C^{1,1}(\bar{\Omega})$ consists of all Lipschitz functions in $C^1(\bar{\Omega})$. Note that $C^{1,1}$ can be viewed as intermediate between C^1 and C^2.

that

1. $\Omega_1 \subset \Omega \subset \Omega_2$

2. Ω_i has a C^2 boundary $\partial\Omega_i, i = 1, 2$.

3. $d(\partial\Omega_1, \partial\Omega_2) \leq \varepsilon$,

where $d(\partial\Omega_1, \partial\Omega_2)$ denotes the distance from $\partial\Omega_1$ to $\partial\Omega_2$. For details and proof of this lemma the reader is referred to Grisvard (1985), p.147.

Lemma 4.1.2 allows us to approximate a general convex domain by domains with C^2 boundaries. Thus we can approximate our domain Ω in Theorem 4.1.1 by a sequence of bounded convex open subsets $\Omega_k, k = 1, 2, \cdots$ of $\mathbb{R}^n$ with C^2 boundaries such that $\Omega \subset \Omega_k$ and $d(\partial\Omega_k, \partial\Omega)$ tends to zero as $k \to \infty$. Then, we consider the solution $u_k \in H^2(\Omega_k)$ of the Robin problem in each Ω_k, i.e.

$$-\Delta u_k + u_k = f \qquad \text{in} \quad \Omega_k\,, \tag{4.1.4}$$

$$\frac{\partial u_k}{\partial n} + \beta u_k = 0 \qquad \text{on} \quad \partial\Omega_k. \tag{4.1.5}$$

Such a solution u_k exists by the following theorem:

Theorem 4.1.3. Let Ω_k be a bounded open subset of $\mathbb{R}^n$ with $C^{1,1}$ boundary and $\beta > 0$. Then for every $f \in L^2(\Omega_k)$, there exists a unique $u \in H^2(\Omega_k)$ which is a solution of the system (4.1.4)-(4.1.5).

Proof. See Grisvard (1985), pp. 125-126.

Subsection 4.1.2: A Priori Estimate

Here, we shall establish the following theorem which plays an important role in this section.

Theorem 4.1.4. (A priori estimate) Let Ω_k be a convex, bounded open subset of $\mathbb{R}^n$ with a C^2 boundary $\partial\Omega_k$, and let β be a positive constant. Then we have

$$\|u\|_{2,\Omega_k} \leq \sqrt{6}\| - \Delta u + u\|_{0,\Omega_k} \qquad (4.1.6)$$

for all $u \in H^2(\Omega_k)$ such that $\partial u/\partial n + \beta u = 0$ on $\partial\Omega_k$. Here, $\|\cdot\|_{m,\Omega_k}$ denotes the norm associated with $H^m(\Omega_k)$.

Proof. We first try to obtain the estimate of u and the estimate of the first derivatives of u in the $H^2(\Omega_k)$-norm by integrating $(-\Delta u + u)u$ by parts. Therefore we have

$$\int_{\Omega_k} (-\Delta u + u)u dx = \sum_{i=1}^{n} \int_{\Omega_k} \left|\frac{\partial u}{\partial x_i}\right|^2 dx + \int_{\Omega_k} |u|^2 dx + \beta \int_{\partial\Omega_k} |u|^2 d\sigma\,.$$

Since β is a positive constant, it follows that $\beta \int_{\partial\Omega_k} |u|^2 d\sigma \geq 0$ and applying the Cauchy-Schwarz inequality, we have

$$\sum_{i=1}^{n} \int_{\Omega_k} \left| \frac{\partial u}{\partial x_i} \right|^2 dx + \int_{\Omega_k} |u|^2 dx \leq \| -\Delta u + u\|_{0,\Omega_k} \|u\|_{0,\Omega_k} .$$

Thus, it follows from this inequality that

$$\int_{\Omega_k} |u|^2 dx \leq \| -\Delta u + u\|_{0,\Omega_k} \|u\|_{0,\Omega_k} , \tag{4.1.7}$$

and

$$\sum_{i=1}^{n} \int_{\Omega_k} \left| \frac{\partial u}{\partial x_i} \right|^2 dx \leq \| -\Delta u + u\|_{0,\Omega_k} \|u\|_{0,\Omega_k} . \tag{4.1.8}$$

Inequality (4.1.7) can be simplified to

$$\|u\|_{0,\Omega_k} \leq \| -\Delta u + u\|_{0,\Omega_k} . \tag{4.1.9}$$

We now use this inequality in order to simplify inequality (4.1.8). Hence, we obtain

$$\sum_{i=1}^{n} \int_{\Omega_k} \left| \frac{\partial u}{\partial x_i} \right|^2 dx \leq \| -\Delta u + u\|_{0,\Omega_k}^2 . \tag{4.1.10}$$

We now estimate the second derivatives of u in the $H^2(\Omega_k)$-norm. We apply the identity in Theorem A.0.19 to $\mathbf{v} = \nabla u$. We observe that

$$\Delta u = \operatorname{div} \mathbf{v} \qquad \text{in} \quad \Omega_k , \tag{4.1.11}$$

and that

$$-\frac{\partial u}{\partial n} = -\mathbf{v}\cdot n = \beta u \quad \text{on} \quad \partial\Omega_k\,. \tag{4.1.12}$$

We also have $\mathbf{v}_T = \nabla_T u := \nabla u + \dfrac{\partial u}{\partial n}\cdot n$ which is the projection of the gradient operator on the tangent hyperplane (see Theorem A.0.19). Thus we get

$$\int_{\Omega_k} |\Delta u|^2 dx - \sum_{i,j=1}^{n} \int_{\Omega_k} \left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right|^2 dx \;=\; 2\beta\langle\nabla_T u, \nabla_T u\rangle$$

$$-\int_{\partial\Omega_k} \left\{B(\nabla_T u, \nabla_T u) + (\mathrm{tr}B)(\nabla u)^2\right\} d\sigma\,, \tag{4.1.13}$$

where B is the bilinear form and tr$B$ is the trace of the bilinear form B described in Theorem A.0.19. Since Ω_k is assumed to be convex, then B is nonpositive (see the note at the end of Theorem A.0.19), we thus obtain that

$$\int_{\Omega_k} |\Delta u|^2 dx - \sum_{i,j=1}^{n} \int_{\Omega_k} \left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right|^2 dx \geq 2\beta\langle\nabla_T u, \nabla_T u\rangle\,. \tag{4.1.14}$$

Thus

$$\sum_{i,j=1}^{n} \int_{\Omega_k} |\frac{\partial^2 u}{\partial x_i \partial x_j}|^2 dx \leq \int_{\Omega_k} |\Delta u|^2 dx - 2\beta\langle\nabla_T u, \nabla_T u\rangle \tag{4.1.15}$$

For the second term in the right-hand side of (4.1.15), we can rewrite the bracket as an integral, i.e. $2\beta\langle\nabla_T u, \nabla_T u\rangle = 2\beta\displaystyle\int_{\partial\Omega_k} |\nabla_T u|^2 d\sigma$. This term is non-negative since β is positive constant. Therefore we have

$$\sum_{i,j=1}^{n} \int_{\Omega_k} \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|^2 dx \leq \int_{\Omega_k} |\Delta u|^2 dx = \|\Delta u\|_{0,\Omega_k}^2 = \|\Delta u - u + u\|_{0,\Omega_k}^2$$

$$\leq (\| - \Delta u + u\|_{0,\Omega_k} + \|u\|_{0,\Omega_k})^2$$

$$\leq 2\| - \Delta u + u\|_{0,\Omega_k}^2 + 2\|u\|_{0,\Omega_k}^2 .$$

Hence, applying inequality (4.1.9), we obtain

$$\sum_{i,j=1}^{n} \int_{\Omega_k} \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|^2 dx \leq 2\| - \Delta u + u\|_{0,\Omega_k}^2 + 2\| - \Delta u + u\|_{0,\Omega_k}^2 .$$

Thus

$$\sum_{i,j=1}^{n} \int_{\Omega_k} \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|^2 dx \leq 4\| - \Delta u + u\|_{0,\Omega_k}^2. \tag{4.1.16}$$

Thus adding up inequalities (4.1.9), (4.1.10), and (4.1.16) gives

$$\|u\|_{2,\Omega_k} \leq (4 + 1 + 1)^{1/2} \| - \Delta u + u\|_{0,\Omega_k} ,$$

as desired. □

Note that $\{u_k\}$ is a bounded sequence in $H^2(\Omega_k)$. Moreover, we can restrict the u_k to our domain Ω in Theorem 4.1.1 and obtain a bounded sequence in $H^2(\Omega)$. Thus using the **Alaoglu** compactness theorem (Theorem A.0.12) we can extract a subsequence u_k that converges weakly to some $u \in H^2(\Omega)$, i.e.

$$u_k|_\Omega \rightharpoonup u \quad \text{in} \quad H^2(\Omega), \quad \text{as} \quad k \to \infty\,, \tag{4.1.17}$$

this means that

$$\int_\Omega \nabla u_k \cdot \nabla v dx \longrightarrow \int_\Omega \nabla u \cdot \nabla v dx\,. \tag{4.1.18}$$

Subsection 4.1.3: Passage to Limit

First let $V \in H^1(\mathbb{R}^n)$. Since Ω is a bounded convex open subset of $\mathbb{R}^n$ with Lipschitz continuous boundary $\partial\Omega$, we can therefore set $v = V|_\Omega \in H^1(\Omega)$ in the weak form of (4.1.4)-(4.1.5). It is obvious that $V|_{\Omega_k} \in H^1(\Omega_k)$ (see Theorem A.0.10). We observe that equations (4.1.4) and (4.1.5) imply

$$\int_{\Omega_k} \nabla u_k \cdot \nabla V dx + \int_{\Omega_k} u_k V dx + \beta \int_{\partial\Omega_k} u_k V d\sigma_k = \int_{\Omega_k} f V dx\,. \tag{4.1.19}$$

We now consider the limit of equation (4.1.19) when $k \to \infty$. We first want to show that

$$\int_{\Omega_k} \nabla u_k \cdot \nabla V dx \longrightarrow \int_\Omega \nabla u \cdot \nabla v dx\,, \tag{4.1.20}$$

and that

$$\int_{\Omega_k} u_k V dx \longrightarrow \int_\Omega u v dx\,. \tag{4.1.21}$$

Considering (4.1.20), we have

$$\int_{\Omega_k} \nabla u_k \cdot \nabla V dx - \int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Omega_k \setminus \Omega} \nabla u_k \cdot \nabla V dx + \int_{\Omega} \nabla (u_k - u) \cdot \nabla v dx \,,$$

recalling that $\Omega \subset \Omega_k$ and $v = V|_{\Omega}$. Consequently

$$\left| \int_{\Omega_k} \nabla u_k \cdot \nabla V dx - \int_{\Omega} \nabla u \cdot \nabla v dx \right|$$

$$\leq \|\nabla u_k\|_{0,\Omega_k} \left(\int_{\Omega_k \setminus \Omega} (\nabla V)^2 dx \right)^{1/2} + \|\nabla (u_k - u)\|_{0,\Omega} \|\nabla v\|_{0,\Omega} \,.$$

The right-hand side of this inequality converges to zero due to inequality (4.1.6), (4.1.17), (4.1.18) and the compactness of the injection of $H^2(\Omega)$ in $L^2(\Omega)$. Similarly, we prove that (4.1.21) is satisfied due to inequality (4.1.6), (4.1.17), and the compactness of the injection of $H^2(\Omega)$ in $L^2(\Omega)$.

To complete the proof, we need to show that

$$\int_{\partial\Omega_k} u_k V d\sigma_k \longrightarrow \int_{\partial\Omega} u v d\sigma \,. \tag{4.1.22}$$

Considering (4.1.22), we have

$$\int_{\partial\Omega_k} u_k V d\sigma_k - \int_{\partial\Omega} u v d\sigma = \int_{\partial\Omega_k} (u_k - u) V d\sigma_k + \Big(\int_{\partial\Omega_k} u V d\sigma_k - \int_{\partial\Omega} u V d\sigma \Big) \,.$$

We shall now show that

$$\int_{\partial\Omega_k} (u_k - u) V d\sigma_k \longrightarrow 0\,, \tag{4.1.23}$$

and that

$$\int_{\partial\Omega_k} uV d\sigma_k \longrightarrow \int_{\partial\Omega} uV d\sigma\,. \tag{4.1.24}$$

According to Theorem A.0.10, we have, in (4.1.23),

$$u_k - u \in H^1(\Omega_k)\,,$$

since $\Omega \subset \Omega_k \subset \mathbb{R}^n$ is bounded convex domain. Thus recalling Theorem 2.2.1 in Chapter 2, we have

$$\int_{\partial\Omega_k} |u_k - u|^2 d\sigma_k \leq \|u_k - u\|_{0,\Omega_k} \|u_k - u\|_{1,\Omega_k}\,,$$

with setting $V|_{\Omega_k} = u_k - u \in H^1(\Omega_k)$. The right-hand side of this inequality tends to zero due to inequality (4.1.6), (4.1.17), and the compactness injection of $H^2(\Omega)$ in $L^2(\Omega)$. Now, the integrand in the left-hand side of (4.1.24), i.e. $\int_{\partial\Omega_k} uV d\sigma_k$, tends to zero due to the same argument above. Therefore, recalling Lebesgue's dominated convergence theorem, it follows that

$$\int_{\partial\Omega_k} uV d\sigma_k \longrightarrow \int_{\partial\Omega} uV d\sigma.$$

Now (4.1.22) follows from (4.1.23) and (4.1.24). This completes the proof of Theorem 4.1.1. $\square$

Section 4.2: Strong solutions of the problem (P)

In this section we show further regularity of solutions of the weak form (3.2.2). In particular, we prove more estimates which lead to results for the strong solutions.

For the following Corollaries and later use, we recall the well-known Sobolev interpolation results (a **Gagliardo-Nirenberg inequality**), e.g. see Adams and Fournier (1977): let $p \in [1, \infty]$, $m \geq 1$, $r \in [p, \infty]$ for $m - \frac{d}{p} \geq 0$, $r \in [p, -\frac{d}{m-(d/p)}]$ for $m - \frac{d}{p} < 0$, and $\mu := \frac{d}{m}(\frac{1}{p} - \frac{1}{r})$. Then there is a constant C depending only on $\Omega \subset \mathbb{R}, p, r$ and m such that

$$\|v\|_{0,r} \leq C \|v\|_{0,p}^{1-\mu} \|v\|_{m,p}^{\mu} \quad \forall v \in W^{m,p}(\Omega)\,.$$

Corollary 4.2.1 Let $v \in H^1(\Omega)$, $s \in \mathbb{R}$, $s \geq 1$ for $d = 1, 2$, and $s \in [1, 3]$ for $d = 3$. Then

$$\|v\|_{0,2s} \leq C \|v\|_0^{1-\mu} \|v\|_1^{\mu}\,.$$

Proof. We use the Gagliardo-Nirenberg inequality with $r = 2s, m = 1$, and $p = 2$ to yield

$$\begin{aligned} \|v\|_{0,2s} &\le C\|v\|_{0,2}^{1-\mu}\|v\|_{1,2}^{\mu} \\ &= C\|v\|_0^{1-\mu}\|v\|_1^{\mu}. \qquad \square \end{aligned}$$

Corollary 4.2.2 Let $v \in H^1(\Omega)$, $s \in \mathbb{R}$, $s \ge 1$ for $d = 1, 2$, and $s \in [1, 2]$ for $d = 3$. Then

$$\|v\|_{0,4s-2} \le C\|v\|_0^{1-\mu}\|v\|_1^{\mu}.$$

Proof. The result follows from Gagliardo-Nirenberg inequality with $r = 4s - 2, m = 1, p = 2$. $\square$

To prove existence of strong solutions we assume:

(S)
$$s \le \begin{cases} 5 & \text{if } d = 1, \\ 3 & \text{if } d = 2, \\ \frac{7}{3} & \text{if } d = 3. \end{cases}$$

We now state the main theorem of this chapter:

Theorem 4.2.3. Let $\Omega \subset \mathbb{R}^d$ be an open bounded convex domain. Let the assumptions (1.1.3) and **(S)** hold, $u_0 \in V$, and $\beta > 0$, then the reaction-diffusion system (3.1.1)-(3.1.3) possesses a unique strong solution u satisfying

$$u \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;V)\,,$$

$$\frac{\partial u}{\partial t} \in L^2(\Omega_T)\,.$$

Furthermore, the map

$$u_0(\cdot) \longmapsto u(\cdot,t)\,,$$

is continuous on V.

Proof. We separate the proof into two parts: existence, and uniqueness and continuity of the strong solution u.

Subsection 4.2.1: Existence

We shall first make a further estimate on du^k/dt, where we have abused notation and no confusing arises. We take the inner product of the ordinary differential system (3.2.12) with du^k/dt, so that

$$\left(\frac{\partial u^k}{\partial t}, \frac{\partial u^k}{\partial t}\right) - \left(\Delta u^k, \frac{\partial u^k}{\partial t}\right) + \left(P^k g(u^k), \frac{\partial u^k}{\partial t}\right) = 0\,. \tag{4.2.1}$$

Note that

$$P^k g(u^k) \cdot \frac{\partial u^k}{\partial t} = \frac{\partial}{\partial t} P^k G(u^k)\,,$$

where $G(u^k) = \sum_{j=0}^{2s-1} \frac{b_j}{j+1}(u^k)^{j+1}$. Thus equation (4.2.1) can be written as follows, after multiplying through by 2 and recalling the passage to limit argument (see Section 3.3),

$$2\int_\Omega |u_t^k|^2 dx + \frac{d}{dt}\int_\Omega |\nabla u^k|^2 dx + \beta \frac{d}{dt}\int_{\partial\Omega} |u^k|^2 d\sigma + 2\frac{d}{dt}\int_\Omega G(u^k)dx = 0. \tag{4.2.2}$$

Since

$$\left|\sum_{j=0}^{2s-2} \frac{b_j}{j+1}(u^k)^{j+1}\right| \le \frac{b_{2s-1}}{2s}|u^k|^{2s} + C, \quad C > 0,$$

which implies

$$\frac{b_{2s-1}}{2s}|u^k|^{2s} - C \le G(u^k) \le \frac{3b_{2s-1}}{2s}|u^k|^{2s} + C\,.$$

Thus, using this inequality, the equation (4.2.2) can be written as follows

$$2\int_\Omega |u_t^k|^2 dx + \frac{d}{dt}\int_\Omega |\nabla u^k|^2 dx + \beta\frac{d}{dt}\int_{\partial\Omega} |u^k|^2 d\sigma + \frac{b_{2s-1}}{s}\frac{d}{dt}\int_\Omega |u^k|^{2s} dx \le 2C|\Omega|\,, \tag{4.2.3}$$

where $C > 0$. Integrating both sides of (4.2.3) from 0 to t gives

$$\begin{aligned} 2\int_0^t \|u_t^k\|_0^2\, ds + |u^k(t)|_1^2 + \beta\|u^k(t)\|^2_{L^2(\partial\Omega)} + \frac{b_{2s-1}}{s}\|u^k(t)\|^{2s}_{0,2s} \\ \le 2C|\Omega|t + |u_0^k|_1^2 + \beta\|u_0^k\|^2_{L^2(\partial\Omega)} + \frac{b_{2s-1}}{s}\|u_0^k\|^{2s}_{0,2s} \\ \le 2C|\Omega|t + |u_0^k|_1^2 + \beta C\|u_0^k\|_1^2 + \frac{b_{2s-1}}{s}\|u_0^k\|_1^{2s}\,, \end{aligned} \tag{4.2.4}$$

after using the trace embedding theorems (Theorem A.0.9) and Corollary 4.2.1. Recall that $u_0 \in V$, so the boundedness of the terms $|u_0^k|_1^2 \equiv |P^k u_0|_1^2$, $\|u_0^k\|_1^2 \equiv \|P^k u_0\|_1^2$, and $\|u_0^k\|_1^{2s} \equiv \|P^k u_0\|_1^{2s}$ follows from Lemma 3.2.2 and Lemma 3.2.3. Then we have

$$u^k \quad \text{is uniformly bounded in} \quad L^\infty(0,T;V)\,. \tag{4.2.5}$$

Moreover, inequalities (4.2.4) and (4.2.5) lead to

$$\frac{\partial u^k}{\partial t} \quad \text{is uniformly bounded in} \quad L^2(\Omega_T)\,. \tag{4.2.6}$$

We now make further estimates by taking the inner product of the ordinary differential system (3.2.12) with $-\Delta u^k$ to show that $u^k \in H^2(\Omega)$ for a.e. $t \in (0,t)$. This gives

$$\left(\frac{\partial u^k}{\partial t}, -\Delta u^k\right) + (\Delta u^k, \Delta u^k) - (P^k g(u^k), \Delta u^k) = 0\,, \tag{4.2.7}$$

and hence

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u^k|^2 dx + \frac{\beta}{2}\frac{d}{dt}\int_{\partial\Omega} |u^k|^2 d\sigma + \int_\Omega |\Delta u^k|^2 dx = \int_\Omega P^k g(u^k)\cdot \Delta u^k dx\,. \tag{4.2.8}$$

We now deal with the term on the right-hand side of this equation separately. From the passage to the limit argument (see Section 3.3), the Cauchy-Schwarz inequality, the so-called Young's inequality with $\varepsilon = 1$, and Corollary 4.2.2

we have

$$\int_\Omega P^k g(u^k)\cdot \Delta u^k dx = \int_\Omega g(u^k)\cdot \Delta u^k dx$$

$$\le \|g(u^k)\|_0 \|\Delta u^k\|_0$$

$$\le \frac{1}{2}\|g(u^k)\|_0^2 + \frac{1}{2}\|\Delta u^k\|_0^2$$

$$\le \frac{C}{2}\|u^k\|_{0,4s-2}^{2s-1} + \frac{1}{2}\|\Delta u^k\|_0^2 + C|\Omega|$$

$$\le C\|u^k\|_0^{(2s-1)(1-\mu)}\|u^k\|_1^{(2s-1)\mu} + \frac{1}{2}\|\Delta u^k\|_0^2 + C|\Omega|$$

$$= C\|u^k\|_0^{2s-3}\|u^k\|_0^{2((\frac{1}{2}-s)\mu+1)}\|u^k\|_1^{2(s-\frac{1}{2})\mu} + \frac{1}{2}\|\Delta u^k\|_0^2 + C|\Omega|\,.$$

Applying the Young's inequality (3.1.4) on the right-hand side of the above inequality (with $m = ((\frac{1}{2}-s)\mu+1)^{-1}, n = ((s-\frac{1}{2})\mu)^{-1} > 1$, which is easy to check from assumption **(S)**) gives:

$$\int_\Omega P^k g(u^k)\cdot \Delta u^k dx$$

$$\le C_2(s,\mu,\varepsilon)\|u^k\|_0^\gamma\|u^k\|_0^2 + C_1(s,\mu,\varepsilon)\|u^k\|_1^2 + \frac{1}{2}\|\Delta u^k\|_0^2 + C|\Omega|$$

$$= C_2(s,\mu,\varepsilon)\|u^k\|_0^{\gamma+2} + C_1(s,\mu,\varepsilon)\|u^k\|_0^2 + C_1(s,\mu,\varepsilon)|u^k|_1^2 + \frac{1}{2}\|\Delta u^k\|_0^2 + C|\Omega|\,, \tag{4.2.9}$$

where $\gamma = \frac{2s-3}{(\frac{1}{2}-s)\mu+1}$. Thus substituting (4.2.9) in equation (4.2.8) yields, after multiplying through by 2,

$$\frac{d}{dt}\left[\int_\Omega |\nabla u^k|^2 dx + \beta \int_{\partial\Omega} |u^k|^2 d\sigma\right] + \|\Delta u^k\|_0^2 \leq C_1(s,\mu,\varepsilon)\left[|u^k|_1^2 + \beta\int_{\partial\Omega}|u^k|^2 d\sigma\right]$$

$$+C_2(s,\mu,\varepsilon)\|u^k\|_0^{\gamma+2} + C_1(s,\mu,\varepsilon)\|u^k\|_0^2 + 2C|\Omega|\,, \tag{4.2.10}$$

where the additional term $\beta\displaystyle\int_{\partial\Omega}|u^k|^2 d\sigma$ is non-negative. Applying the usual Grönwall lemma to this inequality yields

$$|u^k(t)|_1^2 + \beta\|u^k(t)\|_{L^2(\partial\Omega)}^2 + \int_0^t \|\Delta u^k\|_0^2 ds \leq \exp\left(C_1(s,\mu,\varepsilon)t\right)\left(|u^k(0)|_1^2 + \beta\|u^k(0)\|_{L^2(\partial\Omega)}^2\right)$$

$$+\exp\left(C_1(s,\mu,\varepsilon)t\right)\left(C_2(s,\mu,\varepsilon)\|u^k\|_{L^{\gamma+2}(0,T;L^2(\Omega))}^{\gamma+2} + C_1(s,\mu,\varepsilon)\|u^k\|_{L^2(\Omega_T)}^2 + 2C|\Omega|t\right)$$

$$\leq \exp\left(C_1(s,\mu,\varepsilon)t\right)\left(|u^k(0)|_1^2 + \beta C\|u^k(0)\|_1^2\right)$$

$$+\exp\left(C_1(s,\mu,\varepsilon)t\right)\left(C_2(s,\mu,\varepsilon)C\|u^k\|_{L^\infty(0,T;L^2(\Omega))}^{\gamma+2} + C_1(s,\mu,\varepsilon)\|u^k\|_{L^2(\Omega_T)}^2 + 2C|\Omega|t\right). \tag{4.2.11}$$

Recall that $u_0 \in V$, so the boundedness of the terms $|u_0^k|_1^2 \equiv |P^k u_0|_1^2$ and $\|u_0^k\|_1^2 \equiv \|P^k u_0\|_1^2$ follows from Lemma 3.2.2 and Lemma 3.2.3, respectively. The third and fourth terms on the right-hand side of (4.2.11) are bounded due to Theorem 3.1.1 . Thus, as $u^k(\cdot,t), \Delta u^k(\cdot,t) \in L^2(\Omega)$ for a.e. $t \in (0,T)$, it follows from Section 4.1 (Theorem 4.1.1) that $u^k(\cdot,t) \in H^2(\Omega)$ for a.e.

$t \in (0, t)$, and hence

$$u^k \quad \text{is uniformly bounded in} \quad L^2(0,T;H^2(\Omega))\,. \tag{4.2.12}$$

As in Chapter 3, we use the **Alaoglu** compactness theorem to extract the appropriate subsequences from (4.2.5), (4.2.6), and (4.2.12). Thus we get

$$u \in L^2(0,T;H^2(\Omega)), \quad u \in L^\infty(0,T;V), \quad \frac{\partial u}{\partial t} \in L^2(\Omega_T) \quad .$$

Subsection 4.2.2: Continuity and Uniqueness

In order to show that $u \in C([0,T];V)$, we actually need the following lemma:

Lemma 4.2.4. For some $k \geq 0$, suppose that

$$u \in L^2(0,T;H^{k+1}(\Omega)) \quad \text{and} \quad \frac{du}{dt} \in L^2(0,T;H^{k-1}(\Omega))\,.$$

Then u is continuous from $[0,T]$ into $H^k(\Omega)$, i.e. $u \in C([0,T];H^k(\Omega))$.

Proof. See J. Robinson (2001), pages 191-194.

Note that we have proved in Chapter 3 a similar result to the one above (see Lemma 3.4.1). Here, in our case $k = 1$,

$$H^{k+1}(\Omega) = H^2(\Omega), \quad H^k(\Omega) = V, \quad H^{k-1}(\Omega) = L^2(\Omega) \quad ,$$

so the application of this lemma gives the desired result.

Finally, we prove the unique dependence of a solution of problem **(P)** on the initial data in V. We suppose that there are two solutions u_1 and u_2 of the weak form (3.2.2) with initial conditions $u_1(0), u_2(0) \in V$ respectively. Then, letting $v = -\Delta w := -\Delta(u_1 - u_2)$, we obtain

$$\left(\frac{\partial w}{\partial t}, -\Delta w\right) + (\nabla w, \nabla(-\Delta w)) - \beta \int_{\partial\Omega} w \cdot \Delta w d\sigma = (g(u_1) - g(u_2), \Delta w), \tag{4.2.13}$$

and hence

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla w|^2 dx + \frac{\beta}{2}\frac{d}{dt}\int_{\partial\Omega} |w|^2 d\sigma + \int_\Omega |\Delta w|^2 dx \le \|g(u_1) - g(u_2)\|_0 \|\Delta w\|_0 . \tag{4.2.14}$$

Thus, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla w|^2 dx + \frac{\beta}{2}\frac{d}{dt}\int_{\partial\Omega} |w|^2 d\sigma + \|\Delta w\|_0^2 \le \frac{1}{2}\|g(u_1) - g(u_2)\|_0^2 + \frac{1}{2}\|\Delta w\|_0^2 , \tag{4.2.15}$$

after recalling the so-called Young's inequality with $\varepsilon = 1 > 0$. Now, recalling inequality (3.2.16) and multiplying through by 2 we get

$$\frac{d}{dt}\int_\Omega |\nabla w|^2 dx + \beta\frac{d}{dt}\int_{\partial\Omega} |w|^2 d\sigma + \|\Delta w\|_0^2 \le C(t)^2 \|u_1 - u_2\|_0^2 , \tag{4.2.16}$$

where $C(t)$ is a positive Lipschitz constant of the function g (see (3.4.5)). In particular

$$\frac{d}{dt}\int_\Omega |\nabla w|^2 dx + \beta\frac{d}{dt}\int_{\partial\Omega} |w|^2 d\sigma \le C(t)^2 \|w\|_0^2 . \tag{4.2.17}$$

Thus combining inequalities (3.4.6) and (4.2.17), using the Sobolev embedding theorems and adding a non-negative term $\beta \int_{\partial\Omega} |w|^2 d\sigma$ on the right-hand side gives

$$\frac{d}{dt}\left[\int_\Omega (|w|^2 + |\nabla w|^2)dx + \beta \int_{\partial\Omega} |w|^2 d\sigma\right] \leq (2C(t)+C(t)^2)\left[C\|w\|_1^2 + \beta \int_{\partial\Omega} |w|^2 d\sigma\right]. \tag{4.2.18}$$

Applying the usual Grönwall lemma to this inequality yields

$$\|w(t)\|_1^2 + \beta\|w(t)\|_{L^2(\partial\Omega)}^2 \leq \exp\left(\int_0^t (2C(s) + C(s)^2)ds\right)\left(C\|w(0)\|_1^2 + \beta\|w(0)\|_{L^2(\partial\Omega)}^2\right). \tag{4.2.19}$$

By omitting the last term on the left-hand side of inequality (4.2.19) and using the trace embedding theorems ($\|w(0)\|_{L^2(\partial\Omega)}^2 \leq C\|w(0)\|_1^2$), we thus have

$$\|w(t)\|_1^2 \leq C(1+\beta)\|w(0)\|_1^2 \exp\left(\int_0^t (2C(s) + C(s)^2)ds\right). \tag{4.2.20}$$

Thus if $u_1(0) = u_2(0)$ then $w(0) = 0$ and hence it follows from (4.2.20) that $w(t) = 0$ and hence $u_1(t) = u_2(t)$ for all t. However, if $u_1(0) \neq u_2(0)$, then we have continuous dependence in $V = H^1(\Omega)$. This completes the proof of Theorem 4.2.3. $\square$

Chapter 5

A Semi-discrete Approximation

This chapter is divided into three sections. In the first section, Section 5.1, we introduce the finite element method and give the necessary assumptions on the partitioning of Ω that are required for the numerical analysis. We also give some definitions, properties, and associated spaces that are necessary tools for this chapter and Chapter 6. In Section 5.2 we discretise the weak form (3.2.2) in the finite element space. The existence and uniqueness of the semi-discrete finite element approximations are proven. Finally, in Section 5.3, we estimate the difference between the continuous and semi-discrete solutions.

Section 5.1: Notation and Preliminaries

We shall consider the finite element approximation of the weak form (3.2.2) under the following assumptions on the mesh:

(A) Let $\Omega \subset \mathbb{R}^d$, $d \leq 3$, be a convex (connected) domain if $d = 1$, a convex polygonal domain if $d = 2$ and a convex polyhedral domain if $d = 3$. Let $\mathcal{T}^h$ be a quasi-uniform partitioning of Ω into disjoint open simplices $\{\kappa\}$[8] with $h_\kappa := \text{diam}(\kappa)$ and $h := \max_{\kappa \in \mathcal{T}^h} h_\kappa$, so that $\overline{\Omega} = \bigcup_{\kappa \in \mathcal{T}^h} \overline{\kappa}$. In addition, it is assumed that $\mathcal{T}^h$ is a (weakly) acute partitioning (see, e.g., Barrett and Blowey (2001), p.257); that is for (a) $d = 2$ the sum of opposite angles relative to any side does not exceed π. (b) $d = 3$ the angle between any two faces of the same tetrahedron does not exceed $\pi/2$.

We now introduce the finite element space of piecewise linear basis functions associated with $\mathcal{T}^h$:

$$S^h := \left\{\chi \in C(\overline{\Omega}) \,:\, \chi|_\kappa \text{ is linear } \;\forall \kappa \in \mathcal{T}^h\right\} \subset H^1(\Omega).$$

Let $\{\varphi_j\}_{j=0}^J$ be a basis for S^h, satisfying $\varphi_j(x_i) = \delta_{ij}$, where $\{x_i\}_{i=0}^J$ is the set of nodes of $\mathcal{T}^h$.

Let $\pi^h : C(\overline{\Omega}) \to S^h$ be the interpolation operator such that $\pi^h\chi(x_j) = \chi(x_j)$ for all $j = 0, \cdots, J$. We consider a discrete L^2 inner product on $C(\overline{\Omega})$, defined by

$$(\chi_1, \chi_2)^h := \int_\Omega \pi^h\{\chi_1(x)\,\chi_2(x)\}\,dx = \sum_{j=0}^J M_j\,\chi_1(x_j)\,\chi_2(x_j), \tag{5.1.1}$$

[8]We recall that a simplex κ is a triangle if $d = 2$ and a tetrahedron if $d = 3$.

where $M_j = (\varphi_j, \varphi_j)^h = (1, \varphi_j) > 0$, which is called the lumped mass matrix. Note that the matrix M is diagonal with positive entries (see Strang and Fix (1973), Section 4.2). From the definition of the interpolation operator and (5.1.1) we can show that

$$(\pi^h \chi, \eta)^h = (\chi, \eta)^h \qquad \forall \chi, \eta \in C(\overline{\Omega}) . \tag{5.1.2}$$

Below we recall some well-known results concerning S^h:

The discrete inner product (5.1.1) induces a norm on S^h such that

$$|\chi|_h := \left[(\chi, \chi)^h\right]^{1/2} \qquad \forall \chi \in S^h. \tag{5.1.3}$$

The norm $|\cdot|_h$ is equivalent to $\|\cdot\|_0$, namely,

$$C_1 \|\chi\|_0 \leq |\chi|_h \leq C_2 \|\chi\|_0 \quad \forall \chi \in S^h, \tag{5.1.4}$$

(see, e.g., Raviart (1973)).

We also need the following estimate that helps us to bound the error between continuous solutions and their semi-discrete approximations (see Garvie (2003), Lemma 4.2.7):

$$|(\chi, \eta) - (\chi, \eta)^h| \leq C h^{m+1} \|\chi\|_m \, \|\eta\|_1 \quad \forall \chi, \eta \in S^h,\ m = 0 \text{ or } 1. \tag{5.1.5}$$

We require the well-known interpolation error in $H^2(\Omega)$ (see Theorem 3.1.6 in Ciarlet (1978)):

$$\|(I-\pi^h)\chi\|_0 \le Ch^2|\chi|_2 \qquad \forall \chi \in H^2(\Omega), \tag{5.1.6}$$

$$|(I-\pi^h)\chi|_1 \le Ch|\chi|_2 \qquad \forall \chi \in H^2(\Omega). \tag{5.1.7}$$

For later use, we recall the following inverse inequalities for all $\chi \in S^h$ (see Theorem 3.2.6 in Ciarlet (1978)):

$$\|\chi\|_{0,q} \le Ch^{\frac{d(p-q)}{pq}}\|\chi\|_{0,p}\,, \quad 1 \le p \le q \le \infty, \tag{5.1.8}$$

$$|\chi|_{1,q} \le Ch^{\frac{d(p-q)}{pq}}|\chi|_{1,p}\,, \quad 1 \le p \le q \le \infty, \tag{5.1.9}$$

$$h|\chi|_1 \le C|\chi|_h\,, \tag{5.1.10}$$

and the Sobolev embedding for $d = 1, 2$:

$$\|\chi\|_{0,\infty} \le C\left(\ln\frac{1}{h}\right)^{\frac{(d-1)}{2}}|\chi|_1 \quad \forall \chi \in S^h\,. \tag{5.1.11}$$

We will frequently use the simple Young's inequality that there exists a positive constant C depending only on ε and c such that

$$cab \le Ca^2 + \frac{1}{2\varepsilon}b^2 \qquad \forall a, b, c, \varepsilon > 0. \tag{5.1.12}$$

We now consider a generalization of the norm (5.1.3) on S^h, defined by

$$|\chi^h|_{h,p} := \left(\int_\Omega \pi^h \{ |\chi^h(x)|^p \} dx \right)^{1/p} = \left(\sum_{j=0}^{J} M_j |\chi^h(x_j)|^p \right)^{1/p}, \qquad (5.1.13)$$

for all $\chi^h \in S^h$, $1 \leq p < \infty$. For $p = \infty$, we have

$$|\chi^h|_{h,\infty} := \max_{0 \leq j \leq J} |\chi^h(x_j)| \qquad \forall \chi^h \in S^h. \qquad (5.1.14)$$

We shall consider the following *discrete Minkowski* and *discrete Hölder* inequalities respectively for all $\chi^h, \eta^h \in C(\overline{\Omega})$, and hence for all $\chi^h, \eta^h \in S^h$:

$$|\chi^h + \eta^h|_{h,p} \leq |\chi^h|_{h,p} + |\eta^h|_{h,p}, \quad 1 \leq p < \infty, \qquad (5.1.15)$$

$$|(\chi^h, \eta^h)^h| \leq |\chi^h|_{h,p} |\eta^h|_{h,q}, \quad \frac{1}{p} + \frac{1}{q} = 1 \leq p, q \leq \infty, \qquad (5.1.16)$$

(see Blowey and Garvie (2005), p.626). We also have that $|\chi^h|_{h,p}$ and $|\chi^h|_{h,\infty}$, given by (5.1.13) and (5.1.14) respectively, are norms on S^h.

Now, consider the following Banach spaces

$$L^{h,p}(\Omega) := \left\{ u^h \in S^h \ : \ |u^h|_{h,p} < \infty, \, 1 \leq p \leq \infty \right\}. \qquad (5.1.17)$$

Note that we have embedding results for elements in S^h:

$$|u^h|_{h,q} \leq C |u^h|_{h,p} \qquad \forall u^h \in L^{h,p}(\Omega), \qquad (5.1.18)$$

where $C = |\Omega|^{\frac{p-q}{pq}}$ and $1 \le q \le p \le \infty$. We extend the spaces $L^{h,p}(\Omega)$, given by (5.1.17), to time-dependent finite dimensional spaces $L^{h,p}(\Omega_T)$ with norm

$$|u^h|_{h,p;T} := \left(\int_0^T |u^h(\cdot, t)|_{h,p}^p \, dt \right)^{1/p}, \tag{5.1.19}$$

for all elements in S^h and $1 \le p \le \infty$. The space $L^{h,p}(\Omega_T)$ with the norm (5.1.19) leads to the following embedding results for elements in S^h:

$$|u^h|_{h,q;T} \le C|u^h|_{h,p;T} \qquad \forall\, u^h \in L^{h,p}(\Omega_T)\,, \tag{5.1.20}$$

where $C = (|\Omega|T)^{\frac{p-q}{pq}}$ and $1 \le q \le p \le \infty$.

We require the following lemma:

Lemma 5.1.1. Let $v^h \in S^h$, $r \in \mathbb{R}$, $r \ge 2$ for $d = 1, 2, 3$. Then

$$\|(I - \pi^h)(v^h)^r\|_{0,1} \le \begin{cases} Ch^2|v^h|_1^r & \text{for } d = 1, \\ Ch^2\big(\ln\frac{1}{h}\big)^{\frac{r-2}{r}}|v^h|_1^r & \text{for } d = 2, \\ Ch^{3-\frac{r}{2}}\|v^h\|_1^r & \text{for } d = 3. \end{cases} \tag{5.1.21}$$

Proof. We refer to Imran (2001), pp.36-45.

We use the above lemma to deduce the discrete Sobolev embedding result that will be used in later sections:

Corollary 5.1.2. Let $v^h \in S^h$, $r \in \mathbb{R}$, $r \geq 2$ for $d = 1, 2$, and $r \in [2, 6]$ for $d = 3$. Then

$$|v^h|_{h,r}^r \leq C\|v^h\|_1^r . \tag{5.1.22}$$

Proof. We split $|v^h|_{h,r}^r$ via

$$|v^h|_{h,r}^r \leq \|(I - \pi^h)(v^h)^r\|_{0,1} + \|v^h\|_{0,r}^r .$$

Using (5.1.21) and the Gagliardo-Nirenberg inequality yields

$$|v^h|_{h,r}^r \leq \begin{cases} Ch^2|v^h|_1^r + C\|v^h\|_1^r & \text{for } d = 1, \\ Ch^2(\ln\frac{1}{h})^{\frac{r-2}{r}}|v^r|_1^r + C\|v^h\|_1^r & \text{for } d = 2, \\ (Ch^{3-\frac{r}{2}} + C)\|v^h\|_1^r & \text{for } d = 3, \end{cases} \tag{5.1.23}$$

$$\leq \begin{cases} C(h^2 + 1)\|v^h\|_1^r & \text{for } d = 1, \\ C(h^2(\ln\frac{1}{h})^{\frac{r-2}{r}} + 1)\|v^r\|_1^r & \text{for } d = 2, \\ C(h^{3-\frac{r}{2}} + 1)\|v^h\|_1^r & \text{for } d = 3, \end{cases} \tag{5.1.24}$$

which proves the corollary after noting that the Sobolev embedding theorem for $H^1(\Omega) \hookrightarrow L^r(\Omega)$. $\square$

Note that for proving the existence of solutions we assume $r = 2s$ with $s \geq 1$ for $d = 1, 2$ and $s \in [1, 3]$ for $d = 3$.

Finally, we define P^h to be the L^2 projection operator $P^h : L^2(\Omega) \to S^h$ given by

$$(P^h\eta, \chi^h) = (\eta, \chi^h) \qquad \forall \chi^h \in S^h \,. \tag{5.1.25}$$

This projection satisfies the following estimates (see, e.g., Barrett, Blowey and Garcke (1998)):

$$\|(I - P^h)\eta\|_0 + h|(I - P^h)\eta|_1 \leq Ch^m|\eta|_m \,, \tag{5.1.26}$$

for all $\eta \in H^m(\Omega), m = 1$ or 2.

We now give the following lemma that will be useful for bounding the initial semi-discrete approximations in $H^1(\Omega)$:

Lemma 5.1.3. Let $\eta \in H^1(\Omega)$. Then

$$\|P^h\eta\|_1 \leq C\|\eta\|_1 \,. \tag{5.1.27}$$

Proof. We split $\|P^h\eta\|_1$ via

$$\|P^h\eta\|_1^2 = \|P^h\eta\|_0^2 + |P^h\eta|_1^2 \,. \tag{5.1.28}$$

Due to the nature of projections, the first term on the right-hand side of equation (5.1.28) satisfies the following inequality ($\eta \in H^1(\Omega) \hookrightarrow L^2(\Omega)$):

$$\|P^h\eta\|_0 \le \|\eta\|_0 \le \|\eta\|_1 \,. \tag{5.1.29}$$

Using the estimate (5.1.26) we deduce that

$$\begin{aligned} |P^h\eta|_1 &\equiv |P^h\eta - \eta + \eta|_1 \\ &\le |(I - P^h)\eta|_1 + |\eta|_1 \\ &\le C|\eta|_1 + |\eta|_1 \\ &\le C\|\eta\|_1 \,, \end{aligned} \tag{5.1.30}$$

for all $\eta \in H^1(\Omega)$. Thus from inequalities (5.1.29) and (5.1.30) we obtain the desired result. □

Finally, we will need the following important results that will be required in Section 5.3:

$$\|(I - \pi^h)\chi\|_{0,1} \le Ch^2|\chi|_{2,1}, \qquad \forall \chi \in W^{2,1}(\Omega)\,, \tag{5.1.31}$$

for $d \leq 3$ (see Theorem 5 in Ciarlet and Raviart (1972)), and the equivalence inequality:

$$|\chi|_{2,p} \leq \left(\int_\Omega \sum_{i,j}^{d} \left| \frac{\partial^2 \chi}{\partial x_i \partial x_j} \right|^p dx \right)^{1/p} \leq 2^{1/p} |\chi|_{2,p} , \qquad (5.1.32)$$

for $1 \leq p < \infty$ (see Garvie (2003), p.18).

Section 5.2: Existence and Uniqueness

We introduce the following semi-discrete finite element approximation of the problem **(P)**:

($\mathbf{P}^h$) Find $u^h \in S^h$ such that $u^h(\cdot, 0) = u_0^h(\cdot)$ and for almost every $t \in (0, T)$

$$\left(\frac{\partial u^h}{\partial t}, \chi^h \right)^h + (\nabla u^h, \nabla \chi^h) + \beta \int_{\partial\Omega} u^h \chi^h d\sigma + (g(u^h), \chi^h)^h = 0 , \qquad (5.2.1)$$

for all $\chi^h \in S^h$. Note that since Ω is polygonal or polyhedral the above boundary integral is exact.

We now state the following existence and uniqueness theorem of the semi-discrete approximation:

Theorem 5.2.1. Let the assumptions (1.1.3) and **(A)** hold, $u_0 \in H^1(\Omega)$, and $\beta > 0$, then there exists a unique solution u^h of the problem $(\mathbf{P}^h)$ such that the following stability bounds hold independently of h:

$$u^h \in L^\infty(0,T;H^1(\Omega)) \cap L^{h,2s}(\Omega_T)\,,$$

$$\frac{\partial u^h}{\partial t} \in L^2(\Omega_T)\,.$$

Proof. We separate the proof into three parts: local existence of the semi-discrete approximations, global existence of the semi-discrete approximations, and uniqueness of the solution u^h.

Subsection 5.2.1: Local existence

We first represent u^h as follows

$$u^h(\cdot,t) = \sum_{i=0}^{J} C_i(t)\,\varphi_i(\cdot)\,, \tag{5.2.2}$$

where $C_i(t)$ are to be determined. For the initial approximations we take

$$u_0^h(\cdot) := P^h u_0(\cdot)\,, \tag{5.2.3}$$

with

$$\sum_{i=0}^{J} C_i(0)(\varphi_i, \varphi_j) = (u_0^h, \varphi_j) = (P^h u_0, \varphi_j), \quad j = 0, \cdots, J. \tag{5.2.4}$$

Substituting (5.2.2) and (5.2.3) into the semi-discrete approximation (5.2.1), and taking $\chi^h = \varphi_j$, $j = 0, \cdots, J$, lead to

$$\sum_{i=0}^{J} \frac{dC_i}{dt}(\varphi_i, \varphi_j)^h + \sum_{i=0}^{J} C_i(\nabla\varphi_i, \nabla\varphi_j)$$

$$+\beta \sum_{i=0}^{J} \int_{\partial\Omega} C_i \varphi_i \varphi_j d\sigma = -(g(u^h), \varphi_j)^h. \tag{5.2.5}$$

The nonlinear term on the right-hand side of equation (5.2.5) can be expressed, noting (5.2.2), as follows:

$$(g(u^h), \varphi_j)^h = (g(\sum_{i=0}^{J} C_i\varphi_i), \varphi_j)^h = \int_\Omega \pi^h(g(\sum_{i=0}^{J} C_i\varphi_i) \cdot \varphi_j) dx =: g(C_j)M_j. \tag{5.2.6}$$

Thus from (5.2.5) and (5.2.6) we obtain a system of $(J+1)$ ordinary differential equations (ODEs) in the components C_j:

$$\frac{dC_j}{dt} M_j + \sum_{i=0}^{J} C_i K_{ij} = -g(C_j)M_j - \beta \sum_{i=0}^{J} C_i A_{ij}, \tag{5.2.7}$$

where $M_j = (\varphi_j, \varphi_j)^h$, $K_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$, and $A_{ij} = \langle\varphi_i, \varphi_j\rangle$, for $j = 0, \cdots, J$. We could also write this system as

$$M\frac{d\widehat{C}}{dt} + K\widehat{C} = -M\widehat{g} - \beta A\widehat{C}, \quad P^h\widehat{u}_0 := B\widehat{C}(0)\,, \tag{5.2.8}$$

where $M = \text{diag}\{M_0, \cdots, M_J\}, \{\widehat{C}\}_i := C_i$, $\{\widehat{g}\}_i := g(C_i)$, $\{P^h\widehat{u}_0\}_i := (P^h u_0, \varphi_i)$, $\{B\}_{ij} := (\varphi_i, \varphi_j)$. Since M is a non-singular matrix we can simplify the system (5.2.8) to

$$\frac{d\widehat{C}}{dt} = -(\widehat{g} + L\widehat{C} + \beta N\widehat{C}) = \widehat{\mathcal{G}}(\widehat{C}), \quad \widehat{C}(0) = B^{-1}P^h\widehat{u}_0\,, \tag{5.2.9}$$

where $L = M^{-1}K$ and $N = M^{-1}A$. As $\widehat{\mathcal{G}}$ is a locally Lipschitz function, it follows from the local existence theorem (Theorem A.0.13) that the system of ODEs has a unique solution $\widehat{C}$ on some finite time interval. Hence we have local existence for u^h for some finite time interval $(0, t_h), t_h > 0$.

Subsection 5.2.2: Global existence

To obtain existence of a global solution we first need to derive an *a priori* estimate on u^h (independently of h) and $t_h = t$ (t independent of h). Secondly, we make a further estimate on $\partial u^h/\partial t$ to derive the stability bounds in the theorem.

We now set $\chi^h = u^h$ in the semi-discrete approximation (5.2.1) to yield

$$\left(\frac{\partial u^h}{\partial t}, u^h\right)^h + (\nabla u^h, \nabla u^h) + \beta \int_{\partial\Omega} |u^h|^2 d\sigma + (g(u^h), u^h)^h = 0 \,. \qquad (5.2.10)$$

From (5.1.1) and (3.3.5), we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \pi^h\{|u^h|^2\}dx + \|\nabla u^h\|_0^2 + \beta |u^h|^2_{L^2(\partial\Omega)}$$

$$+\frac{1}{2} b_{2s-1} \int_\Omega \pi^h\{|u^h|^{2s}\}dx \le C_1|\Omega| \qquad (5.2.11)$$

where $C_1 > 0$. In particular, recalling (5.1.13),

$$\frac{d}{dt}|u^h|^2_{h,2} + b_{2s-1}|u^h|^{2s}_{h,2s} \le 2C_1|\Omega| \,. \qquad (5.2.12)$$

Applying the Hölder inequality to the norm $|u^h|_{h,2}$ yields

$$\begin{aligned} |u^h|^2_{h,2} &= \sum_{j=0}^J M_j |u_j^h|^2 \\ &\le \left(\sum_{j=0}^J M_j |u_j^h|^{2s}\right)^{1/s} \left(\sum_{j=0}^J M_j\right)^{1/s'} \\ &= \left(\int_\Omega \pi^h\{|u^h|^{2s}\}dx\right)^{1/s} (|\Omega|)^{1/s'} \,, \end{aligned}$$

where s and s' are conjugate, and noting that $\sum_{j=0}^{J} M_j = (1, \sum_{j=0}^{J} \varphi_j) = (1,1) = |\Omega|$. Now using Young's inequality yields

$$|u^h|_{h,2}^2 \leq \frac{1}{2} b_{2s-1} |u^h|_{h,2s}^{2s} + C b_{2s-1}^{-s'/s} |\Omega| . \qquad (5.2.13)$$

From inequalities (5.2.12) and (5.2.13) we obtain

$$\frac{d}{dt} |u^h|_{h,2}^2 + \frac{1}{2} b_{2s-1} |u^h|_{h,2s}^{2s} \leq -|u^h|_{h,2}^2 + C_2 , \qquad (5.2.14)$$

where $C_2 = (2C_1 + C b_{2s-1}^{-s'/s})|\Omega|$. Applying the usual Grönwall lemma (Theorem A.0.14) to this inequality yields

$$|u^h(t)|_{h,2}^2 + \frac{e^{-t}}{2} \int_0^t |u^h|_{h,2s}^{2s} \, dr \leq |u^h(0)|_{h,2}^2 e^{-t} + C_2(1 - e^{-t}) . \qquad (5.2.15)$$

Recall that $u_0 \in H^1(\Omega) \subset L^2(\Omega)$, so the boundedness of the term $|u^h(0)|_{h,2}^2$ can be obtained as follows:

$$|u^h(0)|_{h,2}^2 \equiv |P^h u_0|_{h,2}^2 \leq C .$$

Thus we have

$$u^h \in L^\infty(0,T; L^{h,2}(\Omega)) \cap L^{h,2s}(\Omega_T) . \qquad (5.2.16)$$

We now make a further estimate by taking $\chi^h = \partial u^h/\partial t$ in the semi-discrete approximation (5.2.1), so that

$$\int_\Omega \pi^h \left\{ \left| \frac{\partial u^h}{\partial t} \right|^2 \right\} dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u^h|^2 dx + \frac{\beta}{2}\frac{d}{dt}\int_{\partial\Omega} |u^h|^2 d\sigma + \frac{d}{dt}\int_\Omega \pi^h \left\{ G(u^h) \right\} dx = 0\,, \tag{5.2.17}$$

after noting that

$$g(u^h)\cdot\frac{\partial u^h}{\partial t} = \frac{\partial}{\partial t}G(u^h) \quad \text{and} \quad G(u^h) = \sum_{j=0}^{2s-1} \frac{b_j}{j+1}(u^h)^{j+1}\,.$$

Using (3.3.5) the equation (5.2.17) can be written as follows

$$\int_\Omega \pi^h \left\{ \left| \frac{\partial u^h}{\partial t} \right|^2 \right\} dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u^h|^2 dx + \frac{\beta}{2}\frac{d}{dt}\int_{\partial\Omega} |u^h|^2 d\sigma$$

$$+\frac{b_{2s-1}}{4s}\frac{d}{dt}\int_\Omega \pi^h \left\{ |u^h|^{2s} \right\} dx \le C_3|\Omega|\,, \tag{5.2.18}$$

where $C_3 > 0$. Integrating both sides of (5.2.18) from 0 to t and multiplying through by 2, gives

$$2\int_0^t \left| \frac{\partial u^h}{\partial t} \right|_{h,2}^2 ds + |u^h(t)|_1^2 + \beta\|u^h(t)\|_{L^2(\partial\Omega)}^2 + \frac{b_{2s-1}}{2s}\,|u^h(t)|_{h,2s}^{2s}$$

$$\le 2C_3|\Omega|t + |u^h(0)|_1^2 + \beta\|u^h(0)\|_{L^2(\partial\Omega)}^2 + \frac{b_{2s-1}}{2s}|u^h(0)|_{h,2s}^{2s}$$

$$\le 2C_3|\Omega|t + |u^h(0)|_1^2 + \beta C\|u^h(0)\|_1^2 + \frac{b_{2s-1}}{2s}\,|u^h(0)|_{h,2s}^{2s}\,. \tag{5.2.19}$$

Recalling that $u_0 \in H^1(\Omega)$ so $|u^h(0)|_1^2 \leq \|P^h u_0\|_1^2 \leq C$ and $|u^h(0)|_{h,2s}^{2s} \leq C\|P^h u_0\|_1^{2s} \leq C$, $s > 1$, which follow from Corollary 5.1.2 (with $r = 2s$) and Lemma 5.1.3, we then have

$$u^h \in L^\infty(0, T; H^1(\Omega)) . \tag{5.2.20}$$

By noting the injection $L^{h,2s}(\Omega) \hookrightarrow L^{h,2}(\Omega)$, the semi-norm bound for $H^1(\Omega)$, and (5.2.16) we thus have

$$\frac{\partial u^h}{\partial t} \in L^2(\Omega_T) . \tag{5.2.21}$$

Subsection 5.2.3: Uniqueness

To prove the unique dependence of a semi-discrete solution we suppose that there are two solutions u_1^h and u_2^h of the semi-discrete approximation (5.2.1). Then letting $\chi^h := w^h = u_1^h - u_2^h$, we obtain

$$\left(\frac{\partial w^h}{\partial t}, w^h\right)^h + (\nabla w^h, \nabla w^h) + \beta \int_{\partial\Omega} |w^h|^2 d\sigma = -(g(u_1^h) - g(u_2^h), w^h)^h , \tag{5.2.22}$$

and hence

$$\frac{1}{2}\frac{d}{dt}|w^h|_{h,2}^2 + |w^h|_1^2 + \beta\|w^h\|_{L^2(\partial\Omega)}^2 \leq |g(u_2^h) - g(u_1^h)|_{h,2}\,|w^h|_{h,2} . \tag{5.2.23}$$

Note that the second and third terms on the left-hand side of this inequality are bounded and non-negative. Thus, recalling inequality (3.2.16), we have

$$\frac{d}{dt}|w^h|_{h,2}^2 \leq 2C(t)|w^h|_{h,2}^2\,, \tag{5.2.24}$$

where

$$C(t) \equiv C(u_2^h, u_1^h) = \max\{|b_i|\}\, C\left(1 + |u_2^h|_{h,2s-2}^{2s-2}\right)\left(1 + |u_1^h|_{h,2s-2}^{2s-2}\right) \tag{5.2.25}$$

is a positive Lipschitz constant of the function g (compare this with the one in (3.4.5)). Now applying the usual Grönwall lemma to inequality (5.2.24) yields

$$|w^h(t)|_{h,2}^2 \leq \exp\left(2\int_0^t C(s)ds\right)|w^h(0)|_{h,2}^2\,.$$

Therefore we have

$$|u_1^h(t) - u_2^h(t)|_{h,2}^2 \leq \exp\left(2\int_0^t C(s)ds\right)|u_1^h(0) - u_2^h(0)|_{h,2}^2\,. \tag{5.2.26}$$

Thus if $u_1^h(0) = u_2^h(0)$, then it follows from (5.2.26) that $u_1^h(t) = u_2^h(t)$ for all t. This completes the proof of Theorem 5.2.1. $\square$

Section 5.3: Error bound

In this section we estimate the error between a solution u of problem $(\mathbf{P})$ and a semi-discrete solution u^h of problem $(\mathbf{P}^h)$, which is optimal in H^1, but sub-optimal in L^2. We shall use the interpolant $\pi^h u$ to estimate the error bound (see, e.g., Barrett and Blowey (1995)). Note that the interpolant $\pi^h u$ is defined only for continuous functions u. Since $u \in H^2(\Omega)$ (see Theorem 4.1.1) is continuous for $d = 1, 2, 3$ (see Thomée and Larsson (1999), p.28) we have that $\pi^h u$ is well-defined.

Throughout this section C represents a generic positive constant. We require the following Sobolev embedding result that will be used for some estimates in the proof of the error bound:

Corollary 5.3.1 Let $v \in H^1(\Omega)$, $s \in \mathbb{R}$, $s \geq \frac{5}{3}$ for $d = 1, 2$, and $s \in [\frac{5}{3}, 2]$ for $d = 3$. Then

$$\|v\|_{0,12s-18} \leq C\|v\|_{0,6s-6} \leq C\|v\|_1 \quad \text{for } \frac{5}{3} \leq s \leq 2, d \leq 3\,,$$

and

$$\|v\|_{0,6s-6} \leq C\|v\|_{0,12s-18} \leq C\|v\|_1 \quad \text{for} \qquad s \geq 2, d \leq 2\,.$$

Proof. Using the Gagliardo-Nirenberg inequality (see Section 4.2) with $m = 1$, $p = 2$, and hence $r = 12s - 18 \in [2, \infty]$ for $d = 1, 2$, $r = 12s - 18 \in$

[2, 6] for $d = 3$, we can easily show the above inequalities. $\square$

We now state the main result of this section:

Theorem 5.3.2 Let the assumptions of Theorem 5.2.1 hold. Then for all $h \leq 1$ we have that

$$\|u - u^h\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u - u^h\|^2_{L^2(0,T;H^1(\Omega))} \leq Ch^2 .$$

Proof. We set $e := u - u^h$, $e^A := u - \pi^h u$, and $e^h := \pi^h u - u^h$. As a result we have that $e := e^A + e^h$. Choosing $v = \chi^h = e^h \in S^h$ and subtracting (5.2.1) from (3.2.2), then for a.e. $t \in (0, T)$ it follows that

$$\left(\frac{\partial u}{\partial t}, e^h\right) - \left(\frac{\partial u^h}{\partial t}, e^h\right)^h + (\nabla e, \nabla e^h) + \beta \int_{\partial\Omega} e \cdot e^h d\sigma$$

$$= (g(u^h), e^h)^h - (g(u), e^h) . \tag{5.3.1}$$

Adding and subtracting the term $\left(\frac{\partial u^h}{\partial t}, e^h\right)$, we can rewrite equation (5.3.1) as follows:

$$\left(\frac{\partial e}{\partial t}, e\right) + (\nabla e, \nabla e) = \left\{\left(\frac{\partial e}{\partial t}, e^A\right)\right\}$$

$$+ \left\{(\nabla e, \nabla e^A)\right\}$$

$$
\begin{aligned}
&+\left\{\left(\frac{\partial u^h}{\partial t}, e^h\right)^h - \left(\frac{\partial u^h}{\partial t}, e^h\right)\right\} \\
&+\left\{\left(g(u^h), e^h\right)^h - \left(g(u), e^h\right)\right\} \\
&+\left\{\beta \int_{\partial\Omega} (u^h - u)\cdot e^h d\sigma\right\} \\
&= \sum_{j=1}^{5} E_j\,,
\end{aligned}
\tag{5.3.2}
$$

where E_j correspond to the brackets $\{\cdot\}$ on the right-hand side of equation (5.3.2), respectively.

Now we bound each term separately. By the Cauchy-Schwarz inequality and (5.1.6) we obtain

$$
\begin{aligned}
E_1 = \left(\frac{\partial e}{\partial t}, e^A\right) &\le \left\|\frac{\partial e}{\partial t}\right\|_0 \|e^A\|_0 \\
&\le Ch^2 \, |u|_2 \left\|\frac{\partial e}{\partial t}\right\|_0 \\
&\le Ch^2 \, \|u\|_2 \left\|\frac{\partial e}{\partial t}\right\|_0 .
\end{aligned}
\tag{5.3.3}
$$

Using the Cauchy-Schwarz inequality, (5.1.7), and the simple Young's inequality (5.1.12) we have

$$
E_2 = (\nabla e, \nabla e^A) \le |e|_1 |e^A|_1 \le Ch \, \|u\|_2 |e|_1
$$

$$\leq Ch^2 \|u\|_2^2 + \frac{1}{2\varepsilon}|e|_1^2 . \tag{5.3.4}$$

To bound the third term, we use the result (5.1.5) and the simple Young's inequality (5.1.12) to obtain

$$E_3 \equiv \left| \left(\frac{\partial u^h}{\partial t}, e^h \right)^h - \left(\frac{\partial u^h}{\partial t}, e^h \right) \right| \leq Ch \left\| \frac{\partial u^h}{\partial t} \right\|_0 \|e^h\|_1$$

$$\leq Ch^2 \left\| \frac{\partial u^h}{\partial t} \right\|_0^2 + \frac{1}{2\varepsilon}\|e^h\|_1^2 , \tag{5.3.5}$$

and from (5.1.6) and (5.1.7) we deduce that $\|e^A\|_1 \leq Ch\|u\|_2$ and hence

$$\|e^h\|_1^2 \leq 2\left(\|e\|_1^2 + \|e^A\|_1^2\right)$$

$$\leq 2\|e\|_0^2 + 2|e|_1^2 + Ch^2 \|u\|_2^2 . \tag{5.3.6}$$

Substituting (5.3.6) in (5.3.5) yields

$$E_3 \leq Ch^2 \left\| \frac{\partial u^h}{\partial t} \right\|_0^2 + C\|e\|_0^2 + \frac{1}{\varepsilon}|e|_1^2 + Ch^2 \|u\|_2^2 . \tag{5.3.7}$$

To deal with

$$E_4 = (g(u^h), e^h)^h - (g(u), e^h) ,$$

we subtract and add the term $(g(u^h), e^h)$ to E_4, and get

$$E_4 \leq |(g(u^h), e^h)^h - (g(u^h), e^h)| + |(g(u^h) - g(u), e^h)| . \tag{5.3.8}$$

We first bound the first term on the right-hand side of this inequality as follows:

$$
\begin{aligned}
|(g(u^h), e^h)^h - (g(u^h), e^h)| &\le \int_\Omega \left|(I-\pi^h) g(u^h)\cdot e^h\right| dx \\
&\le Ch^2 \sum_{i,j}^{d} \int_\Omega \left| \frac{\partial^2 (g(u^h)\cdot e^h)}{\partial x_i \partial x_j} \right| dx\,, \qquad (5.3.9)
\end{aligned}
$$

after noting (5.1.31) and (5.1.32). We take the derivatives of the right-hand side of (5.3.9) as follows: (recalling that u^h and e^h are piecewise linear functions)

$$\frac{\partial}{\partial x_j}(g(u^h)\cdot e^h) = g'(u^h)\frac{\partial u^h}{\partial x_j}\cdot e^h + g(u^h)\cdot \frac{\partial e^h}{\partial x_j}\,, \qquad (5.3.10)$$

$$\frac{\partial}{\partial x_i}\left(g'(u^h)\frac{\partial u^h}{\partial x_j}\cdot e^h\right) = g''(u^h)\frac{\partial u^h}{\partial x_i}\cdot\frac{\partial u^h}{\partial x_j}\cdot e^h + g'(u^h)\frac{\partial u^h}{\partial x_j}\cdot\frac{\partial e^h}{\partial x_i}\,, \qquad (5.3.11)$$

$$\frac{\partial}{\partial x_i}\left(g(u^h)\cdot\frac{\partial e^h}{\partial x_j}\right) = g'(u^h)\frac{\partial u^h}{\partial x_i}\cdot\frac{\partial e^h}{\partial x_j}\,. \qquad (5.3.12)$$

Thus from (5.3.9)-(5.3.12) we obtain

$$
\begin{aligned}
|(g(u^h), e^h)^h - (g(u^h), e^h)| \le Ch^2 \sum_{i,j}^{d} \int_\Omega \{ & |g''(u^h)| \left|\frac{\partial u^h}{\partial x_i}\right| \left|\frac{\partial u^h}{\partial x_j}\right| |e^h| \\
& + |g'(u^h)| \left|\frac{\partial u^h}{\partial x_j}\right| \left|\frac{\partial e^h}{\partial x_i}\right| + |g'(u^h)| \left|\frac{\partial u^h}{\partial x_i}\right| \left|\frac{\partial e^h}{\partial x_j}\right| \} \, dx\,.
\end{aligned}
\qquad (5.3.13)
$$

Using the Hölder inequality, the growth condition of the polynomial g (see (3.2.1)), Corollary 5.3.1, Theorem 5.2.1, the inverse inequality (5.1.8)-(5.1.9), noting that $h^{2-\frac{d}{3}} \leq h$ as $h \leq 1, d \leq 3$, the Young's inequality (5.1.12), and (5.3.6), we therefore obtain

$$
\begin{aligned}
|(g(u^h), e^h)^h - (g(u^h), e^h)| &\leq Ch^2 \sum_{i,j}^{d} \left\{ \|g''(u^h)\|_{0,6} \left\|\frac{\partial u^h}{\partial x_i}\right\|_{0,3} \left\|\frac{\partial u^h}{\partial x_j}\right\|_{0,3} \|e^h\|_{0,6} \right. \\
&+ \|g'(u^h)\|_{0,3} \left\|\frac{\partial u^h}{\partial x_j}\right\|_{0,3} \left\|\frac{\partial e^h}{\partial x_i}\right\|_{0,3} + \left. \|g'(u^h)\|_{0,3} \left\|\frac{\partial u^h}{\partial x_i}\right\|_{0,3} \left\|\frac{\partial e^h}{\partial x_j}\right\|_{0,3} \right\} \\
&\leq Ch^2 \left\{ \left(\|u^h\|_{0,12s-18}^{2s-3} + 1\right) |u^h|_{1,3}^2 \|e^h\|_{0,6} + \left(\|u^h\|_{0,6s-6}^{2s-2} + 1\right) |u^h|_{1,3} |e^h|_{1,3} \right\} \\
&\leq Ch^2 \left\{ \left(\|u^h\|_{1}^{2s-3} + 1\right) |u^h|_{1,3}^2 \|e^h\|_{1} + \left(\|u^h\|_{1}^{2s-2} + 1\right) |u^h|_{1,3} |e^h|_{1,3} \right\} \\
&\leq Ch^2 \left\{ |u^h|_{1,3}^2 \|e^h\|_1 + |u^h|_{1,3} |e^h|_{1,3} \right\} \\
&\leq Ch^{2-\frac{d}{3}} \left\{ |u^h|_1^2 \|e^h\|_1 + |u^h|_1 |e^h|_1 \right\} \\
&\leq Ch \|e^h\|_1 \\
&\leq Ch^2 + \frac{1}{2\varepsilon}\|e^h\|_1^2 \\
&\leq Ch^2 + C\|e\|_0^2 + \frac{1}{\varepsilon}|e|_1^2 + Ch^2 \|u\|_2^2 .
\end{aligned}
\tag{5.3.14}
$$

For the last term on the right-hand side of inequality (5.3.8), we use the Cauchy-Schwarz inequality and (3.2.16) to give

$$|(g(u^h) - g(u), e^h)| \leq \|g(u) - g(u^h)\|_0 \|e^h\|_0 \leq C(t)\|e\|_0 \|e^h\|_0 \,, \qquad (5.3.15)$$

where $C(t) \equiv C(u, u^h) = \max\{|b_i|\}\, C\left(1 + \|u\|_{0,2s-2}^{2s-2}\right)\left(1 + |u^h|_{h,2s-2}^{2s-2}\right)$ is a positive Lipschitz constant of the function g. Since

$$\|e^h\|_0 \leq \|e\|_0 + \|e^A\|_0 \leq \|e\|_0 + Ch^2\, \|u\|_2 \,, \qquad (5.3.16)$$

we have that

$$|(g(u^h) - g(u), e^h)| \leq C(t)\|e\|_0^2 + C(t)Ch^2\, \|u\|_2 \|e\|_0$$

$$\leq C\|e\|_0^2 + Ch^2\, \|u\|_2 \|e\|_0 \,. \qquad (5.3.17)$$

Thus from (5.3.14) and (5.3.17), we have

$$E_4 \leq C\|e\|_0^2 + \frac{1}{\varepsilon}|e|_1^2 + Ch^2\, \|u\|_2^2 + Ch^2\, \|u\|_2 \|e\|_0 + Ch^2 \,. \qquad (5.3.18)$$

To bound E_5, we split it via

$$E_5 = \beta \int_{\partial\Omega} (-e) \cdot e^h \, d\sigma$$

$$= -\beta \int_{\partial\Omega} e^2 \, d\sigma + \beta \int_{\partial\Omega} e \cdot e^A \, d\sigma \,. \qquad (5.3.19)$$

Since the first term on the right-hand side of equation (5.3.19) is non-positive, we thus have

$$\begin{aligned}
E_5 &\leq \beta \int_{\partial\Omega} e \cdot e^A \, d\sigma \\
&\leq C\|e\|_1 \|e^A\|_1 \\
&\leq Ch\|u\|_2 \|e\|_1 \\
&\leq C\varepsilon h^2 \|u\|_2^2 + \frac{1}{2\varepsilon}\|e\|_1^2 \\
&= C\varepsilon h^2 \|u\|_2^2 + \frac{1}{2\varepsilon}\|e\|_0^2 + \frac{1}{2\varepsilon}|e|_1^2 \,,
\end{aligned} \tag{5.3.20}$$

where we have used the trace embedding theorems (Theorem A.0.9), the Cauchy-Schwarz inequality, (5.1.6), (5.1.7), and Young's inequality.

Therefore, from (5.3.2), (5.3.3), (5.3.4), (5.3.7), (5.3.18), and (5.3.20) we have after multiplying through by 2

$$\begin{aligned}
\frac{d}{dt}\|e\|_0^2 + K|e|_1^2 \leq C\|e\|_0^2 + Ch^2 \, \{ \, &\|u\|_2^2 + \|u_t^h\|_0^2 \\
&+ C\|u\|_2\|e\|_0 + \|u\|_2\|e_t\|_0 + 1 \, \} \,,
\end{aligned} \tag{5.3.21}$$

where $K = 2 - \dfrac{6}{\varepsilon}$. Choosing $\varepsilon = 6$ we thus obtain

$$\begin{aligned}
\frac{d}{dt}\|e\|_0^2 + |e|_1^2 \leq C\|e\|_0^2 + Ch^2 \, \{ \, &\|u\|_2^2 + \|u_t^h\|_0^2 \\
&+ C\|u\|_2\|e\|_0 + \|u\|_2\|e_t\|_0 + 1 \, \} \,.
\end{aligned} \tag{5.3.22}$$

Now applying the usual Grönwall lemma to this inequality yields

$$\|e(t)\|_0^2+\int_0^t |e|_1^2 ds \le \exp(Ct)\|e(0)\|_0^2+\exp(Ct)Ch^2\Big\{\|u\|^2_{L^2(0,T;H^2(\Omega))}+\|u_t^h\|^2_{L^2(\Omega_T)}$$

$$+C\|u\|_{L^2(0,T;H^2(\Omega))}\|e\|_{L^\infty(0,T;H^1(\Omega))}+\|u\|_{L^2(0,T;H^2(\Omega))}\|e_t\|_{L^2(\Omega_T)}+t\Big\}\,. \tag{5.3.23}$$

The first term on the right-hand side of (5.3.23) is bounded due to inequality (5.1.26), we thus have

$$\|e(0)\|_0^2 \equiv \|(I-P^h)u_0\|_0^2 \le Ch^2|u_0|_1^2 \le Ch^2\,. \tag{5.3.24}$$

Note that norms in the other terms on the right-hand side of (5.3.23) are bounded due to Theorem 4.2.3 and Theorem 5.2.1. Thus

$$\|e(t)\|_0^2+\int_0^t |e|_1^2\, ds \le \exp(Ct)Ch^2 \le Ch^2\,. \tag{5.3.25}$$

This implies that

$$\|u-u^h\|^2_{L^2(0,T;H^1(\Omega))} \le Ch^2\,, \tag{5.3.26}$$

and that

$$\|u-u^h\|^2_{L^\infty(0,T;L^2(\Omega))} \le Ch^2\,. \tag{5.3.27}$$

From (5.3.26) and (5.3.27) we have that the error bound is optimal in H^1, but sub-optimal in L^2. $\square$

Chapter 6

A Fully Discrete Approximation

In this chapter we prove an error bound for a fully discrete finite element approximation of the weak form (3.2.2). In Section 6.1 we discretise the weak form (3.2.2) in time using backward Euler. We discuss stability estimates that are needed for the following section. We also discuss the existence and uniqueness of the fully discrete approximation. Finally, in Section 6.2, we estimate the error bound between the semi-discrete solution and the fully discrete one which leads to the proof of the error bound between the continuous and fully discrete solutions.

Throughout this chapter, C represents a generic bounded positive constant independent of h and Δt.

Section 6.1: Existence and Uniqueness

Given N, a positive integer, let $\Delta t := T/N$ denote a fixed time step, and

$t_n := n\Delta t$ $(n = 0, 1, \cdots, N)$. For our fully discrete finite element approximation of **(P)**, we split the nonlinearity g as a function of U^n and U^{n-1} to be $g_+(U^n)+g_-(U^{n-1})$ where $g_+(U^n)$ contains the monotone increasing terms of the polynomial g and $g_-(U^{n-1})$ contains the monotone decreasing terms of the polynomial g, so that $g = g_+ + g_-$. We need to introduce the notation $G(s) = G_+(s) + G_-(s)$ where $G_+(s) = \int_0^s g_+(r)dr$ and $G_-(s) = \int_0^s g_-(r)dr$ are convex and concave functions respectively. We shall now give an illustrative example of the above definition:

Example: If we choose $g(u) = u^3 + u^2 - u$, then this can be split into monotone increasing and decreasing terms as follows:

$$g(u) = u^3 + [u]_+^2 + [u]_-^2 - u\,,$$

where $g_+(u) = u^3 + [u]_+^2$ and $g_-(u) = [u]_-^2 - u$.

We therefore consider the following fully-discrete finite element approximation of **(P)**:

$(\mathbf{P}^{h,\Delta t})$ Find $U^n \in S^h$, for $n = 1, \cdots, N$, such that $U^0 = P^h u_0$ and

$$\left(\frac{U^n - U^{n-1}}{\Delta t}, \chi^h\right)^h + (\nabla U^n, \nabla\chi^h) + \beta \int_{\partial\Omega} U^n \chi^h d\sigma$$

$$+(g_+(U^n) + g_-(U^{n-1}), \chi^h)^h = 0 \qquad \forall \chi^h \in S^h\,. \tag{6.1.1}$$

We require the following identity for later use:

$$2s(s-r) = (s-r)^2 + s^2 - r^2 \quad \forall r, s\,. \tag{6.1.2}$$

We now state a theorem of existence, uniqueness, and some estimates of the fully discrete approximation:

Theorem 6.1.1. Let the assumptions of Theorem 5.2.1 hold, and $\Delta t > 0$. Then for all $h > 0$ there exists a unique solution $\{U^n\}_{n=1}^N$ to $(\mathbf{P}^{h,\Delta t})$ such that the following stability bounds hold:

$$\sum_{n=1}^{N} |U^n - U^{n-1}|_{h,2}^2 \leq C\Delta t\,, \tag{6.1.3}$$

$$\sum_{n=1}^{N} \left(|U^n - U^{n-1}|_1^2 + \beta \|U^n - U^{n-1}\|_{L^2(\partial\Omega)}^2 \right) \leq C\,, \tag{6.1.4}$$

$$\max_{n=1,\cdots,N} \left\{ |U^n|_1^2 + \beta \|U^n\|_{L^2(\partial\Omega)}^2 + (G(U^n), 1)^h \right\} \leq C\,. \tag{6.1.5}$$

Proof. We separate the proof into three parts: existence, uniqueness, and stability estimates.

Subsection 6.1.1: Stability estimates

We set $\chi^h = U^n - U^{n-1}$ in the fully discrete approximation (6.1.1) to yield

$$\frac{1}{\Delta t}\big(U^n - U^{n-1}, U^n - U^{n-1}\big)^h + (\nabla U^n, \nabla(U^n - U^{n-1})) + \beta \int_{\partial\Omega} |U^n||U^n - U^{n-1}| d\sigma$$

$$+(g_+(U^n), U^n - U^{n-1})^h + (g_-(U^{n-1}), U^n - U^{n-1})^h = 0\,. \qquad (6.1.6)$$

Note that from convexity (concavity) we have

$$(g_+(U^n), U^n - U^{n-1})^h \geq (G_+(U^n) - G_+(U^{n-1}), 1)^h\,, \qquad (6.1.7)$$

$$(g_-(U^{n-1}), U^n - U^{n-1})^h \geq (G_-(U^n) - G_-(U^{n-1}), 1)^h\,. \qquad (6.1.8)$$

Thus noting the identity (6.1.2) and substituting (6.1.7) and (6.1.8) in to (6.1.6) we have, after multiplying through by $2\Delta t$, that

$$2|U^n - U^{n-1}|_{h,2}^2 + \Delta t\left(|U^n - U^{n-1}|_1^2 + |U^n|_1^2 - |U^{n-1}|_1^2\right)$$

$$+\beta\Delta t\left(\|U^n - U^{n-1}\|_{L^2(\partial\Omega)}^2 + \|U^n\|_{L^2(\partial\Omega)}^2 - \|U^{n-1}\|_{L^2(\partial\Omega)}^2\right)$$

$$+\Delta t\left((G_+(U^n) + G_-(U^n), 1)^h - (G_+(U^{n-1}) + G_-(U^{n-1}), 1)^h\right) \leq 0\,. \qquad (6.1.9)$$

Summing the above inequality over $n = 1, \cdots, m$ for $m \leq N$ and rearranging the terms yields

$$\max_{m=1,\cdots,N}\Big\{\sum_{n=1}^{m}\big(2|U^n - U^{n-1}|_{h,2}^2 + \Delta t|U^n - U^{n-1}|_1^2 + \beta\Delta t\|U^n - U^{n-1}\|_{L^2(\partial\Omega)}^2\big)$$

$$+\Delta t|U^m|_1^2 + \beta\Delta t\|U^m\|_{L^2(\partial\Omega)}^2 + \Delta t(G(U^m), 1)^h\Big\}$$

$$\leq \Delta t|U^0|_1^2 + \beta\Delta t\|U^0\|^2_{L^2(\partial\Omega)} + \Delta t(G(U^0),1)^h\,, \tag{6.1.10}$$

where $G(U^m) = G_+(U^m) + G_-(U^m)$ and $G(U^0) = G_+(U^0) + G_-(U^0)$. Since the function G satisfies the growth condition (see Subsection 4.2.1 after equation (4.2.2)), so we have

$$\begin{aligned}(G(U^0),1)^h &\leq C(|U^0|^{2s},1)^h + C(1,1)^h \\ &= C|U^0|^{2s}_{h,2s} + C(1,1)^h\,.\end{aligned} \tag{6.1.11}$$

Substituting (6.1.11) in to (6.1.10) we have

$$\begin{aligned}&\max_{m=1,\cdots,N}\{\sum_{n=1}^{m}(2|U^n - U^{n-1}|^2_{h,2} + \Delta t|U^n - U^{n-1}|_1^2 + \beta\Delta t\|U^n - U^{n-1}\|^2_{L^2(\partial\Omega)}) \\ &\qquad +\Delta t|U^m|_1^2 + \beta\Delta t\|U^m\|^2_{L^2(\partial\Omega)} + \Delta t(G(U^m),1)^h\,\} \\ &\leq \Delta t|U^0|_1^2 + \beta\Delta t\|U^0\|^2_{L^2(\partial\Omega)} + C\Delta t|U^0|^{2s}_{h,2s} + C\Delta t(1,1)^h \\ &\leq \Delta t|U^0|_1^2 + \beta C\Delta t\|U^0\|_1^2 + C\Delta t\|U^0\|_1^{2s} + C\Delta t \leq C\Delta t\,,\end{aligned} \tag{6.1.12}$$

where we have used the trace embedding theorems (Theorem A.0.9) and Corollary 5.1.2 to give $\|U^0\|^2_{L^2(\partial\Omega)} \leq C\|U^0\|_1^2$ and $|U^0|^{2s}_{h,2s} \leq C\|U^0\|_1^{2s}$, hence using Lemma 5.1.3 gives boundedness of the right-hand side of (6.1.12).

Thus, the bound (6.1.3) follows from the first term on the left-hand side of (6.1.12). The bound (6.1.4) follows from the second and third terms on left-hand side of (6.1.12). Finally, we deduce the bound (6.1.5) from the remaining terms on the left-hand side of (6.1.12).

Subsection 6.1.2: Existence

We prove the existence of the solution of the problem $(\mathbf{P}^{h,\Delta t})$ as follows. The equation (6.1.1) can be considered to be the Euler-Lagrange equation of the following minimization problem:

Find $1 \leq n \leq N$ fixed, find $U \in S^h$ such that

$$\min_{\chi \in S^h} J^h(\chi)\,, \tag{6.1.13}$$

where

$$J^h(\chi) := \frac{1}{2\Delta t}|\chi - U^{n-1}|_{h,2}^2 + \frac{1}{2}|\chi|_1^2 + \frac{\beta}{2}\|\chi\|_{L^2(\partial\Omega)}^2 + (G_+(\chi), 1)^h + (g_-(U^{n-1}), \chi)^h\,.$$

The last term on the right-hand side of this inequality can be bounded below as follows:

$$\begin{aligned}(g_-(U^{n-1}), \chi)^h &\geq -|g_-(U^{n-1})|_{h,2}\,|\chi|_{h,2} \\ &\geq -|g_-(U^{n-1})|_{h,2}\left(|\chi - U^{n-1}|_{h,2} + |U^{n-1}|_{h,2}\right) \\ &\geq -|g_-(U^{n-1})|_{h,2}|\chi - U^{n-1}|_{h,2} - |g_-(U^{n-1})|_{h,2}|U^{n-1}|_{h,2}\end{aligned}$$

$$\geq -\frac{1}{2\Delta t}|\chi - U^{n-1}|_{h,2}^2 - \frac{\Delta t}{2}|g_-(U^{n-1})|_{h,2}^2 - |g_-(U^{n-1})|_{h,2}|U^{n-1}|_{h,2}$$

$$\geq -\frac{1}{2\Delta t}|\chi - U^{n-1}|_{h,2}^2 - C(\Delta t + 1)\,. \tag{6.1.14}$$

Thus we have

$$J^h(\chi) \geq \frac{1}{2}|\chi|_1^2 + \frac{\beta}{2}\|\chi\|_{L^2(\partial\Omega)}^2 + (G_+(\chi), 1)^h - C(\Delta t + 1)\,. \tag{6.1.15}$$

So $J^h(\chi)$ is bounded below in S^h.

Now, let $\mu = \inf_{S^h} J^h(\chi)$ and $\{\chi_n\}$ be a minimising sequence of J^h in S^h (i.e. $\lim_{n\to\infty} J^h(\chi_n) = \mu$). It follows from the above estimate that $\{\chi_n\} \in H^1(\Omega)$, and hence there exists $U \in S^h$ and subsequence $\{\chi_n\}$ such that

$$\chi_n \longrightarrow U \in S^h\,. \tag{6.1.16}$$

Since S^h is closed, then the continuity of J^h gives

$$J^h(\chi_n) \longrightarrow J^h(U) = \mu\,. \tag{6.1.17}$$

Therefore, there exists a solution U to the minimization problem (6.1.13).

Subsection 6.1.3: Uniqueness

To prove uniqueness we suppose that (6.1.1) has two solutions U_1^n and U_2^n for all $n \geq 1$. We use proof by induction. We shall first assume uniqueness of the approximation at time $t_{n-1} := (n-1)\Delta t$ and note that we have uniqueness at time t_0. Thus letting $\chi^h := W^n = U_1^n - U_2^n$, we obtain

$$\frac{1}{\Delta t}(W^n, W^n)^h + (\nabla W^n, \nabla W^n) + \beta \int_{\partial\Omega} |W^n|^2 d\sigma$$

$$+(g_+(U_1^n) - g_+(U_2^n), U_1^n - U_2^n)^h = 0\,, \qquad (6.1.18)$$

and hence,

$$|W^n|_{h,2}^2 + \Delta t |W^n|_1^2 + \beta \Delta t \|W^n\|_{L^2(\partial\Omega)}^2 \leq 0\,, \qquad (6.1.19)$$

after noting $(g_+(U_1^n) - g_+(U_2^n), U_1^n - U_2^n)^h \geq 0$, which follows from the monotonicity of g_+ and the fact that $(g_+(U_1^n), U_2^n - U_1^n)^h \leq (G_+(U_2^n) - G_+(U_1^n), 1)^h$. The second and third terms on the left-hand side of inequality (6.1.19) are bounded and non-negative. Therefore, we have

$$|W^n|_{h,2}^2 \leq 0\,. \qquad (6.1.20)$$

Thus, the uniqueness follows from (6.1.20), i.e. $U_1^n \equiv U_2^n$ for all $n \geq 1$. This completes the proof of Theorem 6.1.1. □

Section 6.2: Error bound

In this section we derive the error estimate between the semi-discrete approximation u^h and the fully discrete approximation U^n. This leads us to easily derive the error estimate between the continuous solution and the fully discrete approximation. We shall first consider the following definitions:

$$U(t) := \left(\frac{t - t_{n-1}}{\Delta t}\right) U^+ + \left(\frac{t_n - t}{\Delta t}\right) U^- , \quad t \in [t_{n-1}, t_n], n \geq 1, \tag{6.2.1}$$

where

$$U^+(t) := U^n, \quad U^-(t) := U^{n-1}, \quad t \in (t_{n-1}, t_n], n \geq 1 . \tag{6.2.2}$$

We also have that for $t \in (t_{n-1}, t_n)$

$$\frac{\partial U}{\partial t} = \frac{U^+ - U^-}{\Delta t} = \frac{U^+ - U}{t_n - t} = \frac{U - U^-}{t - t_{n-1}} . \tag{6.2.3}$$

Using the above we can restate the problem $(\mathbf{P}^{h,\Delta t})$ as follows:

Find $U \in H^1(0, T; S^h)$ such that $U(0) = P^h u_0$ and for a.e. $t \in (0, T)$

$$\left(\frac{\partial U}{\partial t}, \chi^h\right)^h + (\nabla U^+, \nabla \chi^h) + \beta \int_{\partial\Omega} U^+ \chi^h d\sigma + (g_+(U^+) + g_-(U^-), \chi^h)^h = 0, \tag{6.2.4}$$

for all $\chi^h \in S^h$.

Theorem 6.2.1. Let the assumptions of Theorem 6.1.1 hold. Then we have

$$\|u^h - U\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u^h - U^+\|^2_{L^2(0,T;H^1(\Omega))} \leq C\Delta t\,.$$

Proof. Let $E := u^h - U, E^+ := u^h - U^+, E^- := u^h - U^- \in S^h$ for a.e. $t \in (0,T)$. Using these definitions and (6.2.3) we have

$$E^+ - E = U - U^+ = (t - t_n)\frac{\partial U}{\partial t}\,, \tag{6.2.5}$$

$$E^- - E = U - U^- = (t - t_{n-1})\frac{\partial U}{\partial t}\,. \tag{6.2.6}$$

We choose $\chi^h = E^+$ in (5.2.1) and (6.2.4), and subtract (6.2.4) from (5.2.1), then for a.e. $t \in (0,T)$ it follows that

$$\left(\frac{\partial E}{\partial t}, E^+\right)^h + (\nabla E^+, \nabla E^+) + \beta \int_{\partial\Omega} |E^+|^2 d\sigma$$

$$+(g_+(u^h) - g_+(U^+), E^+)^h + (g_-(u^h) - g_-(U^-), E^+)^h = 0\,, \tag{6.2.7}$$

and hence, recalling (3.2.16),

$$\frac{1}{2}\frac{d}{dt}|E|^2_{h,2} + |E^+|^2_1 \leq C_+(t)|E^+|^2_{h,2} + C_-(t)|E^-|_{h,2}|E^+|_{h,2} + \left(\frac{\partial E}{\partial t}, U^+ - U\right)^h$$

$$\leq (C_+(t) + \frac{1}{2}C_-(t)|E^+|^2_{h,2} + \frac{1}{2}C_-(t)|E^-|^2_{h,2} + \left(\frac{\partial E}{\partial t}, U^+ - U\right)^h\,, \tag{6.2.8}$$

where

$$C_+(t) \equiv C(U^+, u^h) = \max\{|b_i|\}\, C\left(1 + |U^+|_{h,2s-2}^{2s-2}\right)\left(1 + |u^h|_{h,2s-2}^{2s-2}\right)\,, \quad (6.2.9)$$

and

$$C_-(t) \equiv C(U^-, u^h) = \max\{|b_i|\} C\left(1 + |U^-|_{h,2s-2}^{2s-2}\right)\left(1 + |u^h|_{h,2s-2}^{2s-2}\right)\,, \quad (6.2.10)$$

are positive Lipschitz constants of the functions g_+ and g_-, respectively. Note that we have

$$\begin{aligned} |E^+|_{h,2} &\leq |E|_{h,2} + |U - U^+|_{h,2} \\ &\leq |E|_{h,2} + \frac{|t - t_n|}{\Delta t}|U^+ - U^-|_{h,2} \\ &\leq |E|_{h,2} + |U^+ - U^-|_{h,2}\,, \end{aligned}$$

which implies

$$\begin{aligned} |E^+|_{h,2}^2 &\leq 2\left(|E|_{h,2}^2 + |U^+ - U^-|_{h,2}^2\right) \\ &\leq 2|E|_{h,2}^2 + 2|U^+ - U^-|_{h,2}^2\,, \end{aligned} \quad (6.2.11)$$

after using the elementary result, $(a+b)^p \leq 2^{p-1}(a^p + b^p), a, b \geq 0, p \geq 1$. In the same way we obtain that

$$|E^-|_{h,2}^2 \leq 2|E|_{h,2}^2 + 2|U^+ - U^-|_{h,2}^2\,. \quad (6.2.12)$$

The last term on the right-hand side of (6.2.8) can be dealt with as follows. By the Hölder inequality (5.1.16), (6.2.3), and the simple Young's inequality (5.1.12) we obtain

$$\left(\frac{\partial E}{\partial t}, U^+ - U\right)^h \le \left|\frac{\partial E}{\partial t}\right|_{h,2} |U^+ - U|_{h,2}$$

$$\le \left(\left|\frac{\partial u^h}{\partial t}\right|_{h,2} + \left|\frac{\partial U}{\partial t}\right|_{h,2}\right) \frac{|t_n - t|}{\Delta t} |U^+ - U^-|_{h,2}$$

$$\le \left(\left|\frac{\partial u^h}{\partial t}\right|_{h,2} + \frac{1}{\Delta t}\left|U^+ - U^-\right|_{h,2}\right) |U^+ - U^-|_{h,2}$$

$$\le \left|\frac{\partial u^h}{\partial t}\right|_{h,2} |U^+ - U^-|_{h,2} + \frac{1}{\Delta t}|U^+ - U^-|^2_{h,2}. \tag{6.2.13}$$

Thus from inequalities (6.2.8)-(6.2.13) and multiplying through by 2 we obtain

$$\frac{d}{dt}|E|^2_{h,2} + 2|E^+|^2_1 \le C(t)|E|^2_{h,2} + C(t)|U^+ - U^-|^2_{h,2}$$

$$+C\left|\frac{\partial u^h}{\partial t}\right|_{h,2} \left|U^+ - U^-\right|_{h,2} + \frac{C}{\Delta t}|U^+ - U^-|^2_{h,2}, \tag{6.2.14}$$

where $C(t) = 4C_+(t) + 4C_-(t)$. Now applying the usual Grönwall lemma to (6.2.14) and noting $E(0) = u_0^h - U^0 = 0$ yields

$$|E(t)|^2_{h,2} + 2\int_0^t |E^+|^2_1 ds \le \exp\left(\int_0^t C(s)ds\right)\int_0^t \Big\{C\left|U^+ - U^-\right|^2_{h,2}$$

$$+C\left|\frac{\partial u^h}{\partial t}\right|_{h,2}\left|U^+ - U^-\right|_{h,2} + \frac{C}{\Delta t}|U^+ - U^-|_{h,2}^2\Bigg\}\, ds\,. \qquad (6.2.15)$$

To bound the first and last terms on the right-hand side of (6.2.15) we use Theorem 6.1.1, (6.1.3), to get

$$C\int_0^t |U^+ - U^-|_{h,2}^2 ds = C\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} |U^n - U^{n-1}|_{h,2}^2 ds \le C(\Delta t)^2\,, \qquad (6.2.16)$$

and

$$\frac{C}{\Delta t}\int_0^t |U^+ - U^-|_{h,2}^2 ds = \frac{C}{\Delta t}\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} |U^n - U^{n-1}|_{h,2}^2 ds \le C\Delta t\,. \qquad (6.2.17)$$

To bound the second term on the right-hand side of (6.2.15) we use the Cauchy-Schwarz inequality and Theorem 5.2.1 to obtain

$$\begin{aligned} C\int_0^t \left|\frac{\partial u^h}{\partial t}\right|_{h,2} |U^+ - U^-|_{h,2} ds &\le C\left(\int_0^t \left|\frac{\partial u^h}{\partial t}\right|_{h,2}^2 ds\right)^{1/2}\left(\int_0^t |U^+ - U^-|_{h,2}^2 ds\right)^{1/2} \\ &\le C\left(\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} |U^n - U^{n-1}|_{h,2}^2 ds\right)^{1/2} \\ &\le C\Delta t\,. \qquad (6.2.18) \end{aligned}$$

Thus from inequalities (6.2.15)-(6.2.18) we conclude that

$$\begin{aligned} |E(t)|_{h,2}^2 + 2\int_0^t |E^+|_1^2 ds &\le C(\Delta t)^2 + C\Delta t \\ &\le C\Delta t\,. \qquad (6.2.19) \end{aligned}$$

This implies that

$$\|u^h - U\|^2_{L^\infty(0,T;L^2(\Omega))} \leq C\Delta t\,, \tag{6.2.20}$$

and that

$$\|u^h - U^+\|^2_{L^2(0,T;H^1(\Omega))} \leq C\Delta t\,, \tag{6.2.21}$$

as desired. $\square$

Remark 6.2.2. As we are using a Backward Euler approximation in time, which is first-order accurate, an optimal error bound between the full and semi discrete approximation would be

$$\|u^h - U^+\|_{L^2(0,T;H^1(\Omega))} \leq C\Delta t\,.$$

However, we have only achieved

$$\|u^h - U^+\|_{L^2(0,T;H^1(\Omega))} \leq C\Delta t^{1/2}\,.$$

We now state the main theorem of this chapter:

Theorem 6.2.3. Let the assumptions of Theorem 6.1.1 hold. Then we have

$$\|u - U\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u - U^+\|^2_{L^2(0,T;H^1(\Omega))} \leq C(\Delta t + h^2)\,.$$

Proof. The result follows from combining the results of Theorem 5.3.2 and Theorem 6.2.1. $\square$

Corollary 6.2.4 Let the assumptions of Theorem 6.2.3 hold. If $\Delta t \leq C_1 h^2$ for some constant C_1, then

$$\|u - U\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u - U^+\|^2_{L^2(0,T;H^1(\Omega))} \leq C h^2 .$$

Proof. The result directly follows from Theorem 6.2.3. $\square$

Chapter 7

Numerical Experiments

In this chapter we shall perform some one-dimensional experiments. In Section 7.1 we give full details of an algorithm for computing the numerical solution. In Section 7.2 we have used the implicit scheme for all simulations. We discuss some computational results in one space dimension.

Section 7.1: Practical Algorithm: One dimensional simulations

In this section we discuss a practical algorithm for solving ($\mathbf{P}^{h,\Delta t}$) in one space dimension on a uniform partition of $\Omega = (0, L)$, for $0 \leq t \leq T$, with mesh points $x_j = jh$, $j = 0, \cdots, J$. Let us expand U^n in terms of the standard nodal basis function φ_i of the finite element space S^h, that is,

$$U^n = \sum_{i=0}^{J} U_i^n \varphi_i \,, \tag{7.1.1}$$

where J is the number of node points and $U_i^n \approx u(ih, n\Delta t)$.

We consider the following Robin boundary conditions:

$$u'(0,t) = \beta u(0,t)\,, \tag{7.1.2}$$

$$u'(L,t) = -\beta u(L,t)\,. \tag{7.1.3}$$

Thus equation (6.1.1) with these boundary data can be presented as follows:

$(\mathbf{P}^{h,\Delta t})$ Find $U^n \in S^h$, for $n = 1, \cdots, N$, such that $U^0 = P^h u_0$

$$\left(\frac{U^n - U^{n-1}}{\Delta t}, \chi^h\right)^h + (\nabla U^n, \nabla \chi^h) + \beta(U_J^n \chi_J^h + U_0^n \chi_0^h)$$

$$+ \left(g_+(U^n) + g_-(U^{n-1}), \chi^h\right)^h = 0 \qquad \forall \chi^h \in S^h\,. \tag{7.1.4}$$

Substituting (7.1.1) in (7.1.4) and taking $\chi^h = \varphi_j$, $j = 0, \cdots, J$ yields

$$\frac{1}{\Delta t}\left(\sum_{i=0}^{J}(U_i^n - U_i^{n-1})\varphi_i, \varphi_j\right)^h + \left(\sum_{i=0}^{J} U_i^n \nabla\varphi_i, \nabla\varphi_j\right)$$

$$+\beta(U_J^n \delta_{j,J} + U_0^n \delta_{j,0}) + \sum_{i=0}^{J}(\varphi_i, \varphi_j)^h \left(g_+(U_i^n) + g_-(U_i^{n-1})\right) = 0, \tag{7.1.5}$$

for $j = 0, \cdots, J$. This is equivalent to

$$\frac{1}{\Delta t} M(\mathbf{U}^n - \mathbf{U}^{n-1}) + K\mathbf{U}^n + \beta(U_0^n, 0, \cdots, 0, U_J^n)^T + M(g_+(\mathbf{U}^n) + g_-(\mathbf{U}^{n-1})) = 0\,, \tag{7.1.6}$$

where $\mathbf{U}^n = (U_0^n, U_1^n, \cdots, U_J^n)^T$, $M_{ij} = (\varphi_i, \varphi_j)^h$ is the lumped mass matrix, and $K_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$ is the stiffness matrix. Thus multiplying this equation through by Δt and M^{-1} we obtain

$$\mathbf{U}^n - \mathbf{U}^{n-1} + \Delta t M^{-1} K \mathbf{U}^n + \Delta t \beta M^{-1}(U_0^n, 0, \cdots, 0, U_J^n)^T + \Delta t(g_+(\mathbf{U}^n) + g_-(\mathbf{U}^{n-1})) = 0\,. \tag{7.1.7}$$

To solve the algebraic nonlinear system (7.1.7), we define the operators $\mathcal{A}$ and $\mathcal{B}$ such that

$$\mathcal{A} : (a,b)^{J+1} \to \mathbb{R}^{J+1}$$

$$\mathcal{A}(\chi) = \Delta t g_+(\chi), \tag{7.1.8}$$

$$\mathcal{B} : S^h \to S$$

$$\mathcal{B}(\chi) = \chi - \mathbf{U}^{n-1} + \Delta t M^{-1} K \chi + \Delta t \beta M^{-1}(\chi_0, 0, \cdots, 0, \chi_J)^T + \Delta t g_-(\mathbf{U}^{n-1})\,, \tag{7.1.9}$$

so that the system (7.1.7) can be written in the following form

$$\mathcal{B}(\mathbf{U}^n) + \mathcal{A}(\mathbf{U}^n) = 0\,. \tag{7.1.10}$$

We recall here that an operator

$$T : D(T) \subseteq X \longrightarrow X$$

is said to be a monotone operator if

$$(Tu - Tv, u - v) \geq 0 \qquad \forall u, v \in D(T)\,,$$

where $(\cdot,\cdot)$ denotes the scalar product on X. One of the most important concepts in the theory of monotone operators is a *maximal monotone* operator. The following basic result on maximal monotone operators is useful in the work that follows:

$$T \text{ is called maximal monotone} \iff \text{it is monotone and } \mathrm{rang}(I+T) = X\,. \tag{7.1.11}$$

This was proved by Minty (1962), pp.343-344.

Lemma 7.1.1. Let the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy the definitions (7.1.8) and $\beta > 0$, then the operator $\mathcal{A}$ is maximal monotone and $\mathcal{B}$ is coercive.

Proof. It follows from the monotonicity of g_+ and (7.1.8) that $\mathcal{A}$ is monotone. Since the range of $I + \mu\mathcal{A} \in \mathbb{R}^{J+1}$, $\forall \mu \in \mathbb{R}^+$, then $\mathcal{A}$ is maximal.

To show $\mathcal{B}$ is coercive: given $\eta = (\eta_0, \cdots, \eta_J)^T, \chi = (\chi_0, \cdots, \chi_J)^T \in S^h$ and denoting by $(\cdot,\cdot)$ the inner product on $\mathbb{R}^{J+1}$ defined by $(\eta,\chi) = \chi^T M \eta$, we get, noting (7.1.9),

$(\mathcal{B}(\eta) - \mathcal{B}(\chi), \eta - \chi)$

$$= \big(\eta - \chi + \Delta t M^{-1}K(\eta - \chi) + \Delta t\beta M^{-1}(\eta_0 - \chi_0, 0, \cdots, 0, \eta_J - \chi_J)^T, \eta - \chi\big)$$

$$= (\eta-\chi, \eta-\chi) + \Delta t(M^{-1}K(\eta-\chi), \eta-\chi) + \Delta t\beta M^{-1}((\eta_0-\chi_0, 0, \cdots, 0, \eta_J-\chi_J)^T, \eta-\chi)$$

$$= (\eta-\chi)^T M(\eta-\chi) + \Delta t(\eta-\chi)^T K(\eta-\chi) + \Delta t\beta(\eta_0-\chi_0, 0, \cdots, 0, \eta_J-\chi_J)I(\eta-\chi) . \tag{7.1.12}$$

We first deal with the second term on the right-hand side of (7.1.12) as follows:

Define $Z = \sum_{j=0}^{J} \eta_j \varphi_j$ and $X = \sum_{k=0}^{J} \chi_k \varphi_k$, we have

$$\Delta t|Z - X|_1^2 = \Delta t(\eta - \chi)^T K(\eta - \chi) \geq 0 . \tag{7.1.13}$$

The last term on the right-hand side of (7.1.12) can be dealt as follows:

$$\Delta t\beta(\eta_0 - \chi_0, 0, \cdots, 0, \eta_J - \chi_J)I(\eta - \chi) = \Delta t\beta\left((\eta_0 - \chi_0)^2 + (\eta_J - \chi_J)^2\right) \geq 0 . \tag{7.1.14}$$

Thus it follows from (7.1.12)-(7.1.14) that

$$(\mathcal{B}(\eta) - \mathcal{B}(\chi), \eta - \chi) \geq (\eta - \chi)^T M(\eta - \chi) , \tag{7.1.15}$$

as desired. $\square$

To solve the system (7.1.10), we adapt the algorithm of Lions and Mercier (1979), who consider the case when $\mathcal{A}$ and $\mathcal{B}$ are two general maximal monotone operators. Copetti and Elliott (1992) adapted this algorithm in the case when

$\mathcal{A}$ is a general maximal monotone operator and $\mathcal{B}$ is a coercive operator. Barrett and Blowey (1997) have adapted this algorithm where there are two Lagrange multipliers present.

Multiplying the system (7.1.10) by $\mu \in \mathbb{R}^+$, adding $\mathbf{U}^n$ to both sides, and rearranging the terms yields

$$\mathbf{U}^n + \mu\mathcal{A}(\mathbf{U}^n) = \mathbf{U}^n - \mu\mathcal{B}(\mathbf{U}^n)\,. \tag{7.1.16}$$

Now, we define

$$\mathbf{Z}^n = \mathbf{U}^n - \mu\mathcal{B}(\mathbf{U}^n)\,. \tag{7.1.17}$$

$$\mathbf{X}^n = 2\mathbf{U}^n - \mathbf{Z}^n = \mathbf{U}^n + \mu\mathcal{B}(\mathbf{U}^n)\,. \tag{7.1.18}$$

For n fixed, a natural iteration to find $\mathbf{U}^n$ satisfying (7.1.10) is as follows: Find $\mathbf{U}^{n,j+\frac{1}{2}}$ such that

$$\mathbf{U}^{n,j+\frac{1}{2}}(x_i) + \mu\mathcal{A}(\mathbf{U}^{n,j+\frac{1}{2}}(x_i)) = \mathbf{Z}^{n,j}(x_i) \quad \text{for } i = 0, \cdots, J\,, \tag{7.1.19}$$

and set

$$\mathbf{X}^{n,j+1} = 2\mathbf{U}^{n,j+\frac{1}{2}} - \mathbf{Z}^{n,j}\,. \tag{7.1.20}$$

Then find $\mathbf{U}^{n,j+1}$ such that

$$\mathbf{U}^{n,j+1} + \mu\mathcal{B}(\mathbf{U}^{n,j+1}) = \mathbf{X}^{n,j+1}\,, \tag{7.1.21}$$

and set

$$\mathbf{Z}^{n,j+1} = 2\mathbf{U}^{n,j+1} - \mathbf{X}^{n,j+1}. \tag{7.1.22}$$

We note that solving (7.1.21) is equivalent to solving

$$\begin{aligned}\mathbf{X}^{n,j+1} &= \mathbf{U}^{n,j+1} + \mu\mathcal{B}(\mathbf{U}^{n,j+1})\\ &= \mathbf{U}^{n,j+1} + \mu[\mathbf{U}^{n,j+1} - \mathbf{U}^{n-1} + \Delta t M^{-1}K\mathbf{U}^{n,j+1}\\ &+\Delta t\beta M^{-1}(U_0^{n,j+1}, 0, \cdots, 0, U_J^{n,j+1})^T + \Delta t g_-(\mathbf{U}^{n-1})]. \end{aligned}\tag{7.1.23}$$

Rearranging the above equation gives

$$\begin{aligned}&\left((1+\mu) + \mu\Delta t M^{-1}K\right)\mathbf{U}^{n,j+1} + \mu\Delta t\beta M^{-1}(U_0^{n,j+1}, 0, \cdots, 0, U_J^{n,j+1})^T\\ &= \mathbf{X}^{n,j+1} + \mu\mathbf{U}^{n-1} - \mu\Delta t g_-(\mathbf{U}^{n-1}). \end{aligned}\tag{7.1.24}$$

The square matrices $(1+\mu)I + \mu\Delta t M^{-1}K$ and $\mu\Delta t\beta M^{-1}$ are symmetric positive definite. It follows that the system has a unique solution.

Theorem 7.1.2. For all $\mu \in \mathbb{R}^+$ and $\{\mathbf{U}^{n,0}\} \in S^h$, the sequence $\{\mathbf{U}^{n,j}\}_{j=0}^{\infty}$ generated by algorithm (7.1.19)-(7.1.21) converges to the unique solution $\mathbf{U}^n$ of (7.1.10).

Proof. The proof is the same as that of Imran (2001), pp.111-114, who has adapted the proof of Copetti and Elliott (1992), pp.58-59.

Section 7.2: Numerical Simulations

Numerical simulations in one space dimension were performed with $\Omega = (0, L)$, for $0 \le t \le 20$. In all simulations we take $J = 1000$. We choose $\mu = 0.01$ and set $TOL = 1 \times 10^{-7}$ where $\|U^{n,j+1}(x_i) - U^{n,j}(x_i)\|_\infty < TOL$ is the stopping criterion for (7.1.19)–(7.1.22). Programs were written in Fortran 77 and graphs were generated in Matlab.

For the first example, we consider the Fisher equation

$$\frac{\partial u}{\partial t} = \Delta u + u(1-u), \tag{7.2.1}$$

with boundary conditions corresponding to (7.1.2)-(7.1.3). The equation is posed on the interval $\Omega = (0, L) = (0, 100)$ with $\Delta t = T/M = 20/20480 = 0.0009766$ and $h = 0.1$. Although the reaction term of Fisher's equation is outside the remit of the $g(u)$ which are considered in this thesis, the numerical experiment is still of some interest and verifies the validity of the Fortran programme. In Figure 1, we display the numerical results for the Fisher equation with (a) $\beta = 0$[9], (b) $\beta = 1$, (c) $\beta = 10$, and (d) $\beta = 1000$. The time separation between any two successive plots is 2 time units. Figure 1 shows

[9] the numerical result in (a) is a repeat of the one showed in Gazdag and Canosa (1974), p.453.

clearly that the wave front maintains its shape very accurately between plots regardless of β. Namely, the distance between any two successive profiles appears to be constant. An interesting result in our numerical solution is that the values of the numerical solution at $x = 0$ when $T = 20$ are affected by the values of β (these values are 1.0 in (a), 0.44 in (b), 0.057 in (c) and 0.00057 in (d) of Figure 1). We also did simulations with $TOL = 1 \times 10^{-9}$. The results were very similar. Moreover, the values of the numerical solution at $x = 0$ when $T = 20$ were the same.

In Figure 2 the Fisher equation (7.2.1) supplemented with boundary conditions (7.1.2)-(7.1.3) is posed on the interval $\Omega = (0, 50)$ with $TOL = 1 \times 10^{-7}$, where $\Delta t = 0.000651$ and $h = 0.05$ are decreased. Again, this figure shows clearly that the wave front still maintains its shape very accurately between plots regardless of β. Furthermore, the values of the numerical solution at $x = 0$ when $T = 20$ are almost similar to the values in Figure 1.

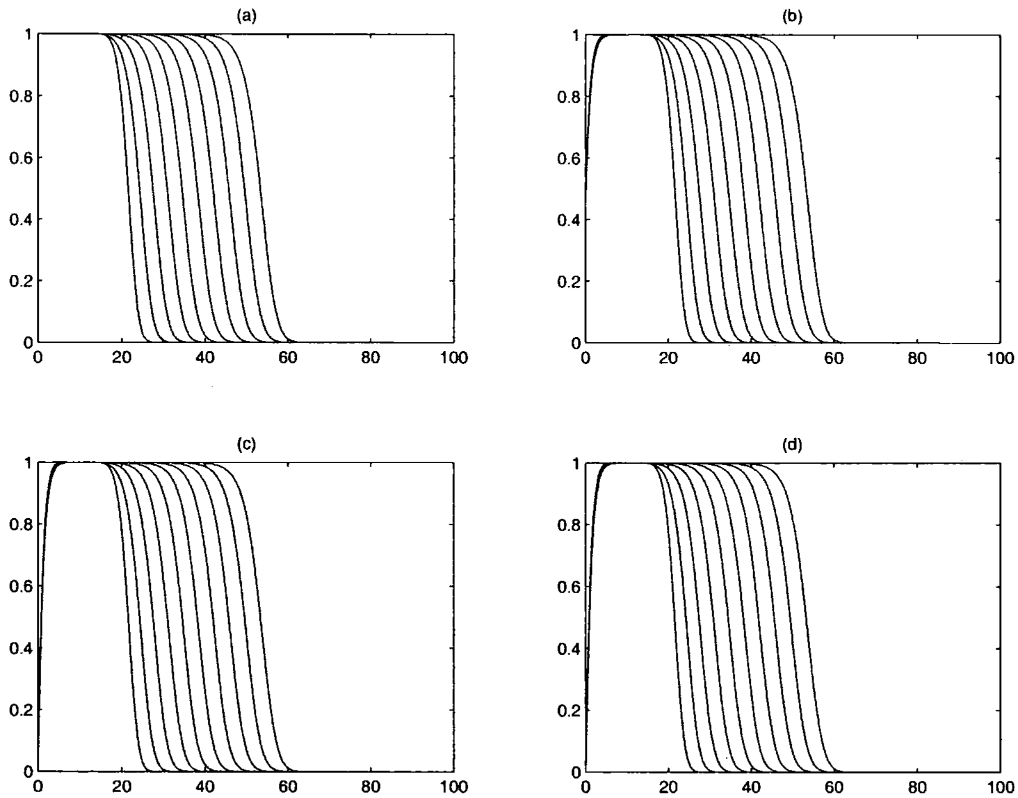

Figure 1: Numerical results for the 1D Fisher equation, obtained using the implicit scheme with $\Delta t = 0.0009766$ and $h = 0.1$. The time separation between any two successive profiles is 2 time units. We compare the results of numerical solution at β's (a) 0, (b) 1, (c) 10 and (d) 1000 when $t = 2, 4, 6, \ldots, 20$.

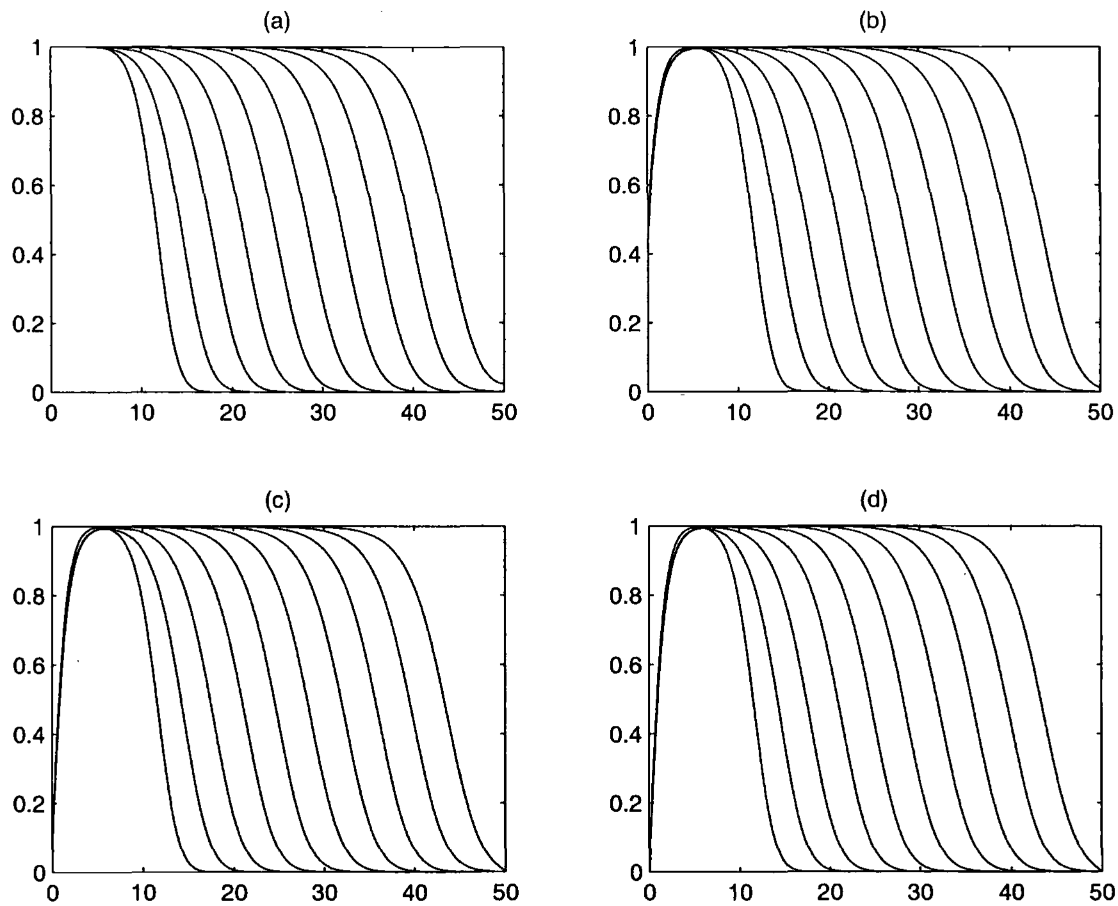

Figure 2: Numerical results for the 1D Fisher equation, obtained using the implicit scheme with $\Delta t = 0.000651$ and $h = 0.05$. The time separation between any two successive profiles is 2 time units. We compare the results of numerical solution at β's (a) 0, (b) 1, (c) 10 and (d) 1000 when $t = 2, 4, 6, \ldots, 20$.

For the second example, we consider the Ginzburg-Landau equation

$$\frac{\partial u}{\partial t} - \Delta u + u(u^2 - 1) = 0\,, \tag{7.2.2}$$

with boundary conditions corresponding to (7.1.2)-(7.1.3). This equation is also known as the Allen-Cahn equation. Here the reaction term of this equation satisfies the conditions on $g(u)$ (see (3.2.1)). Note that with $g(u) = u^3 - u$ we have $g_+(u) = u^3$ and $g_-(u) = -u$ when solving (7.1.4). Note that the solution of a time-independent Ginzburg-Landau equation in the absence of boundary conditions is

$$u(x) = \tanh\left(\frac{x}{\sqrt{2}}\right)\,. \tag{7.2.3}$$

For the one dimensional problem, we supplement (7.2.2) and (7.1.2)-(7.1.3) with the initial condition

$$u(x,0) = u^0(x) = \tanh\left(\frac{x - 0.5L}{\sqrt{2}}\right)\,. \tag{7.2.4}$$

The system is posed on the interval $\Omega = (0, L)$ with $TOL = 1 \times 10^{-7}$, $\Delta t = T/M = 32/20480 = 0.0016$ and different values of h (depending on L). It should be pointed out that the numerical solution is stationary because no iterations are required to go from one time level to the next, i.e., U^n does not change. In plots (a)-(d) of Figure 3, the numerical stationary solution of (7.1.4) at $T = 32$ and the tanh solution $u^0(x,t)$ are displayed, with initial condition corresponding to (7.2.4). Simulations are performed by setting $L = 10$ in (a), $L = 30$ in (b), $L = 60$ in (c) and $L = 300$ in (d). Here the values of β that satisfy the boundary conditions of the tanh solution are -1.8160×10^{-4}, -3.7430×10^{-13}, -3.5026×10^{-26} and -2.0593×10^{-130}, respectively. In each picture of Figure 3 we plot the numerical stationary solution with $\beta = 0.005$, 0.5 and 10, and compare them with the tanh solution. We notice that the numerical stationary solution accurately matches the tanh solution apart from the boundary layers. We also notice that taking $\beta > 0$ and very small makes no difference whereas a large $\beta > 0$ does as is seen in Figure 3(a). However, the effect of β disappears as L increases as is seen in (b)-(d) of Figure 3 since the boundary layers diminish as L increases. We also did simulations with $TOL = 1 \times 10^{-10}$. The results were similar. It is also interesting to realize that we have many mesh points in the interface, that is our approximations are good enough. In Figure 3(d), for instance, the interface occurs in the subinterval $I = (147, 153)$ where $h = 300/1000 = 0.3$.

In Figure 4 we supplement (7.2.2) and (7.1.2)-(7.1.3) with the initial condition (7.2.4). This system is posed on the interval $\Omega = (0, 30)$ with $TOL = 1 \times 10^{-9}$, where h and Δt are reduced progressively together according to the relationship $\Delta t = Ch^2$. This figure shows that the errors in the numerical stationary solutions decrease roughly as $\mathcal{O}(h^2)$ as the space-steps and time-steps are decreased. In other words, we notice from Figure 4 that the approximate solutions converge to the exact solutions of the tanh, which is compatible with what we have got in the theoretical convergence results. We also did simulations with $TOL = 1 \times 10^{-12}$. The results were similar.

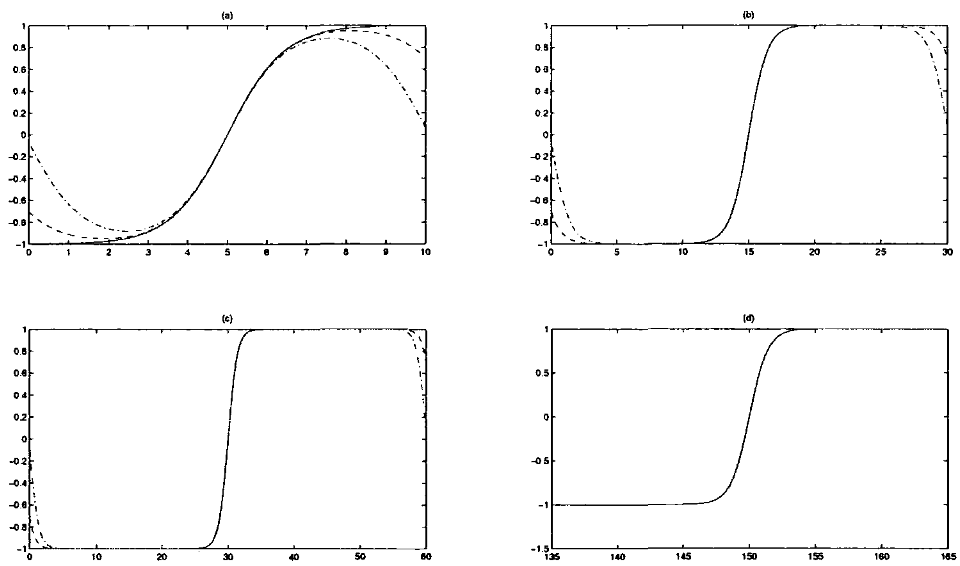

Figure 3: A comparison of the tanh solution (denoted ——) and numerical stationary solutions with $\beta = 0.005$ (denoted $\cdots\cdots$), $\beta = 0.5$ (denoted - - -) and $\beta = 10$ (denoted -·-·-·-). For L equal to (a) 10, (b) 30, (c) 60 and (d) 300. The implicit scheme is applied with $\Delta t = 0.0016$.

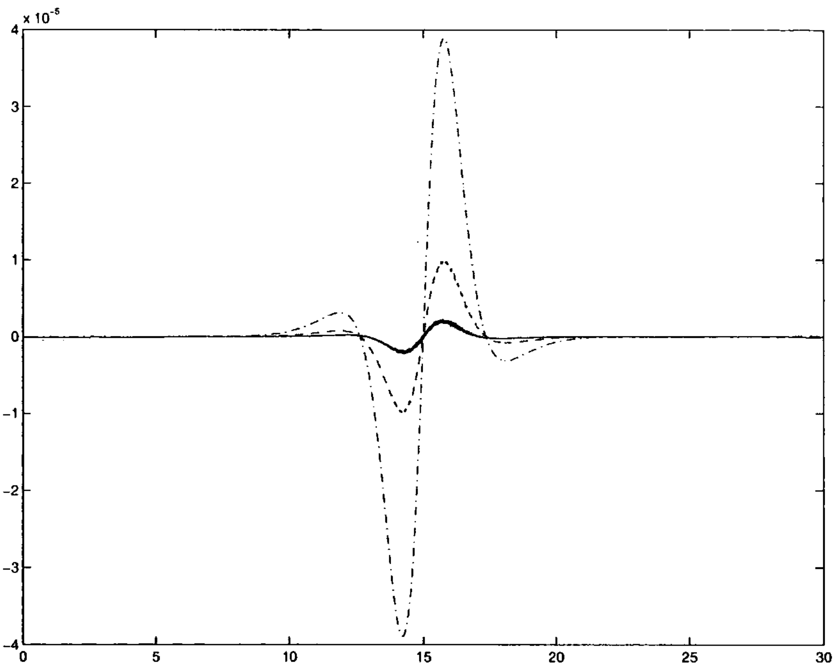

Figure 4: Error between the tanh solution and numerical stationary solutions with $L = 30$ and $\beta = 1 \times 10^{-7}$. The error with $h = 0.04$ and $\Delta t = 0.01$ is denoted by (-·-·-·-), with $h = 0.02$ and $\Delta t = 0.0025$ is denoted by (- - - -), and with $h = 0.01$ and $\Delta t = 0.000625$ is denoted by (——).

In the next experiment, we consider the Ginzburg-Landau equation supplemented with the initial condition

$$u(x,0) = u^0(x) = \sin\left(\frac{7\pi x}{20}\right), \qquad (7.2.5)$$

which was also considered in Elliott and Stuart (1993). The equation is posed on the interval $\Omega = (0,20)$ with $\Delta t = 0.1$ and $h = 0.2$. Figures 5–8 show the evolution of interfaces starting from initial data corresponding to (7.2.5). Figure 5 shows the solution at times $t = 0, 4, 10, 20, 40$ and 80 for $\beta = 1 \times 10^7$ being very large; this was chosen to replicate Elliott and Stuart's experiment results with Dirichlet boundary conditions. We found that $u(0) = 7.0473 \times 10^{-8}$ and $u(20) = 7.0474 \times 10^{-8}$, i.e. $u(0) = u(20) \approx 0$. This means that the Dirichlet boundary conditions are successfully achieved by taking large values of β. Moreover, the small timescale solutions are similar. However, rounding errors result in a different numerical solution, in particular, the numerical stationary solution appears to be the negative version of Elliott and Stuart's solution. Figure 6, Figure 7 and Figure 8 show the solution at times $t = 0, 4, 10$ and 80 for $\beta = 20, 2$ and 0, respectively. In all figures, we notice that the interfaces propagate on short times such as $t = 4$.

We also notice that in Figures 5–8 the growth of the approximate solutions, which initially were equal to 1 or less, never grew to exceed 1.

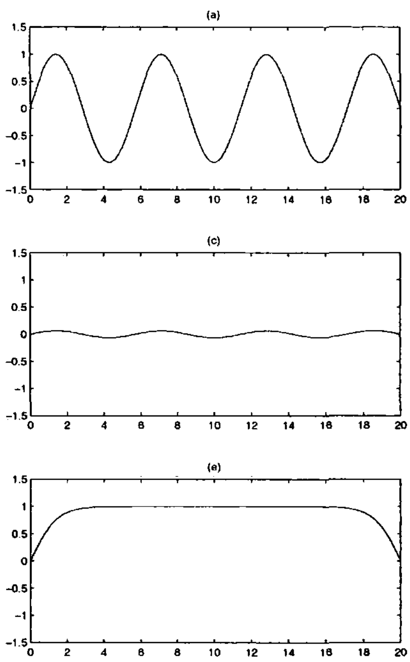

Figure 5: Numerical solutions of the Ginzburg-Landau equation with Robin boundary conditions and $\beta = 1 \times 10^7$. $\Omega = [0, 20]$ and $u^0(x) = \sin\left(\frac{7\pi x}{20}\right)$. For (a) $t = 0$, (b) $t = 4$, (c) $t = 10$ and (d) $t = 20$, (e) $t = 40$ and (f) $t = 80$. The implicit scheme is applied with $\Delta t = 0.1$ and $h = 0.2$.

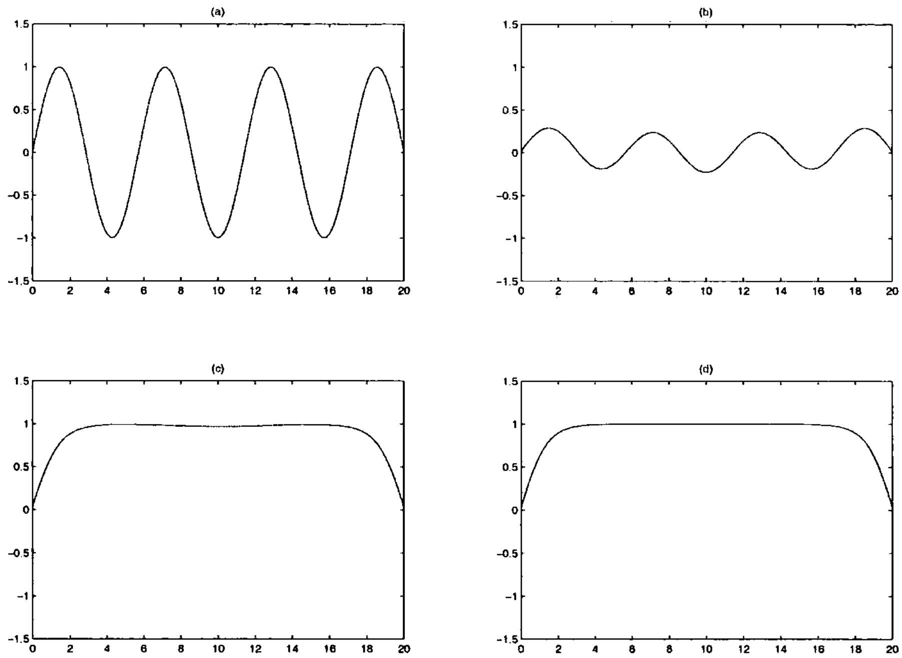

Figure 6: Numerical solutions of the Ginzburg-Landau equation with Robin boundary conditions and $\beta = 20$. $\Omega = [0, 20]$ and $u^0(x) = \sin\left(\frac{7\pi x}{20}\right)$. For (a) $t = 0$, (b) $t = 4$, (c) $t = 10$ and (d) $t = 80$. The implicit scheme is applied with $\Delta t = 0.1$ and $h = 0.2$.

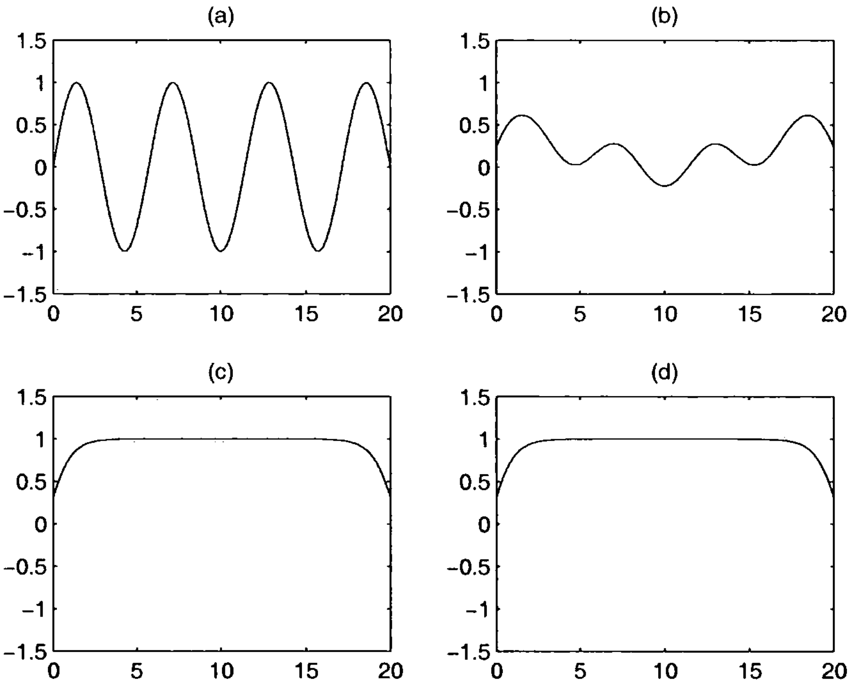

Figure 7: As in Figure 6 but for $\beta = 2$.

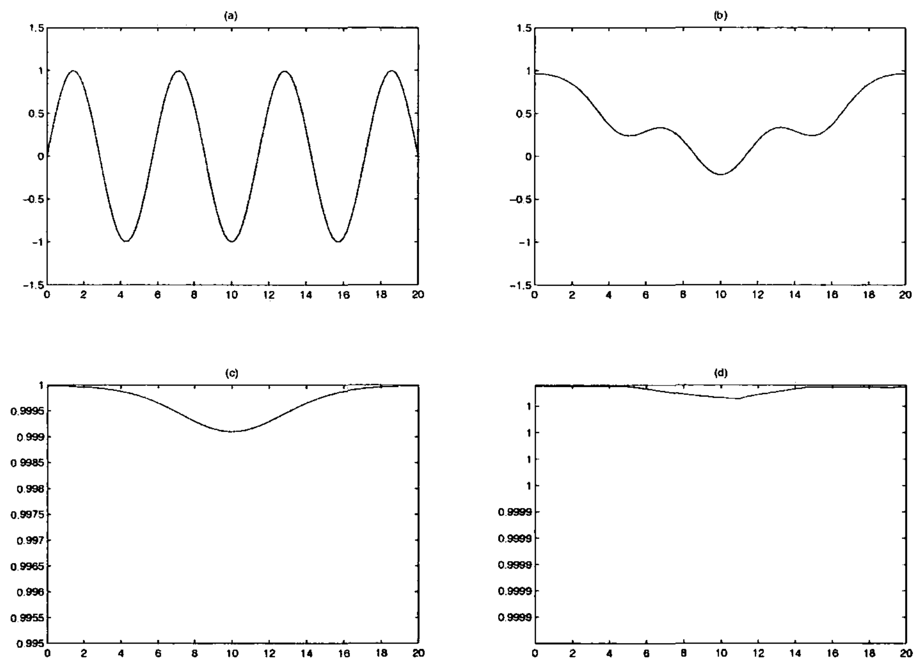

Figure 8: As in Figure 6 but for $\beta = 0$.

Chapter 8

Conclusions

Section 8.1: Summary

We discussed the spectral theory of the Robin boundary value problem. It was shown using the Hilbert-Schmidt theorem that there is an orthonormal basis for $L^2(\Omega)$ and an orthogonal basis for $H^1(\Omega)$ consisting of eigenfunctions of the operator $A = -\Delta + I$ with the Robin boundary condition (see Chapter 2). This work is essential for constructing the Galerkin approximations for the reaction-diffusion equations with the initial and Robin boundary conditions.

In Chapter 3 a generalised class of nonlinear reaction-diffusion equations with initial and Robin boundary conditions has been studied. By using the Faedo-Galerkin method and the Alaoglu compactness theorem, we proved the existence, uniqueness and continuous dependence on initial data of weak solutions of the nonlinear reaction-diffusion problem with the parameter $s > 1$.

The regularity result of the elliptic boundary value problem with Robin boundary conditions was considered in Section 4.1. This result and estimates of weak solutions were used to prove the existence, uniqueness and continuous dependence on initial data of the strong solutions of the nonlinear reaction-diffusion problem (see Section 4.2). The difficulty in this chapter was to prove the existence of strong solutions when the initial data is in $H^1(\Omega)$. This forced us to restrict the values of the parameter s via the assumption **(S)**.

The concept of finite element method and some necessary tools were given in Section 5.1 for analysing a semi and fully discrete approximation. In Section 5.2 the existence, uniqueness and stability estimates of the semi-discrete finite element approximation were proved in $d \leq 3$ space dimensions. An error bound between the semi-discrete and continuous solutions was proved in Section 5.3, which is optimal in H^1 but sub-optimal in L^2.

In Section 6.1 we proved the existence, uniqueness and stability estimates of a fully discrete finite element approximation. An error bound between the fully and semi discrete solutions was proved in Section 6.2, which is not optimal in the sense of Δt.

The error bound between the fully discrete and continuous solutions was proved by combining the results of Theorem 5.3.2 and Theorem 6.2.1. This error bound was optimal in H^1 for the space step, and not optimal in H^1 for the time step.

An algorithm for computing the numerical solutions was given in Section 7.1. We showed that the algorithm converges to the unique solution of the implicit scheme (see Theorem 7.1.2). Simulations in one space dimension were performed using the implicit scheme.

Section 8.2: Future Work

In this thesis we studied the scalar reaction-diffusion equations:

$$\frac{\partial u}{\partial t} - \Delta u + g(u) = 0\,,$$

where the nonlinear reaction term $g(u)$ is an odd polynomial satisfying the growth assumption. It would be possible to mimic this study to analyse the reaction-diffusion equations involving a vector function $u = (u_1, \cdots, u_m) \in \mathbb{R}^m$. A particular example is the coupled pair of the standard predator-prey equations:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u + (1 - r^2)u - (\omega_0 - \omega_1 r^2)v\,, \\
\frac{\partial v}{\partial t} &= \Delta v + (\omega_0 - \omega_1 r^2)u + (1 - r^2)v\,,
\end{aligned}$$

with $r^2 = u^2 + v^2, \omega_i > 0$, and the Robin boundary conditions. Analysing this example is recommended for future work.

We could relax the condition of the nonlinear term $g(u)$ by making further assumptions on it such as

$$\textbf{(S1)} \qquad g(\cdot) \in C^2(\mathbb{R}, \mathbb{R}) \text{ and } g(0) = 0\,,$$

$$\textbf{(S2)} \qquad \exists\, \bar{u} > 0 \text{ such that } g(r)/r > 0\, \forall r : |r| > \bar{u}\,,$$

$$\textbf{(S3)} \qquad G''(u) \geq -C_G \text{ where } G(u) = \int g(u)du\,,$$

along with the growth equation (1.1.3) (see, e.g., Elliott and Stuart (1993)). A canonical example of a function with such assumptions is $g(u) = u^3 - u$. We could also investigate the Fisher's equation with an appropriate modification.

In the thesis we showed that all results hold for an open bounded convex domain Ω, which has a Lipschitz continuous boundary $\partial\Omega$. Noting that some results still hold for an open bounded (not necessarily convex) domain with a Lipschitz continuous boundary $\partial\Omega$ (i.e. with $\partial\Omega \in C^{0,1}$). For instance, the results of Theorem 3.1.1 holds if the open bounded domain has a Lipschitz continuous boundary. However, the regularity results of Chapter 4 require that the domain Ω be a bounded convex, which can be approximated by domains with C^2 boundaries (see Lemma 4.1.2). The question is "can we approximate a non-convex domain by domains with C^2 boundaries?". We leave this for future work.

The error bound between the fully and semi discrete solutions would be optimal if it were $\mathcal{O}(\Delta t^2)$, but we do not have the necessary stability to achieve this. However, it might be possible to make the error bound optimal as in Barrett and Blowey (2001), and we leave this for future work.

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Appendix

In this appendix we present some definitions and well know theorems that are related to the work in this thesis. **Theorem A.0.1 (Riesz Representation theorem)**
Let H be a Hilbert space with inner product $(\cdot,\cdot)_H$. Let H' be the dual space of H. Then the two spaces are isometrically isomorphic. Moreover, any continuous linear functional $L \in H'$ acting on H can be represented uniquely as

$$L(v) = (u, v) \qquad \forall v \in H\,,$$

for some $u \in H$. Furthermore, we have

$$\|L\|_{H'} = \|u\|_H\,.$$

According to the Riesz Representation Theorem, there is a natural isometry between H and H'

$$u \in H \longleftrightarrow L_u \in H'\,.$$

For this reason, H and H' are often identified. For example, we can write $H^m(\Omega) \cong H^{-m}(\Omega)$ (although they are completely different Hilbert spaces). This theorem and its proof can be found in several books, see, e.g, Brenner and Scott (2002), pp. 55 - 56.

Theorem A.0.2 (The Lax-Milgram lemma)
Let V be a Hilbert space. If $a(\cdot,\cdot)$ is a bounded and coercive bilinear form on $V \times V$, and L is a bounded linear functional on V, then there exists a unique $u \in V$ such that

$$a(u,v) = L(v) = \langle L, v\rangle_{V',V} \qquad \forall v \in V\,,$$

(see, e.g., Ciarlet (1978), p.8). In addition, we have from the coercivity condition that

$$\|v\|_V \leq \frac{1}{\alpha}\|L\|_{V'}\,,$$

where $\alpha > 0$ is the coercivity constant.

This theorem is a generalization of the Riesz Representation theorem since the Lax-Milgram lemma holds in symmetric and non-symmetric bilinear forms whereas the Riesz Representation theorem holds only in symmetric ones. For the proof, we refer to a standard text on functional analysis.

Theorem A.0.3 (Hilbert-Schmidt theorem)

Let H be an infinite dimensional Hilbert space and let $L : H \rightarrow H$ be a compact, self-adjoint operator. Then there is a sequence of non-zero real eigenvalues $\{z_i\}_{i=1}^{\infty}$ of L such that

$$\cdots \leq |\mu_{i+1}| \leq |\mu_i| \leq \cdots \leq |\mu_1| \,,$$

and

$$\lim_{i \to \infty} \mu_i = 0 \,.$$

Furthermore, if each eigenvalue of L is repeated in the sequence according to its multiplicity, then there exists an orthonormal set $\{z_i\}_{i=1}^{\infty}$ of corresponding eigenfunctions, i.e.

$$Lz_i = \mu_i z_i \,.$$

Moreover, the functions z_i form an orthonormal basis for the range of L and L can be written as

$$Lu = \sum_{i=1}^{\infty} \mu_i (z_i, u) z_i \quad \text{for all } u \in H$$

(see Renardy and Rogers (1993), pp.267-268, Robinson (2001), pp.75-76).

Definition A.0.4 (a Lipschitz condition)

A real valued function f defined on a subset U of the Euclidean spaces

$$f : U \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

is called **Lipschitz continuous** or is said to satisfy a **Lipschitz condition** if there exists a constant $C \geq 0$ such that

$$|f(u_1) - f(u_2)| \leq C\|u_1 - u_2\| \quad \forall u_1, u_2 \in U,$$

where $\|\cdot\|$ denotes the Eucliden norm in $\mathbb{R}^m$. The smallest such C is called the Lipschitz constant of the function f (See Zenisek (1990), p. 4.)

The function is called **locally Lipschitz continuous** if for every u in U there exists a convex compact subset V of the domain U so that f restricted to V is Lipschitz continuous. A continuously differentiable function g is Lipschitz continuous (with $C = \sup|g'(x)|$) if it has bounded first derivative. Thus any C^1 function is locally Lipschitz, as continuous functions on a locally compact space are locally bounded (Arnol'd (1992), pp.272-273).

Theorem A.0.5 (Green's identity)
Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. If $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, then

$$\int_\Omega \Delta u \, v \, dx = -\int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, d\sigma \, ,$$

where n is the outward unit normal to $\partial\Omega$ (see, for example, Rodrigues (1987), p. 76).

Theorem A.0.6 (Sobolev spaces results)
The Sobolev spaces $W^{m,p}(\Omega)$, $m \in \mathbb{N}$, associated with the appropriate norms satisfy the following:

(i) $W^{m,p}(\Omega), 1 \leq p \leq \infty$ is a Banach space (Renardy and Rogers (1993), p.206).

(ii) If $1 \leq p < \infty$, then $W^{m,p}(\Omega)$ is separable (Renardy and Rogers (1993), p.206, Adams (1975), p.47).

(iii) If $1 < p < \infty$, then $W^{m,p}(\Omega)$ is reflexive (Adams (1975), p.47).

Definition A.0.7 (Continuous/Compact embedding)
We say that the normed vector space X has *continuous embedding* into the normed vector space Y, if
(i) X is a vector subspace of Y, and
(ii) the identity operator I defined on X into Y by $Ix = x$ for all $x \in X$ is continuous.

By the *compact embedding* we mean that the identity operator I is compact. (see Adams (1975), p.9).

Theorem A.0.7 (Sobolev embedding theorems)
Let $\Omega \subset \mathbb{R}^d$ be bounded with Lipschitz boundary, $m \in \mathbb{N}$ and $p \in [1, \infty]$. Then, the following mappings represent continuous embeddings

$$\text{(i)} \quad W^{m,p}(\Omega) \to L^{p'}(\Omega), \quad \tfrac{1}{p'} = \tfrac{1}{p} - \tfrac{m}{d}, \text{ if } m < \tfrac{d}{p},$$

$$\text{(ii)} \quad W^{m,p}(\Omega) \to L^{q}(\Omega), \quad q \in [1, \infty), \text{ if } m = \tfrac{d}{p},$$

$$\text{(iii)} \quad W^{m,p}(\Omega) \to C^{0,m-\frac{d}{p}}(\overline{\Omega}), \quad \text{if } \tfrac{d}{p} < m < \tfrac{d}{p} + 1,$$

$$\text{(iv)} \quad W^{m,p}(\Omega) \to C^{0,\alpha}(\overline{\Omega}), \quad 0 < \alpha < 1, \text{ if } m = \tfrac{d}{p} + 1,$$

$$\text{(v)} \quad W^{m,p}(\Omega) \to C^{0,1}(\overline{\Omega}), \quad \text{if } m > \tfrac{d}{p} + 1.$$

For proofs we refer to Adams (1975), Chapter 5.

More general is the Sobolev embedding theorem:

$$W^{m,p}(\Omega) \to W^{k,q}(\Omega)$$

with $k < m$ and $q > p$ (See Brenner and Scott (2002), p. 32). In this thesis, we will also take advantage of compact embeddings of Sobolev spaces:

Theorem A.0.8 (Kondrasov embedding theorems)

Let $\Omega \subset \mathbb{R}^d$ be bounded with Lipschitz boundary, $m \in \mathbb{N}$ and $p \in [1, \infty]$. Then, the following mappings are compact embeddings

(i) $W^{m,p}(\Omega) \to L^q(\Omega), 1 \leq q \leq p', \frac{1}{p'} = \frac{1}{p} - \frac{m}{d}$, if $m < \frac{d}{p}$,

(ii) $\quad W^{m,p}(\Omega) \to L^q(\Omega), \quad q \in [1, \infty)$, if $m = \frac{d}{p}$,

(iii) $\quad W^{m,p}(\Omega) \to C^0(\overline{\Omega}), \quad$ if $m > \frac{d}{p}$,

(see Adams (1975), Chapter 6).

Theorem A.0.9 (Trace embedding theorems)

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $\partial\Omega$ is C^1, and $1 \leq p < \infty$. Then there exists a bounded linear operator

$$\gamma : W^{1,p}(\Omega) \longrightarrow L^p(\partial\Omega)$$

such that

$$\gamma u = u|_{\partial\Omega} \quad \text{if} \quad u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}),$$

and

$$\|\gamma u\|_{L^p(\partial\Omega)} \leq C\|u\|_{1,p} ,$$

for each $u \in W^{1,p}(\Omega)$ with the associated norm $\|\cdot\|_{1,p}$, and the constant C depending only on p and Ω. (See Evans (1998), pp. 257-261).

Theorem A.0.10 (Sobolev extension theorem)

Let $\Omega \subset \mathbb{R}^d$ be bounded with Lipschitz boundary, $m \in \mathbb{N}$, and $p \in [1, \infty]$. Then, there exists a bounded linear extension operator $E : W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^d)$, i.e., $Eu|_{\Omega} = u$ for all $u \in W^{m,p}(\Omega)$ and there exists a constant $C \geq 0$ such that for all $u \in W^{m,p}(\Omega)$

$$\|Eu\|_{W^{m,p}(\mathbb{R}^d)} \leq C\|u\|_{W^{m,p}(\Omega)} .$$

This result is proved, e.g., in Stein (1970).

Moreover, if $u \in W^{m,p}(\mathbb{R}^d), 1 \leq p \leq \infty$, then for any domain $\Omega \subset \mathbb{R}^d$ the natural **restriction** Ru of u to Ω

$$Ru(x) := u(x) \qquad \text{for a.e. } x \in \Omega$$

is well-defined in $W^{m,p}(\Omega)$. However, it is in general not possible to continuously extend a function $u \in W^{m,p}(\Omega)$ to a function in $W^{m,p}(\mathbb{R}^d)$ when Ω does not have a Lipschitz boundary.

Theorem A.0.11 (Time-Dependent Sobolev spaces results)

Let X and Y be Banach spaces. The time-dependent Sobolev spaces $L^p(0,T;X)$ with appropriate norms satisfy

(i) $L^p(0,T;X), (1 \leq p \leq \infty)$, is a Banach space.

(ii) $L^p(0,T;X), (1 \leq p < \infty)$, is separable if and only if X is separable.

(iii) $L^p(0,T;X), (1 < p < \infty)$, is reflexive if X is reflexive.

(iv) If X is a reflexive (or separable) Banach space and $(1 \leq p < \infty)$ then $[L^p(0,T;X)]' \cong L^{p'}(0,T;X')$ where $\frac{1}{p} + \frac{1}{p'} = 1$ and the symbol "$\cong$" means " isometrically isomorphic".

(v) The continuous injection $X \hookrightarrow Y$ implies $L^q(0,T;X) \hookrightarrow L^p(0,T;Y)$ if $1 \leq p \leq q \leq \infty$.
These results are collected from Zenisek (1990), Kreyszig (1978) and Malek (1996), and they were summarised in Garvie's thesis, p.133.

Theorem A.0.12 (Alaoglu compactness theorem)

(i) Suppose X is a separable Banach space and let $\{f^k\}$ be a bounded sequence in the dual space X'. Then f^k has a subsequence that is weak* convergent in X' (See, e.g., Robinson (2001), p.105).

(ii) Suppose X is a reflexive Banach space and let $\{x^k\}$ be a bounded sequence in X. Then x^k has a subsequence that converges weakly in X (See, e.g., Robinson (2001), p.106).

Theorem A.0.13 (Picard's existence theorem: local existence) Consider the initial value problem:

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0, \quad t \in [t_0 - \alpha, t_0 + \alpha].$$

Suppose $f(t, y(t))$ is bounded, Lipschitz continuous in y, and continuous in t. Then, for some value $\varepsilon > 0$, there exists a unique solution $y(t)$ to the initial value problem within the range $[t_0 - \varepsilon, t_0 + \varepsilon]$.

Theorem A.0.14 (Grönwall lemma: differential form) Let y, h, g be three locally integrable functions on $]t_0, +\infty[$ that satisfy

$$\frac{dy}{dt} \leq g(t)y + h(t) \quad \text{for } t \geq t_0.$$

Then,

$$y(t) \leq y(t_0) \exp\left(\int_{t_0}^{t} g(\tau)d\tau\right) + \int_{t_0}^{t} h(s) \exp\left(-\int_{t}^{s} g(\tau)d\tau\right) ds,$$

(see Temam (1997), p. 90).

In particular, if $g = -1$, $h(t) = C - f(t)$, where C is a constant and $f(t)$ is a non-negative function, and

$$\frac{dy}{dt} \leq -y + C - f(t) \quad \text{a.e. in } [0, T],$$

then

$$y(T) + \int_0^T e^{t-T} f(t) dt \leq y(0)e^{-T} + C(1 - e^{-T}).$$

Since $e^{t-T} \geq e^{-T}$ for all $t \in [0, T]$ we have

$$y(T) + e^{-T} \int_0^T f(t) dt \leq y(0)e^{-T} + C(1 - e^{-T}).$$

Note that, in the thesis, we have often used the last inequality. For more general result of Grönwall's lemma the reader may refer to Emmrich's paper (1999), p.7. Moreover, the auther gave many versions of the Grönwall lemma in that paper.

Definition A.0.15 (Strong convergence)
Let X be a normed vector space. Then $x_n \to x$ in X means that a sequence $\{x_n\}$ converges *strongly* to an element x in X. This type of convergence is also called *norm convergence*. Note that we use "$\to$" to denote strong con-

vergence.

Definition A.0.16 (Weak convergence)
Let X be a Banach space. Then $x_n \rightharpoonup x$ in X means that a sequence $\{x_n\}$ converges *weakly* to x in X if $f(x_n)$ converges to $f(x)$ for every $f \in X'$. Note that we use "$\rightharpoonup$" to denote weak convergence.

Definition A.0.17 (Weak* convergence)
Let X be a Banach space. Then $f_n \overset{*}{\rightharpoonup} f$ in X' means that a sequence $\{f_n\}$ converges *weakly** to f in X' if $f(x_n)$ converges to $f(x)$ for every $x \in X$.

Theorem A.0.18 (Weak and Weak* convergence results)
Let X be a Banach space and X' its dual. Let $\{x_n\}$ be a sequence in X and $\{f_n\}$ a sequence in X'. Then

(i) $x_n \rightharpoonup x$ in X if and only if $\langle f, x_n\rangle \to \langle f, x\rangle \quad \forall f \in X'$.

(ii) $f_n \overset{*}{\rightharpoonup} f$ in X' if and only if $\langle f_n, x\rangle \to \langle f, x\rangle \quad \forall x \in X$.

(iii) $x_n \to x$ (strong) implies $x_n \rightharpoonup x$ (weak).

(iv) If $x_n \rightharpoonup x$ in X, then $\|x_n\|_X$ is bounded and $\|x\|_X \le \liminf \|x_n\|_X$.

(v) The weak and strong convergences are equivalent in finite dimensions.

For the above results we refer to Rodrigues (1987), pp. 55-56.

(vi) The weak and weak* limits are unique (see, e.g., Renardy and Rogers (1993), p. 203).

Theorem A.0.19

Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary, and let $\mathbf{v} \in H^1(\Omega)^n$. Then, we have

$$\int_\Omega |\operatorname{div} \mathbf{v}|^2 \, dx - \sum_{i,j=1}^{n} \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} \, dx = -2\langle \mathbf{v}_T, \nabla_T(\mathbf{v} \cdot n) \rangle$$

$$- \int_{\partial\Omega} \left\{ B(\mathbf{v}_T, \mathbf{v}_T) + (\mathrm{tr} B)[\mathbf{v} \cdot n]^2 \right\} d\sigma \,, \qquad \text{(A.7.17)}$$

where $\mathbf{v} \cdot n$ is the component of $\mathbf{v}$ in the direction of n while we denote by $\mathbf{v}_T$ the projection of $\mathbf{v}$ on the tangent hyperplane to $\partial\Omega$ (i.e. $\mathbf{v}_T = \mathbf{v} - (\mathbf{v} \cdot n) \cdot n$). In the same way, we denote by ∇_T the projection of the gradient operator on the tangent hyperplane (i.e. $\nabla_T u = \nabla u - \dfrac{\partial u}{\partial n} \cdot n$). Here B is a bilinear form

$$\xi, \eta \longmapsto - \sum_{\mathbf{j},\mathbf{k}} \frac{\partial \mathbf{n}}{\partial \mathbf{s_j}} \cdot \tau_{\mathbf{k}} \xi_{\mathbf{j}} \eta_{\mathbf{k}}$$

where ξ and η are the tangent vectors to $\partial\Omega$, whose components are $\{\xi_1, \cdots, \xi_{n-1}\}$ and $\{\eta_1, \cdots, \eta_{n-1}\}$, respectively, in the basis (the unit tangent vectors) $\{\tau_1, \cdots, \tau_{n-1}\}$ and the arc lengths $\{s_1, \cdots, s_{n-1}\}$. In other words,

$$B(\xi, \eta) = -\frac{\partial \mathbf{n}}{\partial \xi} \cdot \eta$$

where $\partial/\partial\xi$ denotes differentiation in the direction of ξ. The trace of the bilinear form above, $\mathrm{tr}B$, is as follows

$$\mathrm{tr}B = -\sum_{j=1}^{n-1} \frac{\partial n}{\partial s_j} \cdot \tau_j .$$

Note that a convex domain can be approximated by sequences of domains with a C^2 boundary. Such domains with a C^2 boundary give the definition of the bilinear B to be negative, i.e. $B(\xi, \eta) = -\frac{\partial \mathbf{n}}{\partial \xi} \cdot \eta$ (see Grisvard (1985), pp.132-138).