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Smooth Supergravity Solutions and String States

Owen Francis Madden

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A Thesis presented for the degree of
Doctor of Philosophy

Centre for Particle Theory
Department of Mathematical Sciences
University of Durham
England

July 2005

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Dedicated to

Mum and Dad

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Abstract

In this thesis we study smooth supergravity solutions and their relation to string theory in two different contexts; quotient spaces and asymptotically flat solitonic solutions.

We classify discrete cyclic quotients of $p+1$-dimensional anti-de Sitter space. These provide interesting models for string propagation where a non-perturbative description is available. We establish which quotients have well-behaved causal structures, and of those containing closed timelike curves, which have interpretations as black holes. We explain the relation to previous investigations of quotients of asymptotically flat spacetimes and plane waves, and of black holes in AdS.

We construct smooth non-supersymmetric soliton solutions with D1-brane, D5-brane and momentum charges in type IIB supergravity compactified on $T^4 \times S^1$. Such solutions have been conjectured to be related to black hole microstates. The solutions are obtained by considering a known family of $U(1) \times U(1)$ invariant metrics, and studying the conditions imposed by requiring smoothness. We discuss the relation of our solutions to states in the CFT describing the D1-D5 system, and describe various interesting features of the geometry.

We show that the solutions describing charged rotating black holes in five-dimensional gauged supergravities found recently by Cvetič, Lü and Pope [1, 2] are completely specified by the mass, charges and angular momentum, demonstrating that an apparent non-uniqueness is a coordinate artefact.

Declaration

The work in this thesis is based on research carried out at the Centre for Particle Theory, the Department of Mathematical Sciences, Durham University, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and is all my own work unless referenced in the text. The work in chapter 3 was performed in collaboration with José Figueroa-O'Farrill, Simon Ross and Joan Simón [3] apart from section 3.1 which was performed in collaboration with Simon Ross [4] and section 3.5 which is independent work. The work in chapter 4 was performed in collaboration with Vishnu Jejjala, Simon Ross and Gina Titchener [5]. The work in chapter 5 was performed in collaboration with Simon Ross [6].

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Chapter 1

Introduction

Einstein's general theory of relativity is one of the most elegant mathematical descriptions of natural phenomena ever produced. The fundamental tenet of relativity is that the distribution of matter actually determines the geometry of space-time, gravity is then a manifestation of this space-time curvature. Relativity is capable of describing with great accuracy and precision almost every aspect of observed gravitational physics.

Nevertheless, relativity seems incomplete from a theoretical perspective. In the 1960's Penrose, Hawking and Geroch used global methods of analysis to establish a number of theorems which showed that singularities are a generic feature of classical general relativity [7–9]. If certain reasonable assumptions hold then these singularities are expected to arise in two situations of physical relevance. First, sufficiently massive objects such as large stars can undergo gravitational collapse to form a black hole. The matter is condensed into a singularity which is conjectured to be censored from external observers behind an event horizon. Second, by running the collapse argument backwards in time relativity predicts that our expanding universe has an initial 'big bang' singularity.

These predictions signal the breakdown of relativity as a physical theory. To elaborate, a space-time is defined to be singular if it is geodesically incomplete [10]. This implies that in finite proper time observers can reach boundaries of space-time beyond which we cannot evolve the dynamical equations of the theory. Usually we encounter additional problems as we approach these singularities such as divergent curvature, energy densities and tidal forces. Our current belief is that in the neighbourhood of a potential singularity quantum gravitational effects step in to resolve these difficulties.

At present, string theory provides the most developed, though still far from complete, description of gravitational quantum phenomena (See [11–14] for an introduction). String theory is based on the deceptively simple premise that at Planckian scales where the quantum effects of gravity are conjectured to be important, particles are actually one-dimensional extended objects. The usual particles emerge as string excitations and the known forces are described by the geometric splitting and joining of these strings. Unlike theories of point particles, consistent quantization severely constrains the properties of potential string theories. At the perturbative level, this demands supersymmetric theories that live in ten space-time dimensions. In the low energy limit supergravity is recovered which is the supersymmetric extension of standard general relativity.

Once one goes beyond perturbation theory one finds that the spectrum contains extended objects called D-branes [15]. Through the open strings, which have their endpoints confined to the branes, the D-branes realize gauge theories on their worldvolume. On the other hand, they have another low energy interpretation as a gravitational background for closed string propagation. The relation between these two descriptions has motivated conjectured dualities which connect gauge fields and gravitational theories.

The most developed example of such a duality is the AdS $/CFT$ correspondence [16]. This conjecture relates string theory in a spacetime where the non-compact part is asymptotically anti-de Sitter space (AdS), a space of constant negative curvature, to a conformal field theory (CFT) defined on a space isomorphic to the boundary of AdS. The relation between the two theories is a "duality" in the sense that there is a parameter such that in the region where it is small one description is in the perturbative regime (weakly coupled) and the other is strongly coupled, while the opposite is true when this parameter is large. The original example proposed by Maldacena was an equivalence between type IIB string theory compactified on $AdS_5 \times S^5$ and four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory [17].

In short, this equivalence includes a precise map between the states (and fields) on the string side and the local gauge invariant operators on the $\mathcal{N} = 4$ Yang Mills side, as well as a correspondence between the correlators in both theories [18, 19]. Since, as of yet, we do not have a reliable definition of non-perturbative type IIB string theory, it is difficult to prove this correspondence directly. Perhaps a more appropriate viewpoint is to actually take $\mathcal{N} = 4$ super Yang-Mills theory as the definition of non-perturbative type IIB string theory on the $\mathrm{AdS}_5 \times S^5$ background.

To gain insight into the stringy resolution of singularities localized in time such as those that occur inside black holes and at the big bang we need a better understanding of string theory in time-dependent settings. Lorentzian orbifolds of flat space are simple time-dependent solutions that provide (at least to leading order) consistent time-dependent backgrounds for string theory. Thus these can provide good toy models to study cosmological singularities in a controlled setting. Although some progress has been achieved [23–38], the indications are that perturbative string theory breaks down on space-times which include space-like and null singularities. To capture the physics of such singularities, a complete non-perturbative description of string theory appears to be necessary. A classification of smooth quotients of Minkowski spacetime was given in [39] recovering previous results on fluxbranes [40–44][1] and uncovering the existence of an interesting non-static smooth quotient—the nullbrane—which can be understood as a desingularisation of the parabolic orbifold [54], the supersymmetric toy model for a Big Crunch-Big Bang transition singularity, by the introduction of a new scale (modulus) that smoothes the singularity.

Since the AdS/CFT correspondence is conjectured to provide a fundamental, non-perturbative description of string theory with asymptotically AdS boundary conditions, it could be employed to relate the time-dependence to the dual field theory. Thus it is natural to wish to extend these investigations to consider strings on orbifolds of $\mathrm{AdS}_{p+1} \times \mathrm{S}^q$. Since AdS, like Minkowski space, is a maximally symmetric space, it has a large isometry group which can lead to interesting examples of quo-

[1]Related work on the physics of fluxbranes can be found in [45–53].

tients. In addition, it is well-known that a black hole geometry can be constructed from a quotient of AdS_3 [55, 56]. Such an extension was initiated in [57], where an AdS version of the isometry involved in the null brane quotient was constructed[2].

In this thesis we take a systematic approach to this question, classifying all the physically distinct quotients of AdS_{p+1} by one-parameter subgroups of its isometry group. The classification of quotients of AdS_3 was thoroughly explored in [56]. This was extended to AdS_4 in [68]. In chapter 3 we extend this to general dimensions, and in particular address the case of AdS_5, of great interest for string theory. (This question has also been explored independently by Figueroa-O'Farrill and Simón [69]). We focus on the causal structure of the quotients and the symmetry preserved under quotienting. As $AdS_{p+1} \times S^q$ backgrounds are maximally supersymmetric, it is also natural to study the question of how much supersymmetry was preserved by the quotient and in [69] there is a detailed analysis of this question and the related issue of the existence of a spin structure on the quotient.

Many of the quotients classified in chapter 3 contain closed timelike curves and while there may be some interest in studying such quotients, we shall nevertheless concentrate our attention on those quotients for which there is a well-founded expectation that they will provide good backgrounds for string propagation. We therefore study in detail quotients that can be given a simple physical interpretation: smooth quotients with a well-behaved causal structure. We take a conservative definition of well-behaved causal structure, aiming to find space-times that are stably causal. This means the spacetime admits no closed timelike curves even when the light cones are perturbed slightly (as they presumably are when quantum effects are switched on). The causally regular quotients fall into two categories; there are quotients where an action on AdS alone is well behaved and there are quotients where we need to add a suitable action on the transverse sphere to avoid closed null curves. Since smooth quotients are free of orbifold fixed points we can learn much about their geometry and physical interpretation by choosing a good global coordinate system. We systematically construct such coordinates by demanding that the

[2]Some other work concerning orbifolds of AdS can be found in [58–67].

causal structure, preserved symmetries and action of the quotient are made explicit. Disappointingly, all such causally regular quotients have a causal Killing vector ∂_v thus do not provide models for the study of time-dependence. We also briefly study those quotients which can be given a black hole interpretation following [55, 56].

Smooth supergravity solutions may also be relevant in another context. Bekenstein and Hawking showed that in order for black holes to be consistent with the laws of thermodynamics they should be viewed as thermodynamical systems with a temperature and an entropy [70–72]. The temperature is directly related to the black body radiation emitted by the black hole, whereas the entropy is given by

$$S_{BH} = \frac{A}{4G} \tag{1.1}$$

with G Newton's constant and A the area of the black hole horizon. The usual principles of statistical mechanics then suggest that there should be e^S microstates of the black hole for given macroscopic parameters. This is puzzling because in four dimensions the geometry of a black hole is uniquely determined by just its mass, charge and angular momentum. Early attempts to find the black hole microstates were based on looking for small perturbations in the metric and other fields while demanding smoothness at the horizon. No such perturbations were found; black holes have 'no hair' [73]. There was, however, a suggestion that pure states would be dual to geometries which were not smooth at the event horizon [74].

The fact that these differences are not visible in the classical description might not seem a problem as it seems reasonable that the $e^{S_{BH}}$ microstates only differ within a Planck sized neighbourhood of the singularity. However this leads to another potential difficulty, the so-called information paradox [75]. By taking a semi-classical approach Hawking showed that vacuum modes near the *horizon* evolve into particle pairs; one member of the pair falls into the hole and reduces its mass, while the other escapes to infinity as 'Hawking radiation'. If the information about the microstate resides 'near' the singularity then the outgoing radiation is unable to encode the details of the microstate, and when the hole has completely evaporated we cannot recover the information contained in the matter which went in to make the hole. This is a violation of the unitarity of quantum mechanics, and thus a severe contradiction with the way we understand evolution equations in physics.

String theory has made tremendous advances in understanding the microscopic origins of this black hole entropy [76, 77]. In the original calculations, two different dual descriptions of a supersymmetric object were considered: a weakly-coupled description in terms of perturbative strings and D-branes, and a strongly-coupled description as a classical black hole solution. The picture of this black hole, as a background for the perturbative string, is essentially the same as in semiclassical general relativity. We have a singularity in spacetime that is shielded by a horizon with essentially empty space in between them. The entropy was successfully reproduced by counting the degenerate supersymmetric vacua in the dual perturbative D-brane description. This picture did not provide any understanding of where the microstates were in the strong-coupling black hole picture.

The AdS $/CFT$ correspondence provided a deeper understanding of the counting of black hole entropy in string theory. Placing black holes in AdS space amounts to a study of the boundary theory at finite temperature, the black holes were identified with the thermal ensemble in the dual CFT [18, 19]. The microstates were fundamentally thought of as states in the CFT, and it did not appear that they could be thought of as living somewhere in the black hole geometry. The evolution of the states in the CFT is unitary. Certain states can be identified with classical geometries, but as has been emphasised in e.g. [18, 78], the CFT provides a fully quantised description, and reproducing the behaviour of the CFT from a spacetime point of view will in general involve a sum over bulk geometries.

In a series of papers, Mathur and his collaborators have challenged this conventional picture of a black hole in string theory (see [79] for a review). They argue that the black hole geometry is merely a coarse grained description of the spacetime, and that each of the $e^{S_{BH}}$ microstates can be identified with a perfectly regular geometry with neither horizon nor singularity [80,81]. The black hole entropy is a result of averaging over these different geometries, which produces an 'effective horizon', which describes the scale at which the $e^{S_{BH}}$ individual geometries start to differ from each other. They further argue that if a system in an initial pure state undergoes gravitational collapse, it will produce one of these smooth geometries, and the real space-time does not have a global event horizon, thus avoiding the information paradox. Thus the idea is that stringy effects modify the geometry of spacetime at the event horizon, rather than being confined to Planck or string distances from the singularity. In this picture the black hole interior is radically different from the naive picture suggested by classical gravity. There are similarities with the complementarity ideas [82], but unlike that picture, there is no obvious sense in which the spacetime as seen by an infalling observer will be different. It is difficult to see how the singularity behind the black hole's event horizon can arise from a coarse graining over non-singular geometries.[3]

Evidence for this proposal comes from studying a bound system of D1 and D5 branes, which is the simplest string theoretic object with entropy. Smooth asymptotically flat geometries in this D1-D5 system have been constructed that can be identified with individual microstates in the CFT on the worldvolume of the branes. The degeneracy of Ramond-Ramond (RR) ground states in this theory gives a microscopic entropy which scales as $\sqrt{n_1 n_5}$; this was found to match a suitable counting in a supertube description in [83, 84]. However, this entropy is not large enough to correspond to a black hole with a macroscopic horizon. It is therefore important to extend the identification to states that carry a third charge n_p, momentum along the string. These states have a microscopic degeneracy$\sqrt{n_1 n_5 n_p}$, and were used in [77] in the calculation of the black hole entropy. Recently, Giusto, Mathur, and Saxena have identified smooth geometries corresponding to some of these states [85, 86], although the geometries constructed so far correspond to very special states, the spectral flows of the RR groundstates studied earlier.[4] The overall evidence for the picture of black holes advanced by these authors is interesting but not yet compelling.

In chapter 4 we will extend these investigations to find more general solitonic solutions in supergravity, and to identify corresponding CFT states. Whether or not

[3]Although it may be that most measurements in the dual CFT find it difficult to distinguish between regular geometries and the conventional semi-classical picture of a black hole [87].

[4]Three-charge states were previously studied in the supertube description [88, 89]in [90, 91]. Other supersymmetric three-charge solutions have been found in [92–96], but the regular solutions have not yet been identified or related to CFT states.

the picture of black holes advanced by Mathur and collaborators proves to be correct, these solitonic solutions can be viewed as interesting supergravity backgrounds in their own right. It is particularly interesting that we can find completely smooth non-supersymmetric solitons.

We find these solutions by generalising an analysis previously carried out for special cases in [85, 86, 97, 98]. We start with the nonextremal rotating three-charge black holes given in [99], and systematically search for values of the parameters for which the solution is smooth and free of singularities. We find that if we allow non-supersymmetric solutions, there are two integers m, n labelling the soliton solutions. Thus, we find new non-supersymmetric solitons. Further solutions, some of which are smooth, can be constructed by orbifolds of this basic family. Some of the supersymmetric orbifolds have not been previously studied.

The AdS $/CFT$ correspondence provides an incentive to study AdS supergravity solutions in the hope they can furnish information on the dual gauge theory. Several gauge field theory phenomena such as confinement, confinement/deconfinement phase transitions, and conformal anomalies, have been shown to be encoded in the semi-classical physics of asymptotically AdS black holes [18]. Solutions of five-dimensional gauged supergravity play an important role in this context. If we have IIB supergravity on $AdS_5 \times S^5$ then rotation on the five sphere has three independent angular velocities, corresponding to the Cartan sub algebra of SO(6). On dimensional reduction to $D = 5$ gauged supergravity this Cartan sub algebra corresponds to three $U(1)$ gauge fields. This reduction is believed to be a consistent truncation, meaning that all classical solutions of the five-dimensional theory can be uplifted to IIB solutions. The $U(1)$ charges correspond to three R-charges in the dual $\mathcal{N} = 4$ CFT.

Recently Cvetič, Lü and Pope found asymptotically AdS non-extremal black hole solutions in five dimensional gauged supergravity. They began with the three $U(1)$ charges set equal [1] then extended this to the case of distinct charges [2]. The arresting property of these solutions is that they seem to violate the spirit of the no hair theorem, i.e. they carry an additional parameter besides the mass, charges and angular momentum. In chapter 5 we show that this parameter can be removed by a coordinate transformation and redefinition of parameters. Thus, the apparent hair in these solutions is unphysical.

Chapter 2

Preliminaries

In this chapter we survey some results and concepts that will be useful in chapter 3. In section 2.1 we introduce the concept of a discrete quotient, reviewing the parabolic orbifold as an example. In section 2.2 we outline the properties of Anti-de Sitter space. In section 2.3 we discuss the BTZ black hole and describe how it originates as a quotient of AdS.

2.1 Orbifolds of flat space

When we talk about an orbifold, we mean a quotient space obtained by identifying points in a manifold under some discrete symmetry group. To be more precise, if we have a manifold with a metric defined on it, the isometries of the metric define some Lie group G with associated Lie algebra $\mathfrak{g}$. An element $X \in \mathfrak{g}$ of this Lie algebra defines a one parameter subgroup Γ of G by

$$\Gamma = \{\exp(tX) | t \in \mathbb{R}\}. \tag{2.1}$$

X also defines a Killing vector ξ_X whose orbits are the integral curves of Γ. The topology of Γ is either $\mathbb{R}$ or S^1, depending on whether or not $\exp(tX)$ is the identity element in G for some nonzero t.

Every one-parameter subgroup Γ gives rise to an infinite family (indexed by the subgroup itself) of discrete cyclic subgroups Γ_γ generated by an element $\gamma \in \Gamma$. Quotienting a manifold M on which G acts by the action of Γ_γ consists of identifying points of M which are related by the action of γ. Since $\gamma = \exp(lX)$, quotienting by Γ_γ consists of identifying points in M which are related by flowing along the integral curve of the Killing vector ξ_X corresponding to X for a time l. That is, the curves joining two points on M that are on the same orbit will be closed in the quotient space. Immediately we see that if ξ_X is timelike the resulting geometry will have closed timelike curves.

We say that Γ_γ acts freely if for all $x \in M$ and $\gamma \in \Gamma_\gamma$, $\gamma x = x$ implies that $\gamma = 1$. Free actions have no fixed points, points for which $\gamma x = x$, for any $\gamma \in \Gamma_\gamma$ where $\gamma \neq 1$. If Γ_γ acts freely on M then the quotient space M/Γ_γ will be a manifold. If Γ_γ does not act freely (i.e. some elements in Γ_γ have fixed points) then M will fail to be a manifold at precisely the fixed points of Γ_γ. Such points are called orbifold singularities. It is these singularities which distinguish orbifolds from ordinary manifolds.

Thus taking a quotient by a discrete cyclic subgroup we obtain an orbifold M/Γ_γ which is locally isometric to M but can have very different global properties. Orbifolds in which some Euclidean directions are quotiented by a discrete subgroup of the isometry group have been extensively studied [100–103]. String theory defined on such orbifolds has new light states (the so called twisted sectors) which are confined to the orbifold fixed points. These twisted sector states resolve the conical singularities in many cases.

In this thesis we focus on quotients generated by the action of some vector field acting non-trivially in time. As a warm-up for the work of chapter 3 we will briefly survey an interesting time-dependent background that can be constructed as a Lorentzian orbifold of Minkowski spacetime; the so called parabolic orbifold studied in [27, 28, 30, 31]. Since this spacetime is locally flat, it is an exact classical solution to string theory and string propagation is relatively easy to study. The Killing vector generating this quotient is a null rotation

$$\xi = -J_{12} + J_{23} = +(x^1 - x^3)\partial_2 + x^2(\partial_1 + \partial_3), \tag{2.2}$$

it consists of a boost in the x^2 direction plus a rotation in the 23-plane such that the rapidity of the boost and the angle of the rotation have the same norm. If we choose coordinates for $\mathbb{R}^{1,2}$ such that $x^\pm = x^1 \pm x^3$ the action of the generator of the orbifold is:

$$\begin{pmatrix} x^+ \\ x^2 \\ x^- \end{pmatrix} \mapsto \begin{pmatrix} x^+ + 2tx^2 + t^2x^- \\ x^2 + tx^- \\ x^- \end{pmatrix} \tag{2.3}$$

where t is the affine parameter of the orbit. This action has a line of fixed points at $x^- = x^2 = 0$. Since $||\xi||^2 = (x^-)^2$, the quotient does not introduce closed timelike curves into the geometry. However, there will be closed lightlike curves corresponding to $x^- = 0$. The orbifold action leaves one spinor invariant and thus preserves one half of the spacetime supersymmetries. A more illuminating set of coordinates is

$$\begin{aligned} x^- &= y^-, \\ x^2 &= y^- y, \\ x^+ &= y^+ + y^-(y)^2, \end{aligned} \tag{2.4}$$

in terms of which the metric becomes

$$ds^2 = -dy^+dy^- + (y^-)^2dy^2. \tag{2.5}$$

Notice that the transformation (2.4) breaks down at $y^- = x^- = 0$ corresponding to the fixed points of the action. These coordinates are useful because they are adapted to the action of the quotient, that is $\xi = \partial_y$. The orbifold identification is then simply $y \sim y + t$ which makes the physical interpretation of the orbifold clear. The spacetime (2.5), may be visualized as two cones (parametrized by y^- and y) with a common tip at $y^- = 0$, crossed with the real line (for y^+). y plays the role of an 'angular variable' and the null coordinate y^- plays the role of a 'radial variable.' As a function of the 'light-cone time' y^- we have a big crunch of the y circle at $y^- = 0$ which is followed by a big bang. Thus we have a supersymmetric toy model of a cosmological singularity in which string propagation is under some control.

A natural next step is to embed such a scenario in string theory and analyse whether the twisted sectors located at $x^- = 0$ manage to resolve such a singularity. Unfortunately, unlike in Euclidean orbifolds, it turns out that this singular geometry suffers from an instability. The addition of even a single particle causes the entire universe to collapse into a spacelike singularity. Therefore the resolution of the singularities is not accessible in perturbation theory [29–32].

A closely related smooth quotient is the nullbrane introduced in [39]. As a byproduct of our investigations of the quotients of Anti-de Sitter space, we were led to realise that there is a rich structure of symmetries in the nullbrane which has not been fully exploited in previous work on these solutions. We address this issue in Appendix A.

2.2 Anti-de Sitter space

Anti-de Sitter space is the unique maximally symmetric solution of Einstein's equation with constant negative curvature. $(p+1)$-dimensional anti-de Sitter space AdS_{p+1} can be represented as the hyperboloid

$$-x_1^2 - x_2^2 + \sum_{i=1}^{p} x_i^2 = -l^2, \tag{2.6}$$

embedded in the flat $(p+2)$-dimensional space with metric

$$ds^2 = -dx_1^2 - dx_2^2 + \sum_{i=1}^{p} dx_i^2. \tag{2.7}$$

l^2 is known as the radius of curvature of AdS and is related to the cosmological constant by $\Lambda = -\frac{p(p-1)}{2l^2}$.[1] The quadric (2.6) has the isometry group $O(2,p)$ by construction. When we embed AdS_{p+1} in string theory, the presence of fluxes means we have to specify its orientation which restricts this isometry group to $\mathrm{SO}(2,p)$. Any Killing vector, ξ, in $\mathfrak{so}(2,p)$ can be written in terms of a basis J_{ab} of $\mathfrak{so}(2,p)$ as $\xi = \omega^{ab} J_{ab}$, where $\omega^{ab} = -\omega^{ba}$ and

$$J_{12} = x_2\partial_1 - x_1\partial_2, \quad J_{1i} = x_1\partial_i + x_i\partial_1, \quad J_{2i} = x_2\partial_i + x_i\partial_2, \quad J_{ij} = x_i\partial_j - x_j\partial_i. \tag{2.8}$$

A convenient global coordinate system on AdS_{p+1} is defined in terms of the embedding coordinates by

$$\begin{aligned} x_1 &= \cosh\chi \sin\tau, \\ x_2 &= \cosh\chi \cos\tau, \\ x_m &= \sinh\chi\, \hat{x}_m, \quad m = 3, \ldots, p+2, \end{aligned} \tag{2.9}$$

[1] For the remainder of this chapter and in chapter 3 we set $l = 1$.

where the $\hat{x}_m$ are embedding coordinates for an S^{p-1}, $\sum_m \hat{x}_m^2 = 1$. The metric in this coordinate system is

$$g = -\cosh^2\chi d\tau^2 + d\chi^2 + \sinh^2\chi d\Omega_{p-1}. \tag{2.10}$$

The explicit symmetries of this form of the metric are the time-translation

$$J_{12} = \partial_\tau, \tag{2.11}$$

and the SO(p) symmetries of the sphere,

$$J_{mn} = \hat{x}_m\partial_{\hat{x}_n} - \hat{x}_n\partial_{\hat{x}_m}, \quad m,n = 3,\ldots,p+2. \tag{2.12}$$

The other basis Killing vectors are

$$\begin{aligned} J_{1m} &= \cos\tau\tanh\chi\hat{x}_m\partial_\tau + \sin\tau\hat{x}_m\partial_\chi + \sin\tau\coth\chi(\delta_{mn} - \hat{x}_m\hat{x}_n)\partial_{\hat{x}_n}, \\ J_{2m} &= -\sin\tau\tanh\chi\hat{x}_m\partial_\tau + \cos\tau\hat{x}_m\partial_\chi + \cos\tau\coth\chi(\delta_{mn} - \hat{x}_m\hat{x}_n)\partial_{\hat{x}_n}, \end{aligned} \tag{2.13}$$

where $m,n = 3,\ldots,p+2$.

With $\chi \geq 0$ and $0 \leq \tau < 2\pi$ the solution (2.9) covers the hyperboloid once. The hyperboloid is not simply connected: it has topology $S^1 \times \mathbb{R}^{p+1}$, with the S^1 representing closed timelike curves in the τ direction. To obtain a causal spacetime we unwrap the circle S^1 (i.e. take $-\infty < \tau < \infty$ with no identifications) and obtain the universal covering of the hyperboloid without closed timelike curves. In this thesis, when we refer to AdS_{p+1}, we refer to this universal covering space.

What is the symmetry group of this simply connected AdS_{p+1}? The isometry group of the quadric (2.6), SO($2,p$), has a maximal connected compact subgroup SO(2) × SO(p). The SO(2) factor is generated by the Killing vector J_{12} whose orbits are the closed timelike curves on the hyperboloid. In AdS_{p+1} these curves are not closed hence the Killing vector does not generate a circle subgroup but an $\mathbb{R}$ subgroup. Therefore the symmetry group of AdS_{p+1} is the infinite cover $\widetilde{\mathrm{SO}(2,p)}$ of SO($2,p$), in which the above SO(2) subgroup is unwrapped to an $\mathbb{R}$ subgroup. As explained in chapter 3, this technical point does not alter the classification of discrete cyclic quotients of AdS.

In the context of the AdS/CFT correspondence we are interested in the conformal structure of AdS. If we introduce a coordinate θ related to χ by $\tan\theta = \sinh\chi$ where

$(0 \leq \theta < \frac{\pi}{2})$

$$ds^2 = cos^2\theta(-d\tau^2 + d\theta^2 + \sin^2\theta d\hat{x}_i^2). \tag{2.14}$$

Conformally rescaling the metric by $1/cos^2\theta$ it becomes

$$ds^2 = -d\tau^2 + d\theta^2 + \sin^2\theta d\hat{x}_i^2. \tag{2.15}$$

This is the metric of the Einstein static universe. However the coordinate θ runs $(0 \leq \theta < \frac{\pi}{2})$ rather than $(0 \leq \theta < \pi)$. Namely AdS_{p+1} can be conformally mapped into one half of the Einstein static universe. The conformal compactification is a convenient way to describe the asymptotic regions of AdS. The striking feature of AdS is that it has a time-like boundary at $\theta = \frac{\pi}{2}$, with topology $S^p \times \mathbb{R}$. On this boundary the isometries of AdS act like the conformal group acts in p dimensions.

2.2.1 Poincaré coordinates

Another useful set of AdS coordinates are Poincaré coordinates, let us define $\{y^\mu, z\}$ $\mu = 2, \ldots, p+1$ in terms of the flat embedding coordinates in $\mathbb{R}^{2,p}$ introduced in (2.6) by

$$\begin{aligned} x^\mu &= \frac{1}{z} y^\mu \quad \mu = 2, \ldots, p+1 \\ x^1 &= \frac{1}{2z}\left[z^2 + (1 + \eta_{\mu\nu} y^\mu y^\nu)\right] \\ x^{p+2} &= \frac{1}{2z}\left[z^2 - (1 - \eta_{\mu\nu} y^\mu y^\nu)\right] \ . \end{aligned} \tag{2.16}$$

In these coordinates, the AdS_{p+1} metric is

$$g = \frac{1}{z^2}(\eta_{\mu\nu} dy^\mu dy^\nu + dz^2). \tag{2.17}$$

The explicit symmetries in this form of the metric are the Poincaré symmetries acting on the slices of constant z. Using the identities

$$\frac{\partial x^\mu}{\partial y^\nu} = \frac{1}{z}\delta^\mu_\nu \quad , \quad \frac{\partial x^{p+2}}{\partial y^\nu} = \frac{\partial x^1}{\partial y^\nu} = \eta_{\nu\mu} x^\mu \quad , \quad x^1 - x^{p+2} = \frac{1}{z} \ ,$$

we see that these are related to the usual $\mathfrak{so}(2,p)$ basis by

$$\begin{aligned} P_\mu &= \partial_{y^\mu} \ \longrightarrow \ -(J_{1\mu} - J_{\mu p+2}) \\ L_{\mu\nu} &= y_\mu \partial_{y^\nu} - y_\nu \partial_{y^\mu} \ \longrightarrow \ J_{\mu\nu} \ . \end{aligned} \tag{2.18}$$

Timelike translations in the Poincaré patch correspond to a timelike null rotation in global AdS. On the other hand, spacelike translations in the Poincaré patch correspond to a spacelike null rotation. Finally, Lorentz transformations in the Poincaré patch are mapped to Lorentz transformations in $\mathbb{R}^{2,p}$.

The other symmetries in $\mathfrak{so}(2,p)$ are realised as conformal symmetries acting on the slices of constant z together with a suitable ∂_z component:

$$J_{1\mu} + J_{\mu p+2} = -\eta_{\sigma\nu} y^\sigma y^\nu \partial_{y^\mu} + 2y_\mu y^\nu \partial_{y^\nu} + 2zy_\mu \partial_z, \tag{2.19}$$

$$J_{1p+2} = -y^\mu \partial_{y^\mu} - z\partial_z. \tag{2.20}$$

These coordinates will be useful for understanding the relation between certain global AdS quotients and the near horizon limit of the corresponding discrete quotients of brane geometries in supergravity.

2.3 The BTZ black hole

$2+1$ dimensional Einstein gravity has no local degrees of freedom, consequently all vacuum solutions of Einstein's equations in $2+1$ dimensions have constant curvature. It was therefore surprising when Bañados, Teitelboim and Zanelli discovered $2+1$ dimensional solutions with a negative cosmological constant that could be interpreted as black holes [55]. These solutions are given by

$$ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2} dt^2 + \frac{r^2 dr^2}{(r^2 - r_+^2)(r^2 - r_-^2)} + r^2 \left[d\phi - \frac{r_- r_+}{r^2} dt \right]^2 \tag{2.21}$$

where $0 \leq \phi < 2\pi$. This describes a rotating black hole with inner and outer horizons $r_\pm$ given by

$$r_\pm^2 = \frac{M}{2} \left(1 \pm \left(1 - \frac{J^2}{M^2} \right)^{\frac{1}{2}} \right). \tag{2.22}$$

There is a singularity at $r = 0$ in the sense that timelike geodesics end there. Since this space is locally isometric to AdS this is not a curvature singularity. The curvature tensor is everywhere regular. For these solutions to describe black holes, we need

$$M > 0 \quad |J| \leq M. \tag{2.23}$$

As M grows negative one encounters unphysical solutions with a naked conical singularity. There is an important exceptional case. When one reaches $M = -1$ and $J = 0$ the singularity disappears. This is global AdS_3 which is seperated from the continuous black hole spectrum by a mass gap of one unit. This state can not be continuously deformed into the vacuum, because the deformation would require going through a sequence of naked singularities not included in configuration space.

The fact that (2.21) only differs from AdS_3 in its global properties suggests it can be obtained as a quotient of AdS_3. These solutions were extensively studied in [56] where the identifications corresponding to the black hole and its extremal limits were explicitly identified. The Killing vector which generates the non-extremal (i.e $M > |J| > 0$) BTZ black hole solution is

$$\xi = r_+ J_{14} - r_- J_{23}. \tag{2.24}$$

This has norm

$$|\xi|^2 = r_+^2(x_1^2 - x_4^2) + r_-^2(x_2^2 - x_3^2). \tag{2.25}$$

Since ξ is timelike in some regions of AdS_3 the quotient of global AdS_3 generated by this Killing vector will have closed timelike curves. The way to construct the black hole is to excise the regions where $|\xi|^2 < 0$, and consider the quotient just of the remaining portion of AdS_3. The resulting geometry will be causally regular by construction, but will clearly not be geodesically complete, having a 'singularity' corresponding to the boundary of the excised region. In [56] the authors explicitly constructed coordinates covering the regions of AdS_3 with $|\xi|^2 > 0$ corresponding to (2.21) with $-\infty < \phi < \infty$. The surface $r = 0$ corresponds to $|\xi|^2 = 0$. It is the identification $\phi \sim \phi + 2\pi$ that makes the black hole. If ϕ is not a compact variable, one simply has a portion of Anti-de Sitter space and the horizon is just that of an accelerated observer. On the other hand after the identification extending the spacetime beyond the surface $r = 0$ would produce closed timelike curves.

One limit of the BTZ solution that we will study in chapter 3 is $M = J = 0$

$$ds^2 = -r^2 dt^2 + \frac{dr^2}{r^2} + r^2 d\phi^2. \tag{2.26}$$

The Killing vector generating this spacetime is a null rotation

$$\xi = J_{13} - J_{34}. \tag{2.27}$$

The norm $|\xi|^2 = (x_1 - x_4)^2$ is positive semi-definite being zero at $x_1^2 = x_4^2$ corresponding to $r = r_+ = 0$ in the coordinates of equation (2.26). Thus this spacetime has closed null curves on the horizon. In section 3.4.1 we show that by adding a suitable action on a transverse sphere to ξ we can generate a causally regular spacetime as a quotient of $\mathrm{AdS}_{p+1} \times S^p$.

Chapter 3

Quotients of anti-de Sitter space

In this chapter we study the quotients of $p+1$-dimensional anti-de Sitter space by one-parameter subgroups of its isometry group. We initiate this task by classifying the possible physically distinct quotients. Having identified the potentially interesting quotients we examine their properties paying close attention to causal structure. Since we are particularly interested in smooth quotients we develop a formalism for discussing their geometry and physical interpretation.

An important ancillary result of [56] relevant to our work is the classification of the one-parameter subgroups of $SO(2,2)$, this provides a menu of the possible discrete cyclic quotients of AdS_3. This classification was extended to $SO(2,3)$ by Holst and Peldan [68] in an attempt to extend the BTZ type solutions to AdS_4. In section 3.1 we show that the classification of physically distinct one-parameter subgroups of $SO(2,p)$ extends very naturally from the case $p = 2$ to higher p. The subgroups considered in [56] all have higher-dimensional generalisations, whose analysis is directly related to the analysis in the case of AdS_3. There are only two further physically distinct possibilities, one of which appears for all $p \geq 3$, and the other of which appears for all $p \geq 4$. The prototype example of the former was discussed in [68], and the latter contains the null brane-like quotient discussed in [57]. We discuss the Killing vectors on the sphere in section 3.2.

In Section 3.4 we study in detail causally well-behaved quotients. We find that there are two types of quotients with well-behaved causal structures. First, there are quotients where an action on AdS alone is well behaved. These are generalisations of the two cases studied previously:

i. self-dual orbifolds of AdS_3 [104, 105] and their higher-dimensional generalisations, having no analogue in asymptotically flat configurations; and

ii. the AdS analogue of the flat nullbrane construction [57], consisting of a double null rotation action on $SO(2,p)$ $p \geq 4$. This is the near horizon geometry of a stack of D3-branes in the nullbrane vacuum for $p = 4$ and a stack of M5-branes in the same vacuum for $p = 6$.

We give a comprehensive discussion of the structure of these quotients, extending previous results. For the double null rotation, we construct a new symmetry-adapted coordinate system, and find interesting relations to compactified plane waves. We comment on related issues in the nullbranes in appendix A.

Secondly, there are quotients where the norm of the AdS isometry is non-negative, but not always positive, so the pure AdS action would have singularities or closed null curves. These can be removed by a suitable action on the transverse sphere if the latter is odd-dimensional. This second type is qualitatively new. These non-trivial actions on AdS can be divided into three categories:

i. discrete quotients by rotations in AdS, the higher-dimensional analogues of the AdS_3 conical defects;

ii. discrete quotients by a null rotation, whose description in the Poincaré patch corresponds to a spacelike translation (in pure AdS_3, these would give rise to the massless BTZ black hole [56]) and whose sphere deformations are the near horizon limit of brane configurations in fluxbrane vacua classified in [106, 107]; and

iii. discrete quotients defined by an everywhere null vector field in AdS_p ($p \geq 3$), whose description in the Poincaré patch corresponds to a 'translation' along a lightlike direction. Once more, when deformed by a non-trivial action on a transverse sphere, this corresponds to the near horizon counterpart of the corresponding quotients classified in [106, 107].

In general our quotient spaces preserve some symmetries of AdS_{p+1}. A quotient of AdS_{p+1} inherits all Killing vectors in $\mathfrak{so}(2,p)$ which commute with the Killing vector generating the quotient. Thus, there is the possibility that we can construct interesting backgrounds by taking a further discrete quotient along one of these preserved Killing vectors. In section 3.5 we consider this issue and uncover the existence of a smooth causally regular quotient generated by two commuting double null rotations.

It is important to stress that any of these string theory backgrounds are related to many others through U-duality and by Kaluza–Klein reductions from or liftings to M-theory. We shall not pursue this possibility in this thesis, even though it is natural to wonder about the dual incarnations of our backgrounds.

Our emphasis differs from [56, 68] in that we are more interested in quotients which are causally regular. However in section 3.6 we turn our attention to BTZ type quotients. We confirm and elucidate the conclusion of [68], that for $p > 2$, the only locally AdS_{p+1} black hole solution is the higher-dimensional generalisation of the non-rotating BTZ black hole, discussed previously in [118, 119]. We explain the origin of this restriction in general. We discuss the relation to other recent work and comment on the proper interpretation of another solution presented in [119].

3.1 One parameter subgroups of $\mathrm{SO}(2,p)$

We wish to classify physically distinct quotients of AdS_{p+1} by one-parameter subgroups of $\widetilde{\mathrm{SO}(2,p)}$.[1] Annoyingly, it cannot be embedded in a matrix group; that is, it does not admit any finite-dimensional faithful linear representations. Crucially, however, $\widetilde{\mathrm{SO}(2,p)}$ has two features in common with its quotient $\mathrm{SO}(2,p)$. First of all, they share the same Lie algebra $\mathfrak{so}(2,p)$ and furthermore, since conjugation by central elements is trivial, the adjoint action of $\widetilde{\mathrm{SO}(2,p)}$ on $\mathfrak{so}(2,p)$ factors through

[1]We will generally have in mind the quotient by a discrete subgroup, to construct another $p+1$-dimensional spacetime; the prototypical example is the BTZ black hole [55,56]. It is also interesting to consider the Kaluza-Klein reduction along such a direction to construct an p-dimensional spacetime. For the purposes of classification, we can treat these two kinds of quotients together.

$\mathrm{SO}(2,p)$. Similarly, the action of the spin cover $\widetilde{\mathrm{Spin}(2,p)}$ of $\widetilde{\mathrm{SO}(2,p)}$ on the spinor representations factors through $\mathrm{Spin}(2,p)$. These happy facts allow a complete analysis of one-parameter subgroups and also the determination of the supersymmetry preserved by a quotient.

As explained, for example, in [39], if Γ and Γ' are conjugate subgroups of isometries of a space M, then their quotients M/Γ and M/Γ' are isometric, the isometry being induced from the isometry of M which conjugates Γ into Γ'. Therefore to classify such quotients M/Γ, it is enough to classify subgroups up to conjugation. A one parameter subgroup is determined by a Killing vector ξ in the Lie algebra $\mathfrak{so}(2,p)$. Since $\xi = \omega^{ab} J_{ab}$, the classification of physically distinct ξ is equivalent to classifying antisymmetric matrices ω^{ab} up to conjugation by elements of $\mathrm{SO}(2,p)$, that is,

$$\omega' \sim \omega \qquad \text{iff} \qquad \omega'^a_{\ b} = (T^{-1})^a_{\ c}\, \omega^c_{\ d}\, T^d_{\ b} \tag{3.1}$$

for some $T^a_{\ c} \in SO(2,p)$. As explained in [56,68], if we slightly extend the equivalence relation, so that $\omega' \sim \omega$ for $T^a_{\ c} \in O(2,p)$, then the problem is equivalent to the familiar problem of classifying the matrices up to similarity. The distinct matrices are then classified by their eigenvalues and the dimensions of the irreducible invariant subspaces associated with them. This extension of the equivalence relation implies that we will not distinguish between Killing vectors which differ by a sign reversal of some of the embedding coordinates, that is Killing vectors that have different orientations. Since the metric in AdS_{p+1} is invariant under orientation reversing transformations, the geometrical interpretation of these different discrete quotients will be identical. However a distinction will arise in the sign of the fluxes that stabilise the classical configurations. This fact can certainly have consequences concerning the supersymmetry preserved by the discrete quotients.

Since the classification reduces to the study of the eigenvalues and eigenspaces of the matrix $\omega^a_{\ b}$, we can 'build up' the general matrix from the different eigenspaces. We will therefore first consider the different possibilities for invariant subspaces consistent with the signature of spacetime, and then use these possible invariant subspaces as building blocks to construct all the possible inequivalent matrices ω_{ab}, and hence classify the different quotients. In the following we shall say that the matrix ω_{ab} is of type k if its highest dimensional irreducible invariant subspace is of dimension k.

The calculations are simplified by observing that as a consequence of the fact that ω_{ab} is real and antisymmetric, its eigenvalues come in groups: if λ is an eigenvalue of $\omega^a{}_b$ then $-\lambda$ is an eigenvalue of $\omega^a{}_b$, and similarly if λ is an eigenvalue then so is λ^*. Another useful fact is that if v^a and u^a are eigenvectors of $\omega^a{}_b$ with respective eigenvalues λ and μ, so that

$$\omega^a{}_b v^b = \lambda v^a, \qquad \omega^a{}_b u^b = \mu u^a, \tag{3.2}$$

then $v^a u_a = 0$ unless $\lambda + \mu = 0$. Note that v^a etc. are vectors in $\mathbb{R}^{2,p}$; the indices on ω_{ab}, v^a etc are raised and lowered with the metric η_{ab} on $\mathbb{R}^{2,p}$. Thus, we see that $\mathbb{R}^{2,p}$ decomposes into a product of orthogonal eigenspaces, but each such subspace is associated not with a single eigenvalue λ but with the pair of eigenvalues $\lambda, -\lambda$. We will now study the properties of these orthogonal eigenspaces.

Let us first discuss the cases with non-degenerate eigenvalues. The simplest case is when the eigenvalue is zero; then there is a single eigenvector v^a, which is orthogonal to all other eigenvectors, and by the non-degeneracy of the metric must then have $v^a v_a \neq 0$. We can rescale v^a to set $v^a v_a = 1$, which we will refer to as $\lambda^{(0,1)}$, or $v^a v_a = -1$, which we will refer to as $\lambda^{(1,0)}$. These cases correspond physically to a direction in $\mathbb{R}^{2,p}$ which is not affected by the identification.

The next possibility is a pair of real eigenvalues, $a, -a$, $a \geq 0$. Then we have

$$\omega_{ab} l^b = a l_a, \quad \omega_{ab} m^b = -a m_a. \tag{3.3}$$

The only non-zero inner product is $l_a m^a = 1$. To construct an orthonormal basis, we take

$$v_1 = \frac{1}{\sqrt{2}}(l + m), \quad v_2 = \frac{1}{\sqrt{2}}(l - m). \tag{3.4}$$

We then have $v_1 \cdot v_1 = 1$, $v_2 \cdot v_2 = -1$, so this subspace has signature $(-+)$. We denote this by $\lambda^{(1,1)}$; it corresponds physically to a boost in some $\mathbb{R}^{1,1}$ subspace of $\mathbb{R}^{2,p}$.

If we have a pair of imaginary eigenvalues,

$$\omega_{ab} k^b = ibk_a, \quad \omega_{ab} k^{*b} = -ibk^*_a, \tag{3.5}$$

$b \geq 0$, the only non-zero inner product is $k_a k^{*a} = 1$. Now we need to construct the orthonormal basis in a slightly different way, because we need to respect the fact that the action of ω_{ab} on $\mathbb{R}^{2,p}$ is real-valued. We can set

$$v_1 = \frac{1}{\sqrt{2}}(k + k^*), \quad v_2 = \frac{i}{\sqrt{2}}(k - k^*). \tag{3.6}$$

We then have $\omega^a{}_b v_1^b = b v_2^a$, $\omega^a{}_b v_2^b = -b v_1^a$. We have $v_1 \cdot v_1 = 1$, $v_2 \cdot v_2 = 1$, so this subspace has signature $(++)$, which we denote by $\lambda^{(0,2)}$. On the other hand, we could have chosen

$$v_1 = \frac{i}{\sqrt{2}}(k + k^*), \quad v_2 = \frac{1}{\sqrt{2}}(k - k^*). \tag{3.7}$$

This also gives a real action, but now $v_1 \cdot v_1 = -1$, $v_2 \cdot v_2 = -1$, so this subspace has signature $(--)$, which we denote by $\lambda^{(2,0)}$. These two cases correspond physically to rotations in $\mathbb{R}^2$ subspaces of $\mathbb{R}^{2,p}$.

The final possibility is a complex eigenvalue, which gives us the four eigenvalues $\lambda, -\lambda, \lambda^*, -\lambda^*$ (so we can take $\lambda = a + ib$ for $a, b \geq 0$). We have

$$\omega_{ab} l^b = \lambda l_a, \quad \omega_{ab} m^b = -\lambda m_a, \tag{3.8}$$

$$\omega_{ab} l^{*b} = \lambda^* l^*_a, \quad \omega_{ab} m^{*b} = -\lambda^* m^*_a. \tag{3.9}$$

The non-vanishing inner products are $l \cdot m = 1$ and $l^* \cdot m^* = 1$, so l, m and l^*, m^* span two orthogonal two-dimensional spaces; however, we need to mix them to obtain a real basis. If we define

$$v_1 = \frac{1}{2}[(l + l^*) + (m + m^*)], \quad v_2 = \frac{1}{2}[(l + l^*) - (m + m^*)], \tag{3.10}$$

$$v_3 = \frac{i}{2}[(l - l^*) + (m - m^*)], \quad v_4 = \frac{i}{2}[(l - l^*) - (m - m^*)], \tag{3.11}$$

Then we will see that ω_{ab} acts on the v_i with real coefficients, and they span a space of signature $(--++)$, which we denote $\lambda_c^{(2,2)}$.

Now we turn to the higher-dimensional irreducible invariant subspaces. If we have a k-dimensional subspace associated to the eigenvalue zero, then we can pick a basis of vectors m_i, $i = 1, \ldots, k$ such that

$$\omega_{ab} m_1^b = 0, \quad \omega_{ab} m_i^b = m_{(i-1)a} \text{ for } i \neq 1. \tag{3.12}$$

We can then observe that $m_1^a m_{(i-1)a} = m_1^a \omega_{ab} m_i^b = 0$ for $i = 1, \ldots, k$. We then need $m_1^a m_{ka} \neq 0$ for consistency with the non-degenerate metric. We can also use (3.12) to show

$$m_{ia} m_j^a = m_{ia} \omega^{ab} m_{(j+1)b} = -m_{(i-1)a} m_{(j+1)}^a, \tag{3.13}$$

and

$$m_{ia} m_{(i-1)}^a = m_{ia} \omega^{ab} m_{ib} = 0 \tag{3.14}$$

by antisymmetry of ω_{ab}. Now imagine k is even. Then these two relations taken together imply that

$$m_{ka} m_1^a = \pm m_{(k/2)a} m_{(k/2+1)}^a = 0, \tag{3.15}$$

in contradiction with the non-degeneracy of the metric. Hence there cannot be k-dimensional invariant subspaces associated with a zero eigenvalue for k even. For k odd, (3.13) implies

$$m_{ia} m_j^a = (-1)^{i+1} m_{1a} m_k^a \tag{3.16}$$

for $i + j = k + 1$. We can also set all other inner products to zero by a suitable redefinition of the basis m_i^a. We can then define an orthonormal basis by

$$v_{2i-1} = \frac{1}{\sqrt{2}}(m_i + m_{k+1-i}), \quad v_{2i} = \frac{1}{\sqrt{2}}(m_i - m_{k+1-i}) \text{ for } i = 1, \ldots, \frac{k-1}{2}, \tag{3.17}$$

and $v_k = m_{k+1/2}$. We then have $v_{2i-1} \cdot v_{2i-1} = -v_{2i} \cdot v_{2i}$, and we can choose $v_k \cdot v_k$ to be ± 1, so the subspace spanned by these vectors has either $(k-1)/2$ negative signature directions and $(k+1)/2$ positive signature ones, or $(k+1)/2$ negative signature directions and $(k-1)/2$ positive signature ones. The only possibilities which are consistent with embedding as a subspace in $\mathbb{R}^{2,p}$ are $\lambda^{III(1,2)}$ and $\lambda^{III(2,1)}$, and $\lambda^{V(2,3)}$ with signature $(- - + + +)$.

If we have a k-dimensional invariant subspace with a real eigenvalue, we must have a pair of them; we can define a basis such that the action of ω_{ab} is

$$\omega_{ab} l_1^b = a l_{1a}, \quad \omega_{ab} l_i^b = a l_{ia} + l_{(i-1)a} \text{ for } i = 2, \ldots, k, \tag{3.18}$$

and

$$\omega_{ab} m_1^b = -a m_{1a}, \quad \omega_{ab} m_i^b = -a m_{ia} + m_{(i-1)a} \text{ for } i = 2, \ldots, k. \tag{3.19}$$

By repeatedly using these relations, we can show that $l_i \cdot l_j = 0$ and $m_i \cdot m_j = 0$ for all i,j. We can also show $m_1 \cdot l_i = 0$ for $i \neq k$; we then need $m_1 \cdot l_k \neq 0$ for non-degeneracy. As in the case of a zero eigenvalue, we learn that

$$m_i \cdot l_j = (-1)^{i+1} m_1 \cdot l_k, \tag{3.20}$$

for $i + j = k + 1$, and we can set all other inner products to zero by a suitable redefinition of the basis. An orthonormal basis is then formed by taking

$$v_{2i-1} = \frac{1}{\sqrt{2}}(l_i + m_{k+1-i}), \quad v_{2i} = \frac{1}{\sqrt{2}}(l_i - m_{k+1-i}) \text{ for } i = 1, \ldots, k. \tag{3.21}$$

We then have $v_{2i-1} \cdot v_{2i-1} = -v_{2i} \cdot v_{2i}$, so the subspace spanned by these vectors has an equal number of negative and positive signature directions. The only possibility consistent with being a subspace of $\mathbb{R}^{2,p}$ is $\lambda_r^{II(2,2)}$, which has signature $(--++)$.

If we have a k-dimensional invariant subspace with an imaginary eigenvalue, we must again have a pair of them; we can define a basis such that the action of ω_{ab} is

$$\omega_{ab} k_1^b = ibk_{1a}, \quad \omega_{ab} k_i^b = ibk_{ia} + k_{(i-1)a} \text{ for } i = 2, \ldots, k, \tag{3.22}$$

and

$$\omega_{ab} k_1^{*b} = -ibk_{1a}^*, \quad \omega_{ab} k_i^{*b} = -ibk_{ia}^* + k_{(i-1)a}^* \text{ for } i = 2, \ldots, k. \tag{3.23}$$

By repeatedly using these relations, we can show that $k_i \cdot k_j = 0$ and $k_i^* \cdot k_j^* = 0$ for all i,j. We can also show $k_1 \cdot k_i^* = 0$ for $i \neq k$; we then need $k_1 \cdot k_k^* \neq 0$ for non-degeneracy. As in the case of a zero eigenvalue, we learn that

$$k_i \cdot k_j^* = (-1)^{i+1} k_1 \cdot k_k^* \tag{3.24}$$

for $i + j = k + 1$, and we can set all other inner products to zero by a suitable redefinition of the basis. The action of ω becomes real if we define new vectors $w_i = \frac{1}{\sqrt{2}}(k_i + k_i^*)$ and $x_i = \frac{i}{\sqrt{2}}(k_i - k_i^*)$. There is then a technical difference between even and odd dimensions: in even dimensions, the non-zero inner products are $w_i \cdot x_j$ for $i + j = k + 1$, and an orthonormal basis is formed by taking

$$v_{2i-1} = \frac{1}{\sqrt{2}}(w_i + x_{k+1-i}), \quad v_{2i} = \frac{1}{\sqrt{2}}(w_i - x_{k+1-i}) \text{ for } i = 1, \ldots, k, \tag{3.25}$$

We then have $v_{2i-1} \cdot v_{2i-1} = -v_{2i} \cdot v_{2i}$. Thus, in even dimensions, we have a subspace with an equal number of positive and negative directions, and the only possibility in $\mathbb{R}^{2,p}$ is $\lambda_i^{II(2,2)}$, which has signature $(--++)$. In odd dimensions, the non-zero inner products are $w_i \cdot w_j = x_i \cdot x_j$ for $i+j = k+1$, and an orthonormal basis is formed by

$$v_{2i-1} = \frac{1}{\sqrt{2}}(w_i + w_{k+1-i}), \quad v_{2i} = \frac{1}{\sqrt{2}}(w_i - w_{k+1-i}) \text{ for } i = 1, \ldots, \frac{k-1}{2}, \tag{3.26}$$

$$v_k = w_{\frac{k+1}{2}}, \quad v_{k+1} = x_{\frac{k+1}{2}} \tag{3.27}$$

$$v_{2i-1} = \frac{1}{\sqrt{2}}(x_i + x_{k+1-i}), \quad v_{2i} = \frac{1}{\sqrt{2}}(x_i - x_{k+1-i}) \text{ for } i = \frac{k+3}{2}, \ldots, k. \tag{3.28}$$

We then have $v_{2i-1} \cdot v_{2i-1} = -v_{2i} \cdot v_{2i}$ except for $i = \frac{k+1}{2}$; $v_k \cdot v_k = v_{k+1} \cdot v_{k+1}$. The subspace thus either has $k-1$ positive and $k+1$ negative directions or vice-versa. The only possibility in $\mathbb{R}^{2,p}$ is $\lambda^{III(2,4)}$, which has signature $(--++++)$. In the special case $b = 0$, which will be important later, $\lambda^{III(2,4)}$ reduces to a pair of $\lambda^{III(1,2)}$—that is, to a pair of null rotations in independent subspaces. Finally, we could consider invariant subspaces of dimension k associated with complex eigenvalues. We will not give the details here, as it does not lead to any cases that fit inside $\mathbb{R}^{2,p}$. The subspace associated with the set of four complex eigenvalues always has at least $2k$ negative directions.

We have now calculated the possible invariant subspaces that can occur in our ω_{ab}.[2] Let us consider how we can assemble these blocks to form an $p+2$ dimensional matrix ω_{ab}. For p even (which includes the case $p = 4$ which we are particularly interested in), the possibilities are

- Type I

$$\begin{array}{lll} \mathbb{C} & \lambda_c^{(2,2)} & + \frac{p-2}{2}\lambda^{(0,2)}, \\ \mathbb{R} & 2\lambda^{(1,1)} & + \frac{p-2}{2}\lambda^{(0,2)}, \\ \mathbb{I} & \lambda^{(2,0)} & + \frac{p}{2}\lambda^{(0,2)}. \end{array}$$

Where the coefficient in front of a λ corresponds to the number of times that type of invariant subspace appears.

[2]Naturally, the same classification can be applied for the Lorentz group SO$(1,p)$ in $\mathbb{R}^{1,p}$; in that case, the only possible subspaces are $\lambda^{(0,1)}$, $\lambda^{(1,0)}$, $\lambda^{(1,1)}$, $\lambda^{(0,2)}$, and $\lambda^{III(1,2)}$, corresponding to trivial directions, boosts, rotations and null rotations respectively.

- Type II

$$\begin{array}{llll} \mathbb{R} & \lambda_r^{II(2,2)} & +\frac{p-2}{2}\lambda^{(0,2)}, \\ \mathbb{I} & \lambda_i^{II(2,2)} & +\frac{p-2}{2}\lambda^{(0,2)}. \end{array}$$

- Type III

$$\begin{array}{llllll} \mathbb{I} & & \lambda^{III(2,4)} & +\frac{p-4}{2}\lambda^{(0,2)}, & & \\ 0 & (a) & \lambda^{III(1,2)} & +\lambda^{(0,1)} & +\lambda^{(1,1)} & +\frac{p-4}{2}\lambda^{(0,2)}, \\ 0 & (b) & \lambda^{III(2,1)} & +\lambda^{(0,1)} & +\frac{p-2}{2}\lambda^{(0,2)}, & \\ 0 & (c) & \lambda^{III(1,2)} & +\lambda^{(1,0)} & +\frac{p-2}{2}\lambda^{(0,2)}. & \end{array}$$

- Type V

$$\lambda^{V(2,3)} \quad +\lambda^{(0,1)} \quad +\frac{p-4}{2}\lambda^{(0,2)}.$$

To discuss the physics of these different cases, we need a convenient representative of each case. It is easy to construct suitable representatives; in most cases, this is a minor generalisation of the analysis of [56, 68]. For $I_{\mathbb{C}}$ this is [3]

$$\xi = b_1(J_{12} - J_{34}) + a(J_{14} - J_{23}) + b_2 J_{56} + b_3 J_{78} + \cdots + b_{\frac{p}{2}} J_{p+1p+2}, \tag{3.29}$$

with $a, b_i \geq 0$. The norm of this Killing vector is

$$\begin{aligned} \xi_\mu \xi^\mu &= (a^2 - b_1^2)(1 + \|\boldsymbol{x}_\perp\|^2) - 4ab_1(x_1x_3 + x_2x_4) \\ &\quad + b_2^2(x_5^2 + x_6^2) + b_3^2(x_7^2 + x_8^2) + \cdots + b_{\frac{p}{2}}^2(x_{p+1}^2 + x_{p+2}^2). \end{aligned} \tag{3.30}$$

Thus, this Killing vector can be everywhere spacelike for $b_1 = 0$, but its norm is unbounded from below for $b_1 \neq 0$. For type $I_{\mathbb{R}}$ we have

$$\xi = a_1 J_{14} + a_2 J_{23} + b_1 J_{56} + \cdots + b_{\frac{p-2}{2}} J_{p+1p+2}, \tag{3.31}$$

[3] Recall that we have identified Killing vectors differing by conjugation by $O(2,p)$; if we only identified under conjugation by $SO(2,p)$, we should take $b_i, i \geq 2$ to run over the reals, and $\xi = b_1(-J_{12} - J_{34}) + a(-J_{14} - J_{23}) + b_2 J_{56} + b_3 J_{78} + \cdots + b_{\frac{p}{2}} J_{p+1p+2}$ and $\xi = b_1(-J_{12} + J_{34}) + a(J_{14} + J_{23}) + b_2 J_{56} + b_3 J_{78} + \cdots + b_{\frac{p}{2}} J_{p+1p+2}$ for $a, b_1 \geq 0$ would also count as distinct cases. Similar remarks apply in the other cases to follow.

with norm

$$\xi_\mu \xi^\mu = a_1^2(x_1^2 - x_4^2) + a_2^2(x_2^2 - x_3^2) + b_1^2(x_5^2 + x_6^2) + \cdots + b_{\frac{p-2}{2}}^2(x_{p+1}^2 + x_{p+2}^2). \quad (3.32)$$

This is everywhere spacelike for $a_1 = a_2$ (using $\eta^{ab}x_a x_b = -1$), which is equivalent to type $I_{\mathbb{C}}$ with $b_1 = 0$. For $a_1 \neq a_2$ the norm is unbounded from below. For type $I_{\mathbb{0}}$ we have

$$\xi = b_1 J_{12} + b_2 J_{34} + b_3 J_{56} + \cdots + b_{\frac{p+2}{2}} J_{p+1p+2}, \quad (3.33)$$

with norm

$$\xi_\mu \xi^\mu = -b_1^2(1 + \|\boldsymbol{x}_\perp\|^2) + b_3^2(x_5^2 + x_6^2) + \cdots + b_{\frac{p+2}{2}}^2(x_{p+1}^2 + x_{p+2}^2). \quad (3.34)$$

For $b_1 = 0$, this is spacelike away from the axis $x_i = 0, i \geq 2$, where the Killing vector degenerates, so this axis is a line of fixed points. For type $II_{\mathbb{R}}$ we have

$$\xi = a(J_{14} - J_{23}) + J_{12} + J_{13} - J_{24} - J_{34} + b_1 J_{56} + \cdots + b_{\frac{p-2}{2}} J_{p+1p+2}, \quad (3.35)$$

with norm

$$\begin{aligned} \xi_\mu \xi^\mu &= a^2(1 + \|\boldsymbol{x}_\perp\|^2) + 4a(x_1 + x_4)(x_3 + x_2) \\ &\quad + b_1^2(x_5^2 + x_6^2) + \cdots + b_{\frac{p-2}{2}}^2(x_{p+1}^2 + x_{p+2}^2). \end{aligned} \quad (3.36)$$

For $a = 0$, this is spacelike except on the subspace $x_i = 0, i \geq 4$, where the Killing vector is null. For type $II_{\mathbb{0}}$, we have

$$\xi = (b_1 + 1)J_{12} + (b_1 - 1)J_{34} + J_{13} - J_{24} + b_2 J_{56} + \cdots + b_{\frac{p}{2}} J_{p+1p+2}, \quad (3.37)$$

with norm

$$\begin{aligned} \xi_\mu \xi^\mu &= -b_1^2(1 + \|\boldsymbol{x}_\perp\|^2) \\ &\quad + 2b_1\left((x_1 + x_4)^2 + (x_2 + x_3)^2\right) \\ &\quad + b_2^2(x_5^2 + x_6^2) + \cdots + b_{\frac{p-2}{2}}^2(x_{p+1}^2 + x_{p+2}^2). \end{aligned} \quad (3.38)$$

For $b_1 = 0$, this is the same as type $II_{\mathbb{R}}$ with $a = 0$ (as one would expect). For type $III_{\mathbb{0}}$ we have

$$\xi = b(-J_{12} + J_{34} + J_{56}) + J_{15} - J_{35} + J_{26} - J_{46} + b_2 J_{78} + \cdots + b_{\frac{p-2}{2}} J_{p+1p+2}, \quad (3.39)$$

with norm

$$\xi_\mu \xi^\mu = -b^2(1+\|\boldsymbol{x}_\perp\|^2) - 4b(x_6(x_1-x_3)+x_5(x_4-x_2)) + (x_1-x_3)^2 + (x_4-x_2)^2$$
$$+b_2^2(x_7^2+x_8^2) + \cdots + b^2_{\frac{p-2}{2}}(x_{p+1}^2 + x_{p+2}^2). \quad (3.40)$$

This is everywhere spacelike if $b=0$. For type $III_{0(a)}$ we have

$$\xi = aJ_{26} + J_{13} - J_{34} + b_1 J_{78} + \cdots + b_{\frac{p-4}{2}} J_{p+1p+2}, \quad (3.41)$$

with norm

$$\xi_\mu \xi^\mu = (x_1+x_4)^2 + a^2(x_2^2 - x_6^2) + b_1^2(x_7^2+x_8^2) + \cdots + b^2_{\frac{p-4}{2}}(x_{p+1}^2 + x_{p+2}^2), \quad (3.42)$$

for type $III_{0(b)}$ we have

$$\xi = J_{12} - J_{23} + b_1 J_{56} + b_2 J_{78} + \cdots + b_{\frac{p-2}{2}} J_{p+1p+2}, \quad (3.43)$$

with norm

$$\xi_\mu \xi^\mu = -(x_1+x_3)^2 + b_1^2(x_5^2+x_6^2) + \cdots + b^2_{\frac{p-2}{2}}(x_{p+1}^2 + x_{p+2}^2), \quad (3.44)$$

and for type $III_{0(c)}$ we have

$$\xi = J_{13} - J_{34} + b_1 J_{56} + b_2 J_{78} + \cdots + b_{\frac{p-2}{2}} J_{p+1p+2}, \quad (3.45)$$

with norm

$$\xi_\mu \xi^\mu = (x_1+x_4)^2 + b_1^2(x_5^2+x_6^2) + b_2^2(x_7^2+x_8^2) + \cdots + b^2_{\frac{p-2}{2}}(x_{p+1}^2 + x_{p+2}^2). \quad (3.46)$$

This last case is spacelike everywhere away from the subspace $x_1 + x_4 = 0$, $x_i = 0, i \geq 4$, where it is null. Finally, for type V we have

$$\xi = J_{12} + J_{13} - J_{24} + J_{15} - J_{34} - J_{45} + b_1 J_{78} + \cdots + b_{\frac{p-4}{2}} J_{p+1p+2}, \quad (3.47)$$

with norm

$$\xi_\mu \xi^\mu = (x_4 - x_1)^2 - 4x_5(x_2+x_3)$$
$$+b_1^2(x_7^2+x_8^2) + \cdots + b^2_{\frac{p-4}{2}}(x_{p+1}^2 + x_{p+2}^2). \quad (3.48)$$

When p is odd, the possibilities are slightly different:

- Type I

$$\begin{array}{llll} \mathbb{C} & \lambda_c^{(2,2)} & +\lambda^{(0,1)} & +\frac{p-3}{2}\lambda^{(0,2)}, \\ \mathbb{R} & 2\lambda^{(1,1)} & +\lambda^{(0,1)} & +\frac{p-3}{2}\lambda^{(0,2)}, \\ \mathbb{I} & \lambda^{(2,0)} & +\frac{p-1}{2}\lambda^{(0,2)} & +\lambda^{(0,1)}, \\ \mathbb{R}(0) & \lambda^{(1,1)} & +\lambda^{(1,0)} & +\frac{p-1}{2}\lambda^{(0,2)}. \end{array}$$

- Type II

$$\begin{array}{llll} \mathbb{R} & \lambda_r^{II(2,2)} & +\lambda^{(1,0)} & +\frac{p-3}{2}\lambda^{(0,2)}, \\ \mathbb{I} & \lambda_i^{II(2,2)} & +\lambda^{(0,1)} & +\frac{p-3}{2}\lambda^{(0,2)}. \end{array}$$

- Type III

$$\begin{array}{llll} \mathbb{I} & \lambda^{III(2,4)} & +\lambda^{(0,1)} & +\frac{p-5}{2}\lambda^{(0,2)}, \\ 0\ (a) & \lambda^{III(1,2)} & +\lambda^{(1,1)} & +\frac{p-3}{2}\lambda^{(0,2)}, \\ 0\ (b) & \lambda^{III(2,1)} & +\frac{p-1}{2}\lambda^{(0,2)}, & \end{array}$$

- Type V

$$\lambda^{V(2,3)} \quad +\frac{p-3}{2}\lambda^{(0,2)}.$$

The physical interpretation of each block is now clearer. As mentioned above $\lambda^{(0,1)}$, $\lambda^{(1,0)}$, $\lambda^{(1,1)}$, $\lambda^{(0,2)}$, $\lambda^{(2,0)}$ correspond to trivial directions, boosts and rotations respectively. We have two distinct null rotations: a null rotation $\lambda^{III(1,2)}$ involving two spacelike directions and a null rotation $\lambda^{III(2,1)}$ involving two timelike directions. There are three types of non-trivial four-dimensional elementary blocks: a linear combination of timelike and spacelike null rotations deformed by the addition of a linear combination of boosts $\lambda_r^{II(2,2)}$, a different deformation $\lambda_i^{II(2,2)}$ involving the addition of a timelike rotation and a spacelike rotation, and finally a linear combination $\lambda_c^{(2,2)}$ of two actions involving a timelike and spacelike rotation on one side and a linear combination of boosts on the other side. There is only one five-dimensional elementary block, $\lambda^{V(2,3)}$, which can be interpreted as the linear combination of a timelike null rotation and two spacelike null rotations sharing the time direction and one of the spacelike directions. The last elementary block, $\lambda^{III(2,4)}$, appears in six dimensions, and it consists of a double spacelike null rotation acting on orthogonal subspaces, deformed by a simultaneous rotation in the plane formed by the two timelike directions and two orthogonal spacelike planes.

For the cases which occur for both even and odd p, the difference between the two cases is just that for either even or odd p, there is a direction which does not participate in the quotient; that is, they differ by a factor of $\lambda^{(0,1)}$. It is therefore not worth repeating the expressions for the Killing vectors in these cases for p odd. For the one new case, type $I_{\mathbb{R}(0)}$, the Killing vector is

$$\xi = aJ_{23} + b_1 J_{45} + b_2 J_{67} + \cdots + b_{\frac{p-1}{2}} J_{p+1p+2}, \tag{3.49}$$

with norm

$$\xi_\mu \xi^\mu = a^2(x_2^2 - x_3^2) + b_1^2(x_4^2 + x_5^2) + b_2^2(x_6^2 + x_7^2) + \cdots + b_{\frac{p-1}{2}}^2(x_{p+1}^2 + x_{p+2}^2). \tag{3.50}$$

For $a = 0$, this is the same as type $I_\emptyset$ with $b_1 = 0$ in odd dimension. It is spacelike away from $x_i = 0, i \geq 3$, which is an axis where the Killing vector degenerates.

Recall that we identified Killing vectors differing by conjugation by $O(2,p)$. When considering the amount of supersymmetry preserved by a quotient we need to consider the $SO(2,p)$ classification of [69]. The corresponding Killing vectors are given in table 3.1, notice that now some of the blocks come in pairs $\lambda_\pm$. It can be checked that one element of the pair is always mapped into the other by an orientation-reversing transformation. It should be stressed that in table 3.1, we have not constrained the range of the different parameters appearing in these Killing vectors. For a complete discussion concerning these constraints, we refer the reader to [69].

This completes the basic classification of different one-parameter subgroups of $SO(2,p)$. We have observed that the Killing vector describing each distinct type of quotient naturally decomposes into an $\mathfrak{so}(2,n)$ Killing vector, with $n \leq 4$, and a series of $\mathfrak{so}(2)$ rotations in independent planes. In section 3.4 we will construct a coordinate system which makes this decomposition explicit. Then the action of a given quotient on AdS_{p+1} can be simply expressed in terms of the action of the

Block	Killing Vector
$\lambda^{(0,2)}$	bJ_{34}
$\lambda^{(1,1)}$	aJ_{13}
$\lambda^{(2,0)}$	bJ_{12}
$\lambda^{III(1,2)}$	$J_{13} - J_{34}$
$\lambda^{III(2,1)}$	$J_{12} - J_{23}$
$\lambda^{II(2,2)}_{r\pm}$	$\pm J_{12} + J_{13} \mp J_{24} - J_{34} + a(J_{14} \mp J_{23})$
$\lambda^{II(2,2)}_{i\pm}$	$\pm(b+1)J_{12} + (b-1)J_{34} + J_{13} \mp J_{24}$
$\lambda^{(2,2)}_{c\pm}$	$b(\pm J_{12} - J_{34}) + a(J_{14} \mp J_{23})$
$\lambda^{V(2,3)}$	$J_{12} - J_{24} + J_{13} - J_{34} + J_{15} - J_{45}$
$\lambda^{III(2,4)}_{\pm}$	$J_{15} - J_{35} \pm J_{26} - J_{46} + b(\mp J_{12} + J_{34} + J_{56})$

Table 3.1: The Subspaces as orientation preserving Killing vectors.

corresponding quotient on AdS_3 (or AdS_4 or AdS_5) subspaces of the AdS_{p+1} together with rotations in an orthogonal sphere. In addition, the action of the quotient on the boundary of AdS_{p+1} for $p > 2$ ($p > 3$, $p > 4$ respectively) is also expressed in terms of the action on the bulk of the lower-dimensional space. This observation will be used extensively in the study of the physics of these quotients.

The main purpose of this section was to explore the extension of the classification of one-parameter quotients of AdS_{p+1}, discussed in [56, 68] for the cases $p = 2, 3$, to the general case. This extension proved to be reasonably direct. Perhaps surprisingly, there was little novelty in the general analysis; almost all the cases that appear for general p have appeared already for $p = 2$ [56] or 3 [68]. The one exception, type III_0, extends a particular quotient considered for the case $p = 4$ in [57].

3.2 Infinitesimal isometries of spheres

Here we set up the notation to describe the Killing vectors on spheres. For this purpose, we find it convenient to identify the q-sphere of radius R with the quadric traced by

$$\sum_{i=1}^{q+1} x_i^2 = R^2 \tag{3.51}$$

in $\mathbb{R}^{q+1}$. This has the virtue that the isometry group of the quadric, $\mathrm{O}(q+1)$ acts linearly in the ambient Euclidean space. We shall restrict this group to the subgroup $\mathrm{SO}(q+1)$ which preserves the orientation.

The conjugacy theorem for Cartan subalgebras of $\mathfrak{so}(q+1)$ allows us to bring any Killing vector ξ_S on S^q to the form

$$\xi_S = \sum_{i=1}^{r} \theta_i \, R_{2i-1,2i} \ , \tag{3.52}$$

where $r = \lfloor \frac{q+1}{2} \rfloor$, R_{ij} stands for a rotation in the ij-plane and the θ_i are real parameters specifying the rotation angles. This still leaves the freedom to conjugate by the Weyl group, which we can fix by arranging the parameters in such a way that

$$\theta_1 \geq \theta_2 \geq \cdots \geq |\theta_r| \ .$$

For odd-dimensional spheres, Killing vectors with all $\theta_i \neq 0$ are everywhere nonvanishing, whereas in even-dimensional spheres every vector field, Killing or not, has a zero.

It will be convenient in what follows to construct a coordinate system for S^q adapted to a given Killing vector ξ_S; that is, one in which $\xi_S = \partial_\psi$. Let us describe in detail the case of even-dimensional spheres. First, rewrite (3.51) as

$$\sum_{i=1}^{r} |z_i|^2 + (x_{2r+1})^2 = R^2 \ , \tag{3.53}$$

in which we introduce r complex coordinates for the two-planes where the action of (3.52) may be non-trivial. A natural way to solve (3.53) is by

$$\begin{aligned} x_{2r+1} &= R \cos\theta \\ z_i &= R \sin\theta \, \rho_i \, e^{i\varphi_i} \quad \text{where} \quad \sum_{i=1}^{r} \rho_i^2 = 1 \ . \end{aligned} \tag{3.54}$$

It is clear that in coordinates $\{\theta, \rho_i, \varphi_i\}$,

$$\xi_S = \sum_{i=1}^{r} \theta_i \, \partial_{\varphi_i} \ ,$$

whence by a linear transformation in the space $\{\varphi_i\}$ we can rewrite ξ_S as ∂_ψ. Indeed, assume $\theta_1 \neq 0$, and consider

$$\begin{aligned} \psi &= \theta_1^{-1} \varphi_1 \ , \\ \tilde{\varphi}_i &= \varphi_i - \theta_i \, \theta_1^{-1} \varphi_1 \quad i = 2, \ldots, r \ . \end{aligned} \tag{3.55}$$

By construction, ξ_S becomes ∂_ψ.

The case of odd-dimensional spheres follows formally from the above by setting $\theta = \pi/2$ in the above expressions.

3.3 Causal properties of AdS_{p+1} quotients and their deformations

In Section 3.1 we reviewed the classification of one-parameter subgroups of isometries of AdS_{p+1}. The small number of elementary blocks notwithstanding, the taxonomy of inequivalent discrete quotients increases quickly with dimension due to the possibility of combining the action of different blocks acting in orthogonal subspaces of $\mathbb{R}^{2,p}$. Lack of spacetime prevents us from discussing all possible quotients in detail. Our primary criterion will be that a quotient should have a well-behaved causal structure.

We can divide our quotients into three different subsets according to whether

- the norm of the associated Killing vector field is non-negative (Table 3.2);
- the norm can take negative values, but is bounded below (Table 3.3); and
- the norm can take arbitrarily negative values (Table 3.4).

The quotients generated by Killing vectors with unbounded norm clearly contain closed timelike curves corresponding to the very orbits of the Killing vector in regions where it is timelike. Furthermore, even when we consider quotients of $\mathrm{AdS}_{p+1} \times \mathrm{S}^q$

Type of Quotient
$I_{\mathbb{0}}$ *if* $b_1 = 0$
$I_{\mathbb{R}}$ *if* $\lvert a_1\rvert = \lvert a_2\rvert$
$II_{\mathbb{R}\pm}$ *if* $a = 0$
$II_{\mathbb{0}\pm}$ *if* $b_1 = 0$
$III_{0(c)}$
$III_{\mathbb{0}\pm}$ *if* $b_1 = 0$

Table 3.2: Quotients generated by everywhere non-timelike Killing vectors.

Type of Quotient
$I_{\mathbb{0}}$ *if* n is even *and* $\lvert b_i\rvert \geq b_1 > 0$ *for all* i
$II_{\mathbb{0}\pm}$ *if* $\lvert b_i\rvert \geq \lvert b_1\rvert \geq 0$ *for all* i

Table 3.3: Quotients generated by somewhere timelike Killing vectors which have norm bounded below.

by adding a nontrivial action on the sphere, the resulting Killing vector will still be timelike somewhere, so the quotients will still have closed timelike curves. Therefore the only way in which these quotients will enter into our discussion is in asking whether any of them lead to 'black hole' spacetimes.

The quotients generated by the Killing vectors in Table 3.3 also clearly contain closed timelike curves. However if we consider quotients of $\mathrm{AdS}_{p+1} \times \mathrm{S}^q$, the Killing vector can be made everywhere spacelike by adding a suitable action on an odd-dimensional sphere. Since odd-dimensional spheres admit Killing vectors whose norm is pinched away from zero, whence the total Killing vector

$$\xi = \xi_{\mathrm{AdS}} + \xi_{\mathrm{S}} \tag{3.56}$$

Type of Quotient
$I_{\mathbb{R}}$ *unless* $\|a_1\| = \|a_2\|$
$I_{\mathbb{R}(0)}$
$I_{\mathbb{I}}$ *unless* n is even *and* $\|b_i\| \geq b_1 > 0$ *for all* i
$I_{\mathbb{C}\pm}$
$II_{\mathbb{I}\pm}$ *unless* $\|b_i\| \geq \|b_1\| \geq 0$ *for all* i
$II_{\mathbb{R}\pm}$ *unless* $a = 0$
$III_{0(a)}$
$III_{0(b)}$
$III_{\mathbb{I}\pm}$ *unless* $b_1 = 0$
V

Table 3.4: Quotients generated by Killing vectors with norm unbounded below.

may be spacelike even if ξ_{AdS} is not. This can only happen if the norm of ξ_{AdS} is bounded below, since the norm of ξ_{S} is bounded above by compactness of S^q.

The property of being spacelike everywhere is a necessary condition for the absence of closed causal curves, but it is certainly not sufficient (see [115] for another example where it fails to be sufficient and a statement of a sufficient condition, and [116] for a discussion on this topic and its relation with U-duality). Indeed, we will show presently that even when ξ is everywhere spacelike, if ξ_{AdS} is timelike in some region $D \subset \mathrm{AdS}_{p+1}$, then any discrete cyclic quotient associated to $\xi = \xi_{\mathrm{AdS}} + \xi_{\mathrm{S}}$ will have closed timelike curves in the region $(D \times \mathrm{S}^q)/\Gamma$ of the quotient. The key point in the argument is to exploit the fact that the sphere has bounded diameter in order to construct a timelike curve between two points identified by the action of Γ which, as in [115], is different from the integral curve of ξ.

Let us first illustrate this construction with a simple example, which is depicted in Figure 3.1.

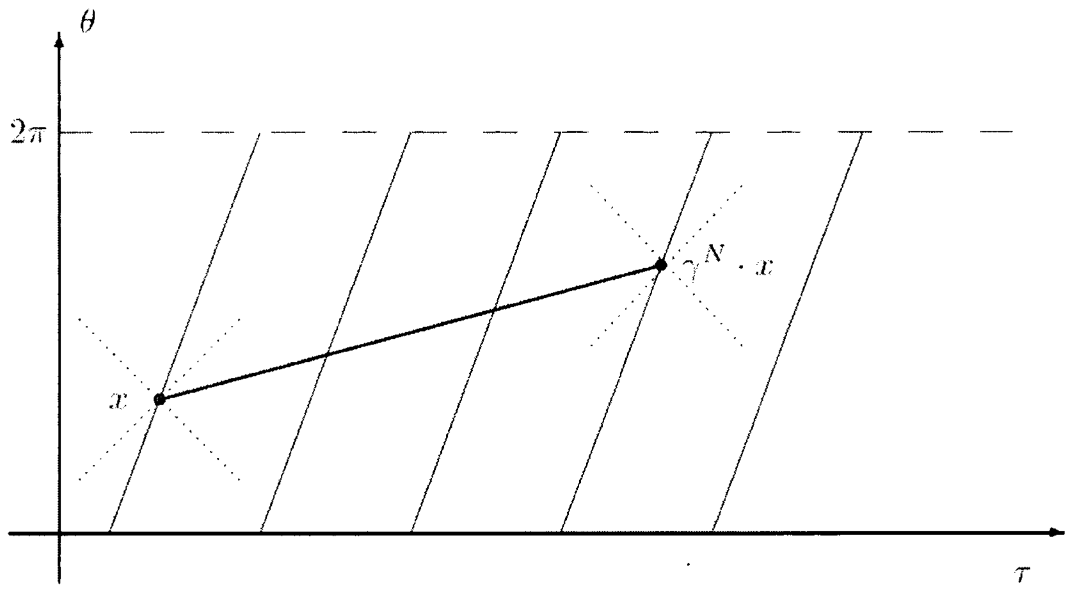

Figure 3.1: Closed timelike curve in a discrete quotient of the Lorentzian cylinder. The dotted lines represent the "lightcones" at x and at $\gamma^N \cdot x$. Notice that although the orbit of ξ is spacelike, the straight line between x and $\gamma^N \cdot x$ is timelike.

Let $C = (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}$ denote a Lorentzian cylinder coordinatised by (θ, τ) and flat metric $d\theta^2 - d\tau^2$. Let $\xi = \partial_\theta + \alpha\partial_\tau$ be a spacelike Killing vector, so that $\alpha^2 < 1$. The integral curve of ξ through a point (θ_0, τ_0) is the curve

$$t \mapsto (\theta_0 + t, \tau_0 + \alpha t) \ .$$

Let us define an action of $\mathbb{Z}$ on C, generated by the operation of flowing along the integral curves of ξ for a time $\ell > 0$:

$$(\theta, \tau) \mapsto (\theta + \ell, \tau + \alpha\ell) \ .$$

Consider the two points (θ, τ) and $(\theta + N\ell, \tau + \alpha N\ell)$, which are identified in the quotient $C/\mathbb{Z}$. The geodesic joining this point to (θ, τ) is the straight line

$$t \mapsto ([\theta + tN\ell], \tau + \alpha N\ell) \ ,$$

where $[-]$ denotes the residue modulo 2π. The norm of the velocity of this curve is therefore

$$[N\ell]^2 - N^2\alpha^2\ell^2 \leq 4\pi^2 - N^2\alpha^2\ell^2 ,$$

which is clearly negative for N large enough. This curve is therefore a closed timelike curve in the quotient $C/\mathbb{Z}$.

Now let us go back to the general case. Let $\gamma = \exp(\ell X)$ for some $X \in \mathfrak{g}$ and $\ell > 0$, and let $\xi = \xi_{\mathrm{AdS}} + \xi_{\mathrm{S}}$ be the Killing vector corresponding to X, with ξ_{AdS} timelike in some nonempty region $D \subset \mathrm{AdS}_{p+1}$. Let $x \in D \times \mathrm{S}^q$. Since the norms of each component ξ_{AdS} and ξ_{S} are separately conserved along the integral curves of ξ, these belong to $D \times \mathrm{S}^q$, and hence so does $\gamma \cdot x$. For those Killing vectors with AdS component in Table 3.3, the associated discrete cyclic groups Γ have infinite order, so we can consider points x and $\gamma^N \cdot x$ for N arbitrarily large, which will give rise to the same point in the quotient. We will construct a curve

$$c : [0, N\ell] \to \mathrm{AdS}_{p+1} \times \mathrm{S}^q$$

between $c(0) = x = (x_{\mathrm{AdS}}, x_{\mathrm{S}})$ and $c(N\ell) = \gamma^N \cdot x = ((\gamma^N \cdot x)_{\mathrm{AdS}}, (\gamma^N \cdot x)_{\mathrm{S}})$ which will be timelike for N sufficiently large and hence becomes a closed timelike curve in the quotient.

The curve c is uniquely specified by its two components: c_{AdS} on AdS_{p+1} and c_{S} on S^q. We will take c_{AdS} to be the integral curve of ξ_{AdS}, and c_{S} to be a minimum-length geodesic between x_{S} and $(\gamma^N \cdot x)_{\mathrm{S}}$. Let L denote the diameter of the sphere; that is, the supremum of the geodesic distances between any two points. Then the arc-length along c_{S} satisfies

$$\int_0^{N\ell} \|\dot{c}_{\mathrm{S}}\| dt = N\ell \|\dot{c}_{\mathrm{S}}\| \leq L ,$$

where the equality is because $\|\dot{c}_{\mathrm{S}}\|$ is constant along c_{S} and the inequality is because c_{S} is length-minimising. Therefore,

$$\|\dot{c}\|^2 = \|\dot{c}_{\mathrm{AdS}}\|^2 + \|\dot{c}_{\mathrm{S}}\|^2 \leq \|\xi_{\mathrm{AdS}}\|^2 + \frac{L^2}{N^2\ell^2} ,$$

which is negative in $D \times \mathrm{S}^q$ for N large enough.

Let us remark that this argument applies to any Freund–Rubin background of the form $\mathrm{AdS} \times N$, or more generally $M \times N$, with M Lorentzian admitting such isometries, at least when N is complete. Indeed, the supergravity equations of motion force N to be Einstein with positive scalar curvature. By the Bonnet–Myers theorem (see, e.g., [108, Section 9.3]), if N is complete, then it has bounded diameter.

This leaves the cases in Table 3.2, where the AdS Killing vector is nowhere timelike. It is clear that the above argument for closed timelike curves fails in this case. One should note that this still does not directly imply the absence of closed timelike curves; however, we will see in the next section that there are in fact no closed timelike curves in any of these cases.

We should also note that in the cases where the Killing vector is null somewhere, namely $I_{\parallel}$ with $b_1 = 0$, $III_{0(c)}$ and $II_{\mathbb{R}\pm}$ with $a = 0$, we can use a similar argument to see that *some* quotients of $\mathrm{AdS}_{p+1} \times \mathrm{S}^q$ still produce closed causal curves. The point is that if we choose ℓ such that $\exp(\ell x_{\mathrm{S}}) \in \mathrm{SO}(q+1)$ has order N, then x and $x' = \gamma^N \cdot x$ can be null separated, as $x'_{\mathrm{S}} = x_{\mathrm{S}}$, and the separation in the AdS factor is null if $||\xi_{\mathrm{AdS}}|| = 0$ at x. Physically, this corresponds to deforming by a rotation with rational angles on S^q.

Clearly, however, deformations for which γ_{S} does not have finite order do exist, and will not lead to closed causal curves by any of our arguments above. Hence, we should discuss all the cases listed in Table 3.2 in the next section, as they can all give rise to causally non-singular quotients.

3.4 Causally non-singular quotients

In this section, we shall discuss in detail the geometry of the discrete quotients that are free of closed causal curves. These are based on the subgroups listed in table 3.2, conveniently deformed when necessary by some non-trivial action on an odd sphere leaving no invariant directions, so that the full Killing vector field (3.56) is spacelike everywhere.

Before initiating such a task, we would like to comment on the general philosophy that we shall apply in each of the particular geometries to be discussed. We know the Killing vector describing each distinct type of quotient naturally decomposes into an $\mathfrak{so}(2,n)$ Killing vector, with $n \leq 4$, and a series of $\mathfrak{so}(2)$ rotations in independent planes. Thus given any Killing vector in table 3.2, we can study the geometry of the corresponding discrete quotient in different dimensional AdS spacetimes, starting with the minimal $(2, n)$ signature in the embedding space $\mathbb{R}^{(l,m)}$. Besides that, we can also study further deformations on the sphere sector of the discrete quotient. It is therefore natural to start our analysis in the lowest dimensional $\text{AdS}_{p+1} \times \text{S}^q$ spacetime allowing our causally non-singular quotients, and afterwards, extend such an analysis to higher dimensions.

This latter extension is entirely straightforward. Indeed, given some adapted coordinate system describing the action of ξ_{AdS} in AdS_{n+1}, it is very simple to construct an adapted coordinate system describing the action of the same Killing vector field in AdS_{p+1} with $p > n$. This is just obtained by considering the standard AdS_{n+1} foliation of AdS_{p+1} given in terms of the embedding coordinates by

$$\begin{aligned} x^i &= \cosh\chi\,\hat{x}^i \quad i = 1, \ldots, n+2 \\ x^m &= \sinh\chi\,\hat{x}^m \quad m = 1, \ldots p-n \end{aligned} \tag{3.57}$$

where χ is non-compact and $\{\hat{x}^i\}$ satisfy the quadric defining relation giving rise to AdS_{p+1}, whereas $\{\hat{x}^m\}$ parametrise an S^{p-n-1} sphere of unit radius. For $p = n+1$, the range of χ is given by $-\infty < \chi < +\infty$, whereas for $p - n \geq 2$, it is simply given by $\chi \geq 0$. The metric description of AdS_{p+1} in the AdS_{n+1} foliation defined in (3.57) is

$$g_{\text{AdS}_{p+1}} = (\cosh\chi)^2\, g_{\text{AdS}_{n+1}} + (d\chi)^2 + (\sinh\chi)^2\, g_{\text{S}^{p-n-1}} \,. \tag{3.58}$$

The foliation given by (3.58) also gives us an interesting description of the asymptotic boundary. If we assume $p - n \geq 2$, taking the limit $\chi \to \infty$ and conformally rescaling by a factor of $e^{-2\chi}$, we can describe the asymptotic boundary in terms of an $\text{AdS}_{n+1} \times \text{S}^{p-n-1}$ metric,[4]

$$g_\partial = g_{\text{AdS}_{n+1}} + g_{\text{S}^{p-n-1}} \,. \tag{3.59}$$

[4]For $p - n = 1$, we would have $-\infty < \chi < \infty$, and conformally rescaling by a factor of $e^{-2|\chi|}$ as we take the limit $|\chi| \to \infty$, we would get a description of the boundary in terms of two AdS_p patches, each covering one of the hemispheres of the S^{p-1} in the usual Einstein static universe $\mathbb{R} \times \text{S}^{p-1}$ description of the boundary of AdS_{p+1}.

To see the relation of this coordinate system to the usual Einstein static universe description of the conformal boundary, let us write the AdS_{n+1} metric in global coordinates,

$$g_{AdS_{n+1}} = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, g_{S^{n-1}} . \qquad (3.60)$$

Then defining $\cos\theta = 1/\cosh\rho$, we can rewrite (3.59) as

$$g_\partial = \frac{1}{\cos^2\theta}(-dt^2 + d\theta^2 + \sin^2\theta g_{S^{n-1}} + \cos^2\theta g_{S^{p-n-1}}). \qquad (3.61)$$

This shows that the metric in (3.59) is indeed conformal to the Einstein static universe metric on $\mathbb{R} \times S^{p-1}$, where we are writing the S^{p-1} as an S^{p-n-1} fibred over an S^n. The coordinates of (3.59) cover all of the Einstein static universe apart from the $\mathbb{R} \times S^{n-1}$ submanifold where $\cos\theta = 0$, which is conformally rescaled to become the boundary of the AdS_{n+1} factor in (3.59).

If there is a global adapted coordinate system for the action of ξ_{AdS} on AdS_{n+1}, we can use the above foliation to construct an adapted coordinate system for the action on AdS_{p+1}. If we deform the action by $\lambda^{(0,2)}$ blocks, these will act as rotations of the S^{p-n-1} factor in the above foliation.

When we consider the deformation of our AdS quotient by some non-trivial action on the transverse sphere, we have two approaches to the construction of an overall adapted coordinate such that the total Killing vector $\xi = \partial_\varphi$ for some coordinate φ. In most of the cases we consider[5], there is a globally well-defined adapted coordinate on AdS_{p+1} such that $\xi_{AdS} = \partial_\phi$. As noted in Section 3.2, there is always a global adapted coordinate system for the Killing vectors in the sphere, in which ξ_S acts by a simple "translation", i.e. $\xi_S = \partial_\psi$. Consequently, the full generator of the discrete quotient is

$$\xi = \partial_\phi + \gamma\partial_\psi, \qquad (3.62)$$

By a linear transformation, $\varphi = \phi, \psi' = \psi - \gamma\phi$, we are able to write $\xi = \partial_\varphi$. This coordinate system is very convenient for studying the causal structure and asymptotic structure of the resulting quotient, so this is the technique we shall mostly employ.

[5]The only exceptions are where the AdS Killing vector has fixed points.

Unfortunately, there are examples where there is no such global adapted coordinate system on AdS. The example of this type we shall be concerned with is the quotient by a Killing vector with a single $\lambda^{III(1,2)}$ block. In this case, we need to use a different technique, exploiting the existence of adapted coordinates on the sphere. The full Killing vector field (3.56) can always be written as

$$\xi = \partial_\psi + \xi_{\text{AdS}} \,. \tag{3.63}$$

We can therefore write ξ as a dressed version of its "translation" component according to

$$\xi = U\,\partial_\psi\,U^{-1} \qquad \text{where} \quad U = \exp\left(-\psi\,\xi_{\text{AdS}}\right) \,. \tag{3.64}$$

Consequently, if the original coordinate system was given by $\{\psi, z^l\}$, where z^l stand for all the remaining coordinates describing the manifold $\text{AdS}_{p+1} \times \text{S}^q$, it is natural to change coordinates to an adapted coordinate system defined by

$$y = U\,z \,, \tag{3.65}$$

which indeed satisfies the property $\xi\,y = 0$, so that $\{y^l\}$ are good coordinates for the space of orbits. Equivalently, $\xi = \partial_\psi$ in the coordinates (3.65). Thus, we obtain an adapted coordinate system on the full quotient for any AdS Killing vector. For the case at hand, we split the coordinates $\{z^l\}$ appearing in the above discussion into $\{z^l\} = \{\tilde{\varphi}_i, \vec{x}\}$, where $\{\vec{x}\}$ stand for the embedding coordinates of AdS_{p+1} in $\mathbb{R}^{2,p}$. Since ξ_{AdS} is a Lorentz transformation in $\mathbb{R}^{2,p}$, its action on $\vec{x}$ can be defined by

$$\xi_{\text{AdS}}\,\vec{x} = B\,\vec{x} \,, \tag{3.66}$$

where B is a $(p+2) \times (p+2)$ constant matrix. Thus, $\vec{y}(\psi, \vec{x}) = e^{-\psi\,B}\,\vec{x}$, so that

$$d\vec{x} = e^{\psi\,B}\left(d\vec{y} + B\,\vec{y}\,d\psi\right) \,. \tag{3.67}$$

One can now compute the metric in adapted coordinates $\{\psi, \tilde{\varphi}_i, \vec{y}\}$. This can be written as

$$g = \|\xi_{\text{S}}\|^2\,\left(d\psi + B_1\right)^2 + \tilde{g} + g_{\text{AdS}_{p+1}} + 2\,d\psi \cdot \hat{\xi}_{\text{AdS}} + \|\xi_{\text{AdS}}\|^2\,d\psi^2 \,, \tag{3.68}$$

where the first two terms are just describing the metric on S^q in the adapted coordinate system $\{\psi, \tilde{\varphi}_i\}$ introduced in Section 3.2, and $\hat{\xi}_{\text{AdS}}$ stands for the one-form associated with the Killing vector ξ_{AdS}, that is,

$$\hat{\xi}_{\text{AdS}} = \eta_{ij}\xi^j_{\text{AdS}}\, dy^i = \eta_{ij}\ (B \cdot y)^j\ dy^i\ . \tag{3.69}$$

After these general considerations, we shall now proceed to discuss the different geometries that appear in these discrete quotients of $\text{AdS}_{p+1} \times \text{S}^q$.

3.4.1 Non-everywhere spacelike ξ_{AdS}

Let us first discuss the three cases in which ξ_{AdS} is not always spacelike. The first of these is I_0 with $b_1 = 0$, corresponding to the quotient of AdS_{p+1} by some combination of rotations in orthogonal two-planes $\mathbb{R}^2$ in the embedding space. These quotients produce special cases of the conical defects, which were discussed extensively in, for example [97]. An interesting discussion of the properties of the supersymmetric orbifolds in string theory is also given in [109, 110]. We will not discuss this case further here, except to note that it is for these quotients where the existence of a spin structure is not guaranteed. The condition for the existence of a spin structure was discussed in [3].

To consider the other two cases in Table 3.2 which are not always spacelike, $III_{0(c)}$ and $II_{\mathbb{R}\pm}$with $a = 0$, we follow our general strategy, and start by describing the action of $\lambda^{III(1,2)}$ or $\lambda_r^{II(2,2)}$ with $a = 0$ in AdS_3. The action of a more general Killing vector of this form on AdS_{p+1} can then be built up by considering the AdS_3 action deformed by the rotations $\lambda^{(0,2)}$ on the S^{p-3} in the $\text{AdS}_3 \times \text{S}^{p-3}$ foliation of (3.58). We will then add in the deformation on a transverse sphere S^q to obtain an everywhere spacelike quotient.

For the quotient of AdS_3 by $\lambda^{III(1,2)}$, the relevant Killing vector is

$$\xi_{\text{AdS}} = J_{13} - J_{34}. \tag{3.70}$$

This Killing vector is spacelike almost everywhere, $||\xi_{\text{AdS}}||^2 = (x_1 + x_4)^2$. There is a single other Killing vector in $\mathfrak{so}(2,2)$ which commutes with this one, $\xi_1 = J_{12} - J_{24}$. It has norm $||\xi_1||^2 = -(x_1 + x_4)^2$. The most convenient coordinate system for studying this quotient is Poincaré coordinates. The form of the Killing vectors in Poincaré coordinates was reviewed in section 2.2. It is easy to see from those expressions that in the case of $\lambda^{III(1,2)}$, we can orient the coordinates so that $\xi_{\text{AdS}} = \partial_x$ and $\xi_1 = \partial_t$, where the AdS_3 metric in Poincaré coordinates is

$$g_{\text{AdS}_3} = \frac{1}{z^2}\left(-dt^2 + dz^2 + dx^2\right) . \tag{3.71}$$

We see that the effect of the quotient is simply to make the coordinate x periodic. The Killing vector ξ_{AdS} becomes null on the Poincaré horizon $z = \infty$ where this coordinate system breaks down. In terms of the embedding coordinates, this is the surface $x_1 + x_4 = 0$, where $\xi_{\text{AdS}} = x_3(\partial_1 - \partial_4)$. We note that this symmetry has a null line of fixed points at $x_1 + x_4 = x_3 = 0$ (parametrised by $x_1 - x_4$). Away from the fixed points, the identification along ξ_{AdS} will generate closed null curves in the Poincaré horizon. These can be eliminated by deforming this quotient by a suitable action on an odd-dimensional sphere. Since we do not have a good global coordinate system on this quotient, the best way to describe the causally regular deformed quotient will be to use the coordinates adapted to the action on the transverse sphere, as described at the end of the last subsection. We will not give the details of the application of this general technique for this particular case; we just remark that for this case, the matrix B defined in (3.66) is

$$B = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} . \tag{3.72}$$

Following the supersymmetry analysis in [69], it is easy to conclude that for a suitable choice of sphere deformation, the above quotient preserves $\nu = \frac{1}{4}$ of the vacuum supersymmetry, that is, it has four supercharges.

For the case where we introduce a deformation on a transverse S^3, we can interpret the quotient as the near horizon geometry of a D1-D5 system that has been quotiented by the action generated by

$$\xi = \partial_x + \theta_1 R_{12} + \theta_2 R_{34} ,$$

in which x stands for the common direction shared by the D1-D5 system, and R_{ij} stand for rotations transverse to the D1-D5's. In the language developed in [106,107], this asymptotically flat spacetime would correspond to a D1-D5 system in a generic intersection of flux 7-branes vacuum. Whenever $\theta_1 = \pm\theta_2$, it would be interpreted as a D1-D5 system in the flux 5-brane vacuum, which also has four supercharges. Note that the standard supersymmetry enhancement due to the near horizon limit is lost in this quotient, as the generator ∂_x, which does not break any supersymmetry in the asymptotically flat spacetime construction, becomes a null rotation generator from the AdS perspective, which breaks one half of the supersymmetry.

We would also like to understand the boundary of this quotient. In the Poincaré coordinates (3.71), the global AdS boundary is written in terms of an infinite series of flat space patches,

$$g_\partial = -dt^2 + dx^2. \tag{3.73}$$

The action of the Killing vector on the AdS boundary compactifies the spatial coordinate x; it might therefore seem that the quotient will have an infinite sequence of boundaries. However, the Killing vector only has isolated fixed points on the boundary, at the points where the line of fixed points $x_1 + x_4 = x_3 = 0$ meets the boundary. In Poincaré coordinates, these correspond to the points at past and future timelike infinity and at spacelike infinity. The different boundary patches are therefore connected. We can extend the Poincaré coordinates to cover more of the boundary by defining

$$v = t - x, \tan T = t. \tag{3.74}$$

The boundary metric then becomes

$$g_\partial = \frac{1}{\cos^2 T}(-2dv\, dT + \cos^2 T dv^2), \tag{3.75}$$

and the Killing vector we quotient along is $\xi_{\text{AdS}} = \partial_v$. Since we only have a conformal structure on the boundary, we can ignore the overall factor in this metric. In the resulting metric, we see that the direction we quotient along is spacelike except when $T = (n + 1/2)\pi$, where it becomes null. These points correspond to one half of future and past null infinity in the original Poincaré coordinates. This coordinate system covers the whole of the conformal boundary with the exception of a null line corresponding to one half of past and future null infinity in each Poincaré patch. We could construct a similar coordinate system by defining $u = t + x$—it would then cover that half but not the one where $t - x$ remains finite. We can think of the field theory dual to the quotient along a null rotation as living on the cylindrical space described in (3.75), which has closed null curves at $T = (n + 1/2)\pi$.[6] Since the deformation by an action on a transverse sphere does not alter the action on the boundary, it cannot remove these closed lightlike curves in the dual theory.

A more interesting example of a not everywhere spacelike quotient is $\lambda_{r\pm}^{II(2,2)}$ with $a_1 = 0$, where the Killing vector we quotient along is

$$\xi_{\text{AdS}}^{\pm} = \pm(J_{12} - J_{24}) + (J_{13} - J_{34}) \tag{3.76}$$

respectively. Both are null everywhere, $||\xi_{\text{AdS}}^{\pm}||^2 = 0$. From now on, we shall focus on ξ_{AdS}^{+}; there is an analogous discussion and structure for ξ_{AdS}^{-}. There are three other Killing vectors in $\mathfrak{so}(2,2)$ commuting with ξ_{AdS}^{+},

$$\xi_1 = J_{24} + J_{13}, \quad \xi_2 = J_{12} + J_{34}, \quad \xi_3 = J_{14} - J_{23} \,. \tag{3.77}$$

These satisfy

$$[\xi_i, \xi_j] = 2\epsilon_{ijk}\xi_k, \tag{3.78}$$

so they define an $\mathfrak{sl}(2,\mathbb{R})$ symmetry which commutes with ξ_{AdS}^{+}. This $\mathfrak{sl}(2,\mathbb{R})$ structure appears because when we write $\mathfrak{so}(2,2) = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$, the $\lambda_r^{II(2,2)}$ Killing vector lies entirely in one of the $\mathfrak{sl}(2,\mathbb{R})$ factors. A similar structure will reappear for the same reason in our discussion of the self-dual orbifold in section 3.4.2; it was first identified in that context in [104].

We would like to adopt a coordinate system adapted to this symmetry. Since the ξ_i do not commute, we can only adapt our coordinates to one of them. We note that $||\xi_1||^2 = ||\xi_3||^2 = 1$, $||\xi_2||^2 = -1$. Since our interest is in causal structure, it seems natural to adapt the coordinates to the timelike vector ξ_2. We therefore want to construct a coordinate system (t, v, ρ) on AdS_3 such that $\xi_{\text{AdS}}^{+} = \partial_v$ and $\xi_2 = \partial_t$.

[6] There are some obvious similarities between this construction and the Milne coordinate system on the orbifold of flat space by a boost.

This requires

$$\begin{aligned}
&\frac{\partial(x^4-x^1)}{\partial v}=0, \quad \frac{\partial(x^4+x^1)}{\partial v}=-2(x^3-x^2),\\
&\frac{\partial(x^3-x^2)}{\partial v}=0, \quad \frac{\partial(x^3+x^2)}{\partial v}=2(x^4-x^1)\,;\\
&\frac{\partial(x^4-x^1)}{\partial t}=x^3-x^2, \quad \frac{\partial(x^4+x^1)}{\partial t}=x^3+x^2,\\
&\frac{\partial(x^3-x^2)}{\partial t}=-(x^4-x^1), \quad \frac{\partial(x^3+x^2)}{\partial t}=-(x^4+x^1)\ .
\end{aligned} \tag{3.79}$$

A combination which is thus independent of t, v is $(x^4-x^1)^2+(x^3-x^2)^2$. We will choose the ρ coordinate so that this combination is $e^{2\rho}$. A suitable coordinate system satisfying these criteria and the condition $-x_1^2-x_2^2+x_3^2+x_4^2=-1$ defining the AdS_3 embedding is

$$\begin{aligned}
x^4-x^1 &= e^{\rho}\sin t,\\
x^4+x^1 &= -e^{-\rho}\sin t-2v\,e^{\rho}\cos t,\\
x^3-x^2 &= e^{\rho}\cos t,\\
x^3+x^2 &= -e^{-\rho}\cos t+2v\,e^{\rho}\sin t\ .
\end{aligned} \tag{3.80}$$

The inverse coordinate transformation is given by

$$\begin{aligned}
e^{2\rho} &= (x^4-x^1)^2+(x^3-x^2)^2,\\
\tan t &= \frac{x^4-x^1}{x^3-x^2},\\
v &= e^{-2\rho}\left\{\left[(x^3+x^2)+e^{-2\rho}(x^3-x^2)\right]^2+\left[(x^4+x^1)+e^{-2\rho}(x^4-x^1)\right]^2\right\}\ .
\end{aligned} \tag{3.81}$$

Since these give finite values of t, v, ρ for all points in AdS_3, this coordinate system covers the whole spacetime. In terms of these coordinates, the metric is

$$g_{AdS_3}=-dt^2+d\rho^2-2e^{2\rho}dv\,dt\ . \tag{3.82}$$

In this coordinate system, the other two Killing vectors are

$$\begin{aligned}
\xi_1 &= \sin 2t\,\partial_\rho+\cos 2t\,(\partial_t-e^{-2\rho}\partial_v),\\
\xi_3 &= -\cos 2t\,\partial_\rho+\sin 2t\,(\partial_t-e^{-2\rho}\partial_v)\ .
\end{aligned} \tag{3.83}$$

We see that making identifications along the Killing vector ∂_v will produce closed null curves. To eliminate these closed null curves, we should introduce a deformation by a rotation on the transverse sphere. To simplify the discussion, we shall work it out explicitly for a transverse S^3, having in mind the standard way of embedding AdS_3 in type IIB string theory, as the near horizon geometry of the D1-D5 system, giving rise to $AdS_3 \times S^3 \times \mathbb{T}^4$. As discussed in Section 3.2, there are several inequivalent quotients that one can take of S^3. We will focus on a particular quotient which preserves supersymmetry, namely the quotient where $\xi_S = \partial_\psi$ when we write the S^3 metric as

$$g_{S^3} = d\theta^2 + d\psi^2 + d\varphi^2 + 2\cos 2\theta \, d\psi \cdot d\varphi. \tag{3.84}$$

Thus, we consider the quotient along a total Killing vector $\xi = \xi_{\text{AdS}} + \gamma\xi_S = \partial_v + \gamma\partial_\psi$. Since we have a global adapted coordinate system (3.82) on the AdS part of the quotient, it is convenient to construct the global coordinate system on the full $AdS_3 \times S^3$ quotient by defining $\psi' = \psi - \gamma v$. The six-dimensional metric is then

$$g = -dt^2 + d\rho^2 - 2e^{2\rho}dvdt + d\theta^2 + (d\psi' + \gamma dv)^2 + d\varphi^2 + 2\cos 2\theta \, (d\psi' + \gamma dv) \cdot d\varphi. \tag{3.85}$$

The quotient is now along $\xi = \partial_v$. We can see that this is an everywhere spacelike direction; $||\xi||^2 = \gamma^2$. This is a necessary but not a sufficient condition for the absence of closed causal curves, but it is easy to check explicitly that there are no closed causal curves in the bulk of the quotient manifold in this case. As shown in [69], the corresponding type IIB configuration preserves $\nu = \frac{1}{8}$ of the vacuum supersymmetry, that is, it has four supercharges. It is interesting to point out that if we would have considered the action on the three sphere (3.84) generated by $\xi_S = \partial_\varphi$, the corresponding quotient $\xi = \xi_{\text{AdS}} + \gamma\,\xi_S$ would have preserved $\nu = \frac{1}{4}$ of the full type IIB supersymmetry.

It is interesting to note that, like the null rotation, the $\lambda_{r\pm}^{II(2,2)}(a = 0)$ Killing vector also has a simple action in Poincaré coordinates. We can orient the coordinates so that $\xi_{\text{AdS}} = \partial_t + \partial_x$ in the metric (3.71). The additional symmetry $\partial_t - \partial_x$ that is manifest in these coordinates can be written in terms of the $\mathfrak{sl}(2,\mathbb{R})$ Killing vectors (3.77) as the combination $\xi_2 - \xi_1$. Although the Poincaré coordinates are not a global coordinate system for the quotient, they allow us to relate these quotients and quotients of branes in asymptotically flat spacetimes: the $\lambda_{r\pm}^{II(2,2)}(a = 0)$ quotients can be understood as the near horizon geometries of a D1-D5 system quotiented by the discrete action generated by

$$\xi = \pm\partial_t + \partial_x + \theta_1 R_{12} + \theta_2 R_{34} . \tag{3.86}$$

The physical interpretation of these quotients is unclear. They can be supersymmetric, and they are free from closed causal curves. It might be possible to give them some interpretation using a limiting procedure in which one finally identifies bulk points along a "null translation", by infinitely boosting a spacelike translation. In this case, there is still a supersymmetry enhancement since the asymptotically flat quotient has four supercharges.

To discuss the conformal boundary of this quotient, we will use a technique that will be used again in section 3.4.3, and relate the spacetime to a plane wave. If we set $r = e^{-\rho}$, the metric (3.85) becomes

$$g = \frac{1}{r^2}[-2dvdt - r^2dt^2 + dr^2 + r^2(d\theta^2 + (d\psi' + \gamma dv)^2 + d\varphi^2 + 2\cos 2\theta\,(d\psi' + \gamma dv)\cdot d\varphi)]. \tag{3.87}$$

The conformally related metric in square brackets is a symmetric six-dimensional plane wave, written in a polar coordinate system deformed so that ∂_v is a mixture of the null translation symmetry of the plane wave and a rotation in the four transverse spacelike coordinates.

The conformal mapping between an $AdS_3 \times S^3$ space and a plane wave is implicit in previous work [111] which showed that such plane waves can be conformally mapped onto the Einstein static universe. That is, since both spaces are conformally flat, we would expect them to be conformally related. It is interesting to note the relative simplicity of the relation: $AdS_3 \times S^3$ corresponds to the plane wave with the axis $r = 0$ excluded, rescaled by a factor of $1/r^2$.

More important for our present purpose is that the Killing vector we wish to quotient along, ∂_v, annihilates the conformal factor (as does $\xi_2 = \partial_t$), so we can use this conformal map to study the boundary of the quotient spacetime, and not just to study global $AdS_3 \times S^3$. Note that unlike the double null rotation in section 3.4.3, the other Killing symmetries ξ_1 and ξ_2 of this quotient do not also commute with the conformal rescaling. They will hence appear as conformal isometries in the boundary theory.

The conformal boundary of the quotient (3.87) lies at $r = 0$, and has the metric (up to conformal transformations)

$$g_\partial = -2dv\,dt. \tag{3.88}$$

Since v is periodically identified in the quotient, there is a compact null direction through every point in the boundary. As in the null rotation case, these closed null curves in the conformal boundary cannot be removed by a sphere deformation. This fact can explicitly by checked in (3.85). It is interesting to note that we get the same metric on the conformal boundary here as on either of the two boundaries in the self-dual orbifold discussed in the next subsection.

If we regard (3.85) simply as a coordinate system on $\mathrm{AdS}_3 \times S^3$, we can relate this description of the conformal boundary to the usual two-dimensional $\mathbb{R} \times S^1$ Einstein static universe boundary of global $\mathrm{AdS}_3 \times S^3$. In global coordinates, the Killing vector field is given by

$$\xi = (1 + \cos(\tau - \varphi))(\partial_\tau - \partial_\varphi), \tag{3.89}$$

where we are using the global coordinates introduced in section 2.2, and further writing $\hat{x}_3 = \cos\varphi$, $\hat{x}_4 = \sin\varphi$, so that the metric on the boundary reads

$$g_\partial = -d\tau^2 + d\varphi^2 . \tag{3.90}$$

We see that the quotient is along a null direction, and has a single null line of fixed points at $\tau - \varphi = \pi$ (mod 2π). While the coordinate system (3.85) covers all of global $\mathrm{AdS}_3 \times S^3$, it does not cover all of its conformal boundary, as these symmetry-adapted coordinates break down on the fixed points of ξ^+_{AdS}. The coordinates of (3.85) cover all of the boundary apart from this null line. They are related to the global description above in the same way that a symmetric plane wave is related to the Einstein static universe in higher-dimensional cases [111] (in two dimensions, there is no non-trivial plane wave). Thus we see that (3.88) provides a natural description of the asymptotic boundary of the quotient, corresponding to excluding these fixed points in discussing the quotient.

While it is clear that the deformed quotient (3.85) is free of closed causal curves, we can show that this quotient does not preserve the stable causality of the original $AdS_3 \times S^3$ space. If we write (3.85) in the form appropriate for Kaluza-Klein reduction along v,

$$g = -(1+\gamma^{-2}e^{4\rho})dt^2 + d\rho^2 + d\theta^2 + \sin^2 2\theta d\varphi^2 + 2\gamma^{-1}e^{2\rho}dt(d\psi' + \cos 2\theta d\varphi) + (\gamma dv + d\psi' + \cos 2\theta d\varphi - \gamma^{-1}e^{2\rho}dt)^2, \quad (3.91)$$

we see that the lower-dimensional metric obtained by Kaluza-Klein reduction along v will have closed null curves, since the compact circle parametrised by ψ' is null. This implies that there can be no time function τ on $AdS_3 \times S^3$ such that $\mathcal{L}_\xi \tau = 0$, for if there was, the Kaluza-Klein reduced metric would be stably causal, which is inconsistent with the appearance of closed null curves in the latter. Thus, the discrete quotient cannot satisfy the condition of [115], and does not preserve stable causality.

Following the discussion around (3.58), it is straightforward to describe the quotient generated by ξ^+_{AdS} in higher dimensional AdS_{p+1} spaces. By construction, the global symmetries of such a higher dimensional quotient will be the ones discussed before times $SO(p-2)$, corresponding to the rotational symmetry transverse to the subspace where ξ_{AdS} acts. Notice that in this case, the metric on the boundary is conformally equivalent to a plane wave metric,

$$g_\partial = -2\, dv\, dt - r^2 dt^2 + dr^2 + r^2 g_{S^{p-3}} \,. \quad (3.92)$$

In higher dimensions, there exists the possibility to deform the quotient by rotations, i.e. $\lambda^{(0,2)}$. Let us focus on AdS_5, for algebraic simplicity. The metric for AdS_5 in the AdS_3 foliation adapted to the action of ξ^+_{AdS} is given by

$$g_{\text{AdS}_5} = \cosh^2\chi\, \left(-dt^2 + d\rho^2 - 2e^{2\rho}\, dv\, dt\right) + d\chi^2 + \sinh^2\chi\, d\theta^2 \,. \quad (3.93)$$

The deformation consists in acting on the angular direction θ through the generator $\xi = b\partial_\theta$. Thus, it is convenient to introduce the new coordinate $\theta' = \theta - bv$, so that $\xi^+_{\text{AdS}} + \xi = \partial_v$. The metric on the deformed quotient is

$$g_{\text{AdS}_5/\Gamma} = \cosh^2\chi\, \left(-dt^2 + d\rho^2 - 2e^{2\rho}\, dv\, dt\right) + d\chi^2 + \sinh^2\chi\, \left(d\theta' + bdv\right)^2 \,, \quad (3.94)$$

where, once again, $v \sim v + 2\pi$. As expected, the periodic coordinate v becomes everywhere spacelike except at the fixed point of the deformed action. This is just a consequence of the fact that the norm of the deformed Killing vector is $||\xi_{\text{AdS}}+\xi||^2 = b^2[(x^5)^2 + (x^6)^2] = b^2\sinh^2\chi$, which certainly vanishes at the origin of the 56-plane, where the fixed point of ξ lies.

This particular deformation ($b \neq 0$) breaks all the supersymmetry and it can be interpreted as the near horizon geometry of a bunch of parallel and coincident D3-branes quotiented by the action of a null translation plus a rotation. It is certainly possible to turn on supersymmetric deformations in higher dimensional AdS spacetimes. In particular, it is possible to consider families of two parameter deformations corresponding to $\lambda^{(0,2)}(b_1) \oplus \lambda^{(0,2)}(b_2)$ in AdS_7. Whenever $b_1 = \pm b_2$, the quotient will preserve supersymmetry. The corresponding asymptotically flat interpretation would be in terms of parallel and coincident M5-branes quotiented by the action of a null translation plus a certain rotation in $\mathbb{R}^4$. The supersymmetric deformation would correspond to the action having an $\mathfrak{su}(2)$ holonomy.

3.4.2 Self-dual orbifolds and their deformations

The fifth Killing vector appearing in table 3.2, $I_{(\mathbb{R})}$ with $|a_1| = |a_2|$, can be interpreted as the deformation of the self-dual orbifolds of AdS_3, first introduced in [104], and recently discussed in [105]. The norm of ξ_{AdS} is spacelike everywhere. Therefore, one can study these geometries with or without any further non-trivial action on transverse spheres.

As already indicated above, the minimal dimension where this discrete quotient exists is for $p = 2$, i.e. AdS_3. The addition of any rotation parameter b_i would increase this dimension by two. Since the elementary indecomposable block acting on AdS_3 is a linear combination of boosts in $\mathbb{R}^{2,2}$, this discrete quotient does not have an analogue in an asymptotically flat spacetime, in the sense that there is no quotient whose near horizon limit gives rise to these self-dual orbifolds.

The anti-de Sitter action, including the deformation parameters $\{b_1\}$, integrates to the following $\mathbb{R}$-action on $\mathbb{R}^{2,p}$:

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \\ x^{2i+5} \\ x^{2i+6} \end{pmatrix} \mapsto \begin{pmatrix} x^1 \cosh at \pm x^3 \sinh at \\ x^2 \cosh at \pm x^4 \sinh at \\ x^3 \cosh at \pm x^1 \sinh at \\ x^4 \cosh at \pm x^2 \sinh at \\ x^{2i+5} \cos b_i t - x^{2i+6} \sin b_i t \\ x^{2i+6} \cos b_i t + x^{2i+5} \sin b_i t \end{pmatrix}, \quad \forall i \tag{3.95}$$

where we set $a_1 = a$ and $a_2 = \pm a$. Notice that the above action is manifestly free of fixed points for any value of the boost and rotation parameters $\{a, b_i\}$.

In the following, we shall review the main features of the self-dual orbifolds of AdS_3, extending the discussion to uncover their embeddings in higher dimensional anti-de Sitter spacetimes and their deformations both by rotations in anti-de Sitter and non-trivial actions on transverse spheres, afterwards.

Pure AdS

Let us start our discussion by focusing on AdS_3, so that there are no $\lambda^{(0,2)}$ blocks. In this case, as first described in [104], the quotient preserves an $\mathbb{R} \times \mathfrak{sl}(2,\mathbb{R})$ subalgebra of the original $\mathfrak{so}(2,2) = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ isometry algebra. A suitable system of global coordinates adapted to the quotient and the timelike vector in $\mathfrak{sl}(2,\mathbb{R})$ is [104]

$$\begin{aligned} x^1 &= \cosh z \cosh a\phi \cos t - \sinh z \sinh a\phi \sin t, \\ x^2 &= \cosh z \cosh a\phi \sin t + \sinh z \sinh a\phi \cos t, \\ x^3 &= -\cosh z \sinh a\phi \cos t + \sinh z \cosh a\phi \sin t, \\ x^4 &= \pm\left(\cosh z \sinh a\phi \sin t - \sinh z \cosh a\phi \cos t\right) . \end{aligned} \tag{3.96}$$

The sign ambiguity in the last line of (3.96) corresponds to the two distinct cases $a_2 = \pm a_1$ in the $SO(2,p)$ classification reviewed in section 3.1. This illustrates explicitly that these two cases are related by an orientation-reversing symmetry of AdS_3, namely the reflection $x_4 \to -x_4$. It is important to stress that, at this point, the coordinates $\{t, \phi, z\}$ are just some particular global description for AdS_3. All of them are defined in the range $-\infty < t, \phi, z < +\infty$. It is only when we identify points in AdS_3 along some discrete step generated by $\xi_{AdS} = \partial_\phi$ that our discrete quotients will differ from AdS_3 globally, by making the adapted coordinate ϕ a compact variable with period 2π in some normalisation, i.e. $\phi \sim \phi + 2\pi$.

As first proved in [104] for AdS_3, corroborated in [105] and extended to any higher dimensional AdS spacetime in [69], the supersymmetry preserved by these self-dual orbifolds is one–half of the original one.

The metric in adapted coordinates (3.96) looks like

$$g_{sd} = -dt^2 + \beta^2\, d\phi^2 + dz^2 - 2\beta \sinh 2z\, dt\, d\phi\ . \tag{3.97}$$

Thus, it describes a non-static but stationary spacetime. One interesting feature which has not previously been noted is that t is a global time function, since $\nabla_\mu t \nabla^\mu t = -1/\cosh^2 2z$, so the self-dual orbifolds are stably causal, and hence do not contain closed timelike curves. This metric can be interpreted as an S^1 fibration over AdS_2, as the following rewriting indicates

$$g_{sd} = -\cosh^2 2z dt^2 + dz^2 + (\beta\, d\phi - \sinh 2z\, dt)^2\ . \tag{3.98}$$

This quotient was recently analysed in detail in [105], where its isometries, geodesics, asymptotic structure and holography in this background were extensively studied.

An important point to note from that analysis is the structure of the conformal boundaries. It was shown in [105] that the quotient has two disconnected conformal boundaries. If we consider the coordinate transformation

$$\sinh z = \tan\theta \quad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\ ,$$

the metric (3.97) becomes

$$g_{sd} = \frac{1}{\cos^2\theta}\left(\cos^2\theta\,(-dt^2 + \beta^2\, d\phi^2) + d\theta^2 - 4\beta \sin\theta\, dt\, d\phi\right)\ , \tag{3.99}$$

from which we learn that the metric on both conformal boundaries, located at $\theta \to \pm\frac{\pi}{2}$ is given by

$$g_\partial = \pm dt\, d\phi\ . \tag{3.100}$$

Thus, there are closed lightlike curves on the conformal boundary. The appearance of two disconnected boundaries can be further understood by noting that in the adapted coordinates (3.96), the original AdS_3 conformal boundary is covered by four connected patches located at $z \to \pm\infty$ and $\phi \to \pm\infty$. After the discrete identification, two of these patches no longer belong to our space, leaving as a consequence, the existence of two boundaries at $z \to \pm\infty$, being disconnected. These boundaries are causally connected through the bulk, as was shown in [105] by analysing the geodesics in this space.

Unlike the previous cases, this quotient has no natural interpretation as arising from a quotient of an asymptotically flat spacetime. This is related to the fact that the quotient does not take a simple form in Poincaré coordinates. However, Strominger [112] showed that these self-dual orbifolds emerge as the local description of a very–near horizon geometry when focusing on the vicinity of the horizon of an extremal BTZ black hole.

Thus, even though this quotient does not emerge directly from the D1-D5 perspective, it is nevertheless possible to set-up an asymptotically flat spacetime which reproduces the self-dual orbifolds in two steps [105]. This is achieved by adding some momentum along the common direction shared by the D1's and D5's, and taking the standard near horizon limit, keeping the momentum density fixed. One then focuses on the vicinity of the horizon resulting from the previous limit. This procedure generalises the construction in [99] to the D1-D5 system, and it provides an independent way of understanding the DLCQ holography proposed in [105].

Following our general discussion presented at the beginning of section 3.4, it is straightforward to extend the analysis to higher dimensional AdS_{p+1} spaces, for $p \geq 3$. Indeed, we can use the foliation in (3.57) and replace the $\{\hat{x}_i\}$ appearing there with the $R = 1$ version of (3.96). The resulting metric is

$$g_{sd_{p+1}} = (\cosh\chi)^2\, g_{sd} + (d\chi)^2 + (\sinh\chi)^2\, g_{S^{p-3}} \,. \tag{3.101}$$

where g_{sd} is the metric given in (3.97).

This allows us to see that in these higher dimensional cases, the boundary of the quotient will be connected. The point is that the boundary of the quotient in higher dimensions is given in these coordinates by $\chi \to \infty$, as discussed earlier. Thus, the boundary of the higher-dimensional quotients naturally contains a copy of the bulk of the AdS_3 quotient. Since the AdS_3 quotient is connected, this implies that the boundary of the quotient is connected in higher dimensions. It also shows us that unlike the AdS_3 case, in higher dimensions there is a natural non-degenerate metric on the boundary of the quotient.

Deformation by $\lambda^{(0,2)}$

Even though we could discuss the turning on of the deformation parameters b_i in the general case, we shall just briefly mention their main new features in the string theory embeddings described above. This means that we shall concentrate on AdS_5 and AdS_7, since these deformations are not available for AdS_4.

This programme is particularly simple to carry on already in the foliation defined by (3.57). As previously mentioned, $\lambda^{(0,2)}$ blocks correspond to rotations in $\mathbb{R}^2$ planes in the embedding space, and in the coordinates of (3.58), these motions can be globally described as a single "translation" along one of the angular variables of the S^{n-1} factor. The definition of the adapted coordinate system in which $\oplus_i \lambda^{(0,2)}(b_i)$ takes the form of a single "translation" is precisely parallel to the discussion for the transverse S^q given in Section 3.2.

As an example, consider AdS_5. In this case, we can only turn on one parameter, $b_1 = b$. It is clear that rotations in $\mathbb{R}^2$ correspond to motions along the S^1 transverse to the AdS_3 foliation of AdS_5 in (3.58), for $p - n = 2$. If we parameterise this circle by θ, the Killing vector field ξ_{AdS} generating the full action of the deformed discrete quotient is given by

$$\xi_{\text{AdS}} = \partial_\phi + b\,\partial_\theta \,, \tag{3.102}$$

in the adapted coordinates defined by (3.57) and (3.96).

It is now just a matter of applying a linear transformation in the $\{\phi, \theta\}$ plane, which will generate an extra fibration, to rewrite the metric in a globally defined coordinate system adapted to the deformed Killing vector field ξ_{AdS}. This metric is given by

$$g = \cosh^2\chi\, g_{sd} + d\chi^2 + \sinh^2\chi\,(d\theta + bd\phi)^2 \,. \tag{3.103}$$

By construction, this deformation will break all the spacetime supersymmetry.

The techniques for AdS_7 are exactly the same, but there is a richer structure of possibilities since we have an S^3 transverse to the AdS_3 action, which allows us to turn on two inequivalent parameters b_1, b_2

$$b_1 R_{12} + b_2 R_{34} \ ,$$

where R_{ij} stands for a rotation generator in the ij-plane belonging to $\mathbb{R}^4$, where the 3-sphere is embedded as a quadric. Let us describe this 3-sphere in terms of standard complex coordinates

$$\begin{aligned} z_1 &= x^1 + i\,x^2 = \cos\theta\, e^{i(\psi+b)} \ , \\ z_2 &= x^3 + i\,x^4 = \sin\theta\, e^{i(\psi-b)} \ . \end{aligned} \tag{3.104}$$

A supersymmetric quotient [69] is given by the choice $b_1 = -b_2 = \theta_1$. The metric describing the global quotient is given by

$$\begin{aligned} g_{\mathrm{AdS}_7/\Gamma} = \cosh^2\chi\, g_{sd} + d\chi^2 + \sinh^2\chi\, \big(d\theta^2 + (db + \theta_1 d\phi)^2 \\ + d\psi^2 + 2\cos 2\theta\, (db + \theta_1\, d\phi)\cdot d\psi\big) \ . \end{aligned} \tag{3.105}$$

Adding a transverse four-sphere and a constant flux on it, the above configuration is supersymmetric. It actually preserves $\nu = \frac{1}{2}$ of the supersymmetries preserved by the original vacuum. Thus, it has sixteen supercharges. It is worthwhile mentioning that the deformation described by $b_1 = -b_2$ does not break any further supersymmetry. It is a further action that we can consider in our spacetime for free, supersymmetry wise. Contrary to what intuition may suggest, as explained in more detail in [69], the deformation $b_1 = b_2$ breaks all the supersymmetry.

Sphere deformations

Let us start our discussion on sphere deformations of self-dual orbifolds on the embedding of $\mathrm{AdS}_3 \times S^3$ in type IIB. The most general action that we can write down on S^3 is given in terms of two real parameters

$$\xi_S = \theta_1\, R_{12} + \theta_2\, R_{34} \ . \tag{3.106}$$

Because of the freedom that we have to quotient by the action of the Weyl group, we can always choose to work on the fundamental region defined by $\theta_1 \geq |\theta_2|$.

Among all these quotients, only a subset preserve supersymmetry. In particular, if we consider the action generated by $J_{13} \pm J_{24}$ on AdS_3, the only supersymmetric deformations are given by $\theta_1 = \pm\theta_2$, the signs being correlated. Interestingly, such deformations still preserve the same amount of supersymmetry as the self-dual orbifolds themselves. Thus, these supersymmetric deformations are for free, as pointed out in [69], where the reader can also find the explanation for this phenomenon.

The discussion proceeds in an analogous way for higher dimensional AdS spacetimes. If we consider the eleven dimensional configuration $\text{AdS}_4 \times S^7$, their deformations are characterised by four real numbers

$$\xi_S = \theta_1\, R_{12} + \theta_2\, R_{34} + \theta_3\, R_{56} + \theta_4\, R_{78}\ . \tag{3.107}$$

Due to the Weyl group action, we can restrict ourselves to the region defined by $\theta_1 \geq \theta_2 \geq \theta_3 \geq |\theta_4|$. As discussed in [69], there are several loci in this parameter space where supersymmetry is allowed. If $\theta_1 = \theta_2$ and $\theta_3 = -\theta_4$ the quotient preserves $\nu = \frac{1}{4}$. Whenever one of the relations

$$\begin{aligned} \theta_1 - \theta_2 + \theta_3 + \theta_4 &= 0\ , \\ \theta_1 + \theta_2 - \theta_3 + \theta_4 &= 0\ , \\ \theta_1 - \theta_2 - \theta_3 - \theta_4 &= 0\ ; \end{aligned}$$

is satisfied, the supersymmetry will be $\nu = \frac{1}{8}$. Finally, there is enhancement whenever $\theta_1 = \theta_2 = \theta_3 = -\theta_4$, giving rise to $\nu = \frac{3}{8}$.

The discussion for $\text{AdS}_5 \times S^5$ is fairly simple. The action on the 5-sphere is given in terms of three real parameters

$$\xi_S = \theta_1\, R_{12} + \theta_2\, R_{34} + \theta_3\, R_{56}\ . \tag{3.108}$$

The deformation preserves $\nu = \frac{1}{4}$ for $\theta_1 = \theta_2$ and $\theta_3 = 0$. It preserves $\nu = \frac{1}{8}$ if $\theta_1 \pm \theta_2 \pm \theta_3 = 0$, with uncorrelated signs. See [69] for more details.

The only supersymmetric deformation for $\text{AdS}_7 \times S^4$ out of the two parameter family

$$\xi_S = \theta_1\, R_{12} + \theta_2\, R_{34}\ , \tag{3.109}$$

is given by $\theta_1 = \theta_2$, also preserving $\nu = \frac{1}{4}$.

As an explicit example of a supersymmetric deformation of the self-dual orbifold, we shall present one particular example of the above discussion, one embedded in

$AdS_5 \times S^5$. More precisely, we shall focus on $\theta_1 = 2$, $\theta_2 = \theta_3 = 1$. A simple description of this quotient can be obtained by parametrising the 5-sphere in terms of the coordinates

$$
\begin{aligned}
z_1 &= x^1 + ix^2 = \cos\theta_1\, e^{i(b_1+2\psi)} \\
z_2 &= x^3 + ix^4 = \sin\theta_1\, \cos\theta_2\, e^{i(\psi+b)} \\
z_3 &= x^5 + ix^6 = \sin\theta_1\, \sin\theta_2\, e^{i(\psi-b)}\ .
\end{aligned}
\tag{3.110}
$$

One can check that $\xi_S = \partial_\psi$. This is an example in which both ξ_{AdS} and ξ_S are described in terms of adapted coordinates. Thus, by a simple linear transformation, we can easily write the fully adapted ten dimensional metric as

$$
\begin{aligned}
g = \cosh^2\chi\, g_{sd} + d\chi^2 + \sinh^2\chi\, d\theta^2 + d\theta_1^2 + \sin^2\theta_1\, d\theta_2^2 + \cos^2\theta_1\, \left(db_1 + 2(d\psi + d\phi)\right)^2 \\
+ \sin^2\theta_1\, \left((d\psi + d\phi)^2 + db^2 + 2\cos 2\theta_2\,(d\psi + d\phi)\cdot b\right)\ .
\end{aligned}
\tag{3.111}
$$

As can be checked from the review of the results in [69] presented at the beginning of this subsection, this particular example preserves $\nu = \frac{1}{8}$ of the vacuum supersymmetry. Thus, it has four supercharges.

Of course, there is no conceptual difficulty in dealing with deformations that contain both $\lambda^{(0,2)}$ factors on AdS and non-trivial sphere actions. The supersymmetric quotients can also be found in [69].

3.4.3 Double null rotation and its deformations

The last Killing vector appearing in Table 3.2, $III_{0\pm}$ with $b_1 = 0$, can be interpreted as a deformation, with deformation parameters b_i, of the double null rotation discrete quotient considered in [57]. Indeed, it consists of the simultaneous action of two spacelike null rotations in transverse $\mathbb{R}^{1,2}$ subspaces, and a set of rotations with parameters b_i in different transverse $\mathbb{R}^2$ planes. Since the norm of ξ_{AdS} is positive everywhere, even for $b_i = 0\ \forall\ i$, there is no need to deform the previous action by a non-trivial one on a transverse sphere to get an everywhere spacelike Killing vector field ξ in (3.56).

The minimal dimension where such an object exists is for $p = 4$, i.e. AdS_5, in which case there are no $\lambda^{(0,2)}$ blocks. The pure double null rotation discrete quotient has a very natural interpretation in the Poincaré patch: it consists of the combined action of a null rotation plus a spacelike translation. Consequently, it has a very straightforward origin in terms of the geometry of a bunch of parallel D3-branes: the pure double null rotation discrete quotient in AdS_5 is the near horizon geometry corresponding to a bunch of parallel D3-branes whose worldvolume is the nullbrane, i.e. $\mathbb{R}^{1,3}/\mathbb{Z}$, four dimensional Minkowski spacetime modded out by the simultaneous discrete action of a null rotation in $\mathbb{R}^{1,2}$ and a spacelike translation along $\mathbb{R}$, which was first introduced in [39].

The full anti-de Sitter action, including the deformation parameters, integrates to the following $\mathbb{R}$-action on $\mathbb{R}^{2,p}$:

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ x^6 \\ x^{2i+5} \\ x^{2i+6} \end{pmatrix} \mapsto \begin{pmatrix} x^1 - tx^3 + \frac{1}{2}t^2(x^1 - x^4) \\ x^2 - tx^5 + \frac{1}{2}t^2(x^2 - x^6) \\ x^3 + t(x^4 - x^1) \\ x^4 - tx^3 + \frac{1}{2}t^2(x^1 - x^4) \\ x^5 + t(x^6 - x^2) \\ x^6 - tx^5 + \frac{1}{2}t^2(x^2 - x^6) \\ x^{2i+5}\cos\varphi_i t - x^{2i+6}\sin\varphi_i t \\ x^{2i+6}\cos\varphi_i t + x^{2i+5}\sin\varphi_i t \end{pmatrix} , \quad \forall \mathrm{i} \tag{3.112}$$

which is manifestly free of fixed points for any value of the rotation parameters.

Pure AdS

Let us first consider the pure double null rotation in AdS_5. This was analysed in [57]. We will extend this analysis by discussing the isometries preserved by the quotient, constructing suitable adapted coordinate systems, and examining the action on the boundary of AdS. In the process, we will uncover interesting relations to compactified plane waves.

The Killing vector that we quotient along is

$$\xi_{\mathrm{AdS}} = J_{13} - J_{34} + J_{25} - J_{56}. \tag{3.113}$$

Its norm is $||\xi_{\mathrm{AdS}}||^2 = (x_1 + x_4)^2 + (x_2 + x_6)^2$. This is clearly positive semidefinite, and the quadric $-(x_1 + x_4)(x_1 - x_4) - (x_2 + x_6)(x_2 - x_6) + x_3^2 + x_5^2 = -1$ defining the AdS embedding constrains the coordinates so that it is positive definite. There are four linearly independent commuting isometries in $\mathfrak{so}(2,4)$:

$$\begin{aligned}
\xi_1 &= J_{13} - J_{34} - J_{25} + J_{56}, \\
\xi_2 &= J_{15} + J_{23} - J_{36} + J_{45}, \\
\xi_3 &= J_{12} - J_{24} + J_{16} + J_{46}, \\
\xi_4 &= J_{35} - J_{12} + J_{46}.
\end{aligned} \tag{3.114}$$

These Killing vectors have the non-trivial commutation relations

$$[\xi_1, \xi_2] = -2\xi_3, \quad [\xi_1, \xi_4] = 2\xi_2, \quad [\xi_2, \xi_4] = -2\xi_1. \tag{3.115}$$

They therefore form a Heisenberg algebra on which ξ_4 acts as an outer automorphism. The symmetry algebra of the quotient is hence $(\mathfrak{h}(1) \rtimes \mathbb{R}) \oplus \mathbb{R}$. The norms of the Killing vectors are $||\xi_1||^2 = ||\xi_2||^2 = ||\xi_{\rm AdS}||^2$, $||\xi_3||^2 = 0$, $||\xi_4||^2 = -1$.

We want to construct adapted coordinates to describe this quotient; it is convenient for studying causality to adapt them to $\xi_{\rm AdS}$, ξ_3 and ξ_4. Let us therefore seek to choose coordinates $(t, u, \phi, \rho, \gamma)$ so that $\xi_3 = \partial_v$, $\xi_4 = -\partial_t$, and $\xi_{\rm AdS} = \partial_\phi$. This requires

$$\begin{aligned}
&\frac{\partial(x^4 - x^1)}{\partial\phi} = 0, \quad \frac{\partial(x^4 + x^1)}{\partial\phi} = -2x^3, \\
&\frac{\partial(x^6 - x^2)}{\partial\phi} = 0, \quad \frac{\partial(x^6 + x^2)}{\partial\phi} = -2x^5, \\
&\frac{\partial x^3}{\partial\phi} = x^4 - x^1, \quad \frac{\partial x^5}{\partial\phi} = x^6 - x^2, \\
&\frac{\partial(x^4 - x^1)}{\partial v} = 0, \quad \frac{\partial(x^4 + x^1)}{\partial v} = -2(x^6 - x^2), \\
&\frac{\partial(x^6 - x^2)}{\partial v} = 0, \quad \frac{\partial(x^6 + x^2)}{\partial v} = -2(x^4 - x^1), \\
&\frac{\partial x^3}{\partial v} = 0, \quad \frac{\partial x^5}{\partial v} = 0, \\
&\frac{\partial(x^4 - x^1)}{\partial t} = (x^6 - x^2), \quad \frac{\partial(x^4 + x^1)}{\partial t} = (x^6 + x^2), \\
&\frac{\partial(x^6 - x^2)}{\partial t} = -(x^4 - x^1), \quad \frac{\partial(x^6 - x^2)}{\partial t} = -(x^4 + x^1), \\
&\frac{\partial x^3}{\partial t} = x^5, \quad \frac{\partial x^5}{\partial t} = -x^3 .
\end{aligned} \tag{3.116}$$

There are two quantities independent of $\{t, v, \phi\}$: $(x^4 - x^1)^2 + (x^6 - x^2)^2$ and $x^3 \cdot$

$(x^6 - x^2) - x^5 \cdot (x^4 - x^1)$. We will choose coordinates $\{\rho, \psi\}$ so that

$$\begin{aligned}(x^4 - x^1)^2 + (x^6 - x^2)^2 &= e^{2\rho} \\ x^3 \cdot (x^6 - x^2) - x^5 \cdot (x^4 - x^1) &= e^{\rho}\psi \ ;\end{aligned} \tag{3.117}$$

we must take $-\infty < \rho < \infty$ and $-\infty < \psi < \infty$ to obtain coordinates that cover the whole spacetime. A coordinate system satisfying all these conditions is

$$\begin{aligned}
x^4 - x^1 &= e^{\rho} \sin t, \\
x^4 + x^1 &= -e^{\rho}(2\phi\psi + 2v) \cos t - (e^{-\rho} + (\psi^2 + \phi^2)e^{\rho}) \sin t, \\
x^6 - x^2 &= e^{\rho} \cos t, \\
x^6 + x^2 &= e^{\rho}(2\phi\psi + 2v) \sin t - (e^{-\rho} + (\psi^2 + \phi^2)e^{\rho}) \cos t, \\
x^3 &= e^{\rho}(\psi \cos t + \phi \sin t), \\
x^5 &= e^{\rho}(-\psi \sin t + \phi \cos t) \ .
\end{aligned} \tag{3.118}$$

The AdS_5 metric in these coordinates is

$$g_{dnr} = -dt^2 + d\rho^2 + e^{2\rho}(d\psi^2 + d\phi^2 - 2dtdv - 4\psi dtd\phi), \tag{3.119}$$

and the other two Killing vectors are

$$\begin{aligned}\xi_1 &= -\cos 2t \, (\partial_\phi - 2\psi\partial_v) + \sin 2t \, \partial_\psi, \\ \xi_2 &= \sin 2t \, (\partial_\phi - 2\psi\partial_v) + \cos 2t \, \partial_\psi \ .\end{aligned} \tag{3.120}$$

Even though we will not give the explicit details, it is easy to check by working out the inverse coordinate transformation that this coordinate system covers the whole of AdS. Before any identification, the range of all adapted coordinates is non-compact. The double null rotation quotient is simply described by making the coordinate ϕ compact.

We would also like to understand the conformal boundary of this quotient. First, we should note that even though the quotient is free of fixed points in the bulk, its boundary has a continuous line of them. The action generated by $III_{0\pm}$ with $b_1 = 0$ integrates to the real line, so the only possible fixed points are the ones for which ξ_{AdS} vanishes. These points are given by

$$x^4 - x^1 = x^6 - x^1 = x^3 = x^5 = 0 \ .$$

The above does not belong to AdS_5, since they do not satisfy the quadric equation (2.6). This is indeed true for the bulk of AdS (finite non-compact spacelike direction in global AdS), but there is a continuous curve of fixed points on an infinite cylinder of axis, global time τ, and a maximal circle base. To see this, consider the standard global description of AdS_5,

$$x^1 = \cosh\chi\cos\tau,$$
$$x^2 = \cosh\chi\sin\tau,$$
$$x^i = \sinh\chi\,\hat{x}^i \quad i = 3,\ldots,6\ ,$$

where $\{\hat{x}^i\}$ parametrise a 3-sphere of unit radius. It is easy to see that any solution to the fixed point conditions requires $\chi \to \infty$, from which we already learn such points belong to the boundary of AdS_5. It is also clear that $\hat{x}^3 = \hat{x}^5 = 0$. Thus, such fixed points belong to a maximal circle in the $x^4 - x^6$ plane. If the angular variable describing such a maximal circle is b ($0 \leq b < 2\pi$), the continuous line of fixed points is determined by

$$\tau = b \pmod{2\pi}\ .$$

Thus, the action of the quotient is well-defined on the global boundary of AdS (i.e., the Einstein static universe) with a single null line deleted. However, we know that the Einstein static universe with a null line deleted is conformal to a symmetric plane wave [111]. This suggests that the boundary of (3.119) should be described in terms of a plane wave.

Inspired by this, and the analysis of the $I_{\mathbb{R}\pm}$ case in section 3.4.1, let us now make a coordinate transformation $Z = e^{-\rho}$ in (3.119). The metric then becomes

$$g_{dnr} = \frac{1}{Z^2}(-2dtdv - Z^2dt^2 + dZ^2 + d\psi^2 + d\phi^2 - 4\psi dtd\phi), \tag{3.121}$$

where $0 < Z < \infty$ covers the whole of AdS_5. By rescaling the metric by a factor of Z^2, we can conformally map global AdS_5 into the space with metric

$$\bar{g} = -2dtdv - Z^2dt^2 + dZ^2 + d\psi^2 + d\phi^2 - 4\psi dtd\phi, \tag{3.122}$$

with the conformal boundary lying at $Z = 0$. Since $\xi_{\text{AdS}} = \partial_\phi$ annihilates the conformal factor, this embedding commutes with the quotient; we can regard the double null rotation as conformally embedded in (3.122) with ϕ compactified.[7]

Now, the space (3.122) is simply a symmetric plane wave. This can be made obvious by making the further coordinate transformation[8]

$$\begin{aligned} V &= v + \psi\phi, \\ U &= t, \\ X &= \psi \cos t + \phi \sin t, \\ Y &= -\psi \sin t + \phi \cos t, \end{aligned} \tag{3.123}$$

under which the metric becomes

$$\bar{g} = -2dUdV - (X^2 + Y^2 + Z^2)dU^2 + dX^2 + dY^2 + dZ^2. \tag{3.124}$$

This provides an interesting alternative description of the double null rotation, of interest independent of the question of the conformal boundary. As in section 3.4.1, this relation between the symmetric plane wave and AdS is anticipated by previous work, since they are both conformally flat spaces and hence conformally embedded in the Einstein static universe. We see also that AdS covers the half of the plane wave at $Z > 0$, as we would expect, since it covers half the Einstein static universe. What is remarkable is that the isometry we want to quotient along commutes with the conformal rescaling, as noted above. In fact, not only does it do so; all the unbroken symmetries of the double null rotation also do so, since they do not involve ∂_ρ. Thus, they are all symmetries of the conformally related plane wave metric (3.124). If we

[7]Note that this conformal embedding does not provide a true compactification of the spacetime, since (3.122) is itself not compact. As noted above, this represents the necessary exclusion of the fixed points of the quotient in the Einstein static universe.

[8]It is worth noting that there is a simple relation between these and the embedding coordinates for AdS_5: $x^4 - x^1 = (\sin U)/Z$, $x^4 + x^1 = -(V \cos U - (X^2+Y^2+Z^2)\sin U)/Z$, $x^6 - x^2 = (\cos U)/Z$, $x^6 + x^2 = (V \sin U - (X^2 + Y^2 + Z^2)\cos U)/Z$, $x^3 = X/Z$, $x^5 = Y/Z$.

introduce the usual basis for the Killing vectors of the plane wave,

$$
\begin{aligned}
\xi_{e_i} &= -\cos U \partial_{X^i} + X^i \sin U \partial_V, \\
\xi_{e_i^*} &= -\sin U \partial_{X^i} - X^i \cos U \partial_V, \\
\xi_{e_V} &= \partial_V, \\
\xi_{e_U} &= -\partial_U,
\end{aligned}
\tag{3.125}
$$

we can identify the isometries of the double null rotation quotient as

$$
\begin{aligned}
\xi_{\mathsf{AdS}} &= -\xi_{e_1^*} - \xi_{e_2}, \\
\xi_1 &= -\xi_{e_1^*} + \xi_{e_2}, \\
\xi_2 &= -\xi_{e_1} - \xi_{e_2^*}, \\
\xi_3 &= \xi_{e_V}, \\
\xi_4 &= \xi_{e_U} - \xi_{M_{12}}.
\end{aligned}
\tag{3.126}
$$

Thus, the double null rotation is conformally related to a compactification of the plane wave of the type considered in [114].

To return to the question of the conformal boundary of the double null rotation, we see that it is given by the surface at $Z = 0$ in (3.122), with metric

$$g_\partial = -2dtdv + d\psi^2 + d\phi^2 - 4\psi dt d\phi. \tag{3.127}$$

This is itself a compactified plane wave, as can be seen by the application of the coordinate transformation (3.123). One might be puzzled by this result, as one would have expected to find the nullbrane as the conformal boundary of the double null rotation. We demonstrate in appendix A that the nullbrane is in fact related to (3.127) by a further conformal transformation. Thus, (3.127) and the nullbrane describe the same conformal structure on the boundary. The description in terms of the compactified plane wave (3.127) is preferable to the nullbrane for two reasons: First, the nullbrane only covers a part of the boundary [it corresponds to the region $-\pi/2 < t < \pi/2$ in (3.127)], so the former description is more global. Second, the further conformal transformation to the nullbrane does not commute with the symmetry ξ_4 of the double null rotation. If we work with (3.127), all the unbroken symmetries of the bulk spacetime after we perform the quotient are realised as symmetries of the boundary (rather than conformal isometries). This should be a helpful simplification in studying the holographic relation for this spacetime.

The connection to plane waves also makes it easy to identify a time function for the double null rotation. Writing the double null rotation metric (3.121) in the form suitable for Kaluza-Klein reduction along ϕ,

$$g = \frac{1}{Z^2}[-2dvdt - (Z^2 + 4\psi^2)dt^2 + d\psi^2 + (d\phi - 2\psi dt)^2], \tag{3.128}$$

we see that the lower-dimensional spacetime would again be a plane wave (up to conformal factor). Hence, applying the results of [115], where time functions were found for general plane waves, we can deduce that a suitable time function for the nullbrane is

$$\tau = t + \frac{1}{2}\tan^{-1}\left(\frac{4v}{1 + Z^2 + 4\psi^2}\right). \tag{3.129}$$

It is easy to check that

$$\nabla_\mu \tau \nabla^\mu \tau = -\frac{4Z^2}{[(1 + Z^2 + 4\psi^2)^2 + 16v^2]}. \tag{3.130}$$

Thus, τ is a good time function on AdS. Since $\mathcal{L}_{\xi_{\text{AdS}}}\tau = 0$, its existence shows that the double null rotation quotient of AdS preserves the property of stable causality by the general argument of [115].

As recently discussed in [69] [9], the supersymmetry preserved by this double null rotation quotient in AdS_5, and actually in any higher dimensional AdS spacetime embedded in a supergravity theory, is $\nu = \frac{1}{2}$. That is, this configuration has sixteen supercharges. It is interesting to comment on the relation with the single null rotation quotient. In that case, we argued that the standard enhancement of supersymmetry when taking the near horizon geometry was lost after the identification. This may suggest that the same phenomenon is taking place in the double null rotation, since the action generated by the latter is the combination of two commuting null rotations. However, the general solution to the eigenvalue problem

$$N\,\varepsilon = N_1 \cdot N_2\,\varepsilon = 0,$$

[9] In [57], it was claimed that the amount of supersymmetry preserved by the double null rotation quotient was $\nu = \frac{1}{4}$, but as shown in [69], the latter is actually enhanced to $\nu = \frac{1}{2}$.

where N stands for the full double null rotation generator in the spinorial representation, and N_i $i = 1, 2$ stand for nilpotent operators, is not given in terms of the intersection of kernels of the nilpotent operators associated with each of the null rotations, which would give rise to $\nu = \frac{1}{4}$, but there exist non-trivial solutions [69] that enhance supersymmetry to one-half. Thus, in this case, the double null rotation quotient preserves the same amount of supersymmetry as the corresponding asymptotically flat analogue in terms of parallel and coincident D3-branes in the nullbrane vacuum.

Deformation by $\lambda^{(0,2)}$. In order to turn on any deformation parameter, we must consider higher dimensional AdS spacetimes. In particular, it is natural to consider AdS_7, since this is very naturally obtained in M-theory from the near horizon limit of M5-branes. If we denote by b the deformation parameter, the deformed seven dimensional quotient can be written as

$$g_{AdS_7/\Gamma} = \cosh^2\chi\, g_{dnr} + d\chi^2 + \sinh^2\chi\, (d\varphi_1 + bd\phi)^2 \ . \tag{3.131}$$

where g_{dnr} stands for (3.119).

Since we only turned on a single deformation parameter, b, the corresponding seven dimensional quotient, when embedded in string theory, will break supersymmetry. It is certainly possible to construct supersymmetric versions of the latter by deforming the orbifold action with a non-trivial action on S^4.

Sphere deformations

Let us start our discussion on sphere deformations of the double null rotation quotient by focusing on $AdS_5 \times S^5$. The family of deformations is described by (3.108), that is, by three real parameters. As discussed in [69], the only supersymmetric loci in the fundamental region defined by the action of the Weyl group is, besides the origin, given either by $\theta_1 = \theta_2$ and $\theta_3 = 0$, preserving $\nu = \frac{1}{4}$, or by $\theta_1 - \theta_2 \pm \theta_3 = 0$, preserving $\nu = \frac{1}{8}$.

The discussion for $AdS_7 \times S^4$ is analogous. In this case, there exists a two parameter family of deformations, given by (3.109). The only supersymmetric loci in the fundamental region defined by the action of the Weyl group is either the origin, corresponding to the double null rotation quotient itself, or the line $\theta_1 = \theta_2$, which preserves $\nu = \frac{1}{4}$.

As an explicit example of a sphere deformation of the double null rotation quotient, we shall focus on a supersymmetric deformation on $AdS_5 \times S^5$. We will focus on the same sphere action considered in section 3.4.2. As before, we apply the general formalism developed in (3.64) for the full Killing vector $\xi = \xi_{AdS} + \xi_S$. If we introduce adapted coordinates so that $\xi = \partial_\phi$ by defining $\psi' = \psi - \gamma\phi$, the full ten-dimensional metric on the quotient space will be

$$g = g_{dnr} + d\theta_1^2 + \sin^2\theta_1\, d\theta_2^2 + \cos^2\theta_1\, \left(d\varphi_1 + 2(d\psi' + \gamma d\phi)\right)^2 + \sin^2\theta_1\, \left((d\psi' + \gamma d\phi)^2 + d\varphi^2 + 2\cos 2\theta_2\, (d\psi' + \gamma d\phi)\cdot\varphi\right) \ . \quad (3.132)$$

where g_{dnr} denotes the metric on the quotient of AdS_5 given in (3.119).

Again we could consider quotients involving both $\lambda^{(0,2)}$ blocks acting on AdS and sphere deformations. The techniques required to deal with them are exactly the same as those used above. The reader can find an analysis of their supersymmetry in [69].

3.5 Further Identifications

We can produce additional quotient spaces by considering quotients generated by more than one commuting Killing vector. For such a spacetime to be causally regular it is certainly necessary that each of the Killing vectors that we quotient along is everywhere non-timelike. However, this is not sufficient to rule out closed timelike curves. Consider the quotient generated by ξ_1 and ξ_2. Since points in AdS_{p+1} lie on orbits of both ξ_1 and ξ_2, the action of the quotient produces closed curves joining points on any linear combination of these orbits. Thus, in order to construct a causally regular spacetime, we need commuting Killing vectors for which any linear combination is everywhere non-timelike. This is in practice a stringent restriction. We discuss three causally regular quotients of this type, which are all generalisation of quotients discussed in section 3.4.

In $\mathfrak{so}(2,2)$ the only everywhere non-timelike commuting Killing vectors live in seperate $\mathfrak{sl}(2,\mathbb{R})$ factors of $\mathfrak{so}(2,2) = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$. Each factor has three Killing vectors

$$\xi_1^{\pm} = J_{24} \pm J_{13}, \quad \xi_2^{\pm} = J_{12} \pm J_{34}, \quad \xi_3^{\pm} = J_{14} \mp J_{23} , \tag{3.133}$$

which satisfy

$$[\xi_i^{\pm}, \xi_j^{\pm}] = 2\epsilon_{ijk}\xi_k^{\pm}. \tag{3.134}$$

We have $||\xi_1^{\pm}||^2 = ||\xi_3^{\pm}||^2 = 1$ and we can build two other everywhere non-timelike Killing vectors, corresponding to two versions of $\lambda_r^{II(2,2)}$ with $a = 0$, by defining

$$\begin{aligned} \tilde{\xi}^+ &= \xi_1^+ - \xi_2^+, \\ \tilde{\xi}^- &= \xi_2^- - \xi_1^+. \end{aligned} \tag{3.135}$$

Unfortunately no combination of these nowhere timelike Killing vectors gives rise to a causally regular quotient. In [117] it was claimed that the quotient generated by two of the everywhere spacelike Killing vectors, i.e ξ_3^+ and ξ_3^-, is causally regular. However the linear combination $\xi_3^+ + \alpha\xi_3^-$ is not everywhere spacelike

$$||\xi_3^+ + \alpha\xi_3^-||^2 = (1 + \alpha^2) + 2\alpha(x_1^2 - x_2^2 - x_3^2 + x_4^2). \tag{3.136}$$

Hence this quotient space has closed timelike curves. In [117] they worked in coordinates for which $\xi_3^+ = \partial_{\phi_1}$ and $\xi_3^- = -\partial_{\phi_2}$ given by

$$\begin{aligned} x_1 &= \cosh(\phi_1 + \phi_2)\sin(\tau - \frac{\pi}{4}), \\ x_2 &= \cosh(\phi_1 - \phi_2)\cos(\tau - \frac{\pi}{4}), \\ x_3 &= \sinh(\phi_1 + \phi_2)\sin(\tau - \frac{\pi}{4}), \\ x_4 &= \sinh(\phi_1 - \phi_2)\sin(\tau - \frac{\pi}{4}). \end{aligned} \tag{3.137}$$

The AdS_3 metric in these coordinates is

$$g_{AdS_3} = \left(-d\tau^2 + d\phi_1^2 + d\phi_2^2 - 2\sin 2\tau d\phi_1 d\phi_2\right) . \tag{3.138}$$

Where the action of the quotient makes ϕ_1 and ϕ_2 periodic. The closed timelike curves are not apparent in the coordinate system defined by equation (3.137) because these coordinates do not cover the entire AdS_3 spacetime. Indeed for these coordinates

$$x_1^2 - x_2^2 - x_3^2 + x_4^2 = \sin 2\tau. \tag{3.139}$$

Thus they only cover the region free of closed timelike curves. A simple way to see these coordinate are not global is to observe that (3.138) is geodesically incomplete; generic geodesics go off to infinite ϕ_1, ϕ_2 as τ approaches $n\pi$.

In AdS_4 a more interesting quotient becomes possible. Consider the two commuting null rotations

$$\begin{aligned} \xi_1 &= J_{13} - J_{35}, \\ \xi_2 &= J_{14} - J_{45}, \end{aligned} \tag{3.140}$$

now we have

$$||\xi_1 + \alpha\xi_2||^2 = (1 + \alpha^2)(x_1 + x_5)^2. \tag{3.141}$$

The discussion for this case proceeds in an analogous fashion to that of section 3.4.1 for the quotient of AdS_3 by $\lambda^{III(1,2)}$. There is a single other Killing vector in $\mathfrak{so}(2,3)$ which commutes with ξ_1 and ξ_2 given by $\xi_3 = J_{12} - J_{25}$. This has norm $||\xi_3||^2 = -(x_1 + x_5)^2$. The most convenient coordinate system for studying this quotient is Poincaré coordinates, where we can orient the coordinates so that $\xi_1 = \partial_x, \xi_2 = \partial_y$ and $\xi_3 = \partial_t$, where the AdS_4 metric in Poincaré coordinates is

$$g_{\mathrm{AdS}_4} = \frac{1}{z^2}\left(-dt^2 + dz^2 + dx^2 + dy^2\right) . \tag{3.142}$$

We see that the effect of the quotient is simply to make the x and y coordinates periodic. The Killing vectors ξ_1 and ξ_2 become null on the Poincaré horizon $z = \infty$ where this coordinate system breaks down. In terms of the embedding coordinates, this is the surface $x_1 + x_5 = 0$. There are two null planes of fixed points given by $x_1 + x_5 = x_3 = 0$ and $x_1 + x_5 = x_4 = 0$. Away from the fixed points, the identification along ξ_1 and ξ_2 will generate closed null curves in the Poincaré horizon. These can be eliminated by deforming this quotient by a suitable action on an odd-dimensional sphere.

This quotient can be generalised to higher dimensions, for $\mathfrak{so}(2,p)$ there are at most $p-1$ commuting null rotations

$$\begin{aligned}\xi_1 &= J_{13} - J_{3p},\\ \xi_2 &= J_{14} - J_{4p},\\ &\vdots\\ \xi_{p-1} &= J_{1(p-1)} - J_{(p-1)p}\end{aligned} \tag{3.143}$$

where

$$\|\sum_{i=1}^{p-1} \alpha_i \xi_i\|^2 = \sum_{i=1}^{p-1} \alpha_i^2 (x_1^2 + x_p^2). \tag{3.144}$$

Going over to Poincaré coordinates we can orient the coordinates so that $\xi_1 = \partial_{x_1}$, $\xi_2 = \partial_{x_2}$, etc and $J_{12} - J_{2p} = \partial_t$. The AdS_{p+1} metric in Poincaré coordinates is

$$g_{\mathrm{AdS}_{p+1}} = \frac{1}{z^2}\left(-dt^2 + dz^2 + dx_1^2 + dx_2^2 + \ldots + dx_{(p-1)}^2\right) . \tag{3.145}$$

The effect of the quotient is to make the x_i coordinates periodic. Each ξ_i has a region of fixed points on the Poincaré horizon.

In $\mathfrak{so}(2,4)$ there are two commuting double null rotations

$$\begin{aligned}\xi_1 &= J_{13} - J_{34} + J_{25} - J_{56},\\ \xi_2 &= J_{15} + J_{23} - J_{36} + J_{45},\end{aligned} \tag{3.146}$$

With

$$\|\xi_1 + \alpha\xi_2\|^2 = (x_1 + x_4 + \alpha(x_2 + x_6))^2 + (x_2 + x_6 + \alpha(x_1 + x_4))^2, \tag{3.147}$$

which is positive semidefinite being null where $x_1 + x_4 = |x_2 + x_6|$. A suitable action on a odd dimensional sphere will make this causally regular. Seperately the action of each quotient is free of fixed points, however when we consider the combined quotient then we have fixed points whenever $x_1 + x_4 = |x_2 + x_6|$ and $|x_3| = |x_5|$.The action of this quotient preserves a null symmetry corresponding to $\xi_3 = J_{12} - J_{24} + J_{16} + J_{46}$. We can construct a coordinate system adapted to this quotient so that $\xi_1 = \partial_\phi$, $\xi_2 = \partial_\theta$ and $\xi_3 = \partial_v$, which in terms of the embedding coordinates is

$$
\begin{aligned}
x^4 - x^1 &= e^\rho \cos\tau, \\
x^4 + x^1 &= -e^\rho(2\phi\theta + 2v)\sin\tau - (e^{-\rho} + (\theta^2 + \phi^2)e^\rho)\cos\tau, \\
x^6 - x^2 &= e^\rho \sin\tau, \\
x^6 + x^2 &= e^\rho(-2\phi\theta + 2v)\cos\tau - (e^{-\rho} + (\theta^2 + \phi^2)e^\rho)\sin\tau, \\
x^3 &= e^\rho(\phi\cos\tau + \theta\sin\tau), \\
x^5 &= e^\rho(\phi\sin\tau + \theta\cos\tau) \ .
\end{aligned}
\tag{3.148}
$$

The AdS_5 metric in these coordinates is

$$
g_{dnr} = -d\tau^2 + d\rho^2 + e^{2\rho}(d\psi^2 + d\phi^2 - 2d\tau dv - 4\sin 2\tau d\phi d\psi). \tag{3.149}
$$

As anticipated this coordinate system is not global, it breaks down at $\tau = \pm\frac{\pi}{4}$ where the expressions for x^3 and x^5 lose their linear independence.

There is one freely acting causally regular quotient of this type which we will call the extended double null rotation because it shares many features of the double null rotation . The minimal dimension where such an object exists is $p = 6$. The pure extended double null rotation discrete quotient in AdS_7 is the near horizon geometry corresponding to a bunch of parallel M5-branes whose worldvolume is the extended nullbrane, introduced in Appendix B. The Killing vectors that we quotient along are

$$
\begin{aligned}
\xi_1 &= J_{13} - J_{37} + J_{24} - J_{48}, \\
\xi_2 &= J_{15} - J_{57} + J_{26} - J_{68},
\end{aligned}
\tag{3.150}
$$

that is two commuting double null rotations with two common spacelike directions. Their norm is $||\xi_1 + \alpha\xi_2||^2 = (1+\alpha^2)[(x_1+x_7)^2 + (x_2+x_8)^2]$. This is clearly positive semidefinite, and the quadric $-(x_1+x_7)(x_1-x_7)-(x_2+x_8)(x_2-x_8)+x_3^2+x_4^2+x_5^2+x_6^2 = -1$ defining the AdS embedding constrains the coordinates so that it is positive definite. There are six linearly independent isometries in $\mathfrak{so}(2,6)$ commuting with

ξ_1 and ξ_2:

$$\begin{aligned}
\xi_3 &= J_{13} - J_{37} - J_{24} + J_{48}, \\
\xi_4 &= J_{14} + J_{23} - J_{38} - J_{47}, \\
\xi_5 &= J_{15} - J_{57} - J_{26} + J_{68}, \\
\xi_6 &= J_{16} + J_{25} - J_{58} - J_{67}, \\
\xi_7 &= J_{12} - J_{27} + J_{18} + J_{78}, \\
\xi_8 &= J_{34} - J_{12} + J_{56} + J_{78}.
\end{aligned} \tag{3.151}$$

These Killing vectors have the non-trivial commutation relations

$$\begin{aligned}
&[\xi_3, \xi_4] = -2\xi_7, \quad [\xi_3, \xi_8] = 2\xi_4, \quad [\xi_4, \xi_8] = -2\xi_3, \\
&[\xi_5, \xi_6] = -2\xi_7 \quad [\xi_5, \xi_8] = 2\xi_6 \quad [\xi_2, \xi_4] = -2\xi_5.
\end{aligned} \tag{3.152}$$

If we define $\tilde{\xi}_1 = \xi_3 + \xi_5$ and $\tilde{\xi}_2 = \xi_4 + \xi_6$ then $\tilde{\xi}_1$, $\tilde{\xi}_2$, ξ_7 and ξ_8 form a Heisenberg algebra on which ξ_8 acts as an outer automorphism. The symmetry algebra of the quotient is hence $(\mathfrak{h}(1) \rtimes \mathbb{R}) \oplus \mathbb{R}$ which is the same as that of the double null rotation quotient. The norms of the Killing vectors are $||\xi_1||^2 = ||\xi_2||^2 = ||\xi_3||^2 = ||\xi_4||^2 = ||\xi_5||^2 = ||\xi_6||^2$, $||\xi_7||^2 = 0$, $||\xi_8||^2 = -1$.

The construction of adapted coordinates proceeds in a identical fashion to that of the the double null rotation; to study causality we adapt them to ξ_1, ξ_2, ξ_7 and ξ_8. Let us therefore seek to choose coordinates $(t, u, \phi, \psi, \rho, \gamma, \beta)$ so that $\xi_7 = \partial_v$, $\xi_8 = -\partial_t$, $\xi_1 = \partial_\phi$ and $\xi_1 = \partial_\phi$. A global coordinate system satisfying these conditions is

$$\begin{aligned}
x^7 - x^1 &= e^\rho \sin t, \\
x^7 + x^1 &= -e^\rho(2\phi\gamma + 2\psi\beta + 2v)\cos t - (e^{-\rho} + (\gamma^2 + \phi^2 + \psi^2 + \beta^2)e^\rho)\sin t, \\
x^8 - x^2 &= e^\rho \cos t, \\
x^8 + x^2 &= e^\rho(2\phi\gamma + 2\psi\beta + 2v)\sin t - (e^{-\rho} + (\gamma^2 + \phi^2 + \psi^2 + \beta^2)e^\rho)\cos t, \\
x^3 &= e^\rho(\gamma\cos t + \phi\sin t), \\
x^4 &= e^\rho(-\gamma\sin t + \phi\cos t), \\
x^5 &= e^\rho(\beta\cos t + \psi\sin t), \\
x^6 &= e^\rho(-\beta\sin t + \psi\cos t)\ .
\end{aligned} \tag{3.153}$$

where $-\infty < \rho < \infty$,$-\infty < \phi < \infty$ and $-\infty < \psi < \infty$ in order for the coordinates to cover the whole spacetime. The AdS_7 metric in these coordinates is

$$g_{dnr} = -dt^2 + d\rho^2 + e^{2\rho}(d\psi^2 + d\phi^2 + d\beta^2 + d\gamma^2 - 2dtdv - 4\gamma dtd\phi - 4\beta dtd\psi), \quad (3.154)$$

and the other four Killing vectors are

$$\begin{aligned} \xi_3 &= -\cos 2t\,(\partial_\phi - 2\beta\partial_v) + \sin 2t\,\partial_\beta, \\ \xi_4 &= \sin 2t\,(\partial_\phi - 2\beta\partial_v) + \cos 2t\,\partial_\beta, \\ \xi_5 &= -\cos 2t\,(\partial_\psi - 2\gamma\partial_v) + \sin 2t\,\partial_\gamma, \\ \xi_6 &= \sin 2t\,(\partial_\psi - 2\gamma\partial_v) + \cos 2t\,\partial_\gamma\ . \end{aligned} \quad (3.155)$$

The action of our quotient is simply described by making the coordinates ϕ and ψ compact. We are interested in the conformal boundary of this quotient. Since the quotient action integrates to the real line the only possible fixed points are where a linear combination of ξ_1 and ξ_2 vanishes. These points are given by

$$x^7 - x^1 = x^8 - x^2 = x^3 = x^4 = x^5 = x^6 = 0\ .$$

The above does not belong to AdS_7, since they do not satisfy the quadric equation (2.6). However there is a continuous null curve of fixed points on an infinite cylinder of axis, global time τ, and a maximal circle base. Thus, the action of the quotient is well-defined on the global boundary of AdS (i.e., the Einstein static universe) with a single null line deleted. This indicates that the boundary of (3.154) can be described in terms of a plane wave. Following our discussion of the double null rotation quotient, we make a coordinate transformation $Z = e^{-\rho}$ in (3.154). The metric then becomes

$$g_{dnr} = \frac{1}{Z^2}(-2dtdv - Z^2dt^2 + dZ^2 + d\phi^2 + d\gamma^2 + d\psi^2 + d\beta^2 - 4\gamma dtd\phi - 4\beta dtd\psi), \quad (3.156)$$

where $0 < Z < \infty$ covers the whole of AdS_7. By rescaling the metric by a factor of Z^2, we can conformally map global AdS_7 into the space with metric

$$\bar{g} = -2dtdv - Z^2dt^2 + dZ^2 + d\phi^2 + d\gamma^2 + d\psi^2 + d\beta^2 - 4\gamma dtd\phi - 4\beta dtd\psi, \quad (3.157)$$

with the conformal boundary lying at $Z = 0$. Since $\xi_1 = \partial_\phi$ and $\xi_2 = \partial_\psi$ annihilate the conformal factor,we can regard the extended double null rotation as conformally embedded in (3.157) with ϕ and ψ compactified. The space (3.157) is simply a symmetric plane wave. This can be made obvious by making the further coordinate transformation

$$\begin{aligned} V &= v + \gamma\phi + \beta\psi, \\ U &= t, \\ K &= \gamma \cos t + \phi \sin t, \\ L &= -\gamma \sin t + \phi \cos t, \\ M &= \beta \cos t + \psi \sin t, \\ N &= -\beta \sin t + \psi \cos t, \end{aligned} \tag{3.158}$$

under which the metric becomes

$$\bar{g} = -2dUdV - (K^2+L^2+M^2+N^2+Z^2)dU^2+dK^2+dL^2+dM^2+dN^2+dZ^2. \tag{3.159}$$

Again, this relation between the symmetric plane wave and AdS is anticipated by previous work. Since all the unbroken symmetries of the extended double null rotation commute with the conformal rescaling they are all symmetries of the conformally related plane wave metric (3.159) although we will not give their explicit form here. The conformal boundary of the extended double null rotation is given by the surface at $Z = 0$ in (3.157), with metric

$$\bar{g} = -2dtdv + d\phi^2 + d\gamma^2 + d\psi^2 + d\beta^2 - 4\gamma dtd\phi - 4\beta dtd\psi. \tag{3.160}$$

This is itself a compactified plane wave, as can be seen by the application of the coordinate transformation (3.158).

The connection to plane waves also makes it easy to identify a time function for the extended double null rotation. Writing (3.121) in the form suitable for Kaluza-Klein reduction along ϕ,

$$g = \frac{1}{Z^2}[-2dvdt-(Z^2+4\gamma^2+4\beta^2)dt^2+d\gamma^2+d\beta^2+(d\phi-2\gamma dt)^2+(d\psi-2\beta dt)^2], \tag{3.161}$$

we see that the lower-dimensional spacetime would again be a plane wave (up to conformal factor). Hence, applying the results of [115], where time functions were found for general plane waves, we can deduce that a suitable time function for the extended double null rotation is

$$\tau = t + \frac{1}{2}\tan^{-1}\left(\frac{4v}{1+Z^2+4\gamma^2+4\beta^2}\right). \tag{3.162}$$

An interesting issue that we did not address here is the amount of supersymmetry preserved by this quotient.

3.6 Black holes as quotients

As discussed in section 2.3, certain causally ill-behaved quotients can be given an interpretation as an analogue of black holes [55, 56]. The idea is that one can excise regions where closed timelike curves will arise from the original spacetime, and consider the quotient just of the remaining portion of AdS_{p+1}. The resulting geometry will be causally regular by construction, but will clearly not be geodesically complete, having a 'singularity' corresponding to the boundary of the excised region. This singularity is not a curvature singularity in the classical geometry, but extending the spacetime beyond it would introduce causal pathologies; it is therefore expected on the basis of the chronology protection conjecture that quantum corrections will lead to a true singularity at this location. The interesting question is whether this singularity is naked—that is, visible from infinity—or concealed by an event horizon. If it is behind an event horizon, we view the quotient geometry as a black hole, generalising the BTZ solution [55, 56].

In this section, we will study which quotients can lead to black holes of this type. Unlike in the previous section, where deformation on the sphere introduced qualitatively new possibilities, we find that the quotients with a black hole interpretation are the BTZ quotients in AdS_3, and the higher-dimensional generalisation of the non-rotating BTZ quotients, coupled with some action on the sphere.

First, we need to establish what region of the spacetime we remove. In [68], where quotients acting just on the AdS factor were considered, it was argued that we should remove the region where the Killing vector ξ_{AdS} fails to be spacelike. Clearly, the quotient will contain closed timelike curves in this region. However, it is not in general true that all closed timelike curves will pass inside this region. In particular, for cases with $\lambda^{(0,2)}$ components, this does not remove all the closed timelike curves.

Closed timelike curves in the region where ξ_{AdS} is spacelike can be constructed by an argument very similar to that used in section 3.3. As discussed at the beginning of section 3.4, for any of our quotients, we can construct a natural coordinate system (3.57) on the AdS part, in which we decompose AdS_{p+1} in terms of an AdS_{n+1} and a S^{p-n-1} factors, where the Killing vector generating the quotient is $\xi_{AdS} = \xi_{AdS_{n+1}} + \xi_r$, with $\xi_{AdS_{n+1}}$ acting only on the AdS_{n+1} part of the metric (3.58) and containing the non-trivial block or blocks, while the ξ_r is a combination of rotations (the $\lambda^{(0,2)}$ blocks) acting on the unit sphere S^{p-n-1}. Now consider an orbit where ξ_{AdS} is spacelike, but $\xi_{AdS_{n+1}}$ is timelike. As in section 3.3, we can construct a closed curve which follows the orbit of $\xi_{AdS_{n+1}}$ on the AdS_{n+1} factor and a length-minimising geodesic on the S^{p-n-1} factor. There are identified points which are separated by an arbitrarily large timelike distance in the AdS_{n+1} factor; since the separation on S^{p-n-1} is bounded, this closed curve will be timelike for sufficiently large separation on the AdS_{n+1} factor. Obviously, a similar argument applies when we consider the deformation on the transverse sphere; there will be closed timelike curves wherever the norm of the non-trivial blocks taken on their own is timelike.

Thus, it would seem that a natural region to excise is the region where $\xi_{AdS_{n+1}}$ is timelike. That is, the region to excise is determined by the norm of the non-trivial blocks, omitting all the rotations (both $\lambda^{(0,2)}$ and the rotations on transverse spheres). Note however that this is still not sufficient to eliminate the closed timelike curves in all cases. That is, the resulting quotient is not guaranteed to be causally regular. However, this is the only possibility we will consider here. It represents the natural generalisation of the construction of black hole solutions of [55,56] to higher dimensions. We will focus on seeing what black analogues can be constructed by removing this portion of the quotient. We will see that the resulting spacetimes in the black hole examples are in fact free of closed causal curves.

The singularity surface we consider is then where $||\xi_{AdS_{n+1}}||^2 = 0$ in $AdS_{p+1} \times S^q$. Our main concern for the rest of this section is to establish in which cases this singularity surface is naked, and in which cases it is concealed by an event horizon. Since $\xi_{AdS_{n+1}}$ is a Killing field,

$$\nabla_{\xi_{AdS_{n+1}}}(||\xi_{AdS_{n+1}}||^2) = 2\, i_{\xi_{AdS_{n+1}}}\left(\nabla_{\xi_{AdS_{n+1}}}\xi_{AdS_{n+1}}\right) = 0, \tag{3.163}$$

so $\xi_{AdS_{n+1}}$ is always tangent to surfaces defined by $||\xi_{AdS_{n+1}}||^2 =$ constant. Hence, the 'singularity' defined by $||\xi_{\mathrm{AdS}_{n+1}}||^2 = 0$ has a null tangent, and must be a timelike or null surface. We think of such a quotient as an analogue of a black hole if there is a non-trivial event horizon $\dot{J}^-(\mathcal{I}^+)$ in the quotient. Since the singularity surface is timelike or null, this can only happen if the singularity surface divides the future null infinity $\mathcal{I}^+$ of the AdS_{p+1} spacetime into disconnected regions. The behaviour of the Killing vector on the asymptotic boundary of the AdS spacetime is therefore essential in determining if a given case is a black hole or not.

3.6.1 AdS_3 black holes

For the AdS_3 case, the addition of a deformation on the sphere does not significantly modify the analysis of [56]: the only quotients which lead to black holes are the ones whose AdS Killing vector field is associated with the Killing vectors $I_{\mathbb{R}}$, for $|a_1| \neq |a_2|$, and $II_{\mathbb{R}\pm}$ for $a \neq 0$, corresponding to non-extremal and extremal black holes, respectively. These AdS Killing vectors correspond to type I_b and type II_a in the notation of [56] [10]. When embedding these black holes in string theory, it is certainly natural to embed them in type IIB, in terms of $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$, coming from the near horizon of the D1-D5 system. Thus, the most general Killing vector field giving rise to black holes is given by

$$\xi = \xi_{BTZ} + \theta_1 R_{12} + \theta_2 R_{34} , \tag{3.164}$$

where we are using the notation introduced in Section 3.2.

The metric on these solutions is easily constructed. For simplicity, we shall focus again on the deformation for which $\theta_1 = \theta_2 = \gamma$. Let us adopt BTZ coordinates on the AdS space, so that $\xi_{\mathrm{AdS}_3} = \partial_\phi$, and adapted coordinates on the sphere, so that

[10]Note that the $M = J = 0$ black hole solutions of [56], obtained by quotienting by $\lambda^{III(1,2)}$, do not have a generalisation to include rotation on the sphere, as the associated AdS Killing vectors are nowhere timelike, so these give causally regular quotients once a non-trivial ξ_{S^3} is included, as described in section 3.3.

$\xi_S = \partial_\psi$. Then the metric is

$$g = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2}dt^2 + \frac{r^2 dr^2}{(r^2 - r_+^2)(r^2 - r_-^2)} + r^2[d\phi - \frac{r_- r_+}{r^2}dt]^2 + d\theta^2 + d\chi^2 + d\psi^2 + 2\cos 2\theta d\chi d\psi, \quad (3.165)$$

and the quotient introduces the periodic identifications $\phi \sim \phi + 2\pi m, \psi \sim \psi + 2\pi\gamma m$, $m \in \mathbb{Z}$. If we introduce a new coordinate $\tilde{\psi} = \psi - \gamma\phi$, then $\xi = \partial_\phi$ and the metric in fully adapted coordinates is

$$g = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2}dt^2 + \frac{r^2 dr^2}{(r^2 - r_+^2)(r^2 - r_-^2)} + r^2[d\phi - \frac{r_- r_+}{r^2}dt]^2 + \gamma^2 d\phi^2 + 2\gamma d\phi(d\tilde{\psi} + \cos 2\theta d\chi) + d\theta^2 + d\chi^2 + d\tilde{\psi}^2 + 2\cos 2\theta d\chi d\tilde{\psi}. \quad (3.166)$$

Note that the deformation on the sphere does not affect the leading r^2 part of the metric at large distances, so the structure of the asymptotic boundary of the black hole is not changed. From the point of view of Kaluza-Klein reduction over the sphere, this geometry is described as the rotating BTZ black hole with a flat $SU(2)_L \subset SO(4)$ gauge connection $A^3_\phi = \gamma$ turned on, in analogy with previous discussions of conical defects [97]. Since the gauge field has zero stress-energy, it does not modify the three-dimensional metric. Its presence does however modify the supersymmetry conditions [97]. Unlike in the conical defect case, we cannot make non-supersymmetric black hole solutions supersymmetric by adding a deformation on the sphere, as we cannot balance the hyperbolic black hole holonomy by a holonomy in $SU(2)$.

3.6.2 Higher-dimensional black holes

Let us now investigate what happens in higher dimensions. For the excision we are studying, the singularity is determined by the non-trivial part of the AdS action, $\xi_{\mathrm{AdS}_{n+1}}$, and the presence of horizons is determined by considering the intersection of this singularity surface with the AdS boundary. We therefore focus on the AdS part of the story, and only add in the sphere at the end.

We want to know if there is an event horizon in the quotient. Since the location of the singularity is determined by $\xi_{\mathrm{AdS}_{n+1}}$, it is natural to study this using the decomposition (3.57). This considerably simplifies the task of studying the higher-dimensional cases, by relating it to the lower-dimensional classification. It would require considerable work to determine directly from the form of the Killing vectors whether or not event horizons exist. By relating this question to the existence of horizons in lower dimensions, we can avoid most of this work and also gain some valuable insight into the differences between the AdS_3 case and higher dimensions.

For a Killing vector which does not contain a $\lambda^{V(2,3)}$ block, a $\lambda^{III(2,4)}$ block, two $\lambda^{III(1,2)}$ blocks, or a $\lambda^{III(1,2)}$ and a $\lambda^{(1,1)}$ block, we can adapt the coordinate system of (3.58) with $n = 2$; that is, we can decompose AdS_{p+1} in terms of AdS_3 and S^{p-3} factors. The Killing vector then decomposes as $\xi_{AdS} = \xi_{AdS_3} + \xi_r$, where ξ_{AdS_3} acts only on the AdS_3 part of the metric (3.58) and contains the non-trivial block or blocks, while the ξ_r is a combination of rotations (the $\lambda^{(0,2)}$ blocks) acting on the unit sphere S^{p-3}. Furthermore, ξ_{AdS_3} is precisely the Killing vector associated to the same type of quotient in the analysis of [56].

We exploit this decomposition to simplify the problem of finding horizons. We will show that there is a simple condition on the action in AdS_3 which will imply that the singularity is naked in AdS_{p+1}. The existence of a non-trivial event horizon in the quotient spacetime implies that there are points in the singularity surface $||\xi_{AdS_3}||^2 = 0$ which cannot be connected to the same asymptotic region in both the past and the future. Conversely, if a point in AdS with $||\xi_{AdS_3}||^2 = 0$ lies on some timelike curve which lies entirely in the region where $||\xi_{AdS_3}||^2 \geq 0$ in the bulk and starts and ends in some connected component of the region of the boundary where $||\xi_{AdS_3}||^2 > 0$, this point on the singularity will be naked in the quotient. Thus, the existence of such a curve implies the nakedness of the singularity.

Now, in the coordinates (3.58), we can consider the restriction to the AdS_3 factor at some fixed point on the sphere factor that ξ_r acts on, and ask if there is such a curve which in addition stays in this submanifold. This will supply a sufficient condition for nakedness of the singularity which can be expressed in AdS_3 terms. We therefore want to look for a timelike curve in AdS_3 which connects points in the same connected component of the region of the boundary where $||\xi_{AdS_3}||^2 > 0$ through the region where $||\xi_{AdS_3}||^2 \geq 0$ in the bulk, and passing through a point at

$||\xi_{\text{AdS}_3}||^2 = 0$. But this is the same thing as the condition for a naked singularity in AdS_3: cases which do not lead to black holes in AdS_3 do not lead to black holes in higher dimensions either. Horizons can only arise in the cases where there is a horizon in the AdS_3 quotient.

Consider now the cases which give black holes in AdS_3; that is $I_{\mathbb{R}}$ for $|a_1| \neq |a_2|$ and $II_{\mathbb{R}\pm}$ for $a \neq 0$. Consider first the rotating black holes. We will see that there will be no horizons in the higher-dimensional cases. In the quotient of AdS_3, we obtained a solution with an inner horizon and a timelike singularity, so any point on the singularity surface was connected to the boundary to both the past and future, but it was connected to different components of the boundary, so this did not imply the absence of a horizon. In higher dimensions, however, we can describe the asymptotic boundary in terms of an $\text{AdS}_3 \times \text{S}^{p-3}$ metric,

$$g_\partial = g_{\text{AdS}_3} + g_{\text{S}^{p-3}}. \tag{3.167}$$

Since the portion of the bulk of AdS_3 where ξ_{AdS_3} is spacelike is connected, the portion of the boundary of AdS_{p+1} where ξ_{AdS_3} is spacelike will be connected, and hence the curves which link a point on the singularity to the boundary have their endpoints in a single connected component of the region of the boundary where $||\xi_{\text{AdS}_3}||^2 > 0$. Thus, they imply that the singularity is naked in the higher-dimensional quotients, as noted for the case $p = 3$ in [68].

This leaves only the cases where we quotient by a Killing vector with a single $\lambda^{(1,1)}$ factor, which would correspond to a non-rotating black hole in AdS_3. We will see shortly that this case does have non-trivial event horizon for AdS_{p+1}, $p \geq 2$. This is thus the only case involving AdS_3 blocks with an event horizon in higher dimensions.[11]

It remains to consider the Killing vectors containing blocks $\lambda^{V(2,3)}$ and $\lambda^{II(2,4)}$, and the cases containing two $\lambda^{(1,2)}$ blocks or a $\lambda^{(1,2)}$ block and a $\lambda^{(1,1)}$ block. However, these do not lead to any more examples with horizons. For two $\lambda^{(1,2)}$ blocks, this is obvious, as the Killing vector is nowhere timelike. For the $\lambda^{V(2,3)}$ block, we can

[11] We are again excluding the case of $\lambda^{(1,2)}$, corresponding to an $M = 0$ black hole, on the grounds that once we include rotation on the sphere, this will become a causally regular quotient.

observe that it was shown in [68] (where this case is called type V) that there is no horizon in this case in AdS_4; this can easily be extended to show that there is no horizon in higher dimensions by the arguments used above. For a $\lambda^{(1,2)}$ block and a $\lambda^{(1,1)}$ block, we can similarly appeal to the analysis of [68].

For the $\lambda^{(2,4)}$, we analyse the situation in AdS_5, and appeal to the argument set forth above to extend the conclusion to general dimensions. In AdS_5, the Killing vector is

$$\xi_{AdS} = J_{15} - J_{35} \pm J_{26} - J_{46} + b(\mp J_{12} + J_{34} + J_{56}) \ . \tag{3.168}$$

The norm of this Killing vector is

$$||\xi_{AdS}||^2 = -b^2 + 4b(x_6(x_3 - x_1) - x_5(x_4 \mp x_2)) + (x_3 - x_1)^2 + (x_4 \mp x_2)^2, \tag{3.169}$$

where $\{x_1, \ldots, x_6\}$ are the $\mathbb{R}^{2,4}$ embedding coordinates. Adapting a global coordinate system on AdS_5,

$$\begin{aligned} x_1 &= \cosh\rho\cos t, \quad x_2 = \cosh\rho\sin t \ , \\ x_3 &= \sinh\rho\cos\theta\cos\phi, \quad x_4 = \sinh\rho\cos\theta\sin\phi, \\ x_5 &= \sinh\rho\sin\theta\cos\psi, \quad x_6 = \sinh\rho\sin\theta\sin\psi \ , \end{aligned} \tag{3.170}$$

the norm becomes

$$\begin{aligned} ||\xi_{AdS}||^2 = -b^2 &+ 4b\sinh\rho\sin\theta\left[-\cosh\rho\sin(\psi \pm t) + \sinh\rho\cos\theta\sin(\psi - \phi)\right] \\ &+ \cosh^2\rho + \sinh^2\rho\cos^2\theta - 2\cosh\rho\sinh\rho\cos\theta\cos(\phi \pm t) \ . \end{aligned}$$

Thus, we see that the global time dependence of the norm is simply a simultaneous rotation in the two angles ϕ, ψ on the S^3 in AdS_5. Thus, the region of the boundary where the norm of the Killing vector is spacelike is clearly connected, and this case does not give rise to a black hole in any dimension.

Thus, the only quotient with a black hole interpretation for $p > 2$ is the quotient by an AdS Killing vector $\lambda^{(1,1)}(a) \oplus_i \lambda^{(0,2)}(b_i)$. The resulting quotient is the higher-dimensional generalisation of the non-rotating BTZ black hole. Special cases of this solution for $p = 3, 4$ have been discussed before in [68, 118, 119].[12] As above, the

[12]Note that in [68], it was claimed that this does not lead to a black hole for $b_i \neq 0$. This is

natural coordinate system on these quotients in general is the one given by the decomposition (3.57). If we adopt adapted coordinates for the $\lambda^{(1,1)}$ action on the AdS_3 factor, this is

$$g = \cosh^2\chi\left(-(r^2-1)dt^2 + \frac{dr^2}{r^2-1} + r^2 d\phi^2\right) + d\chi^2 + \sinh^2\chi d\Omega_{p-3} \,. \qquad (3.171)$$

where we have re-absorbed the length scale r_+ associated with the black hole by rescaling coordinates, so the period of the angular coordinate ϕ depends on r_+. The quotient makes identifications in ϕ with some twist on the S^{p-3} determined by the b_i. We note that although these are deformations of the higher-dimensional BTZ quotient by rotations, they do not look like rotating black holes in the usual sense: ∂_t is still hypersurface-orthogonal, and there is a single horizon.

The special case where we consider a simple boost, so $b_i = 0$, was considered in detail in [68, 118, 119]. In this case the quotient preserves, in addition to the symmetry associated with ξ, an $SO(1, p-1)$ symmetry in the orthogonal subspace. Various coordinate systems were defined on the quotient which are adapted to make some or all of this symmetry manifest in [118, 119]. We would like to briefly connect to that work by showing how our preferred coordinate system above which makes the AdS_3 structure manifest is connected to one of those coordinate systems.

In [119], "spherical" coordinates were defined, in which the metric takes the form

$$g = (\rho^2-1)\left[-\sin^2\theta dt^2 + d\theta^2 + \cos^2\theta d\Omega_{p-3}\right] + \frac{d\rho^2}{(\rho^2-1)} + \rho^2 d\phi^2. \qquad (3.172)$$

These coordinates are one example of coordinates adapted to the $SO(1, p-1) \times SO(1,1)$ symmetry of this spacetime. They are related to (3.171) by the coordinate transformation

$$\cos\theta = \frac{\sinh\chi}{\sqrt{\rho^2-1}}, \quad \rho = r\cosh\chi \,. \qquad (3.173)$$

It is interesting to note that this shows that the $SO(1,1)$ manifest in (3.172) is precisely the time translation of the BTZ black hole. Note that the spherical coordinates of (3.172) cover more of the spacetime than the BTZ coordinates of (3.171).

because [68] took the singularity surface to be $||\xi_{AdS}||^2 = 0$, which does not eliminate all closed timelike curves in this case. We take the singularity surface to be $||\xi_{AdS_3}||^2 = 0$, cutting out more of the global AdS spacetime; this gives a causally regular spacetime which can be interpreted as a black hole.

This illustrates that while the coordinates we have constructed adapted to the decomposition of the Killing vector in terms of lower-dimensional quotients are useful, they are not the best coordinate system for every purpose.

Another interesting coordinate system on this quotient is the 'de Sitter' coordinates of [119], which were used in [120, 121], where this locally AdS_{p+1} black hole arises as the asymptotic behaviour of the bubble of nothing solution. In that context, it is convenient to adopt a coordinate system in which the metric is

$$g = (1 + R^2)d\phi^2 + \frac{dR^2}{1 + R^2} + R^2 \left[-d\tau^2 + \cosh^2 \tau (d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\Omega_{p-3}) \right] . \quad (3.174)$$

These coordinates are adapted to the same $SO(1, p-1) \times SO(1, 1)$ symmetry as in (3.172). The coordinate transformation relating (3.174) to (3.172) is

$$\rho^2 = 1 + R^2, \quad \cos\theta = \cosh\tau \sin\tilde{\theta} , \quad \tanh t = \frac{\tanh\tau}{\cos\tilde{\theta}} . \quad (3.175)$$

These 'de Sitter' coordinates have the advantage that they cover the whole exterior region of the black hole. They demonstrate that the black hole is not a static solution in higher dimensions; there is no Killing vector which is timelike everywhere outside the black hole event horizon.

As in the three-dimensional case, when we consider the quotient of $AdS_{p+1} \times S^q$, we can write the AdS and sphere factors in adapted coordinates separately, so that $\xi_{AdS} = \partial_\phi$, and $\xi_S = \partial_\psi$. Fully adapted coordinates are then obtained by setting $\tilde{\psi} = \psi - \gamma\phi$, which introduces O(1) cross terms between AdS and sphere coordinates. Again, from the Kaluza-Klein reduced point of view, what we are doing is introducing a flat $SO(q+1)$ gauge connection $A^a_\phi = \gamma$ on the black hole solution above, without modifying the metric.

One other issue deserves remarking on on the subject of black holes: in [119], it was claimed that a rotating black hole solution could be constructed by taking a quotient of AdS_5. We want to point out that this is not the same as the deformation by $\lambda^{(0,2)}$ discussed above; in fact, this quotient is not a black hole. The solution of [119] was given by considering AdS_5 in the coordinates

$$g = \sinh^2 \rho \left[-\cos^2 \theta d\tilde{t}^2 + d\theta^2 + \sin^2 \theta d\psi^2 \right] + d\rho^2 + \cosh^2 \rho d\tilde{\phi}^2, \quad (3.176)$$

and making identifications along $\phi = \tilde{\phi}$ at fixed $t = \frac{r_+\tilde{t} - r_-\tilde{\phi}}{r_+^2 - r_-^2}$. This gives a 'black hole' metric of the form

$$g = \cos^2\theta \left[-\frac{(r^2 - r_+^2)\cdot(r^2 - r_-^2)}{r^2} dt^2 + r^2 \left(d\phi - \frac{r_-}{r_+ \cdot r^2}(r^2 - r_+^2)dt \right)^2 \right] + \frac{r^2 dr^2}{(r^2 - r_+^2)\cdot(r^2 - r_-^2)} + \frac{(r^2 - r_+^2)}{(r_+^2 - r_-^2)}(d\theta^2 + \sin^2\theta d\psi^2) + \frac{r_+^2 (r^2 - r_-^2)}{(r_+^2 - r_-^2)} \sin^2\theta d\phi^2, \tag{3.177}$$

where $r^2 = r_+^2 \cosh^2\rho - r_-^2 \sinh^2\rho$. Since the coordinates $\tilde{t}$ and $\tilde{\phi}$ in (3.176) both parametrise SO(1,1) symmetries (while χ parametrises an SO(2) symmetry), we can easily see that this quotient corresponds to the rotating BTZ black hole type of quotient: that is, to a quotient by a Killing vector formed from $\lambda^{(1,1)}(a_1) \oplus \lambda^{(1,1)}(a_2)$, with $a_1 a_2 \neq 0$. This can be seen explicitly by noting that defining the new coordinates $\chi, \bar{r}$ by

$$\begin{aligned} \sinh^2\chi &= \frac{(r^2 - r_+^2)}{(r_+^2 - r_-^2)} \sin^2\theta \,, \\ \bar{r}^2 - r_-^2 &= \frac{r^2 - r_-^2}{\cosh^2\chi} \,, \end{aligned} \tag{3.178}$$

we can rewrite (3.177) as

$$g = \cosh^2\chi \left[-\frac{(\bar{r}^2 - r_+^2)\cdot(\bar{r}^2 - r_-^2)}{\bar{r}^2} dt^2 + \bar{r}^2 \left(d\phi + \frac{r_-}{r_+ \bar{r}^2}(\bar{r}^2 - r_+^2)dt \right)^2 + \frac{\bar{r}^2 d\bar{r}^2}{(\bar{r}^2 - r_+^2)\cdot(\bar{r}^2 - r_-^2)} \right] + d\chi^2 + \sinh^2\chi d\psi^2 \,, \tag{3.179}$$

showing that the quotient space has a rotating BTZ black hole factor and a circle factor, as expected for this type of quotient. Now, we have argued above that the presence of a rotating BTZ black hole factor implies that the region of the boundary of AdS_5 where the Killing vector we are quotienting along is spacelike is connected. Thus, this quotient cannot lead to an event horizon. The apparent presence of an event horizon in the coordinates (3.177) is attributable to those coordinates not covering the whole of infinity.

Chapter 4

Non-supersymmetric smooth geometries and D1-D5-P bound states

4.1 Introduction and Summary

A radical resolution of the information paradox has been suggested whereby quantum gravity effects do not stay confined to microscopic distances, and the black hole interior is quite different from the naive picture suggested by classical gravity. In this 'fuzzball' picture individual states have no horizon and no singularity, but an effective 'horizon' does arise after 'coarse-graining'.

Support for this conjecture has come from studying the D1-D5 system. The theory considered is type IIB supergravity compactified on $S^1 \times T^4$ with n_5 D5-branes wrapping $S^1 \times T^4$, and n_1 D-strings wrapping the S^1. The low energy physics of the bound states of these branes is described by a $1+1$ dimensional CFT with $c = 6n_1n_5$. This field theory has its fermions periodic around the S^1 and is thus in the Ramond sector. The R ground state of the CFT has a large degeneracy, with entropy $S_{micro} = 2\sqrt{2}\pi\sqrt{n_1n_5}$. The geometry usually claimed to correspond to the

R ground state of the D1-D5 system with no angular momentum is

$$ds^2 = \frac{-dt^2 + dy^2}{\sqrt{(1+\frac{Q_1}{r^2})(1+\frac{Q_5}{r^2})}} + \sqrt{(1+\frac{Q_1}{r^2})(1+\frac{Q_5}{r^2})}dx_i dx_i + \frac{\sqrt{(1+\frac{Q_1}{r^2})}}{\sqrt{(1+\frac{Q_5}{r^2})}}dz_a dz_a. \quad (4.1)$$

At $r \to \infty$ the geometry is flat, at small r the geometry is locally $AdS_3 \times S^3 \times T^4$ and it is singular at $r = 0$. Lunin and Mathur showed [80, 125] that this geometry is not actually produced by any configuration of D1 and D5 charges by constructing a family of smooth geometries corresponding to the whole family of RR ground states in the CFT.

They found such solutions by first mapping this '2-charge system' to another 2-charge system, the FP system, by a set of string dualities. The bound state of the F and P charges corresponds to a fundamental string 'multiwound' n_5 times around S^1, with momentum P being carried on this string as travelling waves. Since the F string has no longitudinal vibrations, it must bend away from its central axis in a transverse direction to carry the momentum. It is possible to write down the supergravity solution corresponding to such a configuration. Dualising back we get a family of geometries for the D1-D5 system characterised by functions of the displacement of the string in its transverse directions. Upon quantisation this family of geometries should yield the $e^{2\sqrt{2}\pi\sqrt{n_1 n_5}}$ states expected from the entropy.

Each D1-D5 bound state has a finite transverse size which modifies the naive metric (4.1) inside a region $r < r_0$, such that each geometry is regular with no horizon. Had the bound state been pointlike, the metric would have ended in a point singularity at $r = 0$. If we 'coarse grain' by drawing a boundary to enclose the region $r < r_0$ where these geometries differ significantly from each other then for the area A of this boundary we find

$$\frac{A}{4G} \sim \sqrt{n_1 n_5} \sim S_{micro}. \quad (4.2)$$

Such an agreement was also found for the one parameter family of 'rotating D1-D5 systems' where the states in the sytem have angular momentum J [126]. As a test of whether this picture of the two-charge system indeed describes the correct physics one can perform dynamical experiments with these different geometries. To this end the collision time for left- and right-moving excitations on the component string was computed in field theory and compared to the time for graviton absorption and re-emission in the supergravity picture; the two are found to match [80, 127].

However, the 2-charge D1-D5 system has a vanishing macroscopic entropy so does not actually correspond to a finite horizon area black hole. The 3-charge system which has D1, D5 and P charges (P is momentum along the S^1) has a horizon radius that is of the same order as other scales in the geometry and therefore has a finite macroscopic entropy. This system exhibits all the physically important properties of black holes so results from this system are expected to extend to black holes in general.

Giusto, Mathur, and Saxena made a first step in this direction when they identified smooth geometries corresponding to the spectral flows of the 2-charge RR groundstates studied earlier [85, 86]. These states correspond to special subsets of the 3-charge CFT. The form of all regular half-BPS $U(1) \times U(1)$ invariant 3-charge solutions has recently been found and it could be that these geometries will account for a significant part of the entropy of the 3-charge black hole [95, 96]. However, these geometries have not as yet been explicitly related to CFT states.

In this chapter we find non-extremal 3-charge solitonic solutions in supergravity and identify their corresponding CFT states. We find that these non-supersymmetric soliton solutions are parametrised by two integers m, n. The previously studied supersymmetric solutions correspond to $m = n + 1$. Further solutions with another integer degree of freedom k are constructed by orbifolds of this basic family.

We identify the basic family of smooth solutions labelled by m, n with states in the CFT constructed by spectral flow from the NSNS vacuum, with $m + n$ units of spectral flow applied on the left and $m - n$ units of spectral flow applied on the right. We find a non-trivial agreement between the spacetime charges in these geometries and the expectations from the CFT point of view. This agreement extends to the geometries constructed as orbifolds of the basic smooth solutions. We have studied the wave equation on these geometries, and we find that as in [86], there is a mismatch between the spacetime result, $\Delta t_{sugra} = \pi R \varrho$, and the expectation from the CFT point of view, $\Delta t_{CFT} = \pi R$. We believe that understanding this mismatch is a particularly interesting issue for further development. Finally, we discuss the appearance of an ergoregion in the non-supersymmetric solutions. We find that the ergoregion does not lead to any superradiant scattering for free fields.

This chapter is organised as follows. In section 4.2, we recall the metric and matter fields for the general family of solutions we consider, and discuss the near-horizon limit which relates asymptotically flat solutions to asymptotically $AdS_3 \times S^3$ ones. In section 4.3, we discuss the constraints required to obtain a smooth soliton solution. We find that there is a basic family of smooth solutions labelled by the radius R of the S^1, the D1 and D5 brane charges Q_1, Q_5, and two integers m, n. Further solutions can be constructed as $\mathbb{Z}_k$ orbifolds of these basic ones; they will be smooth if m and n are both relatively prime to k. We also discuss the asymptotically $AdS_3 \times S^3$ solutions obtained by considering the near-horizon limit. The asymptotically $AdS_3 \times S^3$ solutions corresponding to the basic family of smooth solutions are always global $AdS_3 \times S^3$ up to some coordinate transformation. In section 4.4, we verify that the solutions are indeed smooth and free of closed timelike curves. In section 4.5, we identify the corresponding states in the CFT, identifying the coordinate shift in the global $AdS_3 \times S^3$ solutions with spectral flow. In section 4.6, we discuss the massless scalar wave equation on these solutions, and show that the non-supersymmetric solutions always have an ergoregion.

4.2 General nonextremal solution

We will look for smooth solutions as special cases of the nonextremal rotating three-charge black holes given in [160] (uplifted to ten-dimensional supergravity following [138]). The original two-charge supersymmetric solutions of [97, 98] were found in this way, and the same approach was applied more recently in [85, 86] to find supersymmetric three-charge solutions. In the present work, we aim to find all the smooth solutions within this family.

In this section, we discuss this family of solutions in general, writing the metric in forms that will be useful for finding and discussing the smooth solutions. We will also discuss the relation between asymptotically flat and asymptotically $AdS_3 \times S^3$

solutions. We write the metric as

$$\begin{aligned}
\mathrm{d}s^2 &= -\frac{f}{\sqrt{\tilde{H}_1\tilde{H}_5}}(\mathrm{d}t^2-\mathrm{d}y^2)+\frac{M}{\sqrt{\tilde{H}_1\tilde{H}_5}}(s_p\mathrm{d}y-c_p\mathrm{d}t)^2 \\
&+\sqrt{\tilde{H}_1\tilde{H}_5}\left(\frac{r^2\mathrm{d}r^2}{(r^2+a_1^2)(r^2+a_2^2)-Mr^2}+\mathrm{d}\theta^2\right) \\
&+\left(\sqrt{\tilde{H}_1\tilde{H}_5}+(a_2^2-a_1^2)\frac{(\tilde{H}_1+\tilde{H}_5-f)\cos^2\theta}{\sqrt{\tilde{H}_1\tilde{H}_5}}\right)\cos^2\theta\mathrm{d}\psi^2 \\
&+\left(\sqrt{\tilde{H}_1\tilde{H}_5}-(a_2^2-a_1^2)\frac{(\tilde{H}_1+\tilde{H}_5-f)\sin^2\theta}{\sqrt{\tilde{H}_1\tilde{H}_5}}\right)\sin^2\theta\mathrm{d}\phi^2 \\
&+\frac{M}{\sqrt{\tilde{H}_1\tilde{H}_5}}(a_1\cos^2\theta\mathrm{d}\psi+a_2\sin^2\theta\mathrm{d}\phi)^2 \\
&+\frac{2M\cos^2\theta}{\sqrt{\tilde{H}_1\tilde{H}_5}}[(a_1c_1c_5c_p-a_2s_1s_5s_p)\mathrm{d}t+(a_2s_1s_5c_p-a_1c_1c_5s_p)\mathrm{d}y]\mathrm{d}\psi \\
&+\frac{2M\sin^2\theta}{\sqrt{\tilde{H}_1\tilde{H}_5}}[(a_2c_1c_5c_p-a_1s_1s_5s_p)\mathrm{d}t+(a_1s_1s_5c_p-a_2c_1c_5s_p)\mathrm{d}y]\mathrm{d}\phi \\
&+\sqrt{\frac{\tilde{H}_1}{\tilde{H}_5}}\sum_{i=1}^{4}\mathrm{d}z_i^2
\end{aligned} \tag{4.3}$$

where

$$\tilde{H}_i = f + M\sinh^2\delta_i, \quad f = r^2 + a_1^2\sin^2\theta + a_2^2\cos^2\theta, \tag{4.4}$$

and $c_i = \cosh\delta_i$, $s_i = \sinh\delta_i$. This metric is more usually written in terms of functions $H_i = \tilde{H}_i/f$. Writing it in this way instead makes it clear that there is no singularity at $f = 0$. As the determinant of the metric is

$$g = -r^2\frac{\tilde{H}_1^3}{\tilde{H}_5}\cos^2\theta\sin^2\theta, \tag{4.5}$$

it is clear that the inverse metric is also regular when $f = 0$. The above metric is in the string frame, and the dilaton is

$$e^{2\Phi} = \frac{\tilde{H}_1}{\tilde{H}_5}. \tag{4.6}$$

From [85], the 2-form gauge potential which supports this configuration is

$$\begin{aligned}
C_2 &= \frac{M\cos^2\theta}{\tilde{H}_1}[(a_2c_1s_5c_p-a_1s_1c_5s_p)dt+(a_1s_1c_5c_p-a_2c_1s_5s_p)dy]\wedge d\psi \\
&+\frac{M\sin^2\theta}{\tilde{H}_1}[(a_1c_1s_5c_p-a_2s_1c_5s_p)dt+(a_2s_1c_5c_p-a_1c_1s_5s_p)dy]\wedge d\phi \\
&-\frac{Ms_1c_1}{\tilde{H}_1}dt\wedge dy-\frac{Ms_5c_5}{\tilde{H}_1}(r^2+a_2^2+Ms_1^2)\cos^2\theta d\psi\wedge d\phi.
\end{aligned} \tag{4.7}$$

We take the T^4 in the z_i directions to have volume V, and the y circle to have radius R, that is $y \sim y + 2\pi R$.

Compactifying on $T^4 \times S^1$ yields an asymptotically flat five-dimensional configuration. The gauge charges are determined by

$$Q_1 = M \sinh \delta_1 \cosh \delta_1, \tag{4.8}$$

$$Q_5 = M \sinh \delta_5 \cosh \delta_5, \tag{4.9}$$

$$Q_p = M \sinh \delta_p \cosh \delta_p, \tag{4.10}$$

where the last is the charge under the Kaluza-Klein gauge field associated with the reduction along y. The five-dimensional Newton's constant is $G^{(5)} = G^{(10)}/(2\pi RV)$; if we work in units where $4G^{(5)}/\pi = 1$, the Einstein frame ADM mass and angular momenta are

$$M_{ADM} = \frac{M}{2}(\cosh 2\delta_1 + \cosh 2\delta_5 + \cosh 2\delta_p), \tag{4.11}$$

$$J_\psi = -M(a_1 \cosh \delta_1 \cosh \delta_5 \cosh \delta_p - a_2 \sinh \delta_1 \sinh \delta_5 \sinh \delta_p), \tag{4.12}$$

$$J_\phi = -M(a_2 \cosh \delta_1 \cosh \delta_5 \cosh \delta_p - a_1 \sinh \delta_1 \sinh \delta_5 \sinh \delta_p) \tag{4.13}$$

(which are invariant under interchange of the δ_i). We see that the physical range of M is $M \geq 0$. We will assume without loss of generality $\delta_1 \geq 0$, $\delta_5 \geq 0$, $\delta_p \geq 0$ and $a_1 \geq a_2 \geq 0$.

We also want to rewrite this metric as a fibration over a four-dimensional base space. It has been shown in [123] that the general supersymmetric solution in minimal six-dimensional supergravity could be written as a fibration over a four-dimensional hyper-Kähler base, and writing the supersymmetric two-charge solutions in this form played an important role in understanding the relation between these solutions and supertubes in [126] and in an attempt to generate new asymptotically flat three-charge solutions by spectral flow [124]. The supersymmetric three-charge solutions were also written in this form in [128]. Of course, in the non-supersymmetric case, we do not expect the base to have any particularly special character, but we can still use the Killing symmetries ∂_t and ∂_y to rewrite the metric (4.3) as a fibration of these two directions over a four-dimensional base space.

This gives

$$
\begin{aligned}
ds^2 = & \frac{1}{\sqrt{\tilde{H}_1\tilde{H}_5}}\Big\{-(f-M)\left[d\tilde{t}-(f-M)^{-1}M\cosh\delta_1\cosh\delta_5(a_1\cos^2\theta d\psi+a_2\sin^2\theta d\phi)\right]^2 \\
& +f\left[d\tilde{y}+f^{-1}M\sinh\delta_1\sinh\delta_5(a_2\cos^2\theta d\psi+a_1\sin^2\theta d\phi)\right]^2\Big\} \\
& +\sqrt{\tilde{H}_1\tilde{H}_5}\Big\{\frac{r^2dr^2}{(r^2+a_1^2)(r^2+a_2^2)-Mr^2}+d\theta^2 \\
& +(f(f-M))^{-1}\left[\left(f(f-M)+fa_2^2\sin^2\theta-(f-M)a_1^2\sin^2\theta\right)\sin^2\theta d\phi^2\right. \\
& +2Ma_1a_2\sin^2\theta\cos^2\theta d\psi d\phi \\
& \left.+\left(f(f-M)+fa_1^2\cos^2\theta-(f-M)a_2^2\cos^2\theta\right)\cos^2\theta d\psi^2\right]\Big\}, \qquad (4.14)
\end{aligned}
$$

where $\tilde{t}=t\cosh\delta_p-y\sinh\delta_p$, $\tilde{y}=y\cosh\delta_p-t\sinh\delta_p$.

We can see that this is still a 'natural' form of the metric, even in the non-supersymmetric case, inasmuch as the base metric in the second $\{\}$ is independent of the charges. This form of the metric is as a consequence convenient for studying the 'near-horizon' limit, as we will now see.

In addition to the asymptotically flat metrics written above, we will be interested in solutions which are asymptotically $AdS_3\times S^3$. These asymptotically $AdS_3\times S^3$ geometries can be thought of as describing a 'core' region in our asymptotically flat soliton solutions, but they can also be considered as geometries in their own right. It is relatively easy to identify the appropriate CFT duals when we consider the asymptotically $AdS_3\times S^3$ geometries. To prepare the ground for this discussion, we should consider the 'near-horizon' limit in the general family of metrics.

The near-horizon limit is usually obtained by assuming that $Q_1, Q_5 \gg M, a_1^2, a_2^2$, and focusing on the region $r^2 \ll Q_1, Q_5$. This limiting procedure is easily described if we consider the metric in the form (4.14): it just amounts to 'dropping the 1' in the harmonic functions H_1, H_5, that is, replacing $\tilde{H}_1 \approx Q_1$, $\tilde{H}_5 \approx Q_5$, and also approximating $M\sinh\delta_1\sinh\delta_5 \approx M\cosh\delta_1\cosh\delta_5 \approx \sqrt{Q_1Q_5}$ in the cross terms in the fibration. This gives us the asymptotically $AdS_3 \times S^3$ geometry

$$
\begin{aligned}
\mathrm{d}s^2 \;=\; & \frac{1}{\sqrt{Q_1Q_5}}\Big\{-(f-M)[\mathrm{d}\tilde{t}-(f-M)^{-1}\sqrt{Q_1Q_5}(a_1\cos^2\theta\mathrm{d}\psi+a_2\sin^2\theta\mathrm{d}\phi)]^2 \\
& +f\,[\mathrm{d}\tilde{y}+f^{-1}\sqrt{Q_1Q_5}(a_2\cos^2\theta\mathrm{d}\psi+a_1\sin^2\theta\mathrm{d}\phi)]^2\Big\} \\
& +\sqrt{Q_1Q_5}\Big\{\frac{r^2\mathrm{d}r^2}{(r^2+a_1^2)(r^2+a_2^2)-Mr^2}+\mathrm{d}\theta^2 \\
& +(f(f-M))^{-1}\left[\left(f(f-M)+fa_2^2\sin^2\theta-(f-M)a_1^2\sin^2\theta\right)\sin^2\theta\mathrm{d}\phi^2\right. \\
& +2Ma_1a_2\sin^2\theta\cos^2\theta\mathrm{d}\psi\mathrm{d}\phi \\
& \left.+\left(f(f-M)+fa_1^2\cos^2\theta-(f-M)a_2^2\cos^2\theta\right)\cos^2\theta\mathrm{d}\psi^2\right]\Big\}. \qquad (4.15)
\end{aligned}
$$

This can be rewritten as

$$
\begin{aligned}
\mathrm{d}s^2 \;=\; & -\left(\frac{\rho^2}{\ell^2}-M_3+\frac{J_3^2}{4\rho^2}\right)d\tau^2+\left(\frac{\rho^2}{\ell^2}-M_3+\frac{J_3^2}{4\rho^2}\right)^{-1}d\rho^2+\rho^2\left(d\varphi+\frac{J_3}{2\rho^2}d\tau\right)^2 \\
& +\ell^2d\theta^2+\ell^2\sin^2\theta[d\phi+\frac{R}{\ell^2}(a_1c_p-a_2s_p)d\varphi+\frac{R}{\ell^3}(a_2c_p-a_1s_p)d\tau]^2 \\
& +\ell^2\cos^2\theta[d\psi+\frac{R}{\ell^2}(a_2c_p-a_1s_p)d\varphi+\frac{R}{\ell^3}(a_1c_p-a_2s_p)d\tau]^2, \qquad (4.16)
\end{aligned}
$$

where we have defined the new coordinates

$$\varphi=\frac{y}{R},\quad \tau=\frac{t\ell}{R}, \qquad (4.17)$$

$$\rho^2=\frac{R^2}{\ell^2}[r^2+(M-a_1^2-a_2^2)\sinh^2\delta_p+a_1a_2\sinh 2\delta_p] \qquad (4.18)$$

and parameters

$$\ell^2=\sqrt{Q_1Q_5}, \qquad (4.19)$$

$$M_3=\frac{R^2}{\ell^4}[(M-a_1^2-a_2^2)\cosh 2\delta_p+2a_1a_2\sinh 2\delta_p], \qquad (4.20)$$

$$J_3=\frac{R^2}{\ell^3}[(M-a_1^2-a_2^2)\sinh 2\delta_p+2a_1a_2\cosh 2\delta_p]. \qquad (4.21)$$

Thus, we see that we recover the familiar observation that the near-horizon limit of the six-dimensional charged rotating black string is a twisted fibration of S^3 over the BTZ black hole solution [129].

4.3 Finding solitonic solutions

In general, these solutions will have singularities, horizons, and possibly also closed timelike curves. Let us now consider the conditions for the spacetime to be free of these features, giving a smooth solitonic solution.

Written in the form (4.3), the metric has coordinate singularities when $\tilde{H}_1 = 0$, $\tilde{H}_5 = 0$ or $g(r) \equiv (r^2 + a_1^2)(r^2 + a_2^2) - Mr^2 = 0$. In addition, the determinant of the metric vanishes if $\cos^2\theta = 0$, $\sin^2\theta = 0$, or $r^2 = 0$, which will therefore be singular loci for the inverse metric. The singularities at $\tilde{H}_1 = 0$ or $\tilde{H}_5 = 0$ are real curvature singularities, so we want to find solutions where $\tilde{H}_1 > 0$ and $\tilde{H}_5 > 0$ everywhere. The vanishing of the determinant at $\theta = 0$ and $\theta = \frac{\pi}{2}$ merely signals the degeneration of the polar coordinates at the north and south poles of S^3; these are known to be just coordinate singularities for arbitrary values of the parameters, so we will not consider them further.

The remaining coordinate singularities depend only on r. We can construct a smooth solution if the outermost one is the result of the degeneration of coordinates at a regular origin in some $\mathbb{R}^2$ factor; that is, of the smooth shrinking of an S^1. If this origin has a large enough value of r, we will have $\tilde{H}_1 > 0$ and $\tilde{H}_5 > 0$ there, and we will get a smooth solution. The coordinate singularity at $r^2 = 0$ cannot play this role, as we can shift it to an arbitrary position by defining a new radial coordinate by $\rho^2 = r^2 - r_0^2$. The determinant of the metric in the new coordinate system will vanish at $\rho^2 = 0$.

The interesting coordinate singularities are thus those at the roots of $g(r)$, and the first requirement for a smooth solution is that this function *have* roots. If we write

$$g(r) = (r^2 - r_+^2)(r^2 - r_-^2) \tag{4.22}$$

with $r_+^2 > r_-^2$, then

$$r_\pm^2 = \frac{1}{2}(M - a_1^2 - a_2^2) \pm \frac{1}{2}\sqrt{(M - a_1^2 - a_2^2)^2 - 4a_1^2a_2^2}. \tag{4.23}$$

We see that this function only has real roots for

$$|M - a_1^2 - a_2^2| > 2a_1a_2. \tag{4.24}$$

There are two cases: $M > (a_1 + a_2)^2$, or $M < (a_1 - a_2)^2$. Note that in the former case, $r_+^2 > 0$, whereas in the latter, $r_+^2 < 0$ (which is perfectly physical, since as noted above, we are free to define a new radial coordinate by shifting r^2 by an arbitrary constant).

Assuming one of these two cases hold, we can define a new radial coordinate by $\rho^2 = r^2 - r_+^2$. Since $r^2 dr^2 = \rho^2 d\rho^2$, in this new coordinate system

$$g_{\rho\rho} = \sqrt{\tilde{H}_1 \tilde{H}_5} \frac{d\rho^2}{\rho^2 + (r_+^2 - r_-^2)}, \tag{4.25}$$

and the determinant of the metric is $g = -\rho^2 \frac{\tilde{H}_1^3}{\tilde{H}_5} \cos^2\theta \sin^2\theta$. Thus, in this coordinate system, the only potential problems are at $\rho^2 = 0$ and $\rho^2 = r_-^2 - r_+^2$, that is, at the two roots of the function $g(r)$.

To see what happens at $r^2 = r_+^2$, consider the geometry of the surfaces of constant r. The determinant of the induced metric is

$$g^{(ty\theta\phi\psi)} = -\cos^2\theta \sin^2\theta \tilde{H}_1^{1/2} \tilde{H}_5^{1/2} g(r). \tag{4.26}$$

Thus, at $r^2 = r_+^2$, the metric in this subspace degenerates. This can signal either an event horizon, where the surface $r^2 = r_+^2$ is null, or an origin, where $r^2 = r_+^2$ is of higher codimension. We can distinguish between the two possibilities by considering the determinant of the metric at fixed r and t; that is, in the (y, θ, ϕ, ψ) subspace. This is

$$\begin{aligned} g^{(y\theta\phi\psi)} &= \cos^2\theta \sin^2\theta \left\{ g(r) \left(r^2 + a_1^2 \sin^2\theta + a_2^2 \cos^2\theta + M(1 + s_1^2 + s_5^2 + s_p^2) \right) \right. \\ &\quad + r^2 M^2 (c_1^2 c_5^2 c_p^2 - s_1^2 s_5^2 s_p^2) + M^2 (M - a_1^2 - a_2^2) s_1^2 s_5^2 s_p^2 \\ &\quad \left. + 2M^2 a_1 a_2 s_1 c_1 s_5 c_5 s_p c_p \right\}. \end{aligned} \tag{4.27}$$

This will be positive at $r^2 = r_+^2$ if and only if $M > (a_1 + a_2)^2$. If it is, the constant t cross-section of $r^2 = r_+^2$ will be spacelike, and $r^2 = r_+^2$ is an event horizon. Thus, we can have smooth solitonic solutions without horizons only in the other case $M < (a_1 - a_2)^2$.

To have a smooth solution, we need a circle direction to be shrinking to zero at $r^2 = r_+^2$. That is, we need some Killing vector with closed orbits to be approaching zero. Then by a suitable choice of period we could identify $\rho^2 = 0$ with the origin in polar coordinates of the space spanned by ρ and the angular coordinate corresponding to this Killing vector. The Killing vectors with closed orbits are linear combinations

$$\xi = \partial_y - \alpha \partial_\psi - \beta \partial_\phi, \tag{4.28}$$

so a necessary condition for a circle degeneration is that (4.27) vanish at $r^2 = r_+^2$, so that some linear combination of this form has zero norm there. We can satisfy this condition in two different ways.

4.3.1 Two charge solutions: $a_1 a_2 = 0$

The first possibility is to set $a_2 = 0$, so $a_1 a_2 = 0$. Then for $M < a_1^2$, $r_+^2 = 0$, and we set (4.27) to zero at $r^2 = 0$ by taking one of the charges to vanish. We will focus on setting $\delta_p = 0$, since these solutions will have a natural interpretation in CFT terms. Recall that in string theory, we can interchange the different charges in this solution by U-dualities.

For this choice of parameters, the metric simplifies to

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{\tilde{H}_1 \tilde{H}_5}} [-(f-M)(dt - (f-M)^{-1} M c_1 c_5 a_1 \cos^2\theta d\psi)^2 \\ &\quad + f(dy - f^{-1} M s_1 s_5 a_1 \sin^2\theta d\phi)^2] \\ &\quad + \sqrt{\tilde{H}_1 \tilde{H}_5} \left(\frac{dr^2}{r^2 + a_1^2 - M} + d\theta^2 + \frac{r^2 \sin^2\theta}{f} d\phi^2 + \frac{(r^2 + a_1^2 - M)\cos^2\theta}{f - M} d\psi^2 \right). \end{aligned} \tag{4.29}$$

Since (4.27) vanishes at $r^2 = 0$, the orbits of a Killing vector of the form (4.28) must degenerate there. It is easy to use the simplified metric (4.29) to evaluate

$$\alpha = 0, \quad \beta = \frac{a_1}{M s_1 s_5}. \tag{4.30}$$

That is, if we define a new coordinate

$$\tilde{\phi} = \phi + \frac{a_1}{M s_1 s_5} y, \tag{4.31}$$

the direction which goes to zero is y at fixed $\tilde{\phi}, \psi$. To make $y \to y + 2\pi R$ at fixed $\tilde{\phi}, \psi$ a closed orbit, we require

$$\frac{a_1}{M s_1 s_5} R = m \in \mathbb{Z}. \tag{4.32}$$

Around $r = 0$, we then have

$$ds^2 = \ldots + \sqrt{\tilde{H}_1 \tilde{H}_5} \left(\frac{dr^2}{a_1^2 - M} + \frac{r^2 dy^2}{M^2 s_1^2 s_5^2} \right) + \ldots \tag{4.33}$$

This will be regular if we choose the radius of the y circle to be

$$R = \frac{M s_1 s_5}{\sqrt{a_1^2 - M}}. \tag{4.34}$$

Thus, the integer quantisation condition fixes

$$m = \frac{a_1}{\sqrt{a_1^2 - M}}. \tag{4.35}$$

With this choice of parameters, the solution is completely smooth, and $\theta, \tilde{\phi}, \psi$ are the coordinates on a smooth S^3 at the origin $r = 0$. We recover the smooth supersymmetric solutions of [97, 98] for $m = 1$.

From the CFT point of view, it is natural to regard the charges Q_1, Q_5 and the asymptotic radius of the circle R as fixed quantities. We can then solve (4.34) and (4.35) to find the other parameters, giving us a one integer parameter family of smooth solutions for fixed Q_1, Q_5, R. The integer (4.35) determines a dimensionless ratio a_1^2/M, while the other condition (4.34) fixes the overall scale (a_1, say) in terms of Q_1, Q_5, R.

4.3.2 Three charge solutions

Solutions with all three charges non-zero can be found by considering $a_1 a_2 \neq 0$. Setting (4.27) to zero at $r^2 = r_+^2$ implies that

$$M = a_1^2 + a_2^2 - a_1 a_2 \frac{(c_1^2 c_5^2 c_p^2 + s_1^2 s_5^2 s_p^2)}{s_1 c_1 s_5 c_5 s_p c_p} \tag{4.36}$$

and hence that

$$r_+^2 = -a_1 a_2 \frac{s_1 s_5 s_p}{c_1 c_5 c_p}. \tag{4.37}$$

The Killing vector which degenerates is (4.28) with[1]

$$\alpha = -\frac{s_p c_p}{(a_1 c_1 c_5 c_p - a_2 s_1 s_5 s_p)}, \quad \beta = -\frac{s_p c_p}{(a_2 c_1 c_5 c_p - a_1 s_1 s_5 s_p)}. \tag{4.38}$$

The associated shifts in the ϕ, ψ coordinates are hence

$$\tilde{\psi} = \psi - \frac{s_p c_p}{(a_1 c_1 c_5 c_p - a_2 s_1 s_5 s_p)} y, \quad \tilde{\phi} = \phi - \frac{s_p c_p}{(a_2 c_1 c_5 c_p - a_1 s_1 s_5 s_p)} y, \tag{4.39}$$

and $y \to y + 2\pi R$ at fixed $\tilde{\phi}, \tilde{\psi}$ will be a closed orbit if

$$\frac{s_p c_p}{(a_1 c_1 c_5 c_p - a_2 s_1 s_5 s_p)} R = n, \quad \frac{s_p c_p}{(a_2 c_1 c_5 c_p - a_1 s_1 s_5 s_p)} R = -m \tag{4.40}$$

[1]This choice of parameters is most easily derived by requiring $g_{ty} \to 0$ at $\rho^2 = 0$; having derived it, one can then check that it also gives $g_{yy} \to 0$ at $\rho^2 = 0$ as required.

for some integers n, m. As in the two-charge case, requiring regularity of the metric at the origin fixes the radius of the y circle. We do not give details of the calculation, but simply quote the result,

$$R = \frac{M s_1 c_1 s_5 c_5 (s_1 c_1 s_5 c_5 s_p c_p)^{1/2}}{\sqrt{a_1 a_2}(c_1^2 c_5^2 c_p^2 - s_1^2 s_5^2 s_p^2)}. \tag{4.41}$$

If we introduce dimensionless parameters

$$j = \left(\frac{a_2}{a_1}\right)^{1/2} \leq 1, \quad s = \left(\frac{s_1 s_5 s_p}{c_1 c_5 c_p}\right)^{1/2} \leq 1, \tag{4.42}$$

then the integer quantisation conditions determine these via

$$\frac{j + j^{-1}}{s + s^{-1}} = m - n, \quad \frac{j - j^{-1}}{s - s^{-1}} = m + n. \tag{4.43}$$

Note that this constraint is invariant under the permutation of the three charges. We note that we can rewrite the mass (4.36) as

$$M = a_1 a_2 (s^2 - j^2)(j^{-2} s^{-2} - 1) = a_1 a_2 nm (s^{-2} - s^2)^2, \tag{4.44}$$

so $M \geq 0$ implies $s^2 \geq j^2$ and $nm \geq 0$. Our assumption that $a_1 > a_2$ implies $n \geq 0$, so $m \geq 0$, and (4.43) implies $m > n$.

Thus, in this case, for given Q_1, Q_5, R, we have a two integer parameter family of smooth solutions. It is a little more difficult to make direct contact with the supersymmetric solutions of [85] in this case, since one needs to take a limit $a_1, a_2 \to \infty$, but these would correspond to $m = n+1$, as it turns out that $s = 1$ and $M = 0$ if and only if $m = n + 1$. We can also think of the two-charge solutions in the previous subsection as corresponding to the case $n = 0$. To gain some insight into the values of the parameters for other choices of m, n, we have plotted the dimensionless quantities $a_1/\sqrt{M}, a_2/\sqrt{M}$ for some representative values in figure 4.1. The highest point on the figure corresponds to $m = 2, n = 0$. Increasing n moves diagonally downwards towards the diagonal, and increasing $m - n$ moves down towards $(0, 1)$. For each point, there is a set of orbifolds labelled by k. Solutions with event horizons exist in the region $a_1/\sqrt{M} + a_2/\sqrt{M} < 1$ (off the bottom of the plotted region).

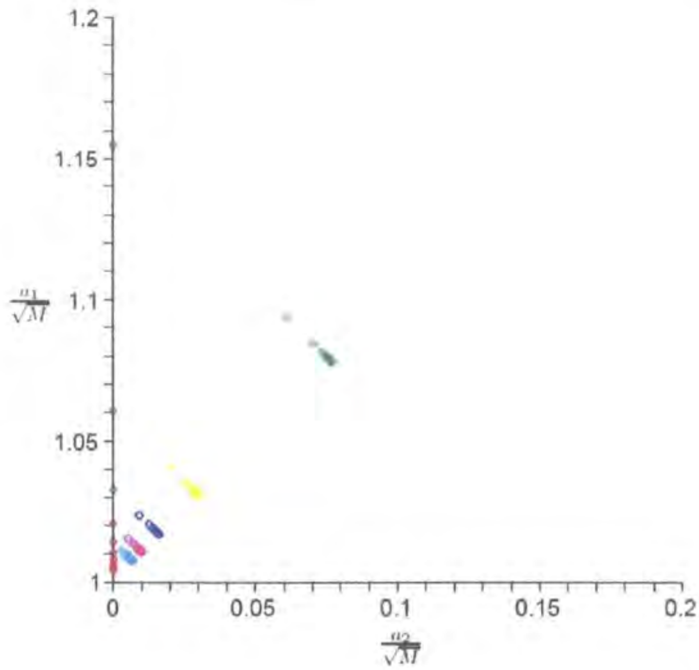

Figure 4.1: The values of the dimensionless quantities $a_2/\sqrt{M}$, $a_1/\sqrt{M}$ for which smooth solitons are obtained are indicated by points.

4.3.3 Orbifolds & more general smooth three-charge solution

So far, we have insisted that the solution be smooth. However, in the context of string theory, we may also consider solutions with orbifold singularities, since the corresponding worldsheet conformal field theory is completely well-defined. In the context of the above smooth solutions, a particularly interesting class of orbifolds is the $\mathbb{Z}_k$ quotient by the discrete isometry $(y, \psi, \phi) \sim (y + 2\pi R/k, \psi, \phi)$.

In the two-charge case, the quotient acts as $(y, \psi, \tilde{\phi}) \sim (y+2\pi R/k, \psi, \tilde{\phi}+2\pi m/k)$ in the coordinates appropriate near $r = 0$. This isometry has a fixed point at $r = 0, \theta = 0$, so the resulting orbifold has a $\mathbb{Z}_k$ orbifold singularity there. In addition, if k and m are not relatively prime, there will be a $\mathbb{Z}_j$ orbifold singularity at $r = 0$ for all θ, where $j = \gcd(k, m)$. The supersymmetric orbifolds corresponding to $m = 1$ have previously been studied [80, 97, 127].

In the three-charge case, the discrete isometry becomes $(y, \tilde{\psi}, \tilde{\phi}) \sim (y+2\pi R/k, \tilde{\psi}-2\pi n/k, \tilde{\phi} + 2\pi m/k)$, and the $\mathbb{Z}_k$ will be freely acting if m and n are relatively prime to k. Thus, we get new smooth three-charged solutions by orbifolding by a k which is relatively prime to m and n. We could have found such solutions directly if we had allowed for the possibility that $y \to y + 2\pi Rk$ is the closed circle at $\rho = 0$, instead of insisting that it be $y \to y + 2\pi R$. We also have orbifolds similar to the two-charged ones if one or both of m and n are not relatively prime to k. In particular, the simple supersymmetric orbifolds studied in [86] correspond to taking $m = kn' + 1$, $n = kn'$ for some integer n'.[2] The preserved supersymmetries in the solutions with $m = n + 1$ correspond to Killing spinors which are invariant under translation in y at fixed ϕ, ψ, so all the orbifolds of cases with $m = n + 1$ will be supersymmetric. In particular, orbifolds where k is relatively prime to n and $n + 1$ will give new smooth supersymmetric solutions.

4.3.4 Asymptotically AdS solutions

In order to understand the dual CFT interpretation of these solutions, it is interesting to see the effect of the constraints (4.41, 4.43) on the asymptotically AdS solution (4.16). Consider first the two-charge case. If we set $a_2 = 0$, $\delta_p = 0$ and insert (4.34, 4.35) in (4.16), we will have

$$\begin{aligned} ds^2 &= -\left(\frac{\rho^2}{\ell^2} + 1\right) d\tau^2 + \left(\frac{\rho^2}{\ell^2} + 1\right)^{-1} d\rho^2 + \rho^2 d\varphi^2 \\ &+\ell^2 \left[d\theta^2 + \sin^2\theta(d\phi + m d\varphi)^2 + \cos^2\theta(d\psi + m d\tau/\ell)^2\right]. \end{aligned} \quad (4.45)$$

Thus, the asymptotically AdS version of the soliton is just global $AdS_3 \times S^3$, with a shift of the angular coordinates on the sphere determined by m.

In the general three-charge case, the interpretation of the dimensionless parameter s changes in the asymptotically AdS solutions: it is now $s = \sqrt{\tanh \delta_p}$. The conditions (4.43) are unaffected, however, and inserting these and the value of the

[2] In [86], other examples where $n \neq kn'$ are obtained by applying STTS duality to these ones. This is possible because while (m, n) are U-duality invariant, k is not, so this transformation can map us to new solutions.

period (4.41) in (4.16), we will have

$$\begin{aligned} ds^2 &= -\left(\frac{\rho^2}{\ell^2}+1\right)d\tau^2 + \left(\frac{\rho^2}{\ell^2}+1\right)^{-1} d\rho^2 + \rho^2 d\varphi^2 \\ &+\ell^2\left[d\theta^2 + \sin^2\theta(d\phi + md\varphi - nd\tau/\ell)^2 + \cos^2\theta(d\psi - nd\varphi + md\tau/\ell)^2\right]. \end{aligned} \tag{4.46}$$

Thus, again, the asymptotically AdS version of the soliton is just global $AdS_3 \times S^3$, with shifts of the angular coordinates on the sphere determined by m, n.

Thus, in the cases where they have a large 'core' region described by an asymptotically AdS geometry, the smooth solitons studied in the first two subsections above approach global $AdS_3 \times S^3$ in this region. As a consequence, the orbifolds studied in the previous section will have corresponding orbifolds of $AdS_3 \times S^3$; some of these orbifolds were discussed in [130,131]. The resulting quotient geometry is still asymptotically $AdS_3 \times S^3$, as can be seen by introducing new coordinates $\varphi' = k\varphi$, $\tau' = k\tau$, $\rho' = \rho/k$. The metric on the orbifold in these coordinates is then

$$\begin{aligned} ds^2 &= -\left(\frac{\rho'^2}{\ell^2}+\frac{1}{k^2}\right)d\tau'^2 + \left(\frac{\rho'^2}{\ell^2}+\frac{1}{k^2}\right)^{-1} d\rho'^2 + \rho'^2 d\varphi'^2 \\ &+\ell^2\left[d\theta^2 + \sin^2\theta\left(d\phi + \frac{m}{k}d\varphi' - \frac{n}{k\ell}d\tau'\right)^2\right. \\ &\left.+\cos^2\theta\left(d\psi - \frac{n}{k}d\varphi' + \frac{m}{k\ell}d\tau'\right)^2\right]. \end{aligned} \tag{4.47}$$

The redefined angular coordinate φ' will have period 2π on the orbifold.

4.4 Verifying regularity

In the previous section, we claim to have found a family of smooth solitonic solutions, by imposing three conditions on the parameters of the general metric. We should now verify that these solutions have no pathologies. In this section, we will use the radial coordinate $\rho^2 = r^2 - r_+^2$ (for the two-charge solutions, $\rho^2 = r^2$), which runs over $\rho \geq 0$.

The first step is to check that $\tilde{H}_1 > 0$, $\tilde{H}_5 > 0$ for all $\rho \geq 0$, as desired. In these coordinates,

$$f = \rho^2 + (a_1^2 - a_2^2)\sin^2\theta + (a_2^2 - a_1 a_2 s^2). \tag{4.48}$$

In the two-charge case, where $a_2 = 0$, the last term vanishes, so $f \geq 0$, and hence $\tilde{H}_1 > 0$, $\tilde{H}_5 > 0$ everywhere. In the more general case, however, the last term is

$$a_2^2 - a_1 a_2 s^2 = -a_1 a_2 (s^2 - j^2) < 0, \tag{4.49}$$

so we do not have $f \geq 0$. Examining $\tilde{H}_1$ directly,

$$\tilde{H}_1 = \rho^2 + (a_1^2 - a_2^2)\sin^2\theta + a_1 a_2 (s^2 - j^2)(s^{-2} j^{-2} s_1^2 - c_1^2), \tag{4.50}$$

so for $\tilde{H}_1 > 0$ everywhere, we need the last factor to be positive. We know $s^2 > j^2$, and we can rewrite the last bracket as

$$(s^{-2} j^{-2} s_1^2 - c_1^2) = \frac{c_1^2}{j^2}\left(s^2 \frac{c_5^2 c_p^2}{s_5^2 s_p^2} - j^2\right) > 0, \tag{4.51}$$

so we indeed have $\tilde{H}_1 > 0$. We can similarly show $\tilde{H}_5 > 0$. Thus, the metric in the $(t, \rho, \theta, \tilde{\phi}, \tilde{\psi}, z^i)$ coordinates is regular for all $\rho > 0$, apart from the coordinate singularities associated with the poles of the S^3 at $\theta = 0, \pi/2$, so the local geometry is smooth.

Next we check for global pathologies. We can easily see that these solutions have no event horizons. The determinant of the metric of a surface of constant ρ, (4.26), is negative for $\rho > 0$. That is, there is a timelike direction of constant ρ for all $\rho > 0$, and hence by continuity there must be a timelike curve which reaches the asymptotic region from any fixed ρ. We will demonstrate the absence of closed timelike curves by proving a stronger statement, that the soliton solutions are stably causal. Using the expression for the inverse metric in appendix C, we can evaluate

$$\partial_\mu t \partial_\nu t g^{\mu\nu} = -\frac{1}{\sqrt{\tilde{H}_1 \tilde{H}_5}}\left[f + M(1 + s_1^2 + s_5^2 + s_p^2) + \frac{M^2(c_1^2 c_5^2 c_p^2 - s_1^2 s_5^2 s_p^2)}{\rho^2 + r_+^2 - r_-^2}\right] < 0, \tag{4.52}$$

so $\partial_\mu t$ is a timelike covector, and t is a global time function for the solitons. Hence the solitons are stably causal, and in particular free of closed timelike curves.

Finally, we should check regularity at $\rho = 0$. In the previous section, we chose R so that the ρ, y coordinates were the polar coordinates in a smooth $\mathbb{R}^2$. If we define new coordinates on this $\mathbb{R}^2$ regular at $\rho = 0$ by

$$x^1 = \rho\cos(y/R), \quad x^2 = \rho\sin(y/R), \tag{4.53}$$

then

$$dy = \frac{1}{(x_1^2 + x_2^2)}(x^1 dx^2 - x^2 dx^1), \tag{4.54}$$

and we need the other $g_{\mu y}$ components in the metric to go to zero at least linearly in ρ for the whole metric to be smooth at $\rho = 0$ once we pass to the Cartesian coordinates x^1, x^2. In fact, we find that the $g_{\mu y}$ go like ρ^2 for small ρ in the $(t, \rho, \theta, \tilde{\phi}, \tilde{\psi}, z^i)$ coordinates.

We also need to verify the regularity of the matter fields. The dilaton is trivially regular, since $\tilde{H}_1 > 0$, $\tilde{H}_5 > 0$, but the Ramond-Ramond two-form requires checking. The non-trivial question is whether the $C_{y\mu}$ go to zero at $\rho^2 = 0$. In fact, in the gauge we used in (4.7), they don't: we find

$$\begin{aligned} C_{y\tilde{\phi}} &= \frac{M s_p c_p s_5 c_5}{a_1 c_1 c_5 c_p - a_2 s_1 s_5 s_p} + O(\rho^2), \\ C_{y\tilde{\psi}} &= \frac{M s_p c_p s_5 c_5}{a_2 c_1 c_5 c_p - a_1 s_1 s_5 s_p} + O(\rho^2), \\ C_{yt} &= \frac{1 + s_1^2 + s_p^2}{s_1 c_1} + O(\rho^2). \end{aligned} \tag{4.55}$$

We can remove these constant terms by a gauge transformation, so the Ramond-Ramond fields are regular at $\rho = 0$. The physical importance of the constant terms is that they correspond to electromagnetic potentials dual to the charges carried by the geometry, and their presence is presumably related to the first law satisfied by these soliton solutions, as in [132].

In summary, we have shown that the two integer parameter family of solutions identified in the previous section are all smooth solutions without CTCs. In the next section, we will explore their relation to the CFT description of the D1-D5-P system.

4.5 Relation to CFT

We have found new smooth solutions by considering the general family of charged rotating black hole solutions (4.3). These are labelled by the radius R, charges (Q_1, Q_5) and three integers (m, n, k). They include the previously known supersymmetric solutions as special cases, and add non-supersymmetric solutions and new supersymmetric orbifold solutions. We would like to see if we can relate these solutions to the CFT description, as was done for the earlier supersymmetric cases in [85, 97, 98].

If we consider the asymptotically $AdS_3 \times S^3$ solutions constructed in section 4.3.4, which describe the 'core' region of the asymptotically flat solitons, we can use the powerful AdS/CFT correspondence machinery to identify the corresponding states in the CFT. The dual CFT for the asymptotically $AdS_3 \times S^3 \times T^4$ spaces with radius $\ell = (Q_1 Q_5)^{1/4}$ is a sigma model with target space a deformation of the orbifold $(T^4)^N / S_N$ [133–135], where

$$N = n_1 n_5 = \frac{\ell^4 V}{g^2 l_s^8}, \tag{4.56}$$

where V is the volume of the T^4. This theory has $c = 6n_1 n_5$. In section 4.3.4, we showed that the corresponding asymptotically AdS solutions for a basic family of solitons were always global $AdS_3 \times S^3$, with a shift on the angular coordinates on the sphere specified by n, m. Following the proposal outlined in [97], we identify the geometries (4.46) with CFT states with charges

$$\begin{aligned} h &= \frac{c}{24}(m+n)^2, \quad & j &= \frac{c}{12}(m+n) \\ \bar{h} &= \frac{c}{24}(m-n)^2, \quad & \bar{j} &= \frac{c}{12}(m-n). \end{aligned} \tag{4.57}$$

Thus, these states have energy

$$E = h + \bar{h} = 2(m^2+n^2)\frac{c}{24} = \frac{1}{2}(m^2+n^2) n_1 n_5, \tag{4.58}$$

and momentum

$$q_p = h - \bar{h} = 4mn\frac{c}{24} = nm n_1 n_5. \tag{4.59}$$

Since the non-compact geometry is global AdS_3, there is a single spin structure on the spacetime. Because of the shifts in the angular coordinates, this spin structure can be either periodic or antiperiodic around φ at fixed ϕ, ψ: it will be periodic if $m+n$ is odd, and antiperiodic if $m+n$ is even. Thus, the geometry is identified with a RR state with the above charges if $m+n$ is odd, and with a NSNS state with these same charges if $m+n$ is even.

These states can be interpreted in terms of spectral flow. Recalling that spectral flow shifts the CFT charges by [136]

$$h' = h + \alpha j + \alpha^2 \frac{c}{24}, \quad j' = j + \alpha \frac{c}{12}, \tag{4.60}$$

$$\bar{h}' = \bar{h} + \beta \bar{j} + \beta^2 \frac{c}{24}, \quad \bar{j}' = \bar{j} + \beta \frac{c}{12}, \tag{4.61}$$

we can see that the required states can be obtained by spectral flow with $\alpha = m+n$, $\beta = m-n$ acting on the NSNS ground state (for which $h = j = 0$, $\bar{h} = \bar{j} = 0$). This spectral flow can be identified with the coordinate transformation in spacetime which relates the (φ, ϕ, ψ) coordinates to the $(\varphi, \tilde{\phi}, \tilde{\psi})$ coordinates. Thus, we see that the non-supersymmetric states corresponding to all the geometries labelled by m, n are constructed by starting with the maximally supersymmetric NSNS vacuum and applying different amounts of spectral flow.

In [97], the special case $m = 1, n = 0$ was discussed. In this case, the spectral flow is by one unit on both the left and the right, and maps the NS vacuum to a R ground state both on the left and the right. We can see the supersymmetry of this state from the spacetime point of view: the covariantly constant Killing spinors in global AdS have the form

$$\epsilon_L^{\pm} = e^{\pm i \frac{\tilde{\phi}_L}{2}} e^{-i\frac{\varphi}{2}} \epsilon_0, \quad \epsilon_R^{\pm} = e^{\pm i \frac{\tilde{\phi}_R}{2}} e^{-i\frac{\varphi}{2}} \epsilon_0, \tag{4.62}$$

so when we shift $\tilde{\phi}_L = \phi_L + \varphi$, $\tilde{\phi}_R = \phi_R + \varphi$, the Killing spinors ϵ_L^+, ϵ_R^+ become independent of φ, corresponding to the preserved Killing symmetries in the R ground state. If we consider $m = n+1$, the spectral flow on the right is by one unit, so ϵ_R^+ is still independent of φ. These are the supersymmetric states considered in [85], which are R ground states on the right, but the more general R states obtained by spectral flowing by $2n+1$ units on the left. Our non-supersymmetric solitons correspond to the more general non-supersymmetric states obtained by spectral flowing the NSNS vacuum by $m-n$ units on the right and $m+n$ units on the left. In [85], an explicit representation for the R sector state obtained by spectral flow by $2r+1$ units was given,[3]

$$|2r+1\rangle_R = (J^+_{-(2r)})^{n_1 n_5} (J^+_{-(2r-4)})^{n_1 n_5} \dots (J^+_{-2})^{n_1 n_5} |1\rangle, \tag{4.63}$$

[3]We use a slightly different notation than [85].

where J^+_{-k} is a mode of the $su(2)$ current of the full CFT which raises h and j by $\Delta h = k$, $\Delta j = 1$, and $|1\rangle$ is the R ground state with $j = +1/2$ obtained by spectral flow from the NS ground state. Similarly, one can give an explicit representation of the NS sector state obtained by spectral flow by $2r$ units, following [137],

$$|2r\rangle_{NS} = (J^+_{-(2r-1)})^{n_1 n_5}(J^+_{-(2r-3)})^{n_1 n_5}\dots(J^+_{-1})^{n_1 n_5}|0\rangle_{NS}. \tag{4.64}$$

The CFT state corresponding to the geometry (4.46) is then $|m+n\rangle_R \times |m-n\rangle_R$ or $|m+n\rangle_{NS} \times |m-n\rangle_{NS}$, depending on the parity of $m+n$.

The situation is more interesting when we consider the orbifolds. The geometries (4.47) should be identified with CFT states with charges

$$\begin{aligned} h &= \frac{c}{24}\left(1 + \frac{(m+n)^2-1}{k^2}\right), \quad j &= \frac{c}{12}\frac{m+n}{k}, \\ \bar{h} &= \frac{c}{24}\left(1 + \frac{(m-n)^2-1}{k^2}\right), \quad \bar{j} &= \frac{c}{12}\frac{m-n}{k}. \end{aligned} \tag{4.65}$$

In the supersymmetric case, when $m = n+1$, $\bar{h} = \frac{c}{24}$, $\bar{j} = \frac{c}{12}\frac{1}{k}$, so these geometries still have the charges of R ground states on the right. This particular R ground state corresponds to the spectral flow of the NS chiral primary state with $\bar{h} = \bar{j} = \frac{c}{24}\frac{k-1}{k}$. However, the charges of the state in the left-moving sector are, in general, not those of a R ground state or even the result of spectral flow on a R ground state. For general m, n, neither sector is the spectral flow of a ground state. Thus, these provide examples of geometries dual to more general CFT states.

To specify the CFT state completely, we need to say if (4.65) are the charges of a RR or a NSNS state. To do so, let us consider the spin structure on spacetime. When m or n is relatively prime to k, there is a contractible circle in the spacetime, and as a result the spin structure is fixed. The contractible circle is $(\varphi', \phi, \psi) \to (\varphi' + 2\pi k, \phi - 2\pi m, \psi - 2\pi n)$. The fermions must be antiperiodic around this circle. For the case where neither m nor n is relatively prime to k, we are not forced to make this choice, but we will assume that we still choose a spin structure such that the fermions are antiperiodic around this circle; this would correspond to the spin structure inherited from the covering space of the orbifold.

In the supersymmetric case $m = n+1$, and more generally for $m+n$ odd, this implies that the fermions are periodic under $\varphi' \to \varphi' + 2\pi k$ at fixed ϕ, ψ. For k odd, this implies the fermions must be periodic under $\varphi' \to \varphi' + 2\pi$, while for k even, they may be either periodic or antiperiodic. Thus, for $m = n + 1$, we can always choose the periodic spin structure for the fermions on spacetime. This spacetime will then be identified with the supersymmetric RR state with the charges (4.65). However, for k even, we can choose the antiperiodic spin structure for the fermions on spacetime; this spacetime will then be identified with a NSNS state with the same charges (4.65). In this latter case, neither the spacetime solution nor the CFT state is supersymmetric.

The situation becomes stranger for $m + n$ even. The antiperiodicity around the contractible cycle implies that the fermions will be antiperiodic under $\varphi' \to \varphi' + 2\pi k$ at fixed ϕ, ψ. If k is odd, this is compatible with a spin structure antiperiodic in φ', but if k is even, there is no spin structure on the orbifold which satisfies this condition. The orbifold cannot be made into a spin manifold. The general conditions for such orbifolds M/Γ to inherit a spin structure from the spin manifold M were discussed in [39]; see also [3] for further discussion relevant to the case at hand. It will be interesting to see how this obstruction for k even, $m + n$ even is reflected in the CFT dual.

In the other cases, we can unambiguously identify the CFT state corresponding to the geometry as the state with charges (4.65) in the sector with the same periodicity conditions on the fermions as in the spacetime (choosing one of the two possible spin structures on spacetime in the case k even, $m + n$ odd). It would be interesting to construct an explicit description of these states, as in the discussion in [85, 86].

Thus, there is a clear CFT interpretation of the asymptotically $AdS_3 \times S^3$ geometries. However, the interesting discovery in this paper is that there are non-supersymmetric asymptotically flat geometries, and we want to ask to what extent these can also be identified with individual microstates in the CFT. Clearly the appropriate CFT states to consider are the ones described above, but does the identification between state and geometry extend to the asymptotically flat spacetimes? In particular, does it make sense to identify the asymptotically flat spacetime with a CFT state in the general case where it does not have a large approximately $AdS_3 \times S^3$

core region, and there is no supersymmetry?[4] We would not in general expect the match to asymptotically flat geometries to be perfect, but there is one non-trivial piece of evidence for the identification of the full asymptotically flat geometries with the CFT states: the form of the charges still reflects the CFT structure. Plugging our parameters into (4.10, 4.12, 4.13) gives

$$Q_p = nm\frac{Q_1Q_5}{R^2}, \tag{4.66}$$
$$J_\phi = -m\frac{Q_1Q_5}{R}, \tag{4.67}$$
$$J_\psi = n\frac{Q_1Q_5}{R}. \tag{4.68}$$

These reproduce the quantisation of the CFT charges in (4.57). In the orbifold case, we replace R by kR, as the physical period of the asymptotic circle is k times smaller, and these values now agree with the charges in (4.65). This seems to us like a very non-trivial consistency check, as it is very difficult to even express the parameters M, a_1, a_2 appearing in the metric (4.3) in terms of Q_1, Q_5 and R and the integers m, n, so there is no reason why we would have expected to get such a simple result automatically. So this appears a good reason to believe properties of the full asymptotically flat geometries are connected to the CFT states. Note, however, that it does not seem to be possible to cast the ADM mass in such a simple form. In the next section, we will also see that the predicted time delay involved in scattering of probes does not quite match CFT expectations.

4.6 Properties of the solitons

We will briefly discuss some properties of these solutions, and their relation to the dual CFT. We first discuss the solution of the massless scalar wave equation in these geometries, following the discussion in [80, 86, 127] closely. We then consider

[4]The CFT state for some of the geometries is in the NSNS sector. We do not regard this as a serious obstruction to an identification at the classical level: we are considering non-supersymmetric geometries, so we can allow the fermions to be antiperiodic around the asymptotic circle in spacetime. At the quantum level, one might worry that these antiperiodic boundary conditions lead to a constant energy density inconsistent with the assumed asymptotic flatness.

the most significant difference between our non-supersymmetric solitons and the supersymmetric cases, the absence of an everywhere causal Killing vector.

4.6.1 Wave equation

It is interesting to study the behaviour of the massless wave equation on this geometry. This is a first step towards analysing small perturbations, and also allows us to address questions of scattering in the geometry which indicate how an exterior observer might probe the soliton. We consider the massless wave equation on the geometry,

$$\Box\Psi = 0. \tag{4.69}$$

It was shown in [138] that this equation is separable. Considering a separation ansatz

$$\Psi = \exp(-i\omega t/R + i\lambda y/R + im_\psi\psi + im_\phi\phi)\chi(\theta)h(r), \tag{4.70}$$

and using the inverse metric given in appendix C, we find that the wave equation reduces to

$$\begin{aligned}&\frac{1}{\sin 2\theta}\frac{d}{d\theta}\left(\sin 2\theta\frac{d}{d\theta}\chi\right)\\&+\left[\frac{(\omega^2-\lambda^2)}{R^2}(a_1^2\sin^2\theta+a_2^2\cos^2\theta)-\frac{m_\psi^2}{\cos^2\theta}-\frac{m_\phi^2}{\sin^2\theta}\right]\chi=-\Lambda\chi,\end{aligned} \tag{4.71}$$

$$\begin{aligned}&\frac{1}{r}\frac{d}{dr}\left[\frac{g(r)}{r}\frac{d}{dr}h\right]-\Lambda h+\left[\frac{(\omega^2-\lambda^2)}{R^2}(r^2+Ms_1^2+Ms_5^2)+(\omega c_p+\lambda s_p)^2\frac{M}{R^2}\right]h\\&-\frac{(\lambda-nm_\psi+mm_\phi)^2}{(r^2-r_+^2)}h+\frac{(\omega\varrho+\lambda\vartheta-nm_\phi+mm_\psi)^2}{(r^2-r_-^2)}h=0,\end{aligned} \tag{4.72}$$

where

$$\varrho=\frac{c_1^2c_5^2c_p^2-s_1^2s_5^2s_p^2}{s_1c_1s_5c_5},\quad \vartheta=\frac{c_1^2c_5^2-s_1^2s_5^2}{s_1c_1s_5c_5}s_pc_p. \tag{4.73}$$

We see that the singularity in the wave equation at $r^2 = r_+^2$ is controlled by the frequency around the circle which is shrinking to zero there. This is a valuable check on the algebra. If we introduce a dimensionless variable

$$x=\frac{r^2-r_+^2}{r_+^2-r_-^2}, \tag{4.74}$$

we can rewrite the radial equation in the form used in [85],

$$4\frac{d}{dx}\left[x(x+1)\frac{d}{dx}h\right]+\left(\sigma^{-2}x+1-\nu^2+\frac{\xi^2}{x+1}-\frac{\zeta^2}{x}\right)h=0, \tag{4.75}$$

where

$$\sigma^2=\left[(\omega^2-\lambda^2)\frac{(r_+^2-r_-^2)}{R^2}\right]^{-1}, \tag{4.76}$$

$$\nu=\left[1+\Lambda-\frac{(\omega^2-\lambda^2)}{R^2}(r_+^2+Ms_1^2+Ms_5^2)-(\omega c_p+\lambda s_p)^2\frac{M}{R^2}\right]^{1/2}, \tag{4.77}$$

$$\xi=\omega\varrho+\lambda\vartheta-nm_\phi+mm_\psi, \tag{4.78}$$

$$\zeta=\lambda-nm_\psi+mm_\phi. \tag{4.79}$$

We can then use the results of [85], where the matching of solutions of this equation in an inner and outer region was carried out in detail, to determine the reflection coefficient. This reflection coefficient can be used to determine the time Δt it takes for a quantum scattering from the core region near $x=0$ to return to the asymptotic region, by expanding $\mathcal{R}=a+b\sum_n e^{2\pi i n\frac{\omega}{R}\Delta t}$. Their matching procedure is valid when

$$\sigma^2\gg 1 \tag{4.80}$$

and

$$\Delta t\gg\frac{R}{(\omega^2-\lambda^2)^{1/2}}. \tag{4.81}$$

Under these assumptions, their matching procedure gives

$$\Delta t=\pi R_s\varrho, \tag{4.82}$$

where R_s is the radius (4.41) for a smooth solution; in the orbifolds, $R=R_s/k$. We note that this is in agreement with their result in the supersymmetric case, as in the limit $\delta_1,\delta_5,\delta_p\to\infty$,

$$\varrho=\frac{s_1^2s_5^2+s_1^2s_p^2+s_5^2s_p^2+s_1^2+s_5^2+s_p^2+1}{s_1c_1s_5c_5}\approx\frac{Q_1Q_5+Q_1Q_p+Q_5Q_p}{Q_1Q_5}=\frac{1}{\eta} \tag{4.83}$$

in the notation of [86].

In the CFT picture, this travel time is interpreted as the time required for two CFT modes on the brane to travel around its worldvolume and meet again. Thus, from the CFT point of view, the expected value is $\Delta t_{CFT}=\pi R_s$. As in [86], there is a 'redshift factor' ϱ between our spacetime result and the expected answer from the CFT point of view. It was argued in [86] that such a factor must appear to make the spacetime result invariant under permutation of the three charges, and it was proposed that this factor could be understood as a scaling between the asymptotic time coordinate t in the asymptotically flat space and the time coordinate appropriate to the CFT. Evidence for this point of view was found by noting that in the cases where the soliton had a large $AdS_3 \times S^3$ core region, the global AdS time τ was proportional to ηt, so $\Delta\tau = \pi R_s$ in accordance with CFT expectations. In our non-supersymmetric case, for fixed m, n, the appropriate limit in which we obtain a large AdS region is the limit $\delta_1, \delta_5 \gg 1$ for fixed δ_p considered in section 4.3.4. We did not see any such scaling between the AdS and asymptotic coordinates there, but $\varrho \approx 1$ in this limit, so this is consistent with the interpretation proposed in [86]. However, we remain uncomfortable with this interpretation. It is hard to argue directly for such a redshift between the CFT and asymptotic time coordinates in the general case where the soliton does not have a large approximately $AdS_3 \times S^3$ core. Indeed, in the dual brane picture of the geometry, where we have a collection of D1 and D5 branes in a flat background, one would naïvely expect the two to be the same. A deeper understanding of this issue could shed interesting light on the limitations of the identification between CFT states and the asymptotically flat geometries.

4.6.2 Ergoregion

Although our soliton solutions are free of event horizons, they typically have ergoregions. These already appear in the supersymmetric three-charge soliton solutions studied in [85, 86], where the Killing vector ∂_t, which defines time-translation in the asymptotic rest frame, becomes spacelike at $f = 0$ if $Q_p \neq 0$. However, in these supersymmetric cases, there is still a causal Killing vector (arising from the square of the covariantly constant Killing spinor), which corresponds asymptotically to the time-translation with respect to some boosted frame. A striking difference in the non-supersymmetric solitons is the absence of any such globally timelike or null

Killing vector field.[5] The most general Killing vector field which is causal in the asymptotic region of the asymptotically flat solutions is

$$V = \partial_t + v^y \partial_y \tag{4.84}$$

for $|v^y| \leq 1$. However, when $f = 0$, the norm of this Killing vector is

$$|V|^2 = \frac{M}{\sqrt{\tilde{H}_1 \tilde{H}_5}} (c_p - v^y s_p)^2 > 0. \tag{4.85}$$

The best we can do is to take $v^y = \tanh \delta_p$, for which this Killing vector is timelike for $f > M$. Note that as a consequence, the two-charge non-supersymmetric solutions also have ergoregions.

In a rotating black hole solution, the existence of an ergoregion typically implies a classical instability when the black hole is coupled to massive fields [139,140]. This instability arises when we send in a wavepacket which has positive energy less than the rest mass with respect to the asymptotic Killing time, but negative energy in the ergoregion. The wavepacket will be partially absorbed by the black hole, but because the absorbed portion has negative energy, the reflected portion will have a larger amplitude. This then reflects off the potential at large distances, and repeats the process. This process causes the amplitude of the initial wavepacket to grow indefinitely, until its back-reaction on the geometry becomes significant.

One might have thought that in the supersymmetric three-charge solitons, the instability would not appear as a consequence of the existence of a causal Killing vector, by a mechanism similar to that discussed in [145] for Kerr-AdS black holes. However, this instability is in fact absent for a different reason, which applies to both supersymmetric and non-supersymmetric solitons. The instability in black holes is a result of the existence of both an ergoregion and an event horizon, so in the solitons, the absence of an event horizon can prevent such an instability from occurring. Indeed, from the discussion of the massless wave equation in the previous section, we can see that the net flux is always zero, and the amplitude of

[5]For the asymptotically AdS spacetimes, there is a globally timelike Killing vector field, given by ∂_t at fixed $\tilde{\psi}$, $\tilde{\phi}$. In (t, y, ψ, ϕ) coordinates, this is of the form $V' = \ell \partial_t - m\partial_\psi + n\partial_\phi$, so it cannot be extended to a globally timelike Killing vector field in the asymptotically flat geometry.

the reflected wave is the same as that of the incident wave. That is, although there is an ergoregion, no superradiant scattering of classical waves occurs in this geometry, and the mechanism that led to the black hole bomb does not apply here. There might be an instability if we considered some interacting theory, as the interactions might convert part of an incoming wavepacket to negative-energy modes bound to the soliton, but we will not attempt to explore this issue in more detail.

Thus, for free fields, there is no stimulated emission at the classical level. We will now show that there is also no spontaneous quantum emission.[6] There is a natural basis of modes for this geometry; for the scalar field, (4.70). To establish which of these modes are associated with creation and which with annihilation operators, we need to consider the Klein-Gordon norm

$$(\Psi,\Psi) = \frac{i}{\hbar}\int_\Sigma d^5x\sqrt{h}n_\mu g^{\mu\nu}(\bar{\Psi}\partial_\nu\Psi - (\partial_\nu\bar{\Psi})\Psi), \tag{4.86}$$

where Σ is a Cauchy surface, say for simplicity a surface $t = t_0$, and n_μ is the normal $n_\mu = \partial_\mu t$. The modes of positive norm, $(\Psi,\Psi) > 0$, correspond to creation operators, while those of negative norm, $(\Psi,\Psi) < 0$, correspond to annihilation operators. Because of the complicated form of the inverse metric (see appendix C), it is difficult to establish explicitly which are which. However, the main point is that we can define a vacuum state by requiring that it be annihilated by the annihilation operators corresponding to all the negative frequency modes in (4.70). This will then be the unique vacuum state on this geometry. Since the modes (4.70) are eigenmodes of both the asymptotic time-translation ∂_t and of the timelike Killing vector in the near-core region,

$$V' = \ell\partial_t - m\partial_\psi + n\partial_\phi, \tag{4.87}$$

these will be the appropriate family of creation and annihilation operators for observers in both regions. That is, these observers who follow the orbits of the Killing symmetries will detect no particles in this state.

Thus, at the level of free fields, the solitons do not suffer from superradiance at either the classical or quantum level.

[6]We thank Don Marolf for pointing out that the argument for non-trivial quantum radiation in the original version of [5] was erroneous, and for explaining the following argument to us.

Chapter 5

Uniqueness of charged Kerr-AdS$_5$ black holes

The question of black hole uniqueness in higher dimensions has been attracting considerable attention since the discovery by Emparan and Reall [147] of an asymptotically flat black ring in five dimensions. This is a solution with a regular event horizon of topology $S^2 \times S^1$, supported against collapse by angular momentum. Since there is a rotating black hole which carries the same mass and angular momentum as this solution, this represents a breakdown of the usual no-hair behaviour: the solution is not uniquely determined by the asymptotic conserved charges. Studies of charged rings have uncovered further examples of non-uniqueness [148–152]. So far, all these examples involve discrete forms of non-uniqueness: the existence of some finite number of solutions with the same asymptotic charges. Although the classical solutions in [149] involved a continuous parameter, this parameter can be physically interpreted in terms of a local charge carried by the ring, so it will be quantized in the fundamental theory. Furthermore, bounds on this charge for the existence of a regular event horizon imply that these solutions give a finite number of solutions with given energy. This is consistent with the expectation that the fundamental quantum theory has a finite number of states of given energy in finite volume. We would expect that in general distinct classical solutions must correspond to different quantum states.

A new example with an apparent continuous non-uniqueness was recently found in [1,2]. They constructed solutions describing rotating charged black holes in five-dimensional gauged supergravity, with the two angular momentum parameters set equal. Studying first a case with a single $U(1)$ gauge field [1], they found solutions with four parameters: the mass, charge, angular momentum, and one additional parameter. In [2], they extended this to a $U(1)^3$ theory with independent charges and found a solution depending on six parameters. The solutions thus appeared to involve a continuous non-uniqueness, which initially appeared to have physical consequences. In light of the previous discussion, we want to understand the physical significance of this extra parameter, to see how the apparent contradiction with our expectation that there should be a finite number of solutions of given energy is resolved. In addition, these solutions appear to provide a first example of non-uniqueness involving black holes with a spherical horizon topology, so it would clearly be interesting to understand them in more detail.

In this chapter, we establish that the extra parameter in the solutions of [1, 2] is unphysical, representing a purely coordinate degree of freedom. In [1], it was observed that if the charge Q vanished, the additional parameter could be removed by a coordinate transformation and redefinition of the other parameters. In section 5.1, we extend this to the case with $Q \neq 0$. We discuss the extremal limits in terms of these parameters, showing how the solutions of [153, 154] are recovered in our parametrization, and briefly discuss the reduction to other known solutions. We also discuss the possibility of discrete non-uniqueness, and show that although the relation between our parametrization of the solutions and the physical mass and charges is non-linear, there is only one black hole solution for given mass, angular momentum and charges. We present a similar argument for the $U(1)^3$ solutions of [2] in section 5.2. Thus, these solutions do not in fact present new examples of non-uniqueness.

5.1 Charged Kerr-de Sitter Black Holes in five dimensions

We consider first the solution of the minimal gauged supergravity obtained in [1], describing a charged rotating black hole with a cosmological constant. The solution can be written in the simplest form by introducing the left-invariant one-forms σ_i on S^3,

$$\sigma_1 = \cos\psi d\theta + \sin\psi\sin\theta d\phi, \quad (5.1)$$

$$\sigma_2 = -\sin\psi d\theta + \cos\psi\sin\theta d\phi, \quad (5.2)$$

$$\sigma_3 = d\psi + \cos\theta d\phi. \quad (5.3)$$

The solution then takes the form

$$\mathrm{d}s^2 = -\frac{r^2W}{4b^2}\mathrm{d}t^2 + \frac{1}{W}\mathrm{d}r^2 + \frac{r^2}{4}(\sigma_1^2+\sigma_2^2) + b^2(\sigma_3 + f\mathrm{d}t)^2, \quad (5.4)$$

$$A = \frac{\sqrt{3}Q}{r^2}(\mathrm{d}t - \frac{1}{2}J\sigma_3), \quad (5.5)$$

where

$$b^2 = \frac{r^2}{4}\left[1 - \frac{J^2Q^2}{r^6} + \frac{2J^2(M+Q)}{r^4}\right], \quad (5.6)$$

$$f = -\frac{J}{2b^2}\left(\lambda\beta r^2 + \frac{2M+Q}{r^2} - \frac{Q^2}{r^4}\right), \quad (5.7)$$

$$W = 1 - \lambda r^2 - \frac{1}{r^2}\left[2\lambda J^2(M+Q) + 2(1-\lambda\beta J^2)^2(M+Q) - 2Q(1-\lambda\beta J^2)\right] + \frac{1}{r^4}\left\{(1-\lambda\beta J^2)^2Q^2 + J^2[\lambda Q^2 + 2(M+Q)]\right\}. \quad (5.8)$$

This is a solution of minimal gauged five-dimensional supergravity. It appears to depend on four parameters, (M, J, Q, β). The first three can be related to the mass, angular momentum and charge, but the physical interpretation of the fourth is obscure: we will show that this solution can in fact be written in terms of only three parameters.

The first step is to transform to a frame in which the metric is non-rotating at infinity, so that it approaches the usual diagonal form of the AdS metric at large distances, by making the shift

$$\tilde{\sigma}_3 = \sigma_3 + 2\lambda\beta J\mathrm{d}t. \quad (5.9)$$

We then have

$$\begin{aligned} ds^2 &= -\frac{r^2W}{4b^2}dt^2 + \frac{1}{W}dr^2 + \frac{r^2}{4}(\sigma_1^2+\sigma_2^2) + b^2(\tilde\sigma_3 + \tilde f dt)^2, && (5.10)\\ A &= \frac{\sqrt{3}Q}{r^2}\left[(1-\lambda\beta J^2)dt - \frac{1}{2}J\tilde\sigma_3\right], && (5.11)\end{aligned}$$

where

$$\tilde f = -\frac{J}{2b^2}\left[\frac{2(M+Q)(1-\lambda\beta J^2) - Q}{r^2} - \frac{Q^2(1-\lambda\beta J^2)}{r^4}\right]. \quad (5.12)$$

It is convenient to exchange β for a new parameter δ (as in [155])

$$\delta = (1-\lambda\beta J^2). \quad (5.13)$$

We then see that the gauge field becomes

$$A = \frac{\sqrt{3}\delta Q}{r^2}dt - \frac{\sqrt{3}QJ}{2r^2}\tilde\sigma_3, \quad (5.14)$$

so the physical gauge charge is related to δQ. We also see that $\tilde f$ involves the parameters only in the combinations $J\delta(M+Q)$, δQ and JQ. This suggests that we define the following new parameters:

$$\begin{aligned} q &= Q\delta, && (5.15)\\ p &= \delta^2(M+Q), \\ j &= \frac{J}{\delta}. && (5.16)\end{aligned}$$

We then find that the solution depends only on the three parameters (q, j, p), and has no remaining dependence on δ:

$$\begin{aligned} ds^2 &= -\frac{r^2W}{4b^2}dt^2 + \frac{1}{W}dr^2 + \frac{r^2}{4}(\sigma_1^2+\sigma_2^2) + b^2(\tilde\sigma_3 + \tilde f dt)^2, && (5.17)\\ A &= \frac{\sqrt{3}q}{r^2}dt - \frac{\sqrt{3}qj}{2r^2}\tilde\sigma_3, && (5.18)\end{aligned}$$

where

$$\begin{aligned} b^2 &= \frac{r^2}{4}\left(1 - \frac{j^2q^2}{r^6} + \frac{2j^2p}{r^4}\right), && (5.19)\\ \tilde f &= -\frac{j}{2b^2}\left(\frac{2p-q}{r^2} - \frac{q^2}{r^4}\right), && (5.20)\\ W &= 1 - \lambda r^2 - \frac{1}{r^2}(2\lambda j^2 p + 2p - 2q) + \frac{1}{r^4}\left[q^2 + j^2(\lambda q^2 + 2p)\right]. && (5.21)\end{aligned}$$

It is perhaps worth noting that this expression for W can be rewritten as

$$W = 1 - 4\lambda b^2 - \frac{1}{r^2}(2p - 2q) + \frac{1}{r^4}(q^2 + 2pj^2). \tag{5.22}$$

Thus, we see that the solution actually only depends on three parameters, corresponding to the three conserved quantities (M, Q, J). The relation of two of the parameters to these conserved quantities is fairly direct: from the form of the gauge potential, we see that

$$Q = q. \tag{5.23}$$

The angular momentum J can be calculated using the Komar integral technique and is given by

$$J = \frac{\pi}{4} j(2p - q), \tag{5.24}$$

so j is analogous to the angular momentum per unit mass parameter a in the usual Kerr solution.

The remaining parameter p is related to the freedom to specify the mass, but in a less simple way. We can define a thermodynamic mass as in [156] by insisting that it satisfy the first law of thermodynamics for charged rotating black holes,

$$\mathrm{d}M = T\mathrm{d}S + 2\Omega_H \mathrm{d}J + \Phi_H \mathrm{d}Q, \tag{5.25}$$

where Φ_H is the co-rotating electric potential evaluated on the horizon. This gives the mass

$$M = \frac{\pi}{4}(3p - 3q - \lambda p j^2). \tag{5.26}$$

In [157] the mass was calculated via the conformal boundary approach of Ashtekar, Das and Magnon and found to be in agreement with this thermodynamic mass. Note that although this looks like a linear relation, if we solve (5.24) for j in terms of J, (5.26) will become a cubic equation for p in terms of M.

The reduction of this solution to previously known metrics was discussed at length in [1]. We will briefly revisit this issue to illuminate our parametrization of the solution. Considering first the known BPS solutions, we note that the form above reduces to the solution in [153] if we set $p = 0$ (after a redefinition of the radial coordinate, $r^2 \to r^2 - q$), so this choice of parameters is well-adapted to this limit, while recovering the solution of [154] requires a more complicated choice: To recover their solution, we write $\lambda = -1/\ell^2$ and set

$$q = \left(1 + \frac{R_0^2}{2\ell^2}\right) R_0^2, \tag{5.27}$$

$$p = 2\left(1 + \frac{R_0^2}{2\ell^2}\right)^2 R_0^2, \tag{5.28}$$

$$j = \frac{\epsilon l R_0^2}{2}\left(1 + \frac{R_0^2}{2\ell^2}\right)^{-1}. \tag{5.29}$$

From the results of [154], we see that for this choice of parameters, the solution has a degenerate horizon (a double root of $W = 0$) at $r = R_0$. The condition that $W = 0$ have a double root at $r = R_0$ in general implies

$$2\lambda j^2 p + 2p - 2q = 2R_0^2 - 3\lambda R_0^4 \tag{5.30}$$

and

$$q^2 + j^2(\lambda q^2 + 2p) = R_0^4 - 2\lambda R_0^2. \tag{5.31}$$

Since these are two conditions on the three parameters (q, p, j), there is a family of solutions with degenerate horizons with one extra parameter, generalising the solution found in [154]. However, from the analysis in [154], we expect that only the solution found there is BPS.

The other simple special cases are when $j = 0, q = 0$ or $\lambda = 0$. When $j = 0$, if we set $p = m + q$, the solution reduces to the RNAdS black hole studied in [158]. When $q = 0$, if we set

$$\lambda = -l^2, \quad j = a, \quad p = \frac{m}{(1 - a^2 l^2)^3} \tag{5.32}$$

the solution reduces to a special case of the five-dimensional Kerr-AdS metric obtained in [159], where the two angular momentum parameters are equal, $a = b$. Relating the form of the metric used in [159] to (5.17) requires a shift of the angular coordinate to make the metric in [159] asymptotically diagonal and a redefinition of the radial coordinate, $r^2 \to r^2 + a^2$. When $\lambda = 0$, the solution is related to a special case of the general solution obtained in [160] where the two angular momentum parameters are equal, $l_1 = l_2 = l$, and the three charge parameters are equal, $\delta_e = \delta_{e1} = \delta_{e2}$. The relation between the parameters is

$$q = m \sinh 2\delta_e, \quad j = l e^{-\delta_e}, \quad p = m e^{2\delta_e}. \tag{5.33}$$

Relating the metric in [160] to (5.17) again requires a redefinition of the radial coordinate, $r^2 \to r^2 + l^2 + 2m\sinh^2\delta_e$.

As discussed in [1], the physically interesting solutions are those with a regular event horizon and no closed timelike curves outside the horizon. That is, we want to consider parameter values for which there is some r_+ such that $W(r_+) = 0$, $W'(r_+) \geq 0$ (so that r_+ is the outer event horizon) and $b^2(r_+) > 0$. Written explicitly in terms of our parameters, these conditions are

$$r_+^4(1 - 4\lambda b_+^2) - 2r_+^2(p - q) + q^2 + 2pj^2 = 0, \tag{5.34}$$

where we have introduced the notation $b_+^2 \equiv b(r_+)^2$,

$$-\lambda r_+^6 + 2r_+^2(\lambda j^2 p + p - q) - 2[q^2 + j^2(\lambda q^2 + 2p)] \geq 0, \tag{5.35}$$

and

$$r_+^6 + 2j^2 p r_+^2 - j^2 q^2 > 0. \tag{5.36}$$

These conditions will impose some constraints on the values of (p, j, q). For example, the requirement that (5.34) have a real positive root for r_+^2 implies that

$$(p - q)^2 \geq (q^2 + 2pj^2)(1 - \lambda b_+^2). \tag{5.37}$$

Unfortunately, although our coordinate transformation and redefinition of parameters simplifies the functions somewhat, it is still difficult to analyse the full set of constraints.

An alternative approach is to use the above relations to replace the parameters j and p in the metric by r_+ and b_+^2, thereby automatically incorporating the constraint $b^2(r_+) > 0$ for the absence of naked closed timelike curves. We can easily determine the relation between j and b_+,

$$j^2 = \frac{r_+^4(4b_+^2 - r_+^2)}{(2pr_+^2 - q^2)} \tag{5.38}$$

and substituting this into (5.34) then gives a quadratic equation to solve for p, determining it in terms of r_+, b_+ and q. The resulting form of the metric is, unfortunately, rather messy and unenlightening, and it would still be necessary to somehow incorporate the constraint that $W'(r_+) \geq 0$, which will restrict the possible values of r_+ and b_+ for given q.

So far we have discussed continuous non-uniqueness. We should also consider the possibility of discrete non-uniqueness[1]. As noted previously, (5.26) gives a cubic equation for p as a function of the physical parameters M, J, Q, so it is possible that we will have more than one black hole solution for a given mass and charge. If we define the dimensionless quantities

$$\begin{aligned} \gamma &= \lambda(2p-q),\ \tilde{M} = \frac{4\lambda}{\pi}M, \\ \tilde{J} &= \frac{4\lambda^2}{\pi}J,\ \tilde{Q} = \lambda Q \end{aligned} \tag{5.39}$$

then (5.24) tells us

$$j = \frac{\tilde{J}}{\lambda\gamma}, \tag{5.40}$$

and substituting this into (5.26) gives

$$\gamma^3 - (2\tilde{M} + 3\tilde{Q})\gamma^2 + \tilde{J}^2\gamma + \tilde{J}^2\tilde{Q} = 0. \tag{5.41}$$

This is a cubic equation in γ so for each (M, J, Q) there are possibly three different solutions. However, for a solution to correspond to a black hole, $W(r_+) = 0$ must have at least one positive real root, at which $b^2(r_+) > 0$. We performed a numerical analysis to check whether there are values of (M, J, Q) for which more than one of the roots of (5.41) satisfy these conditions. We found that at most one root of (5.41) has a black hole interpretation (see fig 5.1, fig 5.2 for representative plots), thus ruling out any discrete non-uniqueness.

5.2 $U(1)^3$ case

In a further paper [2], the previous solution was generalised to a class of non-extremal charged rotating black hole solutions in the five dimensional $U(1)^3$ gauged theory of $\mathcal{N} = 2$ supergravity coupled to two vector multiplets. This theory now has three gauge fields A^i, $i = 1, 2, 3$, and two scalars φ_1, φ_2. The solutions they obtain can be

[1] We thank the referee of [6] for raising this issue

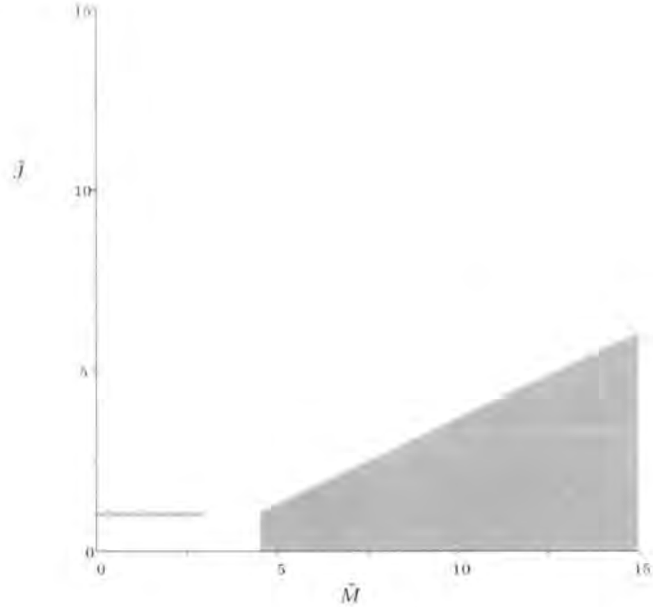

Figure 5.1: Parameters for which we have either one black hole solution (shaded regions) or no solutions with horizons (white regions) for $\tilde{Q} = -1$.

written as

$$ds^2 = -\frac{RY}{f_1}dt^2 + \frac{Rr^2}{Y}dr^2 + \frac{1}{4}R(\sigma_1^2 + \sigma_2^2) + \frac{f_1}{4R^2}(\sigma_3 - 2\frac{f_2}{f_1}dt)^2 \quad (5.42)$$

$$A^i = \frac{\mu}{r^2 H_i}\left[s_i c_i dt - \frac{1}{2}l(c_i s_j s_k - s_i c_j c_k)\sigma_3\right]. \quad (5.43)$$

$$e^{\frac{2}{\sqrt{6}}\varphi_1} = X_3, \quad e^{\sqrt{2}\varphi_2} = \frac{X_2}{X_1}, \quad (5.44)$$

where

$$X_i = \frac{R}{r^2 H_i}, \quad i = 1, 2, 3 \quad (5.45)$$

$$R = r^2\Big(\prod_i^3 H_i\Big)^{\frac{1}{3}}, \quad H_i = 1 + \frac{\mu s_i^2}{r^2}, \quad (5.46)$$

s_i and c_i are shorthand for

$$s_i \equiv \sinh\delta_i, \quad c_i \equiv \cosh\delta_i, \quad (5.47)$$

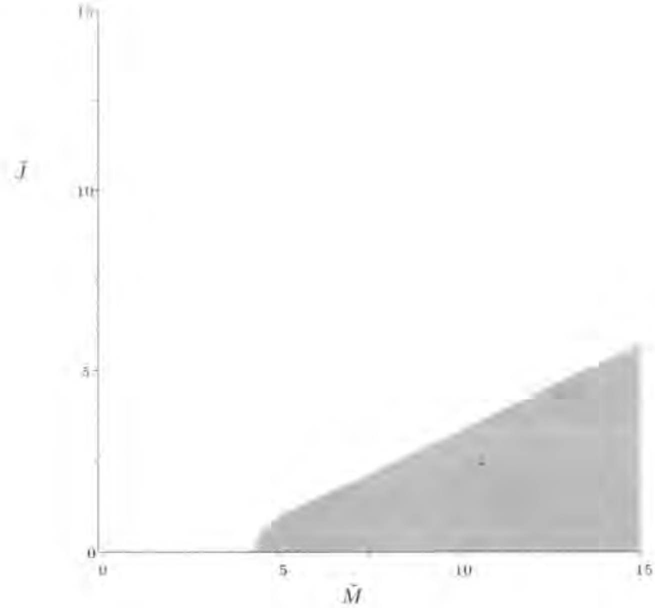

Figure 5.2: Parameters for which we have either one black hole solution (shaded regions) or no solutions with horizons (white regions) for $\tilde{Q} = +1$.

and

$$\begin{aligned} f_1 &= R^3 + \mu l^2 r^2 + \mu^2 l^2 \left[2(\prod_i c_i - \prod_i s_i) \prod_j s_j - \sum_{i<j} s_i^2 s_j^2 \right], \\ f_2 &= \gamma l \lambda R^3 + \mu l (\prod_i c_i - \prod_i s_i) r^2 + \mu^2 l \prod_i s_i, \end{aligned} \tag{5.48}$$

$$\begin{aligned} f_3 &= \gamma^2 l^2 \lambda^2 R^3 + \mu l^2 \lambda \left[2\gamma(\prod_i c_i - \prod_i s_i) - 1 - \gamma^2 l^2 \lambda \right] r^2 + \mu l^2 \\ &\quad - \lambda(1 + \gamma^2 l^2 \lambda) \mu^2 l^2 \left[2(\prod_i c_i - \prod_i s_i) \prod_j s_j - \sum_{i<j} s_i s_j \right] + 2\lambda\gamma\mu^2 l^2 \prod_i s_i, \end{aligned} \tag{5.49}$$

$$Y = f_3 - \lambda(1 + \gamma^2 l^2 \lambda) R^3 + r^4 - \mu r^2. \tag{5.50}$$

This solution seems to depend on six non-trivial parameters $(\mu, \delta_1, \delta_2, \delta_3, l, \gamma)$. Our aim is to show that this depends on only five independent parameters. Again we begin by moving to coordinates in which the metric is asymptotically diagonal by setting

$$\tilde{\sigma}_3 = \sigma_3 + 2\gamma l \lambda \mathrm{d}t. \tag{5.51}$$

We then have

$$\begin{aligned} ds^2 &= -\frac{RY}{f_1}dt^2 + \frac{Rr^2}{Y}dr^2 + \frac{1}{4}R(\sigma_1^2 + \sigma_2^2) + \frac{f_1}{4R^2}(\tilde{\sigma}_3 - 2\frac{\tilde{f}_2}{f_1}dt)^2, & (5.52) \\ A^i &= \frac{\mu}{r^2 H_i}\left\{[s_i c_i + \gamma\lambda l^2(c_i s_j s_k - s_i c_j c_k)]dt + \frac{1}{2}l(c_i s_j s_k - s_i c_j c_k)\tilde{\sigma}_3\right\}, & (5.53) \end{aligned}$$

where

$$\begin{aligned} \tilde{f}_2 &= f_2 - \gamma l \lambda f_1 & (5.54) \\ &= \left[\mu l(\prod_i c_i - \prod_i s_i) - \gamma\mu l^3\lambda\right] r^2 + \mu^2 l \prod_i s_i \\ &\quad -\gamma l\lambda\mu^2 l^2 \left[2(\prod_i c_i - \prod_i s_i)\prod_j s_j - \sum_{i<j} s_i^2 s_j^2\right]. \end{aligned}$$

The radial coordinate used here is different from that used in the previous case: the singularity in this metric will occur at $r^2 = -\mu s_i^2$, where δ_i is the smallest of the charge parameters, and not at $r = 0$ as before. We are therefore motivated to make a change of radial coordinate to a new radial coordinate ρ,

$$r^2 = \rho^2 - \frac{1}{3}\sum_i \mu s_i^2. \quad (5.55)$$

With this new choice of radial coordinate,

$$X_i = \frac{R}{\rho^2 \tilde{H}_i}, \quad (5.56)$$

where

$$R = \rho^2\Big(\prod_i^3 \tilde{H}_i\Big)^{\frac{1}{3}}, \quad \tilde{H}_i = 1 + \frac{\mu(s_i^2 - s_j^2) + \mu(s_i^2 - s_k^2)}{3\rho^2}, \quad (5.57)$$

so the scalar fields, which are determined by the X_i, depend on the parameters only through the combinations $\mu(s_i^2 - s_j^2)$.

As in the previous case, we will find suitable parameters by examining the gauge fields and the function $\tilde{f}_2$ appearing in the asymptotic form of the metric. We are thereby led to define the five independent parameters

$$\begin{aligned} L &= \sqrt{\mu}l, & (5.58) \\ \Gamma &= \sqrt{\mu}\left[(\prod_i c_i - \prod_i s_i) - \gamma l^2\lambda\right], & (5.59) \\ r_i &= \sqrt{\mu}(c_i s_j s_k - s_i c_j c_k). & (5.60) \end{aligned}$$

This may seem an awkward definition, but the r_i in particular have several nice properties:

$$r_i^2 - r_j^2 = \mu(s_i^2 - s_j^2), \tag{5.61}$$

so the combinations $\mu(s_i^2 - s_j^2)$ appearing in the $\tilde{H}_i$, and hence in the scalar fields, can be written as $(r_i^2 - r_j^2)$. Also,

$$\mu\left[s_i c_i + \gamma\lambda l^2(c_i s_j s_k - s_i c_j c_k)\right] = r_j r_k - \Gamma r_i, \tag{5.62}$$

so the gauge fields can be written as

$$A^i = \frac{r_j r_k - \Gamma r_i}{\rho^2 \tilde{H}_i} dt + \frac{L r_i}{2\rho^2 \tilde{H}_i}\tilde{\sigma}_3. \tag{5.63}$$

Finally, the metric in the new coordinates is

$$\mathrm{d}s^2 = -\frac{RY}{f_1}\mathrm{d}t^2 + \frac{R\rho^2}{Y}\mathrm{d}\rho^2 + \frac{1}{4}R(\sigma_1^2 + \sigma_2^2) + \frac{f_1}{4R^2}(\tilde{\sigma}_3 - 2\frac{\tilde{f}_2}{f_1}\mathrm{d}t)^2, \tag{5.64}$$

and after a certain amount of calculation, it is possible to rewrite f_1, f_2 and Y as

$$f_1 = R^3 + L^2\left(\rho^2 - \frac{1}{3}\sum_i r_i^2\right), \tag{5.65}$$

$$\tilde{f}_2 = L\left(\Gamma\rho^2 - \frac{1}{3}\Gamma\sum_i r_i^2 + r_1 r_2 r_3\right), \tag{5.66}$$

$$\begin{aligned} Y &= -\lambda R^3 + \rho^4 + \Big(\frac{1}{3}\sum_i r_i^2 - \lambda L^2 - \Gamma^2\Big)\rho^2 + \frac{1}{3}\Gamma^2\sum_i r_i^2 - 2\Gamma r_1 r_2 r_3 \\ &\quad + \frac{1}{3}\lambda L^2 \sum_i r_i^2 + L^2 + \left[\frac{5}{18}(\sum_i r_i^2)^2 - \frac{1}{2}\sum_i r_i^4\right]. \end{aligned} \tag{5.67}$$

The solution therefore depends only on five independent parameters, and a metric will be uniquely fixed by specifying the mass, angular momentum and three gauge charges. That is, this case is qualitatively the same as in the previous section. A useful consistency check is that when the three charges are equal, this reduces to the previous metric on setting

$$L = \sqrt{2pj}, \quad \Gamma = -\frac{1}{\sqrt{2p}}(2p - q), \quad r_i = \frac{q}{\sqrt{2p}}. \tag{5.68}$$

The conserved quantities for this solution were calculated in [132] and found to be

$$\begin{aligned} M &= \frac{\pi}{8}(3\Gamma^2 - \sum_i r_i^2 - \lambda L^2), \\ J &= \frac{\pi}{4}\Gamma L, \\ Q_i &= \frac{\pi}{4}(r_j r_k - \Gamma r_i), \end{aligned} \tag{5.69}$$

where the mass is defined to satisfy the first law of thermodynamics. There is still the possibility of some discrete non-uniqueness in this case, but we doubt this possibility is realised in practice.

Chapter 6

Discussion

In chapter 3 we classified the possible discrete cyclic quotients of AdS_{p+1}. A feature of this classification is that the majority of the classes exist for all $p \geq 2$. The description of the quotient in the majority of cases is thus a simple generalisation of the AdS_3 quotients. There are two special classes which appear in higher dimensions: one for $p \geq 3$ and one for $p \geq 4$. As we increase p above 4 we only have the freedom to add $SO(2)$ rotations to our existing actions. However when we go to higher dimensions we can get new examples by considering quotients generated by more than one Killing vector. We analysed in detail the properties of those quotients with a regular causal structure. One disappointment was that none of these quotients can be considered time-dependent, they all possess a causal Killing vector.

Our general approach to selecting coordinates was to decompose our Killing vector into an $\mathfrak{so}(2,k)$ part, and a series of $\mathfrak{so}(2)$ rotations in independent planes then adapt the coordinates according to the unbroken symmetries in $\mathfrak{so}(2,k)$. Following the work on further identifications in section 3.5, it might have been enlightening to instead adapt coordinates to the unbroken symmetries of $\mathfrak{so}(2,p)$ when we analyse a quotient in higher than minimal dimension.

An arresting feature of the double null rotation quotients is that they are conformally related to a compactified plane wave. One could attempt to utilise this plane wave description in studying the relation between AdS and the field theory dual. Specifically when we break conformal symmetry in the CFT by going to finite temperature, we would hope to find a black hole solution in the bulk which asymptotically approaches AdS_5 in these coordinates. The everywhere timelike Killing vector of the double null rotation ensures this would be a rotating black hole with the two rotation parameters set equal, i.e. the solutions studied in chapter 5 with some restrictions on the parameters. Similar comments apply to the extended double null rotation and the asymptotically AdS_7 black holes found in [161].

In chapter 4 we found new non-supersymmetric soliton solutions in the D1-D5 system, and identified corresponding states in the CFT. The existence of these solitons, and the fact that they can be identified with states in the dual CFT, might be regarded as further evidence for the description of black holes advanced by Mathur and collaborators. However, it is still questionable whether we can really describe a black hole in this way. First of all, the three-charge states described so far are very special. The orbifolds we consider provide examples where the CFT state is not the spectral flow of a RR ground state, but the geometries we consider all have a $U(1) \times U(1)$ invariance. It is unclear whether the techniques used to date can be extended to obtain even the geometries corresponding to spectral flows of the more general RR ground states of [80, 126], let alone to reproduce the full $e^{\sqrt{n_1 n_5 n_p}}$ states required to explain the black hole entropy. The much more difficult dynamical questions — how the appearance of a global event horizon in gravitational collapse can always be avoided, for example — have not yet been tackled. Nonetheless, the study of these smooth geometries offers a new perspective on the relation between CFT and spacetime, and it is interesting to see that their existence does not depend on supersymmetry.

There are two corresponding classes of issues for further investigation: further study of the geometry itself, and elucidating the relation to the dual CFT. In the first category, the classical stability of these solitons as solutions in IIB supergravity should be checked. As we discussed in section 4.6.2, although they have ergoregions, the usual black hole bomb instability will be absent at least for free fields, as there is no net flux in a scattering off the geometry. It would be interesting to study stability more generally; in particular, it would be interesting to know if the geometry suffers from a Gregory-Laflamme [144] type instability if we make the torus in the z^i directions large.

It would be interesting to study these solutions as backgrounds for perturbative string theory. They provide new examples of smooth asymptotically flat geometries that do not have a global timelike Killing symmetry, of a rather different character from those presented in [146]. The existence of supersymmetric special cases may be a simplifying feature.

The most important direction of future work to elucidate the relation of these geometries to the dual CFT is to construct explicit CFT descriptions of the states dual to the generic orbifold spacetimes and study their properties from the CFT point of view. The charges for the dual states found in (4.65) show that these states are not simply the spectral flow of some chiral primary, so they do not maximize the R-charge for given conformal dimension. They should therefore be closer to representing the 'typical' behaviour of a CFT state (although they are clearly still very special) and we expect there will be new tests of the relation between geometry and CFT to be explored. It will also be interesting to see what happens in the CFT when we consider the orbifolds with $m+n$ even, k even, where the spacetime is not a spin manifold.

Another important basic issue from this point of view is to understand the appearance of stationary geometries dual to non-supersymmetric states coupled to bulk modes. We would have expected that the CFT states would decay by the emission of bulk closed string modes. Even in the simple cases where the near-core geometry is global $AdS_3 \times S^3$, the corresponding CFT state carries comparable numbers of left and right-moving excitations, which we would expect can interact to produce bulk gravitons. This physics does not seem to be represented in our dual geometries. It will be important to study the decay of these non-supersymmetric states in more detail, and to try to understand the relation to the soliton.

It would be interesting to try to find asymptotically AdS_5 generalizations of these solitons, by taking the solutions of chapter 5 and systematically searching the parameter space. We expect regular solutions to be possible in this case following the studies of black holes in gauged supergravities in [132]. In AdS it might be possible to find non-supersymmetric solitons with a globally timelike Killing vector. This is known to be possible for some Kerr black holes in AdS [145, 159].

In chapter 5, we showed that the extra parameter in the solutions of [1, 2] is unphysical. Thus, these solutions do not in fact present new examples of non-uniqueness. However, as stressed in [1,2], they do provide interesting testing grounds for the AdS/CFT correspondence, and generalise known solutions in interesting ways. In particular, they provide non-extreme versions of the interesting supersymmetric asymptotically AdS solution found in [154].

Appendix A

Symmetry-adapted coordinates for nullbranes

One possible resolution of the singularity that appears in the parabolic orbifold is to add a compact spacelike transverse direction to the action of the $U(1)$ subgroup used to identify points. The resulting orbifold is the null brane introduced in [39]. As a byproduct of our investigations of the quotients of Anti-de Sitter space in this paper—most particularly, the studies of the double null rotations in section 3.4.3—we were led to realise that there is a rich structure of symmetries in the nullbrane which has not been fully exploited in previous work on these solutions. The nullbrane is a quotient of flat $\mathbb{R}^{1,3}$ by a combination of a null rotation and a translation [39],

$$\xi = \partial_4 - J_{12} + J_{23} = \partial_4 + (x^1 - x^3)\partial_2 + x^2(\partial_1 + \partial_3), \tag{A.1}$$

where x^1 is the timelike coordinate and $\{x^2, x^3, x^4\}$ are spacelike ones. The norm of this Killing vector is $||\xi||^2 = (x^1 - x^3)^2 + 1$, so it is spacelike everywhere. This quotient was shown to be free of closed causal curves in [39]. Furthermore, the presence of ∂_4 removes all fixed points from the action of ξ. The translation does not alter the supersymmetry anlysis from the parabolic orbifold hence the null brane also breaks half the spacetime supersymmetries. There are three Killing vectors in the $\mathfrak{so}(1,3) \ltimes \mathbb{R}^4$ Poincaré algebra on $\mathbb{R}^{1,3}$ which commute with this ξ,

$$\begin{aligned}\xi_1 &= -\partial_4 - J_{12} + J_{23},\\ \xi_2 &= \partial_2 - (J_{14} + J_{34}),\\ \xi_3 &= \partial_1 + \partial_3 \;.\end{aligned} \tag{A.2}$$

These have norms $||\xi_1||^2 = ||\xi_2||^2 = ||\xi||^2$ and $||\xi_3||^2 = 0$. The only non-trivial commutation relation is $[\xi_1, \xi_2] = -2\xi_3$. The coordinates defined on the nullbrane in [39] do not make any of these additional symmetries manifest. We will now construct an adapted coordinate system which makes the ξ_2 and ξ_3 symmetries manifest: that is, we want $\xi = \partial_{\bar\phi}$, $\xi_2 = \partial_{\bar\psi}$ and $\xi_3 = \partial_{\bar v}$. This requires

$$\begin{gathered}\frac{\partial x^1}{\partial\bar\phi} = \frac{\partial x^3}{\partial\bar\phi} = x^2, \frac{\partial x_2}{\partial\bar\phi} = x^1 - x^3, \frac{\partial x^4}{\partial\bar\phi} = 1,\\ \frac{\partial x^1}{\partial\bar\psi} = \frac{\partial x^3}{\partial\bar\psi} = x^2, \frac{\partial x_2}{\partial\bar\psi} = 1, \frac{\partial x^4}{\partial\bar\psi} = x^1 - x^3,\\ \frac{\partial x^1}{\partial\bar v} = \frac{\partial x^3}{\partial\bar v} = 1 \;.\end{gathered} \tag{A.3}$$

Since $x^1 - x^3$ is independent of $\bar\phi, \bar\psi, \bar v$, we will choose to define coordinates so that $x^1 - x^3 = \bar u$. A suitable coordinate system is

$$\begin{aligned}x^1 + x^3 &= 2\bar\phi\bar\psi + \bar u(\bar\phi^2 + \bar\psi^2) + 2\bar v,\\ x^1 - x^3 &= \bar u,\\ x^2 &= \bar\psi + \bar u\bar\phi,\\ x^4 &= \bar\phi + \bar u\bar\psi \;.\end{aligned} \tag{A.4}$$

In these coordinates, the flat metric is

$$g = -2d\bar u d\bar v + (1 + \bar u^2)(d\bar\psi^2 + d\bar\phi^2) + 4\bar u d\bar\phi d\bar\psi \;. \tag{A.5}$$

The nullbrane is constructed by compactifying the $\bar\phi$ coordinate. The determinant of the metric is $-\det g = (1 - \bar u^2)^2$, so this coordinate system breaks down at $\bar u = \pm 1$, where the expressions for x^2 and x^4 lose their linear independence. Thus, although these are symmetry-adapted coordinates, they do not provide global coordinates for the spacetime.

It is interesting to note that in these coordinates, the solution resembles a plane wave written in Rosen coordinates. For the uncompactified solution, this is not unexpected; flat space is a trivial plane wave. The interesting observation is that the compactification of ϕ preserves this structure. By a slight change in the coordinate system, we can make a more direct relation to a non-trivial plane wave, and at the same time obtain global coordinates. Instead of (A.4), we set

$$\begin{aligned} x^1 + x^3 &= 2\phi\psi + u(\phi^2 + \psi^2) + 2v, \\ x^1 - x^3 &= u, \\ x^2 &= \psi + u\phi, \\ x^4 &= \phi - u\psi \ . \end{aligned} \tag{A.6}$$

The flat metric is now

$$g = -2dudv + (1 + u^2)(d\psi^2 + d\phi^2) - 4\psi d\phi du \ . \tag{A.7}$$

The determinant of the metric is $-\det g = (1+u^2)^2$, so this is now a global coordinate system.

The price we pay is that the symmetry ξ_2 is no longer manifest; on the other hand, this form treats the two Killing vectors ξ_1 and ξ_2 more symmetrically. In these coordinates, $\xi = \partial_\phi$, $\xi_3 = \partial_v$, while the other two Killing vectors are

$$\begin{aligned} \xi_1 &= -\frac{1-u^2}{1+u^2}\partial_\phi + \frac{2u}{1+u^2}\partial_\psi + 2\psi\frac{1-u^2}{1+u^2}\partial_v \ , \\ \xi_2 &= \frac{2u}{1+u^2}\partial_\phi + \frac{1-u^2}{1+u^2}\partial_\psi - 2\psi\frac{2u}{1+u^2}\partial_v \ . \end{aligned} \tag{A.8}$$

The inverse coordinate transformation is

$$\begin{aligned} u &= x^1 - x^3, \\ \phi &= \frac{x^4 + (x^1 - x^3)x^2}{[1 + (x^1 - x^3)^2]}, \\ \psi &= \frac{x^2 - (x^1 - x^3)x^4}{[1 + (x^1 - x^3)^2]}, \\ 2v &= (x^1 + x^3) - \frac{(x^1 - x^3)}{1 + (x^1 - x^3)^2}((x^2)^2 + (x^4)^2) \\ &\quad - \frac{2}{[1 + (x^1 - x^3)^2]^2}(x^4 + (x^1 - x^3)x^2)(x^2 - (x^1 - x^3)x^4). \end{aligned} \tag{A.9}$$

The advertised relation to the plane wave can be seen if we now set $u = \tan U$. Then

$$g = \frac{1}{\cos^2 U}[-2dUdv + d\psi^2 + d\phi^2 - 4\psi d\phi dU] \ . \tag{A.10}$$

The metric in square brackets is a conformally flat plane wave. Furthermore, the symmetry $\xi = \partial_\phi$ that we quotient along annihilates the conformal factor, so we can think of the nullbrane as conformally related to a compactified plane wave. The plane wave nature of this solution can be instantly recognised after the further coordinate transformation

$$\begin{aligned} V &= v + \psi\phi, \\ X &= \psi \cos U + \phi \sin U, \\ Y &= -\psi \sin U + \phi \cos U, \end{aligned} \tag{A.11}$$

which brings the metric to the form

$$g = \frac{1}{\cos^2 U}[-2dUdV - (X^2 + Y^2)dU^2 + dX^2 + dY^2]. \tag{A.12}$$

This form makes little of the symmetry explicit. The Killing vector we are quotienting along is

$$\xi = \sin U \partial_X + \cos U \partial_Y + (X \cos U - Y \sin U)\partial_V, \tag{A.13}$$

and the other symmetries of the quotient are

$$\begin{aligned} \xi_1 &= \sin U \partial_X - \cos U \partial_Y + (X \cos U + Y \sin U)\partial_V, \\ \xi_2 &= \cos U \partial_X + \sin U \partial_Y + (-X \sin U + Y \cos U)\partial_V, \\ \xi_3 &= \partial_V. \end{aligned} \tag{A.14}$$

Note that not only does ξ annihilate the conformal factor; so do the other isometries. Thus, all the isometries of the nullbrane are related to isometries of the conformally related compactified plane wave. We can recognise them as

$$\begin{aligned} \xi &= -\xi_{e_1^*} - \xi_{e_2}, \\ \xi_1 &= -\xi_{e_1^*} + \xi_{e_2}, \\ \xi_2 &= -\xi_{e_1} - \xi_{e_2^*}, \\ \xi_3 &= \xi_{e_V}, \end{aligned} \tag{A.15}$$

where we write the isometries of the plane wave in the usual basis

$$\begin{aligned} \xi_{e_i} &= -\cos U \partial_{X^i} + X^i \sin U \partial_V, \\ \xi_{e_i^*} &= -\sin U \partial_{X^i} - X^i \cos U \partial_V, \\ \xi_{e_V} &= \partial_V, \\ \xi_{e_U} &= -\partial_U. \end{aligned} \tag{A.16}$$

Thus, the quotient of the plane wave that is conformally related to the nullbrane is of the type considered in [114]. The additional symmetry ξ_{e_U} that would be present in the plane wave is broken by the conformal factor. As we saw in section 3.4.3, this is precisely the additional symmetry that appears in the double null rotation.

As in section 3.4.3, in addition to exposing this relation to the plane waves, the global coordinates (A.10) allow us to easily find a global time function for the nullbrane, hence demonstrating that it is a stably causal solution. We first rewrite the nullbrane metric in a form suitable for Kaluza-Klein reduction along ϕ,

$$g = \frac{1}{\cos^2 U}[-2dUdv - 4\psi^2 dU^2 + d\psi^2 + (d\phi - 2\psi dU)^2]. \tag{A.17}$$

We see that Kaluza-Klein reduction will give a plane wave metric in one dimension lower (up to conformal factor). Hence, applying the results of [115], a suitable time function for the nullbrane is

$$\tau = U + \frac{1}{2}\tan^{-1}\left(\frac{4v}{1+4\psi^2}\right). \tag{A.18}$$

It is easy to check that

$$\nabla_\mu \tau \nabla^\mu \tau = -\frac{4\cos^2 U}{[(1+4\psi^2)^2 + 16v^2]} = -\frac{4(1+u^2)}{[(1+4\psi^2)^2+16v^2]}. \tag{A.19}$$

Thus, τ is a good time function on flat space, and since $\mathcal{L}_\xi \tau = 0$, the nullbrane is stably causal by the general argument of [115].

Appendix B

The Extended nullbrane

During the work of section 3.5 we uncovered a new causally regular smooth quotient of flat space. This extended nullbrane is a quotient of $\mathbb{R}^{1,5}$ by 2 commuting combinations of null rotations and translations,

$$\begin{aligned} \xi_1 &= \partial_5 - J_{12} + J_{24}, \\ \xi_2 &= \partial_6 - J_{13} + J_{34}. \end{aligned} \tag{B.1}$$

Any linear combination of these is always spacelike $||\xi_1 + \alpha\xi_2||^2 = (1+\alpha^2)[1 + (x_1 + x_4)^2]$. There are five linearly independent isometries in the $\mathfrak{so}(1,5) \ltimes \mathbb{R}^6$ Poincaré algebra on $\mathbb{R}^{1,5}$ which commute with ξ_1 and ξ_2,

$$\begin{aligned} \xi_3 &= -\partial_5 - J_{12} + J_{24}, \\ \xi_4 &= \partial_2 - (J_{15} + J_{45}), \\ \xi_5 &= -\partial_6 - J_{13} + J_{34}, \\ \xi_6 &= \partial_3 - (J_{15} + J_{45}), \\ \xi_7 &= \partial_1 + \partial_4 \ . \end{aligned} \tag{B.2}$$

These have norms $||\xi_1||^2 = ||\xi_2||^2 = ||\xi_3||^2 = ||\xi_4||^2 = ||\xi_5||^2 = ||\xi_6||^2$ and $||\xi_7||^2 = 0$. The only non-trivial commutation relations are $[\xi_3, \xi_4] = -2\xi_7$, $[\xi_5, \xi_6] = -2\xi_7$. We construct global adapted coordinates in exactly the same fashion as for the nullbrane. A system of coordinates $(v, u, \phi, \psi, \beta, \gamma)$ satisfying $\xi_1 = \partial_\phi$, $\xi_2 = \partial_\psi$ and $\xi_7 = \partial_v$ is given by

$$\begin{aligned} x^1 + x^4 &= 2\phi\psi + u(\phi^2 + \psi^2) + 2v, \\ x^1 - x^4 &= u, \\ x^2 &= \beta + u\phi, \\ x^5 &= \phi - u\beta, \\ x^3 &= \gamma + u\psi, \\ x^6 &= \psi - u\gamma \ . \end{aligned} \tag{B.3}$$

The flat metric is now

$$g = -2dudv + (1+u^2)(d\psi^2 + d\phi^2 + d\beta^2 + d\gamma^2) - 4\beta d\phi du - 4\gamma d\psi du \ . \tag{B.4}$$

It will come as no surprise that the extended nullbrane is conformally related to a compactified plane wave. This can be seen explicitly by making the coordinate transformation

$$\begin{aligned} V &= v + \psi\phi, \\ \tan U &= u, \\ W &= \beta\cos U + \phi\sin U; \\ X &= -\beta\sin U + \phi\cos U, \\ Y &= \gamma\cos U + \psi\sin U, \\ Z &= -\gamma\sin U + \psi\cos U, \end{aligned} \tag{B.5}$$

which brings the metric to the form

$$g = \frac{1}{\cos^2 U}[-2dUdV - (W^2 + X^2 + Y^2 + Z^2)dU^2 + dW^2 + dX^2 + dY^2 + dZ^2]. \tag{B.6}$$

This relation enables us to prove stable causality for the extended null brane by a replication of the argument for the nullbrane. The time function is given by

$$\tau = U + \frac{1}{2}\tan^{-1}\left(\frac{4v}{1 + 4\gamma^2 + 4\beta^2}\right) . \tag{B.7}$$

As a mathematical exercise the extended nullbrane can be generalised to a quotient of higher dimensional flat space by adding a commuting nullbrane generator for each extra two spacelike directions.

Appendix C

Inverse 3-charge metric

To calculate the inverse metric, it is convenient to start from the fibred form of the metric (4.14), construct a corresponding orthonormal frame, and invert that. For this reason, it is simpler to give the inverse metric in terms of the boosted coordinates $\tilde{t} = t\cosh\delta_p - y\sinh\delta_p$, $\tilde{y} = y\cosh\delta_p - t\sinh\delta_p$.

The inverse metric is

$$g^{\tilde{t}\tilde{t}} = -\frac{1}{\sqrt{\tilde{H}_1\tilde{H}_5}}\left(f + M + M\sinh^2\delta_1 + M\sinh^2\delta_5 + \frac{M^2\cosh^2\delta_1\cosh^2\delta_5 r^2}{g(r)}\right), \tag{C.1}$$

$$g^{\tilde{t}\tilde{y}} = \frac{1}{\sqrt{\tilde{H}_1\tilde{H}_5}}\frac{M^2\sinh\delta_1\sinh\delta_5\cosh\delta_1\cosh\delta_5 a_1 a_2}{g(r)}, \tag{C.2}$$

$$g^{\tilde{t}\phi} = -\frac{1}{\sqrt{\tilde{H}_1\tilde{H}_5}}\frac{M\cosh\delta_1\cosh\delta_5 a_2(r^2+a_1^2)}{g(r)}, \tag{C.3}$$

$$g^{\tilde{t}\psi} = -\frac{1}{\sqrt{\tilde{H}_1\tilde{H}_5}}\frac{M\cosh\delta_1\cosh\delta_5 a_1(r^2+a_2^2)}{g(r)}, \tag{C.4}$$

$$g^{\tilde{y}\tilde{y}} = \frac{1}{\sqrt{\tilde{H}_1\tilde{H}_5}}\Bigg(f + M\sinh^2\delta_1 + M\sinh^2\delta_5 + \frac{M^2\sinh^2\delta_1\sinh^2\delta_5(r^2+a_1^2+a_2^2-M)}{g(r)}\Bigg), \tag{C.5}$$

$$g^{\tilde{y}\phi} = -\frac{1}{\sqrt{\tilde{H}_1\tilde{H}_5}}\frac{M\sinh\delta_1\sinh\delta_5 a_1(r^2+a_1^2-M)}{g(r)}, \tag{C.6}$$

$$g^{\tilde{y}\psi} = -\frac{1}{\sqrt{\tilde{H}_1\tilde{H}_5}}\frac{M\sinh\delta_1\sinh\delta_5 a_2(r^2+a_2^2-M)}{g(r)}, \tag{C.7}$$

$$g^{rr} = \frac{1}{\sqrt{\tilde{H}_1\tilde{H}_5}}\frac{g(r)}{r^2}, \tag{C.8}$$

$$g^{\theta\theta} = \frac{1}{\sqrt{\tilde{H}_1\tilde{H}_5}}, \tag{C.9}$$

$$g^{\phi\phi} = \frac{1}{\sqrt{\tilde{H}_1\tilde{H}_5}}\left(\frac{1}{\sin^2\theta} + \frac{(r^2+a^2)(a_1^2-a_2^2)-Ma_1^2}{g(r)}\right), \tag{C.10}$$

$$g^{\phi\psi} = -\frac{1}{\sqrt{\tilde{H}_1\tilde{H}_5}}\frac{Ma_1a_2}{g(r)}, \tag{C.11}$$

$$g^{\psi\psi} = \frac{1}{\sqrt{\tilde{H}_1\tilde{H}_5}}\left(\frac{1}{\cos^2\theta} + \frac{(r^2+a^2)(a_2^2-a_1^2)-Ma_2^2}{g(r)}\right). \tag{C.12}$$

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