A Sufficient Condition for Weakly Acyclic games with Applications

zhao, guo and Chai, Yingming

Southwest Minzu University, Southwest Minzu University

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Guo Zhao, Yingming Chai*

School of Mathematics, Southwest Minzu University, Chengdu 610064, China

Abstract

The class of weakly acyclic games captures many practical application domains, and is particularly relevant for multi-agent distributed control problems. However, reliably checking weak acyclicity is extremely computationally intractable (PSPACE-complete) in the worst case. The present paper identifies sufficient conditions for weak acyclicity by means of the transitive closure of individual conditional preference, which can be constructed in terms of better-reply improvement paths. This pure-ordinal approach leads to a novel connection between weak acyclic games and better-reply secure games. Specifically, a better-reply secure game is weakly acyclic if the better reply dynamics does not possess a dense orbit (in addition to the quasi-concavity of individual preferences as well as the usual convexity and compactness assumptions on strategy sets). These results give a partial answer to an open problem of finding applicable and tractable conditions for weak acyclicity, posed by Fabrikant, Jaggard, and Schapira [11]. (JEL C72, C78, D01)

Keywords: pure-strategy Nash equilibrium, weakly acyclicity, better reply dynamics, better reply security

1. Introduction

Convergence to a pure-strategy Nash equilibrium is an important objective in a large variety of application domains. Ideally, this might be achieved via simple and natural dynamics, e.g., better-reply or best-reply dynamics, in which players myopically make a better or best reply to the strategy profile last chosen by all other players (Friedman and Mezzetti [13]; Kukushkiny, Takahashiz, and Yamamorix [14]; Cabrales and Serrano [15]; Kukushkin [16]). Reply based dynamics play a fundamental role in the field game-theoretic learning (Swenson et al. [1]; Heinrich et al. [2]; Fudenberg and Levine [3]; Nisan et al. [4]). However, Hart and Mas-Colell [5] have proved that no reply based dynamics can converge to Nash equilibrium in general games, due to the lack of a basic informational condition for dynamics (i.e.,"uncoupled").

Obviously, a necessary condition for better-/best-reply dynamics to converge to a Nash equilibrium regardless of the initial state is that the game is weakly acyclic (Young [6]), in which there exist some better-/best-reply improvement path to a Nash equilibrium for every initial state. The class of weakly acyclic games captures many practical application domains, including potential games, dominance-solvable games, coordination games, and games with strategic complementarities (Marden et al. [7]; Arieli and Young [8]). Also, weakly acyclic games capture distributed environments, as evidenced by today's protocol for routing on the Internet (Engelberg and Schapira [9]). Further, weakly acyclic games are particularly relevant for multi-agent cooperative control problems, such as consensus and dynamic sensor coverage (Marden, Arslan, and Shamma [10]).

Note that whether a game is weakly acyclic only depends on the structure of the game that determines the better reply dynamics. Unfortunately, It turns out that reliably checking

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*Corresponding author

Email addresses: zhaoguo@swun.edu.cn (Guo Zhao), ymchai@swun.edu.cn (Yingming Chai)

weak acyclicity is extremely computationally intractable (PSPACE-complete) even in succinctly-described games(Mirrokni and Skopalik [12]). With little hope of finding efficient ways to consistently check weak acyclicity, Fabrikant, Jaggard, and Schapira [11] raised the open question of finding sufficient conditions for weak acyclicity. They show that "unique subgame stability" is sufficient for weak acyclicity. However, the connection between weak acyclicity and unique subgame stability is not immediately practicable.

Here we give a sufficient condition for weak acyclicity by means of the transitive closure of conditional preference (Rosen [17]), which can be constructed in terms of better-reply improvement paths (Friedman and Mezzetti [13]; Kukushkiny, Takahashiz, and Yamamorix [14]; Cabrales and Serrano [15]; Kukushkin [16]). More precisely, we show that a strategic-form game is weakly acyclic if every chain in the transitive closure of conditional preference has a strict upper bound. The powerful Zorn's Lemma (Davey and Priestley [18]) is applied to obtain weak acyclicity in a straightforward manner.

Our approach is entirely ordinal and all results are stated in terms of individual preference relations over the joint strategy space. No utility concept need be hypothesized, and no topological structure need be imposed. Also the convexity assumption of the preference relation is not needed.

This approach not only leads to sufficient conditions for weak acyclicity in general strategic-form games, it helps to better connects weakly acyclic games with better-reply secure games (Reny [20]; Reny [21]). Specifically, we show that the better-reply security of a game implies weak acyclicity under better reply if in addition the strategy spaces are compact and convex, the payoffs are bounded and quasiconcave in the owner's strategy, and, roughly speaking, the better reply dynamics has no dense orbit. This result is of interest in that better-reply secure games are discontinuous in general and include continuous games as special case.

The rest of the paper proceeds as follows. Section 2 builds a strategic-form game on the basis of conditional preferences. In section 3, we identify sufficient conditions for weak acyclicity on the basis of the Zorn's Lemma. Then, we establish a novel connection between weak acyclicity and better reply security. Section 4 concludes this paper with some remarks.

2. Strategic-Form Games Based on Conditional Preferences

In strategic-form games, each player $i \in \{1, 2, \cdots, n\}$ has a *strategy* set S_i. The Cartesian product of the strategy sets of all the players makes up the *strategy space* of the game, denoted by $S = S_1 \times S_2 \times \cdots S_n$. We use the usual subscript $-i$ to denote "all players other than i". In particular, $S_{-i} = \prod_{j \neq i} S_j$ and $x = (x_i, x_{-i}) \in S$.

2.1. Individual Preferences

Each player $i \in \{1, 2, \cdots, n\}$ has an *individual preference* $\preceq^i$ on the strategy space $S = S_1 \times S_2 \times \cdots S_n$. Strategy profile x is preferred to or indifferent to strategy profile y will be symbolized by $y \preceq^i x$, or equivalently, $x \succeq^i y$. Without loss of generality, the preference ordering $\preceq^i$ is assumed to be a *quasi-order* (or *pre-order*) on the strategy space, i.e., a binary relation over the strategy space that is reflexive and transitive.

(i) *Reflexivity*: $x \preceq^i x$ for all $x \in S$.

(ii) *Transitivity*: $x \preceq^i y$ and $y \preceq^i z$ imply $x \preceq^i z$ for all $x, y, z \in S$.

From individual preference $\preceq^i$, a strict order can be derived:

$$x \prec^i y \Leftrightarrow x \preceq^i y \text{ and not } y \preceq^i x. \tag{1}$$

A quasi-order that is anti-symmetric is called a *partial order*.

(iii) *Anti-symmetry*: $x \preceq^i y$ and $y \preceq^i x$ imply $x = y$ for all $x, y \in S$.

Partially ordered sets occur everywhere in mathematics. However, in the discussion of nets and directed limits, it is not always so convenient to assume anti-symmetry property (see Gierz et al. [22]). We begin, therefore, with the more general assumptions of quasi-order to accommodate bounded rationality of players(Simon [26]).

A pure-strategy profile $x^* \in S$ is a *Nash equilibrium* if, for **no** player i there exists $x_i \in S_i$ such that

$$x^* = (x_i^*, x_{-i}^*) \prec^i (x_i, x_{-i}^*). \tag{2}$$

A special case of economic interest is a profile of strategies whereby each player's strategy is optimal independent of the other players' strategies. A pure-strategy profile $x^* S$ is a *dominant strategy equilibrium* if, for all player i

$$(y_i, x_{-i}) \preceq^i (x_i^*, x_{-i}^*), \forall y_i \in S_i, \forall x_{-i} \in S_{-i}. \tag{3}$$

2.2. Conditional Preferences

Within our framework, the concept of *conditional preference* plays a central role in the existence of pure-strategy Nash equilibrium, as well as the convergence of better reply dynamics.[1]

With the individual preference ordering $(S, \preceq^i)$ for each player, we associate a conditional preference relation $\trianglelefteq^i$ on $S = S_1 \times S_2 \times \cdots S_n$

$$x \trianglelefteq^i y \Leftrightarrow x \preceq^i y \text{ and } x_{-i} = y_{-i}. \tag{4}$$

Notice that there exist conditional-preference relations only between strategy profiles that differ in exactly one coordinate. In the literature, x_i is said to be a *better reply* of player i against $x_{-i} = y_{-i}$ than y_i (Young [6]; Reny [20]).

Remark 2.1. By definition, a better-reply improvement must be strict in that a switch to the same outcome is not an improvement unless there is no incentive for any player to deviate unilaterally from it. A better reply could be interpreted as a special case of Simon [26] *satisficing* behavior, with the aspiration level given by the payoff resulting from the player's *status quo* action (Friedman and Mezzetti [13]).

To proceed, the set-theoretic union of these conditional preferences give rise to a binary relation $\trianglelefteq$ on $S = S_1 \times S_2 \times \cdots S_n$

$$x \trianglelefteq y \Leftrightarrow \exists i \in \{1, 2, \cdots, n\}, x \preceq^i y \text{ and } x_{-i} = y_{-i}. \tag{5}$$

Notice that for given strategy profile x, there can be different players who have better replies than x by making improvement **exactly once**. This flexibility is useful in investing the connection between better-reply secure games and weakly acyclic games (see Section 3.3 for details).

For convenience, we call $(S, \trianglelefteq)$ the *conditional preference*, which describes the interaction pattern for the game as a whole. The asymmetric part of conditional preference $\trianglelefteq$ gives rise to a relation $\triangleleft$ of *strictly better reply*:$x \triangleleft y \Leftrightarrow x \trianglelefteq y$ and not $y \trianglelefteq x$.

If each individual preference $\preceq^i$ is replaced with conditional preference $\trianglelefteq$, the set of Nash equilibria remains intact. Actually, a pure-strategy profile $x^* \in S$ is a Nash *equilibrium* if there is **no** $y \in S$ satisfying $x^* \triangleleft y$.

However, the conditional preference $\trianglelefteq$ lacks the property of transitivity, which is the most obvious property of a binary relation conducive to the existence of maximal elements. To circumvent this difficulty, we consider the *transitive closure* of the conditional preference . By definition (Rosen [17]), the transitive closure, denoted by $\sqsubseteq$, of $(S, \trianglelefteq)$ is the smallest transitive relation on S that contains as a subset.

[1] Historically, conditional preferences are used to distinct between private and social preferences (Gibbard [23]), or to characterize dependency relationships among different attributes (Boutilier et al. [24]). Stirling and Felin [25] introduced conditional preference structure into game theory that permits players to modulate their preference orderings as functions of the preferences of other players. However, their model is based on conditional utility and conditional probability, rather than preference and order structures.

It is well-known that the transitive closure $(S, \sqsubseteq)$ can be constructed in terms of better-reply improvement paths. The concept transitive closure may be augmented to be consistent with infinite paths. A better-reply improvement path of length m is a sequence $(x^1, x^2, \cdots, x^m)$ of elements $x^k \in S$ such that $x^k \triangleleft x^{k+1}$ whenever x^{k+1} is well defined. Formally, $x \sqsubseteq y$ in S whenever there exists a better reply improvement path (of any length, possibly infinite) from x to y. The transitive closure $(S, \sqsubseteq)$ consists of all ordered pairs (x, y) such that $(S, \sqsubseteq)$.

Proposition 2.2. *If each individual preference $(S, \preceq^i)$ is a quasi-ordered set, then the transitive closure $(S, \sqsubseteq)$ of the conditional preference is a quasi-order.*

Remark 2.3. Geometrically, if each S_i is a subset of $\mathbb{R}$ and hence $S = S_1 \times S_2 \times \cdots S_n$ is a subset of an Euclidean space of dimension n then at any point in strategy space the vector aligned with player i's better reply must be parallel to the ith coordinate axis. Consequently, at any point in strategy space the vectors corresponding to different players' better replies must be mutually orthogonal.

3. Weak Acyclicity in terms of Conditional Preferences

In this section, we first characterize Nash equilibrium as a maximal element of the transitive closure of the conditional preference. Then we identify sufficient conditions for weak acyclicity on the basis of the Zorn's Lemma (Davey and Priestley [18]). Finally, we establish a novel connection between weak acyclicity and better reply security.

3.1. Characterization of Nash Equilibrium

We shall characterize Pure-strategy Nash equilibrium in terms of maximal elements with respect to the transitive closure of the conditional preference.

Formally, a strategy profile x is said to be maximal in $(S, \sqsubseteq)$ if there is **no** element $y \neq x$ in S for which $x \sqsubseteq y$.

Proposition 3.1. *Assume that the individual preference $(S, \succeq^i)$ is a quasi-ordered set for each player $i \in \{1, 2, \cdots, n\}$. Then:*

(1) If $x^* \in S$ is a Nash equilibrium, then x^* is a maximal element in $(S, \sqsubseteq)$.

(2) If each induced preference on $(S_i, x_{-i}) \stackrel{def}{=} \{(y_i, x_{-i}) | y_i \in S_i\}$ is inductively ordered, i.e., every chain has an upper bound in (S_i, x_{-i}), then any maximal element in $(S, \sqsubseteq)$ is a Nash equilibrium.

Proof. (1) Suppose $x^* = (x_1^*, x_2^*, \cdots, x_n^*)$ is a Nash equilibrium. Then for no player i there exists $x_i \in S_i$ such that $x^* = (x_i^*, x_{-i}^*) \prec^i (x_i, x_{-i}^*)$. We proceed to argue by contradiction. Suppose instead that x^* is not a maximal element in $(S, \sqsubseteq)$. Then there exists $z \neq x^*$ in S satisfying $x^* \sqsubseteq z$. By definition, there exist a better-reply improvement path $(y^1, y^2, \cdots, y^k, \cdots)$ from $x^* = y^1$ to z. As a result, we have $x^* = y^1 \triangleleft^i y^2$ for some player i, or equivalently, player i strictly prefers $y^1 = (y_i^1, x_{-i}^*)$ to $x^* = (x_i^*, x_{-i}^*)$. This contradicts the definition of Nash equilibrium as desired.

(2) Suppose $x^* = (x_1^*, x_2^*, \cdots, x_n^*)$ is a maximal element in $(S, \sqsubseteq)$. We show $x^* = (x_i^*, x_{-i}^*)$ is a Nash equilibrium. If not, then there exists player i strictly prefers strategy $y_i \in S_i$ to strategy x_i^* against x_{-i}^*, that is,

$$x^* = (x_i^*, x_{-i}^*) \prec^i (y_i, x_{-i}^*) \stackrel{def}{=} y. \tag{6}$$

Now consider the chain $x^* \sqsubseteq y$ in $(S, \sqsubseteq)$. By assumption, $(S_i, x_{-i}^*) \stackrel{def}{=} \{(y_i, x_{-i}^* | y_i \in S_i\}$ is inductively ordered, so there exists a upper bound $z = (z_i, x_{-i}^*)$ such that $x^* \sqsubseteq y \sqsubseteq z$. This contradicts the assumption that $x^* = (x_1^*, x_2^*, \cdots, x_n^*)$ is a maximal element in $(S, \sqsubseteq)$. □

Next, we shall characterize dominant strategy equilibrium in terms of the greatest element with respect to the transitive closure of the conditional preference. Formally, a strategy profile $x \in S$ is said to be *greatest element* if $y \sqsubseteq x$ for all $y \in S$ and no $y \neq x$ in S such that $x \sqsubseteq y$.

Table 1: Matching Pennies.

	Player 2					
		h		t		x
	H	$(\underline{2},0)$	$\longrightarrow$	$(0,\underline{2})$	$\longleftarrow$	$(0,0)$
		$\uparrow$		$\downarrow$		$\downarrow$
Player 1	T	$(0,\underline{2})$	$\longleftarrow$	$(\underline{2},0)$	$\longrightarrow$	$(\frac{1}{2},\frac{1}{2})$
		$\downarrow$		$\uparrow$		$\downarrow$
	X	$(\frac{1}{2},0)$	$\longrightarrow$	$(1,\frac{1}{2})$	$\longrightarrow$	$(\underline{3},\underline{3})$

Proposition 3.2. *Assume that each individual preference $(S,\preceq^i)$ is a quasi-ordered set. If a pure-strategy profile $x^* \in S$ is a dominant strategy equilibrium, then x^* is greatest element of $(S,\sqsubseteq)$.*

Proof. Now that $x^* = (x_1^*, x_2^*, \cdots, x_n^*)$ is a dominant strategy equilibrium, then for all player i

$$(x_i^*, y_{-i}) \succeq^i (y_i, y_{-i}), \forall y_i \in S_i, \forall y_{-i} \in S_{-i}. \tag{7}$$

We show $y \sqsubseteq x^*$ for all $y = (y_1, y_2, \cdots, y_n) \in S$. In fact, by the definition of dominant strategy we have

$$\begin{aligned} x^* &= (x_1^*, x_2^*, x_3^*, \cdots, x_n^*) \\ &\succeq^1 (y_1, x_2^*, x_3^*, \cdots, x_n^*) \\ &\succeq^2 (y_1, y_2, x_3^*, \cdots, x_n^*) \\ &\succeq^3 (y_1, y_2, y_3, \cdots, x_n^*) \\ &\succeq^i \cdots \\ &\succeq^n (y_1, y_2, y_3, \cdots, y_n) = y. \end{aligned} \tag{8}$$

So $y \sqsubseteq x^*$ as desired. □

It is natural to wonder whether the greatest element of $(S,\sqsubseteq)$ is a dominant strategy equilibrium. Unfortunately, the answer is no, as indicated by Example below.

Example 3.3. Matching Pennies (with Uncertainty)

Players 1 and 2 simultaneously announce heads (H) or tails (T) or uncertain (X). The corresponding payoff matrix is given in figure 2. The game has a unique Nash equilibrium (X,x), but no dominant strategy equilibrium.

The conditional preference of player 1 is depicted as $\uparrow$ or $\downarrow$ in figure 2, directing to the preferred strategy against player 2's strategy.

Player 2's conditional preference is depicted as $\longleftarrow$ or $\longrightarrow$ in figure 2, pointing to the preferred strategy against player 1's strategy.

Taken together, the strategy space equipped with the order $\sqsubseteq$ has a greatest element (X,x), which is a Nash equilibrium but not a dominant strategy equilibrium.

It is routine to check that this game is weakly acyclic under better reply (Fabrikant, Jaggard, and Schapira [11]), but there exists a best reply cycle

$$(\mathrm{T},\mathrm{h}) \preceq^1 (\mathrm{H},\mathrm{h}), (\mathrm{H},\mathrm{h}) \preceq^2 (\mathrm{H},\mathrm{t}), (\mathrm{H},\mathrm{t}) \preceq^1 (\mathrm{T},\mathrm{t}), (\mathrm{T},\mathrm{t}) \preceq^2 (\mathrm{T},\mathrm{h})$$

3.2. Sufficient Conditions for Weak Acyclicity

We have characterized pure-strategy Nash equilibrium as a maximal element of the transitive closure of the conditional preference. To guarantee weak acyclicity, we shall invoke the powerful Zorn's Lemma (Davey and Priestley [18]).

Definition 3.4. (Strict Upper Bound) Let C be a subset of $(S, \sqsubseteq)$. Then C is said to be a chain if it is linearly ordered, i.e., $x \sqsubseteq y$ or $y \sqsubseteq x$ for all $x, y \in S$. A strategy profile $x \in S$ is said to be an upper bound of C if $y \sqsubseteq x$ for all $y \in C$. An upper bound $x \in S$ is said to be strict if x does not belong to any closed cycle of C (which is always the case if $(S, \sqsubseteq)$ is a partial order).

Theorem 3.5. *Assume that, for each player $i \in \{1, 2, \cdots, n\}$, the individual preference $(S, \preceq^i)$ is a quasi-order. If every chain in $(S, \sqsubseteq)$ has a strict upper bound in S, then the game is weakly acyclic.*

Proof. Since the individual preference $(S, \preceq^i)$ is a quasi-ordered set, the strategy space equipped with the transitive closure $\sqsubseteq$ of the conditional preference is a quasi-order according to Proposition 2.2.

Now fix a strategy profile x in $(S, \sqsubseteq)$, we have to show that there exists a better-reply improvement path that starts from x and ends in a pure strategy Nash equilibrium.

To this end, consider the set Ω of all chains that starts from x in $(S, \sqsubseteq)$. It's routine to check that the set Ω is partially ordered by the set inclusion relation $\subseteq$. We show that $(\Omega, \subseteq)$ contains a maximal chain that starts from x in $(S, \sqsubseteq)$ by applying Zorn's Lemma in a standard way. To this end, consider any linearly ordered subset in $(\Omega, \subseteq)$

$$L : C_1 \subseteq C_2 \subseteq \cdots \subseteq C_\alpha \subseteq \cdots. \tag{9}$$

Then the union $\bigcup_\alpha C_\alpha$ is itself a chain starting from x, and is by definition an upper bound in $(\Omega, \subseteq)$ for the linearly ordered subset L. As a result, every linearly ordered subset in $(\Omega, \subseteq)$ has an upper bound. Zorn's Lemma asserts that there is a maximal chain C_0 that starts from x and has a strict upper bound x^0 in $(S, \sqsubseteq)$ by hypothesis.

We show that x^0 is a Nash equilibrium by contradiction. If not, there exists a player i who prefers strategy $y_i \in S_i$ to strategy x_i^0 against x_{-i}^0, that is,

$$x^0 = (x_i^0, x_{-i}^0) \preceq^i (y_i, x_{-i}^0) \stackrel{def}{=} y^0. \tag{10}$$

Then we can construct a new chain $C_0 \cup \{x^0, y^0\}$ in $(S, \sqsubseteq)$, which also starts from x and has a strict upper bound z^0 by hypothesis. This violates the maximality of C_0. So x^0 is indeed a pure-strategy Nash equilibrium as desired. ☐

Remark 3.6. The condition that every chain in $(S, \sqsubseteq)$ has a strict upper bound is a significantly weaker assumption. A remarkable merit of this chain condition is that it does not require convexity and/or compactness of the strategy space, or the semi-continuity and/or quasi-concavity of individual payoff functions.

Remark 3.7. Note that Zorn's Lemma is equivalent to the Axiom of Choice over Zermelo-Fraenkel Set Theory (See Jech [19]). However, Zorn's Lemma is non-constructive in mathematics and its statement is not intuitive. So one does not know initially just what choices are to be made and in what order. However, the order in which players update their actions is essentially irrelevant in determining whether the better reply dynamics converge to Nash equilibrium (Heinrich et al. [2]). In fact, when the playing sequence is random, the better reply dynamics converge to a pure Nash equilibrium if one exists in almost all (large) games. It is the order-theoretic property of the better reply dynamics that matters.

3.3. Connections between Weak Acyclicity and Better-Reply Security

It is useful to provide several sufficient conditions for a game to satisfy our chain condition for weak acyclicity. Also, it is worthwhile to compare the present results with classical results on the existence of pure-strategy Nash equilibrium for general strategic-form games. In this subsection, we shall establish a novel link between weak acyclic games and better-reply secure games (Reny [20]).

For simplicity, in this subsection we will assume that each player's preference $\preceq^i$ can be represented by a Neumann-Morgenstern utility function $u_i : S \longrightarrow \mathbb{R}$.

Definition 3.8. (Better-Reply Security, Reny [20]): A game with payoff function $u_i : S \longrightarrow \mathbb{R}$ for each player $i \in \{1, 2, \cdots, n\}$ is better-reply secure if whenever $u^* = (u_1^*, u_2^*, \cdots, u_n^*\}$ is the limit of the vector of payoffs corresponding to some sequence of strategies converging to nonequilibrium strategy x^*, some player $i \in \{1, 2, \cdots, n\}$ can secure a payoff strictly above u_i^* at $x^* \in S$, i.e., there exists $\bar{x}_i \in S_i$ such that $u_i(\bar{x}_i, x'_{-i}) > u_i^*$ for all x'_{-i} in some open neighborhood of x^*_{-i}.

Better-reply security is satisfied in many economic games, and is easy to verify. For example, games with continuous payoff functions are better-reply secure, since any better reply will secure a payoff strictly above a player's inferior nonequilibrium.

Theorem 3.9. *Suppose that for each player $i \in \{1, 2, \cdots, n\}$ the strategy set S_i is a compact convex subset of a Hausdorff topological vector space, the payoff function $u_i(x) = u_i(x_i, x_{-i})$ is bounded and is quasi-concave in x_i for each $x_{-i} \in S_{-i}$. If the game is better reply secure, and for every chain in $(S, \sqsubseteq)$, its closure has no interior point with respect to the Hausdorff topology, then every chain in $(S, \sqsubseteq)$ has a strict upper bound in S.*

The idea behind the theorem is quite straightforward: the flexibility of the securing strategies may offer an escape route out of the closed cycle of any improvement path. Therefore, eventually we will reach an equilibrium under the assumption of better-reply security.

Our proof is based on the following crucial properties concerning better-reply secure games (for proof see Reny [20]).

Lemma 3.10. *For each player i and every $x \in S$, define the lower envelope of each player's payoff function $u_i(x)$ as follows*

$$\underline{u}_i(x) = \sup_{\mathcal{N}(x_{-i})} \inf_{x'_{-i} \in \mathcal{N}(x_{-i})} u_i(x_i, x'_{-i}),$$

where the supremum is taken over all open neighborhoods $\mathcal{N}(x_{-i})$ of x_{-i}. Then under the conditions of Theorem 3.9 we have

(i) *$\underline{u}_i(x)$ is real valued and $\underline{u}_i(x) \leq u_i(x)$ for every $x \in S$;*

(ii) *For fixed $x_i \in S_i$, $\underline{u}_i(x_i, \cdot)$ is lower semicontinuous on S_{-i}, i.e., $\underline{u}_i(x_i, x^*_{-i}) \leq \sup_m \inf_{k \geq m} \underline{u}_i(x_i, x^k_{-i})$ for any sequence x^k_{-i} converging to x^*_{-i} in S_{-i}.*

(iii) *If a pair (x^*, u^*) is in the closure of the graph of the vector payoff function $u = (u_1(x), u_2(x), \cdots, u_n(x))$, and $\sup_{x_i \in S_i} \underline{u}_i(x_i, x^*_{-i}) \leq u_i^*$ for all player i, then x^* is a pure-strategy Nash equilibrium.*

Proof. (Proof of Theorem 3.9) Let C be a chain of better replies in $(S, \sqsubseteq)$. We have to show that a strict upper bound for C exists in $(S, \sqsubseteq)$.

To begin with, suppose that the chain C contains a greatest element with respect to $\sqsubseteq$. Then this greatest element is by definition a strict upper bound for the chain.

To continue, assume that the chain C contains no greatest element with respect to $\sqsubseteq$. Following an approach of Kukushkin [27] in the study of potential games, we define the set $A(x) \overset{def}{=} \{y \in C | x \sqsubseteq y\}$, which is not empty for all $x \in C$. Let $\overline{A(x)}$ be its closure with respect to the Hausdorff topology and define $B = \bigcap_{x \in C} \overline{A(x)}$. Now that C is a chain, all the sets $A(x)$, and hence $\overline{A(x)}$ too, contain one another. That is,

$$x \sqsubseteq y \Rightarrow A(x) \supseteq A(y) \; and \; \overline{A(x)} \supseteq \overline{A(y)}. \tag{11}$$

Because the pure strategy set S_i is compact for each player $i \in \{1, 2, \cdots, n\}$, the strategy space $S = S_1 \times S_2 \times \cdots S_n$ is compact according to the famous Tychonoff product theorem. As a result, $B = \bigcap_{x \in C} \overline{A(x)}$ is not empty. By construction, B is contained in the closure $\overline{C}$. Further, B has no interior point because the closure $\overline{C}$ has no interior point with respect to the Hausdorff topology.

Now let $b \in B$. By transitivity of $\sqsubseteq$, it follows that $x \sqsubseteq b$ for all strategy profile $x \in C$. We want to construct a strict upper bound for the chain C in $(S, \sqsubseteq)$ on the basis of b. We proceed by analyzing whether $b \in B$ belong to an closed cycle of C:

Case 1. If b does not belong to any closed cycle of C. Then b itself is by definition a strict upper bound for the chain C By transitivity of $\sqsubseteq$.

Case 2. If b belong to a closed cycle of C. Then the set B is by construction a closed cycle and all $y \in C$ satisfying $b \sqsubseteq y$ must be included in the closed cycle $C \cap B$.

Obviously, b is not an equilibrium since it belongs to a closed cycle in this case. As a result, some player i strictly prefers strategy $x_i^0 \in S_i$ to strategy b_i against b_{-i}, that is,

$$u_i(b) = u(b_i, b_{-i}) < u(x_i^0, b_{-i}). \tag{12}$$

In terms of $\sqsubseteq$, we obtain $b = (b_i, b_{-i}) \sqsubseteq (x_i^0, b_{-i}) \stackrel{def}{=} b^0$.

From now on fix this player i. The remainder of the proof will be broken into two cases:

Case 2.1. If there exists some better reply $x_i^t \in \{y_i \in S_i | b \prec^i (y_i, b_{-i}\} \stackrel{def}{=} better_i(b)$ such that $b^t \stackrel{def}{=} (x_i^t, b_{-i}) \notin C$, then b^t is a strict upper bound for the chain C in $(S, \sqsubseteq)$ as desired.

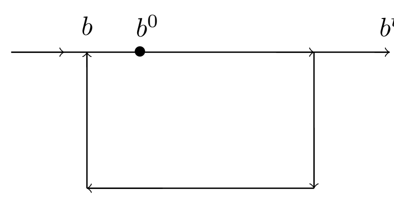

Fig. 1.

Case 2.2. Assume instead that for **any** better reply $x_i^t \in better_i(b)$, we have $b^t \stackrel{def}{=} (x_i^t, b_{-i}) \in C$. Consequently, every such b^t belongs to the closed cycle $C \cap B$ since $b = (b_i, b_{-i}) \sqsubseteq b^t$.

Because strategy space $S = S_1 \times S_2 \times \cdots S_n$ is compact and the payoff function $u_i(x)$ is bounded, it follows that $\sup_{x_i \in S_i} u_i(x_i, b_{-i})$ exists. Now let $u_i^* = \sup_{x_i \in S_i} u_i(x_i, b_{-i})$ and consider any sequence $b^k \stackrel{def}{=} (x_i^k, b_{-i})$, where $x_i^k \in better_i(b)$, such that $u_i(b^k)$ converges to u_i^*. Correspondingly, let x_i^* be the limit point of the sequence x_i^k in S_i and define $b^* \stackrel{def}{=} (x_i^*, b_{-i})$. So defined, we have $b \sqsubseteq b^k \sqsubseteq b^*$ for all $k = 1, 2, \cdots$ in $(S, \sqsubseteq)$.

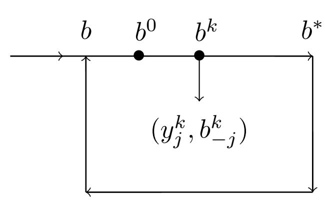

Fig. 2.

Further, because the payoff function $u_i(x_i, b_{-i})$ is quasi-concave in x_i, the better reply correspondence $better_i(b)$ is a convex subset of S_i. As a result, the entire line segment that connects b and $b^* \overset{def}{=} (x_i^*, b_{-i})$ is included in the closed cycle $C \cap B$. To complete the proof, we consider the following two cases:

Case 2.2.1. Assume that some other player $j \neq i$ has a profitable deviation slightly from some $b^k = (b_j^k, b_{-j}^k)$, where $k < \infty$. That is, there exists y_j^k in a sufficiently small neighborhood $\mathcal{N}(b_j^k)$ of b_j^k such that

$$u_j(b^k) = u_j(b_j^k, b_{-j}^k) < u_j(y_j^k, b_{-j}^k). \tag{13}$$

Then we can get a better-reply improvement path in $(S, \sqsubseteq)$ as follows

$$b \sqsubseteq b^k = (b_j^k, b_{-j}^k) \sqsubseteq (y_j^k, b_{-j}^k). \tag{14}$$

Because $j \neq i$, the vector aligned with player j's better reply $b^k \trianglelefteq^j (y_j^k, b_{-j}^k)$ must be linearly independent to the line segment that connects b and $b^* \overset{def}{=} (x_i^*, b_{-i})$ (see Remark 2.3). Because the entire line segment that connects b and b^* is included in the closed cycle $C \cap B$, it follows that $(y_j^k, b_{-j}^k) \notin C$. (Recall that by assumption every chain in $(S, \sqsubseteq)$ has no interior point with respect to the Hausdorff topology.)

By transitivity of $\sqsubseteq$, (y_j^k, b_{-j}^k) is a strict upper bound for the chain C in $(S, \sqsubseteq)$ in this case.

Case 2.2.2. Assume instead that no player $j \neq i$ can profitably deviate slightly from b^k for all $k = 1, 2, \cdots$. We shall show this case is impossible by contradiction argument.

Let $u^* = (u_1^*(x), u_2^*(x), \cdots, u_n^*(x))$ be the limit of the vector of payoffs $u^k =$

$(u_1(b^k), u_2(b^k), \cdots, u_n(b^k))$ corresponding to the sequence $b^k \to b^*$. Then we have $u_j^* = \sup_m \inf_{k \geq m} u_j(b^k)$ for all player j. Notice that the pair (b^*, u^*) is in the closure of the graph of the vector payoff function.

To obtain a contradiction, we shall show that $b^* \in C$ is a Nash equilibrium, which contradicts our assumption that b^* belongs to the closed cycle $C \cap B$.

Since $u_i^* = \sup_{x_i \in S_i} u_i(x_i, b_{-i}^*)$ for the fixed player i, we have

$$\sup_{x_i \in S_i} \underline{u}_i(x_i, b_{-i}^*) \leq \sup_{x_i \in S_i} u_i(x_i, b_{-i}^*) = u_i^*. \tag{15}$$

By Lemma 3.10 it suffices to prove $\sup_{x_j \in S_j} \underline{u}_j(x_j, b_{-j}^*) \leq u_j^*$ for all player $j \neq i$.

To this end, notice that in this case no player $j \neq i$ can profitably deviate slightly from b^k for all $k = 1, 2, \cdots$. So we have for each player $j \neq i$ that there exists a neighborhood $\mathcal{N}(b_j^k)$ of b_j^k such that

$$u_j(y_j^k, b_{-j}^k) < u_j(b_j^k, b_{-j}^k) = u_j(b^k), \forall y_j^k \in \mathcal{N}(b_j^k). \tag{16}$$

Or equivalently, for every player $j \neq i$, the payoff function $u_j(\cdot, b_{-j}^k)$ attains a strict local maximum at $b_j^k \in C$ for all $k = 1, 2, \cdots$.

Since the payoff function $u_j(\cdot, b_{-j}^k)$ is quasi-concave for all $b_{-j}^k, k = 1, 2, \cdots$, a strict local maximum must be a strict global maximum. As a result, b_j^k is a best reply against b_{-j}^k for all player $j \neq i$. Consequently, $u_j(b_j^k, b_{-j}^k) \geq u_j(x_j, b_{-j}^k)$ for all $x_j \in S_j$.

According to Lemma 3.10, for fixed $x_j \in S_j$, the function $\underline{u}_j(x_j, \cdot)$ is lower semicontinuous on S_{-j}. Now that the sequence b^k converges to b^* in S and hence b_{-j}^k converging to b_{-j}^* in S_{-j}, we have for fixed $x_j \in S_j$ that

$$\begin{aligned}\underline{u}_j(x_j, b^*_{-j}) &\leq \sup_m \inf_{k\geq m} \underline{u}_j(x_j, b^k_{-j}) \\ &\leq \sup_m \inf_{k\geq m} u_j(x_j, b^k_{-j}) \\ &\leq \sup_m \inf_{k\geq m} u_j(b^k_j, b^k_{-j}) = u^*_j. \end{aligned} \tag{17}$$

Taken supremum we have $\sup_{x_j\in S_j} \underline{u}_j(x_j, b^*_{-j}) \leq u^*_j$ for all player $j \neq i$. Consequently, $b^* \in C$ is a Nash equilibrium. This contradicts the assumption that b^* belongs to the closed cycle $C \cap B$. □

Remark 3.11. Note that the closure of every chain in $(S, \sqsubseteq)$ has no interior point is a topological property of the better reply dynamics. Loosely stated, if the better reply dynamics does not possess a dense orbit then every chain in $(S, \sqsubseteq)$ has no interior point. On the other hand, the existence of a dense orbit plays an important role in the definition of Chaos (Hirsch, Smale, and Devaney [28]).

4. Concluding Remarks

The class of weakly acyclic games captures many practical application domains, and includes potential games and dominance-solvable games as special cases. Unfortunately, reliably checking weak acyclicity is extremely computationally intractable (PSPACE-complete) in the worst case. For this reason it is useful to provide sufficient conditions for weakly acyclicity. In this paper, we identify sufficient conditions for weak acyclicity on the basis of the transitive closure of conditional preference.

Our approach is entirely ordinal and all results are stated in terms of individual preference relations over the joint strategy space. As it turns out, the chain condition on the transitive closure of conditional preference itself is a significantly weaker assumption. A remarkable merit of the chain condition is that it does not require convexity and/or compactness of the strategy space, or the semi-continuity and/or quasi-concavity of individual payoff functions.

Further, this ordinal approach helps to better connects better-reply secure games (Reny [20]) with weakly acyclic games. Specifically, we show that the better-reply security of a game implies weak acyclicity under better reply if in addition the game is compact, convex, bounded, quasi-concave in the owner's strategy, and, roughly speaking, the better reply dynamics has no dense orbit. This result is of interest in that better-reply secure games are discontinuous in general and include continuous games as special case.

These results give a partial answer to an open problem of finding applicable and tractable conditions for weak acyclicity, posed by Fabrikant, Jaggard, and Schapira [11]. It's a pity that our condition for weak acyclicity is expressed in terms of the transitive closure of individual conditional preference (rather than individual preference), and its proof makes use of Zorn's Lemma. In view of this, a straightforward and elementary proof is needed, and the possibility of improving or generalizing the result is to be considered in the future.

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