Essays on business cycles in Korea

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ESSAYS ON BUSINESS CYCLES IN KOREA

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Abstract

Over the last two decades, a large amount of macroeconomic research has been directed towards the business cycle theory. Until now, however, most literature has focused on the business cycle research in the developed economies like the United States and European countries. On the other hand, the studies on emerging markets are very limited. From this perspective, this thesis attempts to contribute to the current literature by investigating the main characteristics of Korean business cycles, comparing its findings with the previous findings in the developed countries, and then drawing out the economic or policy implications.

Another motivation of this thesis arises from the experience of the Korean financial crisis in the later half of 1997, which had a catastrophic impact on the Korean economy. Although it is not fully agreed on what caused the Korean financial crisis, there are a large amount of literature on this subject. However, what have not been dealt in literature is whether the Korean financial crisis brought out any noticeable changes on the fluctuations of key macroeconomic variables such as output, consumption, worked hours and asset wealth in Korea. One possible excuse for this scarcity is the limitation of data span, namely too short to analyze the difference between the pre- and post-crisis periods. It appears that as a decade passed, this is the right time to investigate this issue. Therefore, across three essays, this issue will be discussed with a high priority.

With these two purposes in mind, this thesis starts by investigating the relationship among consumption, financial wealth and labor income with Korean data. This issue is discussed in the framework of the vector error correction model by applying the full information maximum likelihood approach suggested by Johansen (1988, 1995). The main finding from this

analysis is that only financial wealth shows the sizable and statistically significant error correction behavior. This finding is also confirmed by the permanent and transitory component decomposition, which shows that only fluctuations of financial wealth are largely associated with transitory components while consumption and labor income are mainly governed by the permanent shocks during the examined period. By comparing the pre- and post-crisis periods, we also find that although there were several policy and institutional changes during the crisis, most adjustment to the long run relation has been done by financial wealth across the two sample periods.

The second and last essays explore Korean business cycles by estimating the micro-founded dynamic stochastic general equilibrium models, using maximum likelihood and Bayesian approaches, respectively. Based on the estimation, the second essay finds that the estimated DSGE model outperforms VAR models in predicting hours worked but it has difficulty in predicting other key macrovariables. In addition, although the volatility of the economy has decreased in the recent, the Korean financial crisis seemed not to change the deep parameters in the model. Finally, by comparing the second moments from the HP filtered data with those from the simulated data, this essay finds that the estimated model successfully reproduces the relative volatility of consumption and hours worked as well as the pattern of contemporaneous correlations of output with consumption, investment and hours worked.

The last essay extends the baseline model in the second essay by introducing money through cash-in-advance constraints, that is, firms should borrow cash to pay wages in advance while households have to hold cash to purchase consumption goods. After estimating two versions of cash-in-advance models, namely a baseline cash-in-advance model and a limited participation model, this essay shows that the limited participation model is better to match up the stylized facts in Korean business cycles. In particular, it successfully captures the rise of output in response to an expansionary money shock. Finally, the comparison between the pre- and post crisis periods shows a sharp decline of money shock and a slight decline of productivity shock.

Declaration

The material contained in this thesis has not been submitted in support of an application for another degree or qualification in this or any other University.

Dedicated to

MY BELOVED JIMIN LIM

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Chapter 1

INTRODUCTION

The main focus of the current thesis is to examine the business cycles in Korea and to draw out the economic or policy implications. Currently, most literature has focused on the research on business cycles in the developed countries like the United States and European countries. On the other hand, the studies on emerging market business cycles are very limited. Even when this is the case, most literature has worked with Latin American business cycles (Kydland and Zarazaga 2002; Garcia-Cicco, Pancrazi, and Uribe 2006; Aguiar and Gopinath 2007). From this perspective, this thesis attempts to contribute to the current literature by investigating Korean business cycles and comparing our findings with the previous findings in the developed countries.

During this research, one of the most important aspects in Korean economy needs addressing is the Korean financial crisis in the later 1997, which had a catastrophic impact on the Korea, causing the worst recession since the Korean War. Although it is not fully agreed on what caused the Korean financial crisis, there are a large amount of literature on this subject. Now many academic scholars and policymakers agree with the view that it was caused by a combination of the sudden reversal of capital flows fueled by the Southeast Asian currency crisis and the misguided microeconomic policies such as the too-big-to-fail polices on large firms and a loose supervision on the non-banking sector.[1]

However, what have not been much researched in literature is whether the Korean financial crisis brought out any noticeable changes on the fluctuations of key macroeco-

[1] See, for example, Radelet and Sachs (1998), Stiglitz (1999), Hahm and Mishkin (2000), Park and Song (2001), and Danthine and Donaldson (2005).

nomic variables such as output, consumption, worked hours and asset wealth in Korea. Therefore, this thesis also aims to fill this gap in literature. To this end, the thesis is organized in five chapters, including this introduction. Next we describe briefly the content of each chapter.

Chapter 2 examines the relation among consumption, financial wealth and labor income in Korea. To understand the link among those variables has been an ongoing subject of academic economists as well as policymakers. A great number of studies have been done in this field and the main interest was the response of consumption expenditure to the change in household's wealth, so-called wealth effect on consumption. Following the methodology suggested by Lettau and Ludvigson (2004), this chapter, however, studies the relation among three variables from a different angle by decomposing the movements of three variables into trend (i.e., permanent) components and cyclical (i.e., transitory) components.

Chapter 2 begins from the notion suggested by Lettau and Ludvigson (2001) that there must be an equilibrium relation among consumption, financial wealth and labor income in theory. To examine this relation empirically in Korea, this chapter estimates a 3-dimensional vector error correction model (simply VECM) which incorporates the cointegrating relation into the vector autoregressive (hereafter VAR) approach. The VECM approach has been popular because it provides a useful framework to separate the long-run and short-run information in the data. More importantly, the presence of r cointegrating relations in a p-dimensional VAR model implies $p-r$ common trends in the system and thus it can help separate the persistent trends (i.e, permanent components) from the temporary cycles (i.e., transitory components) as in Stock and Watson (1988) and Gonzalo and Granger (1995).

From this background, this chapter attempts to answer the following questions. First, how strong is the wealth effect on consumption? Second, which variable mainly adjusts to regain the long run relation among consumption, financial wealth and labor income in Korea? Third, how much fraction of fluctuations of each variable are attributable to the permanent and transitory shocks, respectively? That is, which shocks are the primary causes of variability in three endogenous variables? In addition, how do the system's endogenous variables respond dynamically to the transitory shock? Lastly, did the Korean financial crisis bring out any meaningful differences for this relation?

By answering these questions, Chapter 2 expects not only to present an opportunity to cross-check whether the previous findings in the developed countries are robust even in an emerging economy, but also to propose some practical implications to Korean policymakers.

During the third and fourth chapters, this thesis discusses the Korean business cycles with micro-founded dynamic stochastic general equilibrium (hereafter DSGE) models. A DSGE model is firstly converted into an empirically implementable system of equations by linear approximation and then model parameters are estimated by the maximum-likelihood approach, facilitated by Kalman filter, in Chapter 3 and by the Bayesian approach in Chapter 4. The motivation of this choice is clear. Contrary to the empirical work founded on the reduced-form estimation, micro-founded DSGE models are transparent enough that researchers can clearly understand the relationship of key macroeconomic variables. Apparently, this transparency helps us tell a story on the dynamics of the economy. From the theoretical perspective, another benefit is that DSGE models are free from Lucas critique (1976) and thus meet modern standards of conceptual rigor (Woodford 2003).

Chapter 3 estimates a Hansen's (1985) baseline DSGE model augmented with Vector Autoregression (VAR) measurement errors, which is proposed by Ireland (2004), for Korea economy. By this augmentation, this chapter attempts to combine the advantage of DSGE models with the flexibility of VAR models. The DSGE model's parameters are estimated by maximum likelihood method in the Kalman filtering algorithm. Hence, one contribution of this essay is that these estimated parameters can be utilized for future research in calibrated DSGE models for Korea economy.

As noted by Watson (1994), two important questions must be answered in any business cycle research. First, how do the variables respond to exogenous shocks and how long? Second, are the business cycles largely the result of supply shocks like productivity shocks? In other words, what are the important sources of economic fluctuations? This chapter attempts to answer these questions by taking advantage of the impulse response function and the variance decomposition analysis. In addition, it compares the prediction performance of the micro-founded DSGE model with those of pure time series models in order to check the robustness of the current DSGE model. Finally, by comparing the second moments for the data and the simulation, this chapter also attempts to evaluate the estimated model's goodness of fit.

Chapter 4 extends the baseline DSGE model in Chapter 3 by introducing money through cash-in-advance constraints, that is, firms should borrow cash to pay wages in advance while households have to hold cash to purchase consumption goods. Following the literature, this chapter starts by setting up two versions of cash-in-advance models: a baseline cash-in-advance (hereafter CIA) model and a limited participation (hereafter LP) model (see, for instance, Christiano and Eichenbaum 1992; Nason and Cogley 1994). Then, following Schorfheide (2000), we estimate these models using the Bayesian approach.

The two models share the similar structure but there is a key difference between them regarding information structure. While in the CIA model households make a decision on saving after they observe the current realizations of the exogenous shocks, in the LP model households make a decision on saving before they observe the exogenous shocks following Lucas (1990) and Fuerst (1992). The important implication of this distinction becomes clear when we compare the effect of a monetary shock on the short-term interest rate in the two models.

There are two important stylized facts which any plausible monetary DSGE models should account for. First, although many monetary economists expect that an unexpected increase in the money supply reduces the short-term interest rate, a simple correlation analysis between those two variables persistently confirms that an expansionary monetary policy tends to be accompanied by the rise of interest rate rather than the fall of interest rate. This is referred to as the liquidity puzzle (Leeper and Roush 2003). This is potentially a puzzle in the sense that an increase in the money supply would require a decline in the nominal interest rate to persuade the households to hold larger money balances and thus to recover money market equilibrium. However, this essay argues that the liquidity puzzle is not a real puzzle if the liquidity effect is overwhelmed by the anticipated inflation effect in monetary DSGE models, as shown with Korean data. Second important observation needs to be addressed is that an expansionary monetary shock tends to increase the output in the short run but the effect vanishes in the long run.

Based on the estimated parameters, Chapter 4 discusses two models' predictions on theses two stylized facts. As will be clear in model comparison, the LP model is superior to the CIA model in terms of the goodness-of-fit for the data. More importantly, it demonstrates that the LP model is more advantageous to capture two stylized facts of the Korean business cycles. In particular, in response to the positive money shock, the output falls in the CIA model whereas the output rises in the LP model. Then, conditional on the LP model, this chapter compares two sub-sample periods: 1973:Q1 to 1997:Q4 and 1998:Q1 to 2006:Q3 to examine how several institutional and policy changes, such as independence of the central bank and adoption of inflation targeting, during the Korean financial crisis have affected Korea economy.

Finally, before moving on, Bayesian estimation of DSGE models in this chapter are motivated for the following reasons. In general, by combining the prior knowledge in calibration with likelihood from the data, we expect this joint information to alleviate the *dilemma of absurd parameter estimates*, which we often encounter during the maximum likelihood estimation (An and Schorfheide 2007). This is an important point for researchers who would like to study the business cycles in an emerging market economy like Korea. In this case, contrary to the researchers in the developed economies, they do not have luxury of sufficient data. Then, by allowing the prior distributions to play an important role during the estimation, we expect that this additional information will help us sharpen our inference.

Chapter 5 will conclude the thesis.

Chapter 2

THE RELATIONSHIP AMONG CONSUMPTION, WEALTH AND INCOME IN KOREA

2.1 Introduction

In recent years, the linkage between household wealth and consumption has received increased attention both among academic researchers as well as policymakers. This renewed interest is in part due to the dramatic boom and subsequent bust in stock market in the late 1990s and the more recent surge and sub-prime mortgage crisis in housing market across the world. In academics, to understand the link between the consumption and wealth has been an ongoing subject of macroeconomists as well as financial economists. In particular, the response of consumption expenditure to the change in households' wealth, so-called wealth effect on consumption, has been extensively researched since Ando and Modigliani (1963) provided the framework of the life-cycle model. On the other hand, from the perspectives of policymakers, the effect of asset wealth on consumer spending has raised huge discussions of whether the central banks should consider asset prices in setting monetary policy (Greenspan 1999; Trichet 2005).

Modigliani's life-cycle hypothesis states that income varies somewhat predictably over a person's life and that consumers use saving and dissaving to smooth their consumption over their lifetime. Unlike Keynes' current income hypothesis, in which current consumption depends primarily on current income, the life-cycle hypothesis implies that consumers determine their spending level based on not only income but also wealth, because they consume constant fraction of expected entire lifetime human and nonhuman resources. Hence, this hypothesis implies that current consumption must be financed by two sources as in (2.1.1).[1]

$$C_t = aY_t + bW_t + u_t, \tag{2.1.1}$$

where C_t is the consumption level during period t, Y_t is the labor income during period t, W_t is the financial wealth at the beginning of period t and u_t is the error term.

It is well acknowledged that a simple ordinary least squares (hereafter OLS) estimation of (2.1.1) yields spurious correlations between those non-stationary variables. Thus, as stated by Davis and Palumbo (2001), previous studies modify (2.1.1) into (2.1.2), based on the assumption of empirical stationary relation between the logs of consumption and income, to investigate the link between consumption and wealth.

$$c_t = \alpha + \gamma y_t + \beta w_t + \varepsilon_t, \tag{2.1.2}$$

where ε_t is the error term and lower-case letters c_t, y_t, and w_t represent the logarithms of consumption, income and wealth, respectively.[2]

The comparison between Equation (2.1.1) and Equation (2.1.2) reveals that the wealth effects can be measured in the empirical literature in two ways. First, the marginal propensity to consume (hereafter MPC) from household wealth, $b = \frac{\partial C}{\partial W}$, in Equation (2.1.1) measures how much aggregate consumption increases due to the

[1] For example, this equation appears in Equation (6) of Davis and Palumbo (2001).

[2] To derive (2.1.2) from (2.1.1), let us re-scale (2.1.1) first by dividing both sides with the level of income. As a result, we get (2.1.3),

$$\frac{C_t}{Y_t} = a + b\frac{W_t}{Y_t} + \frac{u_t}{Y_t}. \tag{2.1.3}$$

Then, log-linearizing (2.1.3) around the steady state yields

$$\begin{aligned} c_t - y_t &= \ln(a + b\frac{W_t}{Y_t}) \\ &= \ln[a + \exp(\ln b + w_t - y_t)] \\ &\approx \ln(a + b\frac{W}{Y}) + \frac{1}{(a + b\frac{W}{Y})} \times b\frac{W}{Y}[w_t - y_t - (w - y)] \\ &= \mu + \rho(w_t - y_t). \end{aligned}$$

Therefore, $c_t \approx \mu + \rho w_t + (1-\rho)y_t$, where $\rho = \frac{1}{(a+b\frac{W}{Y})} \times b\frac{W}{Y}$ and $\mu = \ln(a + b\frac{W}{Y}) - \rho(w - y)$.

increase in asset wealth. So it is usually expressed as the number of pence spent out of each pound. Second, the elasticity of consumption with respect to wealth, $\beta = \frac{\partial C}{\partial W} \times \frac{W_t}{C_t}$, in Equation (2.1.2) is the percentage change in consumer spending brought about by each percentage change in wealth. Thus, two measures are connected as $\beta = b \times \frac{W_t}{C_t}$.

Based on this background, the primary objective of the present essay is to explicitly examine the relationships among aggregate consumption, financial wealth and labor income using Korea data. To this end, this chapter estimates a 3-dimensional vector autoregressive (hereafter VAR) model which emphasizes modeling of the dynamic interactions of the variables of interest (Sims 1980). It is well understood that the standard OLS estimation like (2.1.2) requires a questionable assumption of weak exogeneity of labor income and financial wealth. In contrast, in VAR models all observed variables are typically treated as a priori endogenous. Therefore, it is often argued that VAR approach provides a flexible and tractable framework for capturing the dynamic structure of many time series variables (Lütkepohl 2006).

More specifically, this chapter estimates a cointegrated VAR model, namely vector error correction model (hereafter VECM), which explicitly takes into account the cointegrating relations among the variables, following Engle and Granger (1987), Stock and Watson (1993), and Johansen (1995) among others. Then, the VECM approach enables us to separate the short-run dynamics of three macro-variables from their long-run dynamics. In this case, this long-run relationship, namely cointegration relation, is also of interest on its own because it can be often associated with the economic equilibrium. More importantly, the presence of r cointegrating relations in a p-dimensional VAR model implies $p - r$ common trends in the system and thus it can help separate the persistent trends (i.e, permanent components) from the temporary cycles (i.e., transitory components) as in Stock and Watson (1988) and Gonzalo and Granger (1995).

Then, what is the theoretical reason to believe that there is a cointegrating relation among consumption, asset wealth and income? Recently, Lettau and Ludvigson (2001, 2004) provide a straightforward theoretical rationale by building off the forward-looking model of consumer behavior initially developed in Campbell and Mankiw (1989) and Campbell (1993). In addition, they find that, in the U.S., the transitory departure from the long-run relationship between consumption, asset wealth and income is recovered by the adjustment of asset wealth rather than those of consumption and income. So, this deviation has a predictive power on the changes of asset wealth but neither on the changes of consumption nor the changes of income. Similar findings are reported for the U.K. (Fernandez-Corugedo, Price, and Blake 2003) and for Australia (Fisher and Voss 2004).

With this background in mind, this chapter aims to answer the following questions by employing Lettau and Ludvigson's methodology to the Korean economic data. First, how strong is the wealth effect on consumption? Second, which variable mainly adjusts to regain the long run relation among consumption, financial wealth and labor income in Korea? Third, how much fraction of fluctuations of each variable are attributable to the permanent and transitory shocks, respectively? That is, which shocks are the primary causes of variability in the endogenous variables? In addition, how do the system's endogenous variables respond dynamically to the transitory shock? Lastly, did the Korean financial crisis make any difference for this relation?

By answering these questions, this essay expects not only to present an opportunity to cross-check whether the previous findings in the developed countries are robust even in an emerging market, but also to propose some practical implications to Korean policymakers. To this end, the current chapter employs the Johansen's (1995) full information maximum likelihood estimation method.

The remainder of the chapter is organized as follows. Literature review on the responsiveness of consumption to wealth change is briefly presented in the next section. Subsequently, Section 2.3 focuses on the theoretical background of equilibrium relation among consumption, financial wealth and labor income. Section 2.4 first provides data descriptions and their properties. Then, it also presents methodology used in the empirical analysis. In Section 2.5, this essay reports the empirical findings and their economic implications. Section 2.6 concludes this chapter.

2.2 Literature Review

Although some researchers raise a doubt over the presence of wealth effect on consumption,[3] most macroeconomists agree with the existence of wealth effect since the intertemporal budget constraint clearly implies that the increase in wealth must lead to

[3]For example, Morck, Shleifer, Vishny, Shapiro, and Poterba (1990) claim that asset returns are correlated with future output growth and thus are just a leading indicator of aggregate consumption. Analogously, Poterba, Samwick, Shleifer, and Shiller (1995) argue that future expectation is the main driving force to influence on both consumer spending and wealth.

an increase in consumer spending today or in the future (Poterba 2000). Accordingly, it seems that there must be a significant statistical relationship between wealth and spending.

The bulk of the studies in this field have focused on the U.S. economy. The traditional macroeconometric models estimate that a dollar's increase in household wealth boosts the consumption expenditure by 3 to 7 cents per year. Brayton and Tinsley (1996) report that the marginal propensity to consume (MPC) out of stock wealth (i.e., equity wealth) is 3.0 cents and the MPC out of non-stock wealth is 7.5 cents to a dollar increase. In addition, Davis and Palumbo (2001) report that the long-run wealth effect out of total wealth ranges from 3 to 6 cents to a dollar in different specifications. They also find the slightly lower MPC out of stock wealth than that out of total wealth.

In this line of research, Ludvigson and Steindel (1999) report that the MPC out of stock wealth is 4.0 cents and the MPC out of non-stock wealth is 3.8 cents using the simple OLS estimation of a traditional life-cycle model. As noted in their article, however, these coefficients are not reliable because of non-stationary data as well as the presence of endogeneity bias (i.e., not only the effect of an increase in wealth on consumption, but also the effect of an increase in consumption on wealth). Therefore, following Stock and Watson (1993), they use a dynamic least squares (hereafter DLS) procedure which removes the effects of regressor endogeneity in the least squares estimator. According to DLS estimation results, a one dollar increase in wealth typically leads to a three-to-four cent trend increase in consumer expenditure. However, they do not find a significant short-run effect, and conclude that the change in wealth in the current quarter does not lead the significant change in consumption expenditure over one or more quarter later.

On the other hand, Case, Quigley, and Shiller (2005) compare the importance of stock market wealth and housing market wealth upon consumption and find a considerably higher impact of housing wealth change upon consumption change. They also find an evidence that the consumption response to changes in housing wealth is asymmetric. Their results show that increases in housing market wealth have positive and significant impact upon household consumption expenditure but that declines in housing market wealth have little effect upon consumption.

However, more recently, Lettau and Ludvigson (2004) find that most of the variation in household net worth in the U.S. is generated by transitory innovations, while variation in aggregate consumption is dominated by permanent shocks. For this reason, they argue that only a very small fraction of the variation in asset holdings is associated with the variation in aggregate consumption and that conventional estimates of the wealth effect possibly overstate the response of consumption to wealth changes.[4] They also argue that only the permanent changes in wealth are related with consumption growth. From this perspective, they conclude that the traditional measure of wealth effect, i.e., marginal propensity to consume out of wealth, cannot relevantly quantify the response of consumption to wealth changes.

For other developed countries, Boone, Giorno, and Richardson (1998) inspect the stock market wealth effect in the G7 countries. They report that a dollar increase in stock wealth leads to 4.5 cents in consumption growth for the U.S. Then, by calibration, they present the wealth effect for other countries, 4.4 for Canada, 4.0 for the U.K., 1.5 for Japan and Italy, 0.8 for Germany, 0.7 for France.[5] For Australian economy, Dvornak and Kohler (2007) find that both housing wealth and stock market wealth have a significant effect on consumption expenditure. They present that the MPC out of stock wealth as 6 to 9 cents in the long run and the MPC out of housing wealth as around 3 cents.

In the recent, following Lettau and Ludvigson (2001, 2004), Fernandez-Corugedo et al. (2003) report the MPC out of asset holdings in the U.K. as around 5 pence to one pound increase in wealth. While Tan and Voss (2003) present the MPC for Australia as 4 cents,[6] Hamburg, Hoffmann, and Keller (2005) report that a one Euro increase in asset wealth leads to a $4 \sim 5$ Euro cent increase in consumption spending for Germany.

On the other hand, due to the limitation of available data, most research on emerging market uses the price data as proxies for wealth. For example, Funke (2004) examines the relationship between private consumption and stock markets for 16 emerging markets and finds a small, but statistically significant link between private consump-

[4]In their research, a variance-decomposition analysis shows that the vast majority of quarterly fluctuations in asset values are attributable to transitory innovations that display virtually no association with consumption, contemporaneously, or at any future date.

[5]For example, the MPC for Canada is calculated by $MPC_{Cananda} = MPC_{US} \times \frac{\text{(equity as \% of disposable income in Canada)}}{\text{(equity as \% of disposable income in the U.S)}}$.

[6]They estimate the consumption equation using a single equation error correction model (ECM) rather than VECM. As critically noted by Lettau, Ludvigson, and Bárczi (2001) and Fernandez-Corugedo et al. (2003) among others, however, only when wealth and income are weakly exogenous, a single ECM is able to produces a consistent estimate on consumption.

tion growth and stock returns.[7] His estimations indicate that a 10% increase in stock prices is associated with an increase in private consumption between 0.2% and 0.4%.

Given these findings for other countries, the next section discusses the theoretical rationale for the presence of cointegrating relation among consumption, asset wealth and labor income.

2.3 Theoretical Framework

This section presents a general framework linking macroeconomic variables (i.e., consumption, asset wealth, and labor income with financial variable (i.e., expected returns on wealth) following Lettau and Ludvigson (2001, 2004). The starting point of their framework is Campbell and Mankiw (1989).

Consider a representative agent economy in which a consumer is deciding how much to consume today versus how much to save for the future in order to achieve the highest level of lifetime satisfaction. In Campbell and Mankiw (1989), the consumer faces an intertemporal budget constraint of (2.3.1), which shows how the available source for consumption evolves.

$$W_{t+1} = (1 + R^w_{t+1})(W_t - C_t), \tag{2.3.1}$$

where real total wealth, W_t, is defined as the sum of observable tangible asset wealth and unobservable intangible human wealth at the beginning of period t, and R^w_{t+1} is the real simple net return on aggregate wealth from t to $t+1$, and C_t is aggregate consumption. Note that labor income does not appear explicitly in wealth accumulation equation (2.3.1) because of the assumption that the market value of tradable human wealth is already included in aggregate wealth.

Starting from (2.3.1), Campbell and Mankiw (1989) and Campbell (1993) derive a useful expression of the loglinear consumption-wealth ratio of (2.3.2) by taking a first-order Taylor expansion of both sides of (2.3.1).[8] Appendix 2.A shows how (2.3.2)

[7] His sample includes eight Asian (India, Indonesia, Korea, Malaysia, Pakistan, the Philippines, Thailand, Turkey), six Latin American (Argentina, Brazil, Chile, Columbia, Mexico, Venezuela), and two African (Nigeria, Zimbabwe) countries.

[8] Campbell and Viceira (2002) present an excellent summary of this line of research.

can be derived solely from (2.3.1).

$$c_t - w_t = E_t \sum_{j=1}^{\infty} \rho^j (r^w_{t+j} - \Delta c_{t+j}) + \frac{\rho \chi}{1-\rho}, \tag{2.3.2}$$

where $c_t = \ln C_t$, $w_t = \ln W_t$, $r^w_{t+j} = \ln(1 + R^w_{t+j})$, χ is a constant, and ρ is the ratio of investment to total wealth at the steady state, i.e., $\rho = \frac{W-C}{W} < 1$.

This is a log-linear version of the infinite horizon budget constraint (2.3.1). One attractive feature of (2.3.2) is that no theory of consumer behavior (i.e., utility function) is needed in the model. Although (2.3.2) is not a solution for consumption, it has an important implication in linking macroeconomics and financial economics. The implication of (2.3.2) is that higher consumption-wealth ratio today will use up wealth and thus future consumption possibilities unless it is offset by higher future rates of return on invested wealth.[9] So the aggregate consumption-wealth ratio is a function of expected future returns on market portfolio and a forward-looking agent consumes today based on the expected future returns conditioning on information set at time t.

However, a critical obstacle of taking (2.3.2) into the data is that human wealth is not observable, so are therefore total wealth and return on total wealth.[10] In order to make (2.3.2) operational, this essay employs Lettau and Ludvigson's (2001, 2004) empirical approximation of the consumption-wealth ratio in terms of a cointegrating relation among consumption, asset wealth and labor income. This section only briefly introduces their approach, but Appendix 2.B illustrates the detailed derivation procedure.

First, notice that total wealth (W_t) is composed of asset wealth (A_t), and human wealth (H_t).

$$W_t = A_t + H_t. \tag{2.3.3}$$

If we denote the average share of asset wealth in total wealth as ω, and average share

[9]If the optimization behavior is considered, the interpretation has different meanings. For example, if we assume a quadratic utility function, zero subjective discount rate and zero interest rate, we arrives at Hall (1978)'s random walk hypothesis of $C_t = C_{t-1} + \varepsilon_t$. In this case, the expectation that consumption rises in the future implies that the current marginal utility of consumption is larger than the expected future marginal utility of consumption. Hence, the consumers adjust their current consumption to the point where consumption is not expected to change (Romer 2006, p. 354).

[10]Furthermore, theoretical log consumption, c_t, which includes not only non-durables and service but also the streams of durable consumption, is not observable in reality, either. This issue will be discussed in Section 2.4.

of human wealth as $(1-\omega)$, the log of total wealth can be approximated as a linear combination of asset wealth and human wealth,

$$w_t \approx \omega a_t + (1-\omega)h_t, \tag{2.3.4}$$

where $a_t = \ln A_t$ and $h_t = \ln H_t$.

Second, we can express the real simple gross return on total wealth as an exact linear combination of the returns on asset wealth and human wealth as (2.3.5)

$$(1+R^w_{t+1}) = \omega(1+R^a_{t+1}) + (1-\omega)(1+R^h_{t+1}), \tag{2.3.5}$$

where R^a_{t+1} and R^h_{t+1} are the simple net returns on asset wealth and human wealth, respectively. Then, as shown in Campbell (1996, p. 308), Equation (2.3.5) can be approximated with the corresponding log returns r_{t+1}'s into (2.3.6),

$$r^w_{t+1} \approx \omega r^a_{t+1} + (1-\omega)r^h_{t+1}. \tag{2.3.6}$$

At this stage, if we omit the unimportant constant, Equation (2.3.2) can be reformulated as (2.3.7) by making use of (2.3.4) and (2.3.6),

$$c_t - \omega a_t - (1-\omega)h_t = E_t \sum_{j=1}^{\infty} \rho^j \left[\omega r^a_{t+j} + (1-\omega)r^h_{t+j} - \Delta c_{t+j}\right]. \tag{2.3.7}$$

A real challenge to take (2.3.7) into the data is that we cannot observe either the log of human wealth (h_t) or the log return on it (r^h_{t+1}). To circumvent this observability problem, Lettau and Ludvigson (2001, 2004) follow Campbell's (1996) suggestion that the observable labor income can be thought of as the dividend on human wealth.[11] Therefore, rather than attempting to search for the proxies for these variables, they try to link unobservable stock variable (i.e., human wealth H_t), to observable flow variable (i.e., aggregate labor income Y_t), by assuming that labor income is the product of a simple net return to human wealth and the human wealth. Under this circumstance,

[11] On the other hand, if we simply assume that return is not time-varying and income follows a random-walk, it can be shown that income is the constant fraction of human wealth.

H_t and Y_t are related as (2.3.8),

$$(1 + R^h_{t+1})H_t = E_t(H_{t+1} + Y_{t+1}) \Leftrightarrow H_t = E_t\left[\frac{(H_{t+1} + Y_{t+1})}{(1 + R^h_{t+1})}\right], \tag{2.3.8}$$

where R^h_{t+1} is a simple net return to human wealth, H_t is human wealth at the beginning of period t and Y_t is labor income during period t.[12]

Then, by applying the transversality condition and log-linearizing (2.3.8), the log of human wealth can be expressed as follows,

$$h_t = \kappa + y_t + z_t, \tag{2.3.9}$$

where $h_t = \ln H_t$, $y_t = \ln Y_t$, κ is a constant, and z_t is a mean zero stationary process. So, the unobservable human wealth can be captured by the observable labor income.

At last, replacing h_t in (2.3.7) by (2.3.9) and ignoring the unimportant constant yield testable Equation (2.3.10),

$$c_t - \omega a_t - (1-\omega)y_t = E_t\sum_{j=1}^{\infty}\rho^j\left[\omega r^a_{t+j} + (1-\omega)r^h_{t+j} - \Delta c_{t+j}\right] + (1-\omega)z_t, \tag{2.3.10}$$

where ω is the average share of asset holdings in total wealth ($= \frac{A}{W}$), ρ is the average share of investment to total wealth ($= \frac{W-C}{W}$), r^a_{t+j} and r^h_{t+j} are the returns on asset holdings and human wealth from the period $t+j-1$ to the period $t+j$, respectively, lastly z_t is a mean zero stationary process.

Note that (2.3.10) is a testable equation because it does not include the unobservable total wealth and the return on total wealth. More importantly, this implies that there is a trivariate cointegrating relation involving consumption, asset wealth and labor income. Since all the terms on the right-hand side of (2.3.10) are found to be stationary,[13] and ρ is less than one, $c_t - \omega a_t - (1-\omega)y_t$ on the left-hand side also should be stationary, that is, c_t, a_t and y_t must be cointegrated in theory.

[12] Note that (2.3.8) holds exactly the same interpretation of the present value model for the stock price,

$$P_t = E_t\left[\frac{(P_{t+1} + D_{t+1})}{(1 + R_{t+1})}\right].$$

That is, H_t is interpreted as the marketable present value of human wealth and its dividend during the current period is Y_t.

[13] It means that the returns on asset and human wealth are stationary and consumption and labor income are integrated of order one. See Appendix 2.B, for details.

This cointegration or long-run relation among consumption, asset wealth and labor income implied in (2.3.10) is of particular interest. The reason is that this cointegrating relationship uniquely identifies the average share of asset holdings in total wealth, and the average share of human wealth in total wealth. This provides a crucial information for policy makers on how average consumers in Korea allocate their resources between the observable asset wealth and the unobservable human wealth.[14]

Under this theoretical background, the next section presents data description, estimation methodology and the related preliminary test results.

2.4 Econometric Framework and Preliminary Test Results

2.4.1 Data and Unit Root Tests

2.4.1.1 Data Descriptions

The Korean data set covers the period of between the first quarter of 1980 and the third quarter of 2005 with a total of 103 observations for each series. These are gathered from the Bank of Korea (BOK) and Korean National Statistics Office (KNSO) on the quarterly basis. To obtain the per capita real consumption, real net wealth and real labor income, firstly corresponding aggregate time series are divided by the population of age 15 and over. In particular, asset net wealth comes from the Flow of Funds Accounts. Following the previous studies, this essay uses seasonally adjusted time series for real consumption and real labor income in BOK's database to eliminate the irrelevant movements in the current research.

Contrary to consumption and labor income, households' asset net wealth data are provided only in nominal terms. Therefore, for data consistency, nominal asset net

[14]Moreover, Equation (2.3.10) implies that the positive departure from this long-run relationship at time t must reflect a rational agent's expectation at time t on either higher future returns on nonhuman and/or human wealth if he or she is not planning to reduce future consumption growth. Put differently, if consumption is mainly affected by persistent shocks and asset wealth is mainly affected by transitory shocks, the deviations from the long-run relationship must forecast the returns on wealth. See Cochrane (2008) for rather a comprehensive literature review for connection between asset returns and macroeconomic variables.

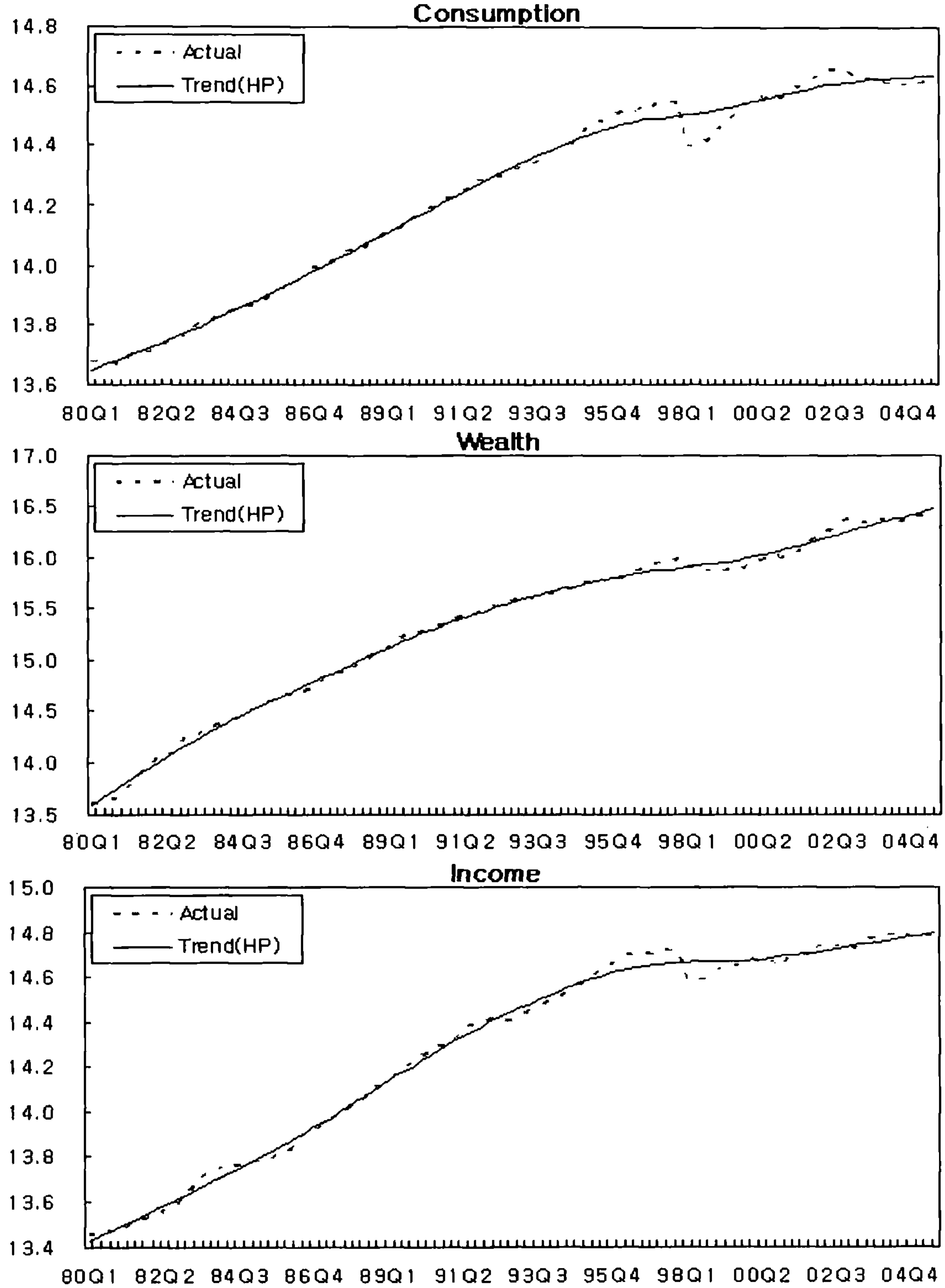

Figure 2.1: Total Consumption, Financial Wealth, and Income

wealth series are deflated by the quarterly GDP deflator (GDPD).[15] Figure 2.1 shows movements of the three variables in logarithm. It presents HP-filtered trends and actual data for those variables. As the current essay uses the quarterly data, the HP-filtered data are obtained using $\lambda = 1600$, relative weight on smoothness, following Hodrick and Prescott's suggestion. Two features draw our attention. First, during the examined period, although growth has been the dominant macroeconomic feature in Korea, the fluctuations of three variables are also noticeable. From this perspective, the choice of VECM approach in this essay seems to be relevant to account for the characteristics of those variables in Korea since VECM allows us to disentangle the short-run dynamics from the long-run dynamics and thus to analyze those two dynamics more clearly. Second, there are the blips in three time series which reflect the effect of Korean financial crisis in the late 1997.

Before proceeding further, it will be convenient to clarify the notations. This essay denotes the log data as the lower letters.

c_t : log of real per capita total consumption over period t,

a_t : log of real per capita net financial wealth at the beginning of period t,

y_t : log of real per capita income over period t,

$\boldsymbol{\beta}'\mathbf{x}_t$: deviation from the long-run relationship over period t, i.e., $c_t - \beta_0 - \beta_a a_t - \beta_y y_t$.

So, Δc_t, Δa_t and Δy_t indicate the growth rate of each variable, respectively.

With respect to the data, there is an important issue on total consumption expenditure. In their articles, Lettau and Ludvigson use the modified total consumption by inflating the expenditure on non-durables and services excluding shoes and clothing into the mean level of total consumption. The underlying rationale for this choice is that the log consumption of non-durables and services is proportional to the log total flow of consumption over the examined period. However, this assumption has been severely criticized by Rudd and Whelan (2006) because the total real consumption

[15]For example, the real asset net wealth at 1980:Q1 is calculated as follows,

$$\text{Nominal asset wealth at 1980:Q1} \times \frac{\text{GDPD 2000 average}}{\text{GDPD 1980:Q1}}.$$

Thus, real asset wealth is constructed using the same GDP deflator that was used to construct real consumption and labor income.

have consistently grown faster than real consumption of non-durables and services in the U.S.[16]

More importantly, Rudd and Whelan (2006) argue that the intertemporal budget constraint (2.3.1) clearly states that durable goods must be included either in asset wealth or in consumption but not in both measures. It means that if the asset wealth contains durable goods like the case in Lettau and Ludvigson, the consumption must not include durable goods. From this perspective, Rudd and Whelan (2006) suggest a simple strategy for this problem: defining c_t as total real consumption and a_t as asset wealth excluding durables. Since the asset wealth in the Flow of Funds Accounts in Korea does not include the durables, this essay uses households' total consumption as c_t following this claim.

Finally, before regression analysis is implemented, there is a crucial preliminary step to distinguish between a stationary variable, which exhibits a significant tendency to mean reversion, and a non-stationary variable, which exhibits a high degree of time persistence (i.e., insignificant mean reversion). As well documented in Nelson and Plosser (1982), the reason is that many macroeconomic time series often contain stochastic trends and, in this case, regression results with non-stationary (or integrated) variables may be spurious.[17] The following section, therefore, will present the stationarity test results for c_t, a_t, and y_t.

2.4.1.2 Tests for the Presence of a Unit Root in c_t, a_t, and y_t

In order to determine the order of integration of c_t, a_t, and y_t, this essay carries out two popular kinds of stationarity tests for those variables: Augmented Dickey-Fuller (hereafter ADF) test (Dickey and Fuller 1979) and KPSS test (Kwiatkowski, Phillips, Schmidt, and Shin 1992).[18] Note first that the existence of a unit root in c_t, a_t, and

[16]Alternatively, if the total flow of consumption, which includes the consumption service out of the stock of durables, is proportional to the non-durable consumption, Lettau and Ludvigson's measure for consumption is correct as long as the returns include the imputed rental income from durables. However, this is more problematic because the total flow of consumption cannot be even observable.

[17]Consider a regression $y_t = \beta_0+\beta_1 x_t+\varepsilon_t$, in which x_t and y_t are independent random walk processes. Then the true value of β_1 is 0. Nonetheless, the Monte Carlo study by Granger and Newbold (1974) reveals that the limiting distribution of $\hat{\beta}_1$ is such that $\hat{\beta}_1$ converges to a function of Brownian motions. They warn that the regression using non-stationary variables can produce a seemingly significant but spurious result with a high R^2 (coefficient of determination).

[18]The integration in a stochastic process is defined as follows.
A stochastic process $\{x_t\}$ is said to be integrated of order d, i.e., $\{x_t\} \sim I(d)$, if it need to be differenced d times in order to achieve the stationarity.

y_t and the non-existence of a unit root in Δc_t, Δa_t, and Δy_t jointly imply that those variables in level are all I(1). Then, the ADF test for the presence of a unit root defines the null hypothesis as the presence of a unit root (i.e., non-stationarity), whereas the KPSS test specifies it as the stationarity of the data. For example, to determine whether a stochastic process $\{x_t\} \sim$ I(1) or $\{x_t\} \sim$ I(0), the null and alternative hypotheses are constructed as in Table 2.1.

Table 2.1: Null and Alternative Hypotheses in ADF and KPSS Tests

ADF test		KPSS test
$x_t \sim I(1)$	H_0	$x_t \sim I(0)$
$x_t \sim I(0)$	H_1	$x_t \sim I(1)$

Note: $I(1)$ denotes that x_t is integrated of order one (existence of a unit root) and $I(0)$ indicates that x_t is stationary.

Notice that, by taking the null hypothesis as a (trend-) stationary process and the unit root as an alternative, KPSS test is therefore consistent with a conservative testing strategy. Kwiatkowski et al. (1992) claim that one should use the tests that place the hypothesis we are interested in as the alternative one. Under this conservative strategy, if we reject the null hypothesis, we can be more confident that the series has indeed a unit root.

Maddala and Kim (1998) recommend this joint procedure because the power of unit root tests such as the ADF and the Phillips-Perron's (1988) tests is often argued to be very low when the process is stationary but with a root close to the non-stationary boundary. Thus, tests which take the null hypothesis as a presence of a unit root often find a unit root in the process, even though the process is indeed stationary.

Table 2.2 presents the ADF test results for a unit root in c_t, a_t, and y_t and the

To understand the implication of integration, consider a univariate random walk process with drift, $x_t = \alpha + x_{t-1} + \varepsilon_t$, where ε_t is a mean zero i.i.d process. Solving x_t in terms of α and an initial condition x_0 yields $x_t = x_0 + \alpha t + \sum_{i=1}^{t} \varepsilon_i$. Note that removing a linear trend does not make x_t weakly stationary since there is a stochastic trend $\sum_{i=1}^{t} \varepsilon_i$. The presence of component $\sum_{i=1}^{t} \varepsilon_i$, which accumulates shocks from 1 to t (discrete integration), implies that one-time shock changes the level of x permanently thereafter. Note that the first difference of x_t, Δx_t, does not includes that summation term. Now since Δx_t only includes α and a stationary ε_t, it is clearly integrated of order zero. See, for details, Enders (2004) and Hamilton (1994).

corresponding first differenced variables.[19] It strongly favors the argument that each variable is integrated at the same order of one, that it, $I(1)$. The ADF test statistics for three level data are far less negative than the 10% critical value, but those for three first differenced data are far more negative than the 5% critical value.

Table 2.2: ADF Tests for Presence of Unit Roots in c_t, a_t, and y_t

	Lag Length			
	1	2	3	4
c_t	-0.94	-1.13	-1.23	-1.26
a_t	-1.92	-1.90	-1.78	-2.01
y_t	-0.14	-0.50	-0.48	-0.67
Δc_t	-5.81	-4.90	-4.38	-4.55
Δa_t	-4.95	-5.20	-3.61	-4.08
Δy_t	-5.33	-4.79	-3.90	-4.29
$<$ *Critical Values* $>$				
5 percent	-3.45			
10 percent	-3.15			

Note: Figures are t-statistics when the estimated model includes a constant and linear time trend. The critical values come from MacKinnon (1996).

On the other hand, Table 2.3 provides the KPSS stationarity Lagrange Multiplier test results, which corroborate the aforementioned argument.[20] For the variables in level, all figures are sufficiently larger than the 5% critical value and thus we can reject

[19]Given the sustained trend in all variables, the ADF test regression used for these series is,

$$\Delta x_t = \alpha_0 + \alpha_1 x_{t-1} + \alpha_2 t + \sum_{j=1}^{k} \beta_j \Delta x_{t-j} + \varepsilon_t. \tag{2.4.1}$$

In the ADF test of (2.4.1), the null hypothesis is $\alpha_1 = 0$ and lags are included to rectify the higher order correlation in error terms. Notice that since the standard t statistic does not have a limiting normal distribution but the Dickey Fuller (DF) distribution, the Dickey-Fuller critical values must be employed.

[20]Given the sustained trend in variables, the KPSS test uses the following model.

$$x_t = \xi t + r_t + \varepsilon_t$$
$$r_t = r_{t-1} + u_t,$$

where u_t is i.i.d. N(0, σ_u^2). Clearly, it is assumed that the series can be decomposed into the sum of a deterministic trend, a random walk, and a stationary error. The basic intuition of KPSS LM test is that if x_t is a trend stationary process under the null hypothesis, σ_u^2 will be very small.

the null hypothesis of stationarity. For the first differenced data, all figures, except the case of Δy_t with one lag, are larger than the 5% critical value and thus we cannot reject the null hypothesis.

Table 2.3: KPSS Tests for Stationarity of c_t, a_t, and y_t

	Lag Length			
	1	2	3	4
c_t	1.00	0.69	0.53	0.44
a_t	1.15	0.78	0.60	0.49
y_t	1.20	0.82	0.62	0.50
Δc_t	0.12	0.11	0.10	0.10
Δa_t	0.11	0.09	0.09	0.08
Δy_t	0.16	0.14	0.13	0.12
$< Critical\ Values >$				
5 percent	0.15			
10 percent	0.12			

Note: Figures are Lagrange Multiplier statistics when the estimated model includes a constant and linear time trend. The asymptotic critical values come from Kwiatkowski et al. (1992).

Hence, both test results strongly confirm that the existence of a unit root in c_t, a_t, and y_t and the non-existence of a unit root in Δc_t, Δa_t, and Δy_t. Thus, these test results jointly imply that those variables in level are all I(1).

In principle, the evidence that c_t, a_t, and y_t are all I(1) non-stationary implies that three shocks in the system (ε_{ct}, ε_{at}, and ε_{yt}) have permanent effects on the levels of these variables. Hence, there are three unit roots in the system. However, the presence of cointegrating relations, which will be discussed in the next section, implies that there are certain linear combinations among c_t, a_t, and y_t are I(0). For example, if there is one cointegrating relation and two unit roots in the system, the following regression is not spurious.

$$c_t = \beta_0 + \beta_a a_t + \beta_y y_t + \zeta_t. \tag{2.4.2}$$

Then, the above regression result can be naturally thought of as an economic equilibrium. Accordingly, three variables in level themselves wander arbitrarily far up and down, but they do not deviate too much from this long-run relation.

2.4.2 Methodology and Cointegration Tests

2.4.2.1 Vector Autoregressive Approach

The purpose of this section is to discuss how the cointegrated VAR approach can be applied to examine the dynamic relations among consumption, financial wealth and labor income. The unrestricted VAR model, which is essentially only a reformulation of the covariances of the data, has been frequently argued that it offers a powerful tool in data description and forecasting to macroeconomists without relying heavily on the theoretical model (Juselius 2006; Lütkepohl 2006).

However, the possibility of combining long-run and short-run information in the data by exploiting the cointegration relation in the restricted VAR model explains why the VAR methodology is still so popular in applied economics. In particular, given the theoretical rationale in Section 2.3 for cointegration among consumption, financial wealth and labor income and the empirical findings of their integrating properties, the cointegrated VAR model seems to provide a flexible framework to take the theoretical questions into the statistical models by imposing testable restrictions on the parameters.

Then, the problem is that how to link an unrestricted VAR model to a cointegrated VAR model. To achieve this goal, it is instructive to reparameterize an unrestricted VAR in terms of differences, lagged differences, and levels of endogenous variables. This reparameterized model is called as a vector error correction model (hereafter VECM).

Note first that, by assuming that there are no deterministic terms, an unrestricted simple 3-dimensional kth order VAR model for consumption, financial wealth and labor income can be estimated as (2.4.3),

$$\mathbf{x}_t = \mathbf{\Pi}_1\mathbf{x}_{t-1} + \mathbf{\Pi}_2\mathbf{x}_{t-2} + \cdots + \mathbf{\Pi}_k\mathbf{x}_{t-k} + \epsilon_t, \tag{2.4.3}$$

where $t = 1, \cdots, T$ and ϵ_t is an independently and normally distributed mean zero process vector, $\mathbf{x}_t$ is a 3-dimensional endogenous variable column vector (i.e. $[c_t\ a_t\ y_t]'$), and $\mathbf{\Pi}_i$'s are 3 by 3 coefficient matrices. In this case, therefore, it is assumed that $\mathbf{\Pi}_{k+1} = \mathbf{\Pi}_{k+2} =, \cdots, = \mathbf{\Pi}_{T-1} = \mathbf{0}$.

As shown in Appendix 2.C, the unrestricted VAR(k) model in level of (2.4.3) can be reparameterized as the VECM($k-1$) (2.4.4) in terms of differences, lagged differences, and levels of endogenous variables,

$$\begin{aligned}\Delta \mathbf{x}_t &= \mathbf{\Pi}\mathbf{x}_{t-1} + \mathbf{\Gamma}_1 \Delta \mathbf{x}_{t-1} + \cdots + \mathbf{\Gamma}_{k-1}\Delta \mathbf{x}_{t-(k-1)} + \epsilon_t \\ &= \mathbf{\Pi}\mathbf{x}_{t-1} + \sum_{i=1}^{k-1} \mathbf{\Gamma}_i \Delta \mathbf{x}_{t-i} + \epsilon_t, \end{aligned} \tag{2.4.4}$$

where $\mathbf{\Gamma}_i = -(\sum_{j=i+1}^{k} \mathbf{\Pi}_j)$ and $\mathbf{\Pi} = -(\mathbf{I} - \sum_{i=1}^{k} \mathbf{\Pi}_i)$.[21]

Now, it is clear that a VECM of (2.4.4) is nothing but a reparameterization of a VAR model of (2.4.3). As noted in Johansen (1995, p. 89), however, this reformulation has a clear advantage in the sense that it makes the coefficient matrices more interpretable. In (2.4.4), given $\mathbf{x}_t \sim I(1)$, the effects of levels are isolated in $\mathbf{\Pi}$ and the effects of short-term dynamics are separated in $\mathbf{\Gamma}_1, \ldots, \mathbf{\Gamma}_{k-1}$. In other words, the long-run properties of the system are summarized by matrix $\mathbf{\Pi}$ which is an accumulation of coefficient matrices in an unrestricted VAR model.

Notice that according to $rank(\mathbf{\Pi}) = r$, (2.4.4) can be classified into three different cases.[22]

1. If $r = 3$ (i.e., full rank), then $\mathbf{x}_t$ must be weakly stationary and standard inference of the unrestricted VAR applies. However, as we already know that $\mathbf{x}_t \sim I(1)$ from Section 2.4.1.2, this is not the case of interest.[23]

2. If $r = 0$ (i.e., $\mathbf{\Pi} = \mathbf{0}$), then $\mathbf{x}_t$ is non-stationary. Since there are 3 different trends (i.e., 3 unit roots) in $\mathbf{x}_t$, each variable is non-stationary with its own individual

[21] Note that this reparameterization can easily incorporates the case of presence of deterministic terms such as constants and linear trends by introducing a new variable vector $\mathbf{y}_t = \boldsymbol{\mu}_t + \mathbf{x}_t$, where $\boldsymbol{\mu}_t = \boldsymbol{\mu}_0 + \boldsymbol{\mu}_1 t$. Appendix 2.E clarifies this case.

Another equivalent representation of (2.4.3) can be found as,

$$\begin{aligned}\Delta \mathbf{x}_t &= \mathbf{\Upsilon}_1 \Delta \mathbf{x}_{t-1} + \mathbf{\Upsilon}_2 \Delta \mathbf{x}_{t-2} + \cdots + \mathbf{\Pi}\mathbf{x}_{t-k} + \epsilon_t \\ &= \sum_{i=1}^{k-1} \mathbf{\Upsilon}_i \Delta \mathbf{x}_{t-i} + \mathbf{\Pi}\mathbf{x}_{t-k} + \epsilon_t, \end{aligned} \tag{2.4.5}$$

where $\mathbf{\Upsilon}_i = -(\mathbf{I} - \sum_{j=1}^{i} \mathbf{\Pi}_j)$ and $\mathbf{\Pi} = -(\mathbf{I} - \sum_{i=1}^{k} \mathbf{\Pi}_i)$. Notice that $\mathbf{\Pi}$ in (2.4.5) and $\mathbf{\Pi}$ in (2.4.4) are exactly the same.

[22] The rank (r) is the number of linearly independent row in matrix $\mathbf{\Pi}$. Thus, if there is only one independent row, only one stationary combination is possible.

[23] When $\mathbf{x}_t \sim I(1)$, $\Delta \mathbf{x}_t \sim I(0)$. Then, it is inconsistent that the right hand side of (2.4.4) includes non-stationary processes while the left hand side of (2.4.4) is stationary. Hence, $\mathbf{\Pi}$ cannot have a full rank and there must be some linear combinations which cancel out the unit root process.

stochastic trend. In this case, the VAR model in levels can be reformulated as a VAR model in first differences with no loss of long-run information.

3. If $0 < r < 3$, the system is non-stationary but there are r stationary long-run relations among three endogenous variables. The next section will discuss this case in details.

2.4.2.2 Tests for Presence of Cointegrating Relations

Section 2.4.1.2 showed that consumption, financial wealth and labor income are all I(1) and then claimed that a simple OLS regression using these non-stationary variables may be spurious. However, Granger (1981) and Engle and Granger (1987) point out that in certain circumstances, a linear combination (i.e., weighted average) of two or more non-stationary series may be stationary since such variables share a common stochastic trend. In this special case, the non-stationary time series set is said to be cointegrated in Engle-Granger's terminology.

More precisely, cointegration can be defined as follows,

Let us assume that all variables in a vector of $\mathbf{x}_t$ *is integrated of order* d*. Then,* $\mathbf{x}_t$ *is said to be cointegrated, briefly,* $CI(d,b)$ *if there exist a non-zero cointegrating vector* $\boldsymbol{\beta}_i$ *such that* $\boldsymbol{\beta}_i'\mathbf{x}_t$ *is* $I(d-b)$*. If there exist* r *such linearly independent vectors* $\boldsymbol{\beta}_i$*,* $i = 1, \ldots, r$*, then* $\mathbf{x}_t$ *is said to be cointegrated with cointegrating rank* r*. The matrix* $\boldsymbol{\beta} = (\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_r)$ *is called the cointegrating matrix.*

Therefore, cointegration implies that certain linear combinations of the variables are integrated of lower order than the process itself.[24] Given the evidence of $\mathbf{x}_t \sim$ I(1), $\mathbf{x}_t \sim CI(1,1)$ implies that $\boldsymbol{\beta}'\mathbf{x}_t$ is a $r \times 1$ vector of stationary cointegrating relations. Then, Granger's representation theorem asserts that if $\boldsymbol{\Pi}$ has reduced rank $r < 3$ (i.e., at least one unit root in an unrestricted VAR model), there exist $3 \times r$ matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ each with rank r such that $\boldsymbol{\Pi} = \boldsymbol{\alpha}\boldsymbol{\beta}'$ and $\boldsymbol{\beta}'\mathbf{x}_t$ is I(0) (see, Johansen 1995, Theorem 4.2 and Hamilton 1994, p. 582).

[24] The economic implication of cointegration arises from the notion that equilibrium theories in economics involving non-stationary variables require the presence of a stationary combination of those variables (Enders 2004). Hence, the cointegrating relation is construed as a long-run equilibrium relation which acts as an attractor in the system. Then, this equilibrium relation deters such variables from meandering too far away from each other. Even though there are cointegrating relations in the system, estimating with the stationary variables (by differencing) not only fails to capture the important information in data, but also gives rise to a misspecification error.

Often, each column of $\boldsymbol{\beta}$ is called the cointegrating vector and the each element of $\boldsymbol{\alpha}$ is said to be the speed of adjustment towards equilibrium or error-correction loading in the VECM. Thus, under the $I(1)$ hypothesis, (2.4.4) can be replaced by the following cointegrated VAR model,

$$\begin{aligned}\Delta \mathbf{x}_t &= \boldsymbol{\alpha}\boldsymbol{\beta}'\mathbf{x}_{t-1} + \boldsymbol{\Gamma}_1 \Delta \mathbf{x}_{t-1} + \cdots + \boldsymbol{\Gamma}_{k-1}\Delta \mathbf{x}_{t-k+1} + \boldsymbol{\nu} + \epsilon_t \\ &= \boldsymbol{\Pi}\mathbf{x}_{t-1} + \boldsymbol{\Gamma}_1 \Delta \mathbf{x}_{t-1} + \cdots + \boldsymbol{\Gamma}_{k-1}\Delta \mathbf{x}_{t-k+1} + \boldsymbol{\nu} + \epsilon_t, \end{aligned} \tag{2.4.6}$$

where all stochastic components are stationary and $\boldsymbol{\nu}$ includes the linear trend and constant in $\mathbf{x}_t$. Two features are worth noting. First, as shown in Appendix 2.E, (2.4.6) represents a case where a linear trend is present in variables but not in the cointegrating relation. Second, the decomposition of the matrix $\boldsymbol{\Pi}_{3\times 3}$ as the product of two $(3 \times r)$ matrices, $\boldsymbol{\Pi} = \boldsymbol{\alpha}\boldsymbol{\beta}'$, is not unique. For an arbitrary non-singular $(r \times r)$ matrix $\boldsymbol{Q}$, $\boldsymbol{\alpha}^+ = \boldsymbol{\alpha}\boldsymbol{Q}'$ and $\boldsymbol{\beta}^+ = \boldsymbol{\beta}\boldsymbol{Q}^{-1}$ yields $\boldsymbol{\Pi}$.

After the lag length is determined, the Cointegrated VAR modeling requires a preliminary investigation of presence of cointegrating relation, which is theoretically implied in (2.3.10).[25] This is an important step in the sense that test results influence all subsequent empirical analysis. To examine the existence of cointegration among c_t, a_t, and y_t, this essay carries out two popular tests for cointegration: Johansen's two versions of likelihood ratio tests and Engle-Granger's residual-based test.

Johansen's approach (1988, 1995), which is most widely adopted in the empirical literature to detect the number of cointegration in the multivariate settings, allows us to determine the number of cointegrating relations implied in the long-run matrix $\boldsymbol{\Pi} = \boldsymbol{\alpha}\boldsymbol{\beta}'$. The basic intuition of Johansen's approach is to estimate matrix $\boldsymbol{\Pi}$ from an unrestricted VAR model and to test whether we can reject the restrictions implied by the reduced rank of $\boldsymbol{\Pi}$.[26] Based on the eigenvalues of estimated $\boldsymbol{\Pi} = \boldsymbol{\alpha}\boldsymbol{\beta}'$, he proposes two versions of likelihood ratio tests: trace test and maximal eigenvalue test. Appendix 2.D provides the detailed explanations on Johansen's two versions of likelihood ratio tests.

[25]The appropriate lag length must be selected before the number of cointegrating relations is specified because specifying the lag order does not require the knowledge of the cointegration rank whereas specifying the cointegration rank requires the knowledge of the lag length (Lütkepohl 2006, p. 325).

[26]In fact, the number of cointegrating relations is tested in the intermediate step of Johansen's full information maximum likelihood (FIML) estimation procedure (Hamilton 1994, pp. 635–645).

Table 2.4 reports Johansen cointegration test results. Clearly, both trace statistics and maximal eigenvalue statistics unanimously indicate one cointegrating relation at the 5% significance level. Therefore, we conclude that Johansen's two types of statistics provide strong evidence against no cointegrating relationship involving c_t, a_t, and y_t.

Table 2.4: Johansen's Likelihood Tests for Cointegration

H_0	λ_{trace}				λ_{max}			
	H_1	Statistics	5% C.V	1% C.V	H_1	Statistics	5% C.V	1% C.V
$r = 0$	$0 < r \leq 3$	**49.60**	29.80	35.46	$r = 1$	**34.81**	21.13	25.86
$r = 1$	$1 < r \leq 3$	14.79	15.49	19.94	$r = 2$	12.91	14.26	18.52
$r = 2$	$2 < r \leq 3$	1.88	3.84	6.63	$r = 3$	1.88	3.84	6.63

Note: "r" stands for the cointegrating rank, i.e., the number of cointegrating relationships. It is assumed that there a linear trend in each variable but not in the cointegrating relations. A test statistic greater than the specified critical value suggests rejection of the null hypothesis. The bold-typed values indicate the statistical significance at the 5% level.

Note that the test results in Table 2.4 are based on two presumptions: consideration of the deterministic terms and the lag selection. For the first issue, Johansen (1995, pp. 80–84) considered five deterministic trend cases for testing cointegration. In this essay, investigating the existence of cointegrating relations is carried out using Johansen's third case: this allows for a linear trend in each variable but not in the cointegrating relations.[27] This choice is motivated from the statistical and theoretical considerations. First, Figure 2.1 clearly demonstrates that there are sustained deterministic trends in three variables. Second, (2.3.10) implies that the long-run equilibrium relation among c_t, a_t, and y_t does not involves the deterministic trends.

For the second issue, the order of lags is selected by the Schwartz information criterion (SIC) as two. Thus this essay sets up a VAR(2), i.e., VECM(1), structure. In order to minimize the misspecification problem, however, the cointegration tests with different lag orders are also implemented and reported in Table 2.5. It also strongly indicates one cointegrating relation with any lag order choice.[28]

[27]For the possibility of misspecification, other four cases are also tested. The results, which are not reported here, find one cointegration, too.

[28]Only one exception happens when the lag order is zero (i.e.,VECM(0)). In this case, the trace statistic is higher than 5% critical value but the maximal eigenvalue statistic is lower than the 5% critical value. Thus, at the 5% significance level, the former indicates 2 cointegrating vectors while the latter indicates 1 cointegrating vector. It is not uncommon that two LR tests indicate the different

Table 2.5: Johansen's Likelihood Tests for Cointegration with Different Lags

Lag Order	H_0	λ_{trace}			λ_{max}		
		Statistics	5% C.V	1% C.V	Statistics	5% C.V	1% C.V
Lag = 0	$r = 0$	**48.83**	29.80	35.46	**32.26**	21.13	25.86
	$r = 1$	**16.57**	15.49	19.94	13.50	14.26	18.52
	$r = 2$	3.07	3.84	6.63	3.07	3.84	6.63
Lag = 1	$r = 0$	**49.60**	29.80	35.46	**34.81**	21.13	25.86
	$r = 1$	14.79	15.49	19.94	12.91	14.26	18.52
	$r = 2$	1.88	3.84	6.63	1.88	3.84	6.63
Lag = 2	$r = 0$	**35.06**	29.80	35.46	**22.74**	21.13	25.86
	$r = 1$	12.32	15.49	19.94	10.04	14.26	18.52
	$r = 2$	2.28	3.84	6.63	2.28	3.84	6.63
Lag = 3	$r = 0$	**37.97**	29.80	35.46	**25.42**	21.13	25.86
	$r = 1$	12.55	15.49	19.94	10.23	14.26	18.52
	$r = 2$	2.32	3.84	6.63	2.32	3.84	6.63
Lag = 4	$r = 0$	**34.44**	29.80	35.46	**21.44**	21.13	25.86
	$r = 1$	13.00	15.49	19.94	9.73	14.26	18.52
	$r = 2$	3.27	3.84	6.63	3.27	3.84	6.63

Note: "r" stands for the cointegrating rank, i.e., the number of cointegrating relationships. It is assumed that there a linear trend in each variable but not in the cointegrating relations. A test statistic greater than the specified critical value suggests rejection of the null hypothesis. The bold-typed values indicate the statistical significance at the 5% level.

For the robustness check of this finding, this essay also carries out the two-step residual based cointegration test which is proposed by Engle and Granger (1987) and Phillips and Ouliaris (1990). Unlike the Johansen's (1995) full information maximum likelihood (hereafter FIML) approach, the residual based cointegration test does not allow us to test for the number of cointegrating relations. For instance, the null hypothesis of Phillips and Ouliaris (1990) is no cointegrating relation (i.e., a unit root in residual process).

The basic intuition of residual based cointegration test is simple: applying the ADF test to the residuals from static OLS estimation. Consider a simple OLS regression using c_t, a_t, and y_t,

$$c_t = \beta_0 + \beta_a a_t + \beta_y y_t + \varepsilon_t. \tag{2.4.7}$$

Now if the saved residuals ($\hat{\varepsilon}_t$) is found be stationary in the ADF test, it is said the three variables are cointegrated. In this case, the regression equation has a form of $\Delta\hat{\varepsilon}_t = \gamma\hat{\varepsilon}_{t-1} + \sum_{j=1}^{k} \alpha_j \Delta\hat{\varepsilon}_{t-j} + \upsilon_t$ and the null hypothesis is $\gamma = 0$.

Two features about this procedure deserve emphasis. First, the $\{\hat{\varepsilon}_t\}$ sequence is a residual from a regression equation, so there is no need to include an intercept term for the mean. Second, it is not possible to use the Dickey-Fuller tables for critical values. The reason is that the OLS estimation chooses three coefficients ($\hat{\beta}_0$, $\hat{\beta}_a$, and $\hat{\beta}_y$) that minimize the sum of squared residuals, so the residual series tend to be slightly more stationary than true error terms and the procedure is thus skewed toward finding a stationary error process in the ADF test (Campbell and Perron 1991; Enders 2004). The relevant critical values are tabulated by Phillips and Ouliaris (1990) and this essay uses the third case: a linear trend in variables but no trend in cointegrating relation (i.e., Case 3 in Table B.9 in Hamilton 1994).[29]

Table 2.6 reports the residual-based cointegration test results involving three variables up to four lags. Notice that the Schwartz information criterion (SIC) suggests no

number of cointegrating vectors. However, at the 1% significance level, both test statistics indicate 1 cointegrating vector unanimously.

[29] Alternatively, one can compute the critical values from MacKinnon's (1991) approximation formula with allowing trends in the raw data. Based on the Monte Carlo simulation results, MacKinnon (1991) proposes the following response surface regressions, $C_k(p, T_k) = \beta_\infty + \beta_1 T_k^{-1} + \beta_2 T_k^{-2} + e_k$, where $C_k(p, T_k)$ is the critical values for a test at the p per cent level with the sample size T_k, and β's are parameters to be estimated. Then, using the GLS estimates of β's provided by MacKinnon, we could calculate the approximate critical values for several significance levels.

lag as the optimal lag structure for this test. Then, Table 2.6 shows that no cointegration can be rejected at 15% significance level.[30]

Table 2.6: Residual-based Tests for Cointegration

	Lag Length				
	0	1	2	3	4
Statistics	**-3.34**	-3.32	-3.23	-3.12	-3.24
(SIC)	(-5.84)	(-5.79)	(-5.74)	(-5.68)	(-5.65)
< *Critical Values* >					
5 percent	-3.80				
10 percent	-3.52				
15 percent	-3.33				

Note: The critical values of the Phillips and Ouliaris (1990)'s test come from Table B.9 in Hamilton (1994). The bold-typed values indicate the statistical significance at the 15% level.

Based on these findings, this essay chooses one cointegrating relation. This is also consistent with theory. Theory postulates that c_t, a_t, and y_t be cointegrated as shown in (2.3.10), and that there should be one equilibrium relation.

2.4.2.3 Vector Error Correction Model Approach

Now, given the result of one cointegrating relation, we can represent (2.4.6) more specifically as (2.4.8).

$$\Delta \mathbf{x}_t = \boldsymbol{\alpha}\boldsymbol{\beta}'\mathbf{x}_{t-1} + \boldsymbol{\Gamma}_1 \Delta \mathbf{x}_{t-1} + \cdots + \boldsymbol{\Gamma}_{k-1}\Delta \mathbf{x}_{t-(k-1)} + \boldsymbol{\nu} + \boldsymbol{\epsilon}_t \tag{2.4.8}$$

$$\begin{bmatrix} \Delta c_t \\ \Delta a_t \\ \Delta y_t \end{bmatrix} = \begin{bmatrix} \alpha_c \\ \alpha_a \\ \alpha_y \end{bmatrix} [1,\ -\beta_a,\ -\beta_y] \begin{bmatrix} c_{t-1} \\ a_{t-1} \\ y_{t-1} \end{bmatrix} + \boldsymbol{\Gamma}_1 \begin{bmatrix} \Delta c_{t-1} \\ \Delta a_{t-1} \\ \Delta y_{t-1} \end{bmatrix} + \cdots + \begin{bmatrix} \nu_c \\ \nu_a \\ \nu_y \end{bmatrix} + \begin{bmatrix} \varepsilon_{c,t} \\ \varepsilon_{a,t} \\ \varepsilon_{y,t} \end{bmatrix},$$

[30]Therefore, the residual-based test for cointegration seems to support the presence of a cointegrating relation only weakly. However, it is well known that the power of residual-based tests are very limited when there is a large shock in the economy. Indeed, the examined period includes the Korean financial crisis at the end of 1997.

or equivalently,

$$\begin{aligned}\Delta \mathbf{x}_t &= \boldsymbol{\alpha}\boldsymbol{\beta}'(\mathbf{x}_{t-1} - \beta_0) + \boldsymbol{\Gamma}_1 \Delta \mathbf{x}_{t-1} + \cdots + \boldsymbol{\Gamma}_{k-1}\Delta \mathbf{x}_{t-(k-1)} + \tilde{\boldsymbol{\nu}} + \boldsymbol{\epsilon}_t \qquad (2.4.9)\\ &= \boldsymbol{\alpha}[\boldsymbol{\beta}' : -\beta_0]\begin{bmatrix}\mathbf{x}_{t-1}\\ 1\end{bmatrix} + \boldsymbol{\Gamma}_1 \Delta \mathbf{x}_{t-1} + \cdots + \boldsymbol{\Gamma}_{k-1}\Delta \mathbf{x}_{t-(k-1)} + \tilde{\boldsymbol{\nu}} + \boldsymbol{\epsilon}_t\\ &= \boldsymbol{\alpha}\tilde{\boldsymbol{\beta}}'\tilde{\mathbf{x}}_{t-1} + \boldsymbol{\Gamma}_1 \Delta \mathbf{x}_{t-1} + \cdots + \boldsymbol{\Gamma}_{k-1}\Delta \mathbf{x}_{t-(k-1)} + \tilde{\boldsymbol{\nu}} + \boldsymbol{\epsilon}_t\end{aligned}$$

$$\begin{bmatrix}\Delta c_t\\ \Delta a_t\\ \Delta y_t\end{bmatrix} = \begin{bmatrix}\alpha_c\\ \alpha_a\\ \alpha_y\end{bmatrix}[1, -\beta_a, -\beta_y, -\beta_0]\begin{bmatrix}c_{t-1}\\ a_{t-1}\\ y_{t-1}\\ 1\end{bmatrix} + \boldsymbol{\Gamma}_1\begin{bmatrix}\Delta c_{t-1}\\ \Delta a_{t-1}\\ \Delta y_{t-1}\end{bmatrix} + \cdots + \begin{bmatrix}\tilde{\nu}_c\\ \tilde{\nu}_a\\ \tilde{\nu}_y\end{bmatrix} + \begin{bmatrix}\varepsilon_{c,t}\\ \varepsilon_{a,t}\\ \varepsilon_{y,t}\end{bmatrix}.$$

Therefore, the VECM of (2.4.8) is simply a first-differenced VAR($k-1$) model augmented with one lagged error correction term in each equation of the model. Note also the difference between (2.4.8) and (2.4.9). As shown in Appendix 2.E, $\boldsymbol{\nu}$ in (2.4.8) is a mixture of a linear trend and constant in $\mathbf{x}_t$ whereas $\tilde{\boldsymbol{\nu}}$ in (2.4.9) only includes the linear trend in the data. Hence, using the demeaned $\boldsymbol{\beta}'\mathbf{x}_{t-1} = c_{t-1} - \beta_a a_{t-1} - \beta_y y_{t-1}$ in (2.4.8) and using $\tilde{\boldsymbol{\beta}}'\mathbf{x}_{t-1} = c_{t-1} - \beta_0 - \beta_a a_{t-1} - \beta_y y_{t-1}$ in (2.4.9) yield exactly the same results including the constant vector.

At this stage, it is crucial to emphasize the implications of $\boldsymbol{\beta}'\mathbf{x}_{t-1}$ (underlying economic equilibrium) and $\boldsymbol{\alpha}$ (speed of adjustment). Consider, there is a deviation from this long-run relationship in the system at period $t-1$, i.e., the demeaned $\boldsymbol{\beta}'\mathbf{x}_{t-1}$ is not zero. Then, at period t, the agents react to this deviation through $\boldsymbol{\alpha}$ to bring back the variables on the right track such that they satisfy the economic equilibrium.[31]

[31] Alternatively, one may estimate (2.4.8) following Stock and Watson's (1993) two-stage method as in Lettau and Ludvigson (2001, 2004).

In this procedure, the first stage is to estimate the normalized long-run vector $\boldsymbol{\beta}$ according to Stock and Watson (1993)'s dynamic least square (hereafter DLS) methodology as (2.4.10),

$$c_t = \beta_0 + \beta_a a_t + \beta_y y_t + \sum_{i=-k}^{k} b_{a,i}\Delta a_{t-i} + \sum_{i=-k}^{k} b_{y,i}\Delta y_{t-i} + \varepsilon_t, \qquad (2.4.10)$$

From the DLS estimation of (2.4.10), the cointegrating residual (i.e., consumption-wealth ratio) can be computed as $\boldsymbol{\beta}'\mathbf{x}_t = c_t - \hat{\beta}_0 - \hat{\beta}_a a_t - \hat{\beta}_y y_t$. Then, in the second stage, OLS estimation of a 3-dimensional first differenced VAR(k-1) model augmented with an exogenous variable of $\boldsymbol{\beta}'\mathbf{x}_{t-1}$ renders (2.4.8).

However, this chapter sticks to the Johansen's method as it is a full information maximum likelihood estimation method and thus it should dominate the two stage method.

2.4.2.4 Permanent Shocks and Transitory Shocks in VECM

One of the main problems in macroeconomics is to characterize the fluctuations of key macro-variables such as output and consumption by separating trends from cycles. In the academic literature, this issue has been associated with decomposition of permanent and transitory shocks. A permanent shock is by definition a shock that has a long lasting effect on the level of the variable, whereas the effect of a transitory shock disappears sooner or later. That is, the trend is non-stationary as a result of permanent shocks while the cycle is stationary as a result of transitory shocks. Thus, for example, if c_t and y_t are primarily governed by the permanent shocks, these shocks will be accumulating in the levels of those variables, and we can conclude that most fluctuations are associated with trends.

As shown in Appendix 2.F, the presence of one cointegration in $\mathbf{x}_t$ of I(1) implies that there are at most two unit roots in Stock and Watson's (1988) common trend representation. Then, following Gonzalo and Granger (1995), one can decompose the fluctuations of variables into the permanent processes and the transitory processes as follows.

First, given one cointegrating relation, recall the following relation which is shown in Johansen (1995, p. 39),

$$\boldsymbol{\beta}_{\perp}(\boldsymbol{\alpha}'_{\perp}\boldsymbol{\beta}_{\perp})^{-1}\boldsymbol{\alpha}'_{\perp} + \boldsymbol{\alpha}(\boldsymbol{\beta}'\boldsymbol{\alpha})^{-1}\boldsymbol{\beta}' = \mathbf{I}_3, \tag{2.4.11}$$

where $\boldsymbol{\alpha}_{\perp}$ and $\boldsymbol{\beta}_{\perp}$ are 3×2 orthogonal complements of $\boldsymbol{\alpha}_{3\times1}$ and $\boldsymbol{\beta}_{3\times1}$, respectively, such that $\boldsymbol{\beta}'\boldsymbol{\beta}_{\perp} = [0,\ 0]$, $\boldsymbol{\alpha}'\boldsymbol{\alpha}_{\perp} = [0,\ 0]$, rank $(\boldsymbol{\alpha} : \boldsymbol{\alpha}_{\perp}) = 3$ and rank $(\boldsymbol{\beta} : \boldsymbol{\beta}_{\perp}) = 3$. One can verify (2.4.11) by pre-multiplying either $\boldsymbol{\beta}'$ or $\boldsymbol{\alpha}'_{\perp}$ on each side.

Then, as shown in Gonzalo and Granger (1995), given the assumption that the part of $\mathbf{x}_t$ that is not explained by the I(1) common factors can only have a transitory effect on $\mathbf{x}_t$, post-multiplying $\mathbf{x}_t$ on each side of (2.4.11) yields,

$$\begin{aligned}
\mathbf{x}_t &= \boldsymbol{\beta}_\perp(\boldsymbol{\alpha}'_\perp\boldsymbol{\beta}_\perp)^{-1}\boldsymbol{\alpha}'_\perp\mathbf{x}_t + \boldsymbol{\alpha}(\boldsymbol{\beta}'\boldsymbol{\alpha})^{-1}\boldsymbol{\beta}'\mathbf{x}_t \\
&= \boldsymbol{A}_1\boldsymbol{\alpha}'_\perp\mathbf{x}_t + \boldsymbol{A}_2\boldsymbol{\beta}'\mathbf{x}_t \\
\begin{bmatrix} c_t \\ a_t \\ y_t \end{bmatrix} &= \left[\begin{pmatrix} A_{1,11} & A_{1,12} \\ A_{1,21} & A_{1,22} \\ A_{1,31} & A_{1,32} \end{pmatrix} \begin{pmatrix} A_{2,11} \\ A_{2,21} \\ A_{2,31} \end{pmatrix}\right] \left[\begin{matrix} \begin{pmatrix} \alpha'_{\perp,11} & \alpha'_{\perp,12} & \alpha'_{\perp,13} \\ \alpha'_{\perp,21} & \alpha'_{\perp,22} & \alpha'_{\perp,23} \end{pmatrix} \\ \begin{pmatrix} 1 & \beta'_{\perp,12} & \beta'_{\perp,13} \end{pmatrix} \end{matrix}\right] \begin{bmatrix} c_t \\ a_t \\ y_t \end{bmatrix} \\
\mathbf{x}_t &= \boldsymbol{A}\begin{bmatrix} \boldsymbol{\alpha}'_\perp \\ \boldsymbol{\beta}' \end{bmatrix}\mathbf{x}_t = \boldsymbol{A}\boldsymbol{G}\mathbf{x}_t. \qquad (2.4.12)
\end{aligned}$$

Clearly, in the second line, the first part in the right hand side must be I(1), i.e., permanent process since the second part is I(0). Now it is straightforward to notice that I(1) processes $\mathbf{x}_t$ can be decomposed into the common I(1) factors $\boldsymbol{\alpha}'_\perp\mathbf{x}_t$ and the I(0) cointegrating relation $\boldsymbol{\beta}'\mathbf{x}_t$. That is, any cointegrated I(1) $\mathbf{x}_t$ can be understood as the linear combinations of I(1) and I(0) processes.[32]

This decomposition is useful in understanding the dynamics of the cointegrated VAR system of c_t, a_t, and y_t. For example, if only a_t has a large portion of transitory components, then this implies that most adjustments to the new shocks are carried out by a_t. Since the presence of cointegrating relation implies the equilibrium in the system, naturally, these decompositions of c_t, a_t, and y_t allow us to check which variable mainly works for recovering the cointegrating relation.

Furthermore, as illustrated in Appendix 2.F, Gonzalo and Ng (2001) show that the permanent and transitory shock vector $\boldsymbol{u}_t$ can be constructed as $\boldsymbol{G}\boldsymbol{\epsilon}_t$.[33] However, since $\boldsymbol{u}_t$ is not orthogonal in general, meaningful impulse responses and variance decomposition analyses are not possible. Therefore, they propose to reformulate the unorthogonalized structural shocks $\boldsymbol{u}_t$ into the orthogonalized structural shocks $\boldsymbol{\eta}_t$ using the lower triangular matrix $\boldsymbol{H}$, where $\boldsymbol{H}$ comes from the Choleski decomposition of covariance matrix of $\boldsymbol{u}_t$. With the mutually uncorrelated structural shocks, the cur-

[32]Among the various methods to compute $\boldsymbol{\alpha}_\perp$, Gonzalo and Ng (2001) recommend that one use the eigenvectors associated with the 2 smallest eigenvalues of the matrix $\hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\alpha}}'$. In addition, it is suggested that insignificant estimates of $\hat{\boldsymbol{\alpha}}$ be to zero before computing $\hat{\boldsymbol{\alpha}}_\perp$ because the permanent factor in the model is very sensitive to small variations in $\hat{\boldsymbol{\alpha}}$.

[33]Gonzalo and Granger (1995) define the permanent and transitory shocks as follows,

$$\lim_{h\to\infty}\frac{\partial E_t(\mathbf{x}_{t+h})}{\partial u_t^P} \neq 0 \quad \text{and} \quad \lim_{h\to\infty}\frac{\partial E_t(\mathbf{x}_{t+h})}{\partial u_t^T} = 0,$$

where E_t is the conditional expectation with respect to the information set at time t.

rent essay investigates which variable among c_t, a_t, and y_t is mostly sensitive to the transitory shock. In addition, we will assess the relative importance of permanent and transitory shocks in fluctuations of those variables.

2.5 Empirical Results

2.5.1 Long-term Dynamics of the Cointegrated System

By implementing the Johansen's method, we obtain the following cointegrating vector among consumption, financial wealth and labor income using the demeaned data.

$$\hat{\beta}' = [1, \ -\hat{\beta}_a, \ -\hat{\beta}_y]' = [1, \ \underset{(0.04)}{-0.10}, \ \underset{(0.12)}{-0.63}]' \tag{2.5.1}$$

where the standard errors appear in parentheses below the coefficient estimates and the cointegrating coefficient on consumption is normalized to one.

Note that all coefficients are correctly signed and statistically significant. From (2.5.1), two different interpretations can be made. The first interpretation is that, in the long-run, 10% increase in household financial wealth leads to around 1% consumption increase and 10% increase in labor income is followed by 6% increase in consumption. In other words, marginal propensity to consume (MPC) out of financial wealth is 0.02, that is, about 2 Korea won consumed out of an additional 100 Korea won in financial wealth and that marginal propensity to consume out of labor income is 0.53, that is 53 Korea won to every 100 Korea won increase in labor income.[34] This finding is similar to those from other countries in Section 2.2.

Alternative but more important interpretation on (2.5.1) emerges from (2.3.10). Equation (2.3.10) states that two estimated parameters on a_t and y_t represent the average shares of financial wealth and human wealth in total wealth, i.e., ω and $(1-\omega)$, respectively. In this respect, (2.5.1) says that the total wealth of average consumer in Korea is composed of 14% of non-human wealth and 86% of human wealth.

In this case, it is worth noting that the sum of two coefficients ($\hat{\beta}_a$ and $\hat{\beta}_y$) is not unity. It seems to be problematic because theory implies that it should be unity

[34] At the end of 2005, financial wealth to consumption ratio is 6.04 and labor income to consumption ratio is 1.18. Since (2.5.1) is obtained by the log data, 0.10 and 0.63 represent the elasticity. Then $\varepsilon_A(=0.10) = \frac{\partial C}{\partial A}(=MPC_A) \times \frac{A_t}{C_t}(=6.04)$ and $\varepsilon_Y(=0.63) = \frac{\partial C}{\partial Y}(=MPC_Y) \times \frac{Y_t}{C_t}(=1.18)$.

as shown in (2.3.10).[35] However, this discrepancy seems to reflect the limitation of consumption measurements, as pointed out by Lettau and Ludvigson (2001, 2004). For instance, we can only observe the actual consumption expenditure, whereas we cannot do the actual flow of consumption especially in durable goods. If this is the case, (2.5.1) implies that the observed consumption expenditure data is about three fourths of the theoretical consumption on average.[36]

2.5.2 Which Variable Recovers the Equilibrium in the System?

More interesting dynamics of c_t, a_t, and y_t arise when we examine the short-term relations among those variables. This is the main content of the current section. Based on lag exclusion test and Schwartz criterion for selecting the optimal lag structure, Table 2.7 presents the estimates for the short-term dynamics in a 3-dimensional VECM with one lag using the Johansen's full information maximum likelihood (FIML) estimation framework. Notice that, as discussed in Section 2.4, this is equivalent to the VAR(2) model.

Table 2.7 reports the estimates the estimates and their standard errors for the short-term dynamics of c_t, a_t, and y_t. In this case, the log differences (i.e., growth rate) in consumption, financial wealth, and labor income are regressed on their own lags, a constant and the deviation from the long-run relationship in the previous period [i.e., cointegrating residual $\hat{\boldsymbol{\beta}}'\mathbf{x}_{t-1} = c_{t-1} - \hat{\beta}_a a_{t-1} - \hat{\beta}_y y_{t-1} - mean(\hat{\boldsymbol{\beta}}'\mathbf{x}_{t-1})$]. Notice that if $\hat{\boldsymbol{\beta}}'\mathbf{x}_{t-1}$ is not zero, at least one of variable must act to recover the long-term relation among those variables in the system.

[35] In the preliminary research, we tested whether the theoretical restriction held in (2.3.10) using the likelihood ratio test about the cointegrating vector as in Hamilton (1994, pp.648–650). The likelihood ratio statistic is 4.30, which is larger than 3.84, the 5% critical value for $\chi^2(1)$. Thus, the null hypothesis that the sum of $\hat{\beta}_a$ and $\hat{\beta}_y$ is unity is rejected. In addition, when we impose this restriction, we obtain rather absurd coefficients estimates as $\hat{\beta}_a = -0.08$ and $\hat{\beta}_y = 1.08$, which mean 10% increase in asset leads to 0.8% decrease in consumption while 10% increase in income leads to 10.8% increase in consumption. In the following, we do not restrict the cointegrating vector.

[36] Let us denote the observed consumption as c_t^{obs} and the theoretical consumption as c_t. If $c_t = \gamma c_t^{obs}$ holds on average, $c_t - \omega a_t - (1-\omega)y_t$ implies $\gamma[c_t^{obs} - \frac{1}{\gamma}\omega a_t - \frac{1}{\gamma}(1-\omega)y_t]$. In this case, (2.5.1) implies $\gamma = 1.36$ and $\omega = 14\%$.

Another measurement issue is associated with labor income. It is well acknowledged that a large portion of capital income is associated with labor services. Thus, in the preliminary research, this essay also estimated parameters with gross national income (GNI) data instead of labor income data. In this case, the coefficients of y_t increase significantly. However, since this practice seems to incur more severe problems, the following analysis sticks to use labor income data.

In addition, Table 2.7 presents those when CPI inflation rate is augmented into the model. The motivation of this augmentation is that although the cointegrating relation is derived from the real budget constraint of (2.3.1), the nominal variable like inflation rate may have an important role on the movements of real variables if the price and the wage are not fully indexed. As shown in the adjusted R^2 statistics, this augmentation seems to contribute to the rise in explanatory power for consumption and income growth equations.

Table 2.7: Short-term Dynamics of c_t, a_t, and y_t (1980:1-2005:3)

	EQUATION			EQUATION		
	Δc_t	Δa_t	Δy_t	Δc_t	Δa_t	Δy_t
Δc_{t-1}	0.00	0.02	0.05	-0.05	0.04	0.01
	(0.14)	(0.20)	(0.15)	(0.33)	(0.21)	(0.15)
Δa_{t-1}	0.03	-0.14	-0.01	0.02	-0.15	-0.03
	(0.07)	(0.10)	(0.07)	(0.07)	(0.10)	(0.07)
Δy_{t-1}	**0.31**	**0.40**	0.22	**0.32**	0.36	0.22
	(0.13)	(0.19)	(0.14)	(0.13)	(0.19)	(0.14)
$\hat{\beta}'\mathbf{x}_{t-1}$	-0.04	**0.26**	0.02	0.002	**0.24**	0.06
	(0.03)	(0.05)	(0.03)	(0.04)	(0.05)	(0.04)
Δp_t	-	-	-	-0.06	-0.01	-0.06
				(0.04)	(0.05)	(0.04)
$\bar{R}^2$	0.09	0.23	0.03	0.10	0.22	0.05

Notes: The values in parentheses denote standard errors. Significant coefficients at the 5% level are highlighted in bold face. $\hat{\beta}'\mathbf{x}_{t-1}$ is the estimated cointegrating residual and the term Δp_t denotes the CPI inflation rate.

Several features are worth noting in Table 2.7. First of all, in the table, it is important to check whether the signs of elements of $\hat{\alpha}$ vector in the last row (i.e., the coefficients of $\hat{\beta}'\mathbf{x}_{t-1}$) are opposite to those of corresponding $\hat{\beta}$ in (2.5.1). If this is the case, we can think that all three variables are exhibiting error correcting behaviors to recover the long-run relation.[37] In the left three equations, at first sight, the signs of the three loadings $\hat{\alpha}_c$ (-0.04), $\hat{\alpha}_a$ (0.26) and $\hat{\alpha}_y$ (0.02) imply that when new exogenous shocks hit the system, all the three variables act to pull the processes back towards the long-run equilibrium. On the other hand, the signs of three loadings (0.002, 0.24, and 0.06) in the right three equations imply that consumption does not show error correcting behavior.

Notice that, however, only $\hat{\alpha}_a$ is sizable and statistically significant. Thus we can conclude that only a_t shows the error correction behavior. On the other hand, $\hat{\alpha}_c$ and $\hat{\alpha}_y$ in both approaches are not statistically different from zeroes. Thus, c_t and y_t do not respond to the discrepancy from the long-run relation and a_t does all of the adjustment. Under this circumstance, c_t and y_t are called to be *weakly exogenous* in the sense that they are influencing the long-run stochastic path of a_t while they are not influenced by a_t. In other words, consumption and income have an impact on the economy, but they experienced negligible feedback from the economy.

To examine the robustness of this argument, Table 2.8 reports the short-term dynamics among three variables with different lag orders in VECMs. Apparently, all estimates of $\hat{\alpha}_c$ and $\hat{\alpha}_y$ in VECMs are weakly exogenous in any lag selections from 0 to 4 in the table. On the other hand, the speeds of adjustment in the financial wealth equations are sizable and statistically significant in most cases.

Finally, in Table 2.7, it seems that the previous income growth has a predicting power on the current consumption in consumption equation in both models. At first glance, it may be interpreted that the current increase of labor income is followed by the increases of consumption in the next period.[38] However, one must be cautious for this interpretation because it is valid only if the residual of the cointegrating vector

[37] On the other hand, if some of them move in the same direction, we can think there is an overshooting (Juselius 2006, p. 122). However, even this is the case, a necessary condition for presence of a *long-run relation* is that at least one of the variables exhibit a *short-run adjustment* to it.

[38] If the previous change of labor income has a significant impact on current change of consumption, one may argue that this is consistent with Keynesian consumption function which is often found in the empirical literature.

However, this result can also be consistent with Friedman's (1957) permanent income hypothesis, which asserts that an increase of the current income is associated with an increase of consumption only to the extent that it reflects an increase in permanent income. Therefore, if a large portion of change in consumption and labor income is associated with permanent movements and a large portion of financial wealth change is related with transitory movements, this result is also consistent with what the permanent income hypothesis predicts.

Table 2.8: Tests for Weakly Exogeneity of c_t, a_t, and y_t

	Lag Order				
	0	1	2	3	4
I. Models without inflation rate					
α_c	-0.03	-0.04	-0.07	-0.06	-0.11
	(0.03)	(0.03)	(0.04)	(0.05)	(0.05)
α_a	**0.24**	**0.26**	**0.22**	**0.26**	**0.22**
	(0.05)	(0.05)	(0.06)	(0.06)	(0.07)
α_y	0.02	0.02	-0.02	-0.01	-0.06
	(0.04)	(0.03)	(0.05)	(0.05)	(0.06)
II. Models with inflation rate					
α_c	0.04	0.002	-0.03	-0.01	-0.09
	(0.03)	(0.04)	(0.04)	(0.04)	(0.06)
α_a	**0.19**	**0.24**	**0.15**	**0.20**	0.15
	(0.04)	(0.05)	(0.06)	(0.06)	(0.08)
α_y	0.05	0.06	0.05	0.05	0.01
	(0.03)	(0.04)	(0.05)	(0.05)	(0.07)

Notes: The values in parentheses denote standard errors. Significant coefficients at the 5% level are highlighted in bold face.

remains unchanged.[39]

2.5.3 Decomposition of Permanent and Transitory Components in c_t, a_t, and y_t

The previous section documents that Korean households adjust their financial wealth in lieu of consumption and labor income when there is a deviation from the long-run relation (i.e., $\hat{\beta}'\mathbf{x}_{t-1} \neq 0$) in the economy. This finding seems to imply that most fluctuations of consumption and labor income are associated with trends, i.e., permanent components whereas fluctuations of financial wealth are largely associated with cycles, i.e., transitory components.

The main purpose of this section is to examine this argument more formally using the decomposition of permanent (i.e., trend) and transitory (i.e., cyclical) components of each variable (Gonzalo and Granger 1995; Gonzalo and Ng 2001). Accordingly, we can directly check that how much fluctuations of each variable are related with permanent and transitory factors.

Following Gonzalo and Granger's (1995) identification scheme of I(1) common factors in the cointegrated system, Figure 2.2 plots the permanent components of consumption, financial wealth and labor income, along with the actual values for each variable. It is clear from the figure that only financial wealth contains a large portion of transitory movements. On the other hand, the transitory movements of consumption and labor income appear to be very limited. In particular, the actual values of labor income is quite indistinguishable from its trend. In fact, actual labor income and consumption are highly correlated with their permanent components whereas financial wealth is not much correlated.[40]

It is also of high interest to economists how much fraction of the forecast error variance of Δc_t, Δa_t, and Δy_t is attributable to permanent and transitory shocks. To that end, the permanent and transitory shocks are orthogonalized in this essay following Gonzalo and Ng (2001). Table 2.9 presents the fraction of j step ahead forecast error variance in consumption, labor income and financial wealth growth that is attributable to the permanent and transitory shocks. In this practice, Gonzalo and

[39] During the discussion, Professor Martin Ellison gratefully pointed out this point.

[40] The correlations between the permanent components and the actual time series are measured using the growth rates of each variable. The correlations for consumption and labor income are 0.97 and 0.99, respectively, but for financial wealth it is only 0.52.

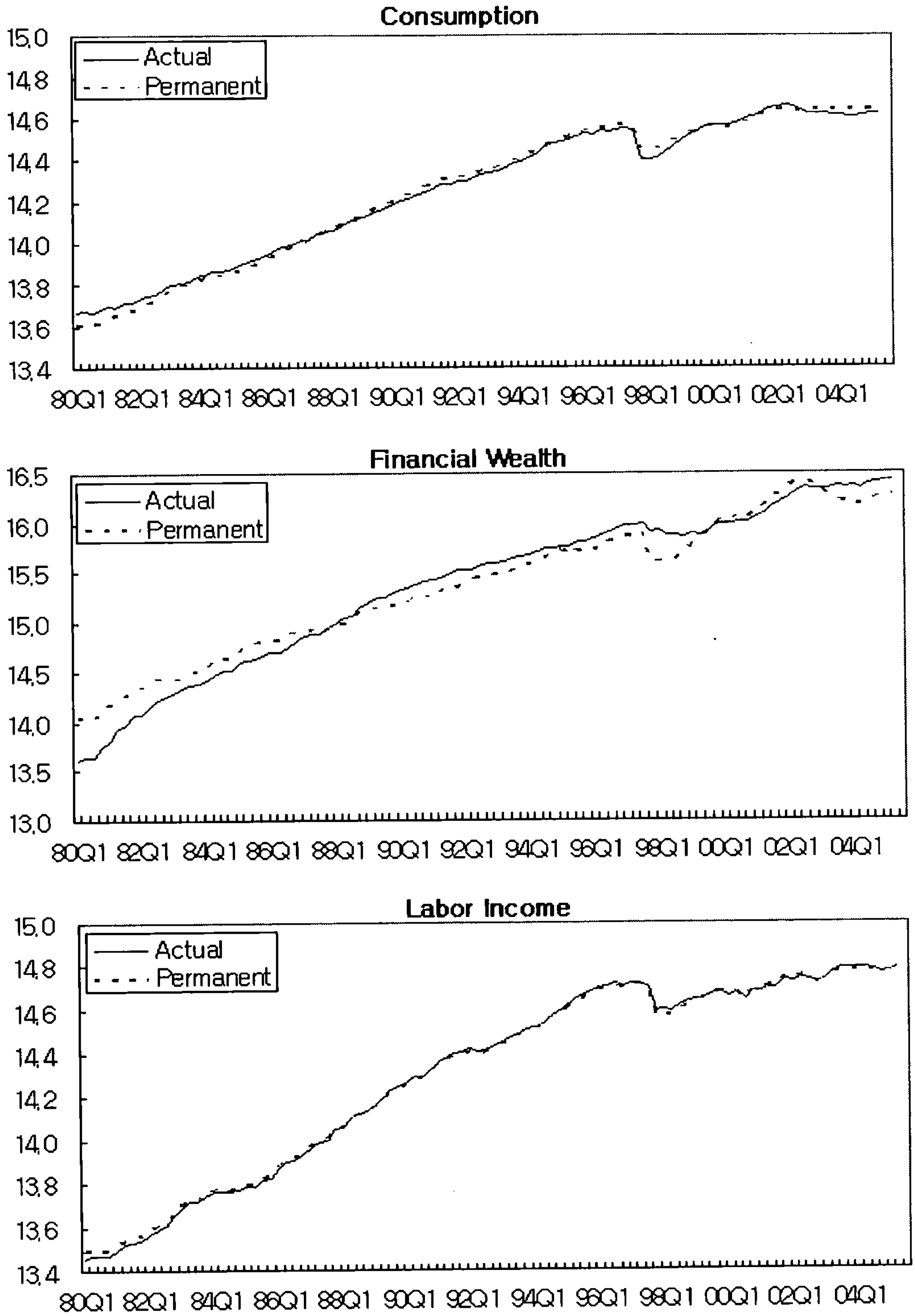

Figure 2.2: Permanent Components and Actual Values of c_t, a_t, and y_t

Ng (2001) suggest that the parameters in α should be fixed to be zeroes during the decomposition of permanent and transitory shocks if they are not different from zeroes at the 5% significance level.[41] Since $\hat{\alpha}_c$ and $\hat{\alpha}_c$ in Table 2.7 are not statistically different from zero at the 5% significance level, the upper panel of Table 2.9 reports forecast error variance decompositions when $\hat{\alpha}_c$ and $\hat{\alpha}_y$ are fixed to zeroes. Apparently, only fluctuations of the financial wealth growth are largely accounted for by the transitory shock.

For the comparison purpose, the lower panel of Table 2.9 reports the variance decomposition results when no restrictions are imposed on $\hat{\alpha}$. The results for financial wealth and labor income growths are very similar to restricted case. On the other hand, although consumption growth is still mostly governed by the permanent shock, the importance of transitory shock increases significantly. However, as aforementioned, this result must be interpreted with caution.

[41] They argue that this restriction generates much more stable estimates of the permanent-transitory decomposition.

Table 2.9: Variance Decomposition of c_t, a_t, and y_t

	$\Delta c_{t+j} - E_t\Delta c_{t+j}$		$\Delta a_{t+j} - E_t\Delta a_{t+j}$		$\Delta y_{t+j} - E_t\Delta y_{t+j}$	
Horizon j	P	T	P	T	P	T
	CASE I. $\alpha_c = \alpha_y = 0$					
1	100.00	0.00	58.38	41.62	100.00	0.00
2	99.98	0.02	58.50	41.50	94.38	5.62
4	98.99	1.01	59.48	40.52	92.98	7.02
8	98.45	1.55	61.70	38.30	92.84	7.16
12	98.37	1.63	63.88	36.12	93.16	6.84
16	98.39	1.61	65.95	34.05	93.55	6.45
24	98.51	1.49	69.73	30.27	94.30	5.70
	CASE II. α_c **and** α_y **estimated**					
1	55.58	44.42	59.16	40.84	93.17	6.83
2	68.03	31.97	59.57	40.43	97.13	2.87
4	74.56	25.44	62.50	37.50	98.70	1.30
8	80.14	19.86	68.38	31.62	99.36	0.64
12	83.61	16.39	73.31	26.69	99.57	0.43
16	86.15	13.85	77.28	22.72	99.68	0.32
24	89.59	10.41	82.93	17.07	99.79	0.21

Note: The table reports the fraction of the variance in the j step-ahead forecast error of each variable, which is attributable to innovations in the permanent shocks, 'P', and the transitory shocks, 'T'. Horizons are in quarters.

Finally, based on the orthogonalized permanent and transitory shocks, one can investigate which variable is most sensitive to the transitory shock. Figure 2.3 shows the responses of consumption, labor income and financial wealth to a one-standard-deviation transitory shock over 40 quarters. Clearly, the responses of financial wealth are large and persistent while those of consumption and labor income are negligible, which is consistent with Lettau and Ludvigson's (2004) findings for the U.S.

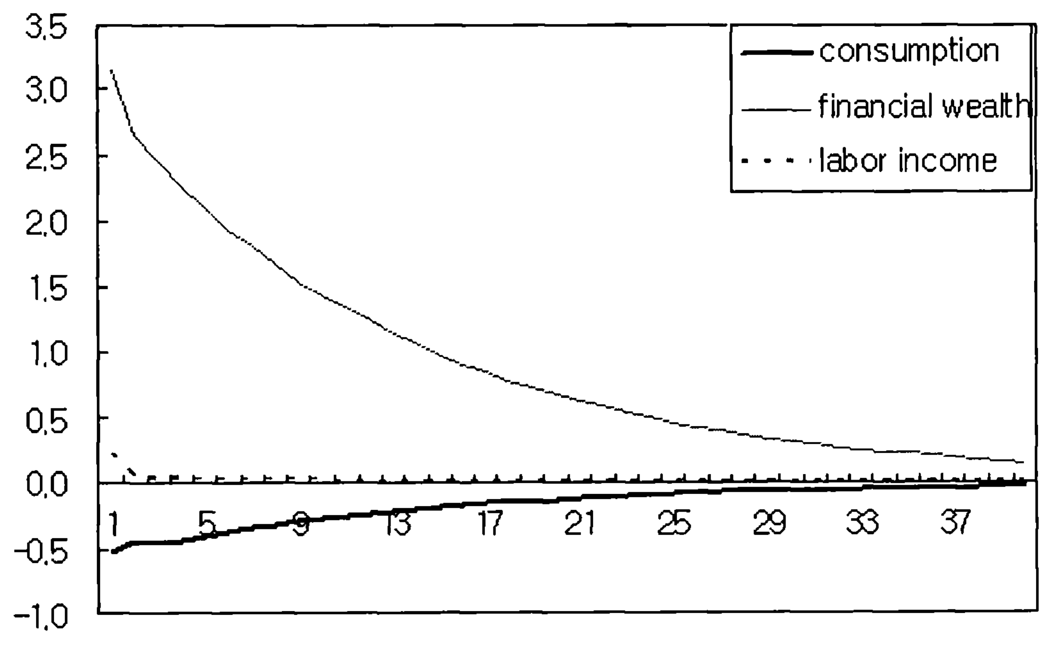

Figure 2.3: Responses to a Transitory Shock

2.5.4 Policy Implications

Until now, we found that only financial wealth is significantly associated with transitory components among three variables and shows the error correcting behavior in the cointegrated system. The second finding can also be interpreted that the deviation from the long-run relation or cointegrating residual has a predicting power on movement of financial wealth in the next period. Then, what is the policy implication of these findings?

For this purpose, first we need to check whether changes in consumption, financial wealth, and labor income are predictable by cointegrating residual over the long-horizon. Table 2.10 reports the forecasting powers of cointegrating residual on growths of consumption, financial wealth, and labor income over horizons j ranging from one to 24 quarters. In the table, Δc_{t+j} (defined as $c_{t+j} - c_t$), Δa_{t+j}, and Δy_{t+j} are regressed on the cointegrating residual $\hat{\beta}'\mathbf{x}_t$ where Δc_t, Δa_t, and Δy_t are controlled.

Note first that consumption is completely unpredictable by the cointegrating residual $\hat{\beta}'\mathbf{x}_t$ at any horizon. In addition, the cointegrating residual has predictive power on the labor income growth over the long horizons but the predictive power is very limited, as implied by low adjusted R^2 statistics. On the other hand, financial wealth appears to be highly predictable by the cointegrating residuals at any horizons. All t statistics for cointegrating residuals are sufficiently high and the adjusted R^2 statistics also high. These results are consistent with the findings from Table 2.7. Since financial wealth instead of consumption and labor income adjusts to recover the long-run relation when a new shock hits the system, the deviation from this long-run relation (i.e., cointegrating residual) must predict the future path of financial wealth growth. These results are consistent with those reported by Lettau and Ludvigson (2001, 2004) for the U.S., by Fernandez-Corugedo et al. (2003) for the U.K. and by Fisher and Voss (2004) for the Australia.[42]

This predictability of financial wealth by the deviation from the long-run relation may provide the important information to the policymakers who are interested in stability of financial markets. That is, the deviation in the previous period is probably a good indicator of how financial market will move in the current period. However, the policy response to this finding should be cautious. The reason is that the movement

[42] For Germany, on the other hand, Hamburg et al. (2005) find the future income predictability by cointegrating residuals.

Table 2.10: Long Horizon Regressions of Δc_{t+j}, Δa_{t+j}, and Δy_{t+j}

Horizon J	Δc_t	Δa_t	Δy_t	$\beta' \mathbf{x}_t$	$\bar{R}^2$
	Panel I: $\sum_{j=1}^{J} \Delta c_{t+j}$ regressed on				
1	0.05	0.00	**0.28**	-0.01	0.08
	(0.13)	(0.07)	(0.13)	(0.04)	
2	0.09	0.03	0.37	-0.02	0.06
	(0.22)	(0.11)	(0.21)	(0.06)	
4	0.11	-0.09	0.51	0.02	0.02
	(0.35)	(0.18)	(0.33)	(0.10)	
8	-0.19	-0.12	0.50	0.06	-0.03
	(0.52)	(0.26)	(0.50)	(0.15)	
16	-1.33	0.08	1.19	0.24	0.03
	(0.68)	(0.35)	(0.67)	(0.18)	
24	-1.38	0.32	**1.90**	**0.83**	0.34
	(0.70)	(0.35)	(0.73)	(0.18)	
	Panel II: $\sum_{j=1}^{J} \Delta a_{t+j}$ regressed on				
Horizon J	Δc_t	Δa_t	Δy_t	$\beta' \mathbf{x}_t$	$\bar{R}^2$
1	0.02	-0.14	**0.39**	**0.27**	0.22
	(0.21)	(0.10)	(0.19)	(0.06)	
2	0.33	0.05	0.47	**0.47**	0.42
	(0.27)	(0.13)	(0.25)	(0.07)	
4	0.28	0.02	**0.92**	**0.99**	0.53
	(0.42)	(0.21)	(0.40)	(0.12)	
8	-0.33	-0.27	**1.44**	**1.90**	0.56
	(0.68)	(0.35)	(0.65)	(0.19)	
16	-1.00	-0.60	1.32	**2.91**	0.60
	(1.00)	(0.51)	(0.99)	(0.27)	
24	-1.95	-0.19	1.66	**3.88**	0.76
	(1.07)	(0.53)	(1.11)	(0.27)	
	Panel III: $\sum_{j=1}^{J} \Delta y_{t+j}$ regressed on				
Horizon J	Δc_t	Δa_t	Δy_t	$\beta' \mathbf{x}_t$	$\bar{R}^2$
1	0.11	-0.05	0.18	0.05	0.04
	(0.15)	(0.07)	(0.14)	(0.04)	
2	-0.02	0.04	**0.46**	0.08	0.09
	(0.23)	(0.11)	(0.22)	(0.06)	
4	-0.24	-0.02	**0.91**	**0.22**	0.11
	(0.37)	(0.19)	(0.36)	(0.10)	
8	-0.27	0.19	0.92	**0.48**	0.13
	(0.60)	(0.30)	(0.57)	(0.17)	
16	-1.18	0.13	1.76	**1.11**	0.23
	(0.95)	(0.49)	(0.94)	(0.25)	
24	-1.43	0.40	2.42	**1.90**	0.41
	(1.22)	(0.61)	(1.28)	(0.31)	

Note: The table reports output from j period regressions of $\Delta x_{t+1} + \cdots + \Delta x_{t+h} = x_{t+h} - x_t$ on Δx_t and the cointegrating residual $\hat{\beta}' \mathbf{x}_t$, where $x \in \{c,\ a,\ y\}$. For each regression, OLS estimates, standard errors (in parentheses) and adjusted R^2 statistics are presented. Significant coefficients at the 5-percent level are highlighted in bold face. The sample spans 1980:Q1 to 2005:Q3.

of financial wealth is largely associated with transitory components whereas those of consumption and labor income are mainly related with permanent components.

Therefore, as long as policymakers regard the long-term stability of the economy as their key responsibility, they must not be too sensitive to the transitory fluctuations of financial wealth since they seem not to be much related with the long-term movements of consumption and labor income in Korea. This finding is therefore consistent with the recent view that many monetary economists and central bankers agree with. In particular, in Korea which adopts the inflation-targeting regime, this result implies that monetary policy should respond to the financial markets only to the extent that movements of financial markets help to forecast inflationary or deflationary pressure in the economy (see, for example, Mishkin 2007, pp. 528–532; Bernanke and Gertler 2001).[43]

2.5.5 Comparison before and after the Korean Financial Crisis

The financial crisis that hit Korea in the late 1997 had a catastrophic impact on the Korean economy, causing Korea's worst recession in the postwar era.[44] Figure 2.4 clearly illustrates what happened since the Korean financial crisis.

Real GDP growth (i.e., thick solid line in the figure), which had been around 7.9% on average between 1971 and 1996, declined to a positive 4.7% in 1997 and then plunged to a negative 6.9% in 1998. In the aftermath of the crisis, unemployment (thin solid line in the figure) rose from pre-crisis levels of 3.4% on average between 1971 and 1996 to 7.0% in 1998. Probably, the most astonishing feature, however, arises when we look at

[43]Notice that this argument does not imply that central banks should not respond the asset price movements. It is clear in the statement of Mishkin (2007, p. 528), *"the issue about how central bankers should respond to asset price movements is not whether they should respond at all, but whether they should respond over and above the response called for by the flexible inflation-targeting framework described."* A stark exception is research by Cecchetti, Genberg, and Wadhwani (2002). Although they do not state that policymakers should target asset prices, they emphasize that central banks can improve macroeconomic performance by reacting to asset price misalignments.

[44]Since, from the viewpoint of macroeconomic fundamentals, Korea looked more or less strong, the Korean financial crisis was such a surprise to the markets and drew a large attention from researchers. It is not fully agreed on what caused the Korean financial crisis. However, it is widely accepted that the primary causes of the Korean financial crisis are 1) the excess short-term foreign borrowing of financial sector and corporate sector and subsequent sudden reversal of capital flows, which was triggered by the Southeast Asian currency crisis, and 2) the poor microeconomic polices such as the too-big-to-fail policies on large firms, which worsened the moral hazard problem by making the lending monitoring mechanism unnecessary and a loose supervision on non-banking sector, which allowed its large exposure to Southern Asian countries (Radelet and Sachs 1998; Stiglitz 1999; Hahm and Mishkin 2000; Park and Song 2001; Danthine and Donaldson 2005).

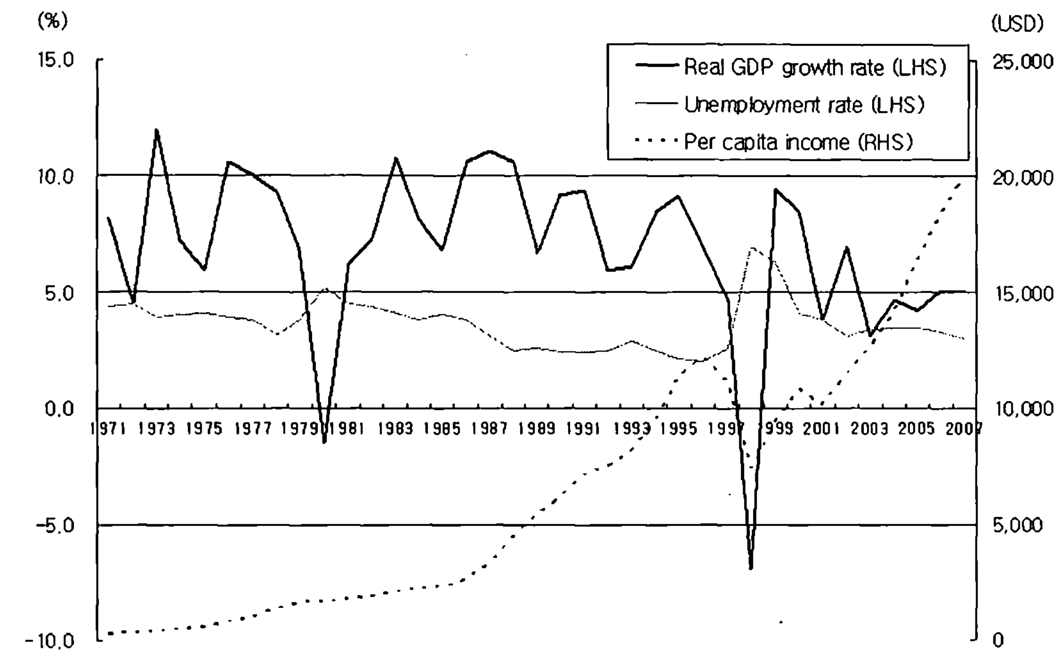

Figure 2.4: Real GDP Growth Rates, Unemployment Rates, and Per Capita Income in US Dollars: 1971-2007

the dotted line, which stands for the per capita income growth in terms of the nominal US dollars. It plummeted to the level of 7,355 US dollars in 1998. Therefore, the per capita income in 1998 declined by 34.2% from the previous year.[45]

Following the policy advices of the International Monetary Fund (IMF), Korea adopted several important policy changes and institutional reforms just after the Korean financial crisis.[46] For example, the exchange rate regime moved from the currency basket system (i.e., partially pegging the values of Korean won with a moving band to currencies of large trading partners such as the U.S. and Japan) to the free-floating exchange rate system on December 16, 1997. The central bank was legally granted more

[45]Partly, this figure reflects the sharp depreciation of the Korean won against US dollar from 954 won per dollar on average in 1997 to 1395 won in 1998. In fact, the foreign exchange rate (i.e., the amount of Korean won needed to buy a dollar) was 843 won at the beginning of 1997 and reached 1962 won on December 23 in 1997. One of the main reasons was that the foreign exchange reserves in Korea dwindled to around $4 billion by late 1997. However, at the end of 2007, the foreign exchange reserves increased to about $262 billion and this raised the public debates on the optimal amount of foreign exchange reserves in Korea.

[46]The Korean government signed an arrangement with the IMF on December 3 for $21 billion in stand-by credit.

independence from the government and it started to primarily use the call rate target (similar to the federal fund rate operating target in the U.S. and the official Bank Rate in the U.K.) as a policy instrument rather than the monetary aggregates. Lastly the Bank of Korea began to publicly announce the inflation target, which is often referred to as the overriding objective.[47]

Against this backdrop, with respect to the short-term dynamics of consumption, financial wealth, and labor income, it is of interest that whether the Korean financial crisis affected the Korean consumers' behaviors. To that end, this section divides the examined period into two subperiods: 1980:Q1 to 1997:Q4 and 1998:Q1 to 2005:Q3.

Table 2.11 reports the estimates and their standard errors for the short-term dynamics of c_t, a_t, and y_t before and after the Korean financial crisis. It also presents those when the CPI inflation rate is included. As shown in the lower panel, this inclusion seems to significantly increase the adjusted R^2 statistics (especially for consumption and income growth equations).

Several features are worth noting from the table. First, in the upper panel, consumption seems to be forward-looking in the Δa equation but backward-looking in the Δc equation. That is, the asset growth equation implies that if households expect an increase in assets at time t, consumption at $t-1$ will rise above the long run trend as shown in a larger error correction term at $t-1$, while the consumption growth equation implies that consumption is influenced by lagged income growth. These two findings seems to be puzzling at least at first glance.[48]

One possible explanation for the dependence of Δc_t on Δy_{t-1} seems to come from the findings in Section 2.5.3. We find there that large portions of changes in consumption and labor income are associated with permanent movements and a large portion of financial wealth change is related with transitory movements. Then, as predicted by Friedman's (1957) permanent income hypothesis, which asserts that an increase of the current income is associated with an increase of consumption only to the extent that it reflects an increase in permanent income, if households observed the increase of income in the last period, they seemed to believe that an appreciable portion of income increase was related with the permanent component. Therefore, it seems that

[47]This can be seen as a reflection of the view that monetary policy cannot affect real quantities in the long run, and that inflation is mostly a monetary phenomenon in the long run (Bernanke and Mishkin 1997).

[48]Dr. Leslie Reinhorn gratefully pointed out this issue.

Table 2.11: Short-term Dynamics of c_t, a_t, and y_t before and after the Financial Crisis

(1980:1-1997:4)	EQUATION			EQUATION		
	Δc_t	Δa_t	Δy_t	Δc_t	Δa_t	Δy_t
Δc_{t-1}	-0.20	0.28	0.19	-0.23	0.30	0.10
	(0.13)	(0.32)	(0.20)	(0.13)	(0.34)	(0.20)
Δa_{t-1}	0.06	-0.08	0.05	0.05	-0.09	0.04
	(0.05)	(0.12)	(0.07)	(0.05)	(0.12)	(0.07)
Δy_{t-1}	**0.26**	0.21	0.18	**0.24**	0.19	0.17
	(0.09)	(0.23)	(0.14)	(0.09)	(0.23)	(0.14)
$\hat{\beta}'\mathbf{x}_{t-1}$	**-0.07**	**0.21**	-0.01	-0.02	**0.17**	0.03
	(0.02)	(0.05)	(0.03)	(0.02)	(0.05)	(0.03)
Δp_t	-	-	-	0.06	0.02	-0.05
				(0.02)	(0.05)	(0.03)
$\bar{R}^2$	0.25	0.19	0.08	0.26	0.13	0.05

(1998:1-2005:3)	EQUATION			EQUATION		
	Δc_t	Δa_t	Δy_t	Δc_t	Δa_t	Δy_t
Δc_{t-1}	0.20	-0.10	0.22	-0.15	-0.33	-0.03
	(0.35)	(0.36)	(0.31)	(0.30)	(0.35)	(0.29)
Δa_{t-1}	-0.15	-0.37	-0.22	-0.14	-0.37	-0.24
	(0.22)	(0.23)	(0.19)	(0.18)	(0.21)	(0.17)
Δy_{t-1}	0.22	0.46	-0.05	0.12	0.36	-0.13
	(0.37)	(0.38)	(0.33)	(0.31)	(0.35)	(0.29)
$\hat{\beta}'\mathbf{x}_{t-1}$	-0.03	**0.51**	0.10	-0.05	**0.48**	0.10
	(0.16)	(0.17)	(0.14)	(0.13)	(0.15)	(0.13)
Δp_t	-	-	-	**-1.07**	**-0.83**	**-0.79**
				(0.29)	(0.34)	(0.28)
$\bar{R}^2$	-0.03	0.21	-0.08	0.30	0.33	0.16

Notes: The values in parentheses denote standard errors. Significant coefficients at the 5% level are highlighted in bold face. The term Δp_t denotes the CPI inflation rate.

households are basically forward-looking as shown in error-correction terms in the Δa equation but changes in past income may give a signal to changes in permanent income and that is why consumption responds to past income as shown in the Δc equation.

Second, in the post-crisis period, although the signs of three loadings in both models imply the error-corrections of those three variables, only a_t shows the statistically significant error correcting behavior but c_t and y_t remain weakly exogenous, as found in the entire period.

Third, the pre-crisis period in the upper panel of Table 2.11 confirms that financial wealth shows a sizeable and statistically significant error-correcting behavior. However, it also provides a conflicting result for error-correcting behavior of c_t. That is, when only the real variables are analyzed as in the left model, it states that consumption also shows the error-correcting behavior even though the magnitude is much less than that of financial wealth. On the other hand, when the nominal variable Δp_t is augmented, it states that consumption does not show the error-correcting behavior.

From these mixed empirical results, at least one thing is clear. Although important policy and institutional changes have taken place during the Korean financial crisis, when new exogenous shocks hit the economy, financial wealth does most work to pull the processes back towards the long-run relation across the two sample periods.

Finally, it is important to notice a caveat for this comparison though. The above comparison is based on the simple analysis of the sign and the statistical significance of each loading. Since the post-crisis period only contains 31 quarter data out of total 103 quarter data, the sample periods are unbalanced and thus the above test result may have a low power to justify the findings especially for the latter sample period.

2.6 Conclusion

Starting from the notion that consumption, financial wealth, and income movements must be cointegrated in theory, this research documents three main empirical findings in Korea. First of all, using the Johansen's method, this study confirms the theoretical prediction on the long-run relationship involving consumption, financial wealth and income in Korea. Two interpretations can be given to this estimation result. The first interpretation is related with wealth effect. This essay finds that the marginal propensity of consume out of financial wealth is 0.02 which is similar to the previous findings in other countries. The more important interpretation, nonetheless, is that average Korean households allocate their wealth between human wealth to non-human wealth at the ratio of about six in the long run.

Second, investigating the short-term dynamics among those variables provides policymakers with the important insights on Korean economy. We found that only financial wealth shows the sizable and statistically significant error correction behavior. Therefore, movements of financial wealth are predictable using the deviations from the long-run relationship. Although this predictability is its own useful information for the policymakers, this essay argues that monetary policy should be cautious to respond to the movements of financial wealth. The reason is that they are largely associated with transitory components while consumption and labor income are mainly governed by the permanent shocks during the examined period. Therefore, large volatility in financial markets need not be followed by large movements in consumption. In fact, households seem to match the smoothness of consumption by adjusting their financial wealth.

Finally, by comparing the Korean consumers' behaviors before and after the Korean financial crisis, this essay finds that although there were several policy and institutional changes during the crisis, most adjustment to the long run relation has been done by financial wealth across the two sample periods. However, it is important to notice a limitation of this test since the sample periods are unbalanced in particular. Therefore, for the future research, it appears interesting and important to investigate this comparison with more rigorous tests.

Appendix to Chapter 2

2.A Derivation of (2.3.2)

Campbell and Mankiw (1989) derive (2.3.2) from (2.3.1) and this appendix shows the detailed steps which are not in their article. To obtain (2.3.2) from (2.3.1), we need to divide (2.3.1) by total wealth at the beginning of period t, W_t, first. Then one gets (2.A.1)

$$\frac{W_{t+1}}{W_t} = (1 + R^w_{t+1})\left(1 - \frac{C_t}{W_t}\right). \tag{2.A.1}$$

Next, taking logs on both sides of (2.A.1) yields

$$\begin{aligned} \ln\left[\frac{W_{t+1}}{W_t}\right] &= \ln(1 + R^w_{t+1}) + \ln\left\{1 - \exp\left[\ln\left(\frac{C_t}{W_t}\right)\right]\right\} \\ \Delta w_{t+1} &= r^w_{t+1} + \ln\big[1 - \exp(c_t - w_t)\big], \end{aligned} \tag{2.A.2}$$

where $r^w_{t+1} = \ln(1 + R^w_{t+1})$, $\exp(\cdot)$ is the exponential operator and lower letters stand for the logarithms of corresponding variables.

Note that a first-order approximation of this second term on the right hand side (around the mean of log consumption-wealth ratio) is

$$\begin{aligned} \ln[1 - \exp(c_t - w_t)] &\approx \ln[1 - \exp(c - w)] - \frac{\exp(c - w)}{1 - \exp(c - w)}\left[c_t - w_t - (c - w)\right] \\ &= \ln\rho - \frac{1 - \rho}{\rho}\big[c_t - w_t - (c - w)\big], \end{aligned} \tag{2.A.3}$$

where c and w are the logs of consumption and total wealth, respectively, at the steady state. Note that we define a new parameter $\rho = 1 - \exp(c - w) = 1 - \frac{C}{W} = \frac{W - C}{W}$, where C and W are the levels of consumption and total wealth at the steady state. Namely, ρ is the investment to total wealth ratio at the steady state.

Then, (2.A.2) can be reformulated as the following form of a difference equation

$$\Delta w_{t+1} \approx r^w_{t+1} + \chi + \left(1 - \frac{1}{\rho}\right)(c_t - w_t), \tag{2.A.4}$$

where $\chi = \ln \rho + \frac{1-\rho}{\rho}(c - w) = \ln \rho + \frac{1-\rho}{\rho} \ln(1 - \rho)$.

In order to derive a long-term version of this budget constraint, the next step is to use the trivial equality.

$$\begin{aligned} \Delta w_{t+1} &= \Delta c_{t+1} - \Delta c_{t+1} + \Delta w_{t+1} \\ &= \Delta c_{t+1} + (c_t - c_{t+1}) + (w_{t+1} - w_t) \\ &= \Delta c_{t+1} + (c_t - w_t) - (c_{t+1} - w_{t+1}). \end{aligned} \tag{2.A.5}$$

Equating the right hand side of (2.A.4) with that of (2.A.5) yields the following difference equation for the consumption-wealth ratio.

$$\begin{aligned} r^w_{t+1} + \chi + \left(1 - \frac{1}{\rho}\right)(c_t - w_t) &= \Delta c_{t+1} + (c_t - w_t) - (c_{t+1} - w_{t+1}) \\ \Leftrightarrow -\frac{1}{\rho}(c_t - w_t) &= \Delta c_{t+1} - (c_{t+1} - w_{t+1}) - \chi - r^w_{t+1} \\ \Leftrightarrow (c_t - w_t) &= -\rho \Delta c_{t+1} + \rho(c_{t+1} - w_{t+1}) + \rho\chi + \rho r^w_{t+1} \\ &= \rho(r^w_{t+1} - \Delta c_{t+1}) + \rho(c_{t+1} - w_{t+1}) + \rho\chi. \end{aligned} \tag{2.A.6}$$

Note that the transversality condition $\lim_{j \to \infty} \rho^j (c_{t+j} - w_{t+j}) = 0$ holds as long as $0 < \rho < 1$ and the consumption-wealth ratio is stationary. Finally, one can solve difference equation (2.A.6) forward as follows,

$$c_t - w_t = \sum_{j=1}^{\infty} \rho^j (r^w_{t+j} - \Delta c_{t+j}) + \frac{\rho\chi}{1 - \rho}. \tag{2.A.7}$$

Equation (2.A.7) holds simply as a consequence of the intertemporal budget constraint and therefore holds *ex-post*, but it also holds *ex-ante*. Accordingly, one can take conditional expectations of both sides of (2.A.7) to obtain (2.3.2).

$$c_t - w_t = E_t \sum_{j=1}^{\infty} \rho^j (r^w_{t+j} - \Delta c_{t+j}) + \frac{\rho\chi}{1 - \rho}. \tag{2.3.2}$$

2.B Derivation of (2.3.10)

By adding an assumption on human wealth to (2.3.2), Lettau and Ludvigson (2001, 2004) derive a long-run relationship as in (2.3.10). This appendix provides the detailed steps which are not in their articles. The derivation starts from the notion that total wealth is composed of asset wealth, A, and human wealth, H.

$$W_t = A_t + H_t. \tag{2.3.3}$$

Then, if we denote average share of asset wealth in total wealth as ω, and average share of human wealth as $(1-\omega)$, then the following result can be obtained by log-linearizing (2.3.3),

$$w_t \approx \omega a_t + (1-\omega)h_t, \tag{2.3.4}$$

where $a_t = \ln A_t$ and $h_t = \ln H_t$.

— **Proof of** (2.3.4) —

$$\begin{aligned} W_t &= A_t + H_t \\ \ln W_t &= \ln(A_t + H_t) \\ w_t &= \ln[\exp(\ln A_t) + \exp(\ln H_t)] \\ &= \ln[\exp(a_t) + \exp(h_t)] \\ &\approx \ln(A+H) + \frac{e^a}{A+H}(a_t - a) + \frac{e^h}{A+H}(h_t - h) \\ &= \mu + \frac{A}{A+H}(a_t) + \frac{H}{A+H}(h_t) \\ &\quad \text{where } \mu = \ln(A+H) - \left[\frac{A}{A+H}(a) + \frac{H}{A+H}(h)\right] \end{aligned}$$

Finally, if we ignore the constant and use the definition of $\omega = \frac{A}{A+H} = \frac{A}{W}$, we obtain (2.3.4).

Second, the simple gross return to total wealth can be exactly decomposed into the simple gross returns to asset wealth and human wealth as follows,

$$(1+R^w_{t+1}) = \omega(1+R^a_{t+1}) + (1-\omega)(1+R^h_{t+1}), \tag{2.3.5}$$

where R^a_t and R^h_t are the simple net returns on asset wealth and human wealth, respectively. Then, as shown in Campbell (1996, p.308), Equation (2.3.5) can be approximated with log returns into (2.B.1),

$$\begin{aligned}(1+r^w_{t+1}) &\approx \omega(1+r^a_{t+1}) + (1-\omega)(1+r^h_{t+1}) \\ &= 1 + \omega r^a_{t+1} + (1-\omega)r^h_{t+1}.\end{aligned} \tag{2.B.1}$$

Now it is trivial to attain (2.3.6) from the above.

$$r^w_{t+1} \approx \omega r^a_{t+1} + (1-\omega)r^h_{t+1}. \tag{2.3.6}$$

At this stage, Equation (2.3.2) can be reformulated as (2.3.7) if we omit the unimportant constant,

$$\begin{aligned}c_t - w_t &= E_t\sum_{j=1}^{\infty}\rho^j(r^w_{t+j} - \Delta c_{t+j}) \\ c_t - \omega a_t - (1-\omega)h_t &= E_t\sum_{j=1}^{\infty}\rho^j(r^w_{t+j} - \Delta c_{t+j}) \quad \{\text{by } (2.3.4)\} \\ &= E_t\sum_{j=1}^{\infty}\rho^j\left[\omega r^a_{t+j} + (1-\omega)r^h_{t+j} - \Delta c_{t+j}\right] \quad \{\text{by } (2.3.6)\}.\end{aligned} \tag{2.3.7}$$

When we try to take (2.3.7) into the data, we, however, are not able to observe either the log of human wealth (h_t) or the log return on it (r^h_{t+1}). To solve this problem, Lettau and Ludvigson (2001, 2004) follow Campbell's (1996) suggestion that observable labor income can be viewed as the dividend on human wealth. Then, H_t can be related with Y_t as (2.3.8),

$$\begin{aligned}(1+R^h_{t+1})H_t &= E_t(H_{t+1} + Y_{t+1}) \\ \Leftrightarrow H_t &= E_t\left[\frac{(H_{t+1} + Y_{t+1})}{(1+R^h_{t+1})}\right]\end{aligned} \tag{2.3.8}$$

where R^h_{t+1} is a simple net return to human wealth.[49] Note that Equation (2.3.8) can be solved forward using the transversality condition of $E_t \sum_{i=1}^{\infty} \left[\prod_{j=1}^{i} (1+R^h_{t+j})^{-1} H_{t+i} \right] = 0$ and that the solution is

$$H_t = E_t \sum_{i=1}^{\infty} \left[\prod_{j=1}^{i} (1 + R^h_{t+j})^{-1} Y_{t+i} \right]. \tag{2.B.2}$$

In order to express unobservable H_t in terms of observable Y_t, we need to divide (2.3.8) by Y_t on both sides and then log-linearize it around the steady state. Without any loss of generality, conditional expectation operator E_t will be omitted during the derivation.

$$\begin{aligned}
\frac{H_t}{Y_t} &= (1 + R^h_{t+1})^{-1} \left(\frac{H_{t+1}}{Y_t} + \frac{Y_{t+1}}{Y_t} \right) \\
h_t - y_t &\approx -r^h_{t+1} + \ln\left[\exp(h_{t+1} - y_t) + \exp(\Delta y_{t+1})\right] \\
&\approx -r^h_{t+1} + \ln\left[\exp(h - y) + e^{\Delta y}\right] + \frac{e^{h-y}}{e^{h-y} + e^{y}} \left[h_{t+1} - y_t - (h - y)\right] \\
&+ \frac{e^{\Delta y}}{e^{h-y} + e^{\Delta y}} \left[\Delta y_{t+1} - \Delta y\right] \\
&= -r^h_{t+1} + \ln\left[\exp(h - y) + 1\right] + \frac{e^{h-y}}{e^{h-y} + 1} \left[h_{t+1} - y_t - (h - y)\right] \\
&+ \frac{1}{e^{h-y} + 1} \left[\Delta y_{t+1}\right],
\end{aligned} \tag{2.B.3}$$

where h and y are the logs of human wealth and labor income at the steady state. Note that the last equality in (2.B.3) holds because $\Delta y = 0$ and $e^{\Delta y} = 1$. Now by collecting unimportant linearization constant in (2.B.3) into k, one can reformulate (2.B.3) as (2.B.4),

$$\begin{aligned}
h_t - y_t &\approx - r^h_{t+1} + k + \frac{e^{h-y}}{e^{h-y} + 1} \left[h_{t+1} - y_t\right] + \frac{1}{e^{h-y} + 1} \left[\Delta y_{t+1}\right] \\
&= - r^h_{t+1} + k + \rho_h \left[h_{t+1} - y_t\right] + (1 - \rho_h) \left[\Delta y_{t+1}\right],
\end{aligned} \tag{2.B.4}$$

where $k = \ln\left[\exp(h - y) + 1\right] - \frac{e^{h-y}}{e^{h-y}+1}(h - y)$ and $\rho_h = \frac{1}{1+e^{y-h}}$.

[49] Note that (2.3.8) holds exactly the same interpretation of the present value model for the stock as,

$$P_t = E_t \left[\frac{(P_{t+1} + D_{t+1})}{(1 + R_{t+1})} \right].$$

Manipulation of the last line of (2.B.4) yields the ratio of human wealth to labor income as (2.B.5),

$$\begin{aligned} h_t - y_t &\approx -r^h_{t+1} + k + \rho_h \left[h_{t+1} - y_t\right] + (1-\rho_h)\left[y_{t+1} - y_t\right] \\ &= -r^h_{t+1} + k + \rho_h \left[h_{t+1} - y_t\right] + \left[y_{t+1} - y_t - \rho_h y_{t+1} + \rho_h y_t\right] \\ &= -r^h_{t+1} + k + \rho_h \left[h_{t+1} - y_{t+1}\right] + \left[y_{t+1} - y_t\right] \\ &= k + \rho_h\left[h_{t+1} - y_{t+1}\right] + \left[\Delta y_{t+1} - r^h_{t+1}\right]. \end{aligned} \tag{2.B.5}$$

Then, the transversality condition $\left[i.e., \lim_{j\to\infty} \rho_h^j (h_{t+j} - y_{t+j}) = 0\right]$ states that the second term in the last line of (2.B.5) will disappear when one solves (2.B.5) forward. Recovering the conditional expectation operator in (2.B.5) produces a log-linear approximation of Equation (2.3.8) as (2.3.9),

$$\begin{aligned} h_t &= \frac{k}{1-\rho_h} + y_t + E_t \sum_{j=1}^{\infty} \rho_h^{j-1}(\Delta y_{t+j} - r^h_{t+j}) \\ &= \kappa + y_t + z_t, \end{aligned} \tag{2.3.9}$$

where $\kappa(=\frac{k}{1-\rho_h})$ is a constant, $z_t = E_t \sum_{j=1}^{\infty} \rho_h^{j-1}(\Delta y_{t+j} - r^h_{t+j})$ and $\rho_h \equiv \frac{1}{1+\exp(y-h)}$. So, the nonstationary component of human wealth is assumed to be captured by labor income.

Finally, plugging (2.3.9) into (2.3.7) and ignoring the unimportant constant yield Equation (2.3.10).

$$c_t - \omega a_t - (1-\omega)\left[\kappa + y_t + z_t\right] = E_t \sum_{j=1}^{\infty} \rho^j \left[\omega r^a_{t+j} + (1-\omega) r^h_{t+j} - \Delta c_{t+j}\right] \quad \{\text{by } (2.3.9)\}$$

$$c_t - \omega a_t - (1-\omega) y_t = E_t \sum_{j=1}^{\infty} \rho^j \left[\omega r^a_{t+j} + (1-\omega) r^h_{t+j} - \Delta c_{t+j}\right] + (1-\omega) z_t, \tag{2.3.10}$$

where ω is the average share of asset wealth in total wealth $(=\frac{A}{W})$ and ρ is the ratio of investment to total wealth $(=\frac{W-C}{W})$ at the steady state. As before, r^a_{t+j} and r^h_{t+j} are the returns on asset wealth and human wealth from $t+j-1$ to $t+j$, respectively. Notice that $z_t = E_t \sum_{j=1}^{\infty} \rho_h^{j-1}(\Delta y_{t+j} - r^h_{t+j})$ is a stationary process if y_t is integrated at the order one. Lastly, $\rho_h = \frac{1}{1+\exp(y-h)}$.

2.C Derivation of (2.4.5) and (2.4.4)

Following Juselius (2006), this section shows how to derive Equation (2.4.5) and (2.4.4) from Equation (2.4.3). This will reveal that three different forms of VAR have the same information on the data. Consider first an unrestricted p-dimensional VAR(k) for the vectors of, possibly nonstationary p variables $\mathbf{x}$,

$$\mathbf{x}_t = \mathbf{\Pi}_1\mathbf{x}_{t-1} + \mathbf{\Pi}_2\mathbf{x}_{t-2} + \cdots + \mathbf{\Pi}_k\mathbf{x}_{t-k} + \epsilon_t, \tag{2.4.3}$$

where $t = 1, \cdots, T$ and ϵ_t is an independently and normally distributed mean zero process vector. In addition, $\mathbf{x}_t$ is a p-dimensional endogenous variable column vector, $\mathbf{\Pi}_i$'s are $p \times p$ coefficient matrices.

Let us start by subtracting $\mathbf{x}_{t-1}$ from both sides of (2.4.3).

$$\Delta\mathbf{x}_t = (\mathbf{\Pi}_1 - \mathbf{I})\mathbf{x}_{t-1} + \mathbf{\Pi}_2\mathbf{x}_{t-2} + \cdots + \mathbf{\Pi}_k\mathbf{x}_{t-k} + \epsilon_t, \tag{2.C.1}$$

where $\mathbf{I}$ is a $p \times p$ identity matrix. Then, adding and subtracting $(\mathbf{\Pi}_1 - \mathbf{I})\mathbf{x}_{t-2}$ on the right hand side of (2.C.1) yields,

$$\Delta\mathbf{x}_t = (\mathbf{\Pi}_1 - \mathbf{I})\Delta\mathbf{x}_{t-1} + (\mathbf{\Pi}_1 - \mathbf{I} + \mathbf{\Pi}_2)\mathbf{x}_{t-2} + \cdots + \mathbf{\Pi}_{k-1}\mathbf{x}_{t-k+1} + \mathbf{\Pi}_k\mathbf{x}_{t-k} + \epsilon_t. \tag{2.C.2}$$

By repeating this procedure until $k-1$, one ends up with the following specification involving levels and first differences,

$$\begin{aligned}\Delta\mathbf{x}_t &= \mathbf{\Upsilon}_1\Delta\mathbf{x}_{t-1} + \mathbf{\Upsilon}_2\Delta\mathbf{x}_{t-2} + \cdots + \mathbf{\Upsilon}_{k-1}\Delta\mathbf{x}_{t-k+1} + \mathbf{\Pi}\mathbf{x}_{t-k} + \epsilon_t \\ &= \sum_{i=1}^{k-1}\mathbf{\Upsilon}_i\Delta\mathbf{x}_{t-i} + \mathbf{\Pi}\mathbf{x}_{t-k} + \epsilon_t,\end{aligned} \tag{2.4.5}$$

where $\mathbf{\Upsilon}_i = -\left(\mathbf{I} - \sum_{j=1}^{i}\mathbf{\Pi}_j\right)$ and $\mathbf{\Pi} = -\left(\mathbf{I} - \sum_{i=1}^{k}\mathbf{\Pi}_i\right)$.

More interesting reformulation of Equation (2.4.3) is (2.4.4), which links the unrestricted VAR to the cointegrated VAR. To derive (2.4.4) from (2.4.3), start by adding and subtracting $\mathbf{\Pi}_k \mathbf{x}_{t-(k-1)}$ on the right hand side of (2.4.3),

$$\begin{aligned}\mathbf{x}_t &= \mathbf{\Pi}_1 \mathbf{x}_{t-1} + \mathbf{\Pi}_2 \mathbf{x}_{t-2} + \cdots + \mathbf{\Pi}_{k-1} \mathbf{x}_{t-(k-1)} + \mathbf{\Pi}_k \mathbf{x}_{t-k} + \epsilon_t \\ &= \cdots + \mathbf{\Pi}_{k-1} \mathbf{x}_{t-(k-1)} + \mathbf{\Pi}_k \mathbf{x}_{t-(k-1)} - \mathbf{\Pi}_k \mathbf{x}_{t-(k-1)} + \mathbf{\Pi}_k \mathbf{x}_{t-k} + \epsilon_t \\ &= \cdots + (\mathbf{\Pi}_{k-1} + \mathbf{\Pi}_k) \mathbf{x}_{t-(k-1)} - \mathbf{\Pi}_k \Delta \mathbf{x}_{t-(k-1)} + \epsilon_t. \end{aligned} \tag{2.C.3}$$

Then, adding and subtracting $(\mathbf{\Pi}_{k-1} + \mathbf{\Pi}_k)\mathbf{x}_{t-(k-2)}$ on the right hand side of (2.C.3) and then repeating this procedure yields,

$$\begin{aligned}\mathbf{x}_t &= \cdots + (\mathbf{\Pi}_{k-2} + \mathbf{\Pi}_{k-1} + \mathbf{\Pi}_k)\mathbf{x}_{t-(k-2)} - (\mathbf{\Pi}_{k-1} + \mathbf{\Pi}_k)\Delta \mathbf{x}_{t-(k-2)} - \mathbf{\Pi}_k \Delta \mathbf{x}_{t-(k-1)} + \epsilon_t \\ &= (\mathbf{\Pi}_1 + \cdots + \mathbf{\Pi}_k)\mathbf{x}_{t-1} - (\mathbf{\Pi}_2 + \cdots + \mathbf{\Pi}_k)\Delta \mathbf{x}_{t-1} - \cdots - \mathbf{\Pi}_k \Delta \mathbf{x}_{t-(k-1)} + \epsilon_t \\ &= (\mathbf{\Pi}_1 + \cdots + \mathbf{\Pi}_k)\mathbf{x}_{t-1} - \left[\sum_{i=1}^{k-1} \Big(\sum_{j=i+1}^{k} \mathbf{\Pi}_j \Big) \Delta \mathbf{x}_{t-i}\right] + \epsilon_t. \end{aligned} \tag{2.C.4}$$

Finally, we end up with (2.4.4),

$$\begin{aligned}\Delta \mathbf{x}_t &= \mathbf{\Pi} \mathbf{x}_{t-1} + \mathbf{\Gamma}_1 \Delta \mathbf{x}_{t-1} + \cdots + \mathbf{\Gamma}_{k-1} \Delta \mathbf{x}_{t-(k-1)} + \epsilon_t \\ &= \mathbf{\Pi} \mathbf{x}_{t-1} + \sum_{i=1}^{k-1} \mathbf{\Gamma}_i \Delta \mathbf{x}_{t-i} + \epsilon_t, \end{aligned} \tag{2.4.4}$$

where $\mathbf{\Gamma}_i = -\big(\sum_{j=i+1}^{k} \mathbf{\Pi}_j\big)$ and $\mathbf{\Pi} = -\big(\mathbf{I} - \sum_{i=1}^{k} \mathbf{\Pi}_i\big)$.

For example, assume that the lag length $k = 2$. Then, (2.4.3) can be reformulated as a VECM form as (2.C.5),

$$\Delta \mathbf{x}_t = -\mathbf{\Pi}_2 \Delta \mathbf{x}_{t-1} + \mathbf{\Pi} \mathbf{x}_{t-1} + \boldsymbol{\mu} + \epsilon_t, \tag{2.C.5}$$

where $\mathbf{\Pi} = -(\mathbf{I} - \mathbf{\Pi}_1 - \mathbf{\Pi}_2)$. Equivalently, (2.4.3) can be represented as,

$$\Delta \mathbf{x}_t = -(\mathbf{I} - \mathbf{\Pi}_1)\Delta \mathbf{x}_{t-1} + \mathbf{\Pi} \mathbf{x}_{t-2} + \boldsymbol{\mu} + \epsilon_t, \tag{2.C.6}$$

where $\mathbf{\Pi} = -(\mathbf{I} - \mathbf{\Pi}_1 - \mathbf{\Pi}_2)$.

Now, it is clear that VECM is nothing but a reparameterization of a VAR model. Hence, the values of the likelihood function are the same in (2.4.3), (2.4.5), and (2.4.4). Notice also that $\mathbf{\Pi}$ in (2.4.5) and $\mathbf{\Pi}$ in (2.4.4) are exactly the same.

2.D Johansen's (1995) Cointegration Test Methodology

This section follows the exposition of Juselius (2006). Notice that Stock and Watson's (1993) two-stage approach has a clear disadvantage in the sense that it cannot capture more than one cointegrating relation. From these motivations, Johansen (1988, 1995) proposes the full information maximum likelihood (FIML) approach, which allows the several cointegrating relations in VECM like (2.4.8). Then, the number of cointegrating relations can be tested in the intermediate step of the FIML estimation procedure by comparing the restricted maximized log-likelihood with the unrestricted maximized log-likelihood.

The procedure starts by concentrating out the short run effects in (2.4.8) following the Frisch-Waugh theorem.[50] That is, we regress $\Delta \mathbf{x}_t$ on first differenced vectors and save the residual vector as $\boldsymbol{R}_{0t}$. Then, we regress $\mathbf{x}_{t-1}$ on first differenced vectors and save the residual vector as $\boldsymbol{R}_{1t}$.

Given these two stacked vectors, we can compute following correlation matrices of the OLS residuals:

$$\boldsymbol{S}_{00} = T^{-1}\sum_{t=1}^{T} \boldsymbol{R}_{0t}\boldsymbol{R'}_{0t}, \quad \boldsymbol{S}_{11} = T^{-1}\sum_{t=1}^{T} \boldsymbol{R}_{1t}\boldsymbol{R'}_{1t}, \quad \text{and} \quad \boldsymbol{S}_{01} = T^{-1}\sum_{t=1}^{T} \boldsymbol{R}_{0t}\boldsymbol{R'}_{1t} = \boldsymbol{S}_{10}.$$

As shown in Juselius (2006, pp. 117–119), these matrices are used in computing the determinant of the **residual covariance matrix, $\hat{\boldsymbol{\Omega}}(\boldsymbol{\beta})$**. Then, under the assumption of multivariate normality, the maximum of the log-likelihood function of concentrated model is given as,

$$\ln \mathscr{L}_{max}(\beta) = -T\frac{1}{2}\ln\left|\hat{\boldsymbol{\Omega}}(\boldsymbol{\beta})\right| - T\frac{3}{2} - T\frac{3}{2}\ln(2\pi). \tag{2.D.1}$$

Notice that since the second and the third terms on the right are the constants, the maximum of the log-likelihood function are solely determined by the minimized log-determinant of the residual covariance matrix.

[50]The Frisch-Waugh theorem states that the multiple regression coefficient of any single variable can also be obtained by first netting out the effect of other variable(s) in the regression model from both the dependent variable and the independent variable. For example, consider a regression model, $y_t = \beta_0 + \beta_1 x_{1t} + \beta_2 x_{2t} + \varepsilon_t$. First, we regress y_t on x_{1t} and save the residual e_{1t}. Second, we regress x_{2t} on x_{1t} and save the residual e_{2t}. Then, regressing e_{1t} on e_{2t} yields the OLS estimate β_2, the clean effect of x_{2t} on y_t.

After some manipulations, one can find the express for $\hat{\boldsymbol{\Omega}}(\boldsymbol{\beta})$ as,

$$\therefore \quad |\hat{\boldsymbol{\Omega}}(\boldsymbol{\beta})| = |\boldsymbol{S}_{00}| \cdot \frac{|\boldsymbol{\beta}'\boldsymbol{S}_{11}\boldsymbol{\beta} - \boldsymbol{\beta}'\boldsymbol{S}_{10}\boldsymbol{S}_{00}^{-1}\boldsymbol{S}_{01}\boldsymbol{\beta}|}{|\boldsymbol{\beta}'\boldsymbol{S}_{11}\boldsymbol{\beta}|}. \tag{2.D.2}$$

Now we can obtain a solution for unnormalized $\boldsymbol{\beta}$ that minimizes $|\hat{\boldsymbol{\Omega}}(\boldsymbol{\beta})|$ by minimizing the second term in (2.D.2). Namely, finding the minimized value of $|\hat{\boldsymbol{\Omega}}(\boldsymbol{\beta})|$ yields the ML estimate for $\boldsymbol{\beta}$. As shown in Hamilton (1994, pp. 639–641), the minimized determinant of the residual covariance matrix can be obtained as,

$$|\hat{\boldsymbol{\Omega}}(\hat{\boldsymbol{\beta}})| = |\boldsymbol{S}_{00}| \prod_{i=1}^{3}(1 - \hat{\lambda}_i), \tag{2.D.3}$$

where the ordered eigenvalues of matrix $\boldsymbol{S}_{11}^{-1}\boldsymbol{S}_{10}\boldsymbol{S}_{00}^{-1}\boldsymbol{S}_{01}$ are $\hat{\lambda}_1 > \hat{\lambda}_2 > \hat{\lambda}_3$, namely these eigenvalues are the solutions of $|\lambda\boldsymbol{S}_{11} - \boldsymbol{S}_{10}\boldsymbol{S}_{00}^{-1}\boldsymbol{S}_{01}| = 0$.

Clearly from $|\lambda - \boldsymbol{S}_{10}\boldsymbol{S}_{00}^{-1}\boldsymbol{S}_{01}\boldsymbol{S}_{11}^{-1}| = 0$, the eigenvalues $\hat{\lambda}_i$ can be interpreted as the squared canonical correlations between nonstationary part residual $\boldsymbol{R}_{1,t}$ and stationary part residuals $\boldsymbol{R}_{0t}$. Therefore, larger $\hat{\lambda}_i$ implies more stationarity.

Finally, given the minimized value of $|\hat{\boldsymbol{\Omega}}(\hat{\boldsymbol{\beta}})|$, we can reformulate (2.D.1) as,

$$\ln \mathscr{L}_{max}(\hat{\boldsymbol{\beta}}) = -\frac{T}{2}\ln|\boldsymbol{S}_{00}| - \frac{T}{2}\sum_{i=1}^{3}\ln(1 - \hat{\lambda}_i) - T\frac{3}{2} - T\frac{3}{2}\ln(2\pi). \tag{2.D.4}$$

Note that (2.D.4) is the unrestricted maximized log-likelihood. Based on (2.D.4), two versions of **likelihood ratio** (LR) tests for specifying the number of cointegrating relations are widely used in the empirical literature: trace (hereafter λ_{trace}) test and maximal eigenvalue (hereafter λ_{max}) test. The strategy of these two tests is classifying 3 relations into r stationary relations which correspond to the r largest nonzero eigenvalues and the $3 - r$ nonstationary relations which correspond to the $3 - r$ zero eigenvalues, where r is the cointegration rank (i.e., $r = rank(\boldsymbol{\Pi})$ = number of non-zero eigenvalues of matrix $\boldsymbol{S}_{11}^{-1}\boldsymbol{S}_{10}\boldsymbol{S}_{00}^{-1}\boldsymbol{S}_{01}$).

Given the three variables, two tests are testing a sequence of null hypotheses,

$$H_0 \ : \ rank(\boldsymbol{\Pi}) = 0, \quad H_0 \ : \ rank(\boldsymbol{\Pi}) = 1, \quad H_0 \ : \ rank(\boldsymbol{\Pi}) = 2.$$

On the other hand, the corresponding sequence of alternative hypotheses in λ_{max} test are,

$$H_A : rank(\mathbf{\Pi}) = 1, \quad H_A : rank(\mathbf{\Pi}) = 2, \quad H_A : rank(\mathbf{\Pi}) = 3,$$

while the alternative hypotheses in λ_{trace} test are all $H_A : rank(\mathbf{\Pi}) = 3$. Therefore, it is straightforward to notice that the trace test is implemented consecutively while the maximal eigenvalue test is implemented by testing $r + 1$ cointegrating vectors against r cointegrating vectors.

First, in the **trace test**, we continue each test until the null hypothesis cannot be rejected for the first time. For example, we first test $H_0 : rank(\mathbf{\Pi}) = 0$ (i.e., $\mathbf{\Pi} = \mathbf{0}$ and thus $\lambda_1 = \lambda_2 = \lambda_3 = 0$). If this first H_0 is not rejected, we conclude that there are three unit roots and no cointegrating relation in the system. Then, a model in first differences is adopted. If the first H_0 is rejected ($\lambda_1 \neq 0$), we test $H_0 : rank(\mathbf{\Pi}) = 1$. If this second H_0 is not rejected (i.e., $\lambda_2 = \lambda_3 = 0$), we conclude that there are two unit roots and one cointegrating relation. On the other hand, if the second H_0 is rejected (i.e., $\lambda_2 \neq 0$), we test the final $H_0 : rank(\mathbf{\Pi}) = 2$. If this last H_0 is not rejected (i.e., $\lambda_3 = 0$), we conclude that there are one unit roots and two cointegrating relations. However, the last H_0 is also rejected (i.e., $\lambda_3 \neq 0$), we conclude that there is no unit root in the system. Therefore, $\mathbf{x}_t$ is stationary and thus a stationary VAR model for the levels is adopted.[51]

In practice, since H_0 in λ_{trace} test can be interpreted as restrictions on λ_i while H_A has no restriction, λ_{trace} statistics are computed as follows,

$$\begin{aligned}\lambda_{trace}(r) &= 2[\ln \mathscr{L}_A - \ln \mathscr{L}_0] = -T\sum_{i=1}^{3} \ln(1 - \hat{\lambda}_i) - \{-T\sum_{i=1}^{r} \ln(1 - \lambda_i)\} \\ &= -T \sum_{i=r+1}^{3} \ln(1 - \hat{\lambda}_i), \qquad (2.D.5)\end{aligned}$$

where $\ln \mathscr{L}_A$ and $\ln \mathscr{L}_0$ denote log-likelihoods under H_A and H_0, respectively.[52]

Then, for example, if three restrictions $\lambda_1 = \lambda_2 = \lambda_3 = 0$ are correct as under the first H_0, (2.D.4) is equal to zero. Then, λ_{trace} statistic (2.D.5) is the difference

[51] Note that when the second $H_0 : rank(\mathbf{\Pi}) = 1$ in λ_{trace} test is considered separately, the acceptance of the second H_0 does not mean an exact one cointegrating relation. Rather it only means $\lambda_2 = \lambda_3 = 0$ and does not rule out $\lambda_1 = 0$. Therefore, the acceptance of the second H_0 in λ_{trace} test is interpreted as at most one cointegration. Similarly, the sole acceptance of the second H_0 in λ_{max} test does not rule out no cointegration. For this reason, two tests must be implemented sequently.

[52] For example, if only $\lambda_3 = 0$, the maximized log-likelihood $\ln \mathscr{L}_0$ becomes exclusively a function of

between unrestricted maximized log-likelihood under H_A and maximized log-likelihood with three restrictions under H_0,

$$\lambda_{trace}(0) = 2(\ln \mathscr{L}_A - \ln \mathscr{L}_0) = -T\big[\ln(1-\hat{\lambda}_1) + \ln(1-\hat{\lambda}_2) + \ln(1-\hat{\lambda}_3)\big] + 0.$$

And if this test statistic is small, we cannot reject $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Similarly, if two restrictions $\lambda_2 = \lambda_3 = 0$ are correct as under the second H_0, λ_{trace} statistic (2.D.5) becomes,

$$\begin{aligned}\lambda_{trace}(1) &= 2(\ln \mathscr{L}_A - \ln \mathscr{L}_0)\\ &= -T\big[\ln(1-\lambda_1) + \ln(1-\hat{\lambda}_2) + \ln(1-\hat{\lambda}_3)\big] + T\big[\ln(1-\lambda_1)\big]\\ &= -T\big[\ln(1-\hat{\lambda}_2) + \ln(1-\hat{\lambda}_3)\big].\end{aligned}$$

And if the test statistic is small, we cannot reject $\lambda_2 = \lambda_3 = 0$. Finally, if one restriction $\lambda_3 = 0$ is correct as under the third H_0, λ_{trace} statistic (2.D.5) becomes,

$$\begin{aligned}\lambda_{trace}(2) &= 2(\ln \mathscr{L}_A - \ln \mathscr{L}_0)\\ &= -T\big[\ln(1-\lambda_1) + \ln(1-\lambda_2) + \ln(1-\hat{\lambda}_3)\big] + T\big[\ln(1-\lambda_1) + \ln(1-\lambda_2)\big]\\ &= -T\big[\ln(1-\hat{\lambda}_3)\big].\end{aligned}$$

And if the test statistic is small, we cannot reject $\lambda_3 = 0$. Then, we conclude that there are two cointegrating vectors.

On the other hand, λ_{max} statistic, which is the difference between maximized log-likelihood with $r-1$ restrictions (i.e., $r+1$ cointegrating vectors) under H_A and maximized log-likelihood with r restrictions (i.e., r cointegrating vectors) under H_0, is computes as

$$\lambda_{max}(r, r+1) = -T\sum_{i=1}^{r+1}\ln(1-\hat{\lambda}_i) + T\sum_{i=1}^{r}\ln(1-\hat{\lambda}_i) = -T\ln(1-\hat{\lambda}_{r+1}). \quad (2.D.6)$$

the non-zero eigenvalues:

$$\ln \mathscr{L}_0(\beta) = -\frac{T}{2}\ln|S_{00}| - \frac{T}{2}\sum_{i=1}^{2}\ln(1-\lambda_i) - T\frac{3}{2} - T\frac{3}{2}\ln(2\pi).$$

where $r = 0, 1, 2$ is the number of cointegrating relations. Similarly, if λ_{max} statistic is small, we cannot reject the corresponding null hypothesis.

2.E Deterministic Terms in Cointegrated Processes

This section shows that the deterministic terms like constants and linear trends can be easily included in (2.4.4) following Lütkepohl (2006, pp. 256–258). First note that an intercept term in a random walk with drift generates a linear trend in the mean of the process, whereas an intercept term in a stationary autoregressive model simply stands for a constant mean value. To understand the impact of the deterministic terms in VECM, let us consider a process $\mathbf{y}_t = \boldsymbol{\mu}_t + \mathbf{x}_t$, where $\mathbf{x}_t$ is a zero mean VAR(k) process with cointegrated variables and $\boldsymbol{\mu}_t$ denotes for the deterministic term. $\boldsymbol{\mu}_t$ can be understood as $\boldsymbol{\mu}_0 + \boldsymbol{\mu}_1 t$ in general, where $\boldsymbol{\mu}_0$ and $\boldsymbol{\mu}_1$ are fixed 3-dimensional parameter vectors.[53]

Suppose that $\mathbf{x}_t$ can be represented as a VECM with r cointegrating vectors,

$$\begin{aligned}\Delta \mathbf{x}_t &= \boldsymbol{\alpha}\boldsymbol{\beta}' \mathbf{x}_{t-1} + \boldsymbol{\Gamma}_1 \Delta \mathbf{x}_{t-1} + \cdots + \boldsymbol{\Gamma}_{k-1} \Delta \mathbf{x}_{t-k+1} + \boldsymbol{\epsilon}_t \\ &= \boldsymbol{\Pi} \mathbf{x}_{t-1} + \boldsymbol{\Gamma}_1 \Delta \mathbf{x}_{t-1} + \cdots + \boldsymbol{\Gamma}_{k-1} \Delta \mathbf{x}_{t-k+1} + \boldsymbol{\epsilon}_t. \end{aligned} \tag{2.E.1}$$

(**Case I.** $\boldsymbol{\mu}_t = \boldsymbol{\mu}_0$)

If $\boldsymbol{\mu}_t = \boldsymbol{\mu}_0$, we know that $\mathbf{x}_t = \mathbf{y}_t - \boldsymbol{\mu}_0$ and $\Delta \mathbf{x}_t = \Delta \mathbf{y}_t$. From (2.E.1), we get

$$\begin{aligned}\Delta \mathbf{y}_t &= \boldsymbol{\alpha}\boldsymbol{\beta}'(\mathbf{y}_{t-1} - \boldsymbol{\mu}_0) + \boldsymbol{\Gamma}_1 \Delta \mathbf{y}_{t-1} + \cdots + \boldsymbol{\Gamma}_{k-1} \Delta \mathbf{y}_{t-k+1} + \boldsymbol{\epsilon}_t \\ &= \boldsymbol{\alpha} \begin{bmatrix} \boldsymbol{\beta}' & -\boldsymbol{\beta}'\boldsymbol{\mu}_0 \end{bmatrix} \begin{bmatrix} \mathbf{y}_{t-1} \\ 1 \end{bmatrix} + \boldsymbol{\Gamma}_1 \Delta \mathbf{y}_{t-1} + \cdots + \boldsymbol{\Gamma}_{k-1} \Delta \mathbf{y}_{t-k+1} + \boldsymbol{\epsilon}_t \\ &= \boldsymbol{\alpha}\tilde{\boldsymbol{\beta}}' \tilde{\mathbf{y}}_{t-1} + \boldsymbol{\Gamma}_1 \Delta \mathbf{y}_{t-1} + \cdots + \boldsymbol{\Gamma}_{k-1} \Delta \mathbf{y}_{t-k+1} + \boldsymbol{\epsilon}_t \\ &= \begin{bmatrix} \boldsymbol{\Pi} & -\boldsymbol{\Pi}\boldsymbol{\mu}_0 \end{bmatrix} \tilde{\mathbf{y}}_{t-1} + \boldsymbol{\Gamma}_1 \Delta \mathbf{y}_{t-1} + \cdots + \boldsymbol{\Gamma}_{k-1} \Delta \mathbf{y}_{t-k+1} + \boldsymbol{\epsilon}_t \\ &= \tilde{\boldsymbol{\Pi}} \tilde{\mathbf{y}}_{t-1} + \boldsymbol{\Gamma}_1 \Delta \mathbf{y}_{t-1} + \cdots + \boldsymbol{\Gamma}_{k-1} \Delta \mathbf{y}_{t-k+1} + \boldsymbol{\epsilon}_t, \end{aligned} \tag{2.E.2}$$

where $\tilde{\boldsymbol{\beta}}' = \begin{bmatrix} \boldsymbol{\beta}' & -\boldsymbol{\beta}'\boldsymbol{\mu}_0 \end{bmatrix}$, $\tilde{\mathbf{y}}_{t-1} = \begin{bmatrix} \mathbf{y}_{t-1} \\ 1 \end{bmatrix}$, and $\tilde{\boldsymbol{\Pi}} = \begin{bmatrix} \boldsymbol{\Pi} & -\boldsymbol{\Pi}\boldsymbol{\mu}_0 \end{bmatrix}$ is $(3 \times (3+1))$.

[53]The advantage of splitting a process $\mathbf{y}_t$ into the deterministic part and the zero mean stochastic part is that the mean of $\mathbf{y}_t$ involves only the deterministic term. The disadvantage of this strategy is that we cannot observe the stochastic part in reality and we must reexpress $\mathbf{x}_t$ in terms of $\mathbf{y}_t$ (Lütkepohl 2006).

Now it is straightforward to notice that if there is just a constant mean in data, it can be incorporated into the cointegrating relations. Put differently, the constant mean becomes an intercept term in the cointegrating relations. Therefore, by defining an overall intercept term $\boldsymbol{\nu}_0 = -\boldsymbol{\Pi}\boldsymbol{\mu}_0$, (2.E.2) can also be rewritten as

$$\begin{aligned}\Delta\mathbf{y}_t &= \boldsymbol{\nu}_0 + \boldsymbol{\alpha}\boldsymbol{\beta}'\mathbf{y}_{t-1} + \boldsymbol{\Gamma}_1\Delta\mathbf{y}_{t-1} + \cdots + \boldsymbol{\Gamma}_{k-1}\Delta\mathbf{y}_{t-k+1} + \epsilon_t \\ &= \boldsymbol{\nu}_0 + \boldsymbol{\Pi}\mathbf{y}_{t-1} + \boldsymbol{\Gamma}_1\Delta\mathbf{y}_{t-1} + \cdots + \boldsymbol{\Gamma}_{k-1}\Delta\mathbf{y}_{t-k+1} + \epsilon_t. \qquad (2.E.3)\end{aligned}$$

Note that (3×1) intercept vector $\boldsymbol{\nu}_0$ cannot be an arbitrary vector but must satisfy the restriction ($\boldsymbol{\nu}_0 = -\boldsymbol{\Pi}\boldsymbol{\mu}_0$) in order to ensure that the intercept term in this model does not generate a linear trend in the mean of the $\mathbf{y}_t$ variables.

(**Case II.** $\boldsymbol{\mu}_t = \boldsymbol{\mu}_0 + \boldsymbol{\mu}_1 t$)

If $\boldsymbol{\mu}_t = \boldsymbol{\mu}_0 + \boldsymbol{\mu}_1 t$, we have $\mathbf{x}_t = \mathbf{y}_t - \boldsymbol{\mu}_0 - \boldsymbol{\mu}_1 t$ and $\Delta\mathbf{x}_t = \Delta\mathbf{y}_t - \boldsymbol{\mu}_1$. Then, from (2.E.1), we get

$$\Delta\mathbf{y}_t - \boldsymbol{\mu}_1 = \boldsymbol{\alpha}\boldsymbol{\beta}'(\mathbf{y}_{t-1} - \boldsymbol{\mu}_0 - \boldsymbol{\mu}_1(t-1)) + \boldsymbol{\Gamma}_1(\Delta\mathbf{y}_{t-1} - \boldsymbol{\mu}_1) + \cdots + \boldsymbol{\Gamma}_{k-1}(\Delta\mathbf{y}_{t-k+1} - \boldsymbol{\mu}_1) + \epsilon_t, \qquad (2.E.4)$$

or by collecting deterministic terms,

$$\begin{aligned}\Delta\mathbf{y}_t = &\underbrace{\boldsymbol{\mu}_1 - \boldsymbol{\alpha}\boldsymbol{\beta}'\boldsymbol{\mu}_0 - \boldsymbol{\Gamma}_1\boldsymbol{\mu}_1 - \cdots - \boldsymbol{\Gamma}_{k-1}\boldsymbol{\mu}_1}_{constants} + \boldsymbol{\alpha}\boldsymbol{\beta}'\mathbf{y}_{t-1} - \boldsymbol{\alpha}\boldsymbol{\beta}'\boldsymbol{\mu}_1(t-1) + \\ &\boldsymbol{\Gamma}_1\Delta\mathbf{y}_{t-1} + \cdots + \boldsymbol{\Gamma}_{k-1}\Delta\mathbf{y}_{t-k+1} + \epsilon_t \\ = &\boldsymbol{\nu} + \boldsymbol{\alpha}\begin{bmatrix}\boldsymbol{\beta}' & -\boldsymbol{\beta}'\boldsymbol{\mu}_1\end{bmatrix}\begin{bmatrix}\mathbf{y}_{t-1} \\ t-1\end{bmatrix} + \boldsymbol{\Gamma}_1\Delta\mathbf{y}_{t-1} + \cdots + \boldsymbol{\Gamma}_{k-1}\Delta\mathbf{y}_{t-k+1} + \epsilon_t \\ = &\boldsymbol{\nu} + \boldsymbol{\Pi}^+\mathbf{y}^+_{t-1} + \boldsymbol{\Gamma}_1\Delta\mathbf{y}_{t-1} + \cdots + \boldsymbol{\Gamma}_{k-1}\Delta\mathbf{y}_{t-k+1} + \epsilon_t, \qquad (2.E.5)\end{aligned}$$

where $\boldsymbol{\nu} = -\boldsymbol{\alpha}\boldsymbol{\beta}'\boldsymbol{\mu}_0 + (I_3 - \boldsymbol{\Gamma}_1 - \cdots - \boldsymbol{\Gamma}_{k-1})\boldsymbol{\mu}_1$, $\mathbf{y}^+_{t-1} = \begin{bmatrix}\mathbf{y}_{t-1} \\ t-1\end{bmatrix}$, and $\boldsymbol{\Pi}^+ = \boldsymbol{\alpha}\begin{bmatrix}\boldsymbol{\beta}' & -\boldsymbol{\beta}'\boldsymbol{\mu}_1\end{bmatrix}$ is a $(3 \times (3+1))$ matrix.

Now the overall intercept term $\boldsymbol{\nu}$ is determined by $\boldsymbol{\mu}_0$, $\boldsymbol{\mu}_1$, and the other parameters and the trend term is thus incorporated into the cointegrating relations. If we set up the model with unrestricted linear trend term in the form of (2.E.6), this will yields a model which is capable of generating quadratic trends in the means of the variables though this case is limited usefulness.

$$\Delta \mathbf{y}_t = \boldsymbol{\nu} + \mathbf{\Pi}^+ \mathbf{y}_{t-1}^+ + \mathbf{\Gamma}_1 \Delta \mathbf{y}_{t-1} + \cdots + \mathbf{\Gamma}_{k-1} \Delta \mathbf{y}_{t-k+1} + \epsilon_t. \tag{2.E.6}$$

(Case III. Linear trend in data but no linear trend in cointegration)

The most interesting case arises when $\boldsymbol{\beta}'\boldsymbol{\mu}_1 = \mathbf{0}$, namely the trend slope parameter $\boldsymbol{\mu}_1$ is orthogonal to the cointegrating relation matrix. Then, (2.E.5) can be written as follows.

$$\begin{aligned} \Delta \mathbf{y}_t &= \boldsymbol{\nu} + \boldsymbol{\alpha}\boldsymbol{\beta}' \mathbf{y}_{t-1} + \mathbf{\Gamma}_1 \Delta \mathbf{y}_{t-1} + \cdots + \mathbf{\Gamma}_{k-1} \Delta \mathbf{y}_{t-k+1} + \epsilon_t \\ &= \boldsymbol{\nu} + \mathbf{\Pi} \mathbf{y}_{t-1} + \mathbf{\Gamma}_1 \Delta \mathbf{y}_{t-1} + \cdots + \mathbf{\Gamma}_{k-1} \Delta \mathbf{y}_{t-k+1} + \epsilon_t, \end{aligned} \tag{2.E.7}$$

where $\boldsymbol{\nu} = -\boldsymbol{\alpha}\boldsymbol{\beta}'\boldsymbol{\mu}_0 + (I_3 - \mathbf{\Gamma}_1 - \cdots - \mathbf{\Gamma}_{k-1})\boldsymbol{\mu}_1$.

Now it is straightforward to notice that the trend term disappears from the cointegrating relations and the linear trends are generated via the intercept term $\boldsymbol{\nu}$. As long as $\boldsymbol{\mu}_1 \neq \mathbf{0}$, the data have linear trends in their means but the cointegrating relations do not involve linear trends.

Alternatively, we can rewrite (2.E.7) as with (2.E.8).

$$\Delta \mathbf{y}_t = \tilde{\boldsymbol{\nu}} + \boldsymbol{\alpha}\boldsymbol{\beta}'(\mathbf{y}_{t-1} - \boldsymbol{\mu}_0) + \mathbf{\Gamma}_1 \Delta \mathbf{y}_{t-1} + \cdots + \mathbf{\Gamma}_{k-1} \Delta \mathbf{y}_{t-k+1} + \epsilon_t, \tag{2.E.8}$$

where $\tilde{\boldsymbol{\nu}} = \boldsymbol{\nu} + \boldsymbol{\alpha}\boldsymbol{\beta}'\boldsymbol{\mu}_0 = (I_3 - \mathbf{\Gamma}_1 - \cdots - \mathbf{\Gamma}_{k-1})\boldsymbol{\mu}_1$.

This derivation clearly shows why we have to use the demeaned $\boldsymbol{\beta}'\mathbf{x}_{t-1}$ in (2.4.8) and the non-demeaned $\boldsymbol{\beta}'\mathbf{x}_{t-1}$ in (2.4.9). Thus, when the demeaned $\boldsymbol{\beta}'\mathbf{x}_{t-1}$ in (2.4.8) is used, $\boldsymbol{\nu}$ in (2.4.8) has the same meaning of $\tilde{\boldsymbol{\nu}}$ in (2.4.9).

2.F Decomposition of Permanent and Transitory Shocks

Following Juselius (2006) and Lütkepohl (2006), this section will show that common factor trend can be understood as the other side of cointegration in the moving average representation, and then how the permanent and transitory decomposition can be implemented.

2.F.1 Stock and Watson's (1988) Common Trend Representation

Consider a 3-dimensional vector of data $\mathbf{x}_t$, which is I(1). Suppose that $\Delta\mathbf{x}_t$ can be modeled as VECM$(k-1)$ as (2.4.6). Then, the Wold decomposition theorem states that $\Delta\mathbf{x}_t$ can be represented as follows,

$$\Delta\mathbf{x}_t = \boldsymbol{\delta} + \boldsymbol{C}(L)\boldsymbol{\epsilon}_t, \tag{2.F.1}$$

where $C(L)$ is a polynomial in the lag operator and $\boldsymbol{\epsilon}_t \sim i.i.d(\mathbf{0}, \boldsymbol{\Omega})$.[54]

For convenience, in the below, we assume that $\mathbf{x}_t$ is the detrended time series, without loss of generality.[55] Then, as shown in Stock and Watson (1988), there is a common trend representation of $\Delta\mathbf{x}_t$,

$$\begin{aligned}\Delta\mathbf{x}_t &= \boldsymbol{C}(L)\boldsymbol{\epsilon}_t = \big[\boldsymbol{C}(1) + (1-L)\boldsymbol{C}^*(L)\big]\boldsymbol{\epsilon}_t \\ \mathbf{x}_t &= \frac{\boldsymbol{C}(1)}{(1-L)}\boldsymbol{\epsilon}_t + \boldsymbol{C}^*(L)\boldsymbol{\epsilon}_t, \end{aligned} \tag{2.F.2}$$

where $\boldsymbol{C}(1)$ is a constant matrix which determines the long-run property of $\mathbf{x}_t$, and

[54]For example, a simple VAR(2) model with one unit root can be represented as follows,

$$\begin{aligned}\mathbf{x}_t &= \boldsymbol{\Pi}_1\mathbf{x}_{t-1} + \boldsymbol{\Pi}_2\mathbf{x}_{t-2} + \boldsymbol{\mu} + \boldsymbol{\epsilon}_t \\ (\mathbf{I} - \boldsymbol{\Pi}_1 L - \boldsymbol{\Pi}_2 L^2)\mathbf{x}_t &= \boldsymbol{\mu} + \boldsymbol{\epsilon}_t \\ \boldsymbol{\Pi}(L)\mathbf{x}_t &= \boldsymbol{\mu} + \boldsymbol{\epsilon}_t \\ (1-L)\mathbf{x}_t &= \boldsymbol{\Pi}^{-1}(L)(1-L)(\boldsymbol{\mu} + \boldsymbol{\epsilon}_t) \\ \Delta\mathbf{x}_t &= \boldsymbol{C}(L)(\boldsymbol{\mu} + \boldsymbol{\epsilon}_t) \\ &= (\boldsymbol{C}_0 + \boldsymbol{C}_1 L + \boldsymbol{C}_2 L^2 + \cdots)(\boldsymbol{\mu} + \boldsymbol{\epsilon}_t) \\ &= \boldsymbol{C}(L)\boldsymbol{\mu} + \boldsymbol{C}(L)\boldsymbol{\epsilon}_t = \boldsymbol{C}(1)\boldsymbol{\mu} + \boldsymbol{C}(L)\boldsymbol{\epsilon}_t \\ &= \boldsymbol{\delta} + \boldsymbol{C}(L)\boldsymbol{\epsilon}_t.\end{aligned}$$

Note that we can obtain $\mathbf{x}_t$ by the discrete integration as

$$\begin{aligned}\mathbf{x}_t &= \boldsymbol{C}(1)\sum_{i=1}^{t}\boldsymbol{\epsilon}_i + \boldsymbol{C}(1)\boldsymbol{\mu}t + \boldsymbol{C}^*(L)\boldsymbol{\epsilon}_t - \boldsymbol{C}^*(L)\boldsymbol{\epsilon}_0 + \mathbf{x}_0 \\ &= \boldsymbol{C}(1)\sum_{i=1}^{t}\boldsymbol{\epsilon}_i + \boldsymbol{C}(1)\boldsymbol{\mu}t + \boldsymbol{C}^*(L)\boldsymbol{\epsilon}_t + \mathbf{X}_0,\end{aligned}$$

where $\boldsymbol{C}(1)$ is a constant matrix and $\mathbf{X}_0$ is a collection of initial values. It is straightforward that $\mathbf{x}_t$ can be decomposed into common stochastic trend, deterministic trend, stationary process, and initial values.

[55]For instance, let us define a (3×1) data vector $\mathbf{y} = \boldsymbol{\mu}_t + \mathbf{x}_t$, where $\mathbf{x}_t$ is a zero mean VAR(k) process with cointegrated variables and $\boldsymbol{\mu}_t$ stands for the deterministic term. $\boldsymbol{\mu}_t$ can be understood as $\boldsymbol{\mu}_0 + \boldsymbol{\mu}_1 t$ in general, where $\boldsymbol{\mu}_0$ and $\boldsymbol{\mu}_1$ are fixed 3-dimensional parameter vectors.

$C^*(L)$ is a polynomial in the lag operator which determines the short-run property of $\mathbf{x}_t$.[56] Clearly, the second line of (2.F.2) shows that $\mathbf{x}_t$ can be decomposed into permanent factor (unit roots) and transitory factor (I(0)). Then, if we can identify the permanent movements of $\mathbf{x}_t$ (i.e., $C(1)$), we can also identify the transitory movements as the residual.

Juselius (2006, p. 87) and Johansen (1995, Ch.4) show that $C(1)$ is,

$$C(1) = \beta_\perp(\alpha'_\perp \Gamma \beta_\perp)\alpha'_\perp = \tilde{\beta}_\perp \alpha'_\perp, \tag{2.F.3}$$

where $\Gamma = (\mathbf{I} - \Gamma_1 - \cdots - \Gamma_{k-1})$ and $\alpha_\perp$ and $\beta_\perp$ are $3\times(3-r)$ orthogonal complements of $\alpha_{3\times r}$ and $\beta_{3\times r}$, respectively, such that $\beta'\beta_\perp = \mathbf{0}_{3-r}$, $\alpha'\alpha_\perp = \mathbf{0}_{3-r}$, rank $(\alpha, \alpha_\perp) = 3$ and $(\beta, \beta_\perp) = 3$. Then, $\mathbf{x}_t$ can be represented as,

$$\mathbf{x}_t = \tilde{\beta}_\perp \frac{\alpha'_\perp \epsilon_t}{(1-L)} + I(0). \tag{2.F.4}$$

Now, (2.F.4) clarifies an important implication of cointegration in the MA representation. In AR representation β and α are the cointegrating matrix and its loading matrix respectively while in MA representation $\alpha'_\perp$ and $\tilde{\beta}_\perp$ are the common stochastic trends and their loadings. Thus, one cointegration ($r = 1$) in AR representation implies two (i.e., $3 - r$) common trends in MA representation. Note also that the non-stationarity in the process $\mathbf{x}_t$ comes from the cumulative sum of the $3 - r$ combinations $\alpha'_\perp [\sum_{i=1}^t \epsilon_{c,i} \;\; \sum_{i=1}^t \epsilon_{a,i} \;\; \sum_{i=1}^t \epsilon_{y,i}]'$.[57]

2.F.2 Gonzalo and Ng's (2001) Decomposition of Permanent and Transitory Shocks

Gonzalo and Granger (1995) and Gonzalo and Ng (2001) begin their structural decomposition of permanent shocks and transitory shocks from Johansen's (1995, p. 39) *beautiful relation*,

$$\beta_\perp(\alpha'_\perp \beta_\perp)^{-1}\alpha'_\perp + \alpha(\beta'\alpha)^{-1}\beta' = \mathbf{I}_3, \tag{2.F.5}$$

where $\alpha_\perp$ and $\beta_\perp$ are $3\times(3-r)$ orthogonal complements of $\alpha_{3\times r}$ and $\beta_{3\times r}$, respectively, such that $\beta'\beta_\perp = \mathbf{0}_{3-r}$, $\alpha'\alpha_\perp = \mathbf{0}_{3-r}$, rank $(\alpha, \alpha_\perp) = 3$ and $(\beta, \beta_\perp) = 3$.

[56] The first line of (2.F.2) is derived as follows. Let us define $F(L) = C(L) - C(1)$. Then, since $F(1) = 0$, $F(L) = (1-L)C^*(L)$. Finally, $C(L) = C(1) + (1-L)C^*(L)$.

[57] The common driving trends are the variables $\alpha'_\perp \sum_{i=1}^t \epsilon_i$.

Using (2.F.5) we can decompose $\mathbf{x}_t \sim I(1)$ into common trends, $\boldsymbol{\alpha}'_\perp \mathbf{x}_t$, and the cointegration relations, $\boldsymbol{\beta}'\mathbf{x}_t$,

$$
\begin{aligned}
\mathbf{x}_t &= \boldsymbol{\beta}_\perp(\boldsymbol{\alpha}'_\perp\boldsymbol{\beta}_\perp)^{-1}\boldsymbol{\alpha}'_\perp\mathbf{x}_t + \boldsymbol{\alpha}(\boldsymbol{\beta}'\boldsymbol{\alpha})^{-1}\boldsymbol{\beta}'\mathbf{x}_t \\
&= \boldsymbol{A}_1\boldsymbol{\alpha}'_\perp\mathbf{x}_t + \boldsymbol{A}_2\boldsymbol{\beta}'\mathbf{x}_t \\
\begin{bmatrix} c_t \\ a_t \\ y_t \end{bmatrix} &= \begin{bmatrix} \begin{pmatrix} A_{1,11} & A_{1,12} \\ A_{1,21} & A_{1,22} \\ A_{1,31} & A_{1,32} \end{pmatrix} & \begin{pmatrix} A_{2,11} \\ A_{2,21} \\ A_{2,31} \end{pmatrix} \end{bmatrix} \begin{bmatrix} \begin{pmatrix} \alpha'_{\perp,11} & \alpha'_{\perp,12} & \alpha'_{\perp,13} \\ \alpha'_{\perp,21} & \alpha'_{\perp,22} & \alpha'_{\perp,23} \end{pmatrix} \\ \begin{pmatrix} 1 & \beta'_{\perp,12} & \beta'_{\perp,13} \end{pmatrix} \end{bmatrix} \begin{bmatrix} c_t \\ a_t \\ y_t \end{bmatrix} \\
\mathbf{x}_t &= \boldsymbol{A} \begin{bmatrix} \boldsymbol{\alpha}'_\perp \\ \boldsymbol{\beta}' \end{bmatrix} \mathbf{x}_t = \boldsymbol{A}\boldsymbol{G}\mathbf{x}_t.
\end{aligned} \tag{2.F.6}
$$

Clearly, in the second line, the first part in the right hand side is $I(1)$, i.e., permanent process since the second part is $I(0)$. Then, Gonzalo and Ng (2001) show that the $(3-r)\times 1$ vector $\boldsymbol{u}_t^P = \boldsymbol{\alpha}'_\perp\boldsymbol{\epsilon}_t$ and $r \times 1$ vector $\boldsymbol{u}_t^T = \boldsymbol{\beta}'\boldsymbol{\epsilon}_t$ are the permanent and transitory shocks, respectively as long as matrix $\boldsymbol{G}$ is non-singular.[58] Note that in the third line, we assume that there are two common stochastic trends (i.e., two unit roots) and one cointegrating relation.

Now (2.F.2) can be factorized as,

$$
\begin{aligned}
\Delta\mathbf{x}_t &= \boldsymbol{C}(L)\boldsymbol{G}^{-1}\boldsymbol{G}\boldsymbol{\epsilon}_t = \boldsymbol{C}(L)\boldsymbol{G}^{-1}\boldsymbol{u}_t \\
&= \boldsymbol{D}(L)\boldsymbol{u}_t = \big[\boldsymbol{D}(1) + (1-L)\boldsymbol{D}^*(L)\big]\boldsymbol{u}_t,
\end{aligned} \tag{2.F.7}
$$

where

$$
\Delta\mathbf{x}_t = \boldsymbol{D}(L)\boldsymbol{u}_t = \begin{bmatrix} \boldsymbol{D}_{11}(L) & \boldsymbol{D}_{12}(L) & \boldsymbol{D}_{13}(L) \\ \boldsymbol{D}_{21}(L) & \boldsymbol{D}_{22}(L) & \boldsymbol{D}_{23}(L) \\ \boldsymbol{D}_{31}(L) & \boldsymbol{D}_{32}(L) & \boldsymbol{D}_{33}(L) \end{bmatrix} \begin{bmatrix} u_t^{P1} \\ u_t^{P2} \\ u_t^T \end{bmatrix}.
$$

Notice a long-run property of the polynomial matrix $\boldsymbol{D}(L)$ that the last column of $\boldsymbol{D}(1) = \boldsymbol{C}(1)\boldsymbol{G}^{-1} = \big(\boldsymbol{\beta}_\perp(\boldsymbol{\alpha}'_\perp\boldsymbol{\Gamma}\boldsymbol{\beta}_\perp)\boldsymbol{\alpha}'_\perp\big)\begin{bmatrix} \boldsymbol{\alpha}'_\perp \\ \boldsymbol{\beta}' \end{bmatrix}^{-1}$ is full of zeroes because they refer to the responses of $\Delta\mathbf{x}_t$ to the transitory shocks, and by definition, they have no effects on $\Delta\mathbf{x}_t$ or $\mathbf{x}_t$.[59]

[58] Gonzalo and Granger (1995) define the permanent and transitory shocks as follows,

$$
\lim_{h\to\infty} \frac{\partial E_t(\mathbf{x}_{t+h})}{\partial u_t^P} \neq 0 \quad \text{and} \quad \lim_{h\to\infty} \frac{\partial E_t(\mathbf{x}_{t+h})}{\partial u_t^T} = 0,
$$

where E_t is the conditional expectation with respect to the information set at time t.

[59] Namely, $\boldsymbol{D}_{13}(1) = \boldsymbol{D}_{23}(1) = \boldsymbol{D}_{33}(1) = 0$ are the long-run impact of the transitory shock. In

The intuition of this decomposition is that if speed of adjustment α_j is small, it gives the jth variable a large weight in the permanent innovations. This implies that that variable adjust little to recover the equilibrium. Conversely, if speed of adjustment α_j is large, it gives the jth variable a small weight in the permanent innovations. This implies that that variable adjust actively to recover the equilibrium. For example, suppose that

$$\alpha = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \alpha_\perp = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, clearly, the rapid error-correction of second variable implies that there is no common trend in the second variable.

Until now we show that three unorthogonalized shocks are $\boldsymbol{u}_t = G\epsilon_t$ and the permanent movements and transitory movements can be separated. Nonetheless, in order to implement meaningful impulse response analyses, shocks must be mutually uncorrelated. For this purpose, Gonzalo and Ng (2001) propose to use a lower triangular matrix $\boldsymbol{H}$, which is the Choleski decomposition of covariance matrix of $\boldsymbol{u}_t = \boldsymbol{G}\epsilon_t$.[60] Then, (2.F.7) can be reformulated as,

$$\Delta \mathbf{x}_t = \boldsymbol{C}(L)\boldsymbol{G}^{-1}\boldsymbol{H}\boldsymbol{H}^{-1}\boldsymbol{G}\epsilon_t = \boldsymbol{D}(L)\boldsymbol{H}\boldsymbol{H}^{-1}\boldsymbol{u}_t = \tilde{\boldsymbol{D}}(L)\eta_t, \tag{2.F.8}$$

where $\eta_t = \boldsymbol{H}^{-1}$ is the orthogonalized shock vector.

The practical procedure is summarized as follows,

1. Decide the lag length (k) and the number of cointegrating vectors (r).

2. Estimate a VECM(k-1) incorporating the cointegrating relationships. This yields $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$, thus one can construct $\hat{\boldsymbol{\alpha}}_\perp$.

3. Construct $\hat{\boldsymbol{G}} = \left(\hat{\boldsymbol{\alpha}}_\perp \ \hat{\boldsymbol{\beta}}\right)'$ and find permanent and transitory shocks $\hat{\boldsymbol{u}}_t$ as $\hat{\boldsymbol{G}}\hat{\epsilon}_t$.

4. Obtain a lower triangular matrix $\hat{\boldsymbol{H}}$ by applying Choleski decomposition to covariance matrix of $(\hat{\boldsymbol{G}}\hat{\epsilon}_t)$. The orthogonalized permanent and transitory shocks are $\hat{\boldsymbol{H}}^{-1}\hat{\boldsymbol{u}}_t = \hat{\boldsymbol{H}}^{-1}\hat{\boldsymbol{G}}\hat{\epsilon}_t$.

addition, as shown in Fisher and Huh (2006), $\boldsymbol{G}^{-1} = [\boldsymbol{\beta}_\perp(\boldsymbol{\alpha}'_\perp\boldsymbol{\beta})^{-1}, \ \boldsymbol{\alpha}(\boldsymbol{\beta}'\boldsymbol{\alpha})^{-1}]$. It is clear to verify the last column of $\boldsymbol{D}(1)$ is full of zeroes.

[60] Let us denote covariance matrix $\boldsymbol{e}\boldsymbol{e}'/T$ as $\boldsymbol{\Omega}$. Then, since $\boldsymbol{u} = \boldsymbol{G}\epsilon$, the covariance matrix of $\boldsymbol{u}$ is $\Sigma_u = \boldsymbol{G}\boldsymbol{\Omega}\boldsymbol{G}'$. Then, find the lower triangular matrix $\boldsymbol{H}$ from $\boldsymbol{H}\boldsymbol{H}' = \boldsymbol{G}\boldsymbol{\Omega}\boldsymbol{G}'$.

5. Post-multiply $\hat{\boldsymbol{D}}(L) = \hat{\boldsymbol{C}}(L)\hat{\boldsymbol{G}}^{-1}$ by $\hat{\boldsymbol{H}}$ to obtain an estimate of $\tilde{\boldsymbol{D}}(L)$.

Chapter 3

MAXIMUM LIKELIHOOD ESTIMATION OF A REAL BUSINESS CYCLE MODEL IN KOREA

3.1 Introduction

Since the large-scale system-of-equation models in Cowles Commission tradition were severely criticized by not only Lucas critique (1976) and but also Sims critique (1980), dynamic stochastic general equilibrium (henceforth, DSGE) models and vector autoregression (henceforth, VAR) models have become two standard building blocks of modern macroeconomists.

First, a DSGE model, firstly presented by Kydland and Prescott (1982), attempts to explain the movements and co-movements of many of central macroeconomic variables within the laboratories they created, firmly based on microeconomic foundations.[1] For instance, Kydland and Prescott (1982) demonstrated how technology – main source of long-run economic growth – can generate short-term cycles using a fully artificial model

[1]The term, a DSGE model, represents three distinct features. First, it is a general competitive equilibrium model in which we look for a collection of prices and quantities, such that all markets clear simultaneously, while households and firms solve the well-defined optimization problems. Second, it is a dynamic model in which the intertemporal allocation of resources is a key problem of rational agents. Finally, it is a stochastic model in which uncertain exogenous shocks trigger the adjusted processes of the endogenous variables determined by the decision-makers.

economy where consumers maximize their utility, firms maximize their profits and all markets clear. Since then, DSGE models have dominated business cycle research and several important questions are addressed in this framework (see, for example, Cooley 1995).

Work following this tradition usually has relied on simulation approach based on the *calibrated* values of parameters.[2] Then, by comparing moments of the artificial data from simulations in DSGE models with those of the actual data, a researcher is able to answer a variety of "what if?" questions in the laboratory, which is usually impossible in the traditional empirical work (see, for example, King, Plosser, and Rebelo 1988; Cooley 1995).[3] The power of this capacity cannot be overstated. Moreover, contrary to the empirical work based on the reduced form estimation, the micro-founded DSGE models are transparent enough that researchers can clearly understand the relationship of key macroeconomic variables. From the theoretical perspective, however, the most important benefit of DSGE models is that they are free from Lucas critique (1976) and they thus meet modern standards of conceptual rigor (Woodford 2007).

However, these benefits of DSGE models come at the costs though. DSGE models are often criticized because they are not rich enough to match the data in multi-dimensions by ignoring some features of the economy for the idealization and imposing strong restrictions on the actual data. Furthermore, when one attempts to deal with the actual data in a DSGE model, it is hard to apply traditional econometric methods for estimation, hypothesis testing, and forecasting to DSGE models. Thus, Ireland (2004) claims that DSGE models can often produce fragile results which are not robust in different settings at least at first glance.

Second, a VAR model, pioneered by Sims (1980), is proposed as an alternative to Cowles Commission's large-scale macroeconometric system. A larger model in this tradition has been severely criticized not only because many of the restrictions to identify

[2]Compared to the standard estimation, calibration is often criticized as casual or unsophisticated empirical practice since it does not explicitly account for the uncertainty associated with model parameters (Hoover 1995; Hartley, Hoover, and Salyer 1997). On the other hand, King and Rebelo (1999) defend the validity of calibration, where model parameters are assigned values so as to match the model's long-run macroeconomic features with those in the data and to render the behaviors of individuals in the model consistent with empirical microeconomic studies. For details on the calibration methods, see Prescott (1986), Kydland and Prescott (1996) and Cooley (1997).

[3]For example, from the artificial data one can evaluates the properties of the model in terms of 1) standard deviations of various relevant variables, and their relative standard deviations to output, 2) correlation coefficients of various variables with output, and 3) impulse response patterns of variables when shocks bring key variables off the steady state and back (Walsh 2003).

behavioral equations in a large system are inconsistent with dynamic macroeconomic theories but also because even the univariate time series model was often found to be superior to a larger model for the forecasting purpose. The main rationale behind VAR models is that no variables can be treated as being a priori exogenous in the economy populated with forward-looking rational agents whose behaviors are subject to the solution of an intertemporal optimization model (Cooley and LeRoy 1985; Favero 2001).

VAR models also have several advantages. The specification of VAR models does not require detailed prior knowledge about the economic system. Thus, it is often argued that VAR models not only summarize the information in the data successfully but they are also flexible enough for the dynamic analysis of macroeconomic variables. Moreover, from the practical perspective, one can easily estimate parameters of interest, test the statistical hypotheses and generate out-of-sample forecasts using VAR models.

However, because they are not firmly founded in economy theory, VAR models are often criticized to fail to uncover structural relationships among variables. Therefore, even though VAR models are able to fit the data, the parameters they give are not easily interpretable (Hartley et al. 1997). In addition, from the practical points, VAR models often require large samples of observations to estimate the coefficients of interests.

With this background in mind, this essay estimates a small-scale DSGE model for Korean economy by applying the maximum likelihood (ML) estimation methodology developed by Ireland (2004) in which a micro-founded DSGE model is augmented with VAR-structured measurement errors. Thus this approach attempts to couple the flexibility of VAR methodology with the theoretical rigor of DSGE models. More specifically, this joint approach seeks to take advantage of VAR models to capture movements in the data which are not explained by real business cycle theory. Then, contrary to the calibration method, the maximum likelihood estimation of model parameters allows us to exploit the standard econometric tools within the DSGE framework (Ruge-Murcia 2007). For example, we can assess the adequacy of a DSGE model by comparing its performance for forecasting with those of pure time series models.[4]

[4]On the contrary, generalized method of moments (GMM) approach is not designed for model comparison. However, as a limited information method, it loses some efficiency but requires less structure. For instance, it does not depend on any specific assumption on the distribution of shocks in the model.

As acknowledged by Watson (1994), two important questions must be answered in any business cycle research. First, how do the variables respond to exogenous shocks and how long? Second, are the business cycles largely the result of supply shocks like productivity shocks? In other words, what are the important sources of economic fluctuations? This essay also attempts to answer these questions by taking advantage of the impulse response function and the variance decomposition analysis.

For this purpose, the rest of the chapter is structured as follows. Section 3.2 introduces a baseline DSGE model which is in the line of Hansen's (1985) one-sector growth model with indivisible labor. Section 3.3 solves this model by applying Blanchard and Kahn's (1980). Section 3.4 accounts for how the solved system in Section 3.3 can be used to estimate a DSGE model. In this section, the baseline DSGE model is augmented with VAR structured measurement errors to obtain an econometrically conformable DSGE model. The motivation of including measurement errors will become clear after the stochastic singularity problem is discussed. Section 3.5 presents main empirical findings of this chapter. In this section, estimated deep parameters are presented with their standard errors. Then, based on these estimates, the responses of endogenous variables to an external shock and variance decomposition analysis are provided. In addition, we compare the estimated parameters from the pre-financial crisis period with those from the entire period. This section closes by comparing the forecast performance of the current DSGE model with those of pure time series models and by comparing the second moments for the data and the simulation. Finally, Section 3.6 concludes.

3.2 Setting up a Baseline DSGE Model

This essay employs Hansen's (1985) one-sector stochastic growth model with indivisible labor. The reason is that, in the academic literature, this DSGE model is regarded as a superior one in accounting for the stylized facts of the business cycle than Kydland and Prescott's (1982) divisible labor model because most of the variation in total hours worked is due to individuals either working or not working (see, for example, Basu, Marsiliani, and Renström 2004). In this model, since the allocation of a competitive equilibrium is equivalent to the solution of a social planner's problem, one can solve a simpler social planner's problem without taking into account the prices for labor and capital.[5]

Herein, a benevolent social planner chooses aggregate per capita consumption C_t and aggregate per capita hours worked H_t in order to maximize the following expected utility function of a representative household in each period $t = 0, 1, 2, \cdots$,

$$E_0 \sum_{t=0}^{\infty} \beta^t [\ln(C_t) - \gamma H_t], \tag{3.2.1}$$

where β, the subjective discount factor, is positive and less than unity reflecting a preference for current consumption, $-\gamma < 0$ is disutility weight of labor, and E_0 is the expectations operator conditional on information available at time 0. Then, since the utility function is linear in labor across time, the intertemporal elasticity of substitution on labor is infinite (Hansen 1985; Rogerson 1988).[6]

One type of aggregate per capita output Y_t is produced with capital K_t and labor supply H_t according to the Cobb-Douglas production function described by

$$Y_t = A_t K_t^{\alpha} H_t^{1-\alpha}, \tag{3.2.2}$$

where $0 < \alpha < 1$ represents capital's share in output.

The total factor productivity of the economy A_t, which is assumed to be temporary,

[5]The *first welfare theorem* states that if there is a complete set of markets and there are no market imperfections such as distorting taxes and externalities, the decentralized competitive equilibrium allocation is socially optimal in the sense that it solves the social planner's problem (see, for example, Mas-Colell, Whinston, and Green 1995, pp. 549–550).

[6]Let us denote the elasticity of substitution between labor in different periods as,

$$\kappa = \frac{\frac{\Delta(H_{t+1}/H_t)}{H_{t+1}/H_t}}{\frac{\Delta(MU_{H_t}/MU_{H_{t+1}})}{MU_{H_t}/MU_{H_{t+1}}}},$$

where MU_{H_t} is the marginal disutility of labor at period t.

Kydland and Prescott's divisible labor model requires high κ to explain the positive covariance of output and hours worked with only a small movements in the real wage, but this is inconsistent with empirical evidence that an individual's elasticity of substitution is low (Pencavel 1986). To overcome this problem, Hansen (1985) notes that fluctuations in aggregate hours worked are mainly due to variations in the number of people employed (extensive margin) rather than those in the number of hours worked per worker (intensive margin). Thus, by requiring individuals to choose lotteries rather than hours worked, he shows that even with low intertemporal elasticity of individuals who are constrained to either work full time or not at all, the elasticity for the representative agent can be high.

follows the first-order autoregressive process:

$$\ln A_t = (1-\rho)\ln \bar{A} + \rho \ln A_{t-1} + \varepsilon_t, \tag{3.2.3}$$

where $\bar{A} > 0$ and $0 < \rho < 1$. The innovation ε_t, the only shock in this artificial economy, is identically, independently and normally distributed with zero mean and standard deviation σ. It is also assumed that A_t is known at the beginning of period t before the social planner makes a decision. The choice of AR(1) specification for the shock follows the standard practice in the DSGE literature.

During each period, output Y_t can either be consumed or invested, subject to the aggregate resource constraint of the economy

$$Y_t = C_t + I_t. \tag{3.2.4}$$

The capital stock available for production in period $t+1$, K_{t+1}, is determined at the end of period t according to the following accumulation equation

$$K_{t+1} = (1-\delta)K_t + I_t. \tag{3.2.5}$$

where the rate of capital depreciation satisfies $0 < \delta < 1$.

The resource constraint and capital accumulation equation can be combined as,

$$K_{t+1} = (1-\delta)K_t + Y_t - C_t.$$

In the current system, the social planner chooses the infinite sequences of controls $\{Y_t, C_t, I_t, H_t, K_{t+1}\}_{t=0}^{\infty}$ to maximize the utility function (3.2.1) subject to constraints (3.2.2)–(3.2.5) and K_0. Thus, when K_t is given from the previous period $t-1$ and A_t is observed at time t, the social planner determines Y_t by making decision on H_t and divides Y_t between C_t and I_t. Then, next period capital stock K_{t+1} is automatically determined.

3.3 Solving the Model

To find a solution to the dynamic model is to write the non-predetermined endogenous variables in terms of the predetermined variables and the exogenous variables. However, since most DSGE models do not have a known analytical solution, one has to approximate the solutions to these non-linear models. Among the approximation solution methods, in the literature log-linear approximation method has been extensively used following Blanchard and Kahn (1980), King et al. (1988), and Campbell (1994).[7]

Although there has been some criticism on this method, this essay chooses to solve the current DSGE model using the log-linear approximation method for facilitating the implementation of the likelihood-based estimations.[8] As clearly noted in An and Schorfheide (2007), this method is still popular in the literature on the likelihood-based DSGE model estimations at least for two reasons. First, it can be easily linked to a state-space model and thus can be analyzed with the Kalman filtering algorithm. Another reason for its popularity seems to be the computational burden which researchers often encounter when evaluating the likelihood for non-linear solution of a DSGE model.

The log-linear approximation method involves several steps. First, we need to establish the expectational nonlinear system by deriving optimality conditions. Next, this nonlinear system must be linearized around the steady state. Then, applying Blanchard and Kahn's (1980) solution method yields the solution of the linearized system of difference equations.

First of all, as shown in Appendix 3.A.1, there are two optimality conditions for this problem: the intratemporal optimality condition and the intertemporal optimality condition. The intratemporal optimality condition means that the social planner equalizes the marginal disutility of labor to the marginal utility of consumption due to extra unit of labor as shown in (3.3.1),

$$\gamma = \frac{(1-\alpha)Y_t}{H_t} \cdot \frac{1}{C_t}. \tag{3.3.1}$$

On the other hand, the intertemporal optimality condition means that the social planner equalizes the utility loss due to one unit of saving today to discounted expected

[7]Uhlig (1999), Klein (2000) and Sims (2002) provide solution algorithms for the linearized approximate models.

[8]In particular, the linear approximation method has been criticized as it suppresses higher moments which may be important in certain circumstances. For example, Kim and Kim (2003) demonstrate how linearization can generate approximation errors that can yield a reversal of welfare ordering in international business cycle models. In order to overcome this problem, Sims (2000) derives a second-order approximation to the policy function for various models and more recently Schmitt-Grohé and Uribe (2004) provide a set of programs that implements the second-order approximation method. For other solution methods, see for example Aruoba, Fernández-Villaverde, and Rubio-Ramírez (2006).

compensation for additional unit of savings in terms of consumption goods as shown in (3.3.2),

$$1/C_t = \beta E_t(1/C_{t+1})[\alpha(Y_{t+1}/K_{t+1}) + 1 - \delta] \tag{3.3.2}$$

for all $t = 0, 1, 2, \cdots$.

Therefore, the intratemporal optimality condition (3.3.1) equates marginal rate of substitution between consumption and leisure to the marginal product of labor, while the intertemporal optimality condition (3.3.2) equates the marginal rate of intertemporal substitution to the marginal product of capital.

Now we attain a non-linear expectational system with 6 equations for 6 variables: (3.2.2)-(3.3.2) for $Y_t, C_t, I_t, H_t, K_t, A_t$. Then, the next step is to construct a linear approximation to the original non-linear expectational system by log-linearizing 6 equations around the steady state after removing trends in the variables (Campbell 1994). Appendices 3.A.3 to 3.A.5 illustrate these processes. The outcome of this second step is summarized as the following six equations,

$$\hat{y}_t = \hat{a}_t + \alpha\hat{k}_t + (1-\alpha)\hat{h}_t, \tag{3.3.3}$$

$$\hat{a}_t = \rho\hat{a}_{t-1} + \varepsilon_t, \tag{3.3.4}$$

$$(1/\beta - 1 + \delta)\hat{y}_t = [(1/\beta - 1 + \delta) - \alpha\delta]\hat{c}_t + \alpha\delta)\hat{i}_t, \tag{3.3.5}$$

$$\hat{k}_{t+1} = (1-\delta)\hat{k}_t + \delta\hat{i}_t, \tag{3.3.6}$$

$$\hat{c}_t + \hat{h}_t = \hat{y}_t, \tag{3.3.7}$$

$$0 = (1/\beta)\hat{c}_t - (1/\beta)E_t\hat{c}_{t+1} + (1/\beta - 1 + \delta)E_t\hat{y}_{t+1} - (1/\beta - 1 + \delta)\hat{k}_{t+1}, \tag{3.3.8}$$

where we denote the detrended stationary variables with lower letters. Then, we denote a hat over a variable as representing a percentage deviation of that variable from its steady state level, for instance, $\hat{c}_t = \ln(c_t/c)$, etc.

Given the above linearized expectational system of 6 difference equations, we can solve the current system through Blanchard and Kahn's (1980) method. However, Blanchard and Kahn's solution method can be applied when the models are written as the form of (3.3.9),

$$\begin{bmatrix} x_{1,t+1} \\ E_t(x_{2,t+1}) \end{bmatrix} = K \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} + Lz_t, \tag{3.3.9}$$

where $x_{1,t}$ signifies a vector of predetermined variables (i.e., $E_t(x_{1,t+1}) = x_{1,t+1}$), $x_{2,t}$ is a vector of control variables (i.e., $E_t(x_{2,t+1}) + \zeta_{t+1} = x_{2,t+1}$ with ζ_{t+1} denoting an expectation error), and z_t is a vector of exogenous forcing variables.

Therefore, as a preliminary step, we need to convert the above system into the form of (3.3.9). This step requires a system reduction, which involves expressing the model in terms of uniquely determined variables such as $\hat{a}_t$, $\hat{c}_t$, and $\hat{k}_t$ (King and Watson 2002).[9] As shown in Appendix 3.A.6, we can find the conformable difference system as follows,

$$\begin{bmatrix} \hat{k}_{t+1} \\ E_t(\hat{c}_{t+1}) \end{bmatrix} = K \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} + L\hat{a}_t, \tag{3.3.10}$$

where K is a 2×2 matrix and L is a 2×1 vector.

Now, we can apply Blanchard and Kahn (1980)'s procedure to (3.3.10) as shown in Appendix 3.A.7. This procedure begins with a Jordan decomposition of matrix K and then transforms the system with the diagonal matrix, which is composed of eigenvalues of K. Next, in the transformed system, the explosive part is solved forward and the stable part is solved backward. Based on these solutions, we can not only recover the solutions for $\hat{k}_t$ and $\hat{c}_t$, but also find the solutions for $\hat{y}_t$, $\hat{i}_t$ and $\hat{h}_t$.

Lastly, we can summarize the solutions to the current linear system as transition equations of state variables (3.3.11) and optimal policy functions (3.3.12),

$$s_{t+1} = \Xi s_t + \Phi\varepsilon_{t+1} \quad \Leftrightarrow \quad \begin{bmatrix} \hat{k}_{t+1} \\ \hat{a}_{t+1} \end{bmatrix} = \begin{bmatrix} \Xi_{kk} & \Xi_{ka} \\ 0 & \rho \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{a}_t \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \varepsilon_{t+1}, \tag{3.3.11}$$

[9]That is, we can rewrite the whole linear system into the reduced system only using $\hat{a}_t$, $\hat{c}_t$, and $\hat{k}_t$. Note that, (*i*) observations on $\hat{a}_t$, $\hat{c}_t$, and $\hat{k}_t$ are sufficient for determining $\hat{y}_t$ jointly from (3.3.3) through (3.3.8), (*ii*) given $\hat{y}_t$, both $\hat{i}_t$ and $\hat{h}_t$ are superfluous. Hence, all information we need to solve the problem are contained in $\hat{a}_t$, $\hat{c}_t$, and $\hat{k}_t$.

and

$$f_t = \Psi s_t \quad \Leftrightarrow \quad \begin{bmatrix} \hat{y}_t \\ \hat{i}_t \\ \hat{h}_t \\ \hat{c}_t \end{bmatrix} = \begin{bmatrix} \Psi_{yk} & \Psi_{ya} \\ \Psi_{ik} & \Psi_{ia} \\ \Psi_{hk} & \Psi_{ha} \\ \Psi_{ck} & \Psi_{ca} \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{a}_t \end{bmatrix}, \tag{3.3.12}$$

for all $t = 0, 1, 2, \cdots$. Then, given the state of the economy s_t, which traces percentage deviations of two state variables from their steady-state levels, the policy functions (3.3.12) give us the optimal policy to follow. Thus, these are the solutions to the standard linear optimal control problem.

3.4 Model Estimation Methodology

3.4.1 Reinterpreting (3.3.11) and (3.3.12) for Estimation

For the purpose of estimation, notice first that we can observe only three independent endogenous variables due to the aggregate resource constraint (3.2.4) in the model. Thus, this paper uses just three observable variables Y_t, C_t and H_t in estimating the baseline DSGE model. Then, the theoretical linear system in the previous section, composed of transition equations (3.3.11) and policy functions (3.3.12), should be reinterpreted as a state space econometric model, composed of transition equations of hidden state vector (3.4.1) and observation (or measurement) equations (3.4.2)

$$s_{t+1} = A s_t + B \varepsilon_{t+1} \quad \Leftrightarrow \quad \begin{bmatrix} \hat{k}_{t+1} \\ \hat{a}_{t+1} \end{bmatrix} = \begin{bmatrix} A_{kk} & A_{ka} \\ 0 & \rho \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{a}_t \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \varepsilon_{t+1}, \tag{3.4.1}$$

and

$$d_t = C s_t \quad \Leftrightarrow \quad \begin{bmatrix} \hat{y}_t \\ \hat{c}_t \\ \hat{h}_t \end{bmatrix} = \begin{bmatrix} C_{yk} & C_{ya} \\ C_{ck} & C_{ca} \\ C_{hk} & C_{ha} \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{a}_t \end{bmatrix}, \tag{3.4.2}$$

for all $t = 0, 1, 2, \cdots$.

There is a duality between two systems. However, there is a slight but important difference between (3.3.11) and (3.3.12) in the previous section and (3.4.1) and (3.4.2) in the current section. In (3.3.11) and (3.3.12), with $\hat{k}_t$ given from the previous period and $\hat{a}_t$ observed at the current period, the problem is to choose the optimal response of f_t. On the other hand, in (3.4.1) and (3.4.2), the problem is to compute the expected values of a hidden state vector $\boldsymbol{s}_t$ conditional on the observation vector $\boldsymbol{d}_t$. Now, therefore, matrices A and C are the objects to be estimated using the observations on $\hat{y}_t$, $\hat{c}_t$, and $\hat{h}_t$.

3.4.2 Stochastic Singularity Problem

Notice that, in observation equations (3.4.2), the number of driving forces is smaller than that of observable variables. Namely, the model has only one shock – the aggregate technology shock – driving all business cycle fluctuations, but there are three endogenous variables to feed in. This is known as the stochastic singularity problem in estimating DSGE models. As illustrated in Appendix 3.B, the presence of the stochastic singularity makes traditional econometric methods inapplicable to the model's parameter estimations. The reason is that if $\hat{c}_t$ is observed, $\hat{y}_t$ and $\hat{h}_t$ cannot be observed independently (Ingram, Kocherlakota, and Savin 1994; Ruge-Murcia 2007). Since there are three observable endogenous variables in the current baseline model, we need to add at least two more shocks to the model.

To make the current DSGE model empirically implementable, two approaches are suggested in the empirical literature. One strategy is introducing additional structural disturbances – to preferences, investment, monetary and fiscal policy rules – to the DSGE model. Therefore, the models would be specified at a deep enough level that the differential response to different types of shocks would be properly spelt out. This approach is followed by DeJong, Ingram, and Whiteman (2000), Kim (2000), Schorfheide (2000), Ireland (2003), and Bouakez, Cardia, and Ruge-Murcia (2005). In a first-best world, this strategy has its advantages: it gives help to identify sources of aggregate fluctuations beyond the productivity shock and allows for a direct comparison of the relative importance of those additional disturbances in driving aggregate fluctuations. However, the critical shortcoming of this approach is to require too much detailed *ad hoc* assumptions about how the economy evolves.

An alternative strategy to address this problem is attaching **measurement errors** to the observation equations following the tradition of Sargent (1989). This strategy replaces three observation equations in (3.4.2) with three observation equations with measurement errors, which have VAR(1) structure with three innovations. This second approach is utilized by McGrattan (1994), Hall (1996), McGrattan, Rogerson, and Wright (1997) and Ireland (2004).

This essay adopts the latter strategy to avoid *ad hoc* assumptions in the process of estimation. Then, the empirical counterpart to (3.4.1) and (3.4.2) consists of (3.4.1), (3.4.3), and (3.4.4)

$$\boldsymbol{s}_{t+1} = A\boldsymbol{s}_t + B\varepsilon_{t+1} \quad \Leftrightarrow \quad \begin{bmatrix} \hat{k}_{t+1} \\ \hat{a}_{t+1} \end{bmatrix} = \begin{bmatrix} A_{kk} & A_{ka} \\ 0 & \rho \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{a}_t \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \varepsilon_{t+1}, \tag{3.4.1}$$

$$\boldsymbol{d}_t = C\boldsymbol{s}_t + \boldsymbol{u}_t \quad \Leftrightarrow \quad \begin{bmatrix} \hat{y}_t \\ \hat{c}_t \\ \hat{h}_t \end{bmatrix} = \begin{bmatrix} C_{yk} & C_{ya} \\ C_{ck} & C_{ca} \\ C_{hk} & C_{ha} \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{a}_t \end{bmatrix} + \begin{bmatrix} u_{yt} \\ u_{ct} \\ u_{ht} \end{bmatrix}, \tag{3.4.3}$$

and

$$\boldsymbol{u}_{t+1} = D\boldsymbol{u}_t + \boldsymbol{\xi}_{t+1} \quad \Leftrightarrow \quad \begin{bmatrix} u_{yt+1} \\ u_{ct+1} \\ u_{ht+1} \end{bmatrix} = \begin{bmatrix} D_{yy} & D_{yc} & D_{yh} \\ D_{cy} & D_{cc} & D_{ch} \\ D_{hy} & D_{hc} & D_{hh} \end{bmatrix} \begin{bmatrix} u_{yt} \\ u_{ct} \\ u_{ht} \end{bmatrix} + \begin{bmatrix} \xi_{yt+1} \\ \xi_{ct+1} \\ \xi_{ht+1} \end{bmatrix}, \tag{3.4.4}$$

where three innovations in the vector $\boldsymbol{\xi}_t$ are assumed to be serially uncorrelated in each term and normally distributed with mean zero. We further assume that these three innovations are contemporaneously correlated each other but that they are orthogonal to the technology shock, ε_t as follows.

$$E_t(\boldsymbol{\xi}_t\boldsymbol{\xi}_t') = V = \begin{bmatrix} V_y^2 & V_{yc} & V_{yh} \\ V_{yc} & V_c^2 & V_{ch} \\ V_{yh} & V_{ch} & V_h^2 \end{bmatrix}, \tag{3.4.5}$$

and

$$E_t\left(\boldsymbol{\xi}_t\varepsilon_t\right) = \mathbf{0}_{3\times 1}, \tag{3.4.6}$$

for all $t = 0, 1, 2, \cdots$.

The earlier study imposes the diagonal restrictions on matrices D and V. It, thus, assumes that each measurement error follows its own AR(1) process and that ξ_{yt}, ξ_{ct}, and ξ_{ht} are mutually uncorrelated. However, this essay does not impose those restrictions on matrices D and V. The reason is that the residuals from VAR(1) structure of $\boldsymbol{u}_t$ can be interpreted as capturing the movements and co-movements in the data that the real business theory cannot account for (Ireland 2004). In this way, the augmented DSGE model with VAR structure can incorporate the flexibility of VAR models to the micro-founded DSGE model.

3.4.3 Data and Estimation Methodology

As noted in Section 3.4.2, the estimation process requires three actual data: Y_t, C_t and H_t because I_t is redundant. C_t is defined as the real private consumption expenditure per capita in 2000 Korean won and I_t is defined as the real gross private domestic investment per capita, also in 2000 Korean won. Then, Y_t is defined as the sum of C_t and I_t. Hours worked per capita H_t is calculated as follows: average quarterly worked hours of the employed is multiplied by the employment rate. Each series is seasonally adjusted and Y_t and C_t are divided by the age 15 and over population. Data for consumption and investment are taken from the Bank of Korea database whereas population data come from National Statistics Office. Lastly, data for hours worked come from the Ministry of Labor. The data are quarterly, and span the first quarter of 1973 through the third quarter of 2006.

Taking the model to the data involves the isolation of cycles from the original data. To this end, among the several empirical techniques, this essay employs the Hodrick-Prescott's (HP) filter developed by Hodrick and Prescott (1997), which is widely used among researchers.[10] Since the current essay uses the quarterly data, we choose $\lambda = 1,600$, relative weight on smoothness, following Hodrick and Prescott's suggestion. In addition, all three variables are logged before filtering.

This essay attempts to estimate parameters of interest using the maximum likelihood estimation method in the Kalman filter framework. Since the current system can be interpreted as the combination of transition equations for hidden states and measurement equations for observable variables, Kalman's (1960) filtering algorithm is exactly fit for the current system.[11]

[10] In extracting cycles, the band pass (B-P) filter developed by Baxter and King (1999) and the linear detrending are also popular in business cycle literature. In the preliminary research, we also estimated the current model with the linearly detrended data. However, the examined period includes the Korean financial crisis and this generates blips in the data. As the linear detrending seems to be problematic in this case, the current essay uses the HP filtered data. This problem was gratefully pointed out by Professor Martin Ellison.

[11] To make the system conformable for the Kalman filtering algorithm, (3.4.1), (3.4.3), and (3.4.4) need to be rewritten as the state space form. As shown in Appendix 3.C.1, the state space representation can be written as (3.C.1), (3.C.2) and (3.C.3).

The Kalman filter is originally developed to extract the unobservable signal (i.e., state) from the data by eliminating noises in the measurements. To achieve this goal, it is composed of two steps: predicting and updating the state vector conditional on the data. When new information becomes available, this changes our best least square estimates of the state vector. Importantly, during these repetitions, Kalman filter yields maximum likelihood estimates for the model parameters as a by-product. Appendix 3.C presents the detailed maximum likelihood estimation procedure in the Kalman filtering algorithm.

3.5 Empirical Results

In what follows, we first present the estimates of the parameters in the current DSGE model. For the purpose of comparison, we also report those for the restricted DSGE model which is restricted in the sense that matrices D and V in the previous section are assumed to be diagonal. This is followed by the responses of endogenous variables to an external shock and variance decomposition analysis. In addition, we compare the estimated parameters from the pre-financial crisis period with those from the entire period. Then, this section compares the forecasting performance of the current DSGE model with those of the pure time series models and then comparing the forecast. Finally, we report the second moments from the HP filtered data and those from the simulated data to see the model's goodness of fit.

3.5.1 Estimated Parameters

Table 3.1 reports maximum likelihood estimates of 20 parameters: the five structural parameters $\gamma, \alpha, \bar{A}, \rho, \sigma$ from the real business cycle model, the nine elements of the matrix D governing the persistence of the VAR residuals, and the six elements of the variance-covariance matrix V for the VAR residuals. In the preliminary work, we estimated these parameters and obtained a very low $\beta = 0.9045$ and a very high $\delta = 0.1699$. These numbers seem not to be reasonable because the current parameters are estimated from the quarterly data. It is widely acknowledged that DSGE estimations using maximum likelihood method often yield "the dilemma of absurd parameter estimates" (Christensen and Dib 2006; An and Schorfheide 2007).[12] For this reason, Altug (1989) and Ireland (2004) suggest to fix β and δ since this calibration helps to successfully estimate the remaining parameters. Following their suggestion, we therefore set β and δ fixed as 0.99 and 0.025, respectively.

The standard errors are also presented in the last column of Table 3.1. The standard errors are obtained by using a parametric bootstrapping procedure similar to those employed by Malley, Philippopoulos, and Woitek (2007) and Ireland (2007) (see, for more details, Efron and Tibshirani 1993).[13] This procedure is composed of the following three steps. The first step is to simulate the model with the estimated parameters in order to obtain 1000 artificial data sets where each data set contains 135 sample periods. The second step is to run 1000 estimations with those artificial data sets and to save the resulting parameter estimates. The final step is to calculate the standard deviations of individual parameter estimates and to report them as the standard errors of those parameter estimates.

All the estimated structural parameters in the model are quite reasonable and statistically significant. First of all, the estimates $\bar{A} = 28.3960$ and $\gamma = 0.0022$ help match the steady-state values of output, consumption, and hours worked in the model with the average levels of the same variables in the data. The estimated value of α, capital's share in output, is 0.4714, which is larger than the usual estimate from other countries. However, this is highly consistent with actual average capital income ratio, 45.9%, of Korea during the examined period. The estimate $\rho = 0.4495$ implies that the productivity shock is not too persistent as it is below a half. The estimate $\sigma = 0.0063$,

[12]Christensen and Dib (2006) and An and Schorfheide (2007) find that the likelihood function has often several local peaks and the optimization stops at one of the local peaks. Furthermore, even if it reaches a global peak, it might provide the parameter estimate which is at odds with extraneous information. As a consequence, An and Schorfheide (2007) strongly argue that additional information that the researcher have should be incorporated.

[13]Alternatively, one can obtain the standard errors as the square roots of the diagonal elements of the covariance matrix of the maximum likelihood estimators. The covariance matrix can be obtained using the inverse of the negative of the second derivative of the log likelihood function (Hessian) evaluated at estimated parameters $\hat{\boldsymbol{\theta}}_{ML}$ (Kim and Nelson 1999, Ch.2).

$$Cov(\hat{\boldsymbol{\theta}}_{ML}) = \left[-\frac{\partial^2 \ln \mathcal{L}(\boldsymbol{\theta}|\boldsymbol{d}_T, \cdots, \boldsymbol{d}_1)}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\Big|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}_{ML}} \right]^{-1},$$

where $\boldsymbol{\theta}$ is the parameter vector, $\boldsymbol{d}$ is the collection of data, and $\ln \mathcal{L}$ is the log likelihood.

However, it is often argued that this approach is problematic when the likelihood in a nonlinear dynamic model is a highly nonlinear surface. I am grateful for this indication by Professor Martin Ellison.

volatility of technical shock, is a little larger than the magnitude found in the developed countries.[14] This result is intuitively understandable since an emerging economy such as Korea is often regarded as more volatile than developed countries.

Table 3.1: Parameter Estimates and Standard Errors: DSGE augmented VAR

Parameters	Estimates	Standard Errors
γ	0.0022	0.0001
α	0.4714	0.0038
$\bar{A}$	28.3960	0.0003
ρ	0.4495	0.0007
σ	0.0063	0.0007
d_{yy}	-0.1762	0.0310
d_{yc}	1.6987	0.0184
d_{yh}	-1.0373	0.0027
d_{cy}	0.2774	0.0338
d_{cc}	0.3393	0.0189
d_{ch}	0.1721	0.0035
d_{hy}	-0.1396	0.0342
d_{hc}	0.3933	0.0192
d_{hh}	0.2184	0.0030
v_y	0.0267	0.0146
v_c	0.0188	0.0073
v_h	0.0129	0.0019
v_{yc}	0.00043260	0.00071951
v_{yh}	0.00002107	0.00019709
v_{ch}	0.00012985	0.00011379

The other non-structural estimates in Table 3.1 suggest, however, that the productivity shock does not successfully account for some important parts of the data. For instance, the matrix D has three real eigenvalue of -0.8589, 0.7848 and 0.4556. This empirical result implies that measurement errors u_t are also quite persistent and that there is significant amount of part explained by VAR residuals, instead of the productivity shock.

In addition, if the real business cycle theory explains the business cycles of Korea well, the volatility of productivity shock should be sufficiently higher than those of measurement errors. However, as shown in Table 3.1, the volatilities of three measurement

[14]Using the similar model, Ireland (2004) reports 0.0056 for the U.S. economy.

errors (0.0267, 0.0188 and 0.0129) are much higher than that of productivity shock (0.0063). Thus, this result implies that, to an extent, the real business cycle theory has a difficulty to account for the Korean business cycles.

On the other hand, as shown in Appendix 3.C.2.3, we can construct estimates for shocks to productivity and measurement errors ($E_T(\varepsilon_t)$ and $E_T(\boldsymbol{\xi}_t)$) by employing the Kalman smoothing procedure. Given $E_T(\varepsilon_t)$ and $E_T(\boldsymbol{\xi}_t)$, we can verify the orthogonality assumption between ε_t and the vector $\boldsymbol{\xi}_t$, which is made in (3.4.6). This examination is crucial in the sense that when they are orthogonal each other, the variance decomposition analysis becomes meaningful. Table 3.2 reports the correlation between shocks to productivity and measurement errors. The correlation between $E_T(\varepsilon_t)$ and $E_T(\xi_{yt})$ is -0.0137, the correlation between $E_T(\varepsilon_t)$ and $E_T(\xi_{ct})$ is -0.0033, and the correlation between $E_T(\varepsilon_t)$ and $E_T(\xi_{ht})$ is 0.0511. All these numbers are quite low. In addition, as shown in p-values, no correlation is statistically significant even at the 10% significance level. Therefore, the orthogonality assumption between two sources of shocks in (3.4.6) is not violated.

Table 3.2: Correlation between Shocks to Productivity and Measurement Errors

	Correlation	P-values
$E_T(\varepsilon_t), E_T(\xi_{yt})$	-0.0137	0.8749
$E_T(\varepsilon_t), E_T(\xi_{ct})$	-0.0033	0.9701
$E_T(\varepsilon_t), E_T(\xi_{ht})$	0.0511	0.5573

Note: The p-values are computed with *corrcoef* function in MATLAB.

Lastly, for the purpose of comparison, Table 3.3 reports the estimated parameters and their standard errors in the restricted DSGE model. This model is restricted in the sense that contrary to Table 3.1, matrices D and V are constrained to be diagonal. This constraints reflect the implicit assumption that all of the co-movements between the observed variables can be captured by the real business cycle theory and the residuals are not correlated.

The maximum values of the log-likelihood function are 985.63 in the restricted DSGE model and 1,052.31 in the unrestricted DSGE model. Then, the goodness-of-fit between two models can be compared with likelihood ratio test and Akaike information criteria (AIC).[15] With respect to likelihood ratio test, the likelihood ratio statistic 133.36 is significantly larger than the 1% critical value 21.67 of $\chi^2_{0.01}(9)$. That is, the null hypothesis of no difference between models can be sufficiently rejected and thus the restrictions are not binding. According to the AIC, the value (-15.44) in the unrestricted model is significantly less than the value (-14.52) in the restricted model. Hence, these two test statistics jointly imply that the unrestricted DSGE model is superior to the restricted DSGE model in the goodness-of-fit for the data. Therefore, the following analysis will be implemented with the unrestricted model.

Table 3.3: Parameter Estimates and Standard Errors: Restricted DSGE Model

Parameters	Estimates	Standard Errors
γ	0.0022	0.00001
α	0.4699	0.0008
$\bar{A}$	28.9110	0.0014
ρ	0.7328	0.0154
σ	0.0072	0.0008
d_{yy}	0.7393	0.1087
d_{cc}	0.7720	0.0662
d_{hh}	0.0133	0.0050
v_y	0.0265	0.0024
v_c	0.0188	0.0016
v_h	0.0110	0.0015

3.5.2 Impulse Response Analysis

An exogenous shock to the productivity, which leads the technology to deviate from its steady state level, not only directly affects the productivity but is also transmitted to

[15] Let $\ln \mathscr{L}_R$ be the maximum value of the log-likelihood of the data when the parameters are restricted (and reduced in number). On the other hand, let $\ln \mathscr{L}_U$ be the maximum value of the log-likelihood of the data with all the parameters unrestricted. Then the likelihood ratio statistics can be established as,

$$LR = -2(\ln \mathscr{L}_R - \ln \mathscr{L}_U) \sim \chi^2(q),$$

where the degree of freedom (q) is 9 (=20-11).

Alternatively, one can use the Akaike information criterion (AIC) to choose a better model,

$$AIC = -2\frac{\ln \mathscr{L}}{T} + \frac{k}{T},$$

where k is the number of parameters and T is the length of data. Note that the lower value in AIC implies the better fit.

model's four variables – output, consumption, investment and hours worked – through the dynamic structure of the model. That is, after the impulse of exogenous shock initiates a business cycle, the propagation mechanism in the current model will perpetuate that cycle (Lucas 1975). This propagation mechanism can be best understood by impulse response analysis, which traces the effect of a one-time innovation to the productivity on current and future values of endogenous variables.

Now, given the estimated parameters in Section 3.5.1, one can compute the responses of four endogenous variables to an unexpected temporary technology shock to Korea economy. First, the model's solution was given by (3.3.11) and (3.3.12) in Section 3.2.

$$\boldsymbol{s}_{t+1} = \Xi \boldsymbol{s}_t + \Phi \varepsilon_{t+1}, \tag{3.3.11}$$

and

$$\boldsymbol{f}_t = \Psi \boldsymbol{s}_t. \tag{3.3.12}$$

Let us assume that the economy is on the steady-state at time 0 (i.e. $\varepsilon_0 = 0$). Then at time 1 there is one-time temporary productivity shock with the amount of one standard deviation (i.e. $\varepsilon_1 = \sigma = 1.04\%$, and $\varepsilon_2 = \varepsilon_3 = \cdots = 0$). Given that we are starting at the steady-state, $\boldsymbol{s}_0 = \boldsymbol{0}_{2\times 1}$, (3.3.11) and (3.3.12) are specified as

$$\boldsymbol{s}_1 = \Phi \sigma, \tag{3.5.1}$$

and

$$\boldsymbol{f}_1 = \Psi \boldsymbol{s}_1. \tag{3.5.2}$$

So, at time 2,

$$\boldsymbol{s}_2 = \Xi \boldsymbol{s}_1, \tag{3.5.3}$$

and

$$\boldsymbol{f}_2 = \Psi \boldsymbol{s}_2, \tag{3.5.4}$$

and so for $t = 3, 4, \cdots$.

The model's impulse response functions for 40 periods are shown in Figure 3.1.

Note that the responses to a temporary technology shock die out to zero after all as the estimated system is stationary. The impulse response functions based on the estimated

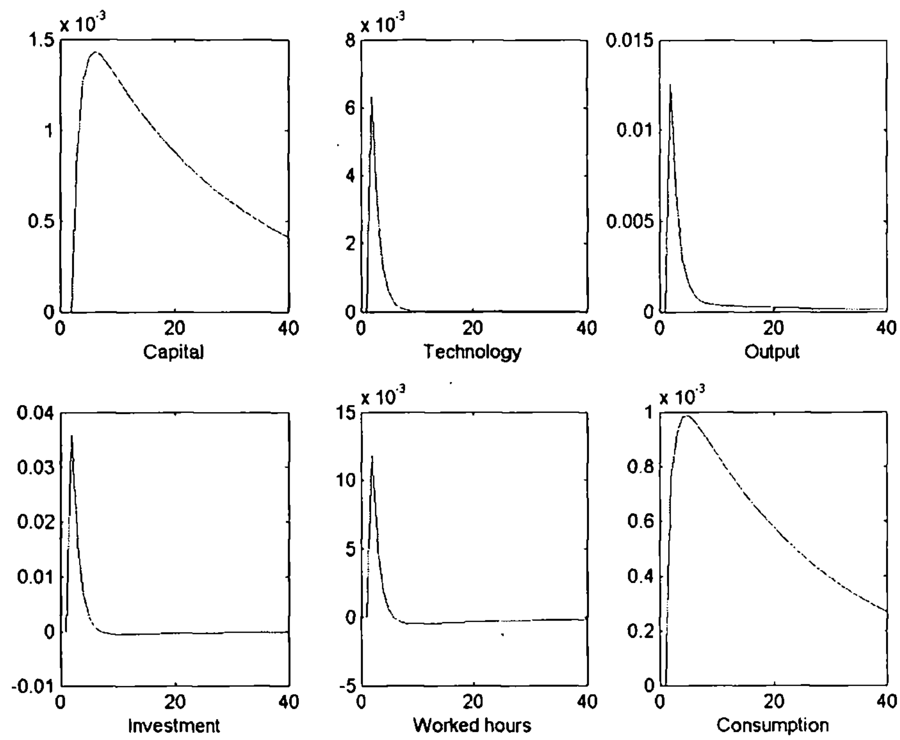

Figure 3.1: Impulse Responses to an Unexpected Temporary Technology Shock

parameters seem to be quite reasonable. First, a positive technology shock in period t represents a higher-than-average growth rate of productivity. Thus, the economy is able to produce more output. Higher productivity raises wages, so labor supply in period t increases as workers find work more profitable than leisure. Thus higher labor input as well as higher productivity raises period t output. On the other hand, although the return to capital increases as well, capital stock cannot be increased at time t when the shock is not expected. This comes as no surprise in this standard model since the current capital level is the result of decision in the previous period.

Next, the increased output in period t is consumed or invested. The allocation decision hinges on consumer's preference and the expected persistence of the productivity shock. A desire to smooth consumption over time decides the portion of the saving out of temporary increased output. In addition, if the productivity shock is expected to be persistent, it will be more profitable to save and invest. Then, this will raise capital stock in period $t + 1$, too.

Due to the propagation mechanism, all macroeconomic variables display autocorrelation and co-movement. Furthermore, the response of investment is higher than that of output and much higher than that of consumption since there is no investment smoothing incentive in this economy. The effects of a temporary shock eventually die out since the productivity process is mean-reverting. In addition, decreasing returns to capital bring investment back to the steady state. Hence this mechanism is stable.

3.5.3 Variance Decomposition Analysis

Another way to look at the implications of the current model is to compute the fractions of the forecast-error variance of the observed output, consumption, investment and hours worked attributable to each type of shock. This procedure addresses how important a particular shock is for explaining the fluctuations in each endogenous variable. Therefore, the current section attempts to separate the variations of each variable into the two component shocks to the system: shocks to the technology and measurement errors.

Note first that since the productivity shock ε_t and the innovations to measurement errors $\boldsymbol{\xi}_t$ are orthogonal as empirically shown in Section 3.5.1, the relative importance of ε_t to the evolution of endogenous variables can be properly evaluated by variance decomposition analysis. Given the knowledge of elements of matrices F, G and Q in (3.C.1), (3.C.2) and (3.C.3) in Appendix 3.C.1, this can be done by dividing the forecast error variance of each variable due to ε_t by the total variance of forecast error of each variable.[16] Table 3.4 presents the proportion of k-step-ahead forecast error variance explained by the technology shock and its standard error. In the table, all statistics are statistically significant. All detailed procedure for variance decomposition is illustrated in Appendix 3.C.2.1.

Several features in the table are worth addressing. First, the productivity shock explains below one fifth of one-quarter-ahead forecast error variance in output. Furthermore, the explanatory power of productivity shock on output decreases as the

[16]On the other hand, by construction, the variations due to three exogenous shocks ξ_{yt}, ξ_{ct} and ξ_{ht} cannot be separated because they are contemporaneously correlated.

Table 3.4: Contribution of Technology Shocks to the Variance of the Forecast Error

Quarter ahead	Percentage of Variance due to Technology	Standard Errors
(A) Output		
1	18.0860	4.8672
4	10.9190	1.7172
8	9.8230	1.3384
12	9.6915	1.3775
20	9.6885	1.4951
40	9.7090	1.6221
∞	9.7149	1.7300
(B) Consumption		
1	0.1600	0.0471
4	0.4364	0.1017
8	0.7618	0.1614
12	1.0261	0.2204
20	1.3760	0.3071
40	1.7014	0.3932
∞	1.7911	0.4331
(C) Investment		
1	32.9860	8.3332
4	22.2170	3.1255
8	20.5070	2.3891
12	20.2680	2.4745
20	20.2330	2.7274
40	20.2470	3.0079
∞	20.2520	3.2543
(D) Hours Worked		
1	45.6720	8.2839
4	40.7090	6.8011
8	39.4880	6.7536
12	39.4200	6.7693
20	39.5470	6.7606
40	39.6950	6.7562
∞	39.7360	6.9869

forecasting horizon increases. Real business cycle theory seems to have more difficulty accounting for output fluctuations over the longer horizons. On the other hand, most of the variance in aggregate output arises from three exogenous shocks to measurement errors, which are assumed to reflect the combined effects of shocks, including monetary and fiscal policy shocks, not present in the baseline RBC theory. Notice also that the last line of panel (A), with $k = \infty$, implies that the technology shock accounts for only 9.7% of the unconditional variance in detrended output.

Second, panel (B) displays how much the variance of detrended consumption is explained by the productivity shock. As shown in this panel, the productivity shock has significant difficulty in explaining the fluctuations of consumption. Although the percentage of variance due to the productivity shock increases along the time periods, all numbers are very low.

Third, the productivity shock seems to explain relatively well the variation of aggregate investment. Notably, it accounts for about one third of the one-quarter-ahead forecast error variance in the investment, though the explanatory power declines as the time horizon increases.

Fourth, the variance of hours worked appears to be quite well accounted for by the productivity shock. For the one-quarter-ahead forecast error variance of hours worked, the real business cycle theory explains about 45% of its variation. In addition, for 4 to 40 periods, it explains about 40% of them consistently. Thus, the productivity shock does a much better job in explaining the behavior of hours worked than those of output and its components in Korean economy.

In sum, although the significant amount of fluctuation in hours worked is caused by the productivity shock, the productivity shock has a limited power to explain the movements of other major macroeconomic variables in Korea. In one sense, these empirical results should not be surprising since the simple model used here omits possibly important shocks for Korean business cycles such as monetary and fiscal policy shocks. In the other sense, the results signal monetary and fiscal policy shocks may also be the important sources in understanding the behavior of business cycles in Korea.

3.5.4 Parameter Stability

Until mid 1980's, the main works of macroeconomists were to set up a very large econometric system and to analyze the effects of monetary and fiscal policy changes in this system based on the implicit assumption of parameter stability in a reduced econometric model. Policymakers also tried to take advantage of the historical correlations among macroeconomic variables based on the same assumption.

This approach in macroeconomics was, however, severely criticized by Lucas (1976) because this econometric model fails to capture the rational expectation of individuals to policy changes. From this perspective, Lucas (1976, p. 41) wrote,

> *Given that the structure of an econometric model consists of optimal decision rules vary systematically with changes in the structure of series relevant to the decision maker, it follows that any change in policy will systematically alter the structure of econometric models.*

Accordingly, the parameters in a reduced-form macroeconometric model cannot be stable over different policy regimes. Lucas critique implies that a policy based on historical relationship between macroeconomic variables is not effective and one cannot evaluate that policy effects properly. As a result, Lucas (1976) argues that a model attempting to analyze the effects of a policy change should be based on the deep structural parameters, which are policy invariant. This irrefutable critique encouraged many economist to establish various DSGE models describing the economy with the structural parameters such as tastes and technology since DSGE models are presumed to free from Lucas critique. Thus, the stability of structural parameters is the fundamental assumption for validity of DSGE models.

In the empirical literature, the non-structural parameter instability in macroeconometric models is well documented, but there is little research on the structural parameter stability. One possible reason for this asymmetry is that calibrated DSGE models just assume the stability of structural parameters rather than estimate structural parameters. Contrary to calibration approach, estimation approach used in this chapter seems to be useful in examining the parameter stability.

From this perspective, Korean economy provides a useful laboratory to examine this fundamental assumption in DSGE models because Korea, often-cited success story like other East Asian tigers, experienced several significant policy changes since the financial crisis in 1997. Among others, the central bank of Korea became independent legally and committed itself explicitly to the straightforward target on inflation, which has been popular since the first adoption by the Reserve Bank of New Zealand. Moreover, the free-floating exchange rate system replaced the currency basket system (i.e., partially pegging the values of Korean won with a moving band to currencies of large trading partners such as the U.S.and Japan) at the end of 1997.

In order to address an interesting question whether the structural parameters in DSGE models are actually stable across the different policy regimes, we initially attempted to estimated the current DSGE model for the two separate periods, namely, pre- and post-crisis periods. Then, we can formally test the structural parameter stability using a Wald statistic, proposed by Andrews and Fair (1988). Due to the short sample period after the crisis, however, we failed to estimate the parameters for the post-crisis period. Under the circumstances, the current essay therefore reports the estimates from the pre-crisis period (1973:1 to 1997:4) with those from the entire period (1973:1 to 2006:3) and only attempts to compare them informally. The breakpoint, end of 1997:4, is motivated by the fact that major changes in Korean monetary and fiscal policies are widely thought to have occurred just after the outbreak of the Korean financial crisis.

Table 3.5 reports estimates for the DSGE model's parameters, along with their standard errors, for the two sample data sets. Three features are worth noting from the table. First, the persistence of productivity shock ρ is a little lower in the pre-crisis period, and this can be interpreted that the explanatory power of the real business cycle theory increased since the Korean financial crisis.

Second, the indicator of volatility of macroeconomy σ is a little higher in the pre-crisis period, which implies that volatility of the economy decreased after the financial crisis. Finally, it seems that the five structural parameter estimates overall are not significantly different across the two periods while the non-structural parameters are different.

However, it is important to note here that in order to formally check, we cannot apply the Wald test for the parameter stability to the current case because it will be misspecified when the examined samples are overlapping and thus dependent.[17]

[17]This issue was gratefully pointed out by Professor Martin Ellison.

Table 3.5: Comparison between Pre-crisis Period and the Entire Period

Parameters	1973:1-1997:4 estimates	Standard errors	1973:1-2006:3 estimates	Standard errors
γ	0.0022	0.00004	0.0022	0.0001
α	0.4724	0.0014	0.4714	0.0038
A	28.0910	0.00003	28.3960	0.0003
ρ	0.3965	0.0024	0.4495	0.0007
σ	0.0068	0.0006	0.0063	0.0007
d_{yy}	-0.0343	0.0170	-0.1762	0.0310
d_{yc}	1.4624	0.0144	1.6987	0.0184
d_{yh}	-0.8302	0.0015	-1.0373	0.0027
d_{cy}	0.3812	0.0047	0.2774	0.0338
d_{cc}	0.0504	0.0096	0.3393	0.0189
d_{ch}	0.4285	0.0046	0.1721	0.0035
d_{hy}	-0.1194	0.0147	-0.1396	0.0342
d_{hc}	0.2722	0.0105	0.3933	0.0192
d_{hh}	0.2260	0.0086	0.2184	0.0030
v_y	0.0227	0.0029	0.0267	0.0146
v_c	0.0137	0.0020	0.0188	0.0073
v_h	0.0116	0.0020	0.0129	0.0019
v_{yc}	0.00023378	0.00008339	0.00043260	0.00071951
v_{yh}	-0.00010759	0.00004459	0.00002107	0.00019709
v_{ch}	0.00004631	0.00004117	0.00012985	0.00011379

3.5.5 Forecast Accuracy

Comparisons of forecast accuracy of the current DSGE model against those of the pure time series models are crucial to assess the DSGE model adequacy. From this perspective, this section compares the accuracy of the unconstrained DSGE model's out-of-sample forecasts with those of restricted DSGE model, VAR(1) and VAR(2). As noted above, the restricted model is the specific case of the DSGE model where the matrices D and V are restricted to be diagonal.

Table 3.6: Tests for Forecast Accuracy : 1997:4 – 2006:3

Quarters Ahead (%)	1	2	3	4
(A) Output				
RMSE: Unrestricted DSGE	4.7684	5.4084	5.4789	4.8606
RMSE: Restricted DSGE	4.5685	5.0004	5.2309	4.6219
RMSE: VAR(1)	4.0987	4.5559	4.8418	4.2889
RMSE: VAR(2)	**4.0298**	**4.4988**	**4.6562**	**3.9813**
(B) Consumption				
RMSE: Unrestricted DSGE	3.5031	3.7919	3.9016	3.7747
RMSE: Restricted DSGE	3.5400	3.9518	4.0925	4.1039
RMSE: VAR(1)	**2.9194**	**3.2668**	**3.3105**	**3.2675**
RMSE: VAR(2)	3.0153	3.4442	3.4687	3.2879
(C) Investment				
RMSE: Unrestricted DSGE	8.7370	9.5242	9.9193	7.8903
RMSE: Restricted DSGE	7.6644	7.8182	8.5590	**6.4125**
RMSE: VAR(1)	7.7122	8.4023	9.3086	7.5353
RMSE: VAR(2)	**7.2893**	**7.5999**	**8.4232**	6.4950
(D) Hours Worked				
RMSE: Unrestricted DSGE	**1.8935**	**2.1311**	**2.1482**	**2.2146**
RMSE: Restricted DSGE	1.9481	2.2069	2.2457	2.2952
RMSE: VAR(1)	1.9630	2.2010	2.2540	2.3246
RMSE: VAR(2)	1.9605	2.4487	2.4878	2.3839

Note: The lowest RMSEs are highlighted in bold faced.

A k-quarter-ahead forecast at time t, $d_{t+k|t}$, is computed as follows. Firstly, one-to-four quarter ahead forecasts are calculated using the data from 1973:1 to 1997:3. Secondly, one-to-four quarter ahead forecasts are calculated using the data from 1973:1

to 1997:4. Thus, one quarter ahead forecasts range from 1997:4 to 2006:3, two quarter ahead forecasts do from 1998:1 to 2006:3, and so forth.

Table 3.6 reports the root mean square errors (RMSEs) for the k-step-ahead prediction and lower values in RMSEs imply better performance in forecasting. In the table, forecasting performances of two DSGE models are inferior to those of the VAR models in most cases. Apparently, fluctuations in output, consumption and investment are better accounted for by the VAR models. However, there is only one exception that the four-quarter-ahead-forecast of the restricted DSGE model for investment is better than those of two VAR models. On the other hand, for hours worked prediction, two DSGE models outperform two VAR models. Note also that the unrestricted DSGE model is superior for the predictions on consumption and hours worked while the restricted DSGE model is superior for the predictions on output and investment.

Finally, Table 3.7 reports the second moments for the HP filtered data and the simulated data.[18] In this case, the estimated unrestricted DSGE model has been simulated 1000 times, with each simulation being 135 periods long, to match the number of observations. Then, we present the average second moments of 1000 simulated data. It seems that the model successfully reproduces the relative volatility of consumption and hours worked as well as the pattern of contemporaneous correlations of output with consumption, investment and hours worked.

Table 3.7: Volatility and Contemporaneous Correlations with Output for Actual Data and Simulated Data : 1973:1 – 2006:3

	Volatility				Corr. with Output	
	Actual		Simulated		Actual	Simulated
	SD%	Relative to Output	SD%	Relative to Output		
Output	4.89	1.00	4.44	1.00	1.00	1.00
Consumption	3.08	0.63	2.91	0.66	0.89	0.87
Investment	9.68	1.98	10.93	2.47	0.96	0.92
Hours Worked	2.12	0.43	1.99	0.45	0.52	0.44

As found in the HP filtered data, the simulated data show that consumption fluctuates significantly less than output and is highly correlated with output. In addition,

[18]This comparison analysis was gratefully motivated by Dr. Leslie Reinhorn.

investment from the simulation fluctuates much more than output and is highly correlated with output. Lastly, worked hours from simulation fluctuate significantly less than output and are moderately correlated with output as shown in the actual data.

3.6 Conclusion

Rather than calibrating a baseline DSGE model, this essay estimates it for Korean economy with HP-filtered data by augmenting Vector Autoregression (VAR) measurement errors. The parameters in the current model are estimated by maximum likelihood method in the Kalman filtering algorithm and they are quite intuitive and statistically significant. Hence, one contribution of this essay is that these estimated parameters can be utilized for future research in calibrated DSGE models for Korea economy.

By comparing the pre-crisis period with the entire examined period, this essay finds that the indicator of volatility of macroeconomy σ is a little higher in the pre-crisis period, which implies that volatility of the economy decreased after the Korean financial crisis. Overall, however, the five structural parameter estimates seem not to be significantly different across the two periods while the non-structural parameters seem to be different. Nonetheless, one caveat should be acknowledged again. This essay does not formally test the parameter stability because the post-crisis period was too short to estimate the model. This issue may be an important subject in future research.

In addition, we find that the forecasting performances of the current DSGE model is worse than those of VAR models in most cases. However the current DSGE model outperforms VAR models in predicting hours worked. Then, using the variance decomposition analysis, we find that the productivity shock is the important source of business cycles in Korea, in particular for hours worked, but it has a difficulty to explain fluctuations of some key variables. Therefore, incorporating other important shocks such as monetary and fiscal policy shocks in the model may help to alleviate this problem.

Finally, by comparing the second moments from the HP filtered data with those from the simulated data, this essay finds that the estimated model successfully reproduces the relative volatility of consumption and hours worked as well as the pattern of contemporaneous correlations of output with consumption, investment and hours worked. From this perspective, the current DSGE model seems to serve as a starting point for further complicated but realistic DSGE models.

Appendix to Chapter 3

3.A Solving a Baseline Model

To find a solution to the current model is to write the non-predetermined endogenous variables in terms of the predetermined variables and exogenous variables. Following Ireland (2004), this appendix shows how we can solve the baseline model.

3.A.1 Dynamic Optimization

The optimization problem of a social planner is to choose paths for $\{Y_t, C_t, I_t, H_t, K_{t+1}\}_{t=0}^{\infty}$ to maximize the utility function (3.2.1) subject to constraints (3.2.2)–(3.2.5) for all $t = 0, 1, 2, \cdots$. Note that the constraints can be simplified by substituting production function (3.2.2) and aggregate resource constraint (3.2.4) into capital accumulation equation (3.2.5). Then the optimality conditions can be obtained using the dynamic programming approach.[19] Let us define the value function $V(K_t, A_t)$ as the maximum present value of utility the planner can achieve if the current state is (K_t, A_t). Then the value function at time 0 can be written as,

[19] Alternatively, following Chow (1997), the problem can be solved by the Lagrangian approach where the social planner solves the following sequential maximization problem,

$$\mathcal{L} = E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \Big\{ \ln(C_\tau) - \gamma H_\tau + \lambda_\tau \big[(1-\delta) K_\tau + A_\tau K_\tau^{\alpha} H_\tau^{1-\alpha} - C_\tau - K_{\tau+1} \big] \Big\}.$$

$$
\begin{aligned}
V(K_0, A_0) = \max_{\{C_t, H_t\}} \quad & E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[\ln C_t - \gamma H_t \right] \right\} \\
s.t. \quad & K_{t+1} = A_t K_t^{\alpha} H_t^{1-\alpha} + (1-\delta) K_t - C_t \quad \text{for all } t \geq 0 \\
& K_0 \text{ given} \\
& \ln A_t = (1-\rho) \ln \bar{A} + \rho \ln A_{t-1} + \varepsilon_t. \qquad (3.A.1)
\end{aligned}
$$

After getting the value function at time t and replacing C_t by $A_t K_t^{\alpha} H_t^{1-\alpha} + (1-\delta)K_t - K_{t+1}$, one can obtain the Bellman equation as,

$$
V(K_t, A_t) = \max_{\{K_{t+1}, H_t\}} \left\{ U\left[A_t K_t^{\alpha} H_t^{1-\alpha} + (1-\delta)K_t - K_{t+1}, H_t\right] + \beta E_t V(K_{t+1}, A_{t+1}) \right\}. \qquad (3.A.2)
$$

Note that the expectation operator E_t appears due to presence of the stochastic productivity process. In period t there are two control variables, K_{t+1} and H_t and two state variables, K_t and A_t. Therefore, given K_t, the planner solves the Bellman equation after observing the exogenous technology shock.

The first order necessary conditions for this problem are

$$
\begin{aligned}
-U_C(C_t, H_t) + \beta E_t V_K(K_{t+1}, A_{t+1}) &= 0, \\
U_C MP_{H,t} + U_H(C_t, H_t) &= 0, \\
V_K(K_t, A_t) &= U_C[MP_{K,t} + (1-\delta)], \qquad (3.A.3)
\end{aligned}
$$

where the subscripts represent the partial derivative with respect to those variables and MP_t denotes the marginal product of each production factor.

Given the assumed functional form, by combining the first and third lines of conditions, one obtains the intertemporal optimality condition for choice between consumption and investment as (3.3.2)

$$
1/C_t = \beta E_t (1/C_{t+1}) [\alpha (Y_{t+1}/K_{t+1}) + 1 - \delta], \qquad (3.3.2)
$$

for all $t = 0, 1, 2, \cdots$.

Thus, in terms of utility, the planner adjusts consumption until the marginal current cost from the marginal increase in investment (i.e., marginal decrease in current consumption) just equals the discounted marginal future benefit from the expected marginal increase in investment (i.e., expected marginal increase in consumption opportunity).

On the other hand, from the second line of conditions, one obtains the intratemporal optimality condition for choice between consumption and leisure as (3.3.1).

$$\gamma = \frac{(1-\alpha)Y_t}{H_t} \cdot \frac{1}{C_t}, \tag{3.3.1}$$

for all $t = 0, 1, 2, \cdots$.

Thus, the planner adjusts labor supply until the marginal benefit in terms of utility from increasing labor supply exactly equal the marginal costs in terms of disutility from increasing labor supply.

3.A.2 Equilibrium Conditions

Now, the equilibrium behaviors of the six variables Y_t, C_t, I_t, H_t, K_t, and A_t are determined by the following six equations,

$$Y_t = A_t K_t^{\alpha} H_t^{1-\alpha}, \tag{3.2.2}$$

$$\ln A_t = (1-\rho)\ln \bar{A} + \rho \ln A_{t-1} + \varepsilon_t, \tag{3.2.3}$$

$$Y_t = C_t + I_t, \tag{3.2.4}$$

$$K_{t+1} = (1-\delta)K_t + I_t, \tag{3.2.5}$$

$$\gamma C_t H_t = (1-\alpha)Y_t, \tag{3.3.1}$$

$$1/C_t = \beta E_t(1/C_{t+1})[\alpha(Y_{t+1}/K_{t+1}) + 1 - \delta], \tag{3.3.2}$$

for all $t = 0, 1, 2, \cdots$.

3.A.3 Removing the Trend in Variables

In order to extract cyclical components from the current variables, this essay employs the Hodrick and Prescott (1997) filter. Let us denote the detrended variables with lower letters. Then, equations (3.2.2)–(3.3.2) can be expressed in terms of transformed detrended variables as follows,

$$y_t = a_t k_t^{\alpha} h_t^{1-\alpha}, \tag{3.A.4}$$

$$\ln a_t = (1-\rho)\ln \bar{A} + \rho \ln a_{t-1} + \varepsilon_t, \tag{3.A.5}$$

$$y_t = c_t + i_t, \tag{3.A.6}$$

$$k_{t+1} = (1-\delta)k_t + i_t, \tag{3.A.7}$$

$$\gamma c_t h_t = (1-\alpha) y_t, \tag{3.A.8}$$

$$1/c_t = \beta E_t(1/c_{t+1})[\alpha(y_{t+1}/k_{t+1}) + 1 - \delta], \tag{3.A.9}$$

for all $t = 0, 1, 2, \cdots$.

3.A.4 The Steady State of the Economy

If there is no shock in the economy, the system converges to the steady state and the six stationary variables $y_t = y, c_t = c, i_t = i, h_t = h, k_t = k$, and $a_t = a$ are constant. Then, we can express these constant values only with underlying structural parameters. First notice that from (3.A.5), $a = \bar{A}$. The other five values are obtained as follows. For convenience, suppose that the steady-state value y is in hand, and use (3.A.9) to solve for k

$$k = \left(\frac{\alpha}{1/\beta - 1 + \delta}\right) y.$$

Use (3.A.7) to solve for i

$$i = \left[\frac{\alpha\delta}{1/\beta - 1 + \delta}\right] y.$$

Use (3.A.6) to solve for c

$$c = y - i = \left\{1 - \left[\frac{\alpha\delta}{1/\beta - 1 + \delta}\right]\right\} y.$$

Use (3.A.8) to solve for h

$$h = \left(\frac{1-\alpha}{\gamma}\right)\frac{y}{c} = \left(\frac{1-\alpha}{\gamma}\right)\left\{1 - \left[\frac{\alpha\delta}{1/\beta - 1 + \delta}\right]\right\}^{-1}.$$

Finally, substitute a,k and h into (3.A.4) to solve for y

$$y = \bar{A}k^{\alpha}h^{1-\alpha} = \bar{A}\left[\left(\frac{\alpha}{1/\beta - 1 + \delta}\right) y\right]^{\alpha}\left[\left(\frac{1-\alpha}{\gamma}\right)\left\{1 - \left[\frac{\alpha\delta}{1/\beta - 1 + \delta}\right]\right\}^{-1}\right]^{1-\alpha}.$$

This messy equation can be simplified as

$$y = \bar{A}^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{1/\beta - 1 + \delta}\right)^{\frac{\alpha}{1-\alpha}}\left(\frac{1-\alpha}{\gamma}\right)\left\{1 - \left[\frac{\alpha\delta}{1/\beta - 1 + \delta}\right]\right\}^{-1}.$$

Notice that the steady-state values y, c, i, h, k, and a depend on the model's underlying structural parameters: $\beta, \gamma, \alpha, \delta$ and $\bar{A}$, but not on ρ and σ.

3.A.5 Log-linear Approximation

To obtain the approximate solutions to the economy, Equations (3.A.4) to (3.A.9) need to be log-linearized around their steady state. We followed the log-linear approximation method proposed by Campbell (1994). Denote a hat over a variable as representing a proportionate deviation of that variable from its steady state level, for instance, $\hat{c}_t = \ln(c_t/c)$, etc. Then, the first-order Taylor approximations to (3.A.4) through (3.A.9) produce

$$\hat{y}_t = \hat{a}_t + \alpha\hat{k}_t + (1-\alpha)\hat{h}_t, \tag{3.3.3}$$

$$\hat{a}_t = \rho\hat{a}_{t-1} + \varepsilon_t, \tag{3.3.4}$$

$$(1/\beta - 1 + \delta)\hat{y}_t = [(1/\beta - 1 + \delta) - \alpha\delta]\hat{c}_t + \alpha\delta\hat{i}_t, \tag{3.3.5}$$

$$\hat{k}_{t+1} = (1-\delta)\hat{k}_t + \delta\hat{i}_t, \tag{3.3.6}$$

$$\hat{c}_t + \hat{h}_t = \hat{y}_t, \tag{3.3.7}$$

$$0 = (1/\beta)\hat{c}_t - (1/\beta)E_t\hat{c}_{t+1} + (1/\beta - 1 + \delta)E_t\hat{y}_{t+1} - (1/\beta - 1 + \delta)\hat{k}_{t+1}, \tag{3.3.8}$$

for all $t = 0, 1, 2, \cdots$.

3.A.6 Preliminary Step for Blanchard and Kahn's (1980) Solution Method

After the nonlinear system of expectational difference equations are approximated as the linear system, we can solve the linearized system through Blanchard and Kahn's (1980) method. However, Blanchard and Kahn's (1980) solution method can be applied when the models are written as the form of (3.3.9),

$$\begin{bmatrix} x_{1,t+1} \\ E_t(x_{2,t+1}) \end{bmatrix} = K \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} + Lz_t, \tag{3.3.9}$$

where $x_{1,t}$ is a vector of predetermined variables (i.e., $E_t(x_{1,t+1}) = x_{1,t+1}$), $x_{2,t}$ is a vector of control variables (i.e., $E_t(x_{2,t+1}) + \zeta_{t+1} = x_{1,t+1}$ with ζ_{t+1} denoting an expectation error), and z_t is a vector of exogenous forcing variables.

Therefore, as a preliminary step, we need to convert the linearized system in Section 3.A.5, which is composed of (3.3.3) through (3.3.8), into a form of (3.3.9). This step requires a system reduction, which involves expressing the model in terms of uniquely determined variables such as $\hat{a}_t$, $\hat{c}_t$, and $\hat{k}_t$ (King and Watson 2002). Thus, we need to find the following form first,

$$\begin{bmatrix} \hat{k}_{t+1} \\ E_t(\hat{c}_{t+1}) \end{bmatrix} = K \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} + L\hat{a}_t. \tag{3.3.10}$$

For this purpose, let us define two vectors: $\boldsymbol{f}_t^0 = [\hat{y}_t \quad \hat{i}_t \quad \hat{h}_t]'$ and $\boldsymbol{s}_t^0 = [\hat{k}_t \quad \hat{c}_t]'$. In addition, let us simplify the notation using $\kappa = 1/\beta - 1 + \delta$ and $\lambda = \delta$. Then, (3.3.3), (3.3.5) and (3.3.7) can be summarized as (3.A.10),

$$\hat{y}_t + (\alpha - 1)\hat{h}_t = \alpha\hat{k}_t + \hat{a}_t, \tag{3.3.3}$$

$$\kappa\hat{y}_t - \alpha\lambda\hat{i}_t = [\kappa - \alpha\lambda]\hat{c}_t, \tag{3.3.5}$$

$$\hat{y}_t - \hat{h}_t = \hat{c}_t, \tag{3.3.7}$$

$$\boldsymbol{A}\boldsymbol{f}_t^0 = \boldsymbol{B}\boldsymbol{s}_t^0 + \boldsymbol{C}\hat{a}_t \tag{3.A.10}$$

$$\begin{bmatrix} 1 & 0 & \alpha-1 \\ \kappa & -\alpha\lambda & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{y}_t \\ \hat{i}_t \\ \hat{h}_t \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \kappa-\alpha\lambda \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \hat{a}_t.$$

Rearrangements of (3.3.6) and (3.3.8) yield (3.A.11),

$$\hat{k}_{t+1} = (1-\delta)\hat{k}_t + \lambda\hat{i}_t, \tag{3.3.6}$$

$$\kappa\hat{k}_{t+1} + (1/\beta)E_t\hat{c}_{t+1} - \kappa E_t\hat{y}_{t+1} = (1/\beta)\hat{c}_t, \tag{3.3.8}$$

$$\boldsymbol{D}E_t\boldsymbol{s}_{t+1}^0 + \boldsymbol{F}E_t\boldsymbol{f}_{t+1}^0 = \boldsymbol{G}\boldsymbol{s}_t^0 + \boldsymbol{H}\boldsymbol{f}_t^0 \tag{3.A.11}$$

$$\begin{bmatrix} 1 & 0 \\ \kappa & 1/\beta \end{bmatrix} E_t \begin{bmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -\kappa & 0 & 0 \end{bmatrix} E_t \begin{bmatrix} \hat{y}_{t+1} \\ \hat{i}_{t+1} \\ \hat{h}_{t+1} \end{bmatrix} = \begin{bmatrix} 1-\delta & 0 \\ 0 & 1/\beta \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} + \begin{bmatrix} 0 & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{y}_t \\ \hat{i}_t \\ \hat{h}_t \end{bmatrix}.$$

Lastly, due to the assumption $|\rho| < 1$, (3.3.4) can be solved forward as

$$E_t(\hat{a}_{t+j}) = \rho^j\hat{a}_t, \tag{3.A.12}$$

for all $j = 0, 1, 2, \cdots$.

Now we have the linearized system in matrix form,

$$\boldsymbol{A}\boldsymbol{f}_t^0 = \boldsymbol{B}\boldsymbol{s}_t^0 + \boldsymbol{C}\hat{a}_t, \tag{3.A.10}$$

$$\boldsymbol{D}E_t\boldsymbol{s}_{t+1}^0 + \boldsymbol{F}E_t\boldsymbol{f}_{t+1}^0 = \boldsymbol{G}\boldsymbol{s}_t^0 + \boldsymbol{H}\boldsymbol{f}_t^0, \tag{3.A.11}$$

$$E_t(\hat{a}_{t+j}) = \rho^j \hat{a}_t. \tag{3.A.12}$$

Next, substituting (3.A.10) and (3.A.12) into (3.A.11) yields (3.3.10),

$$\begin{aligned}(D + FA^{-1}B)E_t s^0_{t+1} &= (G + HA^{-1}B)s^0_t + (HA^{-1}C - FA^{-1}C\rho)\hat{a}_t \\ \Leftrightarrow E_t s^0_{t+1} &= K s^0_t + L\hat{a}_t \\ \begin{bmatrix} \hat{k}_{t+1} \\ E_t\hat{c}_{t+1} \end{bmatrix} &= \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \hat{a}_t, \end{aligned} \tag{3.3.10}$$

where $K = (D + FA^{-1}B)^{-1}(G + HA^{-1}B)$, $L = (D + FA^{-1}B)^{-1}(HA^{-1}C - FA^{-1}C\rho)$ and $s^0_t = [\hat{k}_t \quad \hat{c}_t]'$.

3.A.7 Applying Blanchard and Kahn's (1980) Solution Method to the Linearized System

The Blanchard and Kahn's (1980) solution method begins with a Jordan decomposition of 2×2 matrix K,

$$K = M^{-1}NM, \tag{3.A.13}$$

where

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}.$$

Note that the diagonal elements of N, consisting of eigenvalues of K, are ordered in increasing absolute value from left to right. Thus N_1 lies within the unit circle, and N_2 lies outside of the unit circle in order to guarantee that the system has a unique solution to the model.[20] Moreover, the first and second columns of M^{-1} are eigenvectors corresponding to N_1 and N_2, respectively.

Next, (3.3.10) is pre-multiplied by M, yielding

$$\begin{aligned}\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \hat{k}_{t+1} \\ E_t\hat{c}_{t+1} \end{bmatrix} &= \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} + \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \hat{a}_t \\ \begin{bmatrix} \tilde{k}_{t+1} \\ E_t\tilde{c}_{t+1} \end{bmatrix} &= \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} \tilde{k}_t \\ \tilde{c}_t \end{bmatrix} + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \hat{a}_t \end{aligned} \tag{3.A.14}$$

[20] This is known as the Blanchard-Kahn condition. According to this condition, the system is said to be saddle-path stable when the number of control variables are equal to the number of explosive eigenvalues. If the number of unstable eigenvalues is larger than that of controls, the system is overdetermined and unstable. If the number of unstable eigenvalues is smaller than that of controls, the system is indeterminate since for any given state we have at least one more degree of freedom. Note also that it can be shown that K has one unstable eigenvalue and one stable eigenvalue.

where
$$\begin{bmatrix}\tilde{k}_t\\ \tilde{c}_t\end{bmatrix}=\begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}\begin{bmatrix}\hat{k}_t\\ \hat{c}_t\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}Q_1\\ Q_2\end{bmatrix}=\begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}\begin{bmatrix}L_1\\ L_2\end{bmatrix}.$$

Note that since $N_1 < 1$ and $N_2 > 1$, this transformation decouples the system into the stable equation for $\tilde{k}_t$ and the unstable equation for $\tilde{c}_t$,

$$\tilde{k}_{t+1} = N_1\tilde{k}_t + Q_1\hat{a}_t \quad \text{and} \quad E_t\tilde{c}_{t+1} = N_2\tilde{c}_t + Q_2\hat{a}_t. \tag{3.A.15}$$

From this decoupled system, three more steps are required. First, we need to solve the unstable equation for $\tilde{c}_t$. Second, we need to solve the stable equation for $\tilde{k}_t$. Third, we need to recover the solutions for $\hat{c}_t$ and $\hat{k}_t$ from the solutions for $\tilde{c}_t$ and $\tilde{k}_t$.

First, we solve the unstable equation for $\tilde{c}_t$ by the forward iteration since N_2 lies outside of the unit circle.

$$\begin{aligned}\tilde{c}_t &= \frac{1}{N_2}E_t\tilde{c}_{t+1} - \frac{Q_2}{N_2}\hat{a}_t\\ &= \frac{1}{N_2}E_t\left(\frac{1}{N_2}E_{t+1}\tilde{c}_{t+2} - \frac{Q_2}{N_2}\hat{a}_t\right) - \frac{Q_2}{N_2}\hat{a}_t\\ &= \left(\frac{1}{N_2}\right)^2 E_tE_{t+1}\tilde{c}_{t+2} - \left(\frac{1}{N_2}\right)\frac{Q_2}{N_2}\rho\hat{a}_t - \frac{Q_2}{N_2}\hat{a}_t\\ &= -\frac{Q_2/N_2}{1-\frac{\rho}{N_2}}\hat{a}_t\\ &= \frac{Q_2}{\rho - N_2}\hat{a}_t,\end{aligned} \tag{3.A.16}$$

where in the third line we exploit the Law of Iterated Expectations, which holds that

$$E_t\left[E_{t+1}(\tilde{c}_{t+2})\right] = E_t(\tilde{c}_{t+2}).$$

Then, by combining (3.A.14) and (3.A.16), we can obtain the solution for $\hat{c}_t$,

$$\begin{aligned}\tilde{c}_t &= M_{21}\hat{k}_t + M_{22}\hat{c}_t\\ \left(\frac{Q_2}{\rho - N_2}\right)\hat{a}_t &= M_{21}\hat{k}_t + M_{22}\hat{c}_t\\ \hat{c}_t &= -\frac{M_{21}}{M_{22}}\hat{k}_t + \frac{1}{M_{22}}\left(\frac{Q_2}{\rho - N_2}\right)\hat{a}_t = S_1\hat{k}_t + S_2\hat{a}_t.\end{aligned} \tag{3.A.17}$$

Next, we seek for the solution for the nonexplosive portion of the system, i.e., the solution for $\tilde{k}_t$ by plugging (3.A.17) into (3.A.14),

$$\begin{aligned}\tilde{k}_t &= M_{11}\hat{k}_t + M_{12}\hat{c}_t \\ &= M_{11}\hat{k}_t + M_{12}\left[S_1\hat{k}_t + S_2\hat{a}_t\right] \\ &= \left(M_{11} + M_{12}S_1\right)\hat{k}_t + M_{12}S_2\hat{a}_t. \end{aligned} \tag{3.A.18}$$

Then, by combining (3.A.14) and (3.A.18), we can obtain the solution for $\hat{k}_{t+1}$,

$$\begin{aligned}\tilde{k}_{t+1} &= N_1\tilde{k}_t + Q_1\hat{a}_t \\ \left(M_{11} + M_{12}S_1\right)\hat{k}_{t+1} + M_{12}S_2\rho\hat{a}_t &= N_1\Big[\left(M_{11} + M_{12}S_1\right)\hat{k}_t + M_{12}S_2\hat{a}_t\Big] + Q_1\hat{a}_t \\ \left(M_{11} + M_{12}S_1\right)\hat{k}_{t+1} &= N_1\left(M_{11} + M_{12}S_1\right)\hat{k}_t + \left(Q_1 + N_1M_{12}S_2 - M_{12}S_2\rho\right)\hat{a}_t \\ \hat{k}_{t+1} &= S_3\hat{k}_t + S_4\hat{a}_t, \end{aligned} \tag{3.A.19}$$

where $S_3 = \left(M_{11}+M_{12}S_1\right)^{-1}N_1\left(M_{11}+M_{12}S_1\right) = N_1$ and $S_4 = \left(M_{11}+M_{12}S_1\right)^{-1}\left(Q_1 + N_1M_{12}S_2 - M_{12}S_2\rho\right)$.

Now, we can also rewrite $\left[\hat{y}_t \quad \hat{i}_t \quad \hat{h}_t\right]'$ in terms of the predetermined variable and the exogenous variable using (3.A.10),

$$\begin{aligned}\boldsymbol{f}_t^0 &= A^{-1}B\boldsymbol{s}_t^0 + A^{-1}C\hat{a}_t \\ \begin{bmatrix}\hat{y}_t \\ \hat{i}_t \\ \hat{h}_t\end{bmatrix} &= \underbrace{A^{-1}}_{3\times 3}\underbrace{B}_{3\times 2}\begin{bmatrix}\hat{k}_t \\ \hat{c}_t\end{bmatrix} + A^{-1}\underbrace{C}_{3\times 1}\hat{a}_t \\ &= A^{-1}B\begin{bmatrix}\hat{k}_t \\ S_1\hat{k}_t + S_2\hat{a}_t\end{bmatrix} + A^{-1}C\hat{a}_t \quad \text{[from (3.A.17)]} \\ &= A^{-1}B\begin{bmatrix}1 \\ S_1\end{bmatrix}\hat{k}_t + \left(A^{-1}C + A^{-1}B\begin{bmatrix}0 \\ S_2\end{bmatrix}\right)\hat{a}_t = S_5\hat{k}_t + S_6\hat{a}_t. \end{aligned} \tag{3.A.20}$$

Finally, by collecting all solutions, we can establish two blocks of solutions for the state variables and the non-predetermined variables. Let us denote $\boldsymbol{s}_t = \left[\hat{k}_t \quad \hat{a}_t\right]'$ and $\boldsymbol{f}_t = \left[\hat{y}_t \quad \hat{i}_t \quad \hat{h}_t \quad \hat{c}_t\right]'$. Then, collecting the solutions for the state variables in (3.A.19) and (3.3.4) yields,

$$\boldsymbol{s}_{t+1} = \Xi\boldsymbol{s}_t + \Phi\varepsilon_{t+1} \quad \Leftrightarrow \quad \begin{bmatrix}\hat{k}_{t+1} \\ \hat{a}_{t+1}\end{bmatrix} = \begin{bmatrix}S_3 & S_4 \\ 0 & \rho\end{bmatrix}\begin{bmatrix}\hat{k}_t \\ \hat{a}_t\end{bmatrix} + \begin{bmatrix}0 \\ 1\end{bmatrix}\varepsilon_{t+1}, \tag{3.3.11}$$

for all $t = 0, 1, 2, \cdots$.

Similarly, collecting the solutions for the non-predetermined variables in (3.A.20) and (3.A.17) yields,

$$\boldsymbol{f}_t = \Psi \boldsymbol{s}_t \quad \Leftrightarrow \quad \begin{bmatrix} \hat{y}_t \\ \hat{i}_t \\ \hat{h}_t \\ \hat{c}_t \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} S_5 \end{pmatrix} & \begin{pmatrix} S_6 \end{pmatrix} \\ S_1 & S_2 \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{a}_t \end{bmatrix}, \qquad (3.3.12)$$

for all $t = 0, 1, 2, \cdots$. That is, given the vector $\boldsymbol{s}_t$ which traces the percentage deviations of two state variables from their steady-state levels, the policy functions (3.3.12) give us the optimal policy to follow.

3.B Stochastic Singularity Problem

This section illustrates why the presence of the stochastic singularity makes traditional econometric methods inapplicable to the model's parameter estimations. The stochastic singularity problem in DSGE model estimations arises when the number of shocks are less than the number of observable endogenous variables. To understand this problem, notice first that the current model has only one structural shock – the aggregate technology shock – driving all business cycle fluctuations, but there are three observable endogenous variables (i.e., output, consumption and hours worked) to feed in. Look at the following linearized system.

$$\begin{pmatrix} \hat{y}_t \\ \hat{c}_t \\ \hat{h}_t \end{pmatrix} = \begin{pmatrix} C_{yk} & C_{ya} \\ C_{ck} & C_{ca} \\ C_{hk} & C_{ha} \end{pmatrix} \begin{pmatrix} \hat{k}_t \\ \hat{a}_t \end{pmatrix}, \qquad (3.B.1)$$

where 3×2 matrix C can have at most rank 2 and thus it is singular. At first glance, it seems to be sufficient to provide one more shock to estimate the model.

However, providing one more shock is not enough because $\hat{k}_t$ is the predetermined variable at time t. Namely, it is a function of $\hat{y}_{t-1}$, $\hat{c}_{t-1}$, $\hat{h}_{t-1}$ and $\hat{a}_{t-1}$. Therefore, (3.B.1) can be rewritten as,

$$\hat{y}_t = C_{yk}\hat{k}_t(\hat{y}_{t-1}, \hat{c}_{t-1}, \hat{h}_{t-1}, \hat{a}_{t-1}) + C_{ya}\hat{a}_t, \qquad (3.B.2)$$

$$\hat{c}_t = C_{ck}\hat{k}_t(\hat{y}_{t-1}, \hat{c}_{t-1}, \hat{h}_{t-1}, \hat{a}_{t-1}) + C_{ca}\hat{a}_t, \qquad (3.B.3)$$

$$\hat{h}_t = C_{hk}\hat{k}_t(\hat{y}_{t-1}, \hat{c}_{t-1}, \hat{h}_{t-1}, \hat{a}_{t-1}) + C_{ha}\hat{a}_t. \tag{3.B.4}$$

Then, from (3.B.3), we get the expression for $\hat{k}_t(\hat{y}_{t-1}, \hat{c}_{t-1}, \hat{h}_{t-1}, \hat{a}_{t-1})$ as,

$$\hat{k}_t(\hat{y}_{t-1}, \hat{c}_{t-1}, \hat{h}_{t-1}, \hat{a}_{t-1}) = \frac{\hat{c}_t - C_{ca}\hat{a}_t}{C_{ck}}. \tag{3.B.5}$$

Finally, substituting (3.B.5) into (3.B.2) yields

$$\hat{y}_t = \frac{C_{yk}}{C_{ck}}\hat{c}_t + \left(C_{ya} - \frac{C_{yk}C_{ca}}{C_{ck}}\right)\hat{a}_t. \tag{3.B.6}$$

Immediately apparent from (3.B.6) is that during the estimation if we observe $\hat{a}_t$ and $\hat{c}_t$, we cannot observe an independent $\hat{y}_t$. That is, $\hat{y}_t$ should be observed as the exact linear relationship of $\hat{a}_t$ and $\hat{c}_t$ without noise. For the similar reason, if we observe $\hat{a}_t$ and $\hat{c}_t$, we cannot observe an independent $\hat{h}_t$, either. Therefore, we can find the solutions in the DSGE model as the optimal choice of sequences $\{\hat{y}_t, \hat{c}_t, \hat{i}_t, \hat{h}_t, \hat{k}_{t+1}\}_{t=0}^{\infty}$, but we cannot estimate the coefficient matrix C. In sum, to make the estimation possible, we need at least two more shocks in this system. This is said to be the stochastic singularity problem in DSGE model estimation.

3.C Estimating a DSGE Model by Maximum Likelihood Method

This section demonstrates how maximum likelihood estimation gives rise to the model's parameters using the Kalman filtering algorithm following Hamilton (1994, Ch.13) and Kim and Nelson (1999, Ch.3). In addition, it discusses the model evaluation procedures for forecast error variance decomposition, forecasting performance comparison between models, and the shock estimates generation.

3.C.1 Kalman Filter

The Kalman filter, originally developed by Kalman (1960), is a tool to recursively estimate the state variables given observations.[21] The key idea of the Kalman filter algorithm is predicting and updating the conditional state mean vector and the covariance matrix. When new information becomes available, this changes our best least square estimates of the state vector. These prediction-correction type estimates are optimal in the sense that they minimizes the estimated error covariance. During this procedure, we acquire maximum likelihood estimates for the model parameters as a by-product.

To make the system conformable for the Kalman filtering algorithm, (3.4.1), (3.4.3), and (3.4.4) need to be substituted with the following state space representation.

$$x_{t+1} = Fx_t + \eta_{t+1} \Leftrightarrow \begin{bmatrix} \hat{k}_{t+1} \\ \hat{a}_{t+1} \\ u_{yt+1} \\ u_{ct+1} \\ u_{ht+1} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} A_{kk} & A_{ka} \\ 0 & \rho \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} D_{yy} & D_{yc} & D_{yh} \\ D_{cy} & D_{cc} & D_{ch} \\ D_{hy} & D_{hc} & D_{hh} \end{pmatrix} \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{a}_t \\ u_{yt} \\ u_{ct} \\ u_{ht} \end{bmatrix} + \begin{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varepsilon_{t+1} \\ \begin{pmatrix} \xi_{yt+1} \\ \xi_{ct+1} \\ \xi_{ht+1} \end{pmatrix} \end{bmatrix}, \tag{3.C.1}$$

[21]This is a *filter* in the sense that it extracts the unobservable signal (i.e., state) from the data by eliminating noises in the measurements. This is recursive because only the estimated state from the previous period and the current observations are needed to compute the estimates for the current state. Thus, one does not have to store old observations.

$$d_t = Gx_t \Leftrightarrow \begin{bmatrix} \hat{y}_t \\ \hat{c}_t \\ \hat{h}_t \end{bmatrix} = \left[\begin{pmatrix} C_{yk} & C_{ya} \\ C_{ck} & C_{ca} \\ C_{hk} & C_{ha} \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] \begin{bmatrix} \hat{k}_t \\ \hat{a}_t \\ u_{yt} \\ u_{ct} \\ u_{ht} \end{bmatrix}, \tag{3.C.2}$$

$$Q = E_t(\eta_{t+1}\eta'_{t+1}) \Leftrightarrow \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \sigma^2 \end{bmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} V_y^2 & V_{yc} & V_{yh} \\ V_{yc} & V_c^2 & V_{ch} \\ V_{yh} & V_{ch} & V_h^2 \end{pmatrix} \end{bmatrix}, \tag{3.C.3}$$

where (3.C.1) are the transition equations of hidden state vector, (3.C.2) are the observation (or measurement) equations, and $\eta_{t+1} \sim i.i.d\ N(0, Q)$ where Q is the covariance matrix as in (3.C.3).

In addition, let us introduce the notations of the optimal estimator of x_t based on d_t and its covariance matrix.

$$\begin{aligned} x_{t|t} &= E(x_t|d_t) \\ \Sigma_{t|t} &= E[(x_t - x_{t|t})(x_t - x_{t|t})']. \end{aligned}$$

Some features need to be addressed at this point. First, we cannot observe true x_t. All we can have is only the updated estimate $x_{t|t}$. On the other hand, we do not update d_t but just observe it. Thus Kalman filter predicts and updates the state vector but only predicts the observation vector. Second, there are 25 independent elements to be estimated in this state space form; 12 in F, 6 in G and 7 in Q. Maximizing the likelihood function during the prediction and correction of the state yields the maximum likelihood estimates for these matrices, which are constructed with parameters of interest. Note also that these matrices are time-invariant if d_t are covariance stationary. Third, since eigenvalues of matrix F are all inside the unit circle by the model's assumptions, the process for x_t is covariance-stationary.[22]

[22]Note first in the solution for the stable equation of $\hat{k}_{t+1}$ that $A_{kk} = S_3 = N_1$ in (3.A.19) is lower than unity. In addition, ρ is also lower than unity by assumption. In a upper-triangular matrix like (3.3.11), eigenvalues are just diagonal elements. Thus, eigenvalues of matrix A are all inside the unit circle. In addition, matrix D's three eigenvalues are assumed to be lower than unity. Lastly, since F is a block diagonal matrix, in which eigenvalues are diagonal elements, all moduli of eigenvalues in

3.C.1.1 Starting a Kalman Filter

Given the arbitrary initial values of elements in F, G, and Q (i.e., given the initial values for corresponding structural parameters $\boldsymbol{\theta}_0$), the first step is to predict $\boldsymbol{x}_1$ and Σ_1 based on no information, or equivalently to find $\boldsymbol{x}_{1|0}$ and $\Sigma_{1|0}$. It is straightforward that without data, $\ln \mathscr{L}(\boldsymbol{\theta}_0) = 0$.[23] Since $\boldsymbol{x}_{1|0}$ is a forecast of $\boldsymbol{x}_1$ based on $\boldsymbol{d}_0$ (i.e., no observation), it is just the unconditional mean of $\boldsymbol{x}_1$,

$$\boldsymbol{x}_{1|0} = E(\boldsymbol{x}_1).$$

In addition, $\Sigma_{1|0}$, the mean squared error matrix of $\boldsymbol{x}_{1|0}$ conditional on information up to time 0, is equal to Σ_1, the unconditional mean squared error matrix.

$$\begin{aligned}
\Sigma_{1|0} &= E\Big[(\boldsymbol{x}_1 - \boldsymbol{x}_{1|0})(\boldsymbol{x}_1 - \boldsymbol{x}_{1|0})'\Big] \\
&= E\Big[(\boldsymbol{x}_1 - E(\boldsymbol{x}_1))(\boldsymbol{x}_1 - E(\boldsymbol{x}_1))'\Big] \\
&= \Sigma_1.
\end{aligned}$$

Then the unconditional mean of $\boldsymbol{x}_t$ can be calculated by taking expectations of both sides of (3.C.1),

$$\begin{aligned}
E(\boldsymbol{x}_1) &= E(F\boldsymbol{x}_0) + E(\boldsymbol{\eta}_1) \\
&= F \cdot E(\boldsymbol{x}_0) \\
(I_5 - F) \cdot E(\boldsymbol{x}_1) &= \mathbf{0}_{5\times 1},
\end{aligned}$$

where the second equality comes from $E(\boldsymbol{\eta}_1) = \mathbf{0}$ and the third equality comes from $E(\boldsymbol{x}_0) = E(\boldsymbol{x}_1) = \cdots$ due to the covariance stationarity of $\boldsymbol{x}_t$. Matrix $(I_5 - F)$ is nonsingular, so this equation has the unique solution $E(\boldsymbol{x}_1) = \mathbf{0}$.

$$\boldsymbol{x}_{1|0} = E(\boldsymbol{x}_1) = E(\boldsymbol{x}_0) = \mathbf{0}. \tag{3.C.4}$$

Similarly, the unconditional covariance of $\boldsymbol{x}_t$ can be similarly found by postmulti-

F are lower than unity.

[23] As the log-transformation of likelihood function is monotonic, we can evaluate the likelihood function in the form of logarithm without loss of generality.

plying (3.C.1) with its transpose and taking expectations:

$$\begin{aligned}
E(\boldsymbol{x}_1\boldsymbol{x}_1') &= E[(F\boldsymbol{x}_0 + \boldsymbol{\eta}_1)(\boldsymbol{x}_0'F' + \boldsymbol{\eta}_1')] \\
&= F \cdot E(\boldsymbol{x}_0\boldsymbol{x}_0') \cdot F' + Q \\
&= F \cdot E[\{\boldsymbol{x}_0 - E(\boldsymbol{x}_0)\}\{\boldsymbol{x}_0 - E(\boldsymbol{x}_0)\}'] \cdot F' + Q \\
E[\{\boldsymbol{x}_1 - E(\boldsymbol{x}_1)\}\{\boldsymbol{x}_1 - E(\boldsymbol{x}_1)\}'] &= F \cdot E[\{\boldsymbol{x}_0 - E(\boldsymbol{x}_0)\}\{\boldsymbol{x}_0 - E(\boldsymbol{x}_0)\}'] \cdot F' + Q \\
\Sigma_1 &= F\Sigma_0 F' + Q = F\Sigma_1 F' + Q,
\end{aligned}$$

where the last equality uses $\Sigma_0 = \Sigma_1 = \cdots$ when $\boldsymbol{x}_t$ is covariance-stationary. A closed-form solution to the last result can be obtained in terms of the *vec* operator which stacks the columns vertically. If *vec* operator is applied to both sides,

$$\begin{aligned}
vec(\Sigma_1) &= vec(F\Sigma_1 F') + vec(Q) \\
&= (F \otimes F)_{25\times 25} \cdot vec(\Sigma_1)_{25\times 1} + vec(Q)_{25\times 1} \qquad [\because vec(ABC) = (C' \otimes A) \cdot vec(B)] \\
&= [I_{25} - (F \otimes F)_{25\times 25}]^{-1} \cdot vec(Q) \\
&= vec(\Sigma_{1|0}) = vec(\Sigma_0),
\end{aligned} \tag{3.C.5}$$

where $\otimes$ denotes the Kronecker product.

In sum, we get $\boldsymbol{x}_{1|0}$ as (3.C.4) and $\Sigma_{1|0}$ as (3.C.5). Based on these values, we make predictions on $\boldsymbol{d}_1$ from (3.C.2) as

$$\boldsymbol{d}_{1|0} = G\boldsymbol{x}_{1|0}. \tag{3.C.6}$$

Now we observe $\boldsymbol{d}_1$. The prediction error is

$$\begin{aligned}
\boldsymbol{w}_1 &= \boldsymbol{d}_1 - \boldsymbol{d}_{1|0} \\
&= G(\boldsymbol{x}_1 - \boldsymbol{x}_{1|0}).
\end{aligned}$$

Then, the conditional covariance of the prediction error is

$$\begin{aligned}
\Sigma^{\boldsymbol{d}}_{1|0} &= E(\boldsymbol{w}_1\boldsymbol{w}_1') \\
&= E\left[G(\boldsymbol{x}_1 - \boldsymbol{x}_{1|0})(\boldsymbol{x}_1 - \boldsymbol{x}_{1|0})'G'\right] \\
&= G\Sigma_{1|0}G'.
\end{aligned} \tag{3.C.7}$$

The Gaussian error $\boldsymbol{w}_1$ can then be used to form the likelihood function for $\boldsymbol{d}_1$ as

$$\ln \mathscr{L} = -\frac{3}{2}\ln(2\pi) - \frac{1}{2}\ln|G\Sigma_{1|0}G'| - \frac{1}{2}\boldsymbol{w}_1'(G\Sigma_{1|0}G')\boldsymbol{w}_1. \tag{3.C.8}$$

Therefore, during the prediction stage, maximizing (3.C.8) yields maximum likelihood estimates for F, G, and Q and thus for unknown parameters of the model.

3.C.1.2 Updating State Vector

With new observation $\boldsymbol{d}_1$, the optimal predictor $\boldsymbol{x}_{1|0}$ and its covariance matrix $\Sigma_{1|0}$ are revised using the formula for updating a linear projection and its mean squared error (hereafter MSE) as follows (Hamilton 1994, pp. 379-380, pp. 99-100),[24]

$$\begin{aligned}\boldsymbol{x}_{1|1} &= \boldsymbol{x}_{1|0} + E\left[(\boldsymbol{x}_1 - \boldsymbol{x}_{1|0})(\boldsymbol{d}_1 - \boldsymbol{d}_{1|0})'\right] \cdot E\left[(\boldsymbol{d}_1 - \boldsymbol{d}_{1|0})(\boldsymbol{d}_1 - \boldsymbol{d}_{1|0})'\right]^{-1} \cdot \left[(\boldsymbol{d}_1 - \boldsymbol{d}_{1|0})\right] \\ &= \boldsymbol{x}_{1|0} + E[(\boldsymbol{x}_1 - \boldsymbol{x}_{1|0})(\boldsymbol{x}_1 - \boldsymbol{x}_{1|0})'G'] \cdot E[G(\boldsymbol{x}_1 - \boldsymbol{x}_{1|0})(\boldsymbol{x}_1 - \boldsymbol{x}_{1|0})'G']^{-1} \cdot [\boldsymbol{w}_1] \\ &= \boldsymbol{x}_{1|0} + \Sigma_{1|0}G'[G\Sigma_{1|0}G']^{-1}[\boldsymbol{w}_1]. \end{aligned} \tag{3.C.9}$$

The MSE of this updated projection can be found as,

$$\begin{aligned}\Sigma_{1|1} =& E\left[(\boldsymbol{x}_1 - \boldsymbol{x}_{1|1})(\boldsymbol{x}_1 - \boldsymbol{x}_{1|1})'\right] \\ =& E\left[(\boldsymbol{x}_1 - \boldsymbol{x}_{1|0})(\boldsymbol{x}_1 - \boldsymbol{x}_{1|0})'\right] - E\left[(\boldsymbol{x}_1 - \boldsymbol{x}_{1|0})(\boldsymbol{d}_1 - \boldsymbol{d}_{1|0})'\right] \\ & \times E\left[(\boldsymbol{d}_1 - \boldsymbol{d}_{1|0})(\boldsymbol{d}_1 - \boldsymbol{d}_{1|0})'\right]^{-1} \times E\left[(\boldsymbol{d}_1 - \boldsymbol{d}_{1|0})(\boldsymbol{x}_1 - \boldsymbol{x}_{1|0})'\right] \\ =& \Sigma_{1|0} - \Sigma_{1|0}G'(G\Sigma_{1|0}G')^{-1}G\Sigma_{1|0}. \end{aligned} \tag{3.C.10}$$

[24]The optimal forecast of Y_3 conditional on Y_2 and Y_1 is

$$\begin{aligned}\hat{P}(Y_3|Y_2, Y_1) =& \hat{P}(Y_3|Y_1) + E\left[\{Y_3 - \hat{P}(Y_3|Y_1)\}\{Y_2 - \hat{P}(Y_2|Y_1)\}'\right] \\ & \times E\left[\{Y_2 - \hat{P}(Y_2|Y_1)\}\{Y_2 - \hat{P}(Y_2|Y_1)\}'\right]^{-1} \times \left[Y_2 - \hat{P}(Y_2|Y_1)\right].\end{aligned}$$

Here $\boldsymbol{x}_t = Y_3$, $\boldsymbol{d}_t = Y_2$ and $\boldsymbol{d}_{t-1} = Y_1$. The MSE of this forecast is

$$\begin{aligned}E\left[\{Y_3 - \hat{P}(Y_3|Y_2, Y_1)\}\{Y_3 - \hat{P}(Y_3|Y_2, Y_1)\}'\right] =& E\left[\{Y_3 - \hat{P}(Y_3|Y_1)\}\{Y_3 - \hat{P}(Y_3|Y_1)\}'\right] \\ & - E\left[\{Y_3 - \hat{P}(Y_3|Y_1)\}\{Y_2 - \hat{P}(Y_2|Y_1)\}'\right] \times E\left[\{Y_2 - \hat{P}(Y_2|Y_1)\}\{Y_2 - \hat{P}(Y_2|Y_1)\}'\right]^{-1} \\ & \times E\left[\{Y_2 - \hat{P}(Y_2|Y_1)\}\{Y_3 - \hat{P}(Y_3|Y_1)\}'\right].\end{aligned}$$

3.C.1.3 Prediction Again

For $\boldsymbol{x}_2$, taking expectations on both sides of (3.C.1) conditional on $\boldsymbol{d}_1$ gives rise to expectation of $\boldsymbol{x}_2$ conditional on $\boldsymbol{d}_1$.

$$\begin{aligned}\boldsymbol{x}_{2|1} =& E(\boldsymbol{x}_2|\boldsymbol{d}_1) + E(\boldsymbol{\eta}_2|\boldsymbol{d}_1)\\ =& F\boldsymbol{x}_{1|1}.\end{aligned} \tag{3.C.11}$$

Now let's substitute (3.C.9) into (3.C.11), then we get the recursive structure

$$\begin{aligned}\boldsymbol{x}_{2|1} =& F(\boldsymbol{x}_{1|0} + \Sigma_{1|0}G'[G\Sigma_{1|0}G']^{-1}[\boldsymbol{w}_1])\\ =& F\boldsymbol{x}_{1|0} + F\Sigma_{1|0}G'[G\Sigma_{1|0}G']^{-1}[\boldsymbol{w}_1]\\ =& F\boldsymbol{x}_{1|0} + K_1[\boldsymbol{w}_1],\end{aligned} \tag{3.C.12}$$

where K_1 is Kalman gain matrix which gives weight to new information when one makes new prediction.[25] Thus Kalman gain is similar to the error correction mechanism. This means that old prediction is updated to new prediction when new information $\boldsymbol{w}_1 = (\boldsymbol{d}_1 - \boldsymbol{d}_{1|0})$ becomes available.

Similarly, $\Sigma_{2|1}$ can be obtained by postmultiplying (3.C.1) by its transpose and taking expectation.

$$\begin{aligned}\Sigma_{2|1} =& E\left[(\boldsymbol{x}_2 - \boldsymbol{x}_{2|1})(\boldsymbol{x}_2 - \boldsymbol{x}_{2|1})'\right]\\ =& E\left[(F\boldsymbol{x}_1 + \boldsymbol{\eta}_2 - F\boldsymbol{x}_{1|1})(F\boldsymbol{x}_1 + \boldsymbol{\eta}_2 - F\boldsymbol{x}_{1|1})'\right]\\ =& E\left[\{F(\boldsymbol{x}_1 - \boldsymbol{x}_{1|1}) + \boldsymbol{\eta}_2\}\{F(\boldsymbol{x}_1 - \boldsymbol{x}_{1|1}) + \boldsymbol{\eta}_2\}'\right]\\ =& F\Sigma_{1|1}F' + FE\{(\boldsymbol{x}_1 - \boldsymbol{x}_{1|1})\boldsymbol{\eta}_2'\} + E\{\boldsymbol{\eta}_2(\boldsymbol{x}_1 - \boldsymbol{x}_{1|1})'\}F' + E(\boldsymbol{\eta}_2\boldsymbol{\eta}_2')\\ =& F\Sigma_{1|1}F' + Q \quad [\because E(\boldsymbol{x}_1 - \boldsymbol{x}_{1|1}) \cdot E(\boldsymbol{\eta}_2') = \mathbf{0}_{5\times 5}].\end{aligned} \tag{3.C.13}$$

Now substituting (3.C.10) into (3.C.13) gives rise to a recursive structure

$$\begin{aligned}\Sigma_{2|1} =& F[\Sigma_{1|0} - \Sigma_{1|0}G'(G\Sigma_{1|0}G')^{-1}G\Sigma_{1|0}]F' + Q\\ =& F\Sigma_{1|0}F' - F\Sigma_{1|0}G'(G\Sigma_{1|0}G')^{-1}G\Sigma_{1|0}F' + Q\\ =& F\Sigma_{1|0}F' - K_1G\Sigma_{1|0}F' + Q.\end{aligned} \tag{3.C.14}$$

[25] Alternatively, one can define Kalman gain matrix as $K_1 = \Sigma_{1|0}G'[G\Sigma_{1|0}G']^{-1}$ in updating equation. However, the implication is exactly the same.

On the other hand, new forecast $\boldsymbol{d}_{2|1}$ and the conditional variance of the prediction error $\Sigma^{\boldsymbol{d}}_{2|1}$ are given respectively as

$$\begin{aligned} \boldsymbol{d}_{2|1} &= G_1 \boldsymbol{x}_{2|1} \\ \Sigma^{\boldsymbol{d}}_{2|1} &= G_1 \Sigma_{2|1} G_1'. \end{aligned} \tag{3.C.15}$$

3.C.1.4 Iteration of Predicting-Updating

Given the recursive structure, we then iterate on the following four prediction equations for $\boldsymbol{x}_3, \cdots, \boldsymbol{x}_T$,

$$\begin{aligned} \boldsymbol{x}_{t+1|t} &= F\boldsymbol{x}_{t|t-1} + K_t[\boldsymbol{w}_t] \\ \Sigma_{t+1|t} &= F\Sigma_{t|t-1}F' - K_t G \Sigma_{t|t-1} F' + Q \\ \boldsymbol{d}_{t+1|t} &= G\boldsymbol{x}_{t+1|t}, \end{aligned}$$

and after we observe $\boldsymbol{d}_{t+1}$,

$$\Sigma^{\boldsymbol{d}}_{t+1|t} = G\Sigma_{t+1|t}G'.$$

When the observations are normally distributed, the sample log likelihood function value can be computed as (3.C.16),

$$\ln \mathscr{L} = -\frac{3}{2}\sum_{t=1}^{T} \ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T} \ln |G\Sigma_{t|t-1}G'| - \frac{1}{2}\sum_{t=1}^{T} \boldsymbol{w}_t'(G\Sigma_{t|t-1}G')\boldsymbol{w}_t. \tag{3.C.16}$$

At time t, one maximizes the accumulated $\ln \mathscr{L}$ in (3.C.16) up time t. This gives rise to the maximum likelihood estimates for F, G and Q and thus for the parameters of interest.

Finally, we can update the state vector and its MSE with new information from the new data as follows,

$$\begin{aligned} \boldsymbol{x}_{t|t} &= \boldsymbol{x}_{t|t-1} + K_t \boldsymbol{w}_t \\ \Sigma_{t|t} &= \Sigma_{t|t-1} + K_t \boldsymbol{w}_t, \end{aligned}$$

where K_t is the Kalman gain matrix and $\boldsymbol{w}_t$ is the prediction error for $\boldsymbol{d}_t$.

3.C.2 Model Evaluation

3.C.2.1 Variance Decomposition Analysis

Given the estimated parameter set implied in F, G and Q, one can use forecast error variance decomposition to compute the proportion of the variability of endogenous variables (output, consumption, investment and hours worked) at time $t+s$ due to two orthogonal exogenous shocks to productivity and measurement errors. This enables us to evaluate the relative importance of the shocks in the system.

For this purpose, it is convenient to express this state space form into vector moving average representation, VMA(∞), using a lag operator as long as the eigenvalues of F have modulus less than unity (i.e., $\boldsymbol{x}_t$ is covariance-stationary). Notice first $\boldsymbol{x}_t = F\boldsymbol{x}_{t-1} + \boldsymbol{\eta}_t \rightarrow (I - FL)\boldsymbol{x}_t = \boldsymbol{\eta}_t$. Then,

$$\begin{aligned}
\boldsymbol{x}_t =& (I - FL)^{-1}\boldsymbol{\eta}_t \\
=& (I + FL + \cdots)\boldsymbol{\eta}_t \\
=& \boldsymbol{\eta}_t + F\boldsymbol{\eta}_{t-1} + F^2\boldsymbol{\eta}_{t-2} + \cdots \\
=& \sum_{j=0}^{\infty} F^j \boldsymbol{\eta}_{t-j}.
\end{aligned}$$

This implies

$$\begin{aligned}
\boldsymbol{x}_{t+k} =& \sum_{j=0}^{\infty} F^j \boldsymbol{\eta}_{t+k-j} \\
=& \boldsymbol{\eta}_{t+k} + F\boldsymbol{\eta}_{t+k-1} + F^2\boldsymbol{\eta}_{t+k-2} + \cdots + F^{k-1}\boldsymbol{\eta}_{t+k-(k-1)} \\
& + \underbrace{F^k\boldsymbol{\eta}_{t+k-k} + F^{k+1}\boldsymbol{\eta}_{t+k-(k+1)} + \cdots}_{\boldsymbol{x}_{t+k|t}},
\end{aligned}$$

and the conditional expectation for time $t+k$ at time t is given as

$$\begin{aligned}
\boldsymbol{x}_{t+k|t} =& F^k\boldsymbol{\eta}_{t+k-k} + F^{k+1}\boldsymbol{\eta}_{t+k-(k+1)} + \cdots \\
=& \sum_{j=k}^{\infty} F^j \boldsymbol{\eta}_{t+k-j}.
\end{aligned}$$

Thus, forecast error for a k-step ahead can be acquired as

$$\begin{aligned}
\boldsymbol{x}_{t+k} - \boldsymbol{x}_{t+k|t} =& \sum_{j=0}^{\infty} F^j \boldsymbol{\eta}_{t+k-j} - \sum_{j=k}^{\infty} F^j \boldsymbol{\eta}_{t+k-j} \\
=& \boldsymbol{\eta}_{t+k} + F\boldsymbol{\eta}_{t+k-1} + F^2\boldsymbol{\eta}_{t+k-2} + \cdots + F^{k-1}\boldsymbol{\eta}_{t+k-(k-1)} \\
=& \sum_{j=0}^{k-1} F^j \boldsymbol{\eta}_{t+k-j}.
\end{aligned}$$

Postmultiplying this forecast error with its transpose and taking expectation gives rise to a covariance matrix of state vector conditional on information up to time t, also known as a MSE matrix,

$$\begin{aligned}
\Sigma_{t+k|t} =& E[(\boldsymbol{x}_{t+k} - \boldsymbol{x}_{t+k|t})(\boldsymbol{x}_{t+k} - \boldsymbol{x}_{t+k|t})'] \\
=& E[(\boldsymbol{\eta}_{t+k} + F\boldsymbol{\eta}_{t+k-1} + F^2\boldsymbol{\eta}_{t+k-2} + \cdots)(\boldsymbol{\eta}_{t+k} + F\boldsymbol{\eta}_{t+k-1} + F^2\boldsymbol{\eta}_{t+k-2} + \cdots)'] \\
=& Q + FQF' + F^2QF'^2 + \cdots + F^{k-1}QF'^{k-1} \quad [\because E(\boldsymbol{\eta}_t\boldsymbol{\eta}_s') = 0_{5\times 5} \text{ for } t \neq s].
\end{aligned}$$

On the other hand, unconditional covariance matrix can be obtained using *vec* operator,

$$\begin{aligned}
vec(\Sigma) =& vec(F\Sigma F') + vec(Q) \\
=& (F \otimes F)_{25\times 25} \cdot vec(\Sigma)_{25\times 1} + vec(Q)_{25\times 1} \\
=& [I_{25} - (F \otimes F)]^{-1} \cdot vec(Q).
\end{aligned}$$

Similarly, the conditional variance of the prediction error can be expressed as

$$\begin{aligned}
\Sigma^{\boldsymbol{d}}_{t+k|t} =& E[(\boldsymbol{d}_{t+k} - \boldsymbol{d}_{t+k|t})(\boldsymbol{d}_{t+k} - \boldsymbol{d}_{t+k|t})'] \\
=& E[\boldsymbol{w}_{t+k}\boldsymbol{w}_{t+k}'] \\
=& E\left[G(\boldsymbol{x}_{t+k} - \boldsymbol{x}_{t+k|t})(\boldsymbol{x}_{t+k} - \boldsymbol{x}_{t+k|t})'G'\right] \\
=& G\Sigma_{t+k|t}G',
\end{aligned}$$

and unconditional variance of the prediction error is given as

$$\Sigma^{\boldsymbol{d}} = G\Sigma G'.$$

Let us define $\boldsymbol{\theta}$ as the unknown structural parameter vector, $\boldsymbol{\theta}_0$ as the estimated structural parameter vector, and H (inverse of hessian matrix) as the covariance matrix of these estimated parameters. Then, it is known that asymptotically,

$$\boldsymbol{\theta} \sim N(\boldsymbol{\theta}_0, H).$$

Note that since $\Sigma_{t+k|t} = Q + FQF' + F^2QF'^2 + \cdots + F^{k-1}QF'^{k-1}$ is a nonlinear function of underlying structural parameter vector of $\boldsymbol{\theta}$, $\Sigma^{\boldsymbol{d}}_{t+k|t}(= G\Sigma_{t+k|t}G')$ is also $\Sigma^{\boldsymbol{d}}_{t+k|t} = g(\boldsymbol{\theta})$. Now one can take advantage of the delta method in Appendix 3.D to obtain the approximate covariance matrix of the forecast errors.[26]

$$\Sigma^{\boldsymbol{d}}_{t+k} = \left[\frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'} H \frac{\partial g(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right]_{21\times 21} = \nabla g H \nabla g',$$

where ∇g is the gradient for the nonlinear function.

3.C.2.2 Comparing Forecast Performance between Models

Forecasts for data can be generated using the system of (3.C.1) and (3.C.2). Note first from (3.C.2) that k-period-ahead predictions on observable variables at time t are

$$E(\boldsymbol{d}_{t+k}|\boldsymbol{d}_t) = \boldsymbol{d}_{t+k|t} = G\boldsymbol{x}_{t+k|t}.$$

Hence, in order to make predictions on $\boldsymbol{d}_{t+k|t}$, we need to obtain first $\boldsymbol{x}_{t+k|t} = E(\boldsymbol{x}_{t+k}|\boldsymbol{d}_t)$. From (3.C.1), $\boldsymbol{x}_{t+k|t} = F^k\boldsymbol{x}_{t|t}$. So, forecasts from the DSGE model can be generated as follows

$$E(\boldsymbol{d}_{t+k}|\boldsymbol{d}_t) = G\boldsymbol{x}_{t+k|t} = GF^k\boldsymbol{x}_{t|t}.$$

Next, to evaluate the model's forecast accuracy, this essay uses root mean square error (simply, RMSE), in which low values imply better performance in forecasting.

3.C.2.3 Generating Smoothed Estimates of the Shocks

The estimated model can be used to produce estimates of the shocks to the productivity and measurement errors. Then with these estimates, one can verify the orthogonality assumption between these shocks. Hamilton (1994, pp. 394-397) shows how to generate

[26] Alternatively, one can simply use the bootstrap sampling (or Monte Carlo sampling), which is a randomization technique that permits estimation of parameters with unknown distributions. After generating a number of bootstrap samples using the estimates, one can calculates the standard errors of the estimates as their standard deviations.

a sequence of smoothed estimates for state variables $\{\boldsymbol{x}_{t|T}\}_{t=1}^{T}$, where

$$\boldsymbol{x}_{t|T} = E(\boldsymbol{x}_t|\boldsymbol{d}_T, \boldsymbol{d}_{T-1}, \cdots, \boldsymbol{d}_1).$$

To begin with, construct a sequence $\{J_t\}_{t=1}^{T}$ using Hamilton's equation (13.6.11):

$$J_t = \Sigma_{t|t} F' \Sigma_{t+1|t}^{-1}.$$

Then note that $\{\boldsymbol{x}_{T|T}\}_{t=1}^{T}$ is just the last element of $\{\boldsymbol{x}_{t|t}\}_{t=1}^{T}$. From this terminal condition, the rest of the sequence can be generated recursively using Hamilton's equation (13.6.16):

$$\boldsymbol{x}_{T-j|T} = \boldsymbol{x}_{T-j|T-j} + J_{T-j}(\boldsymbol{x}_{T-j+1|T} - \boldsymbol{x}_{T-j+1|T-j}),$$

for $j = 1, 2, \cdots, T-1$. For example, when $j = 1$,

$$\boldsymbol{x}_{T-1|T} = \boldsymbol{x}_{T-1|T-1} + J_{T-1}(\boldsymbol{x}_{T|T} - \boldsymbol{x}_{T|T-1}).$$

We know all the values on the right, so we can obtain the smoothed estimates for state vector on the left. For $j = 2$,

$$\boldsymbol{x}_{T-2|T} = \boldsymbol{x}_{T-2|T-2} + J_{T-2}(\boldsymbol{x}_{T-1|T} - \boldsymbol{x}_{T-1|T-2}).$$

Again, we can obtain the smoothed estimates for state vector on the left. And its MSE is given by

$$\Sigma_{T-j|T} = \Sigma_{T-j|T-j} + J_{T-j}(\Sigma_{T-j+1|T} - \Sigma_{T-j+1|T-j})J_{T-j}'.$$

Finally, we can construct the estimated shocks as follows,

$$\boldsymbol{x}_{T-j|T} - F\boldsymbol{x}_{T-(j+1)|T},$$

for $j = 0, 1, \cdots, T-1$.

3.D Delta Method

As shown in Greene (2003, pp.913–914), the delta method expands a function of a random variable around its mean using a linear Taylor approximation and then derives the variance of the function. Consider the case in which we want to know the standard error not of z, but rather of a function of z, $g(z)$ such as $2z$ and z^2. If $g(z)$ is a continuous differentiable function, $g(z)$ can be approximated by a linear Taylor series as (3.D.1).

$$g(z) = g(\mu) + (z - \mu)g'(\mu), \tag{3.D.1}$$

where μ is the mean of z and $g'(\cdot) = \frac{dg}{dz}$.

Then, the variance of $g(z)$ can be approximated as (3.D.2).

$$Var\big[g(z)\big] = \big[g'(\mu)\big]^2 Var(z). \tag{3.D.2}$$

Now it is straightforward to notice that the approximate standard errors of $2z$ and z^2 are $2[SE(z)]$ and $2z[SE(z)]$, respectively.

The above result can be extended to the multi-dimensional case and the variance-covariance matrix of the function $\boldsymbol{g}(\boldsymbol{z})$ can be approximated as (3.D.3).

$$Var\big[\boldsymbol{g}(\boldsymbol{z})\big] = \boldsymbol{g}'(\boldsymbol{\mu})Var(\boldsymbol{z})\big[\boldsymbol{g}'(\boldsymbol{\mu})\big]^T, \tag{3.D.3}$$

where $\boldsymbol{z}$ is a $K \times 1$ column vector, $\boldsymbol{g}(\boldsymbol{z})$ is a $J \times 1$ column vector of continuous functions, $\boldsymbol{g}(\cdot)$ is the $J \times K$ matrix of the first derivatives, T is the transpose operator and $Var(\boldsymbol{g}(\boldsymbol{z}))$ is the $J \times J$ variance-covariance matrix of $\boldsymbol{g}(\boldsymbol{z})$.

Chapter 4

BAYESIAN ESTIMATION OF MONETARY BUSINESS CYCLE MODELS IN KOREA

4.1 Introduction

Estimating micro-founded Dynamic Stochastic General Equilibrium (henceforth, DSGE) models has become increasingly popular in the analysis of monetary economics (see, for example, An and Schorfheide 2007; Smets and Wouters 2007; and Rabanal and Rubio-Ramírez 2005). Most literature on monetary DSGE models introduces money to the model through money-in-utility-function or cash-in-advance constraints.[1] The models in this chapter closely follow Christiano and Eichenbaum (1992) and Nason and Cogley (1994) which discuss two versions of monetary DSGE models with cash-in-advance constraints: a baseline cash-in-advance (henceforth, CIA) model and a limited participation (henceforth, LP) model using the calibrated parameters. The form of the cash-in-advance model relies on which purchases are subjective to the CIA requirements. The current two models assume that firms should borrow cash to pay wages in advance while households have to hold cash to purchase consumption goods.[2]

[1]In a money-in-utility (MIU) model, money is valued because it enters the utility function. This can be understood as a short-cut in the sense that money facilitates the transactions by reducing shopping time or by providing liquidity services (Walsh 2003). On the other hand, CIA models seem to be more micro-founded to account for why the agents value money.

[2]On the other hand, Lucas (1982), Svensson (1985), and Cooley and Hansen (1989) apply the CIA constraint to only consumption while Lucas and Stokey (1987), and Cooley and Hansen (1995) apply

In this case, the limited participation is related with the timing of decision-making, which implies that some agents must make a decision before observing the exogenous shocks.[3] However, contrary to the above articles, this essay attempts to estimate those parameters in monetary DSGE models using the Bayesian approach. Thus, our goal is to take the two models to Korean data and to assess the role of shocks, in particular monetary shock, to the business fluctuation in Korean economy.[4]

As well noted in literature (see, Nason and Cogley 1994; Christiano and Eichenbaum 1995; more recently Leeper and Roush 2003), there is a key difference between the CIA model and the LP model regarding information structure. While in the CIA model households make a decision on deposit after they observe the exogenous shocks, in an LP model households make a decision on deposit before they observe those shocks. Accordingly, in the LP model, households have only limited access to financial markets since they cannot readjust their deposit decision at least in the short run while firms have direct access to financial markets. In other words, the portfolio allocation of households is perfectly flexible in the CIA model whereas it is to the some extent rigid in the LP model.[5] This limited participation assumption appears consistent with reality in which most households save some amount of cash into saving accounts which are costly to withdraw until maturity. The important implication of this distinction will be discussed in Section 4.4 when we compare the effect of a monetary shock on the interest rate in the two models.

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it to a subset of consumption goods.

[3]Notice that, in financial economics, limited participation often refers to the limited participation of some agents in the equity markets.

[4]On the other hand, Schorfheide (2000) estimates the CIA model and the portfolio adjustment cost (PAC) model with the U.S. data. See, for details, footnote 5.

[5]On the other hand, Christiano and Eichenbaum (1992, 1995) suggest the portfolio adjustment cost (simply, PAC) model in order to generate a persistent liquidity effects. Although this model has the same information structure of LP model, there is an additional factor to capture the persistency of liquidity effect. The basic idea is to make the financial sector remain more liquid than the real sector for several periods after a monetary shock. To produce the persistent liquidity effect, they assume that adjusting the available fund, which is the money carried from the previous period after making a deposit, is costly by reducing the available time for leisure in the utility function.

Consequently, households increase the available fund by a relatively small amount after the money shock because of adjustment costs. Hence, commercial banks and firms have to absorb a disproportionately large share of the economy's funds for several periods. Indeed, Schorfheide (2000) and Nason and Cogley (1994) compare the CIA model with the PAC model. The problems of PAC model, nonetheless, are two-fold. First, it is very hard to calibrate the values of additional parameters as well as to find any economic rationale for them. Second, and more importantly, the restriction on adjustment seems to be *ad-hoc* in nature (Dow 1995).

Recently, nearly every central bank adopts the short-term interest rate as its instrument and a type of Taylor's (1993) rules as a policy rule in which the central bank responds to only the output gap and the departure of inflation from the target rate by adjusting the short-term interest rate. This now-popular new Keynesian models following Rotemberg and Woodford (1997) and Clarida, Gali, and Gertler (1999) deemphasize the role of money in monetary policy analysis by assuming infinitely elastic money supply. This essay, nonetheless, chooses to explicitly include money in the models following the suggestions of Leeper and Roush (2003) and Smets (2003), rather than assuming a cashless model economy like Woodford (2003) in which money stock is redundant for determining output and inflation once the interest rate is present. The main motivation of this choice is that disappearance of money does not appear to reflect the reality for the following reasons.

First, most central banks which adopt inflation targeting still monitor the movements of money growth when they implement the monetary policy (Goodfriend 2007). Thus, the movements of monetary aggregates are still considered as crucial indicators when they evaluate the economic conditions. The most evident example is that the European Central Bank assigns a prominent role for money and monetary analysis in its two-pillar monetary policy strategy (European Central Bank 2004).[6] From this perspective, in his article (*No Money, No Inflation*) Mervyn King (2002) argues that "the disappearance of money from the models used by economists is, ..., more apparent than real."[7]

[6]The first pillar is "economic analysis", which is relevant for the short-to-medium-term. This analysis "takes account of the fact that price developments over those horizons are influenced largely by the interplay of supply and demand in the goods, services and factor markets." On the other hand, the second pillar is "monetary analysis", which is relevant for the medium-to-long-term. This analysis "exploits the long-run link between money and prices." These two pillars provide "cross-checks" for one another in order to achieve the price stability (ECB, 2004, pp. 55–66). Therefore, money serves as a nominal anchor in the long run and the ECB is also required to use the information in monetary aggregates systematically.

[7]Most relevantly, Robert Lucas (2006) states that "central banks that do not make explicit use of money supply data have recent histories of inflation control that are quite as good as the record of the ECB. I am concerned that this encouraging but brief period of success will foster the opinion, already widely held, that the monetary pillar is superfluous, and lead monetary policy analysis back to the muddled eclecticism that brought us the 1970s inflation."

In addition, he argues that "money supply measures play no role in the estimation, testing, or policy simulation of these (New Keynesian) models. A role for money in the long run is sometimes verbally acknowledged, but the models themselves are formulated in terms of deviations from trends that are themselves determined somewhere off stage. ...This remains an unresolved issue on the frontier of macroeconomic theory. Until it is resolved, monetary information should continue be used as a kind of add-on or cross-check, just as it is in ECB policy formulation today (p. 168)."

In addition, in the New Keynesian models, when the short-term interest rate affects output, the effect of the change in monetary aggregates is not much considered. In reality, however, the central banks cannot affect the short-term interest rate just by promulgating the target rate. In other words, the changes of the short-term interest rate essentially involve those of monetary aggregates. In fact, Leeper and Roush (2003) find evidence that the monetary aggregate as well as the short-term interest rate plays an essential role in the transmission of monetary policy in the United States.

Third, the money supply is not perfectly elastic in reality. For example, when the monetary aggregates rise rapidly, individuals expect the inflation to rise soon. This point is well illustrated in the U.S. experience. That is, the Volcker Fed in the U.S. succeeded in taming the long-lasting inflationary expectation of the public in the early 1980s and vindicated the monetarist message, by targeting the money growth rates (Goodfriend 2007).[8]

Lastly but most importantly, until the early of 1999, the monetary policy instrument in Korea had been monetary aggregates and the role of interest rate in monetary policy had been extremely limited mostly due to the lack of liquid bond markets.

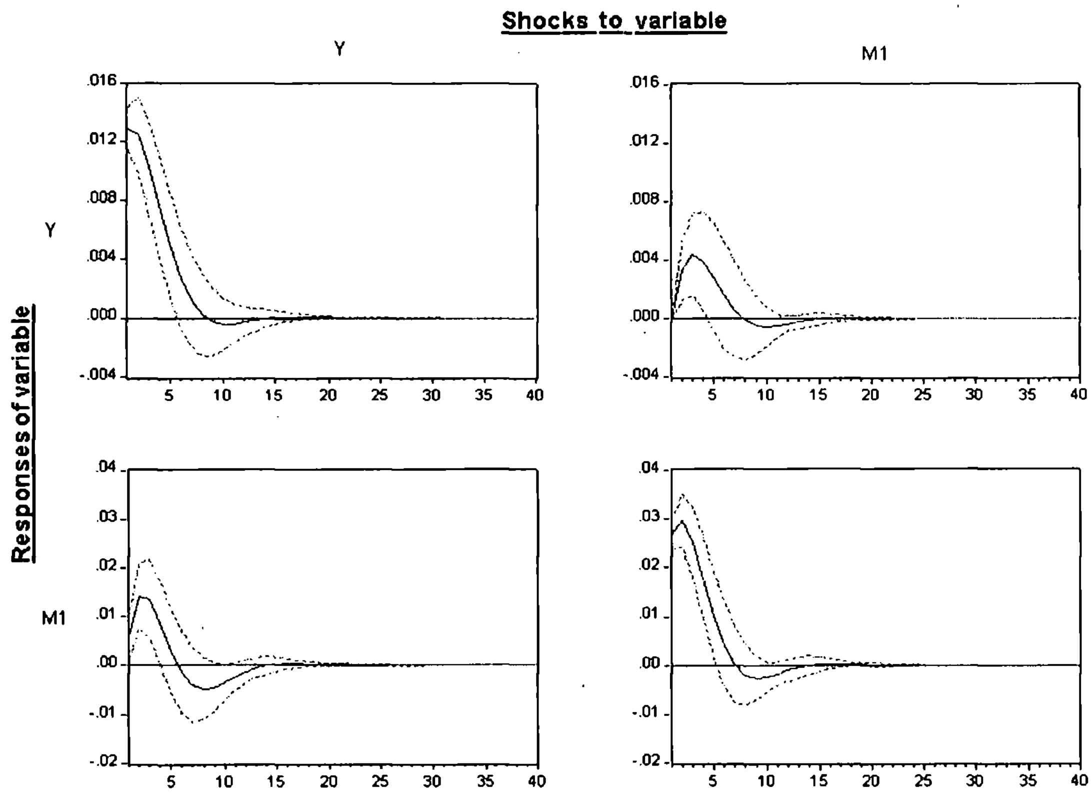
There are two important stylized facts which any plausible monetary DSGE models should account for. First important observation needs to be addressed is that an expansionary monetary shock tends to increase the output. To examine this issue, we estimate a bivariate vector autoregression model as in (4.1.1),

$$\mathbf{x}_t = \boldsymbol{\mu} + \boldsymbol{\Pi}_1 \mathbf{x}_{t-1} + \boldsymbol{\Pi}_2 \mathbf{x}_{t-2}, \tag{4.1.1}$$

where $\mathbf{x}_t$ is a 2 dimensional column vector of the detrended log of real GDP and the detrended log of $M1$ at time t, $\boldsymbol{\mu}$ is a 2 dimensional constant vector, and $\boldsymbol{\Pi}_1$ and $\boldsymbol{\Pi}_2$ are 2×2 coefficient matrices.

Figure 4.1 plots the impulse responses functions of (4.1.1) when the identification is Choleski-type with money ordered last. The upper-right panel clearly shows that output responds to the monetary shock in the short run but the effect vanishes in the long run.

[8]Goodfriend (2007) asserts that the Volcker disinflation justified the main monetarist view. He states that "monetary policy alone — without wage, price, or credit controls , and without supportive fiscal policy — could reduce inflation permanently at a cost to output and employment that, while substantial, was far less than in common Keynesian scenarios."



Source : Bank of Korea
a. The logarithms of M1 and real GDP are detrended using the HP filter and the estimated vector autoregression model includes two lags and a constant based on the Schwartz criterion.
b. Each cell depicts the forty quarter response (percentage deviation from initial level) of the given row variable to a shock to the given column variable (one standard deviation). Impulse responses are orthogonalized recursively in the order shown. Thus, M1 is assumed to be more endogenous than real GDP. Dashed lines represent two standard error bands.

Figure 4.1: Impulse Response Functions of a Bivariate VAR of GDP and $M1$: 1970Q1–2007Q3

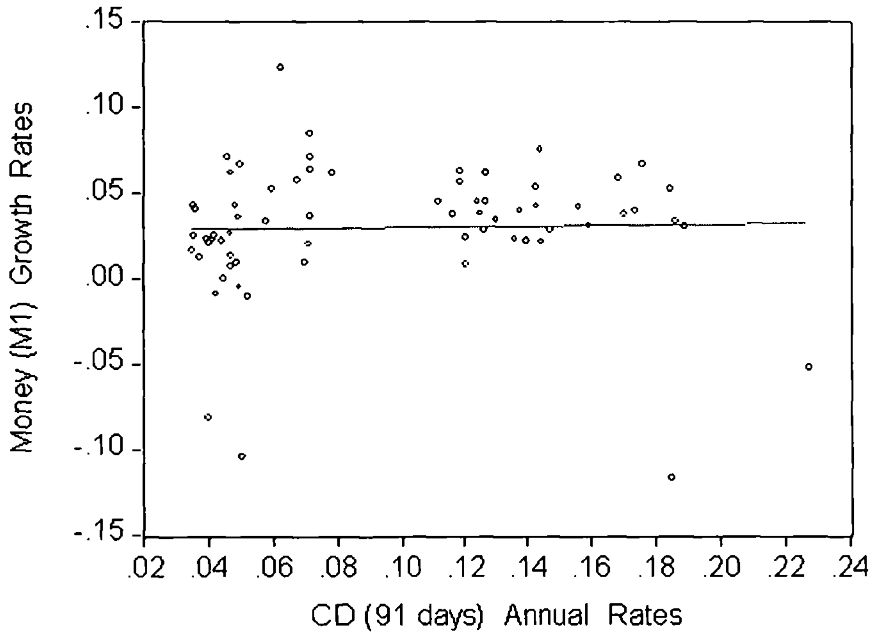
Second, although many monetary economists expect that an unexpected increase in the money supply reduces the short-term interest rate, a simple correlation analysis between the short-term interest rate and the growth rate of monetary aggregates such as $M1$ and $M2$ in the United States and other countries persistently confirms that an expansionary monetary policy tends to be accompanied by the rise of interest rate rather than the fall of it. These results are also supported by the recursive VAR model (Leeper and Gordon 1992; Leeper, Sims, and Zha 1996) and more recently by the identified VAR model (Bernanke, Boivin, and Eliasz 2005). This is referred to be the liquidity puzzle (Leeper and Roush 2003).

Figure 4.2 illustrates the liquidity puzzle for the Korean economy for 1991Q1-2007Q3 using M1 growth rate and the 91-day certificate of deposit rate. This slight but positive correlation is potentially a puzzle in the sense that an increase in the money supply would require a decline in the interest rate to persuade households to hold larger money balances and thus to recover money market equilibrium. Contrary to previous literature, however, this essay argues that the liquidity puzzle is not a real puzzle if the liquidity effect (downward pressure to the interest rate) is overwhelmed by the anticipated inflation effect (upward pressure to the interest rate) in monetary DSGE models.[9]

It might be true that these two stylized fact can be better explained by simple sticky price models (friction in the commodity market) rather than limited participation models (friction in the financial market).[10] However, as Kiley (2000) argues, nominal price rigidities might be less important in much higher inflation country like Korea. For example, menu cost theory, suggested by Akerlof and Yellen (1985) and Mankiw (1985), implies that price stickiness should be smaller in high-inflation countries. In addition, although limited participation models often predict the decline in nominal interest rate, they are also capable of generating the rise of nominal interest rates in response to a positive money supply shock in principle. As Christiano and Eichenbaum (1992) argue, this happens when the liquidity effect dominates the anticipated inflation effect in limited participation models. In fact, as shown in Section 4.6.2, this case seems to be consistent with Korean economy. From this perspective, the current essay

[9]Strongin (1995) and Christiano, Eichenbaum, and Evans (1996) attempt to solve the liquidity puzzle by focusing on narrow monetary aggregates such as nonborrowed reserves.

[10]For example, the first stylized fact can be easily accounted for the sticky price model used in Yun (1996) and Ellison and Scott (2000).



Source : Bank of Korea
Note: The slope of the solid line stands for the correlation between money growth and short-term interest rate.

Figure 4.2: Plot of Money Growth and Short-term Interest Rate

attempts to examine whether a small friction in the financial market is able to explain those two stylized facts in Korean economy. As it is often hard to disentangle which frictions do most work to generate a certain feature of stylized facts, this chapter may serve as a starting point to more complex but realistic models.[11]

This essay discusses the models' prediction on theses two stylized facts after solving the linearized DSGE models. To take the models to the data, this chapter uses the Kalman filter to evaluate the likelihood function of a log-linear approximation of the model and then samples from the posterior distribution using one of the Markov Chain Monte Carlo (henceforth, MCMC) posterior simulator: the random walk metropolis algorithm suggested by An and Schorfheide (2007).[12] Based on these draws one can numerically approximate the relevant moments of the posterior distributions and make inference about model parameters. Then the relative importance of the technology shock and monetary shock and the propagation mechanism can be evaluated, too. Next fitness of each model is assessed based on the Bayes' factor, which is the ratio between the marginal likelihood of CIA model and that of LP model. During this comparison, this essay emphasizes the importance of timing of the deposit decision-making in the two models and demonstrates that the LP model is more advantageous to capture the above two stylized facts in Korean business cycles.

We take the Bayesian approach rather than the Maximum-likelihood approach to estimate DSGE models. The main motivation of this choice is that through the Bayes' theorem, Bayesian estimation of DSGE models naturally links the calibration method (prior information) with maximum likelihood estimation (likelihood from the data) to obtain the posterior distribution of the structural parameters. In this case, prior knowledge can be often obtained from economic theory, long-run averages of aggregate data and empirical findings in microeconomic studies. We expect that prior information helps us to sharpen our inference based on maximum likelihood (ML) estimation in two respects.

[11]For example, Keen (2004) analyzes the impact of a money supply shock in a model with limited participation and sticky price. Christiano, Eichenbaum, and Evans (2005) present a model which combines Calvo-style nominal price and wage contracts with habit formation in preferences for consumption, adjustment costs in investment, variable capital utilization and cash in advance constraint to firms. They argue that this estimated model can accounts for the inflation inertia and output persistence in the U.S.

[12]The Bayesian estimation is programmed in Dynare, a software kindly provided by Michel Juillard and his team.

First of all, as well documented in the DSGE literature using ML estimation, the likelihood often peaks in regions of the parameter space that are contradictory with common observations, leading to the unrealistic parameter estimates (Altug 1989; Ireland 2004; Christensen and Dib 2006).[13] Then, using the informative priors prevents the posterior distribution from peaking at strange points where the likelihood locally peaks. In other words, priors can be seen as weights on the likelihood function in order to give more importance to certain areas of the parameter subspace.

More importantly, the inclusion of priors is also useful to identify the parameters of interests. It is quite common in estimation process that different values of structural parameters yield the same joint distribution for observables. This means that the joint distribution is often flat over a subspace of some structural parameter values.[14] But the weighting of the likelihood with prior densities often leads to adding just enough curvature in the posterior distribution to facilitate numerical maximization (Canova 2007, p.443).

The rest of the chapter is organized as follows. Since the timing assumption yields an important distinction between a CIA model and an LP model, Section 4.2 clarifies time lines of the two models before discussing the models in details. Section 4.3 of the essay first presents the setup of the CIA model and then finds the equilibrium conditions for it. Section 4.4 briefly accounts for the key difference of the LP model and its influence on the equilibrium conditions. More importantly, this section discusses what each model predicts on the movements of interest rate and output qualitatively to an expansionary monetary shock. Section 4.5 discusses the fundamental background for Bayesian inference and then shows that how Bayesian estimation of DSGE models can be implemented. It also provides the rationale on how the prior densities are chosen. Section 4.6 presents the empirical results for two versions of DSGE models: posterior densities for structural parameters, impulse response functions and variance decomposition. In addition, it assesses the goodness-of-fit of the two models based on Bayes' factor and shows that the LP model is better to match up with the stylized facts of business cycles in Korean economy during the examined period. Moreover, based on the LP model, this essay compares two sub-sample periods: 1973:Q1 to 1997:Q4 and

[13] An and Schorfheide (2007) call this problem as "dilemma of absurd parameter estimates."

[14] We may solve this problem by increasing the length of observations in ML estimation. However, when we have to confront the data from an emerging market like Korea, the length is usually very short compared to those of developed economy.

1998:Q1 to 2006:Q3 to investigate how several institutional and policy changes during the Korean financial crisis have affected the economy. Finally, Section 4.7 concludes.

4.2 Time Lines of CIA Model and LP Model

Both CIA and LP models introduce money via cash-in-advance (CIA) constraints to consumption for households and to wage payment for firms.[15] In both models, monetary shocks and productivity shocks are the sources of uncertainty. Under these circumstances, money is valued because it can be used to purchase cash goods (i.e., consumption C_t). Furthermore, the two models have the same four types of participants: households, firms and commercial banks, and a central bank. It is assumed that households own the whole shares of firms and banks and they start period t with money stock M_t which is carried from period $t-1$. That is, the timing of the subscript of M_t is the beginning of period t.

The only difference arises from the information structure in each model. Since timing assumptions yields an important distinction between a CIA model and an LP model, it is convenient to clarify the time lines of the two models before discussing the models in details. Figure 4.3 illustrates the time lines of the two models from the perspectives of three agents. To make them clear, following two subsections will account for these time lines verbally.

4.2.1 Time Line of a CIA Model

1. Households start period t with cash M_t in the economy and firms begin period t with capital K_t.

2. Two exogenous shocks — technology and monetary shocks — realize. In particular, the central bank distributes money to commercial banks. This injected money ($X_t = M_{t+1} - M_t$), which can be used for loans to firms, constitutes the capital rather than the liability on the balance sheets of commercial banks.

[15]In the following setup, while consumption and leisure are cash goods, the investment is credit goods. On the other hand, Cooley and Hansen (1989) treat consumption as the cash good, and investment and leisure as credit goods while Fuerst (1992) treats both consumption and investment as cash goods.

Time Line of a CIA model

	Period t starts	ε_{At} and ε_{mt} realize	Credit Market	Labor Market	Goods Market	Period t end
Households	M_t		$-D_t$	$+W_tN_t$	$-P_tC_t$	$R_tD_t+(F_t+B_t)=M_{t+1}$
Firms	K_t		$+L_t(=W_tN_t)$	$-W_tN_t$	$+P_tC_t$ $P_tI_t=P_tY_t-P_tC_t$	$P_tC_t-(R_tL_t+F_t)=0$ $K_{t+1}=I_t+(1-\delta)K_t$
Banks		X_t	$(X_t+D_t)-L_t=0$			$R_tL_t-(R_tD_t+B_t)=0$

Time Line of an LP model

	Period t starts	Credit Market	ε_{At} and ε_{mt} realize	Credit Market	Labor Market	Goods Market	Period t ends
Households	M_t	$-D_t$			$+W_tN_t$	$-P_tC_t$	$R_tD_t+(F_t+B_t)=M_{t+1}$
Firms	K_t			$+L_t(=W_tN_t)$	$-W_tN_t$	$+P_tC_t$ $P_tI_t=P_tY_t-P_tC_t$	$P_tC_t-(R_tL_t+F_t)=0$ $K_{t+1}=I_t+(1-\delta)K_t$
Banks		$+D_t$	X_t	$(X_t+D_t)-L_t=0$			$R_tL_t-(R_tD_t+B_t)=0$

Figure 4.3: Time Lines of CIA and LP Models

3. In the credit market, households make a deposit (D_t) to commercial banks and firms borrow labor cost ($L_t = W_t N_t$) from commercial banks where W_t is the nominal wage and N_t is the labor input. Thus, $L_t = X_t + D_t$.

4. Households go to the labor market and work, and then receive wage. On the other hand, firms hire labor from households with cash and produce final goods (Y_t). This output can be used as consumption or investment goods.

5. Households move to the goods market and buy the consumption goods with cash. Note that just before this purchase all cash in the economy is held by households, thus $P_t C_t = M_{t+1}$. Firms earn the total revenue ($P_t Y_t$) by selling their final goods in return to cash of households ($P_t C_t$) and in return to investment goods of the other firms ($P_t I_t$). Thus, just after the sale all cash in the economy is held by firms.

6. At the end of period, firms return the loan with interests to commercial banks and then distribute all remaining cash to households as dividends (F_t).

7. Commercial banks return the deposit with interest to households. They also distribute their remaining cash to households as dividends (B_t). Therefore, the balance sheet of commercial banks are zeros on each side.

8. All cash M_{t+1} in the economy is carried by households to the next period while capital stock K_{t+1} in the economy is carried by firms to the next period.

4.2.2 Time Line of an LP Model

1. Households start period t with cash M_t in the economy and firms begin period t with capital K_t.

2. In the credit markets, households make a deposit (D_t) to commercial banks.

3. Two exogenous shocks — technology and monetary shocks — realize. In particular, the central bank distributes money to commercial banks. This injected money ($X_t = M_{t+1} - M_t$), which can be used for loans to firms, constitutes the capital rather than the liability on the balance sheets of commercial banks.

4. Firms borrow labor costs ($L_t = W_t N_t$) from commercial banks in the credit market where W_t is the nominal wage and N_t is the labor input.

5. Households go to the labor market and work, and then receive wage. On the other hand, firms hire labor from households with cash and produce final goods (Y_t). This output can be used as consumption or investment goods.

6. Households move to the goods market and buy the consumption goods with cash. Note that just before this purchase all cash in the economy is held by households, thus $P_t C_t = M_{t+1}$. Firms earn the total revenue ($P_t Y_t$) by selling their final goods in return to cash of households ($P_t C_t$) and in return to investment goods of the other firms ($P_t I_t$). Thus, just after the sale all cash in the economy is held by firms. Thus, just after the sale all cash in the economy is held by firms.

7. At the end of period, firms return the loan with interests to commercial banks and then distribute all remaining cash to households as dividends (F_t).

8. Commercial banks return the deposit with interest to households. They also distribute their remaining cash to households as dividends (B_t). Therefore, the balance sheet of commercial banks are zeros on each side.

9. All cash M_{t+1} in the economy is carried by households to the next period while capital stock K_{t+1} in the economy is carried by firms to the next period.

As shown in time lines, the LP model is different from the CIA model in the point that households cannot adjust their decision on deposit though they can still adjust their labor supply while firms are still able to make all decision after they observe two exogenous shocks (Lucas 1990; Fuerst 1992).

4.3 A Baseline Cash-in-Advance Model

This section is devoted to the exposition of a CIA model. It first presents behaviors of four types of participants in the model, in the order of households, firms, a central bank and commercial banks. Then, the equilibrium conditions of the model and their implications will be discussed.

4.3.1 Households

In the model, households can purchase the single consumption goods C_t only with cash that comes from two sources: money holdings from the previous period M_t^H and current-period wage earnings, $W_t H_t$ where W_t is the nominal wage and H_t is the fraction of time worked. Thus, it is assumed that households receive the wages before they go shopping to the goods market as in Christiano and Eichenbaum (1992), Fuerst, Evans, and Gertler (1995), and Schorfheide (2000).[16]

In addition, they are able to use the credit market to save some cash, D_t^H, into commercial banks before going to the labor market. Hence, the amount of cash which households bring to the goods market is $M_t^H - D_t^H + W_t H_t$. Then, the cash-in-advance constraint for households in the goods markets is,

$$P_t C_t \leq M_t^H - D_t^H + W_t H_t, \tag{4.3.1}$$

where $D_t^H \geq 0$.

At the end of period t, since households own shares of firms, they receive dividends F_t from firms. From commercial banks, they receive dividends B_t and return to deposit $D_t^H R_t^H$. Accordingly, the budget constraint of households, which shows the assumption that households own the whole cash in the economy at the end of period t, can be expressed as (4.3.2),

$$M_{t+1}^H = (M_t^H - D_t^H + W_t H_t - P_t C_t) + R_t^H D_t^H + F_t + B_t, \tag{4.3.2}$$

where R_t^H is the gross interest rate to deposit during period t.

Three important implications of budget constraint (4.3.2) should be noted here. To begin with, R_t^H is applied from the beginning of period t to the end of period t. Thus, it is the intra-temporal interest rate rather than the inter-temporal interest rate. Second, unlike the labor earnings, households cannot use three sources of cash (F_t,B_t and $D_t^H R_t^H$) for consumption until period $t+1$.

Finally, the existence of bank dividends B_t in (4.3.2) implies that money injection by the central bank increases the net worth rather than liabilities on the balance sheets

[16]In contrast, some literature assumed that the current labor earnings cannot be spent until the following period, then inflation acts like a tax on labor earnings (Christiano 1991; Fuerst 1992; Keen 2004; Auray and Fève 2005). Note also that the good servcs either as the investment flow into the capital stock of firms or as the consumption good of households.

of commercial banks at the injection point. If the central bank wants to distribute cash to households and uses commercial banks as a simple funnel for injection, the injected money will be treated as part of liabilities in lieu of net worth on the balance sheets of commercial banks. In this latter case, the balance sheet will be composed of assets and liabilities only. Then, at the end of period t, commercial banks will return households' money from the central bank with interest rather than distribute the dividends. That is, they will return $R_t^H\left[M_{t+1}^H - M_t^H\right]$ to households. As shown in footnote 22 in Section 4.3.5, however, B_t is equal to $R_t^H\left[M_{t+1}^H - M_t^H\right]$ and thus this modification does not alter the implication of the model at all. In the following, we will stick to the assumption that money injection increases the net worth on the balance sheets of commercial banks.

Now, taking as given P_t, W_t, R_t^H, F_t and B_t, households choose real consumption C_t, work effort H_t, non-negative deposit D_t and money holding M_{t+1} to maximize the following expected present value of lifetime utility, subject to (4.3.1) and (4.3.2).[17]

$$\max_{\{C_t, H_t, M_{t+1}^H, D_t^H\}} E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[(1-\phi)\ln C_t + \phi \ln(1-H_t)\right] \right\}, \tag{4.3.3}$$

where discount factor and the relative weight to leisure in utility, β and ϕ, fulfill $0 < \beta < 1$ and $0 < \phi < 1$, respectively and E_0 denotes the expectations operator conditional upon information available at time 0.

4.3.2 Firms

In each and every period, a final good Y_t is produced by the perfectly competitive firms using labor and capital. It is assumed that firms own the capital stock in the economy and households own the shares of firms.

[17]The utility function used here is a special case of the constant relative risk aversion (CRRA) utility (Kydland and Prescott 1982; King, Plosser, and Rebelo 1988; Cooley 1997) which is consistent with balanced growth.

$$U(C_t, 1-H_t) = \frac{[C_t^{1-\phi}(1-H_t)^{\phi}]^{1-\varphi}}{1-\varphi}$$

where $1/\varphi$ is the intertemporal elasticity of substitution in composite commodity consumption (φ is the coefficient of relative risk aversion $= -[C_t^{1-\phi}(1-H_t)^{\phi}]U''/U'$) and ϕ is the share parameter for leisure in the composite commodity and U' and U'' stand for the first and second derivatives of utility with respect to composite consumption. Parameter φ is known as one of the most difficult parameters to calibrate or estimate. In addition, if φ is less than one, consumption and leisure are complements and thus consumption and worked hours move in opposite directions (Dow 1995). These motivate the wide use of the log utility function which arises when $\varphi = 1$.

The constant-return-to-scale output function is given as (4.3.4),

$$Y_t = K_t^{\alpha}(A_t N_t)^{1-\alpha}, \tag{4.3.4}$$

where K_t stands for the capital stock at the beginning of period t, N_t is the labor input, A_t is the state of labor-augmenting technology at period t, and finally $0 < \alpha < 1$ is the capital's share in output.

The state of technology is assumed to be governed by the following random walk process with drift,

$$\ln A_t = \gamma + \ln A_{t-1} + \epsilon_{A,t} \qquad \epsilon_{A,t} \sim N(0, \sigma_A^2), \tag{4.3.5}$$

where γ denotes the steady state of technology growth rate, and $\epsilon_{A,t}$ is the exogenous technology shock and is presumed to be permanent.

In this economy, firms hire labor from households in the competitive labor market. But they must pay the wage bills in the labor market before they go to the goods market for sale, so they also face the cash-in-advance constraint. As firms have no cash when trying to employ labor, they must finance labor cost from commercial banks in the credit market before going to the labor market. This constraint can be expressed as (4.3.6),

$$W_t N_t \leq L_t^F, \tag{4.3.6}$$

where L_t^F is the amount of loan from commercial banks.

On the other hand, at the goods market, firms trade their goods with other firms for investment.[18] Thus, unlike labor input (cash goods), they do not have to borrow cash from commercial banks to finance investment activities (credit goods). The stock of capital evolves according to the following law of motion,

$$K_{t+1} = I_t + (1-\delta)K_t, \tag{4.3.7}$$

where I_t is the real investment and $0 < \delta < 1$ is the depreciation rate of the capital stock.

[18]One may think of this setup as there are two parts in each firm: producing part and purchasing part. And purchasing part buys investment goods from other firms.

When the goods market closes, firms receive value of output (P_tY_t) which is the sum of total cash (P_tC_t) in the economy and the nominal investment (P_tI_t). At the end of the period, firms have no cash again because they pay back the loans with interest ($L_t^F R_t^F$) to commercial banks where R_t^F is the gross interest rate to loan and they distribute the remaining cash to their shareholders (i.e., households) as dividends F_t. This relation can be summarized as (4.3.8),

$$F_t + L_t^F R_t^F = P_tY_t - P_tI_t. \tag{4.3.8}$$

Before proceeding further, it is important notice that F_t should be regarded as the gross return to the initial capital stock instead of the profit. This point will become clear when we discuss Figure 4.4 in Section 4.3.5.

Now, taking as given P_t, W_t and R_t^F, firms – acting in the best interests of their owners – choose infinite sequences of controls $\{N_t, L_t^F, F_t, K_{t+1}\}_{t=0}^{\infty}$ to maximize (4.3.9), subject to constraints (4.3.4) – (4.3.8).[19]

$$\max_{\{F_t, K_{t+1}, N_t, L_t^F\}} E_0 \left\{ \sum_{t=0}^{\infty} \beta^{t+1} \frac{F_t}{C_{t+1}P_{t+1}} \right\}. \tag{4.3.9}$$

4.3.3 Central Bank

The central bank mechanically lets the money stock M_t^G grow at rate $m_t = M_{t+1}^G / M_t^G$. The net money growth rate $\ln m_t$ follows an AR(1) stochastic process as (4.3.10),

$$\ln m_t = (1-\rho)\ln m^* + \rho \ln m_{t-1} + \epsilon_{m,t} \qquad \epsilon_{m,t} \sim N(0, \sigma_m^2), \tag{4.3.10}$$

where $-1 < \rho < 1$ denotes the persistence of money growth rate, $\ln m^*$ is the steady-state of the net growth rate of money, and $\epsilon_{m,t}$ is the exogenous money growth shock

[19] To understand (4.3.9), note first that at the end of period t, the value or price of £1 owned by a household in terms of period $t+1$ goods is $1/P_{t+1}$. The reason of using $1/P_{t+1}$ rather than $1/P_t$ is straightforward. £1 is cannot be used for consumption until period $t+1$. Second, this extra unit of pound will increase the household's utility at period $t+1$ exactly up to $U_{c,t+1}$, where $U_{c,t+1}$ is the households' marginal utility of consumption at time $t+1$. Now we know that the tomorrow marginal utility of today £1 is $U_{c,t+1}/P_{t+1}$. Third, this marginal utility must be discounted to yield the present value of extra unit of pound, so $\beta[U_{c,t+1}/P_{t+1}]$ is the discounted marginal utility of £1 received at the end of period t. Finally, the present value of total utility arising from dividends F_t can be calculated trivially as follows,

$$\beta \frac{U_{c,t+1}}{P_{t+1}} F_t.$$

(or simply money shock) which is assumed to be orthogonal to technology shock.

In the following, a positive money shock means money injection $(X_t = M_{t+1}^G - M_t^G)$ by the central bank. It is noteworthy that in this case the log of money stock M_t follows an AR(2) process.[20]

4.3.4 Commercial Banks

Commercial banks serve two purposes. First, they provide the credit markets by bringing savers (households) and borrowers (firms) together. Second, receiving cash from the central bank, they act as a money injection channel from the central bank to the economy. At the start of period t, technology shock $\epsilon_{A,t}$ is realized, and the monetary injection $X_t(= M_{t+1}^G - M_t^G)$, is fed into commercial banks by the central bank.

Then, in the credit market, commercial banks receive additional cash from households as a form of deposit D_t^B. At this point, all cash in the economy is held by the banks and thus in their balance sheets, cash is listed on the left-hand side while net worth and liabilities are listed on the right-hand side. With these two sources of money, they lend cash loan L_t^B to firms. This balance sheet constraint is expressed as (4.3.11),

$$L_t^B \leq (M_{t+1}^G - M_t^G) + D_t^B, \tag{4.3.11}$$

where the equality holds when the banks lend all cash as a loan.

At the end of the period t, in the credit markets, they receive the $L_t^B R_t^F$ from firms and return $D_t^B R_t^H$ to households, where R_t^F and R_t^H stand for the interest rates for loan and deposit, respectively. Lastly, they distribute all remaining cash position to their owners, households, as dividends B_t. Hence, the budget constraint for banks is

$$B_t + R_t^H D_t^B + L_t^B = R_t^F L_t^B + D_t^B + (M_{t+1}^G - M_t^G), \tag{4.3.12}$$

where the left-hand side represents the uses of fund and the right-hand side represents the sources of fund.

[20]If there is a feedback from the monetary policy to the previous technology shock, the equation can be modified as follows,

$$\ln m_t = (1-\rho)\ln m^* + \rho \ln m_{t-1} + \theta \ln A_{t-1} + \epsilon_{m,t} \qquad \epsilon_{m,t} \sim N(0, \sigma_m^2).$$

Then money growth may be either pro-cyclical (when $\theta > 0$) or counter-cyclical (when $\theta < 0$) and two shocks are correlated. However, since the technology shock is assumed to be permanent in the above model, this modification seems to be less interesting.

Similar to firms, banks act in the interests of the owners (i,.e., households). Then, taking as given M^G_{t+1}, M^G_t, R^F_t and R^H_t, they maximize (4.3.13), subject to constraints (4.3.11) and (4.3.12).

$$\max_{\{B_t, L^B_t, D^B_t\}} E_0 \left\{ \sum_{t=0}^{\infty} \beta^{t+1} \frac{B_t}{C_{t+1} P_{t+1}} \right\}. \tag{4.3.13}$$

That is, they attempt to maximize the present value of total utility arising from dividends B_t. Note that because there are no endogenous state variables, they solve the sequence of static optimization problems.

Finally, the model economy is closed by the following aggregate resource constraint,

$$C_t + I_t \leq Y_t. \tag{4.3.14}$$

4.3.5 Equilibrium Conditions for the CIA Model

This section presents the equilibrium conditions and optimality conditions in the model economy. The detailed derivation process is demonstrated in Appendix 4.A.

For the model economy, note first that a competitive equilibrium requires that all markets clear. To this end, the following aggregate consistency conditions must hold.

$$\begin{aligned} D^H_t &= D^B_t, \\ L^F_t &= L^B_t, \\ M^G_{t+1} &= M^H_{t+1}, \\ H_t &= N_t. \end{aligned} \tag{4.3.15}$$

Next, the goods market clears simply when output equals consumption and investment because, by construction, the output can be used as either consumption goods or investment goods. This implies that (4.3.14) holds with strict equality.

$$C_t + I_t = Y_t. \tag{4.3.16}$$

It is straightforward that the credit markets clear when the balance sheet constraint of commercial banks (4.3.11) holds with equality. The reason is that the commercial bank as a rational agent will not let their loanable funds idle as long as the gross interest rate is larger than one.[21] This is clearly shown in Kuhn-Tucker conditions in Appendix 4.A. At the time of lending, therefore, the balance sheet says that the assets (L_t) is exactly equal to the liabilities (D_t) plus the net worth ($M_{t+1} - M_t$) as shown in (4.3.17),

$$L_t = M_{t+1} - M_t + D_t. \tag{4.3.17}$$

Thus, now cash holdings on the assets side of the balance sheet is replaced by the loans. Then no arbitrage condition in the competitive credit market implies that both R_t^F and R_t^H are equalized as R_t.[22]

Exactly with the same reason, (4.3.6) also holds with equality since firms will borrow the exact amount fund which is required to pay the current wage bills.

$$W_t = L_t/N_t. \tag{4.3.18}$$

The cash-in-advance constraint of households (4.3.1) also holds with equality since households try to economize their money holdings which is required to pay the current consumption. Note that idle money in the goods markets does not produce interest but the deposit in the credit markets does.

Finally, by substituting (4.3.15) into the households' CIA constraint (4.3.19), the firms' CIA constraint (4.3.18) and the banks' balance sheet (4.3.11), we get the following money market clearing condition (4.3.19).

$$\begin{aligned} P_t C_t &= M_t - D_t + W_t N_t \\ &= M_t - D_t + L_t \\ &= M_t - D_t + M_{t+1} - M_t + D_t \\ &= M_{t+1}. \end{aligned} \tag{4.3.19}$$

[21] During the discussion, Professor Basu gratefully indicated this point.

[22] Now, it is clear that whether the injected money is treated as the net worth or as the liabilities on the balance sheet of commercial banks makes no significant difference in analysis. By substituting the aggregate consistency conditions (4.3.15) into (4.3.12) and then using (4.3.17), we can confirm this argument as follows.

$$\begin{aligned} B_t + R_t D_t + L_t &= R_t L_t + D_t + (M_{t+1} - M_t) \\ \Leftrightarrow B_t &= R_t(L_t - D_t) + D_t + (M_{t+1} - M_t) - L_t \\ &= R_t(L_t - D_t) \\ &= R_t(M_{t+1} - M_t). \end{aligned}$$

At this stage, it is instructive to illustrate how money flows in this economy. Thus, Figure 4.4 shows how money flows among households, banks and firms. It can also make the argument in Section 4.3.2 clear that F_t should be regarded as the gross return to the initial capital stock rather than the profit.[23]

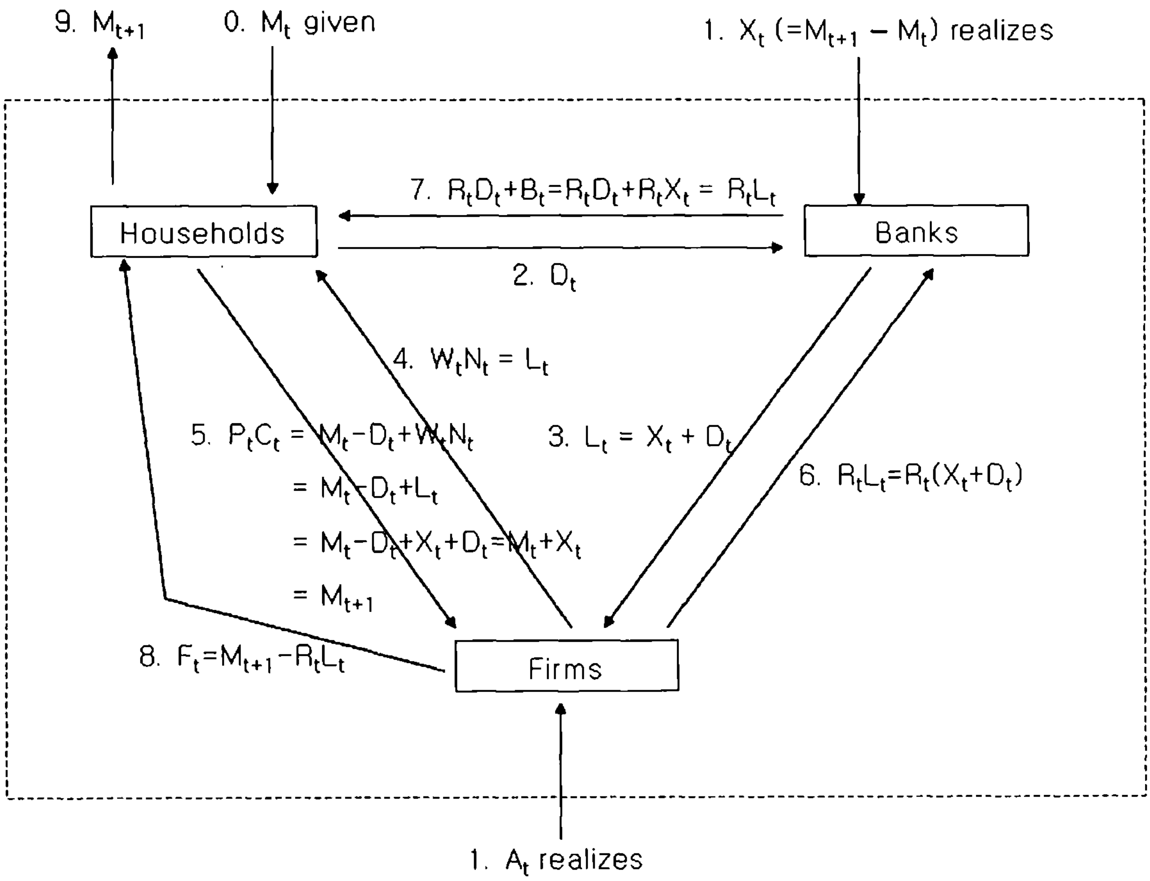


Figure 4.4: Money Flow in the CIA Model

To understand this point, notice first in Figure 4.4 that $F_t = M_{t+1} - R_tL_t = M_{t+1} - R_t(W_tN_t)$. So, F_t is a residual cash after firms repay the loan they owed banks for wage bills. Now let us suppose that there is a market for capital stock. In addition, like the labor, let us assume that firms do not own the capital stock and they must rent it from households with cash which is borrowed from banks, before going to the goods

[23]Note that since firms buy the investment goods with credit during the period, F_t should not be regarded as the gross return to the total capital stock. This point was gratefully pointed out by Dr. Leslie Reinhorn.

market for selling their products. Then, at the end of period, they have to return the capital stock to households and they have to pay back the loan from banks. In this case, it is clear that $M_{t+1} = R_t L_t$ or $F_t = 0$. Finally, since this decline in F_t is exactly matched by the increase in B_t, households own all cash at the end of period. After all, this modification will not alter the model economy much.

In this economy, there are four optimality conditions. Two conditions arise from the maximization problem of households and the other two come from the maximization problem of firms. These four conditions restrict equilibrium paths in the goods, labor, money, and credit markets.

First, the households' intratemporal optimality condition represents how much households are willing to supply labor given nominal wage W_t. Namely, it stands for the *labor supply curve* along which labor supply is an increasing function of the real wage. As (4.3.20) shows, households equate the marginal rate of substitution between leisure and consumption to the marginal product of labor (or equivalently, real wage),

$$\frac{\phi}{1-\phi}\left[\frac{C_t P_t}{1-N_t}\right] = W_t, \tag{4.3.20}$$

where households adjust labor supply until the marginal benefit from increasing labor supply, which is the marginal utility of consumption coming from one extra unit of money multiplied by the nominal wage, exactly equals the marginal costs from increasing labor supply.[24]

One important implication of (4.3.20), which will be crucial to understand the impulse response analysis of the two models, is that labor supply depends not only real wage but also the consumption level. To understand this argument, note first (4.3.20) is obtained from the following equilibrium condition

$$\frac{W_t}{P_t} = -\frac{U_{n,t}}{U_{c,t}},$$

where $U_{n,t} < 0$ is the marginal disutility of labor and $U_{c,t} > 0$ is the marginal utility of consumption.

[24]This condition can be understood as $W_t \frac{U_{c,t}}{P_t} + U_{n,t} = 0$, where $U_{n,t} < 0$ is the marginal disutility of labor. The first term represents the marginal benefit arising from the extra unit of labor supply while the second tern represents the marginal cost of the extra unit of labor supply. It is straightforward that if the W_t cannot be used until period $t+1$, this condition will be altered as $W_t E_t \beta \frac{U_{c,t+1}}{P_{t+1}} + U_{n,t} = 0$.

Now it is clear that the fall of consumption level (i.e., the rise of marginal utility of consumption) will shift the labor supply curve to the right since given the level of labor supply (i.e, the fixed marginal disutility of labor) households are now willing to accept the lower real wage.[25]

Second, the households' intertemporal optimality condition, which is the *Euler equation of households*, requires that given R_t, households' utility loss due to the fall in current consumption (i.e., the increase in the deposit) to match the discounted expected gain due to the rise in future consumption (i.e., the increase in the principal and interest). The intertemporal optimality condition for households can be expressed as (4.3.21),[26]

$$\frac{1}{C_t P_t} = \beta R_t E_t \left\{ \frac{1}{C_{t+1} P_{t+1}} \right\}. \tag{4.3.21}$$

Third, firms own all capital stock of the economy and make a decision on between investment and dividend just before the goods market closes. Then, at the end of period t, they just distribute all remaining cash to households as dividends. When they make a decision, they compare the expected marginal utility of the current dividends with discounted expected marginal utility in the case of investment. This intertemporal optimality condition of firms, which is the *Euler equation of firms*, represents the

[25]If we choose the utility function used in King and Wolman (1999), which is a type of Greenwood, Hercowitz, and Huffman (1988)'s preferences (GHH preferences), the labor supply will be determined only by the real wage. For example,

$$\begin{aligned} U(C_t, N_t) =& \frac{\left[C_t - \frac{\theta}{1+\gamma} N_t^{1+\gamma}\right]^{1-\sigma} - 1}{1-\sigma} \\ U_{c,t} =& \left[C_t - \frac{\theta}{1+\gamma} N_t^{1+\gamma}\right]^{-\sigma} \\ U_{n,t} =& -\theta N_t^{\gamma} [U_{c,t}] \\ \therefore -\frac{U_{n,t}}{U_{c,t}} =& \theta N_t^{\gamma} = \frac{W_t}{P_t}, \end{aligned}$$

where $\theta > 0$ and $\gamma > 0$ and then σ is the degree of risk aversion. Now it is straightforward to see that only real wage determines the labor supply in GHH preference.

[26](4.3.21) can be understood as follows.

$$\frac{1}{P_t} U_{c,t} = \beta R_t E_t \left\{ \frac{1}{P_{t+1}} U_{c,t+1} \right\},$$

where the left hand side stands for the product of the value of today £1 in terms of period t consumption goods and the marginal utility of consumption. Therefore, when households determine the level of deposit, they equate the utility cost of today £1 deposit to the discounted expected utility benefit tomorrow of the gross return.

tradeoff of moving goods across time.

$$E_t \frac{P_t}{C_{t+1}P_{t+1}} = \beta E_t \frac{P_{t+1}}{C_{t+2}P_{t+2}} \left[\alpha K_{t+1}^{\alpha-1} (A_{t+1}N_{t+1})^{1-\alpha} + (1-\delta) \right]. \tag{4.3.22}$$

Notice that one period ahead marginal utility of real consumption is weighted by the purchasing power of money when the dividends are distributed.[27]

Finally, the intratemporal optimality condition for firms represents the *labor demand curve*. Firms will equalize the marginal cost of extra labor to the marginal revenue product of extra labor ($MRP_{n,t}$).

$$W_t R_t = P_t(1-\alpha)K_t^{\alpha} A_t^{1-\alpha} N_t^{-\alpha}, \tag{4.3.23}$$

where the existence of R_t shows that since firms have to pay workers in advance, they borrow $W_t N_t$ from commercial banks before they receive the revenue. One implication of (4.3.23) is that in $\frac{W_t}{P_t} = \frac{MRP_{n,t}}{R_t}$, the rise of the gross return will shift the labor demand to the left.

To estimate the model, one first need to transform the non-stationary variables into stationary variables. It can be shown that when shocks are absent, real variables grow with A_t (except for labor, N_t, which is stationary as there is no population growth), nominal variables grow with M_t and prices with M_t/A_t. Therefore, removing the trends involves the following operations (where lower letters represent stationary variables). For real variables, $Y_t = A_t y_t$, $C_t = A_t c_t$, $I_t = A_t i_t$, $K_{t+1} = A_t k_{t+1}$, and $N_t = n_t$. For nominal variables, $D_t = M_t d_t$, $L_t = M_t l_t$, $W_t = M_t w_t$.[28] And for prices, $P_t = p_t M_t / A_t$.[29] Lastly, since the interest rate is already stationary, we do not need to make R_t stationary.

Solving a dynamic stochastic general equilibrium model means finding the decision rules from a nonlinear system of market clearing conditions and optimality conditions.

[27] For example, suppose that $P_t = 100$, $P_{t+1} = 200$, and $P_{t+2} = 800$. Then, the purchasing power of £1 at time t will fall into a half at time $t+1$ and the purchasing power of £1 at time $t+1$ will fall into a quarter at time $t+2$. Therefore, if $\beta = 1$, this loss of purchasing power should be compensated by the high gross interest rate, which is 2.

[28] So, in this model, there is a dichotomy between the real variables and the nominal variables. However, if we divided these variables by price level, then these variables were affected by the technology level as well as the aggregate money level.

[29] Thus, in this model, the trend in price level can be decomposed into two factors: money supply and the productivity.

After taking a log-linear approximation of equilibrium conditions of the original non-linear model around the deterministic steady state, the model can be solved by the method of undetermined coefficients suggested by Campbell (1994) and Uhlig (1999). Then the responses of endogenous variables to the shocks can be analyzed based on the variables which are expressed in terms of percent deviation from the values at the steady state. Appendix 4.A provides the entire derivation procedure.

4.4 Comparing an LP Model with a CIA Model

4.4.1 Equilibrium Conditions for the LP Model

We now consider the limited participation (LP) model where households cannot adjust their decision on deposit when the shocks are realized but firms are still able to make all decision after the shocks. Since this model is very similar to the CIA model, this alteration of timing only affects the households' optimization problem as follows (Lucas 1990; Fuerst 1992),

$$\begin{aligned} \max_{\{C_t, H_t, M_{t+1}^H, D_{t+1}^H\}} \quad & E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[(1-\phi) \ln C_t + \phi \ln(1-H_t) \right] \right\} \qquad (4.4.1) \\ s.t. \quad & P_t C_t + D_t^H \leq M_t^H + W_t H_t \\ & 0 \leq D_t^H \\ & M_{t+1}^H = (M_t^H - D_t^H + W_t H_t - P_t C_t) + R_t^H D_t^H + F_t + B_t, \end{aligned}$$

where the only difference is that the deposit D_t is no more a control variable and becomes a state variable in the model. As a consequence, now D_t can be regarded to be determined at the end of period $t-1$. This implies that households make a decision on the current period deposit conditional on the previous period information set. Then, this modification affects firms' behavior via the financial market.

Therefore, (4.3.21), which is the intertemporal optimality condition for the household in the CIA model, is replaced by (4.4.2) in the LP model.

$$E_{t-1} \frac{1}{C_t P_t} = \beta E_{t-1} R_t \left\{ \frac{1}{C_{t+1} P_{t+1}} \right\}. \qquad (4.4.2)$$

However, market clearing conditions and the other optimality conditions are not affected. It is noteworthy that while an expansionary monetary shock causes D_t to fall in the CIA model, it drives D_{t+1}, which is determined at the end of period t, up in the LP model. The implications of each model will be discussed in length in the following subsection.

4.4.2 Some Qualitative Implications of Two Models

Before estimating the two versions of monetary DSGE models, we discuss what are the qualitative implications of each model. As noted above, the only difference in setup is the timing of households' decision on deposit. The important implication of this distinction makes clear when we consider the effect of a monetary shock on the interest rate.

As a preliminary, let us start from the simplest case. Suppose for the moment that persistence of money growth rule ρ is zero, that is, money shock is purely transitory. In this case, if there was a money shock in the CIA model, the current and future nominal prices and nominal wages would move in proportion to the change in money. Then, it is clear in (4.3.21) that the interest rate R_t must be constant and the monetary shock is neutral as no real quantity would be affected.[30] Notice that households' deposit D_t must fall. The reason is as follows. Without money injection X_t, the ratio of loanable fund in the economy is D_t/M_t. To maintain this ratio neutral, deposit D_t must drop.[31] Put differently, households know that the increased money supply will lead to an increase in prices, and so will reduce their deposit in order to have more cash on hand for purchasing the consumption goods.

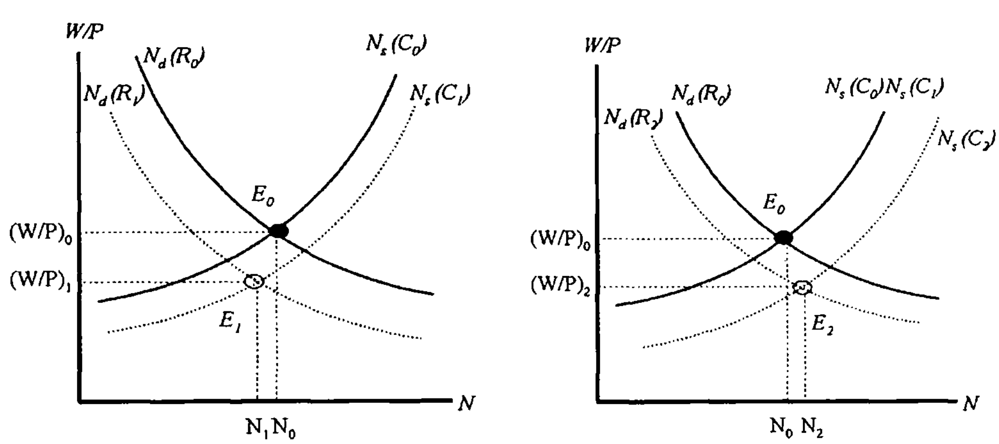
In the LP model, however, households cannot adjust D_t and thus the money injection will lead the ratio of loanable fund in the economy up. To induce firms to absorb this disproportionately large cash, R_t must decline. This is the liquidity effect, which states that excess liquidity works as a downward pressure on R_t. Then, due to the liquidity effect, a positive money shock in the LP model is not neutral any more.

[30] In this special case, money shock will clearly have no effect on present and future real quantities such as consumption, investment, employment and output. The reason is that as soon as households observe an increase in the money supply, they know a jump in present and future prices and wages and thus will adjust deposit to the extent consumption will remain constant. Thus, not only the real interest rate but also the nominal interest interest rate will remain constant (Christiano 1991; Dow 1995).

[31] A simple numerical illustration will make this point clear. Suppose that without money injection, $D_t = £40$ and $M_t = £80$. If there is money injection $X_t = £20$, the ratio of loanable fund rises from $\frac{D_t}{M_t} = 0.5$ to $\frac{D_t+X_t}{M_t+X_t} = 0.6$. Therefore, the household as a rational agent will decrease deposit up to $£10$.

Now consider the general case of $\rho > 0$. In the CIA model, an expansionary monetary shock leads to an unambiguous rise of R_t through the anticipated inflation effect. It is straightforward to check this argument by substituting money market clearing condition (4.3.19) into (4.3.21). Then, (4.3.21) is rewritten as $E_t \frac{M_{t+1}}{M_{t+2}} = \frac{1}{\beta R_t}$. Thus, as long as $\rho > 0$, R_t rises in response to the expansionary money shock and there is no mechanism to account for the liquidity effect. The rise of R_t drives consumption C_t down and investment I_t up because it acts like a tax on the cash goods and a subsidy on the credit goods. Money injection also decreases households' desire to save since they know that the increased money will be distributed as dividends from commercial banks at the end of period. Thus, deposit to commercial banks D_t also falls.

On the other hand, the labor employed in the CIA model is determined by two countervailing effects. First, the rise of R_t will definitely shift the labor demand curve to the left in (4.3.23). Second, the rise of R_t will acts like a tax on the consumption (cash goods) and thus will increase the marginal utility of consumption. Accordingly, the labor supply curve will move to the right in (4.3.20) even though labor supply does not directly involve R_t. Now the amount of labor employed will fall when the left shift of demand curve dominates the right shift of supply curve and vice versa. With plausibly calibrated parameters, Christiano and Eichenbaum (1992, 1995) show that the shift of supply curve is overwhelmed by the shift of demand curve in response to the change of R_t. This case is illustrated in Figure 4.5(a).



(a) CIA Model (b) LP Model

Figure 4.5: Effects of a Positive Money Shock in Labor Market

In the LP model, an expansionary monetary shock has more complicated effects on the economy. First of all, contrary to a CIA model, we do not know whether R_t rises or falls. The reason is that R_t in the LP model is affected not only by the anticipated inflation effect (upward pressure) similar in the CIA model, but also by the liquidity effect (downward pressure).

The liquidity effect arises from the following reason. In the LP model, when an expansionary monetary shock realizes, households cannot decrease D_t. Hence, compared to the CIA model, commercial banks will have more funds to lend.[32] Commercial banks will lend all cash to firms as long as $R_t > 1$. Given the fixed deposit, R_t must drop in order to induce firms to absorb the higher proportion of cash in the economy. Therefore, as Christiano and Eichenbaum (1992) adequately point out, whether R_t rises or falls in the LP model is not a qualitative problem and it should be determined by what data say in the estimated DSGE model in the following sections.[33]

Another complication in the LP model arises in the labor market. Suppose for the moment that the anticipated inflation effect dominates, so R_t rises *slightly* in response to an expansionary monetary shock. Note that if this is supported by the data, there is no *liquidity puzzle*. Then, the labor demand curve will shift *slightly* to the left while the labor supply curve will shift *slightly* to the right. However, there is another factor in the LP model which shifts the labor supply curve to the right. Now as households are unable to reduce their deposit, they realize that they saved too much before the positive money shock. As noted earlier, households decrease deposit D_t in response to the positive money shock if it is allowed. Note that the excess deposit implies the lesser money for current consumption and thus households are more willing to work harder. Therefore, the consumption falls not only because of the inflation tax by the rise of R_t but also because of the rigidity of D_t. The rigidity of D_t drives the consumption to fall further and thus the labor supply curve to shift further to the right.

This point can be easily understood by substituting (4.3.17), (4.3.18) and (4.3.19)

[32] The ratio of loanable fund to total cash in the economy is $\frac{D_t+X_t}{M_t+X_t}$. Therefore, this ratio is higher in the LP model in which D_t cannot drop after the positive money shock.

[33] Christiano and Eichenbaum (1992, p.348) state that "under these circumstances, whether interest rates fall or rise depends on which effect is stronger. Suppose for the moment that the liquidity effect dominates,$\cdots$."

into (4.3.20) as follows.

$$\frac{\phi}{1-\phi}\left[\frac{C_t P_t}{1-N_t}\right] = W_t$$
$$\frac{\phi}{1-\phi}\left[\frac{M_{t+1}}{1-N_t}\right] = \frac{L_t}{N_t}$$
$$\frac{\phi}{1-\phi}\left[\frac{M_{t+1}}{L_t}\right] = \frac{1-N_t}{N_t}$$
$$\frac{\phi}{1-\phi}\left[\frac{X_t+M_t}{X_t+D_t}\right] = \frac{1}{N_t} - 1.$$

In response to a positive money shock, households will decrease D_t if possible. However, in the LP model, households are not allowed to adjust D_t. As a result, $\frac{X_t+M_t}{X_t+D_t}$ declines and thus N_t rises.

Figure 4.5(b) illustrates the case in which the downward pressure from the rise of R_t on equilibrium labor employed is overwhelmed by upward pressure from the large decline of consumption (i.e., the large rise of marginal utility of consumption). If the labor employed increase as in Figure 4.5(b), output Y_t also rises. Then, investment I_t also rises since Y_t rises and C_t falls. Similarly, we can draw the case when the downward pressure on equilibrium labor employed dominates. But this seems not be supported by Korean data as shown in Figure 4.8 of Section 4.6.

Under these circumstances, it is clear that the responses of endogenous variables to an expansionary monetary shock, especially in the LP model, must be evaluated in the framework of estimated DSGE models, which take the data to the models and produce the parameter estimates of interests. Before presenting the empirical findings in Section 4.6, we discuss the estimation methodology for Bayesian inference in the next section.

4.5 Methodology and Prior Densities

This section explains how the solved DSGE models can be estimated via the Bayesian approach and then how the prior densities are actually established for the two models. For the brief background, it is convenient to start from some knowledge of Bayesian inference.

4.5.1 Bayesian Inference

The fundamental idea of Bayesian inference is to yield the posterior density by combining the prior information with the information from the data through the Bayes' rule. Put differently, the posterior density is a compromise between data and prior density. Suppose that we have a data matrix Y which ranges from 1 to T for n variables, and we are interested in the distributions for m parameters. Then, the Bayes' Theorem states

$$\begin{aligned} p(\boldsymbol{\theta}|\boldsymbol{Y}) &= \frac{p(\boldsymbol{Y}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{p(\boldsymbol{Y})} = \frac{p(\boldsymbol{\theta}, \boldsymbol{Y})}{\int p(\boldsymbol{\theta}, \boldsymbol{Y})d\boldsymbol{\theta}} = \frac{\mathscr{L}(\boldsymbol{\theta}|\boldsymbol{Y}) \times p(\boldsymbol{\theta})}{\int p(\boldsymbol{\theta}, \boldsymbol{Y})d\boldsymbol{\theta}} \\ &= \kappa\mathscr{L}(\boldsymbol{\theta}|\boldsymbol{Y}) \times p(\boldsymbol{\theta}) = \kappa\mathcal{K}(\boldsymbol{\theta}|\boldsymbol{Y}) \\ &\propto \mathcal{K}(\boldsymbol{\theta}|\boldsymbol{Y}), \end{aligned} \tag{4.5.1}$$

where $p(\boldsymbol{\theta}|\boldsymbol{Y}) = p(\theta_1, \theta_2, \cdots, \theta_m|\boldsymbol{Y})$ is the joint posterior density of m unknowns, $p(\boldsymbol{Y}) = \int p(\boldsymbol{\theta}, \boldsymbol{Y})d\boldsymbol{\theta}$ is the marginal likelihood, and κ is a normalizing constant because it does not depend on $\boldsymbol{\theta}$. Note that the joint probability distribution of data and unknowns $p(\boldsymbol{\theta}, \boldsymbol{Y}) \equiv \mathcal{K}(\boldsymbol{\theta}|\boldsymbol{Y})$ is often called as the unnormalized joint posterior density (or posterior kernel), and data $\boldsymbol{Y}$ affect the joint posterior density of m unknowns $p(\boldsymbol{\theta}|\boldsymbol{Y})$ only through the likelihood function $p(\boldsymbol{Y}|\boldsymbol{\theta}) \equiv \mathscr{L}(\boldsymbol{\theta}|\boldsymbol{Y})$, which is a function of $\boldsymbol{\theta}$ given $\boldsymbol{Y}$.

Given the parametric form of the joint posterior density $p(\boldsymbol{\theta}|\boldsymbol{Y})$, it is straightforward at least theoretically to obtain the marginal posterior density of the parameters of interest, on which Bayesian inference is grounded. For instance, the marginal posterior density of θ_2 is easily calculated by integrating the joint posterior density of m unknowns with respect to all but θ_2.

$$p(\theta_2|\boldsymbol{Y}) = \int_{\theta_1}\int_{\theta_3}\cdots\int_{\theta_m} p(\boldsymbol{\theta}|\boldsymbol{Y})d\theta_m \cdots d\theta_3 d\theta_1.$$

Also, given the above marginal posterior density of θ_2, it is trivial to calculate the posterior mean as

$$E(\theta_2|\boldsymbol{y}) = \int \theta_2 p(\theta_2|\boldsymbol{Y})d\theta_2.$$

Or equivalently, one can draw enough random samples from the given joint posterior density of all unknowns $\boldsymbol{\theta}$ and then just look at the mean of θ_2 in the draws with ignoring other parameters (see, for example, Lancaster 2004, Ch. 4; Gelman, Carlin, Stern, and Rubin 2004, p. 73).

Then, the problem is how to obtain the joint posterior distributions. Notice in (4.5.1) that obtaining the parametric form of $p(\boldsymbol{\theta}|\boldsymbol{Y})$ requires the optimization of $\mathscr{L}(\boldsymbol{\theta}|\boldsymbol{Y})$ in the numerator as well as the integration for computing $\int p(\boldsymbol{\theta}, \boldsymbol{Y})d\boldsymbol{\theta}$ in the denominator. In the very simple cases, these distributions can be deduced analytically without numerical integration by selecting conjugate priors.[34] As the parameter dimension increases, however, it is almost impossible to detect any analytically tractable form as the conjugate prior. In most cases, therefore, the high dimensions in parameters and the complicate likelihood function due to the mixture of several different distributions, which are very common in DSGE model estimation, do not allow the researcher to exploit the conjugacy (Gamerman and Lopes 2006, pp. 55–58).

Notice that from the last relationship in (4.5.1), it is clear that the joint posterior density of m unknowns $p(\boldsymbol{\theta}|\boldsymbol{Y})$ is different from the unnormalized joint posterior density $\mathcal{K}(\boldsymbol{\theta}|\boldsymbol{Y})$ up to the constant. Accordingly, the analysis based on the latter is equivalent to that based on the former. Therefore, instead of trying to tackle the integration problem to explicitly compute the joint posterior distribution of $p(\boldsymbol{\theta}|\boldsymbol{Y})$, we zero in on the posterior kernel using the posterior simulators such as Markov Chain Monte Carlo (henceforth, MCMC) methods.[35] So MCMC methods can be applied directly to the kernel of the target distribution without knowledge of marginal likelihood.

The rationale of MCMC algorithm is that once the Markov chain has run for a long enough time to wander around the parameter space, in the limit it will converge to the desired stationary posterior kernel (target distribution or equilibrium distribution) of interest. Then once after convergence is reached, the parameter draws of interest are assumed to be drawn from the target density and they are sufficient enough to yield a histogram, which is equivalent to the marginal posterior distribution of each

[34] When the posterior density remains in the same family as the prior density, and the effect of likelihood is to "update" the prior parameters but not to change its functional form, we say that such priors are conjugate with the likelihood (Gelman et al. 2004, pp. 40–43). Then, transformation from prior to posterior only involves a change in the hyperparameters with no additional calculation. The conjugacy is popular because of its mathematical ease and computational convenience. Once the conjugate pair likelihood/prior is found, the posterior is found easily. For example, the Beta (Dirichlet) is the conjugate prior distribution for the parameters of the binomial (multinomial) likelihood. After the development of MCMC algorithms, however, the general agreement is that simple conjugate analysis is of limited practical value.

[35] MCMC algorithm differs from the standard Monte Carlo algorithm in the sense that each draw in the former is mildly dependent on the previous draw while each draw in the latter is independent.

parameter. Given this density, one can examine any features of the posterior in which we are interested (Gill 2002).

4.5.2 Estimation Methodology and Data

After constructing a log-linear approximation to the DSGE model, Bayesian inference on DSGE models can be implemented by Random Walk Metropolis Algorithm with six steps (see, for example, An and Schorfheide 2007; Gelman et al. 2004, Ch. 11).

1. Specify the prior densities for parameters of interest $\boldsymbol{\theta}$.

2. To evaluate the kernel $\mathcal{K}(\boldsymbol{\theta}|\boldsymbol{Y}) = \mathscr{L}(\boldsymbol{\theta}|\boldsymbol{Y})p(\boldsymbol{\theta})$, use a Kalman filter algorithm to obtain the log-likelihood function conditioned on observed data, $\ln \mathscr{L}(\boldsymbol{\theta}|\boldsymbol{Y})$.

3. Given the log-kernel by combining two knowns $\ln \mathcal{K}(\boldsymbol{\theta}|\boldsymbol{Y}) = \ln \mathscr{L}(\boldsymbol{\theta}|\boldsymbol{Y}) + \ln p(\boldsymbol{\theta})$, find a posterior mode $\tilde{\boldsymbol{\theta}}$ which maximizes the log-kernel by using a numerical optimization routine. Note that the obtained posterior mode from the kernel is equivalent to the posterior mode from the posterior distribution, because the kernel differs from the posterior distribution up to the normalizing constant.

4. Draw $\boldsymbol{\theta}^0$ from $\mathcal{N}(\tilde{\boldsymbol{\theta}}, c\boldsymbol{\Sigma}(\tilde{\boldsymbol{\theta}}))$ where c is a jumping scaler by specified by the researcher and $\boldsymbol{\Sigma}(\tilde{\boldsymbol{\theta}})$ is the covariance matrix calculated from the inverse of the negative of the second derivative of the log-kernel function (Hessian) evaluated at $\tilde{\boldsymbol{\theta}}$ as in Kim and Nelson (1999, Ch.2).[36]

5. For $t = 1, \cdots, n_{sim}$ where n_{sim} is the number of simulations,

   (a) Sample a proposal $\boldsymbol{\theta}^*$ from a jumping distribution $\mathcal{N}(\boldsymbol{\theta}^{t-1}, c\boldsymbol{\Sigma}(\boldsymbol{\theta}^{t-1}))$.

   (b) Compute the ratio of the densities, $r_t = \frac{\mathscr{L}(\boldsymbol{\theta}^*|\boldsymbol{Y})p(\boldsymbol{\theta}^*)}{\mathscr{L}(\boldsymbol{\theta}^{t-1}|\boldsymbol{Y})p(\boldsymbol{\theta}^{t-1})} = \frac{\mathcal{K}(\boldsymbol{\theta}^*|\boldsymbol{Y})}{\mathcal{K}(\boldsymbol{\theta}^{t-1}|\boldsymbol{Y})}$.

   (c) Draw a sample u_t from a uniform distribution $U(0,1)$ and set $\boldsymbol{\theta}^t$ as follows

$$\boldsymbol{\theta}^t = \begin{cases} \boldsymbol{\theta}^* & \text{if } r_t \geq 1 \\ \boldsymbol{\theta}^* & \text{if } u_t \leq r_t < 1 \\ \boldsymbol{\theta}^{t-1} & \text{if } r_t < u_t. \end{cases}$$

[36] $\boldsymbol{\Sigma}(\tilde{\boldsymbol{\theta}}) = \left[-\frac{\partial^2 \ln \mathcal{K}(\boldsymbol{\theta}|\boldsymbol{Y})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\Big|_{\boldsymbol{\theta}=\tilde{\boldsymbol{\theta}}}\right]^{-1}$.

6. Given the collection of parameter draws, construct the marginal posterior density for each parameter. Finally report the summarizing statistics such as posterior means, modes, standard errors, and 95% highest posterior density (hereafter, HPD) intervals for parameters. Note that HPD interval, which is based on the height of density function, is correspondent to the frequentist's confidence interval.[37]

Among others, two important features of the algorithm deserve comments. First, the second case in step 5(c) implies that the movement to the lower density point is stochastically allowed during the optimization. This is clear from the fact that each suggested value is accepted with probability $min(r, 1)$. Then, allowing the movement to the downhill is expected to let the chain wander around the wider domain of parameter space since we do not want to be stuck in a local maximum. Thus, we want to sample from the region with highest probability but still want to visit the parameter space as much as possible (Canova 2007, pp. 555–557). It can be shown that the Gibbs sampler is a special case of Metropolis-Hastings algorithm where the probability of accepting the candidate value is always one (Gelman 1992).

Second, the choice of the scaler of covariance matrix of the jumping distribution, c in steps 4 and 5, is highly important at least in the computational aspect. If the scaler is too small, the acceptance rate will be too high and the Markov Chain of candidate parameters will "mix slowly", meaning that the distribution will take a long chain to converge to the target distribution since the chain is likely to get "stuck" around a local maximum. On the other hand, if the scale factor is too large, the acceptance rate will be very low (as the candidates are likely to land in regions of low probability density) and the chain will spend too much time in the tails of the posterior distribution. In this essay, the scaler of covariance matrix of the jumping distribution is fine-tuned to yield the optimal acceptance ratio (around 25%) in a high dimension model suggested by the literature (Roberts et al. 1997; Gelman et al. 2004).

Since there are two shocks in the model, the maximum number of observable variables is two. This is known as the stochastic singularity problem which arises when the rank of policy function matrix is less than the number of observables. Following

[37]On the other hand, based on the quantiles, one may use the credible interval. For instance, $\int_{-\infty}^{q_{0.025}} p(x)dx$ and $\int_{-\infty}^{q_{0.975}} p(x)dx$ are 2.5% quantile and 97.5% quantile, respectively, and they constitute a 95% credible interval.

the previous study by Schorfheide (2000), this essay includes two observable variables: national income per capita Y and GDP deflator P which come from the Bank of Korea at the frequency of the quarter. This choice of two time series is made because Y and P are often regarded as the representative statistic for the real side and the nominal side of macroeconomy, respectively. To match the model's setup, national income is computed as the sum of consumption and private investment. The sample period is 1973:Q1 to 2006:Q3. Rather than using any filtering techniques, only linear trend in the log values of observables is introduced and this trend (γ) is estimated.

After the estimation processes, it is important to check whether the MCMC chains have actually converged to a stationary distribution. Then the convergence implies the realizations come from the target distribution. For this purpose, convergence checks are carried out.[38] In the following, the sampler was run for two parallel chains and the length of each chain was 20,000. In addition, the first half values, referred to as the warm-up or burn-in periods, were discarded. Deletion of these values is required so that the initial chain values might not be from the target distribution (Gamerman and Lopes 2006). The acceptance ratio ranged from 25% to 27% in the chains.

4.5.3 Prior Densities

As noted in Section 4.1, prior distributions play an important role in the estimation of DSGE models. They might not only down-weigh some regions of the parameter space which are absurd, but also add curvature to a likelihood function that is nearly flat in some dimensions of the parameter space. Hence, we expect that the prior information will help us to sharpen the inference.[39] While, in principle, priors can be gleaned from personal introspection to reflect strongly held beliefs about the validity of economic theories, in practice most priors are chosen based on some observations or the previous literature (An and Schorfheide 2007).

[38] Brooks and Gelman (1998) propose a general approach to monitoring convergence of MCMC output through comparing the variation between chains and the variation within each chain. The principle is that if several chains, which will lead us the target density, are run sufficiently long with different initial values, the output from all chains is indistinguishable. From this perspective, in the preliminary research, the sampler was run for one, two, and three parallel chains and the results were very similar.

[39] The role of priors in DSGE model estimation is markedly different from the role of priors in VAR estimation. In the latter case priors are essentially used to reduce the dimensionality of the econometric model and hence the sampling variability of the parameter estimates.

Table 4.1: Prior Distributions for Structural Parameters

Parameters	Mean	S.E.	2.5%	97.5%	Dist.
α	0.4590	0.0700	(0.3240	0.5971)	Beta
β	0.9900	0.0050	(0.9781	0.9973)	Beta
γ	0.0100	0.0050	(0.0002	0.0198)	Normal
m*	1.0300	0.1000	(0.8340	1.2260)	Normal
ϕ	0.7000	0.0500	(0.5978	0.7931)	Beta
ρ	0.5000	0.0900	(0.3249	0.6751)	Beta
Standard deviation of shocks					
σ_A	0.0100	Inf	(0.0000	Inf)	Inv.Gamma
σ_M	0.0100	Inf	(0.0000	Inf)	Inv.Gamma

Note: α = capital's share in output, β = subjective discount factor, γ = steady-state technology (net) growth rate, m^* = steady-state money (gross) growth rate, ϕ = preference parameter between leisure and consumption (i.e., relative weight to leisure), ρ = persistence of money shock, σ_A = standard deviation of technology shocks, and σ_M = standard deviation of money growth shocks.

Table 4.1 lists prior distributions for structural parameters of two versions of DSGE models. As in most literature, it is assumed that all structural parameters are a prior independent of each other. Thus, the joint prior density is simply the product of the marginal densities (DeJong, Ingram, and Whiteman 2000; Schorfheide 2000; An and Schorfheide 2007). Since all the marginal densities integrate to unity, it follows that the joint prior distribution is proper.

The prior densities are specified as follows. Based on Korean data in the examined period, the capital share in output, α, is chosen as the average capital share of 0.459 and its standard error is set as two times of standard deviation. The prior for the subjective discounter factor, β, is set equal to 0.99 meaning the annual real interest rate is 4%. For a prior for γ, we use the average national income growth rate per quarter, 0.01, during the sample period. The priors for the steady-state money growth rate m^* and the autoregressive parameter ρ are obtained by $AR(1)$ model fitting using $M1$. The standard deviation of ρ is set as the two times of the standard error of $\hat{\rho}$.

For the standard deviation of technology shock, we use the MLE estimate in Chapter 3 as a prior. The magnitude of monetary shock is further assumed to be similar to the technology shock. However, since we use the inverse gamma distributions for these priors, they are bounded only to be larger than zero and are allowed to be infinite.[40] Following the previous study, the relative weight to leisure, ϕ, is set equal to 0.7.[41]

Lastly, when we attempted to estimate the model without restriction, this yielded a very unlikely high depreciation rate. So following the suggestions of Christensen and Dib (2006) and Ireland (2004), we impose dogmatic prior over the depreciation rate at 0.025 (i.e., annually 10%).

4.6 Empirical Results

This section presents the model estimates in each model and then discusses which model captures better the stylized facts of Korean economy in responses to the monetary shocks. It also provides the variance decomposition analysis in order to show which shock mainly drives the fluctuations of endogenous variables.

4.6.1 Posterior Densities

In Bayesian inference, the posterior densities contain all the information about the parameters such as location and uncertainty. The posterior densities for the parameters in the CIA model are presented in Table 4.2 and those in the LP model are reported in Table 4.3. Two tables report the posterior modes, means and standard errors for the structural parameters. In addition, the last two columns indicate 95% highest posterior density (HPD) interval for each parameter. Throughout this essay, 2.5% and 97.5% percentiles are chosen for the lower and upper limits of 95% HPD interval. Notice first

[40] The inverse-gamma distribution is widely used for prior of variance since this distribution is the conjugate prior distribution for variance in normal distribution.

[41] Canova (2007, pp. 328–329) gives one rationale of this choice of prior. He argues that when there are two sample sets: one sample set from a developed economy, $\boldsymbol{Y}_a$ and the other from an emerging economy, $\boldsymbol{Y}_d$. Suppose that two sample sets are independent, one may use the posterior density of parameter of interest from a developed economy as the prior for an emerging economy.

Suppose we are interested in a certain parameter ϕ. Then the kernel or unnormalized posterior density, $\mathcal{K}(\phi|\boldsymbol{Y}_d, \boldsymbol{Y}_a)$ can be constructed as follows,

$$\begin{aligned}\mathcal{K}(\phi|\boldsymbol{Y}_d, \boldsymbol{Y}_a) &= p(\boldsymbol{Y}_d, \boldsymbol{Y}_a|\phi)p(\phi) = p(\boldsymbol{Y}_d|\boldsymbol{Y}_a, \phi)p(\phi) = p(\boldsymbol{Y}_d|\phi)p(\boldsymbol{Y}_a|\phi)p(\phi) \\ &\propto p(\boldsymbol{Y}_d|\phi)p(\phi|\boldsymbol{Y}_a),\end{aligned}$$

where $p(\phi)$ is a prior density of ϕ, and $p(\phi|\boldsymbol{Y}_a)$ is the posterior density for ϕ in a developed economy as well as the prior density for ϕ in an emerging market economy. The main reason to use the above method comes from that $\boldsymbol{Y}_d$ is much shorter than $\boldsymbol{Y}_a$. Therefore, it is reasonable to use the posterior in the developed economies as the prior in the emerging markets.

that, for most parameters, the magnitudes of discrepancy between the posterior modes and the posterior means are relatively small.

Table 4.2: Posterior Structural Parameter Densities in the CIA Model

Parameters	Mode	Mean	S.E.	2.5%	97.5%
α	0.4625	0.4632	0.0199	(0.4314	0.4953)
β	0.9907	0.9891	0.0050	(0.9818	0.9971)
γ	0.0089	0.0092	0.0032	(0.0038	0.0144)
m*	1.0342	1.0339	0.0050	(1.0253	1.0426)
ϕ	0.7114	0.7054	0.0500	(0.6250	0.7843)
ρ	0.3646	0.3739	0.0758	(0.2521	0.4924)
Standard deviation of shocks					
σ_A	0.0497	0.0505	0.0038	(0.0442	0.0569)
σ_M	0.0350	0.0355	0.0026	(0.0311	0.0394)

Note: α = capital's share in output, β = subjective discount factor, γ = steady-state technology (net) growth rate, m^* = steady-state money (gross) growth rate, ϕ = preference parameter between leisure and consumption (i.e., relative weight to leisure), ρ = persistence of money shock, σ_A = standard deviation of technology shocks, and σ_M = standard deviation of money growth shocks.

Overall, Tables 4.2 and 4.3 show that the two models produce very similar and reasonable parameter estimates. In particular, the posterior mean of the discount factor, β, is 0.99 in each model, which implies an annualized steady-state real interest rate of 4%. The posterior mean of the steady state growth rate of technology (i.e., aggregate income), γ, is 0.92% (0.95% in the LP model) which implies 3.7% (3.8% in the LP model) growth per year. This number is much higher than the finding of Schorfheide (2000) for the U.S. (0.37% per quarter). This result is expected from the fact that Korea experienced relative high growth during the last three decades. For steady-state money growth rate m^*, the LP model provides a slightly higher value of 1.04, which means 17% growth annually.

The last two rows in each model show the estimated magnitudes of shocks and their standard errors. In each model, the estimated standard deviation of the technology shock is a little larger than that of the monetary shock though they are much higher than the estimates for the U.S. In addition, compared to the LP model, the magnitudes

Table 4.3: Posterior Structural Parameter Densities in the LP Model

Parameters	Mode	Mean	S.E.	2.5%	97.5%
α	0.3832	0.4072	0.0645	(0.3066	0.5222)
β	0.9918	0.9895	0.0044	(0.9822	0.9974)
γ	0.0096	0.0095	0.0028	(0.0048	0.0141)
m*	1.0342	1.0343	0.0049	(1.0264	1.0424)
ϕ	0.7429	0.7325	0.0457	(0.6609	0.8066)
ρ	0.3964	0.3978	0.0674	(0.2956	0.5117)
Standard deviation of shocks					
σ_A	0.0394	0.0426	0.0057	(0.0326	0.0526)
σ_M	0.0321	0.0328	0.0023	(0.0292	0.0368)

Note: α = capital's share in output, β = subjective discount factor, γ = steady-state technology (net) growth rate, m^* = steady-state money (gross) growth rate, ϕ = preference parameter between leisure and consumption (i.e., relative weight to leisure), ρ = persistence of money shock, σ_A = standard deviation of technology shocks, and σ_M = standard deviation of money growth shocks.

of two shocks are larger in the CIA model.

Most importantly, the differences regarding ρ, α, and ϕ have crucial economic implications. First, the estimated persistence of money growth (ρ) is 37.4% (39.8% in LP model). In the qualitative analysis in Section 4.4.2, it is assumed that the two models have the same parameter values. But the estimate $\hat{\rho}$ is higher in the LP model. It implies that the anticipated inflation effect on R_t is larger in the LP model. Remember that the rise of persistence implies more money injection in the next period and this will raise the expected inflation in the next period. The larger anticipated inflation effect in the LP model raises the potential that the anticipated inflation effect dominates the liquidity effect and thus R_t rises in response to an expansionary monetary shock.[42]

Second, posterior mean of capital share of output α is 46.3% conditional on the CIA model but 40.7% conditional on the LP model. To understand the influence of lower α in the LP model, note first that the second derivative of marginal product of

[42]Put differently, as will be clear in the impulse response functions of Figure 4.8, the higher anticipated inflation effect in the LP model is required to capture the rise of R_t in the data in spite of existence of the liquidity effect (i.e., downward pressure to R_t) in response to an expansionary monetary shock. Note also the CIA model does not have a mechanism for the liquidity effect and thus R_t always rises in response to a positive money shock.

labor in (4.3.23) will rise (the absolute value will fall) when α declines from 46.3% to 40.7%.[43] It means that the slope of the labor demand curve is flatter in the LP model.

Third, posterior mean of ϕ, which is the share of leisure in the composite consumption, is 70.5% in the CIA model but 73.3% in the LP model. Notice that when all else are equal, the slope of labor supply curve is determined by $\phi/(1-\phi)$. This ratio is 2.39 in the CIA model and 2.75 in the LP model. Therefore, the slope of the labor supply is steeper in LP model as shown in (4.3.20).

A direct comparison of Tables 4.1 with 4.2 and 4.3 can often provide valuable insights about how much data provide information about the parameters of interest. By and large, the two models' parameter estimates are not much different from the prior densities. However, the CIA model's parameters such as α, β and ϕ seem to be too less respondent to the data. It may reflect that the priors for these parameters are heavily data-based. But more often this implies that the model is misspecified or data do not have much information on these parameters. In these cases, the shape of likelihood function in these parameter dimension is too flat (weak identification problem), and thus the prior specification dominates for those parameters over the data.[44]

Finally, note that all posterior standard errors are smaller than all related prior standard errors since it incorporates the information from the data (Gelman et al. 2004, pp. 36–37).[45]

[43] In (4.3.23),

$$\frac{\partial \frac{W_t}{P_t}}{\partial N_t} = \frac{-\alpha(1-\alpha)K_t^{\alpha}A_t^{1-\alpha}N_t^{-\alpha-1}}{R_t}.$$

Then, it is straightforward to notice that the decline of α (0.463→0.407) leads to the decline of absolute value of $-\alpha(1-\alpha)$ from 0.249 to 0.241.

[44] In both models, discounter factor, β, is not much updated in parameter space during the estimation process. It is noteworthy that a reasonable value of β must be very close to one. Hence, it is often a tight prior used for β estimation. It is quite common to confront quite an unlikely discount factor when we allow the chain to wander around the parameter space widely. This is the identification problem in DSGE models which arises due to a lack of information in observations. As a consequence, different parameter values produce the very similar joint distribution with observed endogenous variables (An and Schorfheide 2007). To verify this point, this essay has estimated the models with alternative priors. As expected, the posterior of β is heavily affected by this modification.

[45]

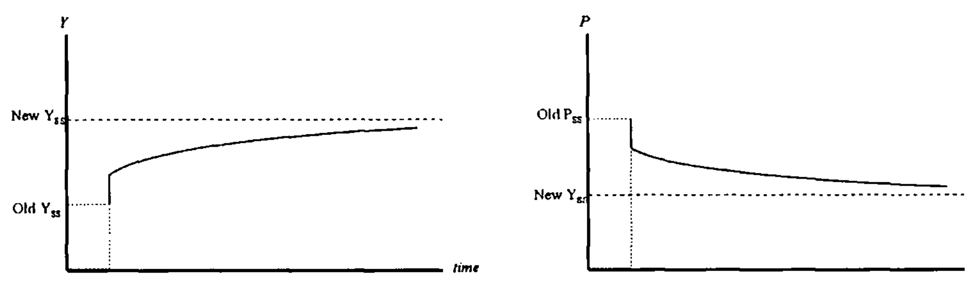
$$\begin{aligned} Var(\theta) &= E(\theta^2) - (E(\theta))^2 \\ &= E(\theta^2) - E\left[(E(\theta|y))^2\right] + E\left[(E(\theta|y))^2\right] - (E(\theta))^2 \\ &= E\left[E(\theta^2|y) - (E(\theta|y))^2\right] + E\left[(E(\theta|y))^2\right] - \left[E(E(\theta|y))\right]^2 \\ &= E\left[Var(\theta|y)\right] + Var\left[E(\theta|y)\right]. \end{aligned}$$

4.6.2 Impulse Response Analysis

Based on these posterior means of the structural parameters in the two models, Figure 4.7 shows the responses to the one standard deviation of permanent technology shock for the first ten years. On the other hand, Figure 4.8 illustrates the responses to the one standard deviation of temporary monetary shock for the first ten years.

4.6.2.1 Responses to a Positive Permanent Technology Shock

Figure 4.7 shows that while the two models produce the quantitatively different responses to a positive permanent technology shock because of the different parameter estimates, they yield the qualitatively quite similar results. Before several features of Figure 4.7 are discussed, note that the permanent technology shock differs from the transitory technology shock in shifting the steady state of the endogenous variables.



(a) Level of Output

(b) Level of Price

Figure 4.6: Responses of Output and Price to a Positive Permanent Technology Shock

First, introducing a permanent technology shock causes an increase in labor employed that declines over time. This will drives the real wage level as well as output level up. However, notice in Figure 4.7 that output declines sharply in response to the shock at period 1. The reason is that output in Figure 4.7 represents the deviation from the steady-state output level and the steady-state output rises permanently during period t. The response of output level is illustrated in Figure 4.6(a). Thereafter, the output level slowly increase because the capital level can only increase steadily. With the same reason, consumption and capital stock initially decline and then increase to the new steady-state over time.

Therefore, the posterior mean of variance conditional on data $E[Var(\theta|y)]$ is smaller than the prior variance $Var(\theta)$ up to the amount of the variance of posterior mean conditional on data $Var[E(\theta|y)]$.

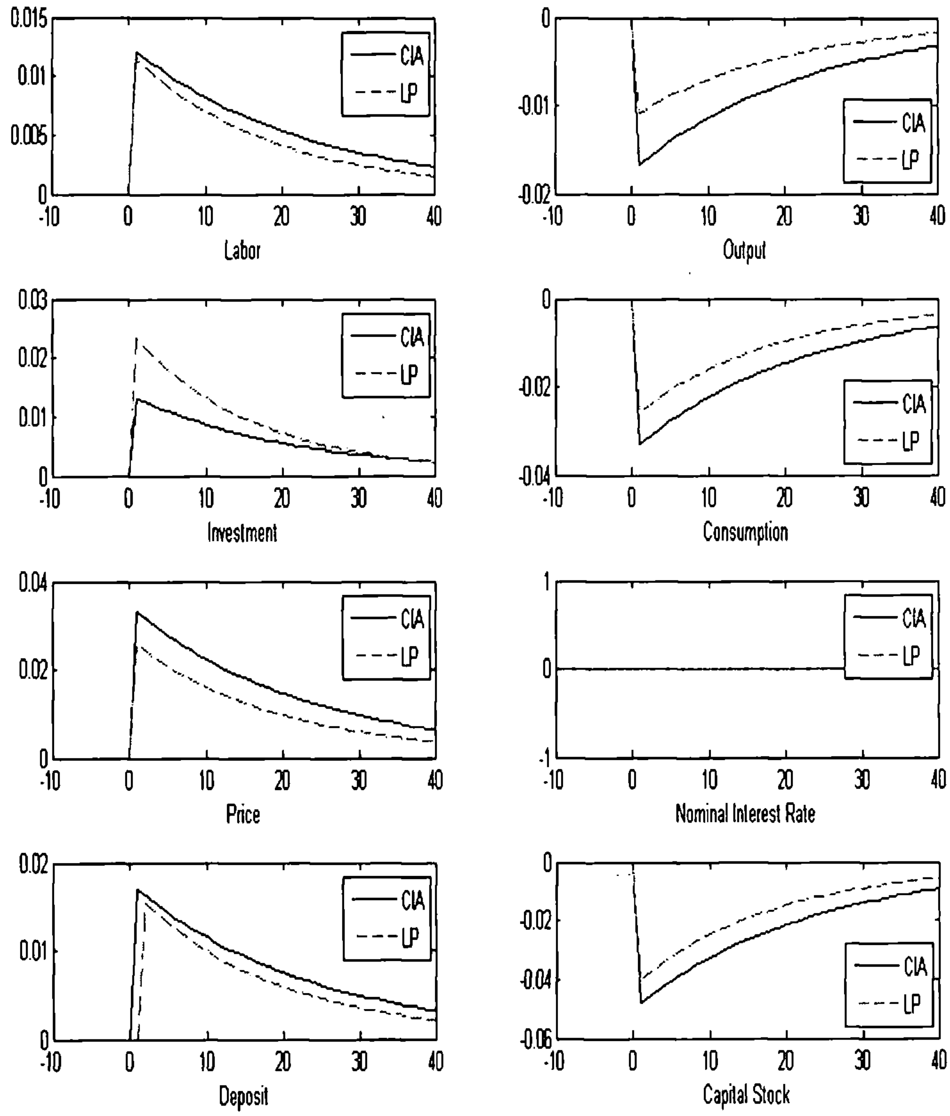


Figure 4.7: Responses to a Positive Permanent Technology Shock

Second, at period 1 the price level also declines as Figure 4.6(b) while the deviation from the steady-state price rises in Figure 4.7. Thereafter, the price level will fall steadily down to the new steady-state level. Finally, the gross interest rate is not affected by this permanent shock in the two models since the technology shock is assumed to be orthogonal to the money shock.

4.6.2.2 Responses to a Temporary Expansionary Money Shock

Figure 4.8 compares the responses in the CIA model to those in the LP model to a temporary expansionary money shock. In the figure, the solid lines represent the responses of endogenous variables in the CIA model and the dashed lines represent those in the LP model. In order to draw a clear distinction between the two models, 95% highest posterior density (HPD) intervals are suppressed in Figure 4.8.

A. Impulse Responses in the CIA Model

In a CIA model, the temporary positive money shock will increase $E_t m_{t+1}$, $E_t m_{t+2}, \cdots$ as long as the persistence is larger than zero. This increases future inflation rate and thus nominal interest rate, via the anticipated inflation effect. Then, since the interest rate acts like a tax on consumption and a subsidy on investment, consumption will fall and investment will rise.

In the labor market, there are two offsetting effects on labor employed from the demand side and supply side. Then, the decline of labor in Figure 4.8 implies that a leftward shift of labor demand curve due to the rise of R_t dominates a rightward shift of labor supply curve due to the decline of consumption (i.e., the rise of marginal utility of consumption). Now, since labor employed has fallen and the capital stock K_t is unchanged when producing output, the current output must also fall. Figure 4.8 exactly shows this case with graphs. However, as noted earlier, this is not consistent with what we observe in reality. So, the CIA model fails to account for the rise of output in response to the positive money shock while it can explain the liquidity puzzle by assuming no liquidity effect.

Finally, money injection also reduces households' desire to save since they know that the increased money will be distributed as dividends from commercial banks at the end of period. Thus, the deposit to commercial banks declines as shown in Figure 4.8.

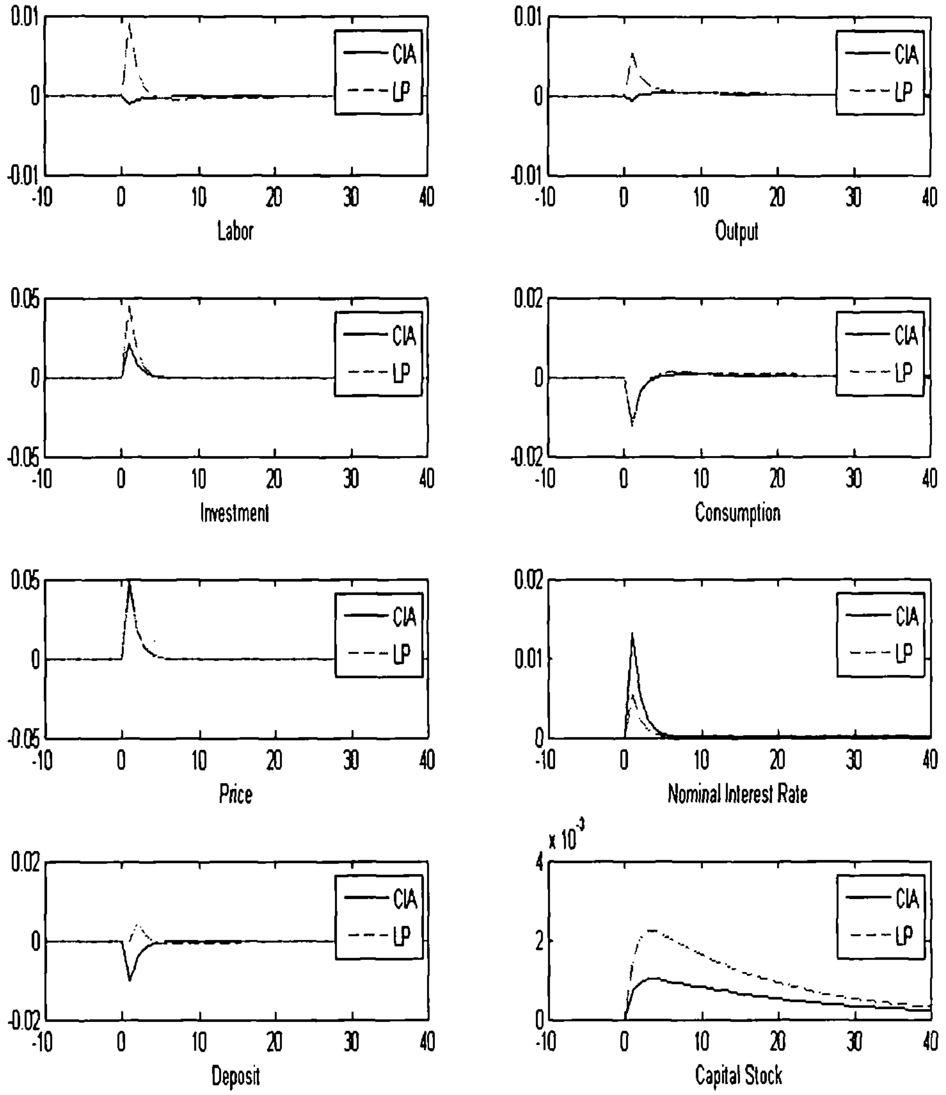


Figure 4.8: Responses to a Temporary Expansionary Money Shock in the CIA and in the LP Models

B. Impulse Responses in the LP Model

Several features need to be mentioned on the responses in the LP model. First and most importantly, gross interest rate R_t still rises though the magnitude of rise is less than that in the CIA model. Unlike the CIA model, LP model has another effect of money injection upon R_t, namely, the liquidity effect. Then, the slight rise of R_t in Figure 4.8 implies that the expected inflation effect on the interest rate dominates the liquidity effect in Korea during the examined period.[46] Namely, when the central bank raises the money growth rate, the estimated model predicts the rise in R_t rather than the fall. Importantly, this result implies that the liquidity puzzle is not indeed a puzzle in Korean economy.

Second, given the rise of R_t in response to an expansionary monetary shock, whether N_t rises or falls hinges primarily on how much C_t drops and the slopes of labor demand and supply curves. Clearly, the rise of R_t will drive C_t down and I_t up. Then, the labor demand curve will shift *slightly* to the left while the labor supply curve will shift *slightly* to the right. However, there is another factor which shifts the labor supply curve to the right. Note that if households were allowed to adjust their deposit, they would reduce their deposit like in the CIA model. But, in the LP model, households are unable to reduce their deposit, so deposit is rigid.

Now after a positive monetary shock occurs, they realize that they saved too much. This decrease of available cash drives consumption to fall further and thus the labor supply curve to shift further to the right. Figure 4.5(b) exactly illustrates this case with graphs. It is noteworthy that since labor employed has increased and the capital stock is unchanged, the current output must rise as shown in the figure.

Finally, deposit increases in the LP model at the end of period t since households realize that they have too much cash due to an expansionary monetary shock. If households were allowed to reduce the deposit, they would reduce their deposit as in the CIA model. To the contrary, the deposit decreases in the CIA model just after households observe the expansionary monetary shock at the beginning of period t. Remember that deposit in the LP model is determined at the end of period t while deposit in the CIA model is determined at the beginning of period t.

[46]To the contrary, in a standard IS/LM model which assumes the price stickiness, only a liquidity effect is present. Thus, an expansionary money shock requires a decline in the interest rate to restore money-market equilibrium.

All in all, the responses of endogenous variables in Figure 4.8 favor the LP model over the CIA model in the sense that the LP model accounts well for two stylized facts in the business cycles in Korean economy: (1) the money growth and the short-term interest rate are positively correlated, and (2) the rise of money growth rate is followed by the rise of real output. Formal comparison based on the Bayes' factor will be discussed in the model comparison section.

4.6.3 Variance Decomposition

This section investigates how much the variations of endogenous variables can be explained by the technology shocks or the monetary shocks using forecast error variance decomposition. Thus, decompositions of the unconditional variance in Tables 4.4 and 4.5 separate the variation of each endogenous variable into two types of shocks in the system.

As expected, the two models seem to support the dominance of technology shock in the Korean business cycles. In the CIA model, as shown in Table 4.4, fluctuations of consumption, capital, labor and output are almost completely (about 99%) explained by the technology shock. This result is consistent with the argument of the long run neutrality of real variables. In addition, money shock has a very limited effect on price, investment and loan movement. On the other hand, the interest rate is completely driven by the monetary shock because of its setup in Equation (4.A.29) and most of movement in nominal wage is also explained by monetary shock. By and large, the channel through which money affects the real economy in the CIA model seems to have a minor role. That is, in the CIA model, monetary propagation mechanism seems to have a very limited ability to capture movements of endogenous variables.

Table 4.5 shows LP model's variance decomposition results. Similar to the CIA model, monetary shocks have a critical role in account for the variations of interest rate and nominal wage. However, some interesting differences can be found from the table. Note first that now money shock in the LP model has substantial effects on loan, investment and price. In particular, investment and its source, loan are to a large extent driven by the monetary shocks. Still most of movements in consumption, capital, labor and output seem to be explained by technology shocks. Nevertheless, for output fluctuation, the importance of money shock has substantially increased from almost zero in the CIA model to 10% in the LP model.

Table 4.4: Contribution of Each Shock to the Variance of the Forecast Error in the CIA Model

Variables	Mean	S.E.	2.5%	97.5%	Mean	S.E.	2.5%	97.5%
	Technology shock				**Monetary shock**			
Consumption	0.9863	0.0071	(0.9768	0.9953)	0.0137	0.0071	(0.0047	0.0232)
Deposit	0.9571	0.0272	(0.9246	0.9950)	0.0429	0.0272	(0.0050	0.0754)
Capital	0.9994	0.0005	(0.9988	0.9999)	0.0006	0.0005	(0.0001	0.0012)
Loan	0.6988	0.0510	(0.6119	0.7713)	0.3012	0.0510	(0.2287	0.3881)
Investment	0.7406	0.1902	(0.4857	0.9636)	0.2594	0.1902	(0.0364	0.5143)
Labor	0.9985	0.0016	(0.9969	0.9998)	0.0015	0.0016	(0.0002	0.0031)
Price	0.8282	0.0476	(0.7629	0.9056)	0.1718	0.0476	(0.0944	0.2371)
Nominal Interest rate	0.0000	0.0000	(0.0000	0.0000)	1.0000	0.0000	(1.0000	1.0000)
Nominal Wage	0.1261	0.0491	(0.0424	0.1952)	0.8739	0.0491	(0.8048	0.9576)
Output	0.9994	0.0006	(0.9988	0.9999)	0.0006	0.0006	(0.0001	0.0012)

Note: Posterior means and their standard errors are reported with 95% highest posterior density (HPD) in parentheses.

Table 4.5: Contribution of Each Shock to the Variance of the Forecast Error in the LP Model

Variables	Mean	S.E.	2.5%	97.5%	Mean	S.E.	2.5%	97.5%
	Technology shock				**Monetary shock**			
Consumption	0.9649	0.0285	(0.9342	0.9973)	0.0351	0.0285	(0.0027	0.0658)
Deposit	0.9768	0.0222	(0.9481	0.9998)	0.0232	0.0222	(0.0002	0.0519)
Capital	0.9942	0.0063	(0.9876	0.9998)	0.0058	0.0063	(0.0002	0.0124)
Loan	0.4754	0.0971	(0.3122	0.6186)	0.5246	0.0971	(0.3814	0.6878)
Investment	0.6254	0.1740	(0.3638	0.8441)	0.3746	0.1740	(0.1559	0.6362)
Labor	0.9239	0.0328	(0.8696	0.9713)	0.0761	0.0328	(0.0287	0.1304)
Price	0.6986	0.1423	(0.5008	0.9342)	0.3014	0.1423	(0.0658	0.4992)
Nominal Interest rate	0.0000	0.0000	(0.0000	0.0000)	1.0000	0.0000	(1.0000	1.0000)
Nominal Wage	0.0731	0.0327	(0.0226	0.1199)	0.9269	0.0327	(0.8801	0.9774)
Output	0.8920	0.1712	(0.7074	0.9993)	0.1080	0.1712	(0.0007	0.2926)

Note: Posterior means and their standard errors are reported with 95% highest posterior density (HPD) in parentheses.

As long as we buy the idea that at least some portion of business fluctuation can be attributed to the monetary shocks as in the literature, the CIA model seems to have a significant deficiency to capture this idea. Therefore, as in the impulse response analysis, the LP model which has a stronger monetary transmission mechanism appears consistent with the reality and data in Korea.

4.6.4 Model Comparison

The impulse response analysis seems to favor the LP model over the CIA model in the sense that the LP model better performs to capture two stylized facts of Korean business cycles: money growth is followed by the rise of interest rate and the rise of output.

However, the formal comparison has not been provided yet. Note that Bayesian estimation of several models yields a natural method of model comparison. The most widely used comparison method in the Bayesian approach is the posterior odds which is computed as the product of prior model odds ratio and the Bayes factor (Gelman et al. 2004, pp. 179–186). However, since we do not have any prior information which model is better, this essay assumes that the two models have equal prior probabilities. Then, the Bayes factor (i.e., the marginal likelihood ratio) can be used for model comparison as (4.6.1).[47]

$$\frac{p(\mathscr{M}_1|\boldsymbol{Y})}{p(\mathscr{M}_2|\boldsymbol{Y})} = \frac{p(\boldsymbol{Y}|\mathscr{M}_1)}{p(\boldsymbol{Y}|\mathscr{M}_2)}\frac{p(\mathscr{M}_1)}{p(\mathscr{M}_2)} \approx \frac{p(\boldsymbol{Y}|\mathscr{M}_1)}{p(\boldsymbol{Y}|\mathscr{M}_2)} = \text{Bayes factor}, \tag{4.6.1}$$

where $\mathscr{M}_1$ and $\mathscr{M}_2$ stand for a CIA model and an LP model, respectively. Note that $p(\boldsymbol{Y}|\mathscr{M}_1)$ is the marginal density of the data conditional on the model $\mathscr{M}_1$ (i.e., likelihood of model $\mathscr{M}_1$) and is equivalent to the marginal likelihood $p(\boldsymbol{Y}) = \int p(\boldsymbol{\theta}, \boldsymbol{Y})d\boldsymbol{\theta}$ (i.e., a normalizing constant).

[47] On the other hand, when one uses the maximum likelihood estimation, model comparison can be implemented via the likelihood ratio. Let us consider two models: $\mathscr{M}_1$ and $\mathscr{M}_2$. Let $p_i(\boldsymbol{Y}|\boldsymbol{\theta}_i)$ denote the data density under model $\mathscr{M}_i$. Since we can calculate the likelihood of each model, the following likelihood ratio can be used,

$$\frac{p_1(\boldsymbol{Y}|\hat{\boldsymbol{\theta}}_1)}{p_2(\boldsymbol{Y}|\hat{\boldsymbol{\theta}}_2)},$$

where $\hat{\boldsymbol{\theta}}_1$ and $\hat{\boldsymbol{\theta}}_2$ are the maximum likelihood estimates in $\mathscr{M}_1$ and $\mathscr{M}_2$, respectively. It is noteworthy that the distinction between the likelihood ratio and the Bayes factor is that the former value is based on the maximized likelihood estimates but the latter value is based on the averages from the density functions.

Theoretically, this marginal likelihood can be computed as (4.6.2).

$$p(\boldsymbol{Y}|\mathscr{M}_1) = \int p(\boldsymbol{Y}, \boldsymbol{\theta}_1|\mathscr{M}_1)d\boldsymbol{\theta}_1 = \int p(\boldsymbol{Y}|\boldsymbol{\theta}_1, \mathscr{M}_1)p(\boldsymbol{\theta}_1|\mathscr{M}_1)d\boldsymbol{\theta}_1, \tag{4.6.2}$$

where $\boldsymbol{\theta}_1$ represents a vector of parameters in the CIA model.

The practical problem is that Equation (4.6.2) cannot be analytically calculated due to the complexity of the integrand. Remember that we did not calculate this marginal likelihood since we focused on the posterior kernel density in the MCMC algorithm. Two approximation methods are widely used to obtain the marginal density of the data conditional on the model. The first one is called Laplace approximation method and the second one is modified harmonic mean method (see, Appendix 4.B).[48]

Table 4.6: Prior and Posterior Model Probabilities

	CIA model	LP model
Prior probability	1/2	1/2
$\ln P(Y\|\mathscr{M})$: log marginal likelihood	N/A	N/A
- Laplace approximation	461.79	470.69
- Harmonic mean	461.79	470.78
Posterior odds P_i/P_{CIA} (where $i = CIA, LP$)	1.0000	7319.5
Posterior probability P_i (where $i = CIA, LP$)	0.0001	0.9999

Note: Based on the Laplace approximation to marginal likelihoods, posterior odds and posterior probability are computed. For example, in the LP model, these two are computed as follows,

$$\text{Posterior odds} = \frac{e^{470.69}}{e^{461.79}} = 7319.5, \text{ and posterior probability} = \frac{e^{470.69}}{e^{461.79} + e^{470.69}} = 0.9999.$$

The model comparison results are summarized in Table 4.6, based on the assumption that the two models are equally possible a priori. As clear in the table, we do not know the exact value of marginal density of the data. But, we have two approximations: Laplace approximation and modified harmonic mean estimator. In Table 4.6, the log values of these two approximations are provided for the CIA model and the LP model. Notice that two approximation methods yield very similar values for the marginal data densities.

Based on Laplace approximation method, the logarithm of the marginal data density is 461.79 in the CIA model and is 470.69 in the LP model. Since these values are

[48]These numbers are obtained from the Dynare program.

logarithms, the Bayes factor shows that the goodness-of-fit of the LP model is about $e^9 (= e^{470-461})$ times better than that of the CIA model. Then as the last row of the table shows, compared to the LP model, the probability that the CIA model is correct is practically zero. Accordingly, we can conclude that Korean data set provides a very strong evidence that favors the LP model.

4.6.5 Comparison between Pre-crisis and Post-crisis Periods

We found that the rigidity of deposit in the LP model helps it be better able to match up with what we observe in the Korean business cycles. This section then turns to the issue on how well the LP model captures the changes in Korean business cycles across times. This essay divides the entire period into 1973:Q1 to 1997:Q4 and 1998:Q1 to 2006:Q3.

The main motivation of this division is that Korea experienced several important policy changes and institutional reform at the end of 1997 due to the Korean financial crisis. For example, the exchange rate regime moved from the currency basket system to the flexible exchange rate system. The central bank was granted more independence from the government and it started to primarily use the call rate target (similar to the federal fund rate operating target in the U.S. and the official Bank Rate in the U.K.) as a policy instrument rather than the monetary aggregates. Lastly the Bank of Korea began to publicly announce the inflation target, which is often referred to as the overriding objective (see Section 2.5.5 in Chapter 2, for more details on the Korean financial crisis). This can be seen as a reflection of the view that monetary policy cannot affect real quantities in the long run, and that inflation is mostly a monetary phenomenon in the long run (Bernanke and Mishkin 1997). We expect that these important changes can be captured in the estimated parameters of the LP model.

Tables 4.7 and 4.8 report the posterior distributions for two sub-sample periods. Two important features draw our attention when the two tables are compared each other. First and most interestingly, the money shock after the crisis declined sharply almost to one third of the pre-crisis level (i.e., 0.0344 $\rightarrow$ 0.0134) while the magnitude of technology shock fell by 20% (i.e., 0.0393 $\rightarrow$ 0.0309). Note also the estimated steady-state growth rate of money supply has declined dramatically from 19.1% to 5.4% at annual basis. This large drop of money supply appears to be one of reasons for the recent inflation stability in Korea.

Table 4.7: Posterior Structural Parameters in the LP Model: 1973:Q1 - 1997:Q4

Parameters	Mode	Mean	S.E.	2.5%	97.5%
α	0.3677	0.4125	0.0625	(0.3048	0.5366)
β	0.9922	0.9903	0.0043	(0.9817	0.9968)
γ	0.0116	0.0114	0.0028	(0.0079	0.0155)
m*	1.0437	1.0434	0.0060	(1.0338	1.0534)
ϕ	0.7407	0.7292	0.0452	(0.6437	0.7919)
ρ	0.3645	0.3678	0.0659	(0.2966	0.4836)
Standard deviation of shocks					
σ_A	0.0331	0.0393	0.0047	(0.0282	0.0487)
σ_M	0.0351	0.0344	0.0027	(0.0314	0.0389)

Note: α = capital's share in output, β = subjective discount factor, γ = steady-state technology (net) growth rate, m^* = steady-state money (gross) growth rate, ϕ = preference parameter between leisure and consumption (i.e., relative weight to leisure), ρ = persistence of money shock, σ_A = standard deviation of technology shocks, and σ_M = standard deviation of money growth shocks.

Table 4.8: Posterior Structural Parameters in the LP Model: 1998:Q1 - 2006:Q3

Parameters	Mode	Mean	S.E.	2.5%	97.5%
α	0.4316	0.4466	0.0679	(0.3417	0.5607)
β	0.9923	0.9900	0.0043	(0.9828	0.9975)
γ	0.0099	0.0100	0.0032	(0.0045	0.0157)
m*	1.0135	1.0135	0.0043	(1.0055	1.0213)
ϕ	0.7087	0.7011	0.0502	(0.6267	0.7897)
ρ	0.4834	0.4875	0.0857	(0.3459	0.6277)
Standard deviation of shocks					
σ_A	0.0280	0.0309	0.0051	(0.0207	0.0400)
σ_M	0.0126	0.0134	0.0015	(0.0106	0.0161)

Note: α = capital's share in output, β = subjective discount factor, γ = steady-state technology (net) growth rate, m^* = steady-state money (gross) growth rate, ϕ = preference parameter between leisure and consumption (i.e., relative weight to leisure), ρ = persistence of money shock, σ_A = standard deviation of technology shocks, and σ_M = standard deviation of money growth shocks.

Figure 4.9 illustrates CPI inflation rate at annual basis for the Korean economy for 1971Q1-2007Q4 where the dashed line stands for the period of 1998Q1. It shows that annual inflation rates have become stabilized around 3% since the financial crisis.

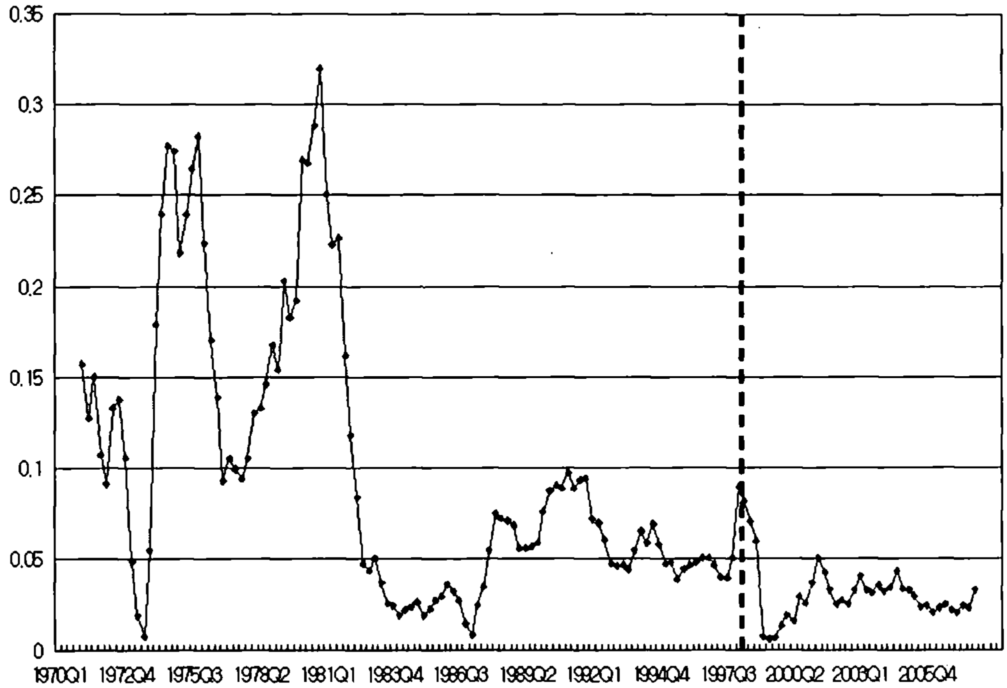


Figure 4.9: CPI Annual Inflation Rates: 1971Q1-2007Q4

Second, as expected, the growth rate of the economy γ fell from 4.7% to 4.1% at annual rate. This result is consistent with the observation that Korean economy experienced the slowdown of the growth since the Korean financial crisis. On the other hand, the persistence of the monetary shock has increased by around a third after the financial crisis.

Lastly, Tables 4.9 and 4.10 present the forecast error variance decompositions for two sub-periods. Overall, they confirm that the importance of money shock has significantly dropped. This seems to reflect the phenomenon that after the financial crisis, the magnitude of money shock declined sharply while the technology shock just slightly decreased.

Table 4.9: Variance Decomposition in the LP Model: 1973:Q1 – 1997:Q4

Variables	Mean	S.E.	2.5%	97.5%	Mean	S.E.	2.5%	97.5%
	Technology shock				**Monetary shock**			
Consumption	0.9467	0.0411	(0.8985	0.9958)	0.0533	0.0411	(0.0042	0.1015)
Deposit	0.9587	0.0398	(0.8992	0.9986)	0.0413	0.0398	(0.0014	0.1008)
Capital	0.9907	0.0103	(0.9818	0.9998)	0.0093	0.0103	(0.0002	0.0182)
Loan	0.3477	0.0929	(0.1988	0.4878)	0.6523	0.0929	(0.5122	0.8012)
Investment	0.5599	0.1513	(0.3682	0.7721)	0.4401	0.1513	(0.2279	0.6318)
Labor	0.8757	0.0547	(0.7965	0.9532)	0.1243	0.0547	(0.0468	0.2035)
Price	0.5778	0.1629	(0.3411	0.8409)	0.4222	0.1629	(0.1591	0.6589)
Nominal Interest rate	0.0000	0.0000	(0.0000	0.0000)	1.0000	0.0000	(1.0000	1.0000)
Nominal Wage	0.0447	0.0235	(0.0117	0.0780)	0.9553	0.0235	(0.9220	0.9883)
Output	0.8207	0.2343	(0.4849	0.9993)	0.1793	0.2343	(0.0007	0.5151)

Note: Posterior means and their standard errors are reported with 95% highest posterior density (HPD) in parentheses.

Table 4.10: Variance Decomposition in the LP Model: 1998:Q1 – 2006:Q3

Variables	Mean	S.E.	2.5%	97.5%	Mean	S.E.	2.5%	97.5%
	Technology shock				**Monetary shock**			
Consumption	0.9847	0.0139	(0.9704	0.9989)	0.0153	0.0139	(0.0011	0.0296)
Deposit	0.9920	0.0079	(0.9825	0.9999)	0.0080	0.0079	(0.0001	0.0175)
Capital	0.9977	0.0026	(0.9951	0.9999)	0.0023	0.0026	(0.0001	0.0049)
Loan	0.7091	0.0961	(0.5621	0.8533)	0.2909	0.0961	(0.1467	0.4379)
Investment	0.6845	0.2364	(0.2795	0.9931)	0.3155	0.2364	(0.0069	0.7205)
Labor	0.9745	0.0134	(0.9574	0.9942)	0.0255	0.0134	(0.0058	0.0426)
Price	0.8692	0.0862	(0.7678	0.9872)	0.1308	0.0862	(0.0128	0.2322)
Nominal Interest rate	0.0000	0.0000	(0.0000	0.0000)	1.0000	0.0000	(1.0000	1.0000)
Nominal Wage	0.2146	0.0936	(0.0836	0.3576)	0.7854	0.0936	(0.6424	0.9164)
Output	0.9809	0.0484	(0.9598	0.9998)	0.0191	0.0484	(0.0002	0.0402)

Note: Posterior means and their standard errors are reported with 95% highest posterior density (HPD) in parentheses.

4.7 Conclusion

Using the Korean data, this chapter estimated two versions of cash-in-advance monetary DSGE models – the baseline cash-in-advance (CIA) and the limited participation (LP) models – in which firms should borrow cash to pay wages in advance while households have to hold cash to purchase consumption goods. In comparison to the CIA model, the central idea of the LP model is that households make a decision on deposit before they observe technology and money shocks in the current period while firms make a decision on borrowing after they observe those shocks. It is shown that this constraint to households leads to the rigidity of the deposit and provides interesting dynamics to the Korean economy.

The main interest of this essay is which model better match up with what we often observe in Korean business cycles. We focused on two important stylized facts in monetary economics. First, the monetary aggregates and the short-term interest rate are usually positively correlated, which is often called the liquidity puzzle. Unlike the previous literature which tried hard to yield the fall of interest rate in response to a positive money shock, this essay has clearly demonstrated that the rise of interest rate, which is widely observed across countries, is consistent with the prediction of the LP model by using the estimated parameters for Korean economy. This case arises when the liquidity effect is overwhelmed by the anticipated inflation effect in the LP model. So this essay finds evidence that the liquidity puzzle is not a real puzzle at least in Korea. Notice that since the CIA model, by construction, has the anticipated inflation effect but does not have a mechanism for the liquidity effect in response to a positive money shock, it is also able to account for this positive correlation between the interest rate and money growth.

Another stylized fact is that an expansionary monetary policy tends to be followed by the increase of output in the short run. For this observation, the two estimated models predict the opposite results. As shown in their impulse response functions based on the estimated parameters, the CIA model wrongly predicts the fall of output in response to the positive money shock while the LP model successfully predicts the rise of output. It is shown that the rigidity of deposit in the LP model has a critical role to match up with this observation.

Therefore, this essay argued that the LP model makes a great deal more sense than the CIA model in accounting for the two stylized facts in the Korean business cycles. In addition, the formal comparison is implemented with the Bayes factor and the result sufficiently favors the LP model over the CIA model.

Finally, based on the LP model, this chapter compared two sub-sample periods: 1973:Q1 to 1997:Q4 and 1998:Q1 to 2006:Q3 to investigate whether several policy changes and institutional reforms since the Korean financial crisis, have made any noticeable changes in the Korean business cycles. We found that the money shock after the crisis declined sharply almost to one third of the pre-crisis level while the magnitude of technology shock fell but not too much.

Clearly, there are some limitations on this essay. This chapter has focused on only two stylized facts in Korean business cycles, but did not attempt to account for other important features of Korean business cycles, for example, the inertia in inflation and the persistence in output in response to an expansionary money shock. Therefore, an important direction for future research is to extend the current models to more complex but realistic ones by embodying more frictions such as Calvo-style nominal price and wage contract, habit formation and adjustment costs in investment, in particular. In addition, estimating a sticky price model and comparing it with a limited participation model seem to be an interesting topic for future research.

Appendix to Chapter 4

4.A Solving CIA Models

Following the outline of model solution in Nason and Cogley (1994), this appendix presents the detailed steps to obtain the solution in the model.

4.A.1 Model Setup

In this model, taking as given P_t, W_t, R_t^H, F_t and B_t, households solve an optimization problem of (4.A.1),

$$\begin{aligned} \max_{\{C_t, H_t, M_{t+1}^H, D_t^H\}} \quad & E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[(1-\phi) \ln C_t + \phi \ln(1-H_t) \right] \right\} \\ s.t. \quad & P_t C_t + D_t^H \leq M_t^H + W_t H_t \\ & 0 \leq D_t^H \\ & M_{t+1}^H = (M_t^H - D_t^H + W_t H_t - P_t C_t) + R_t^H D_t^H + F_t + B_t, \end{aligned} \tag{4.A.1}$$

where both discount factor β and the preference parameter ϕ are located between zero and one.

In the above optimization problem, households start period t with money holding M_t^H which is carried from period $t-1$. Nominal consumption $P_t C_t$ is financed by $(M_t^H - D_t^H)$ and $W_t H_t$, where D_t^H, W_t, and H_t represent the deposit, the nominal wage, and the labor supply, respectively. Note that the leisure endowment is normalized to one. In addition, R_t^H is the gross interest rate to the deposit in dollars (not goods) and F_t and B_t are dividends from firms and commercial banks. Finally, $E_0\{\cdot\}$ is the expectations operator conditional on information available at time 0.

On the other hand, taking as given P_t, R_t^F and W_t, firms solve a maximization problem of (4.A.2),

$$\max_{\{F_t,K_{t+1},N_t,L_t^F\}} \quad E_0\left\{\sum_{t=0}^{\infty}\beta^{t+1}\frac{F_t}{C_{t+1}P_{t+1}}\right\} \tag{4.A.2}$$
$$s.t. \quad F_t + L_t^F(R_t^F - 1) + W_tN_t = P_t\left[Y_t - I_t\right]$$
$$W_tN_t \leq L_t^F$$
$$K_{t+1} = I_t + (1-\delta)K_t,$$

where $0 < \delta < 1$ is the depreciation rate of capital, L_t^F is the loan from commercial banks, I_t is the real investment, Y_t is output, and R_t^F is the gross interest rate to the loan. Note that in this case $\beta^{t+1}\frac{1}{C_{t+1}P_{t+1}}$ can be understood as the discount factor for the log utility of households.[49]

Then, the central bank lets the money stock M_{t+1}^G grow at rate $m_t = M_{t+1}^G/M_t^G$, so the net growth rate of money supply $\ln m_t$ follows an AR(1) stochastic process,

$$\ln m_t = (1-\rho)\ln m^* + \rho\ln m_{t-1} + \epsilon_{m,t} \qquad \epsilon_{m,t} \sim N(0,\sigma_m^2). \tag{4.3.10}$$

where ρ is the persistence of money growth rate, $\ln m^*$ is money growth rate at the steady state.

Taking as given M_{t+1}^G, M_t^G, R_t^F and R_t^H, commercial banks solve a simple static optimization problem (4.A.3) because they do not carry anything to the next period,

$$\max_{\{B_t,L_t^B,D_t^B\}} \quad E_0\left\{\sum_{t=0}^{\infty}\beta^{t+1}\frac{B_t}{C_{t+1}P_{t+1}}\right\} \tag{4.A.3}$$
$$s.t. \quad B_t = (M_{t+1}^G - M_t^G) + (R_t^F - 1)L_t^B - (R_t^H - 1)D_t^B$$
$$L_t^B \leq (M_{t+1}^G - M_t^G) + D_t^B.$$

In the model, firms produce output Y_t by employing labor N_t and capital stock K_t.

$$Y_t = K_t^{\alpha}(A_tN_t)^{1-\alpha}, \tag{4.3.4}$$

where A_t is the state of labor-augmenting technology at period t and $0 < \alpha < 1$ is the capital's share in output. Finally, A_t evolves according to the random walk process

[49] During the discussion, Dr.Renström gratefully indicated this point.

with drift. Thus, technology shock is assumed to be permanent.

$$\ln A_t = \gamma + \ln A_{t-1} + \epsilon_{A,t} \qquad \epsilon_{A,t} \sim N(0, \sigma_A^2), \tag{4.3.5}$$

where γ stands for the steady state technology (net) growth rate.

4.A.2 Dynamic Optimization

This section solves the optimization problems of three agents by applying the Lagrangian method.

[1] Households To solve the households' maximization problem, we set up the Lagrangian function as

$$\begin{aligned}
\mathfrak{L} = E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} \Big\{ \Big[(1-\phi)\ln C_\tau + \phi \ln(1-H_\tau)\Big] + \mu_\tau \Big[M_\tau^H + W_\tau H_\tau - P_\tau C_\tau - D_\tau^H\Big] \\
+ \lambda_\tau \Big[(M_\tau^H + W_\tau H_\tau - P_\tau C_\tau - D_\tau^H) + R_{H,\tau} D_\tau^H + F_\tau + B_\tau - M_{\tau+1}^H\Big] \Big\}
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial \mathfrak{L}}{\partial C_t} = \frac{1-\phi}{C_t} - \lambda_t P_t - \mu_t P_t = 0 \\
&\frac{\partial \mathfrak{L}}{\partial H_t} = -\frac{\phi}{1-H_t} + \lambda_t W_t + \mu_t W_t = 0 \\
&\frac{\partial \mathfrak{L}}{\partial D_t^H} = -\lambda_t - \mu_t + \lambda_t R_t^H = 0 \\
&\frac{\partial \mathfrak{L}}{\partial M_{t+1}^H} = -\lambda_t + \beta E_t(\lambda_{t+1} + \mu_{t+1}) = 0 \\
&0 = \mu_t [M_t^H + W_t H_t - P_t C_t - D_t^H] \\
&0 \leq \mu_t \\
&0 \leq [M_t^H + W_t H_t - P_t C_t - D_t^H].
\end{aligned} \tag{4.A.4}$$

Here λ_t denotes the marginal utility of nominal wealth and μ_t, the Lagrange multiplier related to the CIA constraint on consumption, denotes the marginal value of money due to the liquidity services at period t. The last three lines are the Kuhn-Tucker conditions associated with this constraint: either the CIA constraint does not bind, in which case $\mu_t = 0$ or the CIA constraint binds, in which case $\mu_t \geq 0$.

By linking the first two lines of necessary conditions of (4.A.4), one gets the households' intratemporal optimality condition for choice between consumption and leisure as (4.3.20), which shows the *labor supply schedule*,

$$\frac{W_t}{P_t}\frac{1-\phi}{C_t} = \frac{\phi}{1-H_t} \quad \text{or} \quad \frac{\phi}{1-\phi}\left[\frac{C_tP_t}{1-H_t}\right] = W_t, \tag{4.3.20}$$

for all $t = 0, 1, 2, \cdots$.

On the other hand, by combining the first, third and fourth conditions of (4.A.4), one gets the household's intertemporal optimality condition for choice between consumption and deposit (i.e., consumption Euler equation) as (4.3.21),

$$\frac{1}{C_tP_t} = \beta R_t^H E_t\left\{\frac{1}{C_{t+1}P_{t+1}}\right\}. \tag{4.3.21}$$

Several features in (4.A.4) are worth noting. First, from the third line of condition,

$$R_t^H = \frac{\lambda_t + \mu_t}{\lambda_t} \quad \text{or} \quad i_{H,t} = \frac{\mu_t}{\lambda_t},$$

where $i_{H,t}$ is the net interest rate. Then, as long as $R_t^H > 1$, $\mu_t > 0$.[50] Accordingly, from the fifth condition of (4.A.4), the CIA constraint for households binds and (4.3.1) holds with equality.

$$P_tC_t = M_t^H - D_t^H + W_tH_t. \tag{4.3.1}$$

Then, the first line constraint implies that the marginal utility of consumption of one unit of money is larger than the marginal utility of wealth λ_t by the amount of μ_t when the CIA constraint binds. As Svensson (1985) notes, the wedge arises from the fact that some wealth cannot be used to buy current consumption.

Second, by substituting the third line condition, $\lambda_t = \mu_t/(R_t^H - 1)$ into the first line condition, one gets the opportunity cost of holding money for purchasing goods,

[50] On the other hand, Friedman's rule states that the nominal net interest rate i_t should be zero to achieve the optimal inflation rate, which is equal to negative net real interest rate. The argument is based on the notion that since the social marginal cost of producing money is close to zero, the private opportunity cost of holding money also should be zero at the optimum. Then, from the Fisher relationship, the optimal inflation π_t^* is

$$\begin{aligned} 1 &= (1+\pi_t^*)(1+r_t) \\ \pi_t^* &= -\frac{r_t}{1+r_t} \approx -r_t, \end{aligned}$$

where r_t is the real interest rate.

namely μ_t, as (4.A.5)

$$\frac{U_{c,t}}{P_t} = \lambda_t + \mu_t = \mu_t \frac{R_t^H}{R_t^H - 1}$$
$$\mu_t = \frac{U_{c,t}}{P_t}\frac{R_t^H - 1}{R_t^H} = \frac{U_{c,t}}{P_t}\frac{i_{H,t}}{1 + i_{H,t}}, \tag{4.A.5}$$

where $U_{c,t}$ is the marginal utility of consumption (i.e., $\frac{1-\phi}{C_t}$ in the current model).

Lastly, through the erosion of real money balances, positive nominal interest rate (or inflation) represents a tax on the purchases of the cash good, and therefore higher rates of inflation shift household demand away from the cash good and toward the credit good. In Cooley and Hansen (1989)'s model, this implies that higher inflation increases the demand for leisure. Thus, nominal interest rate imposes an efficiency cost by distorting the consumer's choice between cash and credit goods. However, in the current model, since the consumption and leisure are both cash goods for the household, the choice between them are not distorted by the inflation taxation.[51]

[2] Firms To solve the firms' maximization problem, we set up the Lagrangian function as

$$\mathcal{L} = E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t+1} \Big\{ \frac{F_\tau}{C_{\tau+1}P_{\tau+1}} + \mu_\tau^F \left[L_\tau^F - W_\tau N_\tau\right] + \lambda_\tau^F \Big[P_\tau \left[K_\tau^\alpha (A_\tau N_\tau)^{1-\alpha} + (1-\delta)K_\tau - K_{\tau+1}\right] - L_\tau^F(R_{F,\tau} - 1) - W_\tau N_\tau - F_\tau\Big]\Big\}$$

$$\frac{\partial \mathcal{L}}{\partial F_t} = E_t \frac{1}{C_{t+1}P_{t+1}} - \lambda_t^F = 0 \Leftrightarrow \lambda_t^F = E_t \frac{1}{C_{t+1}P_{t+1}}$$
$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = -\lambda_t^F P_t + \beta E_t \lambda_{t+1}^F P_{t+1} \left[\alpha K_{t+1}^{\alpha-1}(A_{t+1}N_{t+1})^{1-\alpha} + (1-\delta)\right] = 0$$
$$\frac{\partial \mathcal{L}}{\partial N_t} = (1-\alpha)\lambda_t^F P_t K_t^\alpha A_t^{1-\alpha} N_t^{-\alpha} - \lambda_t^F W_t - \mu_t^F W_t = 0$$
$$\frac{\partial \mathcal{L}}{\partial L_t^F} = -\lambda_t^F (R_t^F - 1) + \mu_t^F = 0$$
$$0 = \mu_t^F [L_t^F - W_t N_t], \quad 0 \leq \mu_t^F, \quad \text{and} \quad 0 \leq [L_t^F - W_t N_t]. \tag{4.A.6}$$

[51] As shown in the cash-in-advance constraint in (4.A.2), leisure (or labor) is cash goods in the current model.

Here λ_t^F stands for the households' expected marginal utility of an additional £1 from the current dividend F_t. μ_t^F, Lagrange multiplier related to the CIA constraint on wage bills, denotes the value of money due to the liquidity services at period t. The last line denotes the Kuhn-Tucker conditions associated with this constraint: either the CIA constraint does not bind, in which case $\mu_t^F = 0$ or the CIA constraint binds, in which case $\mu_t^F \geq 0$.

Two things are worth mentioning. First, note the timing of the first line necessary condition. It shows that the marginal utility of nominal wealth, λ_t^F, coming from the current dividend is equal to marginal utility of nominal consumption at time $t+1$ since the dividend at the end of period t cannot be used by households (i.e., owners of firms) until period $t+1$. Second, from the fourth line condition, similar to (4.A.2), as long as nominal net interest rate is positive, $\mu_t^F > 0$. Thus, from the fifth line of (4.A.6), the CIA constraint of firms binds as (4.3.18).

$$W_t N_t = L_t^F. \tag{4.3.18}$$

That is, firms will borrow the exact amount of fund which is required to pay current wage bill.

By combining the first and the second conditions of (4.A.6), one gets the firms' intertemporal optimality condition for choice between dividend and investment (i.e., Euler equation of firms) as (4.3.22),

$$E_t \frac{P_t}{C_{t+1} P_{t+1}} = \beta E_t \frac{P_{t+1}}{C_{t+2} P_{t+2}} \left[\alpha K_{t+1}^{\alpha-1} (A_{t+1} N_{t+1})^{1-\alpha} + (1-\delta) \right]. \tag{4.3.22}$$

Note that the decision on investment and dividend is made just before the good market closes. Thus, in terms of utility, firms adjust dividend until the marginal current cost from the marginal increase in investment (i.e., marginal decrease in current dividend) just equals the discounted marginal future benefit from the expected marginal increase in investment (i.e., expected marginal increase in consumption opportunity).

By combining the third and fourth lines of (4.A.6), one gets the firm's *labor demand schedule* as

$$R_t^F W_t = P_t (1-\alpha) K_t^\alpha A_t^{1-\alpha} N_t^{-\alpha}, \tag{4.3.23}$$

where firms equate the marginal cost of additional labor on the left-hand side with the marginal revenue product of labor on the right-hand side.

Note that now the firms' decision on employment relies on the interest rate R_t^F as well as marginal revenue product of labor $P_t(1-\alpha)K_t^{\alpha}A_t^{1-\alpha}N_t^{-\alpha}$. It is clear that the marginal revenue product of labor is equal to nominal wage times the gross interest rate rather than just nominal wage. Thus, the marginal revenue product of labor should be larger than nominal wage by the amount of $\frac{\mu_t^F}{\lambda_t^F}W_t$ or $(R_t^F-1)W_t$ when the CIA constraint binds.[52]

For firms, since capital holding is credit goods and labor is cash goods, the choice between capital and labor is distorted by the inflation taxation. Inflation acts like a tax on labor input. Thus, higher inflation and thus higher interest rate will increase the demand for investment.

[3] Commercial Banks To solve the optimization problem of commercial banks, we set up the Lagrange function and obtain the first order conditions as shown in (4.A.7). Note that the banks solve a simple static optimization problem at each period, which mean that there is no state variable carried from the previous period.[53] In this economy, therefore, there are four optimality conditions: two from the maximization problem of households and the other two from maximization problem of firms.

$$\mathcal{L} = E_t\sum_{\tau=t}^{\infty}\beta^{\tau-t+1}\Big\{\frac{B_\tau}{C_{\tau+1}P_{\tau+1}} + \lambda_\tau^B\Big[(M_{\tau+1}^G - M_\tau^G) + (R_{F,\tau}-1)L_\tau^B - (R_{H,\tau}-1)D_\tau^B - B_\tau\Big]$$
$$+\mu_\tau^B\Big[(M_{\tau+1}^G - M_\tau^G) + D_\tau^B - L_\tau^B\Big]\Big\} \qquad (4.A.7)$$
$$\frac{\partial\mathcal{L}}{\partial B_t} = E_t\frac{1}{C_{t+1}P_{t+1}} - \lambda_t^B = 0 \Leftrightarrow \lambda_t^B = E_t\frac{1}{C_{t+1}P_{t+1}}$$
$$\frac{\partial\mathcal{L}}{\partial L_t^B} = \lambda_t^B(R_t^F-1) - \mu_t^B = 0$$
$$\frac{\partial\mathcal{L}}{\partial D_t^B} = -\lambda_t^B(R_t^H-1) + \mu_t^B = 0$$
$$0 = \mu_t^B[(M_{t+1}^G - M_t^G) + D_t^B - L_t^B],\quad 0 \le \mu_t^B \text{ and } 0 \le [(M_{t+1}^G - M_t^G) + D_t^B - L_t^B].$$

[52]Alternatively, marginal product of labor should be equal to $R_t^F W_t/P_t$.

[53]Remember that households hold all cash at the beginning of each period. Hence, M_t is a state variable for households but not for commercial banks.

Here λ_t^B denotes the households' expected marginal utility of an additional £1 from the current dividend B_t, and μ_t^B is the Lagrange multiplier related to the balance sheet constraint of banks. The last line represents the Kuhn-Tucker conditions associated with the latter constraint.

Note first that by combining the second and third line conditions in (4.A.7), we can see that banks equalize R_t^F to R_t^H. Let us denote this equilibrium interest rate as R_t. Note also that as long as $R_t > 1$, $\mu_t^B > 0$. Then, from the fourth line in (4.A.7), banks attempt to lend all available cash as follows.

$$L_t^B = M_{t+1}^G - M_t^G + D_t^B. \tag{4.3.11}$$

Now in this competitive equilibrium, the following aggregate consistency conditions must hold.

$$\begin{aligned} D_t^H &= D_t^B, \\ L_t^F &= L_t^B, \\ M_{t+1}^G &= M_{t+1}^H, \\ H_t &= N_t. \end{aligned} \tag{4.3.15}$$

Then, by substituting (4.3.15) into the households' CIA constraint (4.3.19), the firms' CIA constraint (4.3.18) and the banks' balance sheet (4.3.11), we get the following money market clearing condition (4.3.19).[54]

$$\begin{aligned} P_t C_t &= M_t - D_t + W_t N_t \\ &= M_t - D_t + L_t \\ &= M_t - D_t + M_{t+1} - M_t + D_t \\ &= M_{t+1}. \end{aligned} \tag{4.3.19}$$

On the other hand, in LP information structure, only the households' problem is

[54]Hence, it is clear from (4.3.19) that in this model, consumption expenditure velocity of money ($\frac{P_t C_t}{M_{t+1}}$) is equal to one.

modified as follows.

$$\max_{\{C_t,H_t,M^H_{t+1},D^H_{t+1}\}} \quad E_0\left\{\sum_{t=0}^{\infty}\beta^t\left[(1-\phi)\ln C_t+\phi\ln(1-H_t)\right]\right\} \tag{4.4.1}$$

$$\begin{aligned} s.t. \quad & P_tC_t+D^H_t\le M^H_t+W_tH_t \\ & 0\le D^H_t \\ & M^H_{t+1}=(M^H_t-D^H_t+W_tH_t-P_tC_t)+R^H_tD^H_t+F_t+B_t. \end{aligned}$$

Thus, the relevant Lagrangian function becomes

$$\begin{aligned} \mathfrak{L}=E_t\sum_{\tau=t}^{\infty}\beta^{\tau-t}\Big\{&\Big[(1-\phi)\ln C_\tau+\phi\ln(1-H_\tau)\Big]+\mu_\tau\Big[M^H_\tau+W_\tau H_\tau-P_\tau C_\tau-D^H_\tau\Big] \\ &+\lambda_\tau\Big[(M^H_\tau+W_\tau H_\tau-P_\tau C_\tau-D^H_\tau)+R_{H,\tau}D^H_\tau+F_\tau+B_\tau-M^H_{\tau+1}\Big]\Big\} \end{aligned}$$

$$\begin{aligned} &\frac{\partial\mathfrak{L}}{\partial C_t}=\frac{1-\phi}{C_t}-\lambda_tP_t-\mu_tP_t=0 \\ &\frac{\partial\mathfrak{L}}{\partial H_t}=-\frac{\phi}{1-H_t}+\lambda_tW_t+\mu_tW_t=0 \\ &\frac{\partial\mathfrak{L}}{\partial D^H_{t+1}}=E_t\{-\lambda_{t+1}-\mu_{t+1}+\lambda_{t+1}R_{H,t+1}\}=0 \\ &\frac{\partial\mathfrak{L}}{\partial M^H_{t+1}}=-\lambda_t+\beta E_t(\lambda_{t+1}+\mu_{t+1})=0 \\ &0=\mu_t[M^H_t+W_tH_t-P_tC_t-D^H_t] \\ &0\le\mu_t \\ &0\le[M^H_t+W_tH_t-P_tC_t-D^H_t]. \end{aligned} \tag{4.A.8}$$

4.A.3 Conditions

We have 12 equations for 12 unknowns. The unknowns are

$$M_t, A_t, Y_t, K_t, N_t, P_t, R_t, W_t, L_t, C_t, I_t, D_t.$$

The equations are

$$\ln \frac{M_{t+1}}{M_t} = \ln m_t = (1-\rho)\ln m^* + \rho \ln m_{t-1} + \epsilon_{m,t} \qquad \epsilon_{m,t} \sim N(0, \sigma_m^2), \tag{4.3.10}$$

$$\ln A_t = \gamma + \ln A_{t-1} + \epsilon_{A,t} \qquad \epsilon_{A,t} \sim N(0, \sigma_A^2), \tag{4.3.5}$$

$$Y_t = K_t^{\alpha}(A_t N_t)^{1-\alpha}, \tag{4.3.4}$$

$$I_t = K_{t+1} - (1-\delta)K_t, \tag{4.3.7}$$

$$C_t + I_t = Y_t, \tag{4.3.16}$$

$$M_{t+1} - M_t + D_t = L_t, \tag{4.3.17}$$

$$W_t = \frac{L_t}{N_t}, \tag{4.3.18}$$

$$P_t C_t = M_{t+1}, \tag{4.3.19}$$

$$\frac{\phi}{1-\phi}\left[\frac{C_t P_t}{1-N_t}\right] = W_t, \tag{4.3.20}$$

$$\frac{1}{C_t P_t} = \beta R_t E_t \left\{ \frac{1}{C_{t+1} P_{t+1}} \right\}, \tag{4.3.21}$$

$$E_t \frac{P_t}{C_{t+1}P_{t+1}} = \beta E_t \frac{P_{t+1}}{C_{t+2}P_{t+2}} \left[\alpha K_{t+1}^{\alpha-1}(A_{t+1}N_{t+1})^{1-\alpha} + (1-\delta)\right], \tag{4.3.22}$$

$$R_t = P_t(1-\alpha)K_t^{\alpha}A_t^{1-\alpha}N_t^{-\alpha}/W_t. \tag{4.3.23}$$

Note that while (4.3.18) and (4.3.19) represent the CIA constraints, (4.3.20) and (4.3.23) represent the labor supply and demand schedules.

4.A.4 Stationary System

To obtain the stationary system, one needs to transform the non-stationary variables into stationary variables. For this purpose, it is convenient to distinguish nominal variables from real variables in (4.A.3).

- nominal variables : $P_t, M_t, D_t, W_t, R_t, L_t$
- real variables : $Y_t, C_t, I_t, K_t, A_t, N_t$

It can be shown that when shocks are absent, real variables grow with A_t (except for labor, N_t, which is stationary as there is no population growth), nominal variables grow with M_t and prices with M_t/A_t. Therefore, removing the trends involves the following operations (where lower letters represent stationary variables). For real variables, $Y_t = A_t y_t$, $C_t = A_t c_t$, $I_t = A_t i_t$, $K_{t+1} = A_t k_{t+1}$, and $N_t = n_t$. For nominal variables, $D_t = M_t d_t$, $L_t = M_t l_t$, $W_t = M_t w_t$.[55] And for prices, $P_t = p_t M_t/A_t$.[56] Lastly, since the nominal interest rate is already stationary, we just use R_t as the stationary counterpart from now on.

First, from (4.3.10), the stochastic process for money growth rate is already stationary.

$$\ln m_t = (1-\rho)\ln m^* + \rho \ln m_{t-1} + \epsilon_{m,t} \qquad \epsilon_{m,t} \sim N(0, \sigma_m^2). \tag{4.A.9}$$

[55]If we divide these variables by price level, then these variables are affected by the technology level as well as the aggregate money level. So, in this model, there is a dichotomy between the real variables and the nominal variables.

[56]Thus, in this model, the trend of price level can be decomposed into money supply and the productivity.

Second, from (4.3.5), the growth rate of stochastic technology process, which is stationary, is expressed as

$$\frac{A_t}{A_{t-1}} = a_t = e^{(\gamma+\epsilon_{A,t})}. \tag{4.A.10}$$

Third, the production function becomes

$$\begin{aligned} y_t &= \frac{1}{A_t}[k_t^\alpha A_{t-1}^\alpha A_t^{1-\alpha} n_t^{1-\alpha}] \\ &= e^{-\alpha(\gamma+\epsilon_{A,t})} k_t^\alpha n_t^{1-\alpha} = a_t^{-\alpha} k_t^\alpha n_t^{1-\alpha}. \end{aligned} \tag{4.A.11}$$

Fourth, the law of motion for physical capital accumulation becomes

$$\begin{aligned} i_t A_t &= k_{t+1} A_t - (1-\delta) k_t A_{t-1} \\ i_t &= k_{t+1} - (1-\delta) k_t e^{-\alpha(\gamma+\epsilon_{A,t})} = k_{t+1} - (1-\delta)\frac{k_t}{a_t^\alpha}. \end{aligned} \tag{4.A.12}$$

Fifth, the aggregate resource constraint, which is the goods market clearing condition, becomes

$$c_t + i_t = y_t. \tag{4.A.13}$$

Sixth, the credit market clearing condition becomes

$$m_t - 1 + d_t = l_t. \tag{4.A.14}$$

Seventh, the firms borrowing constraint becomes

$$\begin{aligned} w_t M_t &= \frac{l_t M_t}{n_t} \\ w_t &= \frac{l_t}{n_t}. \end{aligned} \tag{4.A.15}$$

Eighth, the money market clearing condition becomes

$$p_t c_t = m_t. \tag{4.A.16}$$

Ninth, the intratemporal optimality condition of the household becomes

$$\frac{\phi}{1-\phi}\left[\frac{c_t A_t \frac{p_t M_t}{A_t}}{1-n_t}\right] = w_t M_t$$
$$\frac{\phi}{1-\phi}\left[\frac{c_t p_t}{1-n_t}\right] = w_t. \tag{4.A.17}$$

Tenth, the intertemporal optimality condition of households becomes

$$\frac{1}{c_t A_t p_t \frac{M_t}{A_t}} = \beta R_t E_t \frac{1}{c_{t+1} A_{t+1} p_{t+1} \frac{M_{t+1}}{A_{t+1}}}$$
$$\frac{1}{c_t p_t} = \beta R_t E_t \frac{1}{c_{t+1} p_{t+1} m_t}. \tag{4.A.18}$$

Eleventh, from (4.3.22), the intertemporal optimality condition of firms, which is Euler equation of the firm in the goods market representing the tradeoff of moving consumption goods across time, becomes

$$\begin{aligned}
E_t \frac{p_t/A_t}{c_{t+1}p_{t+1}m_t} &= E_t\beta \frac{p_{t+1}/A_{t+1}}{c_{t+2}p_{t+2}m_{t+1}}[\alpha k_{t+1}^{\alpha-1} A_t^{\alpha-1}(A_t e^{[\gamma+\epsilon_{A,t+1}]})^{1-\alpha} n_{t+1}^{1-\alpha} + (1-\delta)] \\
E_t \frac{p_t}{c_{t+1}p_{t+1}m_t} &= E_t\beta \frac{p_{t+1}}{c_{t+2}p_{t+2}m_{t+1}} e^{-(\gamma+\epsilon_{A,t+1})}[\alpha k_{t+1}^{\alpha-1} A_t^{\alpha-1}(A_t e^{(\gamma+\epsilon_{A,t+1})})^{1-\alpha} n_{t+1}^{1-\alpha} + (1-\delta)] \\
&= E_t\beta \frac{p_{t+1}}{c_{t+2}p_{t+2}m_{t+1}}[\alpha e^{-\alpha(\gamma+\epsilon_{A,t+1})} k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} + e^{-(\gamma+\epsilon_{A,t+1})}(1-\delta)] \\
&= E_t\beta \frac{p_{t+1}}{c_{t+2}p_{t+2}m_{t+1}}[\alpha e^{-\alpha(\gamma+\epsilon_{A,t+1})} k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} + e^{-(\gamma+\epsilon_{A,t+1})}(1-\delta)] \\
&= E_t\beta \frac{p_{t+1}}{c_{t+2}p_{t+2}m_{t+1}}[\alpha a_{t+1}^{-\alpha} k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} + a_{t+1}^{-1}(1-\delta)],
\end{aligned} \tag{4.A.19}$$

where we used following two relationships

$$\frac{P_t}{C_{t+1}P_{t+1}} = \frac{\frac{p_t M_t}{A_t}}{c_{t+1} A_{t+1} \frac{p_{t+1} M_{t+1}}{A_{t+1}}} = \frac{p_t/A_t}{c_{t+1} p_{t+1} m_t}$$
$$\frac{A_{t+1}}{A_t} = a_{t+1} = e^{(\gamma+\epsilon_{A,t+1})}.$$

Twelfth, the equilibrium nominal gross interest rate in which the marginal revenue product of labor equals the cost of borrowing to pay for that additional unit of labor.

Note that since R_t is stationary, we do not have to make it stationary.

$$\begin{aligned} R_t &= \frac{p_t M_t}{A_t}(1-\alpha)\frac{k_t^\alpha A_{t-1}^\alpha A_t^{1-\alpha} n_t^{-\alpha}}{w_t M_t} \\ &= p_t(1-\alpha)k_t^\alpha \left(\frac{A_t}{A_{t-1}}\right)^{-\alpha} n_t^{-\alpha}/w_t \\ &= (1-\alpha)p_t e^{-\alpha(\gamma+\epsilon_{A,t})} k_t^\alpha n_t^{-\alpha}/w_t \\ &= (1-\alpha)p_t a_t^{-\alpha} k_t^\alpha n_t^{-\alpha}/w_t. \end{aligned} \tag{4.A.20}$$

Lastly in order to link the observables to stationary variables, we reintroduce Y_t and P_t, which are equivalent to Y_t^{obs} and P_t^{obs} respectively, as the observable variables in (4.A.21) and (4.A.22).

$$\begin{aligned} Y_t &= A_t y_t \\ Y_{t-1} &= A_{t-1} y_{t-1} \\ \frac{Y_t}{Y_{t-1}} &= e^{(\gamma+\epsilon_{A,t})}\frac{y_t}{y_{t-1}} = a_t \frac{y_t}{y_{t-1}}. \end{aligned} \tag{4.A.21}$$

$$\begin{aligned} P_t &= \frac{p_t M_t}{A_t} \\ P_{t-1} &= \frac{p_{t-1} M_{t-1}}{A_{t-1}} \\ \frac{P_t}{P_{t-1}} &= m_{t-1} e^{-(\gamma+\epsilon_{A,t})}\frac{p_t}{p_{t-1}} = \frac{m_{t-1}}{a_t}\frac{p_t}{p_{t-1}}. \end{aligned} \tag{4.A.22}$$

Note that (4.A.21) and (4.A.22) implies that the observed real income has the linear trend of γ and that the observed price level has the linear trend of $(\ln m^* - \gamma)$ where $\ln m^*$ is the steady state of money growth rate.

4.A.5 The Steady State of the Economy

If there is no shock in the economy, the system converges to the steady state and the twelve stationary variables, $m, a, k, c, i, p, R, l, d, n, w$, and y are constant. Then, we can express these constant values only with underlying structural parameters.

$$m = m^*, \tag{4.A.23}$$

where m^* is the steady state of money (gross) growth rate.

$$a = e^{\gamma}. \tag{4.A.24}$$

Now, suppose for the time being that we know y. In the end, we will show that y can be expressed in terms of underlying parameters only.

From (4.A.19),

$$\begin{aligned}
1 &= \beta[\alpha e^{-\alpha\gamma}k^{\alpha-1}n^{1-\alpha} + e^{-\gamma}(1-\delta)] = \beta[\alpha\frac{y}{k} + e^{-\gamma}(1-\delta)] \\
k &= \alpha\left[\frac{1}{\beta} - e^{-\gamma}(1-\delta)\right]^{-1} y = \alpha\left[\frac{1}{\beta} - \frac{1-\delta}{a}\right]^{-1} y \\
&= \kappa y,
\end{aligned} \tag{4.A.25}$$

where $\kappa = \alpha\left[\frac{1}{\beta} - \frac{1-\delta}{a}\right]^{-1}$.

From (4.A.12) and (4.A.13),

$$\begin{aligned}
c + k &= y + (1-\delta)e^{-\gamma}k \\
c &= y + [(1-\delta)e^{-\gamma} - 1]k \\
&= \{1 + [(1-\delta)e^{-\gamma} - 1]\kappa\}\, y = \left\{1 + \left(\frac{1-\delta}{a} - 1\right)\kappa\right\} y \\
&= \iota y,
\end{aligned} \tag{4.A.26}$$

where $\iota = \left\{1 + \left(\frac{1-\delta}{a} - 1\right)\kappa\right\}$.

Then, from (4.A.12),

$$i = y - c = [1-\iota]y. \tag{4.A.27}$$

From (4.A.16),

$$p = m^*/c = \frac{m^*}{\iota y}. \tag{4.A.28}$$

From (4.A.18),

$$R = m^*/\beta. \tag{4.A.29}$$

Notice that a monetary shock has an effect on the steady state of nominal interest rate, while a technology shock has no effect.

Then from (4.A.20) and (4.A.29),

$$\begin{aligned}R &= (1-\alpha)p\frac{y}{l}\\ m^*/\beta &= (1-\alpha)\frac{m^*}{\iota y}\frac{y}{l}\\ l &= \frac{\beta(1-\alpha)}{\iota}.\end{aligned} \tag{4.A.30}$$

From (4.A.14)

$$d = l - m^* + 1. \tag{4.A.31}$$

From (4.A.17),

$$\begin{aligned}\frac{\phi}{1-\phi}\left[\frac{cp}{1-n}\right] &= \frac{l}{n}\\ \frac{\phi}{1-\phi}\left[\frac{m^*}{1-n}\right] &= \frac{l}{n}\\ \frac{\phi}{1-\phi}\frac{m^*}{l} &= \frac{1-n}{n} = \frac{1}{n} - 1\\ \left[\frac{\phi}{1-\phi}\frac{m^*}{l} + 1\right]^{-1} &= n\\ n &= \left[\frac{\phi}{1-\phi}\frac{m^*\iota}{\beta(1-\alpha)} + 1\right]^{-1}.\end{aligned} \tag{4.A.32}$$

From (4.A.15),

$$w = \frac{l}{n} = \frac{\beta(1-\alpha)}{\iota}\left[\frac{\phi}{1-\phi}\frac{m^*\iota}{\beta(1-\alpha)} + 1\right]. \tag{4.A.33}$$

To complete the steady state of the model, we must express y in terms of underlying parameters. From (4.A.11),

$$\begin{aligned} y &= e^{-\alpha\gamma} k^{\alpha} n^{1-\alpha} \\ &= e^{-\alpha\gamma} \kappa^{\alpha} y^{\alpha} n^{1-\alpha} \\ &= \left[e^{-\alpha\gamma} \kappa^{\alpha} n^{1-\alpha}\right]^{\frac{1}{1-\alpha}} \\ &= \left(\frac{\kappa}{a}\right)^{\frac{\alpha}{1-\alpha}} n, \end{aligned} \tag{4.A.34}$$

where κ and n are known.

Note that only one endogenous variable, the amount of loan l is independent of the steady-state money growth rate m^*. Moreover, the key ratios such as $\frac{c}{y}$, $\frac{c}{k}$ and $\frac{k}{n}$ are all independent of the steady-state money growth rate. On the other hand, at the steady state, the capital stock, consumption and output will hinge on the money growth rate through the channels of money growth rate on labor supply.

It is clear from (4.A.32) that a permanent expansionary monetary shock (i.e., a rise in the price level) lowers the steady-state labor supply. Then, this lowers output, consumption and capital stock at the steady state. This is the source of the welfare cost of inflation in the CIA model.[57]

4.A.6 Log-linear Approximation

To obtain the approximate solutions to the economy, Equations (4.A.9)–(4.A.20) need to be log-linearized around their values at the steady state following Campbell (1994). Denote a hat over a variable as representing a proportionate deviation of that variable from its steady state level, for instance, $\hat{y}_t = \ln(y_t/y)$, etc. Then, two exogenous shocks, namely technology shocks and money growth shocks, bring the economy off the steady state. The first-order Taylor approximations to (4.A.9)–(4.A.20) produce twelve equations.

$$\hat{m}_t = \rho \hat{m}_{t-1} + \epsilon_{m,t}, \tag{4.A.35}$$

[57]On the other hand, the temporary expansionary monetary shock will tax the cash goods such as consumption and leisure. Therefore, since this will encourage the investment, capital stock will increase temporarily.

$$\hat{a}_t = \epsilon_{A,t}, \tag{4.A.36}$$

$$\hat{y}_t = -\alpha\hat{a}_t + \alpha\hat{k}_t + (1-\alpha)\hat{n}_t, \tag{4.A.37}$$

$$\hat{i}_t = \frac{k}{i}\hat{k}_{t+1} - (1-\delta)\frac{k}{ia^\alpha}\hat{k}_t + (1-\delta)\frac{k}{ia^\alpha}\hat{a}_t, \tag{4.A.38}$$

$$c\hat{c}_t + i\hat{i}_t = y\hat{y}_t, \tag{4.A.39}$$

$$\frac{m}{m-1+d}\hat{m}_t + \frac{d}{m-1+d}\hat{d}_t = \hat{l}_t, \tag{4.A.40}$$

$$\hat{w}_t = \hat{l}_t - \hat{n}_t, \tag{4.A.41}$$

$$\hat{p}_t + \hat{c}_t = \hat{m}_t, \tag{4.A.42}$$

$$\begin{aligned} w\hat{w}_t &= \frac{\phi}{1-\phi}\frac{cp}{1-n}\hat{c}_t + \frac{\phi}{1-\phi}\frac{cp}{1-n}\hat{p}_t + \frac{\phi}{1-\phi}\frac{cp}{(1-n)^2}n\hat{n}_t \\ \hat{w}_t &= \hat{c}_t + \hat{p}_t + \frac{n}{1-n}\hat{n}_t, \end{aligned} \tag{4.A.43}$$

$$\begin{aligned} pm^* cE_t\hat{c}_{t+1} + cm^* pE_t\hat{p}_{t+1} + cpm^* E_t\hat{m}_t &= \beta cpR\hat{R}_t + \beta Rpc\hat{c}_t + \beta Rcp\hat{p}_t \\ E_t\hat{c}_{t+1} + E_t\hat{p}_{t+1} + E_t\hat{m}_t &= \hat{R}_t + \hat{c}_t + \hat{p}_t, \end{aligned} \tag{4.A.44}$$

where we used the relation $cpm^* = \beta Rcp$ at the steady state.

$$\frac{1}{\beta}E_t\left[\hat{c}_{t+2} - \hat{c}_{t+1} + \hat{p}_{t+2} - 2\hat{p}_{t+1} + \hat{p}_t + \hat{m}_{t+1} - \hat{m}_t\right]$$
$$= (\alpha - 1)\left[\frac{1}{\beta} - \frac{1-\delta}{a}\right]\hat{k}_{t+1} + (1-\alpha)\left[\frac{1}{\beta} - \frac{1-\delta}{a}\right]\hat{n}_{t+1} - \left[\frac{1-\delta}{a} + \alpha\left(\frac{1}{\beta} - \frac{1-\delta}{a}\right)\right]\hat{a}_{t+1}, \tag{4.A.45}$$

where we used $\frac{1}{\beta} = \alpha a^{-\alpha}k^{\alpha-1}n^{1-\alpha} + \frac{1-\delta}{a}$.

$$\hat{R}_t = \hat{p}_t - \alpha\hat{a}_t + \alpha\hat{k}_t - \alpha\hat{n}_t - \hat{w}_t. \tag{4.A.46}$$

4.B Two Approximations to Marginal Likelihood of Model

Following Mancini-Griffoli (2007), this section briefly presents two approximation methods to marginal likelihoods.

4.B.1 Laplace Approximation

The first method is to simply assume a functional form of the posterior kernel that we can integrate. The most straightforward and the widely used functional form is the Gaussian and in this case the approximation method is referred to as a Laplace approximation. Then, we would have the following estimator:

$$\begin{aligned} \tilde{p}(\boldsymbol{Y}|\mathscr{M}) =& (2\pi)^{\frac{m}{2}} \left|\boldsymbol{\Sigma}(\tilde{\boldsymbol{\theta}}|\mathscr{M})\right|^{\frac{1}{2}} p(\boldsymbol{Y}|\tilde{\boldsymbol{\theta}},\mathscr{M})p(\tilde{\boldsymbol{\theta}}|\mathscr{M}), \qquad (4.B.1) \\ \text{Or} \quad \ln\tilde{p}(\boldsymbol{Y}|\mathscr{M}) =& \frac{m}{2}\ln(2\pi) + \frac{1}{2}\ln\left|\boldsymbol{\Sigma}(\tilde{\boldsymbol{\theta}}|\mathscr{M})\right| + \ln p(\boldsymbol{Y}|\tilde{\boldsymbol{\theta}},\mathscr{M}) + \ln p(\tilde{\boldsymbol{\theta}}|\mathscr{M}), \end{aligned}$$

where $\tilde{p}(\boldsymbol{Y}|\mathscr{M})$ is a marginal density of data conditional on model $\mathscr{M}$, $\tilde{\boldsymbol{\theta}}$ is the posterior mode and $\boldsymbol{\Sigma}(\tilde{\boldsymbol{\theta}}|\mathscr{M})$ is the covariance matrix calculated from the inverse of the negative of the second derivative of the log-kernel function (Hessian) evaluated at $\tilde{\boldsymbol{\theta}}$. Notice that we already know the posterior mode and can thus evaluate the likelihood at this posterior mode. Hence, the Laplace approximation is very computationally efficient without involving additional numerical calculation.

4.B.2 Modified Harmonic Mean Estimator

Alternatively, to compute the marginal data density, one can follow a numerical approach suggested by Geweke (1999): modified harmonic mean estimator. Note first that a harmonic mean estimator comes from the identity of (4.B.2)

$$\frac{1}{p(\boldsymbol{Y})} = \int \frac{f(\boldsymbol{\theta})}{\mathscr{L}(\boldsymbol{\theta}|\boldsymbol{Y})p(\boldsymbol{\theta})} p(\boldsymbol{\theta}|\boldsymbol{Y})d\boldsymbol{\theta}, \qquad (4.B.2)$$

where $f(\boldsymbol{\theta})$ is chosen to guarantee $\int f(\boldsymbol{\theta})d\boldsymbol{\theta} = 1$. It is clear that the denominator is the unnormalized joint posterior density and the second term is the joint posterior density of m unknowns.

According to Gelfand and Dey (1994), the following relationship is generally satisfied

$$\int \frac{f(\boldsymbol{\theta})}{\mathscr{L}(\boldsymbol{\theta}|\boldsymbol{Y})p(\boldsymbol{\theta})}p(\boldsymbol{\theta}|\boldsymbol{Y})d\boldsymbol{\theta} = E\Big[\frac{f(\boldsymbol{\theta})}{\mathscr{L}(\boldsymbol{\theta}|\boldsymbol{Y})p(\boldsymbol{\theta})}\Big|\boldsymbol{Y}\Big]. \tag{4.B.3}$$

Then, given $f(\boldsymbol{\theta})$, a natural choice of estimator, which is often referred to as the Modified Harmonic Mean Estimator is

$$\hat{p}(\boldsymbol{Y}) = \Big[\frac{1}{n_{sim}}\sum_{s=1}^{n_{sim}}\frac{f(\boldsymbol{\theta}^{(s)})}{\mathscr{L}(\boldsymbol{\theta}^{(s)}|\boldsymbol{Y})p(\boldsymbol{\theta}^{(s)})}\Big]^{-1}, \tag{4.B.4}$$

where $\boldsymbol{\theta}^{(s)}$ is drawn from the posterior $p(\boldsymbol{\theta}|\boldsymbol{Y})$ and n_{sim} is the number of simulations. Note that this is simply the average of the simulated values.

Lastly, for $f(\boldsymbol{\theta})$, Geweke (1999) proposes to use the density of a truncated multivariate normal distribution,

$$\begin{aligned} f(\boldsymbol{\theta}) =& \tau^{-1}(2\pi)^{-m/2}|\boldsymbol{\Sigma}(\boldsymbol{\theta})|^{-1/2}exp\Big[-\frac{1}{2}(\boldsymbol{\theta}-\bar{\boldsymbol{\theta}})'\boldsymbol{\Sigma}(\boldsymbol{\theta})^{-1}(\boldsymbol{\theta}-\bar{\boldsymbol{\theta}})\Big] \\ & \times \chi^2\{(\boldsymbol{\theta}-\bar{\boldsymbol{\theta}})'\boldsymbol{\Sigma}(\boldsymbol{\theta})^{-1}(\boldsymbol{\theta}-\bar{\boldsymbol{\theta}}) \le \chi^2_{1-\tau}(m)\}, \end{aligned} \tag{4.B.5}$$

where $\bar{\boldsymbol{\theta}}$ and $\boldsymbol{\Sigma}(\boldsymbol{\theta})$ are the posterior mean and covariance matrix computed from the Metropolis posterior simulator and $\tau \in (0,1)$. In addition, m is the dimension of the parameter vector and χ^2 is the cumulative density function of χ^2 random variable with m degrees of freedom (An and Schorfheide 2007).

Chapter 5

CONCLUSIONS

This thesis contains three essays on business cycles in Korea that explore the dynamics of macroeconomic variables. Given the limited research on business cycles in the emerging markets, these essays collectively attempt to bridge the gap in the current research by investigating Korean business cycles and comparing our findings with the previous findings in the developed countries. During these studies, one recurring issue is that whether financial crisis in the later half of 1997 brought out any noticeable changes on the fluctuations of key macro-variables in Korea.

Chapter 2 has investigated the relation among consumption, financial wealth and labor income in the framework of vector error correction model (simply VECM). The main findings of this chapter can be summarized as follows. First, using the Johansen's full information maximum likelihood (FIML) estimation method, this research confirmed the Lettau and Ludvigson's (2001) theoretical prediction on the long-run relationship involving those three variables in Korea.

Second, the presence of one cointegrating relation has two crucial implications: 1) at least one variable among three variables should adjust to recover the long run relation, 2) and there are two common trends and one transitory component in the current 3-dimensional system. For the former implication, it turned out that only financial wealth showed the sizable and statistically significant error correcting behavior in the system. For the second implication, this essay finds that most movements in consumption and labor income are closely related with the permanent components while those in financial wealth contain a large portion of transitory component in the current system. These two implications jointly suggest that only financial wealth movements be predictable using the deviation from the long-run relationship. This chapter confirms that this is the case in Korean economy.

The last but probably most important finding in Chapter 2 comes from comparing the Korean consumers' behaviors before and after the Korean financial crisis. From this analysis, we find that although there were several policy and institutional changes during the crisis, most adjustment to the long run relation has been done by financial wealth across the two sample periods. Thus, it did not change the error correcting behaviors of households in the current model.

Chapter 3 and Chapter 4 estimated dynamic stochastic general equilibrium (DSGE) models for Korean economy by the maximum-likelihood approach and the Bayesian approach, respectively. The main findings of Chapter 3 can be recapitulated as follows. First, although DSGE models are often criticized because they are too idealized and thus too restricted to match the data, this chapter finds that in predicting hours worked, the forecasting performances of the current DSGE model is superior to those of VAR models.

Second, using the variance decomposition analysis, we find that the productivity shock is the important source of business cycles in Korea, in particular for hours worked, but it has a difficulty to explain fluctuations of some key variables. Therefore, incorporating other shocks such as monetary and fiscal policy shocks in the model seems to be an important direction for the future research.

Third, although the volatility of economy declined a little since the Korean financial crisis, the structural parameters overall seem not to be significantly different across the crisis while the non-structural parameters seem to be different.

Finally, by comparing the second moments from the HP filtered data with those from the simulated data, this essay finds that the estimated model successfully reproduces the relative volatility of consumption and hours worked as well as the pattern of contemporaneous correlations of output with consumption, investment and hours worked.

Chapter 4 extended the baseline DSGE model in Chapter 3 by introducing money through cash-in-advance (CIA) constraint. In this chapter, we estimated two versions of CIA models: a baseline CIA model and a limited participation (LP) model, in which firms should borrow cash to pay wages in advance while households have to hold cash to purchase consumption goods. The key difference between the two models is that contrary to the CIA model, the LP model assumes that households make a decision on deposit before they observe exogenous shocks while firms still make a decision on borrowing after they observe those shocks. This constraint to households leads to the rigidity of the deposit and provides interesting dynamics to the economy.

This chapter showed that the LP model is better to match up two stylized facts in Korean business cycles: 1) the positive correlation between the short-term interest rate and money growth (i.e., the dominance of the anticipated inflation effect over the liquidity effect), 2) and the positive correlation between the output and money growth. In addition, the formal comparison using the Bayes factor sufficiently favors the LP model over the CIA model.

Finally, based on the LP model, this chapter compared the pre- and post-crisis periods to scrutinize whether several institutional and policy changes since the Korean financial crisis, have made any noticeable changes in Korean business cycles. We found that the money shock declined sharply almost to one third of the pre-crisis level while the magnitude of technology shock fell but not too much.

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