Some properties of partitions, with special reference to primes other than 5,7 and 11

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SOME PROPERTIES OF PARTITIONS, WITH SPECIAL REFERENCE TO PRIMES OTHER THAN 5, 7 AND 11

by

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The work in this thesis
is original and entirely
my own, except where
otherwise stated.

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INTRODUCTION

The work in this thesis follows that done by Atkin and Swinnerton-Dyer [3], and Atkin and Hussain [2]. Constant reference is made to these papers, which we therefore denote by (ASD) and (AH) respectively. All unspecified notation is that of (ASD) together with the following additions.

We write

$$f(z) = \prod_{r=1}^{\infty} (1 - z^r).$$

Then

$$f(y) = P(0) \prod_{a=1}^{(q-1)/2} P(a),$$

$$f(y^q) = P(0),$$

$$1/f(x) = \sum_{n=0}^{\infty} p(n)\, x^n,$$

taking p(0) to be unity. {The above notation, with q = 11, is used in (AH).} Occasionally we need the congruence

$$f^q(y) \equiv f(y^q) \qquad (\text{mod}.\, q),$$

which follows from $(1 - y^r)^q \equiv 1 - y^{qr}$, modulo q. The enclosure of an ordered product of a number of variables in square brackets denotes a summation over all the <u>different</u> terms obtainable by permuting the variables cyclically in a typical term. In such a product one or more of the variables may have degree zero. Square brackets, then, replace capital sigma (which is rather an overworked symbol) as used on page 186 of (AH). It should be pointed out that in such a cyclic sum the number of terms is not necessarily the same as the number of variables. For example the following cyclic sum involving eight variables contains only two terms:

$$[a_1\ a_3\ a_5\ a_7] = a_1\ a_3\ a_5\ a_7 + a_2\ a_4\ a_6\ a_8 .$$

The symbol <b, c, d> is used to denote the following relation, proved in (ASD) (Lemma 4):

$$P^2(b)P(c+d)P(c-d)-P^2(c)P(b+d)P(b-d)+y^{c-d}P^2(d)P(b+c)P(b-c) = 0$$

if none of b, c, d, b $\pm$ c, c $\pm$ d, b $\pm$ d, is divisible by q. Similarly <b, c, d, e> denotes the relation

$$P(b+e)P(b-e)P(c+d)P(c-d)-P(b+d)P(b-d)P(c+e)P(c-e)+$$
$$+y^{c-d}P(b+c)P(b-c)P(d+e)P(d-e) = 0$$

(none of b $\pm$ c, b $\pm$ d, b $\pm$ e, c $\pm$ d, c $\pm$ e, d $\pm$ e, divisible by q). The latter relation may be proved by the method used in (ASD) for the former (which is in fact <b, c, d, 0>), but is however given, in essence, in [14] {equation ($LVII_2$), page 160}. We note that either relation is homogeneous in the P(a).

The thesis is comprised of five Parts, which are to a large extent independent of one another and may in fact be read separately. The contents of these Parts are as follows.

Part 1, throughout which $q = 13$, is divided into four sections (§§ 1 to 4). In § 1 the process employed in § 11 of (AH) to express $\sum_{n=0}^{\infty} p(11n + 6)y^n$ in terms of simple functions of y is used to evaluate $\sum_{n=0}^{\infty} p(13n + 6)y^n$ in a form analogous to Ramanujan's results for $q = 5$ and $q = 7$; more elegance of method is possible in the case of $q = 13$. A secondary consequence of this process is the determination of what is in fact the simplest, non-homogeneous relation between the $P(a)$ for $q = 13$ {equation (1.17)}.* § 2 contains the evaluation of $\sum_{n=0}^{\infty} p(13n + s)y^n$, for all values of $s (s = 0$ to $12)$ except $s = 6$, in a form which, while more complicated than for $s = 6$, involves only simple functions. In actual fact two such forms are given, but these are essentially equivalent. Simple congruences for $\sum_{n=0}^{\infty} p(13n + s)y^n$, $s = 0, 1, 2, \ldots, 12$, such as are given in (ASD) for $q = 5$, $q = 7$, and $q = 11$, are derived in § 3 from the results of § 2. A complete account

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*Neither of the two results of § 1 is new (see text), although such an elementary, algebraic method has not previously been employed. This section is due in its entirety to Dr. Atkin.

of Dyson's rank functions for the cases $q = 5$, $q = 7$, and $q = 11$, is given in (ASD) and (AH). In particular the values of the $r_{bc}(d)$ are obtained for each of these q. We find the values of the $r_{bc}(d)$ for $q = 13$ in § 4, by a method akin to that used for $q = 11$. They are of a somewhat different form than for $q = 11$ and rather more complicated, but are, on the other hand, <u>all</u> of the same nature, similar to that of the expressions given by Theorem 2.2 for $\sum_{n=0}^{\infty} p(13n + s)y^n$ $(s \neq 6)$. In the case of $q = 11$, the $r_{bc}(6)$ and the remaining $r_{bc}(d)$ have values <u>not</u> of the same nature. We note here that in Theorem 4.1, which gives the $r_{bc}(d)$ for $q = 13$, $p(0)$ must be taken to be zero {see (ASD), page 86}. It is of interest to observe that there is a set of linear congruence relations, (4.41), between the $r_{bc}(d)$ for a given value of d when $q = 13$, corresponding to (AH), equations (9.16), for $q = 11$.

Parts 2 (§ 5) and 3(§ 6) contain the evaluations of $\sum_{n=0}^{\infty} p(17n + 5)y^n$ and $\sum_{n=0}^{\infty} p(19n + 4)y^n$ respectively, again by the method used in (AH), § 11, for $\sum_{n=0}^{\infty} p(11n + 6)y^n$. In each case both the process and the result are more elegant than for $q = 11$, but less so than for $q = 13$. Simple congruences for $\sum_{n=0}^{\infty} p(17n + 5)y^n$ and $\sum_{n=0}^{\infty} p(19n + 4)y^n$ are derived from these results. The apparently simplest, non-homogeneous relation between the P(a) for q = 17 and q = 19 is embodied in Theorem 5.1 (third equation) and Theorem 6.1 (fifth equation) respectively.

In part 4 (§ 7) an alternative expression for $\sum_{n=0}^{\infty} p(11n + 6)y^n$ is derived from that given in (AH) {equation (11.9)}, and we then conjecture similar expressions for $\sum_{n=0}^{\infty} p(11n + s)y^n$ (s = 0 to 10) when $s \neq 6$. (Such similarity does not obviously exist in the case of q = 13.) We make no attempt to prove our conjecture, which is almost certainly valid, in this thesis. The form of the expressions concerned is quite different from either of the forms obtained for q = 13 in § 2.* It is worthwhile to note that equation (7.1) is, in effect, what appears to be the simplest, non-homogeneous relation between the P(a) for q = 11, and to pause at this point in order to state together the simplest relations for all prime q as far as q = 19. The relations for q = 5† and q = 7 follow immediately from [7] (Kolberg), equations (4.15) and (5.20) respectively, if, for both q = 5 and q = 7, the g_s of this paper

* Kolberg has obtained certain expressions for $\sum_{n=0}^{\infty} p(5n+s)y^n$, s=0, 1, 2, 3, {[7], equations (4.17) to (4.20)}, and $\sum_{n=0}^{\infty} p(7n+s)y^n$, s=0, 1, 2, 3, 4, 6, {[7], equations (5.23) to (5.27), and (5.29)}. The former decomposition is due originally to Ramanujan [12].

† This relation appears in [12] (Ramanujan).

$\{$defined by $f(x) = \sum_{s=0}^{q-1} g_s x^s$, $g_s = g_s(y)\}$ are expressed in terms of the $P(a)$ by means of (ASD), Lemma 6. We have, remembering that $f(y)/f(y^q) = \prod_{a=1}^{(q-1)/2} P(a)$,

$q = 5$: $f^6(y)/f^6(y^5) = P^5(2)/P^5(1) - 11y - y^2 P^5(1)/P^5(2)$,

$q = 7$: $f^4(y)/f^4(y^7) + 8y = P^2(2)P(3)/P^3(1) + yP^2(3)P(1)/P^3(2) - y^2 P^2(1)P(2)/P^3(3)$,

$q = 11$: $f^3(y)/f^3(y^{11}) = P^2(5)P(4) - y^2 P^2(1)P(3) - yP^2(2)P(5) - yP^2(4)P(1) - yP^2(3)P(2)$,

$q = 13$: $f^2(y)/f^2(y^{13}) = P(2)P(5)P(6)/P(1)P(3)P(4) - 3y - y^2 P(1)P(3)P(4)/P(2)P(5)P(6)$,

$q = 17$: $f^2(y)/f^2(y^{17}) = P(2)P(8)P(6)P(7) - yP(6)P(7)P(1)P(4) - y^2 P(1)P(4)P(3)P(5) - yP(3)P(5)P(2)P(8)$,

$q = 19$: $f(y)/f(y^{19}) = 1/P(2)P(3)P(5) - y/P(1)P(7)P(8) - y^2/P(4)P(6)P(9)$.

The results for $q = 11$, $q = 17$, and $q = 19$, seem to be new.

Parts 1 to 4 involve only elementary algebra. In Part 5 (§ 8) recourse is made to the theory of the elliptic modular functions. We show that there exists, for $q = 13$, a polynomial relation between $xf^2(y)/f^2(x)$ and $x^7 f(y^{13})/f(x)$, of degrees at

most 7 and at most 13 in these variables respectively.*
Then, working for convenience in terms of $y^{-1}f^2(y)/f^2(y^{13})$ and $x^{-7}f(x)/f(y^{13})$ as new variables, we show by elementary algebra that the relation (of degrees at most 7 and at most 13 in the new variables respectively**) is irreducible and that the coefficients involved have, in pairs, a certain symmetry. The relation is evaluated (in terms of the new variables) by comparing coefficients of powers of x in the expansions of the quantities involved, use being made of the symmetry mentioned above to facilitate the calculation. The result could also be obtained by using the expressions for $x^{-7}f(x)/f(y^{13})$ and $y^{-1}f^2(y)/f^2(y^{13})$ in terms of the P(a) {equations (1.1) and (1.17)}, and the homogeneous relations between the P(a) previously described in this Introduction, but this would be comparatively tedious.†

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* It is in fact shown that there is a corresponding result or "modular equation" for all prime q, in which the degree of the function corresponding to $x^7f(y^{13})/f(x)$ is at most q in the cases q = 5, q = 7, and q = 13, and is at most a greater integral multiple of q otherwise. We are indebted to Dr. Morris Newman of the National Bureau of Standards, Washington, D.C., who communicated the proof to us. The relations for q = 5 and q = 7 have been obtained, in essence, by Watson {[15], page 105, formula (3.2), and page 118, (5.2)}, although the former is due originally to Weber {[16], page 256, formula (27)}.

** The degrees are in fact 7 and 13.

† I hope to publish in the near future firstly a paper on the work of Part 5 and secondly, in conjunction with Dr. Atkin, a paper "Some properties of the coefficients of modular forms modulo powers of 13", depending upon the first.

We take this opportunity to observe that it would probably be possible to use the theory of modular functions to obtain expressions for $\sum_{n=0}^{\infty} p(17n + 5)y^n$ and $\sum_{n=0}^{\infty} p(19n + 4)y^n$ more easily than in this thesis, and indeed to obtain corresponding results for still greater values of q (this would otherwise be a very tedious matter), but that a further development of the theory would be needed.

Tabulated values of p(n) (as far as n = 1000), needed at various points in the thesis, are to be found in [5]. The table of the coefficients of powers of f(x) computed by Newman [11] is also required.

Finally, we note that a table of notation {not including that of (ASD) or (AH)} and a list of references are given at the end of the thesis. Some letters occur more than once in the text in different senses (this is purposeful where analogous processes are carried out for different values of q), but the contexts are so different as to give no danger of confusion.

PART I

q = 13 throughout this Part

1. We write

$$\alpha = -x^{-5}P(2)/P(1), \quad \beta = -x^{-6}P(6)/P(3), \quad \gamma = x^{-2}P(5)/P(4),$$

$$\alpha' = -x^{5}P(3)/P(5), \quad \beta' = x^{-7}P(4)/P(2), \quad \gamma' = x^{15}P(1)/P(6);$$

then by (ASD), Lemma 6 (with q = 13) we have

(1.1) $x^{-7}f(x)/f(y^{13}) = \alpha + \beta' + \gamma + \alpha' + \beta + \gamma' + 1.$

In (1.1) we replace x by $\omega_r x$ where ω_r (r = 1 to 13) are the thirteenth roots of unity, and multiply together the thirteen resulting equations, obtaining

(1.2) $y^{-7}f^{14}(y)/f^{14}(y^{13}) = \prod_{r=1}^{13}(\alpha\omega_r^{-5}+\beta'\omega_r^{-7}+\gamma\omega_r^{-2}+\alpha'\omega_r^{5}+\beta\omega_r^{-6}+\gamma'\omega_r^{15}+1).$

Now as ω_r runs through the thirteenth roots of unity so does ω_r^{-3}, so that the product on the right-hand side of (1.2) is equal to

$$\prod_{r=1}^{13}(\alpha\omega_r^{15}+\beta'\omega_r^{-5}+\gamma\omega_r^{-7}+\alpha'\omega_r^{-2}+\beta\omega_r^{5}+\gamma'\omega_r^{-6}+1),$$

and is thus unchanged if α, β', γ, α', β, and γ', are interchanged cyclically. The product is thus a linear combination of terms $[\alpha^{i_1}\beta'^{i_2}\gamma^{i_3}\alpha'^{i_4}\beta^{i_5}\gamma'^{i_6}]$ where i_1 to i_6 are non-negative integers, and considering the left-hand side of (1.2) such terms as occur can only involve x in terms of $y = x^{13}$. Thus if $\alpha^{i_1}\beta'^{i_2}\gamma^{i_3}\alpha'^{i_4}\beta^{i_5}\gamma'^{i_6}$ (or any other term of $[\alpha^{i_1}\beta'^{i_2}\gamma^{i_3}\alpha'^{i_4}\beta^{i_5}\gamma'^{i_6}]$) occurs we must have

(1.3) $-5i_1 - 7i_2 - 2i_3 + 5i_4 - 6i_5 + 15i_6 \equiv 0 \pmod{13}$

(interchanging i_1, i_2, i_3, i_4, i_5, and i_6, cyclically gives the same congruence).

Now, writing

$$a = y^2P^2(1)/P(4)P(5), \qquad a' = y^{-1}P^2(5)/P(6)P(1),$$
$$b = -y^{-1}P^2(3)/P(1)P(2), \qquad b' = -yP^2(2)/P(5)P(3),$$
$$c = -P^2(4)/P(3)P(6), \qquad c' = y^{-1}P^2(6)/P(2)P(4),$$

it is easily verified that

$$\alpha^{13} = b'^{12}\, c^2\, a'^6\, b^7\, c'^4, \qquad \alpha'^{13} = b^{12}\, c'^2\, a^6\, b'^7\, b^4,$$
$$(1.4) \quad \beta^{13} = c'^{12}\, a^2\, b'^6\, c^7\, a'^4, \qquad \beta'^{13} = c^{12}\, a'^2\, b^6\, c'^7\, a^4,$$
$$\gamma^{13} = a'^{12}\, b^2\, c'^6\, a^7\, b'^4, \qquad \gamma'^{13} = a^{12}\, b'^2\, c^6\, a'^7\, b^4.$$

It will be noticed that all of the equations (1.4) may be obtained from any one of them by interchanging a, b', c, a', b, c', and α, β', γ, α', β, γ', cyclically. By (1.4), since $ab'\,ca'\,bc' = -1$,

$$(\alpha^{i_1}\, \beta'^{i_2}\, \gamma^{i_3}\, \alpha'^{i_4}\, \beta^{i_5}\, \gamma'^{i_6})^{13} =$$

$$= (ab'\,ca'\,bc')^{\sigma}\, a^{\sigma_1}\, b'^{\sigma_2}\, c^{\sigma_3}\, a'^{\sigma_4}\, b^{\sigma_5}\, c'^{\sigma_6}$$

where $\sigma = 2i_1 + 4i_2 + 12i_3 + 18i_4 + 16i_5 + 8i_6$, an even integer, and

$$\sigma_1 = 4i_2+7i_3+6i_4+2i_5+12i_6, \qquad \sigma_2 = 4i_3+7i_4+6i_5+2i_6+12i_1,$$
$$\sigma_3 = 4i_4+7i_5+6i_6+2i_1+12i_2, \qquad \sigma_4 = 4i_5+7i_6+6i_1+2i_2+12i_3,$$
$$\sigma_5 = 4i_6+7i_1+6i_2+2i_3+12i_4, \qquad \sigma_6 = 4i_1+7i_2+6i_3+2i_4+12i_5;$$

moreover $\sigma + \sigma_1$ to $\sigma + \sigma_6$ are multiples of 13 by (1.3), hence we arrive at the following:

LEMMA 1.1 Any expression of the form $\alpha^{i_1}\beta'^{i_2}\gamma^{i_3}\alpha'^{i_4}\beta^{i_5}\gamma'^{i_6}$ for which (1.3) holds is of the form $a^{j_1}b'^{j_2}c^{j_3}a'^{j_4}b^{j_5}c'^{j_6}$ where j_1 to j_6 are non-negative integers.

By Lemma 1.1 every term occurring in the right-hand side of (1.2) is of the form $a^{j_1}b'^{j_2}c^{j_3}a'^{j_4}b^{j_5}c'^{j_6}$, and such terms occur in cyclically symmetrical sets of six terms each.

Further, $\Phi(6)$ is the coefficient of x^6 in $1/f(x)$ regarded as a polynomial of degree 12 in x with coefficients involving x in terms of $y = x^{13}$, so that $y^{-6}f^{14}(y)\Phi(6)/f^{13}(y^{13})$ is the coefficient of x^0 in $y^{-7}f^{14}(y)/\{f^{14}(y^{13})(\alpha+\beta'+\gamma+\alpha'+\beta+\gamma'+1)\}$. This is a cyclically symmetric polynomial of degree 12 in α, β', γ, α', β, and γ'; and the terms which give the coefficient of x^0 occur only in symmetrical sets of six expressible as $[a^{j_1}b'^{j_2}c^{j_3}a'^{j_4}b^{j_5}c'^{j_6}]$, as before. (This is not true for the coefficient of any power of x other than 0; the six terms of [a], for example, do not appertain to the same power of x.)

Thus $y^{-7}f^{14}(y)/f^{14}(y^{13})$ and $y^{-6}f^{14}(y)\Phi(6)/f^{13}(y^{13})$ are each equal to a linear combination of terms $[a^{j_1}b'^{j_2}c^{j_3}a'^{j_4}b^{j_5}c'^{j_6}]$. We now write

$$A = yP(2)P(3)/P(4)P(6), \qquad B = -y^{-1}P(4)P(6)/P(1)P(5),$$

$$C = -P(1)P(5)/P(2)P(3); \qquad K = yP(1)P(3)P(4)/P(2)P(5)P(6).$$

Then

(1.5) $$ABC = 1.$$

$\langle 4, 2, 1\rangle$, $\langle 6, 3, 1\rangle$, $\langle 5, 4, 3\rangle$, $\langle 6, 5, 3\rangle$, $\langle 5, 4, 2\rangle$, and $\langle 6, 2, 1\rangle$, give, respectively,

(1.6) to (1.8) $$a = A - K, \quad b = B - K, \quad c = C - K,$$

(1.9) to (1.11) $$a' = A + 1/K, \quad b' = B + 1/K, \quad c' = C + 1/K;$$

all of the equations (1.6) to (1.11) may be obtained from any one of them by interchanging a, b', c, a', b, c', and A, B, C, and $1/K$ $- K$, cyclically. Also, $\langle 5, 3, 2, 1\rangle$ gives

(1.12) to (1.14) $$AB + A + 1 = 0, \quad BC + B + 1 = 0, \quad CA + C + 1 = 0,$$

which equations are equivalent by virtue of (1.5), and $\langle 5, 2, 1\rangle$ gives

$$a + b' = CA,$$

which using (1.6), (1.10), and (1.12) to (1.14), becomes

(1.15) $$A + B + C = -1/K + K - 1,$$

(1.16) $$AB + BC + CA = 1/K - K - 2.$$

We are now in a position to prove

LEMMA 1.2 Any expression of the form $[a^{j_1} b'^{j_2} c^{j_3} a'^{j_4} b^{j_5} c'^{j_6}]$ is equal to a polynomial in $1/K - K$ with integral coefficients.

Using (1.6) to (1.11), any $[a^{j_1} b'^{j_2} c^{j_3} a'^{j_4} b^{j_5} c'^{j_6}]$ can be expressed as a polynomial in A, B, C, $1/K$, and $-K$, with integral coefficients, cyclically symmetric in A, B, C, and $1/K$, $-K$. This polynomial is a linear combination of terms

$\{(1/K)^h + (-K)^h\}[A^\lambda B^\mu C^\nu]$ where h, λ, μ, and ν, are non-negative integers, for if a term $(1/K)^h[A^\lambda B^\mu C^\nu]$ occurs so does the term $(-K)^h[A^\lambda B^\mu C^\nu]$, and vice versa. Further, by Newton's formula for sums of powers of the roots of a polynomial equation in one variable, $(1/K)^h + (-K)^h$ can be expressed as a polynomial in the coefficients of the quadratic equation $z^2 - (1/K - K)z - 1 = 0$ having roots $1/K$ and $-K$, i.e. as a polynomial in $1/K - K$ with integral coefficients. We now assert that any $[A^\lambda B^\mu C^\nu]$ is also equal to a polynomial in $1/K - K$ with integral coefficients. Assume that this is true for all values of λ, μ, and ν, with $\lambda + \mu + \nu \leqslant \tau$ where $\tau \geqslant 1$, and consider any $[A^\lambda B^\mu C^\nu]$ with $\lambda + \mu + \nu = \tau + 1$. If any two of λ, μ, and ν, are non-zero we can express $[A^\lambda B^\mu C^\nu]$ as a linear combination of similar sums with $\lambda + \mu + \nu \leqslant \tau$ by using (1.12) to (1.14); and so by the induction hypothesis it is equal to a polynomial in $1/K - K$ with integral coefficients. Also, using Newton's formula, $[A^\lambda]$ can be expressed as a polynomial in $1/K - K$ with integral coefficients, by (1.5), (1.15), and (1.16).

Thus if our assertion is true for $\lambda + \mu + \nu \leqslant \tau$ it is true for all λ, μ, and ν, with $\lambda + \mu + \nu = \tau + 1$; but it is clearly true for $\tau = 1$, hence it is true for all values of τ by the strong form of mathematical induction. This completes the proof of Lemma 1.2.

Writing

$$F = y^{-1}f^2(y)/f^2(y^{13}),$$

we have shown that F^7 is equal to a linear combination of terms $[a^{j_1} b'^{j_2} c^{j_3} a'^{j_4} b^{j_5} c'^{j_6}]$, and hence, by Lemma 1.2, to a polynomial in $1/K - K$ with integral coefficients. Further, this polynomial is of degree 7 since the lowest powers of y in the expansions of F^7 and $1/K - K$ as ascending power series in y are -7 and -1 respectively. By comparing coefficients of powers of y as far as y^0 we find that

$$F^7 = (1/K - K - 3)^7,$$

or, since F and K are real for real y,

(1.17) $$F = 1/K - K - 3.^*$$

Similarly $y^{-6}f^{14}(y)\,\Phi(6)/f^{13}(y^{13})$ is equal to a polynomial of degree 6 in $1/K-K$ with integral coefficients, or by (1.17), in F. Comparing coefficients as far as y^0 we find that

(1.18) $$yf(y^{13})\,\Phi(6) = 11/F + 36.13/F^2 + 38.13^2/F^3 + 20.13^3/F^4 + 6.13^4/F^5 + 13^5/F^6 + 13^5/F^7$$

on dividing through by F^7. (1.18) was first found by Zuckermann [17], using the theory of the elliptic modular functions.

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* Dr. Atkin points out that this identity is given (in a different notation) on page 326 of [13] (Ramanujan).

2. We shall now find expressions for all the $\Phi(s)$ $(0 \leqslant s \leqslant 12,\ s \neq 6)$. Consider $\Phi(1)$. $y^{-7}f^{14}(y)\Phi(1)/f^{13}(y^{13})$ is the coefficient of x^{8} in $y^{-7}f^{14}(y)/\{f^{14}(y^{13})(\alpha + \beta' + \gamma + \alpha' + \beta + \gamma' + 1)\}$, a cyclically symmetric polynomial in α, β', γ, α', β, and γ'. Thus $y^{-7}f^{14}(y)\Phi(1)x^{8}/ - \alpha\ f^{13}(y^{13})$ is the coefficient of x^{0} in a polynomial in α, β', γ, α', β, and γ', which although not cyclically symmetric, is a linear combination of terms $\alpha^{i_1}\beta'^{i_2}\gamma^{i_3}\alpha'^{i_4}\beta^{i_5}\gamma'^{i_6}$ (the indices here may be presumed non-negative because $-1/\alpha = \beta'\gamma\ \alpha'\ \beta\ \gamma'$), also, for any such term which occurs in the coefficient of x^{0}, (1.3) must hold. Hence, by Lemma 1.1, $y^{-6}f^{14}(y)P(1)\Phi(1)/f^{13}(y^{13})P(2)$ is equal to a linear combination of terms $a^{j_1}b'^{j_2}c^{j_3}a'^{j_4}b^{j_5}c'^{j_6}$. We define $\phi(s)$, the "normalised" form of $\Phi(s)$, in the following six cases:

$$\phi(1) = P(1)\ \Phi\ (1)/P(2),$$

$$\phi(12) = -yP(2)\ \Phi\ (12)/P(4),$$

$$\phi(4) = -P(4)\ \Phi\ (4)/P(5),$$

$$\phi(11) = P(5)\ \Phi\ (11)/P(3),$$

$$\phi(0) = P(3)\ \Phi\ (0)/P(6),$$

$$\phi(8) = -y^{-1}P(6)\ \Phi\ (8)/P(1).$$

Then we have shown that $yf(y^{13})\phi(1)F^{7}$ is equal to a linear combination of terms $a^{j_1}\ b'^{j_2}\ c^{j_3}\ a'^{j_4}\ b^{j_5}\ c'^{j_6}$. We can show, in a similar manner, that this is true if $\phi(1)$ is replaced by $\phi(s)$ for $s = 12, 4, 11, 0$, or 8, if we replace the multiplier $-1/\alpha$ by $-1/\beta'$, $-1/\gamma$, $-1/\alpha'$, $-1/\beta$, or $-1/\gamma'$, respectively. Further, given an expression for any $\phi(s)$ in the above list, we may obtain any other such $\phi(s)$ by interchanging the $\phi(s)$ (in the above order) and a, b', c, a', b, c', cyclically.

We define $\phi(s)$ in the remaining six cases as follows:

$$\begin{aligned}
\phi(10) &= P(3)\,\Phi(10)/P(2),\\
\phi(9) &= -P(6)\,\Phi(9)/P(4),\\
\phi(5) &= -yP(1)\,\Phi(5)/P(5),\\
\phi(2) &= -P(2)\,\Phi(2)/P(3),\\
\phi(3) &= P(4)\,\Phi(3)/P(6),\\
\phi(7) &= y^{-1}P(5)\,\Phi(7)/P(1).
\end{aligned}$$

We may show that the above result holds for these $\phi(s)$ by considering $y^{-7} f^{14}(y)/\{f^{14}(y^{13})(\alpha + \beta' + \gamma + \alpha' + \beta + \gamma' + 1)\}$ multiplied by $\beta'\gamma\alpha'$, $\gamma\alpha'\beta$, $\alpha'\beta\gamma'$, $\beta\gamma'\alpha$, $\gamma'\alpha\beta'$, and $\alpha\beta'\gamma$, instead of $-1/\alpha$, $-1/\beta'$, $-1/\gamma$, $-1/\alpha'$, $-1/\beta$, and $-1/\gamma'$.

Thus we must now examine $a^{j_1}b'^{j_2}c^{j_3}a'^{j_4}b^{j_5}c'^{j_6}$, rather than $[a^{j_1}b'^{j_2}c^{j_3}a'^{j_4}b^{j_5}c'^{j_6}]$. To do this we need certain preliminary results. Using (1.17), (1.15) can be written as

(2.1) $$A + B + C + F + 4 = 0.$$

Multiplying this equation by A, substituting for AB and CA from (1.12) and (1.14), and transposing we obtain

(2.2) $$C = A^2 + (F + 3)A - 2.$$

Substituting this expression for C in (2.1), and transposing we have

(2.3) $$B = -A^2 - (F + 4)A - F - 2.$$

Also, (1.17) can be written in the form

(2.4) $$-K = -1/K + F + 3.$$

Thus, by virtue of (2.2), (2.3), and (2.4), any polynomial in A, B, C, 1/K, and -K, with integral coefficients, can be expressed as a polynomial in A, 1/K, and F, also with integral coefficients. Further, multiplying (2.3) by A, substituting for AB from (1.12), and transposing we obtain

(2.5) $$A^3 = -(F + 4)A^2 - (F + 1)A + 1,$$

and, multiplying (2.4) by 1/K, and transposing we have

(2.6) $$(1/K)^2 = (F + 3)/K + 1.$$

So, by virtue of (2.5) and (2.6), any polynomial in A, 1/K, and F, with integral coefficients, can be expressed as a linear combination of terms

(2.7) $$F^h(e_1A^2/K + e_2A^2 + e_3A/K + e_4A + e_5/K + e_6)$$

where h is a non-negative integer and e_1 to e_6 are positive, negative, or zero, integers. We conclude that any polynomial in A, B, C, 1/K, and -K, with integral coefficients, is equal to a linear combination of terms (2.7).

We note here that by (1.5), (1.15), (1.16), and (1.17), A, B, and C, are the roots of the cubic equation

(2.8) $$z^3 + (F + 4)z^2 + (F + 1)z - 1 = 0;$$

that by (1.17), 1/K and -K are the roots of the quadratic equation

(2.9) $$z^2 - (F + 3)z - 1 = 0;$$

and that (2.5) and (2.6) follow from (2.8) and (2.9) respectively.

Now, using (1.6) to (1.11) any $a^{j_1}b'^{j_2}c^{j_3}a'^{j_4}b^{j_5}c'^{j_6}$ can be expressed as a polynomial in A, B, C, 1/K, and -K, with integral coefficients. Thus we arrive at

LEMMA 2.1 Any expression of the form $a^{j_1}b'^{j_2}c^{j_3}a'^{j_4}b^{j_5}c'^{j_6}$ is equal to a linear combination of terms (2.7). This statement remains valid if in (2.7) A is replaced by any one of A, B, C, and 1/K is replaced by either of 1/K, -K.

The latter sentence follows because of the cyclic properties of our relations.

We note that if we <u>define</u> F by (1.17) then Lemma 1.2 is a consequence of Lemma 2.1, for by Lemma 2.1 any $[a^{j_1}b'^{j_2}c^{j_3}a'^{j_4}b^{j_5}c'^{j_6}]$ is expressible as a linear combination of terms

$$F^h\{e_1(1/K-K)[A^2]+2e_2[A^2]+e_3(1/K-K)[A]+2e_4[A]+3e_5(1/K-K)+6e_6\},$$

and any such term, in view of (1.15), (1.16), and (1.17), is equal to a polynomial in 1/K - K with integral coefficients.

Now, we have shown that $yf(y^{13})\phi(1)F^7$ is equal to a linear combination of terms $a^{j_1}b'^{j_2}c^{j_3}a'^{j_4}b^{j_5}c'^{j_6}$, and hence by Lemma 2.1, to a linear combination of terms (2.7) where, for a reason which will appear in §3, we choose to replace A and 1/K by C and -K respectively. Also, given $\phi(1)$ in terms of C and -K we obtain all the $\phi(s)$ (s = 1, 12, 4, 11, 0, 8) immediately by interchanging $\phi(s)$ (in the order given), and A, B, C, and 1/K, -K, cyclically. We have exactly the same situation for the other six $\phi(s)$ (s = 10, 9, 5, 2, 3, 7) where, again for a reason which will appear in §3, we choose to express $\phi(10)$ in terms of C and -K. Thus if for each of the twelve values of s we choose variables from A, B, C, and 1/K, -K, according to the following tables

s	0	1	4	8	11	12
	A	C	B	B	C	A
	-K	-K	-K	1/K	1/K	1/K

Table 2.1

s	2	3	5	7	9	10
	C	A	B	B	A	C
	1/K	-K	-K	1/K	1/K	-K

Table 2.2

then $yf(y^{13})\phi(s)F^7$ is equal to a linear combination of terms (2.7) in each of which A and 1/K are replaced by variables appropriate to the particular value of s, and for each value of h the coefficients e_1 to e_6 in (2.7) are the same for all the s of one group of six. We find the values of e_1 to e_6

(for each value of h occurring) in the two distinct cases by comparing coefficients, as before.

Consider the case to which Table 2.1 applies. Let H be the highest value of h occurring, i.e. the highest value of h for which e_1 to e_6 are not all zero. Then $yf(y^{13})\phi(12)F^7$ is (without loss of generality) the sum of terms (2.7) with $0 \leqslant h \leqslant H$. Now, since A and 1/K (expanded as ascending power series in y) begin $y + \ldots$ and $y^{-1} + \ldots$ respectively, the lowest power of y occurring in the bracket of (2.7) is -1, and it occurs in the term e_5/K (and in none of the other five terms as it happens). Thus, writing E_1 to E_6 for the e_1 to e_6 appertaining to $h = H$, the lowest power of y in the aggregate of terms (2.7) is $-(H + 1)$ (since F begins $y^{-1}+\ldots$), and it occurs in the term $F^H E_5/K$ (only); but $yf(y^{13})\phi(12)F^7$ begins $-77y^{-5}+\ldots$, hence $E_5 = 0$ if $H + 1 > 5$. Applying this argument to all of the six $\phi(s)$, using the variables indicated in Table 2.1 in each case, we obtain (from s = 0, 1, 4, 8, 11, and 12, respectively)

$$E_6 = 0 \text{ if } H > 6,$$
$$E_2 - E_4 + E_6 = 0 \text{ if } H > 6,$$
$$E_2 = 0 \text{ if } H > 4,$$
$$E_1 = 0 \text{ if } H > 4,$$
$$E_1 - E_3 + E_5 = 0 \text{ if } H > 5,$$
$$E_5 = 0 \text{ if } H > 4;$$

when $s = 1$, or 11, $yf(y^{13})\phi(s)F^7$ is equal to an expression in which the lowest power of y occurs in three terms of the bracket prefixed by F^H. Thus if $H > 6$, E_1 to E_6 (found seriatim) are all zero, but this contradicts the definition of H, hence $H \leqslant 6$. We need only to notice that, from the case $s = 0$ above, $E_6 \neq 0$ if $H = 6$, to conclude that in fact $H = 6$.

It may be shown, by similar reasoning, that for the other group of $\phi(s)$, H is again 6.

For each group of $\phi(s)$ then we need to find the coefficients e_1 to e_6 for each h in the range $0 \leqslant h \leqslant 6$. Comparing coefficients of powers of y for the first 7 powers of y occurring in the expression for $yf(y^{13})\phi(s)F^7$ (for each s of the group in question) we obtain 42 equations relating the 42 unknown coefficients. It turns out that these equations are sufficient to determine the coefficients, in fact, in each of the two cases, the coefficients appear seriatim.

We state the results* in the form:

THEOREM 2.1 We have

---

* In actual fact we checked the values of the coefficients found, in both cases, by comparing the coefficients of the eighth lowest power of y for $s = 8$ and $s = 7$.

$$
\begin{aligned}
yf(y^{13})\phi(12)=1/F \quad & +(-56A/K-33A-1/K+99)/F^2+ \\
& +13(-6A^2/K-3A^2-109A/K-31A-9/K+159)/F^3+ \\
& +13^2(-11A^2/K-4A^2-85A/K-16A-11/K+105)/F^4+ \\
& +13^3(-7A^2/K-3A^2-34A/K-5A-5/K+37)/F^5+ \\
& +13^4(-2A^2/K-A^2-7A/K-A-1/K+7)/F^6+ \\
& +13^4(-3A^2/K-2A^2-8A/K-A-1/K+8)/F^7,
\end{aligned}
$$

$$
\begin{aligned}
yf(y^{13})\phi(9)=(-39A+3)/F \quad & +(-39A^2+11A/K-985A-33/K+264)/F^2+ \\
& +13(2A^2/K-67A^2+13A/K-786A-83/K+348)/F^3+ \\
& +13^2(4A^2/K-46A^2+10A/K-334A-68/K+210)/F^4+ \\
& +13^3(3A^2/K-16A^2+4A/K-82A-28/K+68)/F^5+ \\
& +13^4(A^2/K-3A^2+A/K-11A-6/K+12)/F^6+ \\
& +13^4(2A^2/K-3A^2+A/K-8A-8/K+12)/F^7,
\end{aligned}
$$

and these equations still hold if $\phi(12)$ or $\phi(9)$ is replaced by $\phi(s)$ for values of s occurring in Table 2.1 or Table 2.2 respectively provided that A is replaced by A, B, or C, and 1/K is replaced by 1/K or -K, according to these tables.

It is interesting to compare the powers of 13 occurring in the equations of this theorem with those occurring in the expression for $yf(y^{13})\Phi(6)$ given in (1.18).

We proceed to derive an alternative form of Theorem 2.1.

Writing

$$l = y^2P(3)/P(6)P(5),\quad m = yP(4)/P(5)P(2),\quad n=-y^2P(1)/P(2)P(6),$$

$$l'= yP(2)/P(4)P(1),\quad m'=P(6)/P(1)P(3),\quad n'=-yP(5)/P(3)P(4),$$

we have immediately, from the definitions of A, B, C, and K,

(2.10) $\quad l/l' = m/m' = n/n' = K,$

which equations will be used without explicit mention, and

(2.11) to (2.13) $\quad l/m = A, \quad m/n = B, \quad n/l = C.$

We note that equations (2.10) do <u>not</u> remain valid if $1/K$, $-K$, and l, m', n, l', m, n', are interchanged cyclically, but that (2.10) to (2.13) all remain valid if A, B, C, and $1/K$, $-K$, are interchanged cyclically and l, m', n, l', m, and n', are interchanged according to either

(2.14) $\quad \begin{pmatrix} l & m' & n & l' & m & n' \\ m' & -n & l' & -m & n' & -l \end{pmatrix}$

or

(2.15) $\quad \begin{pmatrix} l & m' & n & l' & m & n' \\ -m' & n & -l' & m & -n' & l \end{pmatrix}.$

Substituting for A, B, and C, from (2.11) to (2.13), in (1.12) to (1.14) we obtain in each case

(2.16) $\quad 1/l + 1/m + 1/n = 0.$

Similarly (2.1) becomes

(2.17) $\quad l/m + m/n + n/l + F + 4 = 0.$

Now, (2.16) may be written as

(2.18) $\quad lm/n = -l - m,$

and (2.17) as

$$l^2/m = -lm/n - Fl - 4l - n$$

which using (2.18) becomes

(2.19) $\quad l^2/m = -Fl - 3l + m - n,$

and using (2.11) this equation may be written as

(2.20) $mA^2 = -Fl - 3l + m - n$

or, dividing through by K,

(2.21) $mA^2/K = -Fl' - 3l' + m' - n'$.

Also we have trivially from (2.11)

(2.22), and (2.23) $mA = l$, $mA/K = l'$.

So, multiplying the first equation of Theorem 2.1 by m, and substituting for mA^2, mA^2/K, mA, and mA/K, from (2.20) to (2.23), we obtain $yf(y^{13})m\phi(12)$ as a sum of terms

(2.24) $F^h(e_1' l + e_2' m' + e_3' n + e_4' l' + e_5' m + e_6' n')$.

We chose to take m with $\phi(12)$ for a reason which will appear in § 3. Now we have seen that the first equation of Theorem 2.1 still holds if we interchange $\phi(1)$, $\phi(12)$, $\phi(4)$, $\phi(11)$, $\phi(0)$, $\phi(8)$, and A, B, C, and $1/K$, $-K$, cyclically. Hence the above equation for $\phi(12)$ still holds if we interchange these $\phi(s)$ cyclically, and interchange l, m', n, l', m, and n', according to (2.14) or (2.15). We obtain a similar result for the other six $\phi(s)$ by multiplying the second equation of Theorem 2.1 by m. Thus multiplying $\phi(s)$ by l', m, n', l, m', and n, when s = 1, 12, 4, 11, 0, and 8, or 10, 9, 5, 2, 3, and 7, respectively, and denoting the result by $\phi'(s)$, so that

$\phi'(1) = y\Phi(1)/P(4)$, $\phi'(10) = yP(3)\Phi(10)/P(4)P(1)$,

$\phi'(12) = -y^2\Phi(12)/P(5)$, $\phi'(9) = -yP(6)\Phi(9)/P(5)P(2)$,

(2.25) $\phi'(4) = y\Phi(4)/P(3)$, $\phi'(5) = y^2P(1)\Phi(5)/P(3)P(4)$,

$\phi'(11) = y^2\Phi(11)/P(6)$, $\phi'(2) = -y^2P(2)\Phi(2)/P(6)P(5)$,

$$\phi'(0) = \Phi(0)/P(1), \qquad \phi'(3) = P(4)\Phi(3)/P(1)P(3),$$

$$\phi'(8) = y\Phi(8)/P(2), \qquad \phi'(7) = -yP(5)\Phi(7)/P(2)P(6),$$

we may re-state Theorem 2.1 in the form:

THEOREM 2.2 We have

$$\begin{aligned} yf(y^{13})\phi'(12) = m/F + (& 6l - m' + 22l' + 99m)/F^2 + \\ +13(& 30l-15m'+3n+52l'+156m+6n')/F^3+ \\ +13^2(& 35l-22m'+4n+39l'+101m+11n')/F^4+ \\ +13^3(& 17l-12m'+3n+13l'+34m+7n')/F^5+ \\ +13^4(& 4l-3m'+n+2l'+6m+2n')/F^6+ \\ +13^4(& 5l-4m'+2n+l'+6m+3n')/F^7, \end{aligned}$$

$$\begin{aligned} yf(y^{13})\phi'(9)=3m/F + (& 3l-33m'+39n-15l'+225m)/F^2+ \\ +13(& 13l-81m'+67n-45l'+281m-2n')/F^3+ \\ +13^2(& 12l-64m'+46n-41l'+164m-4n')/F^4+ \\ +13^3(& 5l-25m'+16n-18l'+52m-3n')/F^5+ \\ +13^4(& l-5m'+3n-4l'+9m-n')/F^6+ \\ +13^4(& l-6m'+3n-5l'+9m-2n')/F^7, \end{aligned}$$

and these equations still hold if $\phi'(12)$ or $\phi'(9)$ is replaced by $\phi'(s)$ for values of s occurring in the first or the second row of the following table respectively provided that l, m', n, l', m, and n', are interchanged according to this table:

s	1	12	4	11	0	8
s	10	9	5	2	3	7
	n'	l	m'	n	l'	m
	-l	m'	-n	l'	-m	n'
	m'	n	l'	m	n'	l
	-n	l'	-m	n'	-l	m'
	l'	m	n'	l	m'	n
	-m	n'	-l	m'	-n	l'

We emphasise that for any particular value of s the equation given in Theorem 2.2 is simply the equation given in Theorem 2.1 multiplied by l, m', n, l', m, or n'; the former equation, of degree 0 in the P(a), becomes an equation of degree -1 in the P(a). Although in Theorem 2.1 each $\Phi(s)$ is expressed in terms of only two variables, such as A and $1/K$, the two variables are different for different values of s. In Theorem 2.2 six variables are needed, but they are the same for all the $\Phi(s)$, and moreover, unlike Theorem 2.1, the expressions are homogeneous in these variables.

3. In this paragraph all congruences are modulo 13. We state and prove:

THEOREM 3.1 We have

$\Phi(0) \equiv 6P(6)\Phi(6)/P(3)-5yP(0)/P(5)$,

$\Phi(1) \equiv 6P(2)\Phi(6)/P(1)+2yP(0)/P(6)$,

$$\Phi(2) \equiv -5P(3)\Phi(6)/P(2)+5P(0)P(5)/P(2)P(4),$$

$$\Phi(3) \equiv 5P(6)\Phi(6)/P(4)+4yP(0)P(3)/P(4)P(5),$$

$$\Phi(4) \equiv -6P(5)\Phi(6)/P(4)+6P(0)/P(2),$$

$$\Phi(5) \equiv -5y^{-1}P(5)\Phi(6)/P(1)+3y^{-1}P(0)P(4)/P(1)P(2),$$

$$\Phi(6) \equiv -2P(0)/f^2(y),$$

$$\Phi(7) \equiv 5yP(1)\Phi(6)/P(5)+2P(0)P(6)/P(3)P(5),$$

$$\Phi(8) \equiv -6yP(1)\Phi(6)/P(6)-4P(0)/P(3),$$

$$\Phi(9) \equiv -5P(4)\Phi(6)/P(6)-6P(0)P(2)/P(1)P(6),$$

$$\Phi(10) \equiv 5P(2)\Phi(6)/P(3)+yP(0)P(1)/P(3)P(6),$$

$$\Phi(11) \equiv 6P(3)\Phi(6)/P(5)+3P(0)/P(4),$$

$$\Phi(12) \equiv -6y^{-1}P(4)\Phi(6)/P(2)+y^{-1}P(0)/P(1).$$

We note that the form of these congruences is analogous to that of the corresponding results for $q = 5$, 7, and 11, given as Theorems 1, 2, and 3, in (ASD). There is a basic difference only in so far as $\Phi_{13}(6) \not\equiv 0$.

Now, the congruence for $\Phi(6)$ follows immediately from (1.18) $\{$since $f(y^{13})=P(0)\}$. Substituting for $\phi'(12)$ from (2.25) in the first equation of Theorem 2.2 we obtain

$$-y^3 f(y^{13})\Phi(12)/P(5) \equiv m/F + (6\,l - m' + 22\,l' + 99m)/F^2,$$

which may be written in the form

$$\Phi(12) \equiv -y^{-1}\,\frac{P(4)}{P(2)}\,\frac{P(0)}{f^2(y)} - \frac{y}{P(1)}\,\frac{P^3(0)}{f^4(y)}\,\frac{6\,l - m' + 22\,l' + 99m}{m\,l'}.$$

Thus, comparing the congruence for $\Phi(12)$ in the theorem with this congruence $\{$using the congruence for $\Phi(6)\}$, we see that the former is valid if

$$y^{-2}f^4(y)/P^2(0) \equiv 1/l - 6/m' - 99/l' - 22/m$$

which equation may be written as

(3.1) $y^{-2}f^4(y)/P^2(0) \equiv -5/l + 3/m' - 6/n + 1/l' - 2/m - 4/n'$,

using (2.16) and (2.16) multiplied through by K. By a similar argument we may show that for each of the other five s of the group containing s = 12 the validity of the congruence in the theorem depends only on the validity of (3.1) multiplied through by some constant. Further, for the remaining six s we find, using the preceding process, that to prove the congruences in the theorem we need again only to show that (3.1) holds. We prove (3.1) as follows.

Writing

$$X = -5/l - 6/n - 2/m$$

we have, multiplying through by l and using (2.11) and (2.12),

$$lX = -5 - 6AB - 2A$$

which using (1.12) becomes

(3.2) $lX = 4A + 1.$

Similarly we may obtain

(3.3) $nX = - 3C - 4,$

(3.4) $mX = - B + 3.$

Multiplying together the last three equations we have

$$lnmX^3 \equiv - ABC + 3[AB] + 4[A] + 1,$$

and by (1.5), (1.15), (1.16), and (1.17), the right-hand side of this equation is congruent to -F so that, squaring both sides of the equation,

$$l^2n^2m^2X^6 \equiv y^{-2}f^4(y)/P^4(0);$$

but from the definitions of l, n, m, and K,

$$l^2n^2m^2 = y^7P(0)K^3/f(y),$$

hence

$$X^6 \equiv y^{-9}f^5(y)/P^5(0)K^3,$$

or since $f^{13}(y) \equiv P(0)$

$$X^2 \equiv y^{-3}f^6(y)/P^2(0)K,$$

where the value of the coefficient of the lowest power of y in the expansion of each side of this equation is examined to determine the appropriate root. By virtue of (1.17) we may write the last equation in the form

$$X^2 \equiv y^{-2}f^4(y)(1/K + 5)^2,$$

whence

$$(3.5) \qquad X \equiv y^{-1}f^2(y)(1/K + 5),$$

where the sign of the coefficient of the lowest power of y on each side of this equation is examined to determine the appropriate root. Now, the right-hand side of (3.1) is congruent to $(5K + 1)(-5/l - 6/n - 2/m)$, i.e. to $(5K + 1)X$, and by (3.5) this is congruent to $y^{-1}f^2(y)(1/K - K - 3)$ which equals $y^{-2}f^4(y)/P^2(0)$ by (1.17). Thus (3.1) holds. This completes the proof of the theorem.

It would be possible to prove Theorem 3.1 by either of the methods used to prove Theorems 1 and 2, and Theorem 3, in (ASD). Indeed the congruences of Theorem 3.1 were originally derived from other more complicated congruences which were found by Dr. Atkin using the method of Theorems 1 and 2. It is because the above congruences for the $\Phi(s)$ were discovered before the identities given by Theorems 2.1 and 2.2 that I was able to assign convenient variables to particular $\Phi(s)$ for the purpose of these two theorems.

4. The values of the $r_{bc}(d)$ for $q = 11$ proved in (AH) were actually found empirically; for $q = 13$ we use a similar method.

Putting $b = 6, 5, 4, 3, 2, 1$, and 0, in equation (6.2) of (ASD) (with $q = 13$), and $b = 0$ and 3 in equation (6.3) of (ASD), we obtain respectively

$$
\begin{aligned}
&S(6) = 0, && S(7) = -S(5), && S(8) = -S(4), \\
(4.1) \qquad &S(9) = -S(3), && S(10) = -S(2), && S(11) = -S(1), \\
&S(12) = -S(0), && S(13) = -f(x)+S(0)+1, && S(16) = x^{-2}f(x)+S(3)+1,
\end{aligned}
$$

and it is easily seen that there are essentially only six distinct $S(b)$, which we take to be $S(0)$ to $S(5)$.

We write

$$N_b = N_b(x) = \sum_{n=0}^{\infty} N(b,\ 13,\ n)x^n,$$

$$N_{bc} = N_b - N_c,$$

so that by (6.10) of (ASD)

(4.2) $$N_{bc}=\sum_{d=0}^{12} r_{bc}(d)x^{d}.$$

Then by (2.13) and (6.1) of (ASD), and (4.1) above,

(4.3)
$$\begin{aligned}
f(x)N_{01} &= \{S(0)+S(13)\}-\{S(1)+S(12)\}=-f(x)+3S(0)-S(1)+1,\\
f(x)N_{12} &= \{S(1)+S(12)\}-\{S(2)+S(11)\}=-S(0)+2S(1)-S(2),\\
f(x)N_{23} &= \{S(2)+S(11)\}-\{S(3)+S(10)\}=-S(1)+2S(2)-S(3),\\
f(x)N_{34} &= \{S(3)+S(10)\}-\{S(4)+S(9)\}=-S(2)+2S(3)-S(4),\\
f(x)N_{45} &= \{S(4)+S(9)\}-\{S(5)+S(8)\}=-S(3)+2S(4)-S(5),\\
f(x)N_{56} &= \{S(5)+S(8)\}-\{S(6)+S(7)\}=-S(4)+2S(5),
\end{aligned}$$

and putting m = 2, 6, 3, 1, 5, and 4, in (6.7) of (ASD) we obtain using (4.1) the following expressions for S(0) to S(5), respectively.

(4.4)
$$S(0)=f(x)\{y^{2}\frac{\Sigma(2,0)}{P(0)}+1\}-g(2)-1+P^{2}(0)\{x\frac{P(3)P(6)}{P(1)P(2)P(5)}-x^{2}\frac{y}{P(3)}-$$
$$-x^{5}\frac{P(4)P(5)}{P^{2}(2)P(6)}-x^{9}y\frac{P(1)P(6)}{P(2)P(4)P(5)}+x^{12}\frac{P(5)}{P(2)P(6)}\},$$

$$S(1)=f(x)\{x^{4}y^{4}\frac{\Sigma(6,0)}{P(0)}\}-g(6)+P^{2}(0)\{-x^{3}\frac{P(3)P(5)}{P(1)P(2)P(6)}+$$
$$+x^{4}y^{2}\frac{P(2)}{P(5)P(6)}-x^{5}\frac{y}{P(4)}-x^{6}y^{3}\frac{P(1)P(2)}{P(5)P^{2}(6)}+x^{9}\frac{P(4)P(5)}{P(2)P(3)P(6)}\},$$

$$S(2)=f(x)\{x^{12}y^{2}\frac{\Sigma(3,0)}{P(0)}\}+g(3)+P^{2}(0)\{-xy^{2}\frac{P(1)}{P(3)P(4)}-x^{4}\frac{P(4)P(5)}{P(1)P(3)P(6)}-$$
$$-x^{8}y\frac{P(1)P(6)}{P^{2}(3)P(4)}+x^{11}\frac{1}{P(2)}+x^{12}\frac{P(2)P(4)}{P(1)P(3)P(5)}\},$$

$$S(3)=f(x)\{-x^{11}y^{-1}-x^{11}\frac{\Sigma(1,0)}{P(0)}\}-g(1)-1+P^{2}(0)\{x^{3}\frac{P(4)}{P(1)P(3)}+x^{7}\frac{y}{P(5)}-$$
$$-x^{10}\frac{P(3)P(5)}{P(1)P(4)P(6)}+x^{11}y^{-1}\frac{P(2)P(4)}{P^{2}(1)P(3)}-x^{12}y^{-1}\frac{P(3)P(6)}{P(1)P(2)P(4)}\},$$

$$S(4)=f(x)\{-xy^{4}\frac{\Sigma(5,0)}{P(0)}\}-g(5)+P^{2}(0)\{-xy^{2}\frac{P(2)P(4)}{P(3)P(5)P(6)}+$$

$$+x^2y\,\frac{P(3)P(6)}{P(2)P^2(5)} + x^3y^3\,\frac{P(1)P(2)}{P(4)P(5)P(6)} - x^6\,\frac{1}{P(1)} + x^{10}\,\frac{P(6)}{P(2)P(5)}\Big\},$$

$$S(5)=f(x)\Big\{-x^8y^3\,\frac{\Sigma(4,0)}{P(0)}\Big\}+g(4)+P^2(0)\Big\{x^4y\,\frac{P(1)P(6)}{P(2)P(3)P(4)} - x^7\,\frac{P(3)P(5)}{P(1)P^2(4)} -$$

$$-x^8\,\frac{y}{P(6)} + x^9\,\frac{P(3)}{P(1)P(4)} + x^{10}y^2\,\frac{P(1)P(2)}{P(3)P(4)P(5)}\Big\}.$$

Now, as with $q = 11$, it is clearly convenient to avoid the terms involving $\Sigma(m, 0)$ which occur in (4.4). For example, from (4.3) and (4.4) N_{01} contains a term

$$-1 + 3\Big\{y^2\,\frac{\Sigma(2,0)}{P(0)} + 1\Big\} - x^4y^4\,\frac{\Sigma(6,0)}{P(0)},$$

i.e., in view of (4.2), $r_{01}(0)$ contains a term $3y^2\,\frac{\Sigma(2,0)}{P(0)} + 2$, and $r_{01}(4)$ contains a term $-y^4\,\frac{\Sigma(6,0)}{P(0)}$. Also, the forms of the $r_{bc}(d)$ for $q = 5, 7$, given in (ASD), and for $q=11$, together with the congruences for the $\bar{\Phi}_{13}(b)$ given in Theorem 3.1, suggest that the values of the $r_{bc}(0)$, for example, will involve either a factor $P(6)/P(3)$ or a factor $y/P(5)$; it is found to be preferable to consider the factors of the former type. We accordingly (following the case of $q = 11$) define $R_{bc}(d)(0 \leqslant d \leqslant 12)$, the "normalised" form of $r_{bc}(d)$, for $q=13$ as shown; clearly, from the definition of $r_b(d)$ and the relation $N(m, q, n) = N(q - m, q, n)$ given in (ASD), we may consider b and c to lie between 0 and 6 inclusive.

$$R_{01}(0) = P(3)\{r_{01}(0)-3y^2\Sigma(2,0)/P(0)-2\}/P(6),$$
$$R_{12}(0) = P(3)\{r_{12}(0)+y^2\Sigma(2,0)/P(0)+1\}/P(6),$$
$$R_{34}(1) = P(1)\{r_{34}(1)-y^4\Sigma(5,0)/P(0)\}/P(2),$$
$$R_{45}(1) = P(1)\{r_{45}(1)+2y^4\Sigma(5,0)/P(0)\}/P(2),$$
$$R_{56}(1) = P(1)\{r_{56}(1)-y^4\Sigma(5,0)/P(0)\}/P(2),$$
$$R_{01}(4) = -P(4)\{r_{01}(4)+y^4\Sigma(6,0)/P(0)\}/P(5),$$
$$R_{12}(4) = -P(4)\{r_{12}(4)-2y^4\Sigma(6,0)/P(0)\}/P(5),$$
$$R_{23}(4) = -P(4)\{r_{23}(4)+y^4\Sigma(6,0)/P(0)\}/P(5),$$
$$R_{45}(8) = -y^{-1}P(6)\{r_{45}(8)-y^3\Sigma(4,0)/P(0)\}/P(1),$$
$$R_{56}(8) = -y^{-1}P(6)\{r_{56}(8)+2y^3\Sigma(4,0)/P(0)\}/P(1),$$
$$R_{23}(11) = P(5)\{r_{23}(11)-\Sigma(1,0)/P(0)-y^{-1}\}/P(3),$$
$$R_{34}(11) = P(5)\{r_{34}(11)+2\Sigma(1,0)/P(0)+2y^{-1}\}/P(3),$$
$$R_{45}(11) = P(5)\{r_{45}(11)-\Sigma(1,0)/P(0)-y^{-1}\}/P(3),$$
$$R_{12}(12) = -yP(2)\{r_{12}(12)+y^2\Sigma(3,0)/P(0)\}/P(4),$$
$$R_{23}(12) = -yP(2)\{r_{23}(12)-2y^2\Sigma(3,0)/P(0)\}/P(4),$$
$$R_{34}(12) = -yP(2)\{r_{34}(12)+y^2\Sigma(3,0)/P(0)\}/P(4),$$

and, for all other values of b and c with c = b + 1,

$$R_{bc}(0) = P(3)r_{bc}(0)/P(6),$$
$$R_{bc}(1) = P(1)r_{bc}(1)/P(2),$$
$$R_{bc}(2) = -P(2)r_{bc}(2)/P(3),$$
$$R_{bc}(3) = P(4)r_{bc}(3)/P(6),$$
$$R_{bc}(4) = -P(4)r_{bc}(4)/P(5),$$
$$R_{bc}(5) = -yP(1)r_{bc}(5)/P(5),$$
$$R_{bc}(6) = r_{bc}(6),$$

$$R_{bc}(7) = y^{-1}P(5)r_{bc}(7)/P(1),$$
$$R_{bc}(8) = -y^{-1}P(6)r_{bc}(8)/P(1),$$
$$R_{bc}(9) = -P(6)r_{bc}(9)/P(4),$$
$$R_{bc}(10) = P(3)r_{bc}(10)/P(2),$$
$$R_{bc}(11) = P(5)r_{bc}(11)/P(3),$$
$$R_{bc}(12) = -yP(2)r_{bc}(12)/P(4),$$

and, for all remaining values of b and c, we use the relations

$$R_{bc}(d)+R_{ce}(d) = R_{be}(d),$$
$$R_{cb}(d) = -R_{bc}(d).$$

It will be noticed that in the above definitions the coefficient of any $r_{bc}(d)$ is precisely the coefficient of $\Phi(d)$ in the definition of $\phi(d)$, given in § 2.

We might now proceed as for q = 11, and use (4.3) and (4.4), together with the congruent form of $1/f(x)$ given by Theorem 3.1, to obtain congruent forms of all the $R_{bc}(d)$, as a first step in the attempt to obtain identical forms. Indeed, it would be possible to find identical forms directly, by using the identical form of $1/f(x)$ given by Theorem 2.1 or Theorem 2.2. However, either of these methods would be extremely tedious, and instead we proceed as follows.

Using (2.13) of (ASD) we determine* each of N_{01} to N_{56}, as a power series in x, as far as x^{142}. In view of (4.2) this

*The divisions by f(x) were carried out by means of a single-length programme on Durham University's Ferranti "Pegasus" computer; further details are given at the end of the thesis (page 90).

gives us every $r_{bc}(d)$, as a power series in y, as far as y^{10}, and it is a simple matter to find the corresponding terminated power series for the $R_{bc}(d)$.

We now seek congruences for the $R_{bc}(d)$, in the following manner. The factor $P(0)/f^2(y)$ occurring in the congruences for the $\Phi(b)$ given in Theorem 3.1, together with the factor $P^2(0)$ occurring in the expressions for the S(b) given in (4.4), suggest that each $R_{bc}(d)$-congruence will involve a factor $P^3(0)/f^2(y)$. Also, the form of the $R_{bc}(d)$-congruences for q = 11, given in [6], and the fact that in (4.4) the terms in the brackets prefixed by $P^2(0)$ are of degree -1 in the P(a), suggest that each $R_{bc}(d)$-congruence will involve a linear combination of l, m', n, l', m, n', and a further variable, the further variable being different only for different values of d and being a multiplicative combination of these quantities, of degree 1. It is obvious that we may consider this further variable to be linearly independent of l, m', n, l', m, and n'.

We find, by comparing coefficients of powers of y in the expansions of the appropriate quantities (the coefficients are of course all integral), that in fact, each $R_{bc}(d)$ appears to be congruent to the product of $P^3(0)/f^2(y)$ and a linear combination of l, m', n, l', m, n', and up to two further variables; the further variables found to suffice are given in the following table.

d	0	1	2	3	4	5	6	7	8	9	10	11	12
	Kl	Kn	n'/K	Kl	Km	Km	-	m'/K	m'/K	l'/K	Kn	n'/K	l'/K
	Kn	Km	-	-	Kl	-	-	-	l'/K	-	-	m'/K	n'/K

Table 4.1

We draw up a list of apparent congruences for all the $R_{bc}(d)$ with $c = b + 1$. The number of terms found in the expansion of each $R_{bc}(d)$ is sufficient to determine and check the 8 (or less) coefficients involved in each such congruence. Inspection of this list reveals no sets of congruent relations between the $R_{bc}(d)$ for different values of d such as are given for $q = 11$ in (9.1) to (9.14) of (AH), so that we cannot hope to find identities for the $R_{bc}(d)$ in the way used for $q = 11$. Instead we adopt the following method.

The form of the identities for the $\Phi(b)$ given in Theorem 3.2 suggests that each $R_{bc}(d)$ may be <u>equal</u> to the sum of <u>two</u> linear combinations of the type already indicated, multiplied by $P^3(0)/f^2(y)$ <u>and</u> $13yP^5(0)/f^4(y)$ respectively. A difficulty now arises: we have not found a sufficient number of terms of any $R_{bc}(d)$ to enable us to determine the 16 (or less) coefficients involved in such an identity. We circumvent this difficulty in a manner sufficiently well illustrated by the following example.

Writing

$$U = P^3(0)/f^2(y), \qquad V = yP^5(0)/f^4(y),$$

so that

(4.5) $\qquad U = FV,$

and noting that for $q = 11$ the numerical values of the coefficients involved in the $R_{bc}(d)$-identities are small, we assume that there is an identity for $R_{01}(0)$ of the form

$$R_{01}(0) = U(-5l-3m-3n-2l'-2m'+3Kn)+$$
$$+13V(f_1l+f_2m+f_3n+f_4l'+f_5m'+f_6n'+f_7Kl+f_8Kn),$$

where the U-term on the right-hand side is our congruent form of $R_{01}(0)$ written so that its coefficients all lie between ± 6 inclusive, and f_1 to f_8 are integers. The numbers of terms found in the expansion of $R_{01}(0)$ is sufficient to determine f_1 to f_8 and check the resulting identity.

In obtaining apparent identities for all the $R_{bc}(d)$ we occasionally find that in the U-bracket a 4, for example, should be a -9; this presents no serious difficulty. Also, we should note that for any particular $R_{bc}(d)$ a certain amount of transfer between U- and V- brackets is possible. For example, in the case of $R_{01}(0)$ we have the relations

(4.6) and (4.7) $\quad U(13l) = 13V(-3l+l'-Kl)$, $U(13n)=13V(-3n+n'-Kn)$,

found by multiplying (1.17) through by l and n respectively and using (4.5).

We state the result, a complete set of conjectural values of the $R_{bc}(d)$ for $q = 13$, in the form of a theorem, and then prove that the values are in fact correct.

THEOREM 4.1 We have the following; for each $R_{bc}(d)$ given, both brackets on the right-hand side involve l, m', n, l', m, n', and the quantities indicated in Table 4.1, only.

$$R_{01}(0) = U(-5l-3m-3n-2l'-2m'+3Kn)+13V(-2l-2m-2n+m'+n'-Kl),$$

$$R_{01}(1) = U(-8l+6m+n+l'+m'-2n'-8Kn)+13V(-l+2m+n+l'-m'-n'-Km-2Kn),$$

$$R_{01}(2) = U(7m-6l'+4m'+4n'+3n'/K)+13V(3m-2l'+m'+n'+n'/K),$$

$$R_{01}(3) = U(6l-9m+3n+m'+7n'-Kl)+13V(l-m+2n-l'+m'+n'+Kl),$$

$$R_{01}(4) = U(3l-m+7n+l'+n'-Kl+6Km)+13V(3n+Kl+2Km),$$

$$R_{01}(5) = U(5l-3m+3n+4l'+n'-5Km)+13V(2l+m+n+n'-2Km),$$

$$R_{01}(6) = U(-l+5m-6n+3l'-m'+2n')+13V(l+m-2n+2n'),$$

$$R_{01}(7) = U(-l-3n+6m'-6n'+2m'/K)+13V(-2n+3m'-n'-m'/K),$$

$$R_{01}(8) = U(-2m-n+3l'-5m'-n'+m'/K)+13V(-2m-n+l'),$$

$$R_{01}(9) = U(3m-10n-l'-2m'+l'/K)+13V(l-3n-l'-m'+n'+l'/K),$$

$$R_{01}(10) = U(8l-8m-2n-m'+6Kn)+13V(2l-4m-n+m'+2Kn),$$

$$R_{01}(11) = U(m+4n+4l'-3m'-4n'-4n'/K)+13V(m+n+l'-2m'-2n'-n'/K),$$

$$R_{01}(12) = U(m-n-6l'+3m'+4n'-3l'/K)+13V(m-n-3l'+m'+n');$$

$$R_{12}(0) = U(4l-m-2n-l'+m'+n'-2Kl-Kn)+13V(l+m-l'-Kn),$$

$$R_{12}(1) = U(7l+m-2n-n'+7Kn)+13V(2l-m-n+Km+2Kn),$$

$$R_{12}(2) = U(-l-4m-5l'+m'+n'+2n'/K)+13V(-l-m-l'+m'+n'),$$

$$R_{12}(3) = U(-4l+6m-4n-m'+n'+2Kl)+13V(-l+m-2n+n'),$$

$$R_{12}(4) = U(6l+m-5n-m'-n'+2Kl-3Km)+13V(l-2n+n'-Km),$$

$$R_{12}(5) = U(-l-3m-7n+4l'+n'+Km)+13V(-m-n+l'+n'+Km),$$

$$R_{12}(6) = U(-l-3m+5n+l'+m'+n')+13V(-l+2n+l'),$$

$$R_{12}(7) = U(-2l+3n-l'+6m'-n'-m'/K)+13V(-l+m+n-n'),$$

$$R_{12}(8) = U(m+n+4l'+2m'+n'-m'/K)+13V(m+n+l'-m'),$$

$$R_{12}(9) = U(-m+9n-3l'-2m'+2n'+2l'/K)+13V(-l+3n-m'+l'/K),$$

$$R_{12}(10) = U(-5l+7m-n+l'+2m'-6Kn)+13V(-l+3m+n+l'-m'-2Kn),$$

$$R_{12}(11) = U(-3n+l'-2m'-6n'-n'/K)+13V(-m-2n-n'),$$

$$R_{12}(12) = U(l-m+n-l'+3m'-3n'-2l'/K+n'/K)+13V(n+m'-n'-l'/K);$$

$$R_{23}(0) = U(5l-m+4n+l'-n'-Kl)+13V(2l-m+n-n'+Kn),$$

$$R_{23}(1) = U(-6l+3m-n+3n'-6Kn)+13V(-2l+2m+n+n'-2Kn),$$

$$R_{23}(2) = U(-2m+6l'-4m'-6n'-4n'/K)+13V(-m+2l'-2m'-2n'-n'/K),$$

$$R_{23}(3) = U(-4l+3m+n+m'-7n'+Kl)+13V(-l+m+l'-m'-2n'),$$

$$R_{23}(4) = U(-3l-5m+m'-Kl-Km)+13V(-m-n'-Km),$$

$$R_{23}(5) = U(-2l+10n-3l'-2n'+3Km)+13V(-l-m+2n-2n'),$$

$$R_{23}(6) = U(-l-4n-l'-4n')+13V(-m-2n-n'),$$

$$R_{23}(7) = U(-2l-2n+l'-m'+5n'+m'/K)+13V(-n+m'+2n'),$$

$$R_{23}(8) = U(-l+m-n-2l'-4m')+13V(-n-l'+n'),$$

$$R_{23}(9) = U(-m-9n+4l'-m'+n')+13V(l-m-2n+l'+m'+n'-l'/K),$$

$$R_{23}(10) = U(7l-5m+3n-m'+5Kn)+13V(2l-2m-l'+m'+2Kn),$$

$$R_{23}(11) = U(-m+3n-6l'-3m'-5n'-m'/K+5n'/K)+13V(-l+n-l'-m'-n'+m'/K+2n'/K),$$

$$R_{23}(12) = U(l+m-n-4l'-3m'-n'+l'/K-2n'/K)+13V(-m-n-l'-m'-n'+l'/K);$$

$$R_{34}(0) = U(-3l-6n+l')+13V(-l-n+l'+n'-Kn),$$

$$R_{34}(1) = U(6l+m+6n-n'-Km+5Kn)+13V(2l-l'-n'+2Kn),$$

$$R_{34}(2) = U(8m+3l'+m'-2n'+n'/K)+13V(l+2m-n+n'/K),$$

$$R_{34}(3) = U(-l-7m-n+3n'-Kl)+13V(-2m+m'+n'),$$

$$R_{34}(4) = U(-5l+3m+5n-l'-m'+5Km)+13V(-2l+n+2Km),$$

$$R_{34}(5) = U(5l+5m-11n-3l'+n'-5Km)+13V(2l+2m-3n-2l'+n'-Km),$$

$$R_{34}(6) = U(3m+4n-2l'+m'+2n')+13V(2m+n-l'),$$

$$R_{34}(7) = U(l+n+l'+3m'-3n')+13V(n+l'-n'),$$

$$R_{34}(8) = U(l-3m+n+6l'+m'+n'-m'/K)+13V(-m+n+2l'-n'-l'/K),$$

$$R_{34}(9) = U(2m+8n+6l'+4m'-4n'-4l'/K)+13V(2m+n-2n'-l'/K),$$

$$R_{34}(10) = U(-4l+2m-4n-2l'+m'-3Kn)+13V(-2l+m-n-2Kn),$$

$$R_{34}(11) = U(-l+m-3n+4l'+5m'+n'+2m'/K-2n'/K)+13V(l+m-n+l'+2m'+n'-m'/K-n'/K),$$

$$R_{34}(12) = U(-l+n-2m'+4n'+3l'/K+n'/K)+13V(m+n-m'+2n'+l'/K);$$

$$
\begin{aligned}
R_{45}(0) &= U(-5l+2m+6n+n'+2Kl)+13V(-l+m+n+Kn),\\
R_{45}(1) &= U(-5l-m+4n-l'-2n'+2Km-3Kn)+13V(-2l-m+n+l'-n'-Kn),\\
R_{45}(2) &= U(l-10m-5l'+3m'+3n'/K)+13V(-l-2m+n-l'+2m'+n'/K),\\
R_{45}(3) &= U(-6l+6m+2n+5n'-Kl)+13V(-l+2m+n+n'-Kl),\\
R_{45}(4) &= U(-3l-2m-7n+l'+n'-7Km)+13V(m-n+n'-Kl-2Km),\\
R_{45}(5) &= U(-6l+10n+n'+5Km)+13V(-3l+3n+l'+2Km),\\
R_{45}(6) &= U(2l-5m-3n-m'+3n')+13V(-2m+m'+n'),\\
R_{45}(7) &= U(l-n+l'-m'-5n'+m'/K)+13V(l-n-n'),\\
R_{45}(8) &= U(4m+5l'-2m'-n'-l'/K)+13V(l+m-n-m'),\\
R_{45}(9) &= U(-2m-5n-2l'-2m'+2l'/K)+13V(-2m-n+l'),\\
R_{45}(10) &= U(-2n+l'+m'+2Kn)+13V(l-n+Kn),\\
R_{45}(11) &= U(l-m+2n+3l'+2n'+m'/k-4n'/k)+13V(-m+n+l'-m'-n'-n'/K),\\
R_{45}(12) &= U(-m-n+6l'+5n')+13V(-m+2l'+m'+n'-l'/K);
\end{aligned}
$$

$$
\begin{aligned}
R_{56}(0) &= U(-6l+m+2n-2l'-n'+Kl)+13V(-2l-n'),\\
R_{56}(1) &= U(2l+2m+2n+l'+2n'-Km+Kn)+13V(l+m+n+n'+Kn),\\
R_{56}(2) &= U(7m+l'-4m'-5n'-4n'/k)+13V(l+m-n-2m'-2n'-n'/K),\\
R_{56}(3) &= U(5l-6m+n-6n'-Kl)+13V(l-2m-2n'),\\
R_{56}(4) &= U(-3l-2m+5n-n'+5Km)+13V(-l-2m+n+l'-n'+Km),\\
R_{56}(5) &= U(3l+6m-6n+5l'-2n'-3Km)+13V(2l+m-2n+l'-m'-Km),\\
R_{56}(6) &= U(4m+n+3l'+m'-4n')+13V(l+m+l'-m'-n'),\\
R_{56}(7) &= U(3l+n-2l'+4m'+7n'-m'/K)+13V(n-l'+m'+n'),\\
R_{56}(8) &= U(-3m-n-l'-3m'+2n'+2l'/K)+13V(-m+n+l'+n'),
\end{aligned}
$$

$R_{56}(9) = U(m+2n+2l'-3m'+4n'+2l'/K)+13V(m+n-m'+2n'+l'/K)$,

$R_{56}(10) = U(-3l+4n+l'-2Kn)+13V(-l+2n+l')$,

$R_{56}(11) = U(m-n-5l'+5m'+6n'+5n'/K)+13V(m-2l'+2m'+2n'+n'/K)$,

$R_{56}(12) = U(-l+m+n+3l'+4m'+n'-4l'/K)+13V(m-l'/K-n'/K)$.

The following relations will be required in the proof of this theorem for systematic simplification of expressions involving l, m', n, l', m, and n'.

(4.8) to (4.10) $lm/n = -l-m$, $mn/l = -m-n$, $nl/m = -n-l$;

(4.11) to (4.13) $l^2/m = -Fl-3l+m-n$, $m^2/n=-Fm-3m+n-l$, $n^2/l = -Fn-3n+l-m$;

(4.14) to (4.16) $l^2/n = Fl+2l-m+n$, $m^2/l = Fm+2m-n+l$, $n^2/m = Fn+2n-l+m$;

(4.17) to (4.19) $Kl = -Fl-3l+l'$, $Km = -Fm-3m+m'$, $Kn = -Fn-3n+n'$,

(4.20) to (4.22) $l'/K = Fl'+3l'+l$, $m'/K = Fm'+3m'+m$, $n'/K = Fn'+3n'+n$;

(4.23) to (4.25) $K^2l = F(3l-Kl)+10l-3l'$, $K^2m=F(3m-Km)+10m-3m'$, $K^2n = F(3n-Kn)+10n-3n'$,

(4.26) to (4.28) $l'/K^2=F(3l'+l'/K)+10l'+3l$,

$m'/K^2=F(3m'+m'/K)+10m'+3m$,

$n'/K^2=F(3n'+n'/K)+10n'+3n$.

(4.8) to (4.16) follow from (2.16) and (2.17), (4.17) to (4.22) from (1.17), and (4.23) to (4.28) from (4.17) to (4.22) respectively; (4.8) and (4.11) have already been given as (2.18) and (2.19) respectively. We shall also need the relations

(4.29) to (4.31) $a = l/m - K$, $b = m/n - K$, $c = n/l - K$,

(4.32) to (4.34) $a' = l'/m' + 1/K$, $b' = m'/n' + 1/K$,

$c' = n'/l' + 1/K$,

arising from (1.6) to (1.11) and (2.11) to (2.13). Of course all of the equations (4.8) to (4.34) remain valid when l, m', n, l', m, and n', are interchanged according to (2.14) or (2.15) and a, b', c, a', b, and c', are interchanged cyclically. Finally, the following will be required

$$
\begin{aligned}
2g(1)-g(2)+1 &= -P^2(0)l'b = P^2(0)(l+l'+m'),\\
2g(2)-g(4)+1 &= P^2(0)mc' = P^2(0)(-m-n+m'),\\
2g(3)-g(6)+1 &= -P^2(0)m'c = P^2(0)(m+m'+n'),\\
2g(4)+g(5) &= P^2(0)n'a = P^2(0)(-n-l'-n'),\\
2g(5)+g(3) &= P^2(0)lb' = P^2(0)(-l-m+l'),\\
2g(6)+g(1) &= P^2(0)na' = P^2(0)(-l-n+n');
\end{aligned}
\tag{4.35}
$$

these relations arise from (ASD), Lemma 8 (with q = 13), and (4.29) to (4.34) above, using (4.8) to (4.10) (divided through by K if necessary).

The proof of Theorem 4.1 is similar to those of (ASD), Theorems 4 and 5, and (AH), Theorem 6. If we write

$$N'_{01} = N_{01}+\{-3y^2\Sigma(2,0)/P(0)-2\}+x^4\{y^4\Sigma(6,0)/P(0)\},$$

$$N'_{12} = N_{12}+\{y^2\Sigma(2,0)/P(0)+1\}+x^4\{-2y^4\Sigma(6,0)/P(0)\}+x^{12}\{y^2\Sigma(3,0)/P(0)\},$$

$$N'_{23} = N_{23}+x^4\{y^4\Sigma(6,0)/P(0)\}+x^{11}\{-\Sigma(1,0)/P(0)-y^{-1}\}+ +x^{12}\{-2y^2\Sigma(3,0)/P(0)\},$$

$$N'_{34} = N_{34}+x\{-y^4\Sigma(5,0)/P(0)\}+x^{11}\{2\Sigma(1,0)/P(0)+2y^{-1}\} + +x^{12}\{y^2\Sigma(3,0)/P(0)\},$$

$$N'_{45} = N_{45}+x\{2y^4\Sigma(5,0)/P(0)\}+x^8\{-y^3\Sigma(4,0)/P(0)\}+ +x^{11}\{-\Sigma(1,0)/P(0)-y^{-1}\},$$

$$N'_{56} = N_{56}+x\{-y^4\Sigma(5,0)/P(0)\}+x^8\{2y^3\Sigma(4,0)/P(0)\},$$

then in view of (4.2) and the definitions of the $R_{bc}(d)$ we have for any fixed values of b and c with $c = b + 1$

(4.36) $$N'=P(6)R_0/P(3)+xP(2)R_1/P(1)-x^2P(3)R_2/P(2)+x^3P(6)R_3/P(4)- -x^4P(5)R_4/P(4)-x^5y^{-1}P(5)R_5/P(1)+x^6R_6+x^7yP(1)R_7/P(5)- -x^8yP(1)R_8/P(6)-x^9P(4)R_9/P(6)+x^{10}P(2)R_{10}/P(3)+ +x^{11}P(3)R_{11}/P(5)-x^{12}y^{-1}P(4)R_{12}/P(2)$$

where for convenience the suffix bc is dropped, and R(d) is written as R_d. Thus writing

$$f(x)N'/P(0) = \sum_{d=0}^{12} t_d x^d$$

we can use (4.36) and the expression for $f(x)/P(0)$ given by (1.1) to find each t_d as a linear combination of R_d in which each R_d occurring is multiplied by some multiplicative combination of the P(a); for example we find that

$$t_2 = -P(2)P(6)(R_o+R_1)/P(1)P(3)-P(3)P(4)R_2/P^2(2)-$$
$$-yP(3)P(6)R_3/P(4)P(5)+y^2P(1)(R_6-R_8)/P(6)+$$
$$+yP(2)P(5)R_{10}/P(3)P(4).$$

If in this example we define T_2, the "normalised" form of t_2, by

$$T_2 = -y^{-2}P(6)t_2/P(1)$$

then we find that

$$T_2=-B(R_o+R_1)/K-BCbR_2-ABc'R_3-R_6+R_8-R_{10}/K,$$

and the coefficient of each R_d in this equation is equal to a simple expression in l, m', n, l', m, and n', as follows:

$-B/K = -m'/n$ by (2.12);

$-BCb = -m(m/n-K)/l$ by (2.12),(2.13), and (4.30),

$= -m(-l/m-n/l-1/k-1)/l$ by (1.17) and (2.17),

$= m'/l-n/l+1$ by (2.16);

$-ABc'= -l'/n-1$ by (2.11),(2.12), and (4.34).

By proceeding in the above manner for all the t_d, suitably normalising the t_d in each case, we arrive at the following:

$$T_0 = y^{-1}t_0 =m(R_0+R_{12})/l+l(R_1+R_{11})/n+n(R_4+R_8)/m+R_6,$$
$$T_1 = y^{-2}P(5)t_1/P(1)=-m'R_0/l +(-l/n-m'/n)(R_1+R_{12})+$$
$$+(-m/n+K)R_2+(-l/m-l/K)(R_5+R_9)+R_7,$$
$$T_2 = -y^{-2}P(6)t_2/P(1)=-m'(R_0+R_1)/n+(m'/l-n/l+1)R_2+$$
$$+(-l'/n-1)R_3-R_6+R_8-R_{10}/K,$$
$$T_3 = -y^{-1}P(6)t_3/P(4)=-l'R_1/n+(-n/m-l'/m)(R_4+R_{11})+$$
$$+(-l/m+K)R_7+(-n/l-1/K)(R_3+R_2)+R_9,$$

$$T_4 = y^{-1}P(3)t_4/P(2) = nR_8/m' + (-m'/l' + n/l')(R_{12}+R_4) + (-n'/l' - 1/K)R_3 + (-m'/n' + K)(R_2+R_5) + R_{10},$$

$$T_5 = y^{-1}P(5)t_5/P(3) = -n'(R_4+R_0)/l + (n'/m - l/m + 1)R_9 + (-m'/l - 1)R_5 - R_6 + R_{11} - R_3/K,$$

(4.37) $$T_6 = -P(2)t_6/P(4) = -l'(R_1+R_4)/m + (l'/n - m/n + 1)R_7 + (-n'/m - 1)R_{10} - R_6 + R_{12} - R_5/K,$$

$$T_7 = P(3)t_7/P(6) = l(R_{11}+R_8)/m' + (-l/n' - m'/n' + 1)R_5 + (n/m' - 1)R_2 - R_6 + R_0 + KR_7,$$

$$T_8 = P(1)t_8/P(2) = n(R_8+R_{12})/l' + (-n/m' - l'/m' + 1)R_3 + (m/l' - 1)R_7 - R_6 + R_1 + KR_9,$$

$$T_9 = -P(2)t_9/P(3) = -n'R_4/m + (-m/l - n'/l)(R_0+R_8) + (-n/l + K)R_9 + (-m/n - 1/K)(R_{10}+R_7) + R_2,$$

$$T_{10} = P(4)t_{10}/P(6) = lR_{11}/n' + (-n'/m' + l/m')(R_8+R_1) + (-l'/m' - 1/K)R_5 + (-n'/l' + K)(R_9+R_{10}) + R_3,$$

$$T_{11} = -P(4)t_{11}/P(5) = m(R_{12}+R_{11})/n' + (-m/l' - n'/l' + 1)R_{10} + (l/n' - 1)R_9 - R_6 + R_4 + KR_2,$$

$$T_{12} = -yP(1)t_{12}/P(5) = mR_{12}/l' + (-l'/n' + m/n')(R_{11}+R_0) + (-m'/n' - 1/K)R_{10} + (-l'/m' + K)(R_7+R_3) + R_5.$$

We observe that, apart from T_0, the T_d fall naturally into two groups of six given by d = 1, 3, 4, 9, 10, 12, and d = 2, 5, 6, 7, 8, 11, respectively, and that with the normalising factors as chosen, interchanging either T_1, T_4, T_3, T_{12}, T_9, and T_{10}, or T_2, T_8, T_6, T_{11}, T_5, and T_7, cyclically corresponds to interchanging R_0, R_8, R_1, R_{12}, R_4, R_{11}, and R_2, R_3, R_7, R_{10}, R_9, R_5, cyclically (leaving R_6 unchanged) if we interchange l, m', n, l', m, and n', according to (2.14) or (2.15); the two groups of six R_d occur naturally in Table 4.1. T_o is invariant under these interchanges. We might have anticipated such a situation as an aid in finding the identities of (4.37) (c f. the proofs of Theorems 2.1 and 2.2).

We now find alternative expressions for the T_d. This time each pair of values of b and c (with c = b + 1) is considered separately, so that we have 78 $T_{bc}(d)$ (in the obvious notation) to determine, viz. $T_{01}(d)$ to $T_{56}(d)$ for d = 0 to d = 12. These expressions are found as in the following examples.

$t_{01}(9)$ (again in the obvious notation) is by definition the coefficient of x^9 in $f(x)N'_{01}/P(0)$, thus we have

(4.38) $t_{01}(9)=P(0)\{-3yP(1)P(6)/P(2)P(4)P(5)-P(4)P(5)/P(2)P(3)P(6)\}$

from the definition of N'_{01}, the expression for $f(x)N_{01}$ given in (4.3), and the values of S(0) and S(1) given in (4.4); of course the terms involving $\Sigma(m,0)$ all disappear. Multiplying (4.38) by $-yP(2)/P(3)$ we obtain

$$\begin{aligned} yT_{01}(9) &= P(0)(3m'a+n'c), \\ &= P(0)(3l'-3m+nn'/l-n) \quad \text{by (4.29) and (4.31)}, \\ &= P(0)(-3m+4l'-m'-n'/K) \end{aligned}$$

by (4.13) (divided through by K) and (1.17). The method of this example applies when $d \neq 0$. When $d = 0$ the procedure is slightly different.

$t_{01}(0)$ is the coefficient of x^0 in $f(x)N'_{01}/P(0)$, and proceeding as in the previous example we obtain

$$t_{01}(0) = \{-3g(2)+g(6)-2\}/P(0).$$

Since $T_{01}(0) = y^{-1}t_{01}(0)$ this equation becomes

$$yT_{01}(0) = P(0)(-1+m+2n+l'-2m'+n')$$

by means of relations (4.35).

A complete set of alternative values of $yT_{bc}(d)/P(0)$ is given in Table 4.2 at the end of this Part (page 46).

By equating our two expressions for each $T_{bc}(d)$ we now have, for any fixed values of b and c, a set of 13 simultaneous linear equations for $R_{bc}(d)(d=0 \text{ to } 12)$. Moreover these equations have a unique solution; this may be seen by proving that a determinant is non-zero, but it is easier to observe that the equations are in fact the necessary and sufficient conditions that $\sum_{d=0}^{12} R_{bc}(d)x^d$ be the quotient of two given power series. Accordingly to prove Theorem 4.1 all that remains is to show that for $(b, c) = (0, 1)$ to $(5, 6)$ respectively the values of the $R_{bc}(d)$ given in the theorem satisfy these equations. In other words we need to show that for each of the 78 $T_{bc}(d)$ the value found by substituting for the $R_{bc}(d)$ from the theorem in the appropriate equation of (4.37) agrees with the value given by Table 4.2. This is tedious but straightforward; we proceed as in the following example.

Consider $T_{01}(1)$ as given by substituting for the $R_{01}(d)$ from the theorem in the second equation of (4.37). Each $R_{01}(d)$ is expressed in the theorem as the sum of two brackets, one multiplied by U and the other by 13V. We write down and simplify {by means of (4.8) to (4.28)} the total contribution of the U-brackets and the total contribution of the V-brackets separately, and combine the resulting two expressions. The contribution of the V-brackets is

$$-m'(-2l-2m-2n+m'+n'-Kl)/l+(-l/n-m'/n)(-l+3m-2l'-Km-2Kn)+$$
$$+(-m/n+K)(3m-2l'+m'+n'+n'/K)+(-l/m-1/K)(3l+m-2n-l'-m'+2n'-2Km+$$
$$+l'/K)+(-2n+3m'-n'-m'/K)$$
$$=(-3l+6m-n-2l'+3m'+2n')+(4Kl+3Km+l'/K-m'/K-2n'/K)+(-l'/K^2)+$$
$$+(2/K^2+3/K-3+K)lm/n+(-1/K^2+2/K)mn/l+(-2/K+2)nl/m+$$
$$+(-1/K^2+1/K-3)l^2/m+(-4/K-2)m^2/n+(2/K+1)l^2/n+(-1/K^2+2/K)m^2/l$$

and this expression, on substituting for l'/K^2, lm/n, mn/l, nl/m, l^2/m, m^2/n, l^2/n, and m^2/l, from (4.8) to (4.16) and (4.23) to (4.28), reduces to

$$F(4l+2m-2l'+6m'-m'/K)+(8l+11m-n-6l'+13m'-3n')+$$
$$+(3Kl+2Km+l'/K-5m'/K+n'/K)$$

which expression, on substituting for each term in the third bracket from (4.17) to (4.22), reduces to

(4.39) $F(l-l'+m'+n'-m'/K)$,

only terms containing a factor F remain. The contribution of the U-brackets is

$$-m'(-5l-3m-3n-2l'-2m'+3Kn)/l+(-l/n-m'/n)(-8l+7m-5l'+4m'+2n'-8Kn-3l'/K)+(-m/n+K)(7m-6l'+4m'+4n'+3n'/K)+(-l/m-1/K)(5l-7n+3l'-2m'+n'-5Km+l'/K)+(-l-3n+6m'-6n'+2m'/K)$$

$$=(-7l+17m+n-5l'+7m'+4n')+(13Kl+7Km-3l'/K+m'/K-n'/K)+(-l'/K^2)+$$
$$+(3/K^3+5/K^2+10/K-7)lm/n+(3/K-3)mn/l+(-1/K+7)nl/m+$$
$$+(-1/K^2-3/K-5)l^2/m+(-4/K^2-11/K-7)m^2/n+(3/K^2+5/K+8)l^2/n+$$
$$+(2/K^2+3/K)m^2/l$$

and this expression, on substituting for l'/K^2, lm/n, mn/l, nl/m, l^2/m, m^2/n, l^2/n, and m^2/l, reduces to

$$F(13l+7m+5l'+14m'+3l'/K+6m'/K)+(28l+35m+3n+9l'+25m'-4n')+$$
$$+(13Kl+7Km+7l'/K+8m'/K-3n'/K)+(-3l'/K^2-3m'/K^2)$$

which expression, on substituting for each term in the third and fourth brackets from (4.17) to (4.22) and (4.23) to (4.28) respectively, reduces to

(4.40) $F(3l'+13m'-3n'+3m'/K)+13(-l+m+l'+2m'-n')$,

only terms containing either a factor F or a factor 13 remain. Multiplying expressions (4.39) and (4.40) by 13V and U respectively, and adding, remembering that FV = U, we obtain the following expression for $T_{01}(1)$

$$FU(3l'+13m'-3n'+3m'/K)+13U(m+3m'-m'/K),$$

and this expression, on substituting for m'/K in the second bracket from (4.21), reduces to

$$FU(3l'-3n'+3m'/K).$$

Since $FU=y^{-1}P(0)$, this is the same as the value of $T_{\emptyset 1}(1)$ given by Table 4.2.

We perform the above verification for each of the 78 $T_{bc}(d)$; the working is always essentially the same as the above, and is therefore omitted. This completes the proof of Theorem 4.1.

As in the case of q = 11, there are certain linear congruence relations (but no identities) between the $r_{bc}(d)$ for a given value of d when q = 13; if we write

$$s_1(d) = r_{01}(d)-6r_{56}(d),$$
$$s_2(d) = r_{12}(d)-5r_{56}(d),$$
$$s_3(d) = r_{23}(d)-4r_{56}(d),$$
$$s_4(d) = r_{34}(d)-3r_{56}(d),$$
$$s_5(d) = r_{45}(d)-2r_{56}(d),$$

we have, modulo 13,

$$s_3(0)-6s_4(0)+5s_5(0) \equiv 0,$$
$$s_2(1)+3s_3(1)-5s_4(1)-5s_5(1) \equiv 0,$$
$$s_4(2) \equiv 0,$$
$$s_1(2)+s_2(2)-5s_3(2)+s_5(2) \equiv 0,$$
$$s_1(3)-s_3(3) \equiv 0,$$
$$s_2(3)+s_3(3)-3s_4(3)-6s_5(3) \equiv 0,$$
$$s_1(4)-4s_2(4)+4s_3(4)-5s_4(4)-6s_5(4) \equiv 0,$$
$$s_1(5) \equiv 0,$$
$$s_2(5)-2s_3(5)-4s_4(5)-2s_5(5) \equiv 0,$$
$$s_1(6)+2s_2(6)-5s_5(6) \equiv 0,$$
$$s_2(6)+5s_3(6)+3s_4(6)+3s_5(6) \equiv 0,$$
$$s_1(7)-3s_2(7)+6s_3(7) \equiv 0,$$
$$s_2(7)-s_3(7)-3s_4(7)-s_5(7) \equiv 0,$$
$$s_1(8)+6s_2(8)-5s_3(8)-5s_4(8)-3s_5(8) \equiv 0,$$
$$s_2(9)-6s_4(9) \equiv 0,$$
$$s_1(9)-4s_3(9)+2s_4(9)-6s_5(9) \equiv 0,$$
$$s_1(10)+3s_2(10)-5s_5(10) \equiv 0,$$
$$s_2(10)+6s_3(10)+5s_4(10)-s_5(10) \equiv 0,$$
$$s_1(11)+5s_2(11)-3s_3(11)-3s_4(11)-3s_5(11) \equiv 0,$$
$$s_1(12)+2s_2(12)+5s_3(12)-5s_4(12)+3s_5(12) \equiv 0.$$

The above congruences with each $r_{bc}(d)$ replaced by the corresponding $R_{bc}(d)$ follow immediately from Theorem 4.1, and for each value of d we simply divide through by the normalising factor contained in the $R_{bc}(d)$ (the coefficients of the $r_{bc}(d)$ in the congruences are such that the terms involving $\Sigma(m, 0)$ disappear).

We may note that since

$$\Phi(d) = \sum_{n=0}^{\infty} p(13n + d) = \sum_{n=0}^{\infty} \sum_{b=0}^{12} N(b, 13, 13n + d)y^n$$

$$= \sum_{n=0}^{\infty} N(0, 13, 13n+d)y^n + 2\sum_{b=1}^{6} \sum_{n=0}^{\infty} N(b,13,13n+d)y^n$$

{using the relation $N(m, q, n) = N(q - m, q, n)$ given in (ASD)}

$$= r_0(d) + 2\sum_{b=1}^{6} r_b(d)$$

$$\equiv r_{01}(d)+3r_{12}(d)+5r_{23}(d)+7r_{34}(d)+9r_{45}(d)+11r_{56}(d) \pmod{13}$$

{using (6.8) and (6.9) of (ASD)}, Theorem 4.1 may be used in an alternative proof of Theorem 3.1.

Table 4.2

$yT_{bc}(d)/P(0)$

b,c / d	0,1	1,2	2,3	3,4	4,5	5,6
0	-l+m+2n+ +l'-2m'+n'	2l-m+m'- -2n'	-l+m-n+ +m'+2n'	-l-m+n- -2m'-n'	l+m+m'+n'	-n-l'-n'
1	3l'-3n'+ +3m'/K	-l'-m'/K	2n'	l-2n'	-2l+2n'	l-n'
2	3m'	-m'	0	-m+l'+m'	2m-2l'-2m'	-m+l'+m'
3	n-m'	-2n+2m'	n	-m'-n'-l'/K	-m'+2n'+ +2l'/K	m'-n'-l'/K
4	-l	3l+m'	-3l-2m'	l+m'	l-m+Kn	-2l+2m- -2Kn
5	3n-3m'-4n'	-n+m'+3n'	-n'	0	0	0
6	-l+l'+n'	2l-2l'-2n'	-l+l'+n'	-l'	2l'	-l'
7	0	0	-l	2l	n+l'	-2l-2n-2l'
8	0	-m-n-n'	2m+2n+2n'	-m-n-n'	-n	2n
9	-3m+4l'- -m'-n'/K	m-3l'+2m'+ +2n'/K	l'-m'-n'/K	0	l'	-2l'
10	0	0	m-n+Kl	-3m+2n- -2Kl	2m-n-n'+ +Kl	m+2n'
11	0	m	l-m+m'	-2l-m-2m'	l+m+m'	0
12	3n	l-2n-Km	-2l+n-l'+ +2Km	l+n+2l'- -Km	-n-l'	0

PART 2

q = 17 throughout this Part

5. We write

$\alpha_1 = -x^{-7}P(2)/P(1)$, $\alpha_2 = -x^{-12}P(6)/P(3)$, $\alpha_3 = x^{28}P(1)/P(8)$, $\alpha_4 = -x^{14}P(3)/P(7)$, $\alpha_5 = x^{-10}P(8)/P(4)$, $\alpha_6 = -x^{-5}P(7)/P(5)$, $\alpha_7 = x^{-11}P(4)/P(2)$, $\alpha_8 = x^{3}P(5)/P(6)$;

then by (ASD), Lemma 6 (with q = 17) we have

$$(5.1)\quad -x^{-12}f(x)/f(y^{17}) = \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+1.$$

In (5.1) we replace x by $w_r x$ where w_r (r = 1 to 17) are the seventeenth roots of unity, and multiply together the seventeen resulting equations, obtaining

$$(5.2)\quad -y^{-12}f^{18}(y)/f^{18}(y^{17}) = \prod_{r=1}^{17} (\alpha_1 w_r^{-7}+\alpha_2 w_r^{-12}+\alpha_3 w_r^{28}+\alpha_4 w_r^{14}+\alpha_5 w_r^{-10}+ +\alpha_6 w_r^{-5}+\alpha_7 w_r^{-11}+\alpha_8 w_r^{3} +1).$$

Now as w_r runs through the seventeenth roots of unity so does w_r^2, so that the product on the right-hand side of (5.2) is equal to

$$\prod_{r=1}^{17} (\alpha_1 w_r^{3}+\alpha_2 w_r^{-7}+\alpha_3 w_r^{-12}+\alpha_4 w_r^{28}+\alpha_5 w_r^{14}+\alpha_6 w_r^{-10}+\alpha_7 w_r^{-5}+\alpha_8 w_r^{-11}+1),$$

and is thus unchanged if α_1, α_2, α_3, α_4, α_5, α_6, α_7, and α_8, are interchanged cyclically. The product is thus a linear combination of terms $[\alpha_1^{i_1}\ \alpha_2^{i_2}\ \alpha_3^{i_3}\ \alpha_4^{i_4}\ \alpha_5^{i_5}\ \alpha_6^{i_6}\ \alpha_7^{i_7}\ \alpha_8^{i_8}]$ where i_1 to i_8 are non-negative integers, and considering the left-hand side of (5.2) such terms as occur can only involve x in terms of $y = x^{17}$. Thus if $\alpha_1^{i_1} \alpha_2^{i_2} \alpha_3^{i_3} \alpha_4^{i_4} \alpha_5^{i_5} \alpha_6^{i_6} \alpha_7^{i_7} \alpha_8^{i_8}$ (or any other term of $[\alpha_1^{i_1} \alpha_2^{i_2} \alpha_3^{i_3} \alpha_4^{i_4} \alpha_5^{i_5} \alpha_6^{i_6} \alpha_7^{i_7} \alpha_8^{i_8}]$) occurs we must have

$$(5.3)\quad -7i_1 - 12i_2 + 28i_3 + 14i_4 - 10i_5 - 5i_6 - 11i_7 + 3i_8 \equiv 0 \pmod{17}$$

(interchanging i_1, i_2, i_3, i_4, i_5, i_6, i_7, and i_8, cyclically gives the same congruence).

Now, writing

$$a_1 = P(1)P(6)/P(2)P(4), \quad a_2 = -y^2P(3)P(1)/P(6)P(5),$$
$$a_3 = y^{-2}P(8)P(3)/P(1)P(2), \quad a_4 = -y^{-1}P(7)P(8)/P(3)P(6),$$
$$a_5 = y^{-1}P(4)P(7)/P(8)P(1), \quad a_6 = P(5)P(4)/P(7)P(3),$$
$$a_7 = -yP(2)P(5)/P(4)P(8), \quad a_8 = yP(6)P(2)/P(5)P(7),$$

it is easily verified that

$$(5.4)\quad \begin{aligned}
\alpha_1^{17} &= a_2^4 a_3^{12} a_4^{11} a_5^9 a_6^5 a_7^{14} a_8^{15}, & \alpha_5^{17} &= a_6^4 a_7^{12} a_8^{11} a_1^9 a_2^5 a_3^{14} a_4^{15},\\
\alpha_2^{17} &= a_3^4 a_4^{12} a_5^{11} a_6^9 a_7^5 a_8^{14} a_1^{15}, & \alpha_6^{17} &= a_7^4 a_8^{12} a_1^{11} a_2^9 a_3^5 a_4^{14} a_5^{15},\\
\alpha_3^{17} &= a_4^4 a_5^{12} a_6^{11} a_7^9 a_8^5 a_1^{14} a_2^{15}, & \alpha_7^{17} &= a_8^4 a_1^{12} a_2^{11} a_3^9 a_4^5 a_5^{14} a_6^{15},\\
\alpha_4^{17} &= a_5^4 a_6^{12} a_7^{11} a_8^9 a_1^5 a_2^{14} a_3^{15}, & \alpha_8^{17} &= a_1^4 a_2^{12} a_3^{11} a_4^9 a_5^5 a_6^{14} a_7^{15}.
\end{aligned}$$

It will be noticed that all of the equations (5.4) may be obtained from any one of them by interchanging a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, and α_1, α_2, α_3, α_4, α_5, α_6, α_7, α_8, cyclically. By (5.4), since $a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 = -1$,

$(\alpha_1^{i_1}\alpha_2^{i_2}\alpha_3^{i_3}\alpha_4^{i_4}\alpha_5^{i_5}\alpha_6^{i_6}\alpha_7^{i_7}\alpha_8^{i_8})^{17} =$

$= (a_1a_2a_3a_4a_5a_6a_7a_8)^{\sigma}\, a_1^{\sigma_1}a_2^{\sigma_2}a_3^{\sigma_3}a_4^{\sigma_4}a_5^{\sigma_5}a_6^{\sigma_6}a_7^{\sigma_7}a_8^{\sigma_8}$

where $\sigma = 10i_1 + 24i_2 + 14i_3 + 26i_4 + 32i_5 + 18i_6 + 28i_7 + 16i_8$, an even integer, and

$$\sigma_1 = 15i_2 + 14i_3 + 5i_4 + 9i_5 + 11i_6 + 12i_7 + 4i_8,$$
$$\sigma_2 = 15i_3 + 14i_4 + 5i_5 + 9i_6 + 11i_7 + 12i_8 + 4i_1,$$
$$\sigma_3 = 15i_4 + 14i_5 + 5i_6 + 9i_7 + 11i_8 + 12i_1 + 4i_2,$$
$$\sigma_4 = 15i_5 + 14i_6 + 5i_7 + 9i_8 + 11i_1 + 12i_2 + 4i_3,$$
$$\sigma_5 = 15i_6 + 14i_7 + 5i_8 + 9i_1 + 11i_2 + 12i_3 + 4i_4,$$
$$\sigma_6 = 15i_7 + 14i_8 + 5i_1 + 9i_2 + 11i_3 + 12i_4 + 4i_5,$$
$$\sigma_7 = 15i_8 + 14i_1 + 5i_2 + 9i_3 + 11i_4 + 12i_5 + 4i_6,$$
$$\sigma_8 = 15i_1 + 14i_2 + 5i_3 + 9i_4 + 11i_5 + 12i_6 + 4i_7;$$

moreover $\sigma + \sigma_1$ to $\sigma + \sigma_8$ are multiples of 17 by (5.3), hence any expression of the form $\alpha_1^{i_1}\alpha_2^{i_2}\alpha_3^{i_3}\alpha_4^{i_4}\alpha_5^{i_5}\alpha_6^{i_6}\alpha_7^{i_7}\alpha_8^{i_8}$ for which (5.3) holds is of the form

$a_1^{j_1}a_2^{j_2}a_3^{j_3}a_4^{j_4}a_5^{j_5}a_6^{j_6}a_7^{j_7}a_8^{j_8}$ where j_1 to j_8 are non-negative integers. Thus every term occurring in the right-hand side of (5.2) is of the form $a_1^{j_1}a_2^{j_2}a_3^{j_3}a_4^{j_4}a_5^{j_5}a_6^{j_6}a_7^{j_7}a_8^{j_8}$, and such such terms occur in cyclically symmetrical sets of eight terms each.

Further, $\Phi(5)$ is the coefficient of x^5 in $1/f(x)$ regarded as a polynomial of degree 16 in x with coefficients involving x in terms of $y = x^{17}$, so that $y^{-11}f^{18}(y)\Phi(5)/f^{17}(y^{17})$

is the coefficient of x^0 in

$y^{-12}f^{18}(y)/\{f^{18}(y^{17})(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+1)\}$. This is a cyclically symmetric polynomial of degree 16 in α_1, a_2, α_3, α_4, α_5, α_6, α_7, and α_8; and the terms which give the coefficient of x^0 occur only in symmetrical sets of eight expressible as $[a_1^{j_1}\ a_2^{j_2}\ a_3^{j_3}\ a_4^{j_4}\ a_5^{j_5}\ a_6^{j_6}\ a_7^{j_7}\ a_8^{j_8}]$, as before. (This is not true for the coefficient of any power of x other than 0; the eight terms of $[a_1]$, for example, do not appertain to the same power of x.)

Thus writing

$$F = y^{-2}f^3(y)/f^3(y^{17})$$

we have the following:

LEMMA 5.1 F^6 and $yf(y^{17})F^6\,\Phi(5)$ are each equal to a linear combination of terms $[a_1^{j_1}\ a_2^{j_2}\ a_3^{j_3}\ a_4^{j_4}\ a_5^{j_5}\ a_6^{j_6}\ a_7^{j_7}\ a_8^{j_8}]$.

We now write

(5.5) to (5.8) $b_1 = a_1a_5$, $b_2 = a_2a_6$, $b_3 = a_3a_7$, $b_4 = a_4a_8$,

so that

(5.9) $$b_1b_2b_3b_4 + 1 = 0.$$

<7, 6, 5, 3> and <8, 4, 2, 1> give, respectively,

(5.10) $$b_1 + b_3 + 1 = 0,$$

(5.11) $$b_2 + b_4 + 1 = 0,$$

while <8, 5, 4, 3>, <8, 7, 5, 2>, <7, 6, 4, 2>, <6, 5, 4, 1>; <5, 3, 2, 1>, <8, 6, 3, 2>, <8, 7, 6, 1>, and <7, 4, 3, 1>, give, respectively, (5.12) to (5.15) $a_1 = b_1 a_2 + 1, \quad a_2 = b_2 a_3 + 1,$

$a_3 = b_3 a_4 + 1, \quad a_4 = b_4 a_5 + 1,$

(5.16) to (5.19) $a_5 = b_1 a_6 + 1, \quad a_6 = b_2 a_7 + 1,$

$a_7 = b_3 a_8 + 1, \quad a_8 = b_4 a_1 + 1.$

It will be observed that each of the equations (5.5) to (5.19) remains valid when b_1, b_2, b_3, b_4, and a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, are interchanged cyclically. We are now in a position to prove

LEMMA 5.2 Any expression of the form $[a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8}]$ is equal to a linear combination of terms $[b_1^{k_1} b_2^{k_2} b_3^{k_3} b_4^{k_4}]$, where k_1 to k_4 are non-negative integers.

Eliminating a_2, a_3, and a_4, from equations (5.12) to (5.15), and using (5.9), we have

(5.20) $$a_1 + a_5 = b_1 b_2 b_3 + b_1 b_2 + b_1 + 1.$$

Multiplying this equation through by a_1, and substituting for $a_1 a_5$ from (5.5), we have

(5.21) $$a_1^2 = (b_1 b_2 b_3 + b_1 b_2 + b_1 + 1) a_1 - b_1.$$

Now, by means of (5.13) to (5.19), each of the a_1 to a_8 can be expressed in the form

(5.22) $$Pa_1 + Q,$$

where P and Q are polynomials in b_1 to b_4 with integral coefficients. (We could of course have used any other of the

a_1 to a_8 here instead of a_1.) It follows that any expression of the form $a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8}$ may be expressed as a polynomial in a_1, the coefficients being polynomials in b_1 to b_4 (with integral coefficients). In view of (5.21) this means that any $a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8}$ is equal to an expression of the form (5.22).

Now in $[a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8}]$ the term $a_5^{j_1} a_6^{j_2} a_7^{j_3} a_8^{j_4} a_1^{j_5} a_2^{j_6} a_3^{j_7} a_4^{j_8}$, obtained under the interchanges (a_1,a_5), (a_2,a_6), (a_3,a_7), and (a_4, a_8), also occurs. Further b_1 to b_4 are not affected by these interchanges; so that the sum of the two terms of $[a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8}]$ under discussion is equal to an expression of the form

$$P(a_1 + a_5) + 2Q,$$

using the cyclic properties of our relations. But by (5.20) this expression is equal to a linear combination of terms $b_1^{k_1} b_2^{k_2} b_3^{k_3} b_4^{k_4}$. Hence Lemma 5.2 follows, since clearly (again using the cyclic properties of our relations) the other three pairs of terms of $[a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8}]$ correspond to the other three terms of each $[b_1^{k_1} b_2^{k_2} b_3^{k_3} b_4^{k_4}]$.

We further write

$$\lambda = b_1 b_3 + b_2 b_4,$$
$$\mu = b_1^2 b_2 b_3 + b_2^2 b_3 b_4 + b_3^2 b_4 b_1 + b_4^2 b_1 b_2,$$

and prove the following:

LEMMA 5.3 Any expression of the form $[b_1^{k_1} b_2^{k_2} b_3^{k_3} b_4^{k_4}]$ is equal to

$$S(\lambda) + \mu T(\lambda),$$

where $S(\lambda)$ and $T(\lambda)$ are polynomials in λ with integral coefficients.

By (5.10) and (5.11) any expression of the form $b_1^{k_1} b_2^{k_2} b_3^{k_3} b_4^{k_4}$ can be expressed as a linear combination of terms $b_1^{l_1} b_2^{l_2}$ where l_1 and l_2 are non-negative integers. Clearly then, performing a cyclic summation, any $[b_1^{k_1} b_2^{k_2} b_3^{k_3} b_4^{k_4}]$ is equal to a linear combination of terms $[b_1^{l_1} b_2^{l_2}]$, and we need only consider the latter expression, rather than the former.

Writing

$$c_1 = b_1b_3, \quad c_2 = b_2b_4,$$

we have by multiplying (5.10) and (5.11) through by b_1 and b_2 respectively

(5.23) $$b_1^2 = - b_1 - c_1,$$

(5.24) $$b_2^2 = - b_2 - c_2.$$

In view of (5.23) and (5.24) any $b_1^{l_1} b_2^{l_2}$ may be expressed in the form

$$A + Bb_1 + Cb_2 + Db_1b_2,$$

where A, B, C, and D, are polynomials in c_1 and c_2 with integral coefficients. Then, since c_1 and c_2 are not affected by the interchanges (b_1, b_3) and (b_2, b_4), $b_3^{l_1} b_4^{l_2}$ is equal to

$$A + Bb_3 + Cb_4 + Db_3b_4.$$

Hence, using (5.10) and (5.11), we have

$$b_1^{l_1} b_2^{l_2} + b_3^{l_3} b_4^{l_4} = E + D(b_1b_2 + b_3b_4), \tag{5.25}$$

where $E = 2A - B - C$.

Now, using the definitions of c_1 and c_2, the definition of λ, and (5.9), may be written as

$$c_1 + c_2 = \lambda, \tag{5.26}$$

$$c_1c_2 = -1, \tag{5.27}$$

respectively. From these two equations we derive

$$c_1^2 = \lambda c_1 + 1, \tag{5.28}$$

$$c_2^2 = \lambda c_2 + 1. \tag{5.29}$$

In view of (5.27), (5.28), and (5.29), any polynomial in c_1 and c_2, with integral coefficients, may be expressed in the form

$$G + Hc_1 + Ic_2,$$

where G, H, and I, are polynomials in λ with integral coefficients. Hence we may write (5.25) in the form

$$b_1^{l_1} b_2^{l_2} + b_3^{l_1} b_4^{l_2} = (G + Hc_1 + Ic_2)+(G' + H'c_1 + I'c_2)(b_1b_2+b_3b_4),$$

where G', H', and I', are also polynomials in λ with integral coefficients. Further since interchanging b_1, b_2, b_3, and b_4, cyclically corresponds to interchanging c_1 and c_2, and leaving λ unchanged, we also have

$$b_2^{l_1} b_3^{l_2} + b_4^{l_1} b_1^{l_2} = (G + Hc_2 + Ic_1)+(G' + H'c_2 + I'c_1)(b_2b_3+b_4b_1).$$

Thus, adding the last two equations, and using (5.26) and the definitions of c_1 and c_2, we obtain

(5.30) $[b_1^{l_1}\ b_2^{l_2}] = 2G + H\lambda + I\lambda + G'[b_1b_2]+H'[b_1^2b_2b_3]+I'[b_1b_2b_3^2].$

But

$$[b_1b_2] = (b_1 + b_3)(b_2 + b_4) = 1$$

by (5.10) and (5.11), and

(5.31) $\mu + [b_1b_2b_3^2] = [b_1^2b_2b_3]+[b_1b_2b_3^2] = (b_1b_3+b_2b_4)[b_1b_2]=\lambda.1.$

Hence (5.30) becomes

$$[b_1^{l_1}\ b_2^{l_2}] = (2G+H\lambda+I\lambda+G'+I'\lambda) + \mu(H'-I'),$$

and since both brackets on the right-hand side of this equation are polynomials in λ with integral coefficients, Lemma 5.3 follows.

We have the following relation between λ and μ:

(5.32) $\mu^2 - \lambda\mu + \lambda^3 + 4\lambda^2 + 4\lambda + 15 = 0.$

Since μ^2 is certainly of the form $[b_1^{k_1}\ b_2^{k_2}\ b_3^{k_3}\ b_4^{k_4}]$ we know by Lemma 5.3 that a relation of the above form exists, and the coefficients in the equation are found by comparing coefficients of powers of y in the expansions of the appropriate quantities as power series in y;{cf. the proof of (AH), equation (8.13).} We give a direct proof also: we have

$$\mu^2 - \lambda u = -[b_1^2b_2b_3][b_1b_2b_3^2],$$

using (5.31),

$$= -\{c_1(b_1b_2+b_3b_4)+c_2(b_2b_3+b_4b_1)\}\{c_2(b_1b_2+b_3b_4)+c_1(b_2b_3+b_4b_1)\}$$

$$= -c_1c_2([b_1^2\ b_2^2]+4b_1b_2b_3b_4)-(c_1^2 + c_2^2)[b_1b_2^2\ b_3],$$

$$= [b_1^2\ b_2^2] - 4 - (\lambda^2 + 2)[b_1b_2^2\ b_3]$$

by (5.9), (5.26) and (5.27). But

(5.33) $[b_1^2\ b_2^2] = (b_1^2 + b_3^2)(b_2^2 + b_4^2),$

$$= (1 - 2b_1b_3)(1 - 2b_2b_4)$$

using (5.10) and (5.11),

$$= -2\lambda - 3$$

using (5.9); and

(5.34) $[b_1b_2^2\ b_3] = b_2b_4(b_1^2 + b_3^2) + b_1b_3(b_2^2 + b_4^2)$

$$= b_2b_4(1 - 2b_1b_3) + b_1b_3(1 - 2b_2b_4)$$

$$= \lambda + 4.$$

Equation (5.32) follows.

Now, by Lemmas 5.1, 5.2, and 5.3, F^6 and $yf(y^{17})F^6\Phi(5)$ are each equal to an expression of the form $S(\lambda) + \mu T(\lambda)$. Since the lowest powers of y in the expansions of F^6, λ, and μ, as power series in y, are -12, -2, and -3, respectively, we assume a form for F^6 with $S(\lambda)$ of degree 6 and $G(\lambda)$ of degree 4. We find the 12 coefficients involved in these two polynomials by comparing coefficients of y^{-12}, $y^{-11}, \ldots, y^{-2}$, and y^0, (they appear seriatim), and check the values obtained by comparing coefficients of y^{-1}. The resulting expression for F^6 is found, using (5.32), to be a perfect cube, and in fact we have

(5.35) $$F^2 = \lambda^2 - 20\lambda - 56 + 8\mu,$$

since F, λ, and μ, are real for real y. Similarly, in the case of $yf(y^{17})F^6\,\Phi(5)$, $S(\lambda)$ and $T(\lambda)$ are of degrees 5 and 4 respectively, and we find the 11 coefficients involved by comparing coefficients of y^{-11}, y^{-10}, ..., y^{-2}, and y^o, (again they appear seriatim), and check the values obtained by comparing coefficients of y^{-1}; we obtain

(5.36)
$$yf(y^{17})F^6\,\Phi(5)=-834\lambda^5+31236\lambda^4-34498\lambda^3+126757\lambda^2-14022\lambda- -112984+\mu(-7\lambda^4+9756\lambda^3-69280\lambda^2+162020\lambda-164885).$$

The equations (5.32), (5.35), and (5.36), for q = 17, are of course analagous to (AH), equations (8.13), (11.7), and (11.9), for q = 11.

We now write

$$\delta = b_1b_2 - b_2b_3 + b_3b_4 - b_4b_1.$$

Then

(5.37)
$$\delta^2 = [b_1^2\,b_2^2] - 2[b_1b_2^2b_3]+4b_1b_2b_3b_4, = -4\lambda - 15$$

by (5.9), (5.33), and (5.34). Also, by (5.35) and (5.37),

$$F^2\delta^2 = (-4\lambda - 15)(\lambda^2 - 20\lambda - 56 + 8\mu),$$

and, using (5.32), it is easily verified that the right-hand side of this equation is equal to

$$(-2\mu + 9\lambda + 30)^2;$$

hence we have

(5.38) $$F\delta = -2\mu + 9\lambda + 30,$$

where the sign of the coefficient of the lowest power of y in the expansion of each side of this equation is examined to determine the appropriate root. Thus, instead of λ and μ, we may take δ and F, as new variables; in fact from (5.37) and (5.38) we have

(5.39) $$\lambda = -(\delta^2 + 15)/4,$$

(5.40) $$\mu = -(4F\delta + 9\delta^2 + 15)/8.$$

Substituting for λ and μ from (5.39) and (5.40) in (5.35) we obtain the following relation between δ and F:

(5.41) $$(\delta^2 - 17)^2 = 16F(F + 4\delta).$$

Also, substituting for λ and μ in (5.36) we obtain $yf(y^{17})F^6\,\Phi(5)$ as a polynomial in δ and F. Further since (5.41) is a quartic in δ, this polynomial is equal to another polynomial in δ and F of degree 3 in δ; in fact we have

(5.42) $$\begin{aligned} 8yf(y^{17})F^6\,\Phi(5) = {} & \delta^3(84.17^2F^3 + 20.17^5F) + \\ & +\delta^2(115.17F^4+316.17^4F^2+17^7) + \\ & +\delta(28F^5+2476.17^3F^3+32.17^6F) + \\ & +(6677.17^2F^4 + 124.17^5F^2 - 9.17^7) \end{aligned}$$

{it is of course obvious from the form of (5.39), (5.40), and (5.41), that the right-hand side of this equation must be a function of δ^2, $F\delta$, and F^2, only}.

We further write

$m_1 = -\,yP(2)P(8)P(3)P(5) - yP(1)P(4)P(6)P(7)$, $\quad n_1 = -y^2P(1)P(4)P(2)P(8)$,
$m_2 = P(6)P(7)P(2)P(8) - y^2P(3)P(5)P(1)P(4)$, $\quad n_2 = P(3)P(5)P(6)P(7)$.

Then

(5.43) and (5.44) $\quad m_1/n_1 = b_1 - b_3,\ m_2/n_2 = b_2 - b_4;$

(5.45) $$n_1n_2 = -y^2 f(y)/f(y^{17});$$

(5.46) and (5.47) $\quad n_2/n_1 = b_1b_3, \quad n_1/n_2 = -b_2b_4.$

Also,

$$m_1^2/n_1^2 = (b_1-b_3)^2 = (b_1+b_3)^2 - 4b_1b_3 = 1 - 4n_2/n_1,$$

using (5.10), (5.43), and (5.46), i.e.

(5.48) $$m_1^2 = n_1^2 - 4n_1n_2,$$

and correspondingly we may obtain

(5.49) $$m_2^2 = n_2^2 + 4n_1n_2.$$

In terms of these new functions we have

(5.50) $$\delta = (b_1 - b_3)(b_2 - b_4) = -y^{-2} f(y^{17}) m_1 m_2 / f(y)$$

by (5.43), (5.44), and (5.45), and also

(5.51) $$\delta^2 = -4\lambda - 15 = -4(b_1b_3+b_2b_4)-15 = -4(n_2/n_1-n_1/n_2)-15,$$

by (5.37), (5.46) and (5.47). Now (5.41) may be written in the form

$$16(F + 2\delta)^2 = \delta^4 + 30\delta^2 + 289,$$

but by (5.51) the right-hand side of this equation is equal to

$$16(n_1/n_2 + n_2/n_1)^2,$$

hence we have

$$F + 2\delta = -(n_1/n_2 + n_2/n_1),$$

where the sign of the coefficient of the lowest power of y on each side of this equation is examined to determine the appropriate root. Now the right-hand side of this equation is equal to

$$y^{-2}f(y^{17})(m_1^2 + m_2^2)/f(y)$$

by (5.45), (5.48), and (5.49). Thus using (5.50) we have

(5.52) $y^2 f(y)F/f(y^{17}) = f^4(y)/f^4(y^{17}) = (m_1 + m_2)^2$,

whence

(5.53) $$f^2(y)/f^2(y^{17}) = m_1 + m_2,$$

where again care is taken to select the appropriate root. Further, in view of (5.50) and (5.52) the right-hand side of (5.41) is equal to

$$16y^{-4}f^2(y)(m_1-m_2)^2/f^2(y^{17}),$$

whence, taking the appropriate square root of this expression,

(5.54) $$\delta^2-17=4y^{-2}f(y)(-m_1+m_2)/f(y^{17}).$$

We note that elimination of δ from equations (5.50) and (5.54) gives

(5.55) $m_1^2\, m_2^2 + 4y^2f^3(y)(m_1-m_2)/f^3(y^{17})-17y^4f^2(y)/f^2(y^{17})=0.$

Making a slight change in notation for convenience, we now re-state (5.53), (5.55), (5.50), (5.42), and (5.41), in order, as follows.

THEOREM 5.1 If we write

$$M_1 = f^2(y^{17})\{-yP(2)P(8)P(3)P(5)-yP(1)P(4)P(6)P(7)\}/f^2(y),$$
$$M_2 = f^2(y^{17})\{P(6)P(7)P(2)P(8)-y^2P(3)P(5)P(1)P(4)\}/f^2(y),$$

then we have

$$M_1 + M_2 = 1,$$

$$M_1^2\, M_2^2 + 4(M_1 - M_2)/F - 17/F^2 = 0,$$

where $F = y^{-2}f^3(y)/f^3(y^{17})$; and if we further write

$$\varepsilon = -M_1M_2,$$

then we have

$$8yf(y^{17})\,\Phi(5) = \varepsilon^3(84.17^2 + 20.17^5/F^2) + \\ +\varepsilon^2(115.17.+316.17^4/F^2+17^7/F^4)+ \\ +\varepsilon(28+2476.17^3/F^2+32.17^6/F^4) + \\ +(6677.17^2/F^2+124.17^5/F^4-9.17^7/F^6),$$

where, from the last three equations but one, there is the following relation between ε and F

$$(\varepsilon^2 - 17/F^2)^2 = 16(4\varepsilon + 1)/F^2.$$

§ We conclude this Part by deriving the following simple congruence

$$(5.56)\quad \Phi(5)\equiv f^2(y^{17})f^5(y)\{7P(3)P(5)P(6)P(7)+6y^2P(1)P(2)P(4)P(8)\} \pmod{17}.$$

Since the only term on the right-hand side of (5.42) without a factor 17 is $28\delta F^5$, we have

$$(5.57)\qquad yf(y^{17})F\,\Phi(5) \equiv -5\delta \pmod{17}.$$

But from (5.51)

$$\delta^2 \equiv -4(n_2 + 4n_1)^2/n_1n_2 \pmod{17},$$

and using (5.45)

$$-1/n_1n_2=y^{-2}f(y^{17})/f(y)\equiv y^{-2}f^{16}(y) \pmod{17}$$

since $f^{17}(y) \equiv f(y^{17})$ (mod. 17), so that, taking the appropriate square root,

$$(5.58)\qquad \delta\equiv 2y^{-1}f^8(y)(n_2+4n_1) \pmod{17}.$$

(5.56) follows immediately, from (5.57), (5.58), and the definitions of n_1, n_2, and F.

PART 3

q = 19 throughout this Part

6. We write

$$\alpha_1 = -x^{-8}P(2)/P(1),\ \alpha_2 = x^{-13}P(4)/P(2),\ \alpha_3 = x^{-14}P(8)/P(4),$$
$$\alpha_4 = x^{20}P(3)/P(8),\ \alpha_5 = -x^{-15}P(6)/P(3),\ \alpha_6 = x^{-3}P(7)/P(6),$$
$$\alpha_7 = -x^{7}P(5)/P(7),\ \alpha_8 = -x^{-10}P(9)/P(5),\ \alpha_9 = -x^{36}P(1)/P(9);$$

then by (ASD), Lemma 6 (with q = 19) we have

(6.1) $-x^{-15}f(x)/f(y^{19}) = \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+1.$

In (6.1) we replace x by $w_r x$ where w_r (r = 1 to 19) are the nineteenth roots of unity, and multiply together the nineteen resulting equations, obtaining

(6.2) $$-y^{-15}f^{20}(y)/f^{20}(y^{19}) = \prod_{r=1}^{19}(\alpha_1 w_r^{-8}+\alpha_2 w_r^{-13}+\alpha_3 w_r^{-14}+\alpha_4 w_r^{20}+$$
$$+\alpha_5 w_r^{-15}+\alpha_6 w_r^{-3}+\alpha_7 w_r^{7}+\alpha_8 w_r^{-10}+\alpha_9 w_r^{36}+1).$$

Now as w_r runs through the nineteenth roots of unity so does w_r^5, so that the product on the right-hand side of (6.2) is equal to

$$\prod_{r=1}^{19}(\alpha_1 w_r^{36}+\alpha_2 w_r^{-8}+\alpha_3 w_r^{-13}+\alpha_4 w_r^{-14}+\alpha_5 w_r^{20}+\alpha_6 w_r^{-15}+\alpha_7 w_r^{-3}+\alpha_8 w_r^{7}+\alpha_9 w_r^{-10}+1),$$

and is thus unchanged if α_1, α_2, α_3, α_4, α_5, α_6, α_7, α_8, and α_9, are interchanged cyclically. The product is thus a linear combination of terms $[\alpha_1^{i_1}\ \alpha_2^{i_2}\ \alpha_3^{i_3}\ \alpha_4^{i_4}\ \alpha_5^{i_5}\ \alpha_6^{i_6}\ \alpha_7^{i_7}\ \alpha_8^{i_8}\ \alpha_9^{i_9}]$ where i_1 to i_9 are non-negative integers, and considering the left-hand side of (6.2) such terms as occur can only involve x in terms of $y = x^{19}$. Thus if $\alpha_1^{i_1} \alpha_2^{i_2} \alpha_3^{i_3} \alpha_4^{i_4} \alpha_5^{i_5} \alpha_6^{i_6} \alpha_7^{i_7} \alpha_8^{i_8} \alpha_9^{i_9}$ (or any other term of $[\alpha_1^{i_1} \alpha_2^{i_2} \alpha_3^{i_3} \alpha_4^{i_4} \alpha_5^{i_5} \alpha_6^{i_6} \alpha_7^{i_7} \alpha_8^{i_8} \alpha_9^{i_9}]$) occurs we must have

$$(6.3)\quad -8i_1-13i_2-14i_3+20i_4-15i_5-3i_6+7i_7-10i_8+36i_9 \equiv 0 \pmod{19}$$

(interchanging $i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8$, and i_9, cyclically gives the same congruence).

Now, writing

$a_1=y^{-1}P(6)P(7)/P(2)P(9)$, $a_2=y^{-2}P(7)P(5)/P(4)P(1)$,

$a_3=-y^{-1}P(5)P(9)/P(8)P(2)$, $a_4=-P(9)P(1)/P(3)P(4)$,

$a_5=y^{3}P(1)P(2)/P(6)P(8)$, $a_6=-yP(2)P(4)/P(7)P(3)$,

$a_7=-P(4)P(8)/P(5)P(6)$, $a_8=yP(8)P(3)/P(9)P(7)$,

$a_9=-y^{-1}P(3)P(6)/P(1)P(5)$,

it is easily verified that

$$(6.4)\quad \begin{aligned}
&\alpha_1^{19}=a_2^{16}a_3a_4^2a_5^7a_6^{13}a_7^5a_8^3a_9^{12}, &&\alpha_2^{19}=a_3^{16}a_4a_5^2a_6^7a_7^{13}a_8^5a_9^3a_1^{12},\\
&\alpha_3^{19}=a_4^{16}a_5a_6^2a_7^7a_8^{13}a_9^5a_1^3a_2^{12}, &&\alpha_4^{19}=a_5^{16}a_6a_7^2a_8^7a_9^{13}a_1^5a_2^3a_3^{12},\\
&\alpha_5^{19}=a_6^{16}a_7a_8^2a_9^7a_1^{13}a_2^5a_3^3a_4^{12}, &&\alpha_6^{19}=a_7^{16}a_8a_9^2a_1^7a_2^{13}a_3^5a_4^3a_5^{12},\\
&\alpha_7^{19}=a_8^{16}a_9a_1^2a_2^7a_3^{13}a_4^5a_5^3a_6^{12}, &&\alpha_8^{19}=a_9^{16}a_1a_2^2a_3^7a_4^{13}a_5^5a_6^3a_7^{12},\\
&\alpha_9^{19}=a_1^{16}a_2a_3^2a_4^7a_5^{13}a_6^5a_7^3a_8^{12}.
\end{aligned}$$

It will be noticed that all of the equations (6.4) may be obtained from any one of them by interchanging

a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, and α_1, α_2, α_3, α_4, α_5, α_6, α_7, α_8, α_9, cyclically. By (6.4), since

$a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7\ a_8\ a_9 = -1,$

$$(\alpha_1^{i_1}\ \alpha_2^{i_2}\ \alpha_3^{i_3}\ \alpha_4^{i_4}\ \alpha_5^{i_5}\ \alpha_6^{i_6}\ \alpha_7^{i_7}\ \alpha_8^{i_8}\ \alpha_9^{i_9})^{19} =$$

$$= (a_1a_2a_3a_4a_5a_6a_7a_8a_9)^{\sigma}\ a_1^{\sigma_1}\ a_2^{\sigma_2}\ a_3^{\sigma_3}\ a_4^{\sigma_4}\ a_5^{\sigma_5}\ a_6^{\sigma_6}\ a_7^{\sigma_7}\ a_8^{\sigma_8}\ a_9^{\sigma_9}$$

where $\sigma = 30i_1+32i_2+2i_3+34i_4+10i_5+28i_6+24i_7+8i_8+20i_9$, an even integer, and

$$\sigma_1 = 12i_2+3i_3+5i_4+13i_5+7i_6+2i_7+i_8+16i_9,$$
$$\sigma_2 = 12i_3+3i_4+5i_5+13i_6+7i_7+2i_8+i_9+16i_1,$$
$$\sigma_3 = 12i_4+3i_5+5i_6+13i_7+7i_8+2i_9+i_1+16i_2,$$
$$\sigma_4 = 12i_5+3i_6+5i_7+13i_8+7i_9+2i_1+i_2+16i_3,$$
$$\sigma_5 = 12i_6+3i_7+5i_8+13i_9+7i_1+2i_2+i_3+16i_4,$$
$$\sigma_6 = 12i_7+3i_8+5i_9+13i_1+7i_2+2i_3+i_4+16i_5,$$
$$\sigma_7 = 12i_8+3i_9+5i_1+13i_2+7i_3+2i_4+i_5+16i_6,$$
$$\sigma_8 = 12i_9+3i_1+5i_2+13i_3+7i_4+2i_5+i_6+16i_7,$$
$$\sigma_9 = 12i_1+3i_2+5i_3+13i_4+7i_5+2i_6+i_7+16i_8;$$

moreover $\sigma + \sigma_1$ to $\sigma + \sigma_9$ are multiples of 19 by (6.3), hence any expression of the form $\alpha_1^{i_1}\ \alpha_2^{i_2}\ \alpha_3^{i_3}\ \alpha_4^{i_4}\ \alpha_5^{i_5}\ \alpha_6^{i_6}\ \alpha_7^{i_7}\ \alpha_8^{i_8}\ \alpha_9^{i_9}$ for which (6.3) holds is of the form $a_1^{j_1}\ a_2^{j_2}\ a_3^{j_3}\ a_4^{j_4}\ a_5^{j_5}\ a_6^{j_6}\ a_7^{j_7}\ a_8^{j_8}\ a_9^{j_9}$ where j_1 to j_9 are non-negative integers. Thus every term occurring in the right-hand side of (6.2) is of the form $a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8} a_9^{j_9}$, and such terms occur in cylically symmetrical sets of nine terms each.

Further, $\Phi(4)$ is the coefficient of x^4 in $1/f(x)$ regarded as a polynomial of degree 18 in x with coefficients involving x in terms of $y = x^{19}$, so that $y^{-14}f^{20}(y)\Phi(4)/f^{19}(y^{19})$ is the coefficient of x^0 in
$y^{-15}f^{20}(y)/\{f^{20}(y^{19})(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7+\alpha_8+\alpha_9+1)\}$. This is a cyclically symmetric polynomial of degree 18 in α_1, α_2, α_3, α_4, α_5, α_6, α_7, α_8, and α_9; and the terms which give the coefficient of x^0 occur only in symmetrical sets of nine expressible as $[a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8} a_9^{j_9}]$, as before. (This is not true for the coefficient of any power of x other than 0; the nine terms of $[a_1]$, for example, do not appertain to the same power of x.)

Thus writing

$$F = y^{-3}f^4(y)/f^4(y^{19})$$

we have the following:

LEMMA 6.1 F^5 and $yf(y^{19})F^5\,\Phi(4)$ are each equal to a linear combination of terms $[a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8} a_9^{j_9}]$.

We now write

(6.5) to (6.7) $b_1 = a_1a_4a_7$, $b_2 = a_2a_5a_8$, $b_3 = a_3a_6a_9$;

(6.8) to (6.10) $c_1 = a_1a_4+a_4a_7+a_7a_1$, $c_2 = a_2a_5+a_5a_8+a_8a_2$,
$c_3 = a_3a_6+a_6a_9+a_9a_3$;

(6.11)to(6.13) $d_1 = a_1+a_4+a_7$, $d_2 = a_2+a_5+a_8$, $d_3 = a_3+a_6+a_9$;

so that

(6.14) $$b_1 b_2 b_3 + 1 = 0.$$

<9, 6, 5, 3>, <9, 7, 6, 1>, <7, 5, 2, 1>, <9, 5, 4, 2>, <9, 8, 4, 1>, <8, 3, 2, 1>, <6, 4, 3, 2>, <8, 7, 6, 4>, and <8, 7, 5, 3>, give, respectively,

(6.15) to (6.17) $a_1 a_4 = a_3+1, \quad a_2 a_5 = a_4+1, \quad a_3 a_6 = a_5+1,$

(6.18) to (6.20) $a_4 a_7 = a_6+1, \quad a_5 a_8 = a_7+1, \quad a_6 a_9 = a_8+1,$

(6.21) to (6.23) $a_7 a_1 = a_9+1, \quad a_8 a_2 = a_1+1, \quad a_9 a_3 = a_2+1.$

It will be observed that each of the equations (6.5) to (6.23) remains valid when b_1, b_2, b_3, and c_1, c_2, c_3, and d_1, d_2, d_3, and a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, are interchanged cyclically. We are now in a position to prove

LEMMA 6.2 Any expression of the form $[a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8} a_9^{j_9}]$ is equal to a linear combination of terms $[b_1^{k_1} b_2^{k_2} b_3^{k_3} c_1^{k_4} c_2^{k_5} c_3^{k_6} d_1^{k_7} d_2^{k_8} d_3^{k_9}]$, where the square bracket in this case denotes a summation of the <u>three</u> different terms obtained by interchanging b_1, b_2, b_3, and c_1, c_2, c_3, and d_1, d_2, d_3, <u>separately</u>, and k_1 to k_9 are non-negative integers.

By eliminating a_3 and a_9 from equations (6.15), (6.21), and (6.23), we obtain

(6.24) $$a_2 = a_1^2+(b_1 - d_1)a_1,$$

and clearly this equation remains valid when a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, and b_1, b_2, b_3, and d_1, d_2, d_3, are interchanged cyclically. Thus, by means of (6.24) and the eight similar equations, each of the a_1 to a_9 can be expressed as a polynomial in b_1, b_2, b_3, d_1, d_2, d_3, and a_1, with integral coefficients; and hence any expression of the form $a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8} a_9^{j_9}$ is equal to such a polynomial. (We could of course have used any other of the a_1 to a_9 here instead of a_1.) But (in view of the definitions of b_1, c_1, and d_1) a_1 (and a_4 and a_7) satisfies a cubic equation with coefficients in terms of b_1, c_1, and d_1. Hence any $a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8} a_9^{j_9}$ may be expressed in the form

$$Pa_1^2 + Qa_1 + R,$$

where P, Q, and R, are polynomials in b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, and d_3, with integral coefficients.

Now in $[a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8} a_9^{j_9}]$ the terms

$a_4^{j_1} a_5^{j_2} a_6^{j_3} a_7^{j_4} a_8^{j_5} a_9^{j_6} a_1^{j_7} a_2^{j_8} a_3^{j_9}$ and

$a_7^{j_1} a_8^{j_2} a_9^{j_3} a_1^{j_4} a_2^{j_5} a_3^{j_6} a_4^{j_7} a_5^{j_8} a_6^{j_9}$, obtained under the cyclic interchanges (a_1, a_4, a_7), (a_2, a_5, a_8), and (a_3, a_6, a_9), also occur. Further b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, and d_3, are not affected by these interchanges; so that the sum of the three terms of $[a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8} a_9^{j_9}]$ under discussion is equal to an expression of the form

$$P(a_1^2 + a_4^2 + a_7^2)+Q(a_1 + a_4 + a_7)+3R,$$

using the cyclic properties of our relations. Since from the definitions of c_1 and d_1

$$a_1 + a_4 + a_7 = d_1,$$

$$a_1^2 + a_4^2 + a_7^2 = d_1^2 - 2c_1,$$

this expression is equal to a linear combination of terms $b_1^{k_1} b_2^{k_2} b_3^{k_3} c_1^{k_4} c_2^{k_5} c_3^{k_6} d_1^{k_7} d_2^{k_8} d_3^{k_9}$. Hence Lemma 6.2 follows, since clearly (again using the cyclic properties of our relations) the other two triplets of terms of $[a_1^{j_1} a_2^{j_2} a_3^{j_3} a_4^{j_4} a_5^{j_5} a_6^{j_6} a_7^{j_7} a_8^{j_8} a_9^{j_9}]$ correspond to the other two terms of each $[b_1^{k_1} b_2^{k_2} b_3^{k_3} c_1^{k_4} c_2^{k_5} c_3^{k_6} d_1^{k_7} d_2^{k_8} d_3^{k_9}]$.

We now prove

LEMMA 6.3 Any expression of the form $[b_1^{k_1} b_2^{k_2} b_3^{k_3} c_1^{k_4} c_2^{k_5} c_3^{k_6} d_1^{k_7} d_2^{k_8} d_3^{k_9}]$ is equal to a linear combination of terms $[b_1^{k_1'} b_2^{k_2'} b_3^{k_3'}]$, where k_1' to k_3' are non-negative integers.

Clearly it will be sufficient to show that c_1, c_2, c_3, d_1, d_2, and d_3, can all be expressed as polynomials in b_1, b_2, and b_3, with integral coefficients. For then any $b_1^{k_1} b_2^{k_2} b_3^{k_3} c_1^{k_4} c_2^{k_5} c_3^{k_6} d_1^{k_7} d_2^{k_8} d_3^{k_9}$ may be expressed as a linear combination of terms $b_1^{k_1'} b_2^{k_2'} b_3^{k_3'}$, and Lemma

6.3 follows from cyclic considerations.

We have

(6.25) to (6.27) $c_1 = d_3+3, \quad c_2 = d_1+3, \quad c_3 = d_2+3,$

the first of which is (6.15) + (6.18) + (6.21), in the obvious notation, and

(6.28) to (6.30) $b_1^2=b_3+c_3+d_3+1, \quad b_2^2=b_1+c_1+d_1+1,$
$b_3^2=b_2+c_2+d_2+1,$

the first of which is (6.15).(6.18).(6.21). Substituting for c_1, c_2, and c_3, in (6.28) to (6.30) from (6.25) to (6.27), and solving the resulting equations for d_1, d_2, and d_3, we obtain

(6.31) $$2d_1=-b_1^2+b_2^2+b_3^2-b_1-b_2+b_3-4,$$

(6.32) $$2d_2=-b_2^2+b_3^2+b_1^2-b_2-b_3+b_1-4,$$

(6.33) $$2d_3=-b_3^2+b_1^2+b_2^2-b_3-b_1+b_2-4.$$

We now show that

(6.34) $$b_1+b_2+b_3+2 = 0.$$

Then the right-hand side of (6.31) is equal to

$$(b_1+b_2+b_3)^2-2b_1-2(b_1b_2+b_2b_3+b_3b_1)-(b_1+b_2+b_3)+2b_3-4$$

$$= -2b_1^2-2(b_1b_2+b_2b_3+b_3b_1)+2b_3+2,$$

and since the latter expression has a factor 2 we have d_1, and hence c_2 by (6.26), as a polynomial in b_1, b_2, and b_3, with integral coefficients; clearly from cyclic considerations the same is true of d_2, d_3, c_1, and c_3, and we have the Lemma.

(6.34) is proved as follows. We have

$$b_1d_1 = c_3+2d_3+3,$$

which is (6.15).(6.18) + (6.18).(6.21) + (6.21).(6.15). Substituting for c_3 from (6.27) and then for d_1, d_2, and d_3, from (6.31) to (6.33), the resulting equation simplifies to

$$b_1^3-b_1b_2^2-b_3^2b_1+4b_1^2+b_2^2-b_3^2+b_1b_2-b_3b_1+3b_1+b_2-3b_3 = 0,$$

and of course we may interchange b_1, b_2, and b_3, cyclically in this equation to obtain two other similar relations. Adding all three equations we arrive at

$$[b_1^3]-[b_1b_2^2]-[b_1^2b_2]+4[b_1^2]+[b_1] = 0.$$

But it is easily verified that the left-hand side of this equation is equal to

$$([b_1]+2)([b_1^2]-2[b_1b_2]+2[b_1]-3),$$

using (6.14); and the second of these two factors, expanded as a power series in y, begins $4y^{-2}+ \ldots$ and is therefore non-zero. Thus we arrive at the relation (6.34), and complete the proof of Lemma 6.3.

We further write

$$\lambda = [b_1b_2],$$

$$\mu = [b_1^2b_2],$$

and prove the following:

LEMMA 6.4 Any expression of the form $[b_1^{k_1'} b_2^{k_2'} b_3^{k_3'}]$ is equal to

$$S(\lambda) + \mu T(\lambda),$$

where $S(\lambda)$ and $T(\lambda)$ are polynomials in λ with integral coefficients.

By (6.34) any expression of the form $b_1^{k_1'} b_2^{k_2'} b_3^{k_3'}$ can be expressed as a linear combination of terms $b_1^{l_1} b_2^{l_2}$ where l_1 and l_2 are non-negative integers. Clearly then, performing a cyclic summation, any $[b_1^{k_1'} b_2^{k_2'} b_3^{k_3'}]$ is equal to a linear combination of terms $[b_1^{l_1} b_2^{l_2}]$, and we need only consider the latter expression, rather than the former.

Now, by (6.14), (6.34), and the definition of λ, b_1 to b_3 are the roots of the cubic equation

$$z^3 + 2z^2 + \lambda z + 1 = 0,$$

so that we have

(6.35) $$b_1^3 = -2b_1^2 - \lambda b_1 - 1,$$

(6.36) $$b_2^3 = -2b_2^2 - \lambda b_2 - 1.$$

In view of (6.35) and (6.36) any $b_1^{l_1} b_2^{l_2}$ may be expressed in the form

$$G + Hb_1 + Ib_2 + Jb_1^2 + Kb_2^2 + Lb_1b_2 + Mb_1^2b_2 + Nb_1b_2^2 + Pb_1^2b_2^2,$$

where G, H, I, J, K, L, M, N, and P, are polynomials in λ with integral coefficients. Then, since λ is not affected when b_1, b_2, and b_3, are interchanged cyclically, we have

(6.37) $[b_1^{l_1}b_2^{l_2}]=3G+(H+I)[b_1]+(J+K)[b_1^2]+L[b_1b_2]+M[b_1^2b_2]+$

$$+N[b_1b_2^2]+P[b_1^2b_2^2].$$

But we have (6.34) and the definitions of λ and μ

(6.38) to (6.40) $[b_1] = -2,\ [b_1b_2]=\lambda,\ [b_1^2b_2] = \mu,$

and

(6.41) $[b_1^2] = [b_1]^2 -2[b_1b_2] = 4 - 2\lambda,$

(6.42) $[b_1b_2^2]=[b_1][b_1b_2]-[b_1^2b_2]-3b_1b_2b_3=-2\lambda-\mu+3,$

(6.43) $[b_1^2b_2^2]=[b_1b_2]^2-2[b_1^2b_2b_3]=[b_1b_2]^2+2[b_1] = \lambda^2-4,$

using (6.14). Hence (6.37) becomes

$[b_1^{l_1}b_2^{l_2}]=\{3G-2(H+I)+(4-2\lambda)(J+K)+\lambda L+(-2\lambda+3)N+(\lambda^2-4)P\}+\mu\{M-N\},$

and since both curly brackets on the right-hand side of this equation are polynomials in λ with integral coefficients, Lemma 6.4 follows.

We have the following relation between λ and μ:

(6.44) $\mu^2+(2\lambda-3)\mu+\lambda^3-12\lambda+17 = 0.$

Since μ^2 is certainly of the form $[b_1^{k_1'}\ b_2^{k_2'}\ b_3^{k_3'}]$ we know by Lemma 6.4 that a relation of the above form exists, and the coefficients in the equation are found by comparing coefficients of powers of y in the expansions of the appropriate quantities as power series in y; {cf. the proof of (AH), equation (8.13).} We give a direct proof also: we have

$$\mu^2+(2\lambda-3)\mu = -[b_1^2b_2][b_1b_2^2]$$

by (6.40) and (6.42),

$$= -[b_1^3b_2^3] + [b_1^3] - 3$$

using (6.14). But

$$[b_1^3b_2^3] = [b_1b_2][b_1^2b_2^2] + [b_1^2b_2] + [b_1b_2^2]$$

using (6.14),

$$= \lambda^3 - 6\lambda + 3$$

by (6.39), (6.40), (6.42), and (6.43); and

$$[b_1^3] = [b_1][b_1^2] - [b_1^2b_2] - [b_1b_2^2],$$

$$= 6\lambda - 11$$

by (6.38), (6.40), (6.41), and (6.42). Equation (6.44) follows.

Now, by Lemmas 6.1, 6.2, 6.3, and 6.4, F^5 and $yf(y^{19})F^5\,\Phi(4)$ are each equal to an expression of the form $S(\lambda) + \mu T(\lambda)$. Since the lowest powers of y in the expansions of F^5, λ, and μ, as power series in y, are -15, -2, and -3, respectively, we assume a form for F^5 with $S(\lambda)$ of degree 7 and $T(\lambda)$ of degree 6. We find the 15 coefficients involved in these two polynomials by comparing coefficients of y^{-15}, y^{-14}, ..., y^{-2}, and y^0, (they appear seriatim), and check the values obtained by comparing coefficients of y^{-1}. The resulting expression for F^5 is found, using (6.44), to be a perfect fifth power, and in fact we have

(6.45) $$F = \mu + 5\lambda + 9,$$

since F, λ, and μ, are real for real y. Similarly, in the case of $yf(y^{19})F^5\Phi(4)$, $S(\lambda)$ and $T(\lambda)$ are of degrees 7 and 5 respectively, and we find the 14 coefficients involved by comparing coefficients of y^{-14}, y^{-13}, ..., y^{-2}, and y^o, (again they appear seriatim), and check the values obtained by comparing coefficients of y^{-1}; we obtain

(6.46) $$\begin{aligned} yf(y^{19})F^5\Phi(4) = &-5\lambda^7+27734\lambda^6-1018027\lambda^5+4089364\lambda^4+10082120\lambda^3-\\ &-61692429\lambda^2+67638607\lambda-319561+\mu(-1155\lambda^5+\\ &+259455\lambda^4-3809331\lambda^3+10287942\lambda^2+2093087\lambda-\\ &-16560108). \end{aligned}$$

The equations (6.44), (6.45), and (6.46), for q = 19, are of course analogous to (AH), equations (8.13), (11.7), and (11.9), for q = 11.

We now write

$$m_1 = yP(1)P(7)P(8), \qquad m_2 = -y^2P(2)P(3)P(5),$$
$$m_3 = P(4)P(6)P(9).$$

Then

(6.47) $$m_1m_2m_3 = -y^3f(y)/f(y^{19});$$

(6.48) to (6.50) $m_1/m_2 = -b_1, \quad m_2/m_3 = -b_2, \quad m_3/m_1 = -b_3.$

Also, in terms of these new functions (6.34) becomes

(6.51) $$m_1m_2^2+m_2m_3^2+m_3m_1^2 = -2y^3f(y)/f(y^{19}),$$

by (6.47) to (6.50). We now prove the following relation

(6.52) $$m_1m_2+m_2m_3+m_3m_1 = yf^2(y)/f^2(y^{19}).$$

Denoting the left-hand side of this equation by X we have

$$X/m_1m_2 = 1-b_3+b_3b_1,$$

$$X/m_2m_3 = 1-b_1+b_1b_2,$$

$$X/m_3m_1 = 1-b_2+b_2b_3,$$

by (6.48) to (6.50). Multiplying together these three equations we obtain

$$X^3/m_1^2m_2^2m_3^2 = -[b_1b_2^2] + 3[b_1b_2] -3[b_1]+6,$$

using (6.14). But by (6.38), (6.39), and (6.42), the right-hand side of this equation is equal to $\mu + 5\lambda + 9$, or by (6.45) to F. Hence

$$X^3 = m_1^2m_2^2m_3^2y^{-3}f^4(y)/f^4(y^{19}) = y^3f^6(y)/f^6(y^{19})$$

using (6.47), and (6.52) follows, since X and f(y) are real for real y. Next we show that

(6.53) $$y^{-2}f(y)(m_1+m_2+m_3)/f(y^{19}) = -\lambda-5.$$

It would be possible to prove this relation by a method similar to that used for (6.52), however the following proof is simpler. Using (6.47) we write (6.52) in the form

$$1/m_1+1/m_2+1/m_3 = -y^{-2}f(y)/f(y^{19}).$$

Then, in view of this relation, the left-hand side of (6.53) is equal to

$$-(m_1+m_2+m_3)(1/m_1+1/m_2+1/m_3),$$
$$= -(m_1/m_2+m_2/m_3+m_3/m_1)-(m_1/m_3+m_2/m_1+m_3/m_2)-3,$$
$$= [b_1]-[b_1b_2]-3$$

by (6.48) to (6.50), and hence is equal to $-\lambda-5$ by (6.38) and (6.39); thus (6.53) is proved. Now, if we write

$$\delta = y^{-2}f(y)(m_1+m_2+m_3)/f(y^{19}), \tag{6.54}$$

then instead of λ and μ we may take δ and F, as new variables, in view of (6.45) and (6.53). In fact from these two relations we have

$$\lambda = -\delta - 5, \tag{6.55}$$

$$\mu = F + 5\delta + 16. \tag{6.56}$$

Substituting for λ and μ from (6.55) and (6.56) in (6.44) we obtain the following relation between δ and F:

$$\delta^3 = F(F + 8\delta + 19). \tag{6.57}$$

Also, substituting for λ and μ in (6.46) we obtain $yf(y^{19})F^5\Phi(4)$ as a polynomial in δ and F. Further since (6.57) is a cubic in δ, this polynomial is equal to another polynomial in δ and F of degree 2 in δ; in fact we have

$$\begin{aligned} yf(y^{19})F^5\Phi(4) &= \delta^2(65.19F^3+1137.19^3F^2+363.19^5F+7.19^7)+ \\ &+\delta(5F^4+2504.19^2F^3+3016.19^4F^2+232.19^6F+19^8)+ \\ &+(2276.19F^4+5431.19^3F^3+717.19^5F^2+24.19^7F+19^8). \end{aligned} \tag{6.58}$$

Making a slight change in notation for convenience, we now re-state (6.47), (6.52), (6.51), (6.54), (6.58), and (6.57), in order, as follows.

THEOREM 6.1 If we write

$$M_1 = y^2 f^3(y^{19})P(1)P(7)P(8)/f^3(y), \quad M_2 = -y^3 f^3(y^{19})P(2)P(3)P(5)/f^3(y),$$

$$M_3 = y f^3(y^{19})P(4)P(6)P(9)/f^3(y),$$

then we have

$$M_1M_2M_3 = -1/F^2,$$

$$M_1M_2+M_2M_3+M_3M_1 = 1/F,$$

$$M_1M_2^2+M_2M_3^2+M_3M_1^2 = -2/F^2,$$

where $F = y^{-3}f^4(y)/f^4(y^{19})$; and if we further write

$$\varepsilon = M_1 + M_2 + M_3,$$

then we have

$$\begin{aligned} y f(y^{19})\Phi(4) = {} & \varepsilon^2(65.19+1137.19^3/F+363.19^5/F^2+7.19^7/F^3)+ \\ & +\varepsilon(5+2504.19^2/F+3016.19^4/F^2+232.19^6/F^3+19^8/F^4)+ \\ & +(2276.19/F+5431.19^3/F^2+717.19^5/F^3+24.19^7/F^4+19^8/F^5), \end{aligned}$$

where, from the last four equations but one, there is the following relation between ε and F

$$\varepsilon^3 = (8\varepsilon + 1)/F + 19/F^2.$$

We conclude this Part by observing that in the last equation but one the only term on the right-hand side without a factor 19 is 5ε, so that, in view of the definitions of ε and M_1 to M_3, we have the following simple congruence, modulo 19,

(6.59) $\Phi(4) \equiv 5f(y^{19})f^{16}(y)\{P(4)P(6)P(9)+yP(1)P(7)P(8)-y^2P(2)P(3)P(5)\}$,

since $f^{19}(y) \equiv f(y^{19})$ (mod. 19).

PART 4

q = 11 throughout this Part

The notation is that of (AH).

7. The following relations, not given in (AH), are needed. They are of a type which has no analogue in the cases

q = 5, 7, and 13.

(7.1) $[rt] = yf^2(y)/f^2(y^{11})$,

(7.2) $y^{-4}f^3(y)[rsu]/f^3(y^{11}) = \lambda + 13$,

(7.3) $y^{-3}f^5(y)[r]/f^5(y^{11}) = -\mu + 6\lambda + 16$,

(7.4) $y^{-7}f^8(y)[rstu]/f^8(y^{11}) = -\lambda^2 - 11\mu + 40\lambda + 7$.

We prove (7.1) as follows. Denoting the left-hand side of the equation by X we have, using the definitions of

α, β, γ, δ, and ϵ,

$$X/rt = \delta\epsilon\alpha\beta + \delta\epsilon\beta + \epsilon\beta + \epsilon + 1,$$

together with the other four equations obtained on interchanging r, s, t, u, v, and α, β, γ, δ, ϵ, cyclically. Multiplying together these five equations we see that $X^5/(rstuv)^2$, i.e. $y^{-10}f^2(y)X^5/f^2(y^{11})$, is equal to a cyclically symmetric polynomial in α, β, γ, δ, and ϵ, with integral

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We may note that in view of the relation (7.1) the factor D in the expressions for the $r_{bc}(d)$ for q = 11 is equal to $y^{-1}f^3(y^{11})/f^2(y)$.

coefficients, i.e. to a linear combination of terms $[\alpha^{\ell}\ \beta^{m}\ \gamma^{n}\ \delta^{p}\ \epsilon^{q}]$, each of which is equal to an expression of the form $Q_1(\lambda) + \mu Q_2(\lambda)$ by (AH), Lemma 9. Using the method employed in (AH) to find the relations (11.7), (11.8), and (11.9), that of comparing coefficients of powers of y in power series expansions, we obtain

$$y^{-10}f^{2}(y)X^{5}/f^{2}(y^{11}) = \lambda\mu - 17\lambda^{2} - 108\mu + 346\lambda - 131.$$

But the right-hand side of this equation is the same as the right-hand side of (AH), equation (11.7). Thus, taking fifth roots, (7.1) follows, since X and f(y) are real for real y.

(7.2), (7.3), and (7.4), may be proved in a manner similar to that used for (7.1), and we omit the details, although it should be pointed out that we now need (AH), equation (8.13) as well as (AH), equation (11.7).

From (7.2), (7.3), and (7.4), together with (AH), equation (11.9), we have the following result:

$$(7.5)\quad \Phi(6) = -11y^{-3}f^{3}(y^{11})[rstu]/f^{4}(y) + 2.11^{2}yf^{6}(y^{11})[r]/f^{7}(y) - 11^{3}f^{8}(y^{11})[rsu]/f^{9}(y) + 11^{4}y^{4}f^{11}(y^{11})/f^{12}(y).$$

We now give conjectural expressions for the other ten $\Phi(s)$ as follows. We write

$\phi(0) = y^{-1}P(4)\,\Phi(0)/P(2)$, $\phi(5) = y^{-1}P(5)\Phi(5)/P(1)$,

$\phi(4) = P(3)\Phi(4)/P(4)$, $\phi(2) = P(1)\Phi(2)/P(2)$,

$\phi(9) = -\,P(5)\Phi(9)/P(3)$, $\phi(1) = -\,P(2)\Phi(1)/P(4)$,

$\phi(7) = -\,yP(1)\Phi(7)/P(5)$, $\phi(8) = -\,P(4)\Phi(8)/P(3)$,

$\phi(10) = -\,P(2)\Phi(10)/P(1)$, $\phi(3) = P(3)\Phi(3)/P(5)$.

Then

THEOREM 7.1 We have

$$\begin{aligned}\phi(0) = {} & (tv)y^{-2}f(y^{11})/f^{2}(y) + \\ & +(-5rstu-53stuv+41tuvr-uvrs+29vrst)y^{-3}f^{3}(y^{11})/f^{4}(y)+ \\ & + 11(-45r+6s+63t-48u+2v)yf^{6}(y^{11})/f^{7}(y)+ \\ & + 11^{2}(-6rsu-32stv+20tur+4uvs+25vrt)f^{8}(y^{11})/f^{9}(y)+ \\ & + 11^{3}(-r/s-3s/t+2t/u-5u/v-4v/r)y^{4}f^{11}(y^{11})/f^{12}(y),\end{aligned}$$

$$\begin{aligned}\phi(5) = {} & (7tv)y^{-2}f(y^{11})/f^{2}(y) + \\ & + (-2rstu-217stuv-10tuvr+15uvrs-6vrst)y^{-3}f^{3}(y^{11})/f^{4}(y)+ \\ & +11(-9r-12s+171t+30u-15v)yf^{6}(y^{11})/f^{7}(y)+ \\ & +11^{2}(6rsu-67stv-20tur+18uvs+19vrt)f^{8}(y^{11})/f^{9}(y)+ \\ & +11^{3}(6r/s+7s/t+10t/u+8u/v+2v/r)y^{4}f^{11}(y^{11})/f^{12}(y),\end{aligned}$$

It is of interest to note that (7.5) is essentially the "right" form for $\Phi(6)$, being equivalent to the equation

$$yf(y^{11})\Phi(6) = 11g_{2} + 2.11^{2}g_{3} + 11^{3}g_{4} + 11^{4}g_{5}$$

given by Atkin [1] in his proof of the Ramanujan congruence for 11^{n}.

and these equations still hold if $\phi(0)$, $\phi(4)$, $\phi(9)$, $\phi(7)$, and $\phi(10)$, or $\phi(5)$, $\phi(2)$, $\phi(1)$, $\phi(8)$, and $\phi(3)$, are interchanged cyclically, so long as r, s, t, u, and v, are also interchanged cyclically.

We hope to prove the above theorem at a later date. The following considerations put its validity beyond any reasonable doubt.

Firstly the definitions of the $\phi(s)$ and the general form of the theorem are analogous for $q = 11$ to the case of $q = 13$ (§.2). Secondly, noting (7.1) and the following relation

$$r/s + s/t + t/u + u/v + v/r = 1$$

which is given in (AH) (page 186) as $[\alpha] = 1$, we point out the correspondence between the expressions for the $\phi(s)$ given in the theorem and the expression for $\Phi(6)$ given by (7.5).

Lastly, in finding the theorem, we assumed that the $\phi(s)$ could be expressed in such a form, and then found <u>and checked</u> the values of the coefficients involved by comparing coefficients of powers of y in power series expansions, in a manner similar to that used for $q = 13$. In fact we made five distinct checks in the case of each of our two sets of coefficients. The powers of 11 which appear in the coefficients serve as an additional check.

PART 5

8. The following theorem is proved in [4] (Theorem 12, pages 95 and 96):

THEOREM 8.1 Suppose that g and h are simple automorphic functions on a group G, such that g has precisely α poles in the fundamental region of G and h has precisely β poles in the fundamental region of G. Then there is a polynomial in u and v, $P(u, v)$, such that $P(g, h) = 0$ and $\deg_u P = \beta$, $\deg_v P = \alpha$.

In our application of this theorem, q is prime and $G = \Gamma_0(q^2)$, where the subgroup $\Gamma_0(n)$ (n a non-zero integer) of the modular group is defined as the group of transformations

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \text{ integral}, \ ad - bc = 1, \ c \equiv 0 \ (\mathrm{mod.}\ n).$$

Also we choose

$$g = g(\tau) = \{\eta(q\tau)/\eta(\tau)\}^s, \quad h = h(\tau) = \eta(q^2\tau)/\eta(\tau),$$

where $\eta(\tau)$, the Dedekind modular form, is defined by

$$\eta(\tau) = \exp(\pi i\tau/12).f(x), \quad x = \exp(2\pi i\tau), \quad \mathrm{im}\,\tau > 0,$$

and $s = s(q)$ is the least positive even integer such that

$$\delta = s(q - 1)/24$$

is integral. Clearly

$$g = x^{\delta} f^s(y)/f^s(x); \qquad h = x^{\Delta} f(y^q)/f(x),$$

where

$$\Delta = (q^2 - 1)/24$$

$\{$and is integral since $(q, 6) = 1\}$. Now, it is shown by Newman in [9] (g and h are precisely as in this paper) that g is an entire modular function* on $\Gamma_0(q)$ $\{$and so on $\Gamma_0(q^2)\}$, h is an entire modular function on $\Gamma_0(q^2)$. Furthermore (see [9]) g has a pole of order δ $\{$in the uniformising variable $z_q = \exp(-2\pi i/q\tau)\}$ at the parabolic vertex $\tau = 0$ and is regular elsewhere throughout the fundamental region of $\Gamma_0(q)$, h has a pole of order Δ at $\tau = 0$ and is regular elsewhere throughout the fundamental region of $\Gamma_0(q^2)$. Since $\Gamma_0(q^2)$ is of index q in $\Gamma_0(q)$, it follows that g has precisely $q\delta$ poles in the fundamental region of $\Gamma_0(q^2)$. Thus by Theorem 8.1 there is a polynomial in u and v, $P(u, v)$, such that $P(g, h) = 0$, $\deg_u P = \Delta$, $\deg_v P = q\delta$.†

From this point onwards q has the value 13. Then $s = 2$, $\delta = 1$, $\Delta = 7$, and we have shown that there is a relation

$$(8.1) \qquad \sum_{\ell=0}^{7} \sum_{m=0}^{13} c(\ell, m) g^{\ell} h^{m} = 0,$$

with coefficients $c(\ell, m)$, not all zero. Replacing g and h by the variables

$$A = A(\tau) = g/h^2 = \{\eta(13\tau)/\eta(169\tau)\}^2 = y^{-1} f^2(y)/f^2(y^{13}),$$

$$b = b(\tau) = 1/h = \eta(\tau)/\eta(169\tau) = x^{-7} f(x)/f(y^{13})$$

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* The term "entire modular function" is not used in [9]; it is defined by Newman in [10] (page 352).

† This result was communicated to us, with the proof, by Dr. Newman.

for convenience, we have $g = A/b^{a}$, $h = 1/b$, and (8.1) becomes

$$(8.2) \qquad \sum_{\ell=0}^{7} \sum_{m=0}^{13} c(\ell, m) A^{\ell} b^{-2\ell-m} = 0.$$

We now examine (for a reason which will appear shortly) the effect of the transformation $\tau \to -1/169\tau$ on equation (8.2). As a special case of the transformation formula (1.4) of [9] we have

$$\eta(-1/\tau) = (-i\tau)^{\frac{1}{2}} \eta(\tau).$$

Whence

$$A(-1/169\tau) = \{\eta(-1/13\tau)/\eta(-1/\tau)\}^{a} = 13\{\eta(13\tau)/\eta(\tau)\}^{a} = 13A/b^{a},$$

$$b(-1/169\tau) = \eta(-1/169\eta)/\eta(-1/\tau) = 13\eta(169\tau)/\eta(\tau) = 13/b,$$

and so, replacing τ by $-1/169\tau$ in (8.2), we obtain

$$(8.3) \qquad \sum_{\ell=0}^{7} \sum_{m=0}^{13} 13^{-\ell-m} c(\ell,m) A^{\ell} b^{m} = 0.^{*}$$

Furthermore, this relation must be irreducible. We prove this in an elementary manner as follows. Consider the more general result

$$(8.4) \qquad \sum_{\ell=0}^{\lambda} \sum_{m=0}^{\mu} d(\ell, m) A^{\ell} b^{m} = 0,$$

as a relation in x. We observe that $A^{\ell}b^{m}$ begins $x^{-13\ell-7m} + \ldots$ and denote by $-t$ the overall lowest power of x in the expansions of those terms $d(\ell, m)A^{\ell}b^{m}$ which actually occur, i.e. for which $d(\ell, m) \neq 0$. Then, since the left-hand side of (8.4) is

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* We may note that $A(\tau) = 13g(-1/169\tau)$ and $b(\tau) = 13h(-1/169\tau)$.

identically zero, x^{-t} must be the initial power of x in the expansions of at least two such terms. In other words there exist distinct integer pairs (ℓ_1, m_1) and (ℓ_2, m_2) such that

$$t = 13\ell_1 + 7m_1 = 13\ell_2 + 7m_2, \quad d(\ell_1, m_1), d(\ell_2, m_2) \neq 0, \tag{8.5}$$
$$0 \leq \ell_1 \leq \lambda, \; 0 \leq \ell_2 \leq \lambda, \; 0 \leq m_1 \leq \mu, \; 0 \leq m_2 \leq \mu.$$

Now $\ell_1 \neq \ell_2$ (otherwise $m_1 = m_2$ also), so that without loss of generality we may take $\ell_1 > \ell_2 (\geq 0)$ (giving $0 \leq m_1 < m_2$). But from (8.5) $\ell_1 \equiv \ell_2$ (mod. 7). Hence $\ell_1 \geq 7$. Similarly $m_2 \geq 13$. Thus, since $d(\ell_1, m_1), d(\ell_2, m_2) \neq 0$, the degrees in A and b of any relation of the form (8.4) must be at least 7 and at least 13 respectively. It follows that (8.3) is irreducible, of degrees 7 and 13 in A and b. Further, taking $\lambda = 7$, $\mu = 13$, so that $\ell_1 \leq 7$, $m_2 \leq 13$, and remembering that, whatever the values of λ and μ, $\ell_1 \geq 7$, $m_2 \geq 13$, we see that in the case of (8.3) $\ell_1 = 7$ and $m_2 = 13$; since $\ell_1 > \ell_2 \geq 0$ and $\ell_1 \equiv \ell_2$ (mod. 7), this means that $\ell_2 = 0$, and similarly $m_1 = 0$, so that $t = 91$. Thus

$$c(7, 0), c(0, 13) \neq 0$$

and $c(\ell, m) = 0$ if $13\ell + 7m > 91$, i.e. $m > 13 - 13\ell/7$, i.e. if

$$m > 13 - 2\ell (0 \leq \ell < 7), \quad m > 0 \; (\ell = 7).$$

It follows that we may rewrite (8.2) and (8.3) respectively as

$$(8.6) \quad c(7, 0)A^7 b^{-14} + \sum_{\ell=0}^{6} \sum_{m=0}^{13-2\ell} c(\ell, m)A^{\ell} b^{-2\ell-m} = 0,$$

$$(8.7) \quad 13^{-7} c(7, 0)A^7 + \sum_{\ell=0}^{6} \sum_{m=0}^{13-2\ell} 13^{-\ell-m} c(\ell, m)A^{\ell} b^{m} = 0.$$

Multiplying (8.6) by $13^{-7} b^{14}$ and writing m for $14-2\ell - m$ in the summation we obtain

$$(8.8) \quad 13^{-7} c(7, 0)A^7 + \sum_{\ell=0}^{6} \sum_{m=1}^{14-2\ell} 13^{-7} c(\ell, 14 - 2\ell - m)A^{\ell} b^{m} = 0.$$

Now in each of equations (8.7) and (8.8) the highest power of A occurring is 7 $\{$since $c(7, 0) \neq 0\}$ and A^7 is present in and only in the initial term. Also, these initial terms are the same and (8.7) is irreducible. It follows, since there can be only one irreducible relation between A and b, that the left-hand sides of the equations must be identical. Hence, equating coefficients of $A^{\ell} b^{m}$, we have

$$c(\ell, 14 - 2\ell - m) = 13^{7-\ell-m} c(\ell, m),$$

and the overlapping of the m-summation ranges means that either side of this equation must be zero whenever $m = 0$, so that in (8.7) $\{$or (8.8)$\}$ we may take $1 \leq m \leq 13 - 2\ell$. Thus, taking $c(0, 7) = -13^7$ (without loss of generality) and writing $d(\ell, m)$ for $13^{-\ell-m} c(\ell, m)$ in (8.7), we arrive at the following.

THEOREM 8.2 Let

$$A = y^{-1}f^{2}(y)/f^{2}(y^{13}), \quad b = x^{-7}f(x)/f(y^{13}).$$

Then there is an irreducible polynomial relation

$$A^{7} = \sum_{\ell=0}^{6} \sum_{m=1}^{13-2\ell} d(\ell, m)A^{\ell}b^{m}$$

with integral coefficients $d(\ell, m)$ which satisfy

$$d(\ell, 14 - 2\ell - m) = 13^{\ell+m-7} d(\ell, m).$$

The last equation of course follows from the corresponding result for the $c(\ell, m)$. The word "integral" is valid as follows. We have seen that, in the polynomial relation of Theorem 8.2, if two or more of the quantities $A^{\ell}b^{m}$ have the same initial power of x, then this power must be -91, and that x^{-91} is the initial power of x in precisely two of these quantities one of which is A^{7}. In other words in the right-hand side no two $A^{\ell}b^{m}$ have the same initial power of x. Thus the $d(\ell, m)$, determined by equating the coefficients of powers of x in the expansions of each side, appear strictly seriatim. Since in the expansion of any $A^{\ell}b^{m}$, including A^{7}, the coefficient of the initial power of x is unity and that of any other power of x integral, it follows that every $d(\ell, m)$ must be integral.

In obtaining the values of the $d(\ell, m)$ only the 28 values such that $\ell + m \geq 7$ need to be calculated; the remainder can then be written down. These 28 values may be obtained by comparing the coefficients of x^{-91}, x^{-90}, ... as far as x^{-49}; 9 of these 43 powers (viz. -78, -71, -65, -64, -58, -57, -52, -51, -50) are not expressible in the form $-13\ell - 7m$ $(0 \leq \ell \leq 6,$ $1 \leq m \leq 13 - 2\ell)$, so that no new $d(\ell, m)$ is obtained, and 6(viz. -72, -66, -60, -59, -54, -53) give, superfluously, $d(\ell, m)$ such that $\ell + m < 7$.*

We find that

$$
\begin{aligned}
A^7 = {} & A^6(11.13b) + \\
& +A^5(36.13b^3-204.13b^2+36.13^2b) + \\
& +A^4(38.13b^5-346.13b^4+126.13^2b^3-346.13^2b^2+38.13^3b) + \\
& +A^3(20.13b^7-222.13b^6+102.13^2b^5-422.13^2b^4+102.13^3b^3 - \\
& \qquad -222.13^3b^2 + 20.13^4b)+ \\
& +A^2(6.13b^9-74.13b^8+38.13^2b^7-184.13^2b^6+56.13^3b^5-184.13^3b^4+ \\
& \qquad +38.13^4b^3-74.13^4b^2+6.13^5b) + \\
& +A(13b^{11}-13^2b^{10}+7.13^2b^9-37.13^2b^8+13^4b^7-51.13^3b^6+13^5b^5- \\
& \qquad -37.13^4b^4+7.13^5b^3-13^6b^2+13^6b)+ \\
& +(b^{13}-13b^{12}+7.13b^{11}-3.13^2b^{10}+15.13^2b^9-5.13^3b^8+19.13^3b^7- \\
& \qquad -5.13^4b^6+15.13^4b^5-3.13^5b^4+7.13^5b^3-13^6b^2+13^6b)
\end{aligned} \tag{8.9}
$$

It turns out then that the $d(\ell, m)$ are all non-zero and that they contain powers of 13 which could not have been anticipated from Theorem 8.2.

---

* In actual fact we examined the coefficients of sufficient of x^{-91}, x^{-90},...,x^{-49}, and of x^{-48}, to enable us to find each of and to make 12 independent checks on the 28 values.

We observe, finally, that while the above result is new, the relation between A and b^2 obtainable by "squaring" (8.9) is given, in effect, by Lehner in [8] (pages 376 and 379).

THE COMPUTER PROGRAMME (SEE PAGE 26)

The programme was written to divide the first of the following two power series by the second

$$u_0 + u_1 x + u_2 x^2 + \dots + u_n x^n + \dots,$$

$$1 + v_1 x + v_2 x^2 + \dots + v_n x^n + \dots,$$

both sets of coefficients being integral. Denoting the quotient power series by

$$w_0 + w_1 x + w_2 x^2 + \dots + w_n x^n + \dots,$$

we have, equating coefficients of powers of x in the first of these series with those in the product of the second and third, and transposing,

$$w_0 = u_0,$$

$$w_1 = u_1 - (v_1 w_0),$$

$$w_2 = u_2 - (v_1 w_1 + v_2 w_0),$$

$$\dots\dots\dots\dots\dots$$

$$w_n = u_n - (v_1 w_{n-1} + v_2 w_{n-2} + \dots + v_n w_0),$$

$$\dots\dots\dots\dots\dots .$$

Thus $w_0, w_1, w_2, \dots, w_n, \dots$ are integral and may be successively found by means of these relations.

We omit the actual programme since its notation is peculiar to "Pegasus" and content ourselves with the following observations. The calculation of the w_n is basically a simple process and indeed the only sub-routines used were a "read" and a "print" routine. As each w_n if found it is both stored and printed; the process terminates at some predetermined value of n (142 in our case), which number forms part of the data. The computer was set to stop immediately if "overflow" occurred at any stage, but in fact this did not happen. The total computer-time taken for the six divisions was well under an hour.

NOTATION

The pages of definition are indicated.

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