Aspects of holographic string theory

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Abstract

In this thesis we study several aspects of gauge/gravity dualities. We start by analyzing the structure of the UV divergences of the Wilson loop for a general gauge/gravity duality. We find that, due to the presence of a nontrivial NSNS B-field and metric, new divergences that cannot be subtracted out by the conventional Legendre transform may arise. We also derive conditions on the B-field and the metric, which when satisfied, the leading UV divergence will become linear, and can be canceled out by choosing the boundary condition of the string appropriately. Our results, together with the result of [15], where the effect of a nontrivial dilaton on the structure of UV divergences in Wilson loop is analyzed, allow us to conclude that Legendre transform is at best capable of canceling the linear UV divergences arising from the area of the worldsheet, but is incapable to handle the divergences associated with the dilaton or the B-field in general. We also solve the conditions for the cancelation of the leading linear divergences generally and find that many well-known supergravity backgrounds are of these kinds, including examples such as the Sakai-Sugimoto QCD model or $\mathcal{N}=1$ duality with Sasaki-Einstein spaces. We also point out that Wilson loop in the Klebanov-Strassler background have a divergence associated with the B-field which cannot be canceled away with the Legendre transform. Moreover, our results indicate that the finiteness of the expectation value of the Wilson loop does not depend on the supersymmetry.

In the next chapter, we propose a definition of the Wilson loop operator in the $\mathcal{N}=1$ β-deformed supersymmetric Yang-Mills theory. Although the operator is not BPS, it has a finite expectation value, result that come from the work in the previous

chapter but also from the field theory calculations at least up to order $(g^2N)^2$. We also derive the general form of the boundary condition satisfied by the dual string worldsheet and find that it is deformed. Finiteness of the expectation value of the Wilson loop, together with some rather remarkable properties of the Lunin-Maldacena metric and the B-field, fixes the boundary condition to be one which is characterized by the vielbein of the deformed supergravity metric. The Wilson loop operators provide natural candidates as dual descriptions to some of the existing D-brane configurations in the Lunin-Maldacena background. We also construct the string dual configuration for a near-1/4 BPS circular Wilson loop operator. The string lies on a deformed three-sphere instead of a two-sphere as in the undeformed case. The expectation value of the Wilson loop operator is computed using the AdS/CFT correspondence and is found to be independent of the deformation.

In the next chapter we focus on a different topic, and find point-like and classical string solutions on the $AdS_5 \times X^5$, where X^5 are the 5-dimensional Sasaki-Einstein manifolds $Y^{p,q}$ and $L^{p,q,r}$. The number of acceptable solutions is limited drastically in order to satisfy the constraints on the parameters and coordinates of the manifolds. We find the energy-spin relations of the above solutions and see that they depend on the parameters of the Sasaki-Einstein manifolds. A discussion on BPS solutions is presented as well.

In the last chapter we present a general discussion on topics which related closely to all previous chapters. Among other things we also give some comments on the form of the Wilson loop operator in the ABJM superconformal Chern-Simons theory.

Declaration

The work in this thesis is based on research carried out at the Centre for Particle Theory and Department of Mathematics, the Department of Mathematical Sciences, University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it all my own work unless referenced to the contrary in the text.

Chapter 1 presents an introduction on aspects of the AdS/CFT and some other gauge/gravity dualities with less supersymmetries. This material is known and can be found in several relevant reviews. Chapter 2 and 3 consists is original work and based on a published work done with collaboration with Prof. Chong-Sun Chu, [63], [24]. Chapter 4 is also original work based on my publication [114]. Chapter 5 consists of a general discussion on topics closely related to chapters 2, 3 and 4. Part of this chapter based on work published in [63].

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Chapter 1

Introduction

1.1 A general description of the ADS/CFT correspondence

The idea that large N gauge theories may have a string theory description was proposed a long time ago [1], but the first concrete proposal was given by Maldacena with the conjectured AdS/CFT correspondence [2–6].

This correspondence relates a conformal field theory in d dimensions and a gravity theory in $d+1$ dimensional Anti de Sitter space. More specifically a type IIB string theory compactified on $AdS_5 \times S^5$ is dual or mathematically equivalent to $\mathcal{N} = 4$ super-Yang-Mills theory. To motivate this relation and make it clearer we can start with a type IIB string theory in flat ten dimensional Minkowski space.

Let us begin by taking N parallel $D3$ branes very close to each other. In this background the perturbative excitations are of two different types. The excitations of the empty space are the closed strings and the excitations of the D3 branes are the open strings which are ending on them. By considering energies lower than the string scale $1/l_s$, only massless string states can be excited and we can write down the effective Lagrangian. The closed string massless states form a gravity supermultiplet in ten dimension and have low energy effective lagrangian of type IIB supergravity. The open string massless states give an $\mathcal{N} = 4$ vector super multiplet in $(3+1)$ dimensions and their low energy effective lagrangian is of $\mathcal{N} = 4$

$U(N)$ super Yang-Mills theory. By taking account of the interactions the complete effective action of the massless modes will be

$$S = S_{bulk} + S_{brane} + S_{int} \ . \tag{1.1}$$

As we said above the S_{bulk} is the action of the ten dimensional supergravity plus some higher derivatives and in the low energy limit $S_{bulk} \rightarrow S_{supergravity}$. Similarly the S_{brane} is the action of the $\mathcal{N} = 4$ super-Yang Mills and in the low energy limit $S_{brane} \rightarrow S_{\mathcal{N}=4}$. The S_{int} term describes the interaction of the bulk modes and the brane modes. But for this term one can see that $S_{int} \propto k \sim g_s \alpha'^2$ where k is the square root of the Newton constant. At the low energy limit $\alpha' \rightarrow 0$ and $k^2 \rightarrow 0$, which states the fact that the gravity and thus supergravity, becomes free at long distances. So, in this limit we get two decoupled systems: the free gravity in the bulk spacetime and the 4 dimensional $\mathcal{N} = 4$ super Yang-Mills gauge theory on the D3 branes which is known to be conformal. Notice also that the Lagrangian (1.1) although it contains only the massless fields, takes into account the effects of integrating out the massive fields.

Now one can consider the same system from a point of view of supergravity description. The D3 brane solution of the supergravity is given by

$$ds^2 = f^{-1/2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + f^{1/2}(dr^2 + r^2 d\Omega_5^2) \ , \tag{1.2}$$

$$F_5 = (1 + *) dt dx_1 dx_2 dx_3 df^{-1} \ , \tag{1.3}$$

$$f = 1 + \frac{R^4}{r^4} \ , \qquad R^4 \equiv 4\pi g_s \alpha'^2 N \ . \tag{1.4}$$

Notice that g_{tt} depends on r, and that means that the energy E_p measured at a point r and the energy E measured at infinity are related by

$$E_p \sim \frac{d}{d\tau} = \frac{1}{\sqrt{-g_{tt}}} \frac{d}{dt} \sim \frac{1}{\sqrt{-g_{tt}}} E = f^{1/4} E \Rightarrow E \sim f^{-1/4} E_p \ . \tag{1.5}$$

So for an observer at infinity, who measures the energy of an object moving from infinity to $r = 0$, would see that the energy of the object reduces. By keeping the energy E_p fixed as $r \rightarrow 0$ the energy observed at infinity E goes to zero, thus we are in the low energy regime. In this picture there are two kinds of low energy excitations from the point of view of the observer at infinity. We can have massless particles propagating in the bulk with very low energies, i.e. big wavelengths, or we can have any other kind of excitation which comes closer to $r = 0$. If one calculate the absorbtion cross section of the waves at large r, can find that it is close to zero since the wavelength of the particle become much more bigger than the typical size of the brane. On the other hand, for the second type of excitations which live close to $r = 0$, can not climb the gravitational potential and escape to infinity.

Hence we see that from the point of view of open strings living on the $D3$ branes and from the point of view of the supergravity description we have two decoupled low energy theories. One of them, the free gravity at large distances is common. Therefore, we can identify the other two low energy systems that appear in both descriptions and we arrive to the conjecture that the $\mathcal{N} = 4$ $U(N)$ super Yang-Mills theory in $3+1$ dimensions is dual to type IIB superstring theory on $AdS_5 \times S^5$.

In the near horizon limit the metric (1.2) becomes

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + R^2\frac{dr^2}{r^2} + R^2 d\Omega_5^2 \ , \tag{1.6}$$

which is the geometry of $AdS_5 \times S^5$. To make things more clear we can change the coordinates $r/R \equiv R/x_0$ and then we get

$$ds^2 = R^2\frac{-dt^2 + d\vec{x}_3^2 + dx_0^2}{x_0^2} + R^2 d\Omega_5^2 \ , \tag{1.7}$$

where the AdS space is in Poincare coordinates and has the same radius R with the sphere.

But, let us explain better how we take the near horizon limit in the supergravity side. In order to be able to consider arbitrary excited string states in the near horizon region, we would like to keep fixed the energies of the objects in string units, and at the same time to take $\alpha' \to 0$, which means that $\alpha' E_p$ stays fixed. For small α' the energy measured from infinity is $E \sim E_p r/\sqrt{\alpha'}$ and to keep it fixed we need to consider r/α' fixed. By defining the new variable $U \equiv r/\alpha'$ the metric (1.6) takes the form

$$ds^2 = \alpha'\left[\frac{U^2}{\sqrt{4\pi g_s N}}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \sqrt{4\pi g_s N}\frac{dU^2}{U^2} + \sqrt{4\pi g_s N}d\Omega_5^2\right] \ . \tag{1.8}$$

This is a metric which will be the starting point in the Wilson loop calculations later.

What we should not forget to mention is the notion of the singletons. If we want to be more precise we should say that the AdS theory in the bulk is describing the $SU(N)$ part of the gauge theory. This is because, a $U(N)$ gauge theory is equivalent to a product of a $U(1)$ vector multiplet and an $SU(N)$ gauge theory up to some $\mathbb{Z}_N$ identifications and in the dual string theory all modes interact with gravity, so there are no decoupled modes. Hence, the excitations we did not consider before, are these that are connecting the throat with the bulk and correspond to the $U(1)$ degrees of freedom. This vector multiplet is related to the center of mass motion of all branes. In case we want to consider a correspondence to the $U(N)$ theory we need to take account of them.

What we should also do, is to verify the basics of the correspondence by checking that the global symmetries are the same in both sides. To start doing that we should know that the parameter N, appears in string theory side as the flux of the five-form Ramond-Ramond field strength on the S^5:

$$\int_{S^5} F_5 = N \ , \tag{1.9}$$

while the g_{YM} is related to g_s and the angle θ to the expectation value of RR scalar χ through

$$\tau \equiv \frac{4\pi i}{g_{YM}^2} + \frac{\theta}{2\pi} = \frac{i}{g_s} + \frac{\chi}{2\pi} \ . \tag{1.10}$$

One can check that the field theory as well the string theory are invariant under an $SL(2,\mathbb{Z})$ acting on τ.

Moreover the IIB string theory on AdS_5 compactified on S^5 has an isometry group $SO(2,4) \times SO(6)$. The $SO(2,4)$ group is the conformal group in $3+1$ dimensions, and this is in agreement with the field theory side since $\mathcal{N} = 4$ super Yang-Mills is known to be conformal. The $SO(6)$ symmetry or the covering group $SU(4)$ since spinors are involving, can be identified with the $SU(4)$ R-symmetry group of the field theory. Hence the isometries of the $AdS_5 \times S^5$ are also symmetries that appear in the field theory side.

In the end of this general discussion on the AdS/CFT we are going to specify the limits of validity of this identification. The supergravity approximation is valid when the curvature of the background is much more larger compared to the string length

$$\frac{R^4}{l_s^4} \sim g_s N \sim g_{YM}^2 N \gg 1 \ . \qquad (1.11)$$

In order to make the string corrections to be small one has to consider $g_s \longrightarrow 0$, which implies that $N \longrightarrow \infty$ in order to keep $\lambda = g_{YM}^2 N$ fixed and large.

On the other hand the SU(N) Yang-Mills gauge theory can be trusted in the perturbative regime when 't Hooft coupling is small

$$g_{YM}^2 N \sim g_s N \sim \frac{R^4}{l_s^4} \ll 1 \qquad (1.12)$$

which is the opposite case of the supergravity approximation of the AdS/CFT. Hence the duality is difficult to be proven since the two different sides are valid in different regions, but at the same time is very useful since we can calculate quantities in different regimes which otherwise would be impossible using only the field theory or the supergravity description.

We can however extend the discussion of the validity of the conjecture more. The conjecture in the limits we described above is in the weakest form, since we did not say anything for the case we go to the full string theory, away from large $g_s N$. A stronger version would be that the AdS/CFT is valid at any $g_s N$, while keeping the limits for N and g_s as above. In this case, the results will agree to α' corrections, but the quantum string corrections governed by g_s may not. The strongest version of the conjecture, which is believed to be true and is the most interesting one, is that the two theories are exactly the same for all values of α' and g_s.

1.2 The Holography

In quantum gravity theory all the physics in a volume can be described in terms of some theory on the boundary. This is the holographic principle statement and can be motivated by the Bekenstein bound. This bound says that the maximum entropy in a region of a space is $S_{max} = Area/G_N$ where the area is that of the boundary of the region. If one apply this relation in the case of the formation of black holes, he can see that the bound is correct otherwise the second law of thermodynamics is violated.

An example of a holography is the AdS/CFT correspondence, since the physics in the bulk of the AdS space can be described by a conformal field theory which lives on the boundary. Here we try to count the number of degrees of freedom in both sides. The area of the boundary of the AdS space is infinite and moreover the field theory has an infinite number of degrees of freedom, so we have to introduce somehow a cutoff on the number of degrees of freedom and then examine how we can obtain similar results in the gravity dual.

To this direction Susskind and Witten [7] noticed that infrared effects in the bulk correspond to ultraviolet effects on the boundary. One way to see that is to start by expressing the metric of the AdS space in Poincare coordinates

$$ds^2 = R^2 \frac{-dt^2 + d\vec{x}^2 + dz^2}{z^2} \ . \tag{1.13}$$

If a wave propagating in this space has a spatial extent λ in the x direction then it should have also in the z direction, since the parameter λ can be eliminated by the $x \longrightarrow \lambda x$ and $z \longrightarrow \lambda z$ transformations. Then we can consider a cutoff

$$z \sim \delta \ , \tag{1.14}$$

where δ is small and corresponds to the ultraviolet cutoff in the field theory. To see that, is convenient to use the metric

$$ds^2 = R^2 \left[- \left(\frac{1+r^2}{1-r^2} \right)^2 dt^2 + \frac{4}{(1-r^2)^2}(dr^2 + r^2 d\Omega^2) \right] \ , \tag{1.15}$$

where the radial position plays the role of some energy scale, since we approach the boundary when we do a conformal transformation that localizes objects in conformal field theory. Since the boundary is at $r = 1$ we can calculate the correlation functions at $r = 1 - \delta$ and then take the limit $\delta \longrightarrow 0$ which corresponds on going to the UV of the field theory. Hence the relation (1.14) is a UV/IR relation.

To proceed to the final stage of finding the equation we are seeking, consider a $U(N)$ gauge theory on a three dimensional sphere with a short distance cutoff δ. So the total degrees of freedom are roughly N^2, which are approximately the number of independent fields, divided by δ^3 which are the partitions of the sphere.

The area of the surface at $r = 1 - \delta$ for $\delta \ll 1$ is

$$\frac{Area}{G_N} = \frac{V_{S^5} R^3 \delta^{-3}}{4G_N} \sim N^2 \delta^{-3} \tag{1.16}$$

since $G_N \sim 1/N^2$. Thus we find that the conformal field theory number of degrees of freedom agrees with the number of physical degrees of freedom.

Very recently there is an attempt to extend the holographic correspondence. In [8] it is claimed that any CFT with a gap and a planar expansion is generated via the AdS/CFT dictionary from a local bulk interaction. The authors arrive to this conjecture by counting arguments on each side and by verifying the conjecture to explicit solutions.

In the next section we are going to introduce the Wilson loop operators and see how one can treat them in the context of the AdS/CFT correspondence.

1.3 Wilson Loops

The Wilson loop is a physical gauge invariant object which can be used to measure the interaction potential between two quarks. The Wilson loop operator can be elementary defined by

$$W(C) = \mathrm{tr}\left[P\exp\left(i\oint_C A\right)\right] , \tag{1.17}$$

where the trace is over some representation of the gauge group and we focus here only on the fundamental representation. The loop C is the loop in four dimensional space were the gauge theory lives.

To understand better the definition, we can consider the path ordered exponential

$$W(y,x;P) = P\ \exp\{i\int_x^y A_\mu(\xi)d\xi^\mu\} \equiv \lim_{n\to\infty}\prod_n e^{iA_\mu(\xi_n^\mu-\xi_{n-1}^\mu)} \tag{1.18}$$

along a curve. By considering the a $U(1)$ gauge field A_μ and a complex scalar field ϕ charged under the $U(1)$ we see that the action of $W(y,x;P)$ to the scalar ϕ, under a gauge transformation gives

$$W(y,x;P)\phi(x) \to e^{i\chi(y)}(W(y,x;P)\phi(x)) , \tag{1.19}$$

which means that the field $\phi(x)$ is parallel transported to the point y. The equation (1.19) follows from the fact that the path ordered exponential we defined is transformed under a gauge transformation $\delta A_\mu = \partial_\mu\chi$, as

$$W(y,x;P) = e^{i\chi(y)}W(y,x;P)e^{-i\chi(x)} . \tag{1.20}$$

However notice that by identifying the x with y and hence producing a closed curve the expression (1.19) is gauge invariant.

For a nonabelian gauge field which is the case of interest, the things are different. Here the gauge transformation is

$$A_\mu \to \Omega(x) A_\mu \Omega^{-1}(x) - i(\partial_\mu \Omega)\Omega^{-1} \ , \tag{1.21}$$

where $\Omega(x) = e^{i\chi(x)}$ is an infinitesimal transformation, for small $\chi(x) = \chi^a T_a$. The gauge transformation is similar to (1.20), hence again $W(y, x; P)$ defines a parallel transport. Since we are in the nonabelian case, when we consider close loop, the path integral is covariant and not gauge invariant. To make it gauge invariant we can take the trace of the object, which is the Wilson loop as defined above.

1.3.1 Wilson loop operator in $N = 4$ Super Yang-Mills

The Wilson loop of the theory $N = 4$ SYM is an operator [11]

$$W_R[C] = \frac{1}{N} \mathrm{Tr}_R \, P \, \exp\left(\oint_C d\tau (iA_\mu \dot{x}^\mu + \varphi_i \dot{y}^i) \right) , \tag{1.22}$$

where A_μ are the gauge fields and φ_i are the six real scalars. The loop C is parametrized by the variables $(x^\mu(\tau), y^i(\tau))$, where $(x^\mu(\tau))$ determines the actual loop in four dimensions, and $(y^i(\tau))$ can be thought of as the extra six coordinates of the ten-dimensional $\mathcal{N} = 1$ super Yang-Mills theory, of which theory is the dimensionally reduced version. R is the representation of the gauge group G. In this chapter we will be interested in the case $G = U(N)$. In (1.22), the coupling to the gauge fields and the scalar fields is controlled by $\dot{x}^\mu$ and $\dot{y}^i$. In particular, Wilson loop operator satisfying the constraint

$$\dot{x}^2 = \dot{y}^2 \tag{1.23}$$

is locally BPS. Moreover it has a finite expectation value.

The derivation of the constraint (1.23) can be achieved using different methods. It can come from the gravity side, by considering the minimal surfaces and appropriate boundary conditions, and we will mention more on this on the next section. Also it can be derived in the gauge theory side, by requiring finite expectation value of the Wilson loop using perturbation theory, or by breaking the gauge group $U(N+1) \to U(N) \times U(1)$ by the Higgs mechanism, and try to get rid off of a term which is not reparametrization invariant after a correlation function calculation. Actually, by using the Higgs mechanism we can derive the whole loop operator.

In order to understand better the derivation we describe first, briefly the Higgs mechanism as done in [13]. In super Yang-Mills theory in 4 dimensions we do not have matter fields to define directly the Wilson loop, so we use the Higgs mechanism and give expectation value to a field where we simultaneously break the $U(N+1) \to U(N) \times U(1)$. This happens by taking the bosonic action for the $U(N+1)$ theory and decomposing the fields as

$$\hat{A}_\mu = \begin{pmatrix} A_\mu & W_\mu \\ W_\mu^\dagger & a_\mu \end{pmatrix}, \qquad \hat{\Phi}_\alpha = \begin{pmatrix} \Phi_\alpha & W_\alpha \\ Y_\alpha & M\theta_\alpha \end{pmatrix}. \tag{1.24}$$

where $\hat{A}_\mu$ and A_μ are the $U(N+1)$ and $U(N)$ gauge fields respectively and similarly with the scalar fields. By calculating a specific correlation function of W's we can find the Wilson loop operator. To reach to the final expression one has to use the reparametrization invariance of an integral and the constraint that must satisfied in order this invariance to hold is the (1.23).

So using the Higgs mechanism we can derive the Wilson loop operator and the Wilson loop constraint in $\mathcal{N}=4$ super Yang-Mills theory. The other way to obtain the constraint (1.23) in field theory is to calculate perturbatively the expectation value of a smooth loop. By keeping the first order terms in g_{YM}^2 and by regularizing the operator with cutoff ϵ, since it is linear divergent we obtain

$$W = 1 + \frac{\lambda}{(2\pi)^2\epsilon} \oint ds|\dot{x}| \left(1 - \frac{\dot{y}^2}{\dot{x}^2}\right) + \text{finite} . \tag{1.25}$$

It is obvious that when the Wilson loop constrain (1.23) holds, the divergent term cancels. There are symmetry arguments based on the dimensional reduction, that any order in the perturbation expansion cancels when the (1.23) satisfied.

Hence we see that when the constraint $\dot{x}^2 = \dot{y}^2$ holds the expectation value of the Wilson loop is finite in $\mathcal{N} = 4$ supersymmetric Yang-Mills. Moreover in this theory, when the constraint satisfied the Wilson loop is locally BPS. Since the scalar field and the gauge boson are in the same supermultiplet, the supersymmetric transformations relates them and it can be seen that the Wilson loop satisfying the constraint is locally half BPS.

Because of this reason it is natural to associate the UV finiteness of the Wilson loop as being due to the existence of local supersymmetry. However later we will prove that this is a simple coincidence and might not hold general for different theories.

1.3.2 Wilson loops in ADS/CFT

The Wilson loop as a boundary operator has his dual description in terms of AdS/CFT. The motivation to what object corresponds comes from QCD, where we expect that the Wilson loop is related to a string running from the quark to antiquark. The quarks are considered very heavy and the distance between them can be considered fixed in time.

The analogous of this picture, would be a string which is on the boundary of AdS. To be more precise [11], start with a gauge group $U(N+1)$ which by giving an expectation value to one of the scalars breaks to $U(N) \times U(1)$. The dual picture is to have a $D3$ brane localized at a point of S^5 and in some radial position U in AdS. From the point of view of $U(N)$ gauge theory we can view the off-diagonal states as massive quarks with mass proportional to the radial direction which also act as a source for the vector fields. By taking infinite mass which means infinite U, we get a non-dynamical source which will correspond to the Wilson loop operator. Hence the string, starts from a $D3$ brane and ends on the boundary of AdS. Since the Wilson loop contour is at the boundary of the AdS where the gauge theory lives, it can be assumed that the contour acts as a boundary for the string. Hence the string world-sheet stretches between the contour C at the boundary to a point at finite distance in AdS.

However, the strings can also be extended on the S^5 parametrized by the coordinates θ^I with $\theta^{I2} = 1$. The θ^I's should be coupled to the six scalars φ^I of the $N=4$ SYM. The scalar fields appear since the string that ending on a p-brane act as a source for the scalar fields and not only of electric field. Additionally, the precise definition of Wilson loop operator which corresponds to the superstring should include also the field theory fermions. However here we ignore the fermion contribution.

The Wilson loop operator in the Euclidean $\mathcal{N}=4$ SYM theory is given by (1.22). To obtain the dual description of the expectation value of the Wilson loop operator, we have to compute the string theory partition function on $AdS_5 \times S^5$ with the condition that the string worldsheet is ending at the loop C which is placed at the boundary of the AdS where the gauge theory lives,

$$\langle W[C]\rangle = \int_{\partial X=C} \mathcal{D}X \, \exp(-\sqrt{\lambda}S[X]) \; , \tag{1.26}$$

for some string action S[X]. When we go to the supergravity regime the leading contribution to this partition function comes from the area of the string worldsheet

$$\langle W\rangle \simeq \exp(-\sqrt{\lambda}A) \; . \tag{1.27}$$

However one must be careful with the boundary condition of the worldsheet ending on the loop, and also to notice that the area defined above is divergent. We expect finite expectation value of the Wilson loop, since a divergence would have implied a mass renormalization on the BPS particle. Moreover, as we said in a previous section the perturbative computation in the field theory shows that the expectation value of the Wilson loop is finite. The solution to this mismatch, comes when we notice the gravity configuration is not fully Dirichlet, so one needs to consider a Legendre transform of the minimal area with respect to the string coordinates obeying Neumann boundary conditions, namely the string coordinates corresponding to θ^I and the radial coordinate u. We are going to present a more extended discussion in the next section since we also need it for later reference.

1.3.3 Wilson Loop and minimal surfaces

The first step should be to define the boundary conditions for the string worldsheet. To do that one should start from the 10 dimensional Yang-Mills theory which live on $D9$ branes. The strings ending on the $D9$ branes obey full Neumann boundary conditions. This means that the Wilson loop in 10 dimensions corresponds to an open string worldsheet with full Dirichlet boundary conditions, since the conditions imposed by the Wilson loop are complementary to the ones imposed on the strings ending on $D9$ branes. To get the 4 dimensional Yang Mills theory we perform a T-duality along the 6 dimensions. The T-Duality changes the 6 Dirichlet boundary conditions for the open string, to 6 Neumann. As a result we will have a string worldsheet with 4 Dirichlet and 6 Neumann boundary conditions.

In order to be able to write these boundary conditions explicitly we have to make a coordinate transformation to the classic metric of the near horizon geometry of N $D3$ branes. The near horizon geometry of N D3-branes is given by the metric

$$\frac{ds^2}{\alpha'} = \frac{U^2}{\sqrt{4\pi g_s N}} \sum_{\mu=0}^{3} dX^\mu dX^\mu + \sqrt{4\pi g_s N}\frac{dU^2}{U^2} + \sqrt{4\pi g_s N} d\Omega_5^2, \tag{1.28}$$

By rescaling the coordinates X^μ by $1/\sqrt{4\pi g_s N}$ and introduce new coordinates $Y^i = \theta^i/U$ $(i = 1, \cdots, 6)$, where θ^i are the coordinates on S^5 and $\theta^2 = 1$, the metric becomes

$$\frac{ds^2}{\alpha'} = \sqrt{4\pi g_s N} Y^{-2} \left(\sum_{\mu=0}^{3} dX^\mu dX^\mu + \sum_{i=1}^{6} dY^i dY^i \right) . \tag{1.29}$$

Where we now have the boundary of the AdS at the boundary of AdS_5 at $Y^i = 0$.

To write down the boundary conditions we are choosing the string world-sheet coordinates to be (σ, τ) such that the boundary is located at $\tau = 0$. It is obvious that the X^μ should be identified 4 dimensional coordinates where the gauge theory lives and hence it is natural to impose Dirichlet conditions on X^μ, so that

$$X^\mu(\sigma_1, 0) = x^\mu(\sigma_1) . \tag{1.30}$$

The remaining 6 string coordinates $Y^i(\sigma^1, \sigma^2)$ should obey Neumann boundary conditions which proposed to be

$$J_1{}^\alpha \partial_\alpha Y^i(\sigma^1, 0) = \dot{y}^i(\sigma^1) , \tag{1.31}$$

where $J_\alpha{}^\beta$ $(\alpha, \beta = 1, 2)$ is the complex structure on the string worldsheet given in terms of the induced metric $g_{\alpha\beta}$,

$$J_\alpha{}^\beta = \frac{1}{\sqrt{g}} g_{\alpha\gamma} \epsilon^{\gamma\beta} . \tag{1.32}$$

The loop constraint is most easily derived using the Hamilton-Jacobi equation. This equation for the area of a minimal surface on a Riemannian manifold with a metric G_{IJ} takes the form,

$$G^{IJ}(\delta A/\delta X^I)(\delta A/\delta X^J) = G_{IJ}\partial_1 X^I \partial_1 X^J . \tag{1.33}$$

After some calculations by using the above boundary conditions we find that there is only one minimal surface that terminates at the boundary of AdS_5 and it requires the constraint (1.23) to hold.

Moreover as we already noticed, we need to use the Legendre transform of the area functional

$$\tilde{A} = A - \oint d\sigma_1 P_i Y^i \,, \tag{1.34}$$

to get rid of the linear divergences since the problem is not fully Dirichlet. The new action has the same equations of motion and still solved by the same minimal surface. After performing some calculations ones get

$$\tilde{A} = \frac{1}{2\pi\epsilon} \oint ds \left(|\dot{x}| - |\dot{y}|\right) + \text{finite} \,, \tag{1.35}$$

which means that when the Wilson loop constraint satisfied we get a finite area for smooth loop.

We will see more details of the above calculations and a more general view of the problem in the next chapters where we will investigate the Wilson loop properties in different gauge/gravity dualities. For now we move on and describe another very interesting aspect of the AdS/CFT duality, which are the semiclassical string solutions.

1.4 Semiclassical String Solutions in AdS/CFT

1.4.1 Brief Overview

When λ approaches infinity the dual gauge theory is strongly coupled and not under best control, which makes the calculations on the gravity side valuable and the correspondence useful. On the other hand, this fact make difficult to realize the concrete connection between the string theory and the gauge theory beyond the supergravity approximation.

Work in this direction was made in [83], where certain gauge theory operators with large R-symmetry charge was proposed to be dual to all type IIB string states in a RR-charged pp-wave background [90], which is a Penrose limit of $AdS_5 \times S^5$ [91]. A generalization came in [92], where with the use of classical solitons and an appropriate quantization of the string theory, it has been shown that there are some semiclassical limits where the string/gauge duality can be reliable. For short strings, where one can approximate AdS_5 by a flat metric near the center, the leading closed string trajectory is reproduced. On the other hand for long strings, the strings feel the metric near the boundary of AdS, and reproduce the logarithmic behavior of the scaling of the wavefunction renormalization and hence the anomalous dimension for operators with "dimension minus spin" equal to two. Based on this work, a further generalization was made by considering multi-spin string states [93,94]. At this time, many papers were published with the aim of finding new close string solutions, for example see [95] and references inside.

Another significant step was the identification of the one loop scalar dilatation operator with the Hamiltonian of integrable $SO(6)$ spin chains [14]. This has brought impressive quantitative agreements between the energy of certain string solutions and the anomalous dimensions for very long operators [96]. Almost one year later, it was shown [97] that the spin chain in a certain subsector, in the limit of a large number of sites, can be described by a sigma model which agrees with the sigma model obtained from the rotating string in the appropriate limit. In the next sections we describe briefly how one can deal with the semiclassical string solution in AdS/CFT.

1.4.2 Introduction

In general to determine the dimensions of local gauge-invariant operators one needs to find the anomalous dimension matrix to all orders in λ and then diagonalize it. One case that this situation is different, is for the BPS operators whose dimension is protected. An other case are the long operators which contain large number of fields under the trace.

For the $\mathcal{N} = 4$ supersymmetric Yang-Mills the closed string states can be classified by the values of the Cartan charges of the symmetry group $SO(2,4) \times SO(6)$. These will be, in the AdS_5 the energy E and the two spins S_1, S_2 and in the S^3 part the three spins J_1, J_2, J_3 and are related each other by the Virasoro constraint. The BPS string states are point-like strings, the near-BPS(BMN) states are nearly pointlike, and the far-from BPS states are represented by extended closed string configurations.

The AdS/CFT duality maps the closed string states in $AdS_5 \times S^5$ to quantum super Yang-Mills states at the boundary, which are single-trace operators. The dictionary says that the energy of the string should be equal to the conformal dimension of the operator, and all the other charges mapped each other trivially, ie. the J charge of the string on the S^5 will be mapped to the $SO(6)$ charge of the operator in field theory.

However, there is a difficulty checking the relation $E = \Delta$ because these two quantities depend on λ and are calculated in completely opposite limits. Generally the perturbative expansion in string side gives $E = \sum_{n=-1}^{\infty} c_n/(\sqrt{\lambda})^n$ while on the SYM side the perturbation theory will give the eigenvalues of the anomalous dimension matrix as $\Delta = \sum_{n=0}^{\infty} a_n \lambda^n$. Of course there is a class of operators, the 1/2 BPS, where the matching can be made easily since their energies and dimensions are protected from corrections. Otherwise the problem is non-trivial and the solution comes by considering the BMN limit.

1.4.3 BMN Limit

One should consider the limit $Q \to \infty$ for the charges Q, and then define a new 'effective' coupling constant as $\tilde{\lambda} = \lambda/Q^2$, and keep it fixed.

In string theory the fraction $Q/\sqrt{\lambda} = 1/\sqrt{\tilde{\lambda}}$ plays the role of a semiclassical parameter and can be taken to be large, implying an energy for these states of the form $E = Q + f(Q, \lambda)$ where $f \to 0$ for $\lambda \to 0$. These semiclassical string states as well as states of small fluctuations near them, should be dual to long SYM operators with large number of fields or derivatives under the trace, and hence have large canonical dimension.

The simplest case is to take a BPS state with large quantum number and consider small fluctuations around it. Then we have a set of near-BPS states characterized by a parameter. So one can consider a pointlike string moving along the geodesic in S^5 (massless geodesic) and take large angular momentum $Q = J$. Then we have $E = J$ and the dual operator in $\mathcal{N} = 4$ SYM is $tr\Phi^j$, for $\Phi = \phi_1 + i\phi_2$. Considering small fluctuations, give small ultrarelativistic closed strings, where their kinetic energy is much larger than their mass. Their dual, is an extension of the previous operators, namely $tr(\Phi^J \ldots)$ where the dots mean a small number of other fields or covariant derivatives. If one calculates the energy of these small fluctuations, finds that is an analytic expression of $\hat{\lambda}$, making possible a direct comparison with the perturbative field theory. It is checked that the terms up to $\hat{\lambda}$ agree precisely with the one and three loop terms in the anomalous dimension of the corresponding operators. However there is no proof yet of why this happens. Moreover, the understanding of why the limits $J \to \infty$ and $\hat{\lambda} \to 0$ and the corresponding ones in field theory, give the same expressions for the energies/dimensions even though in general the limits may not commute, is far for complete so far.

For semiclassical string states with several large spins form non-BPS string states, and their energy E in powers of $\hat{\lambda}$ can be matched with the perturbative expansion of the field theory.

Let us be more precise in that case. The energy of a classical rotating closed string solution is $E = \sqrt{\lambda}\mathcal{E}(w_i)$ with $J_i = \sqrt{\lambda}w_i$ so that $E = E(J_i, \lambda)$. In that case the energy does not have a $\sqrt{\lambda}$ factor in the expansion and is of the form

$$E = J + c_1\frac{\lambda}{J} + c_2\frac{\lambda^2}{J^3} = J(1 + c_1\hat{\lambda} + c_2\hat{\lambda}^2 + \ldots)\ , \tag{1.36}$$

where $J = \sum_{i=1}^{3} J_i$, $\hat{\lambda} = \lambda/J^2$ and c_n are functions of ratios of the spins J_i/J. In the field theory side, one should be able to compare the coefficients c_n to the coefficients in the expression of anomalous dimension of the corresponding SYM operators $\mathrm{tr}(\Phi_1^{J_1}\Phi_2^{J_2}\Phi_3^{J_3} + \ldots)$. However, to compute Δ in general, we need to diagonalize the anomalous dimension matrix defined on a set of long scalar operators. The way to simplify this calculation found in [14], where the one-loop planar dilatation operator in the scalar sector can be interpreted as a Hamiltonian of an integrable $SO(6)$ spin chain and thus can be diagonalized by the Bethe ansatz method. When we find the dimension, one has to expand it, firstly in λ and the in $1/J$. The expansion of the anomalous dimensions should have a similar form to the energy above with the relevant coefficients to agree each other.

In the next section we will describe a deformation of the original AdS/CFT conjecture, since a part of our work is to apply and examine several issues we discussed so far in the marginally deformed AdS/CFT conjecture which involve conformal theories with less supersymmetries.

1.5 Beta deformed theories

Since the original Maldacena conjecture were formulated, there proposed other interesting conjectures. One of them is the Lunin-Maldacena conjecture which obtained from a marginal deformation of the original one.

The beta deformation is a special case of the Leigh-Strassler deformation [21]. In this section we review the β deformed theory and then the gravity dual construction.

1.5.1 Conformal deformations on $\mathcal{N} = 4$ SYM

We are writing the superpotential of $\mathcal{N} = 4$ supersymmetric Yang-Mills in terms of $\mathcal{N} = 1$ superfields,

$$ig\,\mathrm{Tr}(\Phi_1\Phi_2\Phi_3 - \Phi_1\Phi_3\Phi_2)\ . \tag{1.37}$$

Here Φ's are three complex chiral superfields transforming in the adjoint representation of the gauge group and we concentrate on the bosonic part of the action which reads

$$\mathcal{L}_b = \mathrm{Tr}\left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D^\mu\bar{\Phi}^i)(D_\mu\Phi_i) - \frac{g^2}{2}[\Phi_i,\Phi_j][\bar{\Phi}^i,\bar{\Phi}^j] + \frac{g^2}{4}[\Phi_i,\bar{\Phi}^i][\Phi_j,\bar{\Phi}^j]\right) \tag{1.38}$$

where the indices $i, j, k = 1, \ldots, 3$. Moreover, the $SU(N)$ generators are normalized as $\mathrm{Tr}\left(T^aT^b\right) = \delta^{ab}$.

This theory can be marginally deformed to $\mathcal{N} = 1$ super Yang-Mills. Leigh and Strassler [21] found that there is a 3-complex parameter family of marginal deformations which preserves the $\mathcal{N} = 1$ supersymmetry. This family can be written explicitly by replacing the superpotential (1.37) with the

$$ih\,\mathrm{Tr}(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2) + ih'\,\mathrm{Tr}(\Phi_1^3 + \Phi_2^3 + \Phi_3^3)\ , \tag{1.39}$$

where the deformed theory is parametrized by four complex constants h, h', β, τ, with τ being the usual complexified gauge coupling. At classical level the deformation is marginal but at the quantum level is not, since the operators can develop anomalous dimensions. However Leigh and Strassler showed that at the quantum level the deformation is marginal when a constraint

$$\gamma(h, h', \beta, \tau) = 0 \tag{1.40}$$

is satisfied, where γ is the sum of the anomalous dimensions of the three scalar fields. Hence there is a 3-complex dimensional surface defined by (1.40), of conformally invariant $\mathcal{N} = 1$ theories obtained as marginal deformations of $\mathcal{N} = 4$ super Yang-Mills theories.

The deformation used by Lunin-Maldacena is a special case of the previous one which comes by setting $h = g$, $h' = 0$ and β to be real. Then the deformation (1.39) simplified to

$$ig\,\mathrm{Tr}(\Phi_1\Phi_2\Phi_3 - \Phi_1\Phi_3\Phi_2) \rightarrow ig\,\mathrm{Tr}(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2) \ . \tag{1.41}$$

The consequence of setting $h' = 0$ is that the deformed theory preserves a global $U(1) \times U(1)$ symmetry,

$$\begin{aligned} U(1)_1: \qquad & (\Phi_1, \Phi_2, \Phi_3) \rightarrow (\Phi_1, e^{i\varphi_1}\Phi_2, e^{-i\varphi_1}\Phi_3) \\ U(1)_2: \qquad & (\Phi_1, \Phi_2, \Phi_3) \rightarrow (e^{-i\varphi_2}\Phi_1, e^{i\varphi_2}\Phi_2, \Phi_3) \end{aligned} \tag{1.42}$$

which is important in order to find the gravity dual background.

Lunin and Maldacena also noticed that the above deformation can be viewed as arising from a new definition of the product of fields in the $\mathcal{N} = 4$ supersymmetric Yang-Mills Lagrangian

$$f * g \equiv e^{i\pi\beta(Q_1^f Q_2^g - Q_2^f Q_1^g)} fg \tag{1.43}$$

where fg is an ordinary product and $(Q_1^{\text{field}}, Q_2^{\text{field}})$ are the $U(1)_1 \times U(1)_2$ charges of the fields (f or g). The values of the charges for all fields are read from (1.42):

$$\Phi_1 \ : \qquad (Q_1\,, Q_2) = (0\,, -1) \tag{1.44}$$

$$\Phi_2 \ : \qquad (Q_1\,, Q_2) = (1\,, 1) \tag{1.45}$$

$$\Phi_3 \ : \qquad (Q_1\,, Q_2) = (-1\,, 0) \tag{1.46}$$

where of course for the conjugate fields $\bar{\Phi}_i$ the charges are opposite.

Moreover we introduce the β-deformed commutator of fields which will be needed to write the Lagrangian in a more compact form. The commutator defined as

$$[f_i, g_j]_{\beta_{ij}} := e^{i\pi\beta_{ij}} f_i g_j - e^{-i\pi\beta_{ij}} g_j f_i \; . \tag{1.47}$$

and β_{ij} is defined as

$$\beta_{ij} = -\beta_{ji} \, , \quad \beta_{12} = -\beta_{13} = \beta_{23} := \beta \; . \tag{1.48}$$

Using the star-product (1.43), the component Lagrangian of the β-deformed theory (1.41) follows from (1.38) where only the third term in the relevant Lagrangian change, such that in total we have

$$\mathcal{L} = \mathrm{Tr}\left(\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + (D^\mu \bar{\Phi}^i)(D_\mu \Phi_i) - \frac{g^2}{2} [\Phi_i, \Phi_j]_{\beta_{ij}} [\bar{\Phi}^i, \bar{\Phi}^j]_{\beta_{ij}} + \frac{g^2}{4} [\Phi_i, \bar{\Phi}^i][\Phi_j, \bar{\Phi}^j] \right) \, , \tag{1.49}$$

where we have used the definition (1.47). It is also worth mentioning that the β deformation change the fermionic part of the Lagrangian in a similar way, where also one can define star product between the fields.

1.5.2 Supergravity dual solutions generation methods

The solution generation for the dual beta deformed background based on the idea to preserve the global $U(1) \times U(1)$ symmetry. There are many equivalent techniques that one can generate these backgrounds. It seems that the most useful is the TsT transformation, which consists of a T-duality transformation on a coordinate parameterizes one $U(1)$ isometry, followed by a shift of another $U(1)$ coordinate which involves the initial one and where the deformation parameter enters, and finally a T duality follows on the initial coordinate. But let us mention briefly all the generation technics.

The deformed theory is still conformal, so any deformation involved in the backgrounds should be on the $U(1)$ coordinates of the S^5 which correspond to the supersymmetry and not in AdS space which would affect the conformal symmetry. Moreover, the deformation can be always applied to theories with a $U(1) \times U(1)$ global symmetry.

Initially in the paper of Lunin and Maldacena the $SL(2,R)$ transformation was used on the Kähler modulus τ as

$$\tau = B + i\sqrt{g} \rightarrow \tau_\gamma = \frac{\tau}{1+\gamma\tau} \tag{1.50}$$

where g is the metric of the two torus and γ is the deformation parameter in gravity side. This is a solution generating technic and produce an eight dimensional theory on a two torus and is preserving the $U(1) \times U(1)$ global symmetry.

Hence the idea is to write down the B field and the metric of the two torus of the initial background, perform an $SL(2,R)$ transformation as in (1.50) and in the final result identify the real part as the new B field and the imaginary one as the new metric of the torus.

The calculations simplified a lot if one identify the $O(2,2,\mathbb{R})$ [49] which is acting on the background matrix $\tau = g + B$. One has to write down the Γ matrix which contains the deformation parameters according to some rules [54]. Then the new background can be found simply as $E' = g' + B' = E(\Gamma E + I_3)^{-1}$. Moreover, a function G =det$(\Gamma E + I_3)^{-1}$ of the deformed background which depends on the parameter γ, appears due to deformation, as a multiplicative factor to the exponential of dilaton. Using this method we can construct a deformed background in a relatively direct way and we can even extend the method to multiple β deformations.

Another very useful solution generating technic is the TsT transformation, formulated by Frolov [23] and also proposed in the paper of Maldacena. The method is described step by step in Frolov's paper, and using it one can also show that the solutions of the string theory equations of motion in the deformed background are in one-to-one correspondence with those in $AdS_5 \times S^5$ with twisted boundary conditions imposed on the $U(1)$ coordinates. Moreover in this paper were introduced the multi-β deformations, which break all the supersymmetries and hence the resulting β-deformed field theory, which can be constructed by a generalization of the β deformed product (1.43), has no supersymmetry. More explicitly the process consists of a TsT transformation with T-dualities acting on the first angle say ϕ_1 and the shift parameter equal to $\hat{\gamma}^3$ to the torus (ϕ_1, ϕ_2), then a second TsT transformation with the shift parameter equal to $\hat{\gamma}^1$ to the torus (ϕ_2, ϕ_3), and finally a third TsT transformation with the shift parameter equal to $\hat{\gamma}^2$ to the torus (ϕ_3, ϕ_1). The parameters

$\hat{\gamma}$ are directly related to the parameters β of the field theory from (1.52) and the supergravity description is valid in the limit of small curvature $R = (4\pi g_s N)^{1/4} \gg 1$ and

$$R\beta \ll 1, \tag{1.51}$$

with

$$R^2\beta := \hat{\gamma} \quad \text{fixed.} \tag{1.52}$$

Using the β deformations it is useful and convenient to check if certain properties of a theory or the operators are depended on supersymmetry. For example, one can start from $\mathcal{N}=4$ supersymmetric Yang-Mills and investigate some operator properties there; then by reducing to $\mathcal{N}=1$ or to $\mathcal{N}=0$ Yang-Mills by β deformations can see how and if this operator's properties depend on supersymmetry. An example to where this can be applied, is the expectation value of the Wilson loop, and whether or not the UV finiteness of its expectation value depends on the supersymmetry.

A final remark is that when all the deformation parameters are integers, there is no deformation in the field theory, so it is supersymmetric. However the deformation in the dual supergravity background is present. This could be similar to the phenomena of 'supersymmetry without supersymmetry' [51–53] where there exist supersymmetric string vacua for which the corresponding supergravity solution does not have any Killing spinors.

The deformed $AdS_5 \times \tilde{S}^5$ background will be written down in a following chapter where we use it.

1.5.3 Brief Review on the work on the β deformed theories

There is done lot of work in the β deformed theories. Usually the work is in the direction to extend the original AdS/CFT and see how some specific results change in theories with less supersymmetries. Here we are going to present a very short and not complete review with some representative work done in the Lunin-Maldacena conjecture.

In the large N it was noted that there exist many similarities between the deformed and the undeformed theories. This is due to the fact that several times in field theory, the β parameter appear as a factor in exponential divided by N and multiplied by i, and in the large N limit it becomes one and does not affect the results. In [25] it was shown that in perturbation theory there are many similarities between the scattering amplitudes in the (real) β deformed and in the $\mathcal{N}=4$ initial theory. This happens because all amplitudes in the β deformed theory are given by the corresponding $\mathcal{N}=4$ amplitudes multiplied by an overall β dependent phase factor with the form mentioned above.

When one goes to non-perturbative effects, can consider instantons [26]. By considering operators which are the lowest Kaluza-Klein modes on the deformed sphere it is shown that the correlation functions in β deformed $\mathcal{N}=4$ theory is in correspondence with the relevant supergravity results. More precisely the multi-instanton contributions to G_n will reconstruct the moduli forms $f_n(\tau,\bar{t})$ which appear in the effective supergravity action. This agreement is completely non-trivial since the dilaton in the β deformed theory is not anymore constant and the dilaton-axion parameter τ is not equal to the one in the undeformed theory. By doing the calculation one sees that the exponent of the k-instanton action $\exp 2\pi k\tau_0$ becomes $\exp 2\pi k\tau$ which is the desired one.

We can then go to the multi-β deformed non supersymmetric theory [27] and perform the same calculations. These calculations can investigate the role of the supersymmetry in various properties of the theories. So in this paper the authors find that the leading order contributions in Yang-Mills instantons calculated at $g_{YM}^2 N \ll 1$ are in agreement with contributions of D-instantons in the limit $g_{YM}^2 N \gg 1$ even in this non-supersymmetric case. This fact is also true in the original AdS/CFT, and only for the instanton solution, since generally perturbative effects from string theory and gauge theory do not match. This implies the existence of a non-renormalization theorem for the instanton effects [29, 30]. But since the agreement continues to persist in non-supersymmetric case it is a normal clue that the non-renormalization theorem is independent of supersymmetry. However, one should leave open the option that the peculiar structure of the non supersymmetric beta deformed theories somehow can reproduce results that hold in supersymmetric theories. The fact that these results indicate common behavior to the β supersymmetric and non-supersymmetric theories, might not mean that this can be upgraded to general statements.

As one may be able to guess, a lot of work in the Lunin-Maldacena conjecture has been done focusing on the semiclassical string solutions. In [31] some semiclassical string states are compared with scalar operators whose 1-loop anomalous dimensions are described by an integrable spin chain. It is found that the results obtained in the undeformed case can be straightforward generalized to the deformed case, and clear evidence of the existence of integrable structures on the two sides of the duality have been found. In a later paper [32] these calculations are extended in the multi-β deformed non-supersymmetric case. Relative work, where authors calculate the energy of semiclassical string configurations is done in many papers, see for example [33, 34].

Another part of solutions examined in these backgrounds are the giant gravitons [35–37]. The giant gravitons and their stability are examined in the supersymmetric and non-supersymmetric β deformed theory. The interesting part here is that the fields in the DBI and the WZ parts of the action for some $D3$-brane (dual) giant configurations, combine nicely and turn out to give an action independent of the deformation parameter and hence undeformed. For the $D5$-brane dual giants action turn out to be proportional to the inverse of the deformation parameter. In the case discussed here, the branes have a world-volume gauge field strength turned on along the torus, which also has the inverse linear dependence on the deformation parameter. The quantization condition of the $U(1)$ flux requires the deformation parameter to be rational. For these deformation values one can find that in the gauge theory there are additional branches of vacua [38–40] and the gauge theory dual configurations to $D5$-brane dual giants are found to be related to rotating vacuum expectation values in these branches.

Another interesting classical solutions, called magnons have also considered in the β deformed backgrounds where derived the exact dispersion relation for these solutions [28]. By trying to solve the equations of motion one can see that any β-deformation of the S^2-solution of Hoffman and Maldacena will necessarily live on the S^3-sphere. This seems to be a more general fact, that in many cases configurations that live on a S^2 sector and parametrized by two angles of the undeformed sphere, should be modified to be parametrized by three parameters of the sphere in the deformed background.

There are several other papers examining the properties of the β deformed theories. For example in [41] by embedding the spacetime filling $D7$-branes in the deformed background the authors achieve to examine the mesons in these theories and to find the exact mesonic mass spectrum. An other work which generalizes the TsT transformation to $Ts \ldots sT$ is done in [42]. There the multishift deformations considered, and by using similar arguments with Frolov in [23] the authors show that the currents in the deformed and undeformed backgrounds are equal. The resulting background is of course non-supersymmetric in general. Moreover it was shown in [43] that by inspection of the planar diagrams in the β-deformed theory (or even multi-β) shows that the scattering amplitudes to all orders in perturbation theory are the same as in the undeformed $\mathcal{N} = 4$ SYM theory. Furthermore, in the β deformed backgrounds (as well as in non-commutative ones) the basic properties of the quark gluon plasma theories such as: universal ratio between the shear viscosity and the entropy density, jet quenching parameter etc. do not change, and the calculations are almost similar to the undeformed cases. This is due to the fact that most of these properties are most sensitive to the AdS part of the metric than to the S^5 part, which in our case can be deformed. For example in the paper [44] is shown that the jet quenching parameter is modified only in the cases of complex β deformations and this is due to an overall factor in front of the metric.

Other work in beta deformed background can be found in [45–47], or extensions of the β deformations in several other ways [48, 50].

So far we gave a brief overview of the Lunin-Maldacena conjecture. In the next section we are going to introduce a new class of gauge/gravity dualities which also have reduced amount of supersymmetry.

1.6 Sasaki-Einstein dualities

Another famous gauge/gravity correspondence for theories with less supersymmetries, contains a class of backgrounds with at least $\mathcal{N} = 1$ supersymmetry, which are type IIB and with the form $AdS_5 \times X^5$, where X^5 is a Sasaki-Einstein manifold and are dual to superconformal gauge theories called quivers.

A Sasakian manifold is a Riemannian manifold whose metric cone is Kähler. A Kähler manifold is a Hermitian manifold (M, g), say of dimension m, whose Kähler form Ω, defined by

$$\Omega_p(X, Y) = g_p(J_p X, Y) \qquad X, Y \in T_p M, \tag{1.53}$$

where J_p is the almost complex structure, it is closed. This form can be used to prove that a complex manifold is orientable, since the real $2m$ form $\Omega \wedge ... \wedge \Omega$ vanishes nowhere and it serves as a volume element. Additionally, if the manifold M is compact and admits a Ricci flat metric, then its first Chern class must vanish and the manifold is called Calabi-Yau. The Sasaki-Einstein manifolds, are manifolds whose metric cones are Ricci flat and Kähler.

The first non-trivial example in the AdS/CFT correspondence with the use of these manifolds was made in the case of the manifold $T^{1,1}$ [98]. It was noted there that the interactions between the fields can be encoded in quiver diagrams which arising from the low energy excitations of a stack of N $D3$-branes placed at a singular point of the conifold geometry. For this case, which can be described as a coset space manifold of $SU(2) \times SU(2)/U(1)$ and has topology $S^2 \times S^3$, it was proposed the super potential of these theories to be

$$W \propto \varepsilon^{\alpha\beta} \varepsilon^{\dot{\alpha}\dot{\beta}} Tr(A_\alpha B_{\dot{\alpha}} A_\beta B_{\dot{\beta}}) \ , \tag{1.54}$$

since is the only possible choice consistent with the superconformal invariance that preserves a global $SU(2) \times SU(2)$ flavor symmetry. The two $SU(2)$ symmetries act on the two different doublets A_1, A_2 and $B_{\dot{1}}$, $B_{\dot{2}}$. Since $T^{1,1}$ can be seen as a $U(1)$ fibration over the regular Kähler-Einstein manifold $\mathbb{C}P^1 \times \mathbb{C}P^1$, except this $SU(2) \times SU(2)$ symmetry we just mention, should be an additional $U(1)_R$ symmetry which acts to the 4 bifundamental fields with charge $1/2$. Hence the superpotential have a total R-charge 2 and it is a marginal operator.

Significant progress has been made in Sasaki-Einstein backgrounds and their dual field theories, almost a year before the Lunin-Maldacena correspondence was formulated, when it was found that for five-dimensional Sasaki-Einstein manifolds Y, there is an infinite family of inhomogeneous metrics on $Y^{p,q} \cong S^2 \times S^3$, which is characterized by relatively prime positive integers p, q with $0 < q < p$ [99–101]. In this case, there is an effective action of a torus $T^3 \cong U(1)^3$ on the $C(Y^{p,q})$ which preserves the symplectic form on it and the metric, since it is an isometry. The isometry group of these spaces is $SO(3) \times U(1) \times U(1)$ for both p, q odd and $U(2) \times U(1)$ otherwise. In [102] there is an extensive discussion on the geometric features of these manifolds and in a paper [103] that followed some days after, the superconformal quiver gauge theories dual to type IIB string theory on $AdS_5 \times Y^{p,q}$ was proposed.

We are not going to present here an extensive discussion on the conifold field theory. The basics we must know, is that the field theory has a product gauge group $U(N) \times U(N)$, with the matter chiral superfields to live in the bifundamental representations of this gauge group. This means that there are two fields, say A_1 and A_2 transforming in the $(N, \bar{N})$ representation, and two other fields B_1 and B_2 transforming in the $(\bar{N}, N)$ representation.

The spaces $Y^{p,q}$ are of cohomogeneity one, but the correspondence can be generalized to spaces with cohomogeneity two, called $L^{p,q,r}$ spaces [104]. These are characterized by the relative positive coprime integers p, q and r with $0 < p \leq q$, $0 < r < p+q$ and with p, q to be coprime to $s = p+q-r$ and have isometry $U(1) \times U(1) \times U(1)$. The metrics $Y^{p,q}$ are a special case of $L^{p,q,r}$ where $p + q = 2r$.

Moreover, like all theories with at least a $U(1) \times U(1)$ global symmetry, the toric quiver gauge theories and their gravity dual theories admit β deformations [105], which also give space for further analysis. The gravity dual backgrounds can be found by performing a TsT transformation involving two of the angles that parametrize the $U(1)$ directions [23], or by using the T-duality group [49]. An extended discussion for β-deformed Sasaki-Einstein dualities is presented in [106], where giant gravitons are also analyzed.

Hence the use of the Sasaki-Einstein dualities can contribute to the extensive attempt to understand better the AdS/CFT correspondence using theories other than $\mathcal{N} = 4$ supersymmetric Yang-Mills. In a following chapter we investigate semi-classical string solutions in general $Y^{p,q}$ and $L^{p,q,r}$ manifolds. Work in this direction has been done for the very special case of $AdS_5 \times T^{1,1}$ examined in [107–109]. Moreover, a study for the case of BPS massless geodesics and their dual long BPS operators has been done in [111] for $Y^{p,q}$ manifolds and in [112] for $L^{p,q,r}$. Dual giant gravitons have been studied in [110] and recently giant magnons and spiky strings moving in a sector of $AdS_5 \times T^{1,1}$ have been examined in [113].

More details on the metrics of these backgrounds and their general properties will be written in the relevant chapter where we will need them and in the appendix.

We already mention several gauge/gravity dualities with reduced supersymmetry. Moreover, we gave an introduction of how the Wilson loops can be seen in the original Maldacena conjecture. As a next step one can ask several questions; for example when the Legendre transform in other general gauge/gravity dualities can be used to eliminate the linear divergencies in Wilson loop expectation value. Or even to try to propose a Wilson loop operator in other field theories motivated by the gravity results, or to derive it with the field theory methods mentioned above. One more realistic question would be to consider specific Wilson loops in deformed gauge/gravity dualities, eg. generalizations of the 1/4 BPS Wilson loop of $\mathcal{N} = 4$ super Yang-Mills and see if and how the expectation values change in these theories. One other thought would be to find semiclassical string solutions in the Sasaki-Einstein manifolds and investigate the energy-spin relations. In the next chapters we try to address these questions among many other topics.

Chapter 2

UV-divergences of Wilson Loops for Gauge/Gravity Duality

So far there has not been much discussions on the structure of the UV divergences and their cancelation for Wilson loops in more general gauge/gravity correspondence beyond the original $AdS_5 \times S^5$ case. In a general supergravity background where the metric is different from the simple $AdS_5 \times S^5$ one, and where a nontrivial B-field and dilaton could be present, there can be new kind of UV divergences. It is interesting to ask whether the implementation of the Legendre transform can cure all the UV divergences or not. In [15], the effects of a varying dilaton were analyzed by including the Fradkin-Tseytlin term for the dilaton [16]. It was found that new UV-divergent terms proportional to $\sqrt{1/\epsilon}$ and $\log 1/\epsilon$ occurs [1]. Moreover these divergent terms cannot be subtracted away by the application of Legendre transform. A direct subtraction is applied to extract a finite result. However, the subtraction of the log-divergent term is associated with a finite ambiguity and further physical input is needed to fix the supergravity prediction for the expectation value of the Wilson loop. This is unlike the cancelation of the leading linear divergence in the Polyakov action through a quadratic constraint on the loop variables, which

[1]These divergences were computed for the worldsheet associated with the Wilson line operator with fermion bilinear insertion. However it is easy to see that these divergences are common to Wilson loop too.

has a nice geometrical and physical interpretation.

In this chapter, we focus on the gravity dual analysis of the UV divergences from a nontrivial metric and B-field. The main motivation of our work is to provide a general analysis of the kind of UV divergence that may occur in the Wilson loop correspondence and to provide a prescription for their cancelation. We show indeed in general there are new kinds of UV divergences associated with the metric and the B-field that cannot be canceled away by the Legendre transformation. However, when certain asymptotic conditions for the metric and the B-field are satisfied, the leading UV divergence becomes linear and one can cancel out the divergence with the Legendre transform by choosing the open string boundary condition appropriately. Things are different for the B-field. We find that the situation is similar to the dilaton: in general the divergences (if any) associated with the B-field cannot be canceled by the Legendre transformation.

Another motivation of this work is to understand the role of supersymmetry in the holographic correspondence of Wilson loop in a general gauge/gravity duality. In the $\mathcal{N} = 4$ case, the Wilson loop operator (1.22) preserves some amount of local Poincare supersymmetry and is sometimes referred to as "locally BPS". One may wonder if the finiteness of the Wilson loop is related to the preservation of local supersymmetry. Wilson loop operator, being a nonlocal divergent functional, cannot be renormalized by the ordinary R-operation [17] restricted to the local operators. The renormalization properties of Wilson loop with pure glue has been studied in, e.g. [18–20], and it was found that, apart from the conventional wavefunction and coupling renormalization, the only divergence in $W[C]$ is a factor e^{-KL}, where K is a regularization dependent linear divergent constant and L is the length of the loop. This is independent of the form of C and hence the Wilson loop is multiplicative renormalizable. In $\mathcal{N} = 4$ SYM there is no wavefunction renormalization or coupling renormalization, thus the finiteness of the expectation value of the locally BPS Wilson loop means that the multiplicative renormalization factor is finite. As is common in a supersymmetric field theory, it is natural to associate the absence of renormalization of this class of Wilson loop operators with the presence of local supersymmetry, and to suspect that the later is responsible for it. It is thus interesting to consider Wilson loop which preserves less or no local supersymmetry and check if this is correct.

Originally this work motivated from the examination for the Wilson loop correspondence in the Lunin-Maldacena duality [22], which is the topic of the next chapter. For now we mention only that among other results we found there, the absence of the divergence of the minimal surface in supergravity side, is due to some special properties satisfied by the metric and the B-field. Although the operator we proposed is non-BPS, still there is the possibility that the cancelation of the UV divergence is due to the underlying $\mathcal{N} = 1$ supersymmetric dynamics. So a natural question we asked and try to answer in this chapter is in what extension these results hold in general gauge/gravity dualities.

In this chapter, we find that the finiteness of the Wilson loop has nothing to do with supersymmetry at all. As in the $AdS_5 \times S^5$ case, the boundary constraint of the worldsheet has an intermediate interpretation as a constraint on the loop variables of the field theory Wilson loop operator. It is a pure coincidence that this loop constraint also implies a preservation of local Poincare supersymmetry in the $\mathcal{N} = 4$ SYM theory. In general, this condition has nothing to do with preservation of any supersymmetry. In fact, as we will see, the multi-parameters β-deformed supergravity background is an example where the Wilson loop expectation value is finite and where the background is not supersymmetric.

The plan of the chapter is as follows. In section 2.1, we present our analysis of the UV divergence in the supergravity Wilson loop associated with the B-field and the metric. In general the divergence that may arises from the B-field coupling is of a different structure from that in the Legendre transform and so cannot be subtracted away. For background where such divergences are absent, the leading order divergence arises from the area and it can be canceled away using Legendre transform if certain asymptotic conditions are satisfied for the metric and the B-field and if the boundary coordinate of the open string satisfy a certain constraint. As a consistency check, we show that this loop constraint guarantees that the loop equation is satisfied. Subleading divergences could be present in general. We provide a stronger criteria on the supergravity background where the subleading divergences are absent and the Wilson loop is expected to be finite. In section 2.2, we analyze the conditions for the cancelation of leading divergence and show that they can be solved quite generally. Some explicit backgrounds which satisfy these conditions are given as examples. Many of them also satisfy the stronger form of the cancelation conditions and so for these backgrounds, Wilson loop computed using the supergravity description (1.27) is finite. As a final example, we consider the Klebanov-Strassler background and show that the leading linear divergence in the area can be canceled away as usual. However there are subleading divergences of order $(\log\epsilon)^2$ associated with the B-field and this cannot be canceled away with the Legendre transform.

2.1 Structures of UV divergence in the Wilson loop in general supergravity background

2.1.1 Conditions on the supergravity background and the string worldsheet for cancelation of leading order divergence

Consider a general supergravity background. The string worldsheet is sensitive to the metric, NSNS B-field and the dilaton. The structure of UV divergence associated with a varying dilaton has been analyzed in [15] and we will focus on analyzing the effect of a general metric and transverse B field on the UV divergences of the supergravity Wilson loop. Denote the metric in the string frame as

$$ds^2 = G_{\mu\nu}dX^\mu X^\nu + G_{ij}dY^i dY^j, \tag{2.1}$$

where $\mu, \nu = 1, \cdots, m$ denotes the indices of a m-dimensional spacetime; and $i, j = 1, \cdots, n$ denotes the indices of a n-dimensional internal manifold. For this metric to be relevant for a holographic correspondence, we assume that the metric has a (conformal) boundary at $Y = 0$, where $Y := \sqrt{(Y^i)^2}$ is the radial variable and is of length dimension. It is also convenient to introduce the angular variables θ^i where $Y^i = Y\theta^i$ with $\theta^{i2} = 1$. We will assume that in the leading order in Y, the metric have the following asymptotic dependence near the boundary:

$$G_{\mu\nu} = \frac{h_{\mu\nu}}{Y^\alpha} + \cdots, \qquad G_{ij} = \frac{k_{ij}}{Y^\beta} + \cdots, \quad \text{as } Y \to 0 \tag{2.2}$$

for $\alpha, \beta \geq 0$. Here $h_{\mu\nu}, k_{ij}$ are functions of θ^i only and $\cdots$ denotes subleading terms.

Next let us analyze the string boundary condition. Let $(\sigma_1, \sigma_2) = (\tau, \sigma)$ be the worldsheet coordinates. The worldsheet action of the string is

$$I = \int_\Sigma d^2\sigma(\sqrt{\det g} - iB_{ij}\partial_1 Y^i \partial_2 Y^j), \tag{2.3}$$

where $g_{\alpha\beta} = G_{IJ}\partial_\alpha X^I \partial_\beta X^J$ is the induced metric. We note that since the worldsheet is an open one, the B field coupling itself is not invariant under the gauge transformation $\delta B = d\Lambda$. In order to be gauge invariant, the B term should be supplemented with a boundary coupling $\int_{\partial\Sigma} \mathcal{A}$. Without writing this term, we are assuming we are in a gauge where $\mathcal{A} = 0$ and B is the corresponding potential in this gauge. However how to fix this choice of B-field is a subtle issue. Similar subtlety also arise in the computation of Wilson loop expectation value using D3-brane dual where one need to know the form of the RR 4-form potential C_4 used in the WZ coupling of the D3-brane [59]. There a symmetry criteria is used to pick a certain natural form of C_4. We will assume that similar considerations can be applied and the correct form of B field is used in the analysis below.

The equation of motion implies the Hamilton-Jacobi equation

$$G^{ij}(P_i - iB_{ik}\partial_1 Y^k)(P_j - iB_{jl}\partial_1 Y^l) + G^{\mu\nu}P_\mu P_\nu = G_{ij}\partial_1 Y^i \partial_1 Y^j + G_{\mu\nu}\partial_1 X^\mu \partial_1 X^\nu \tag{2.4}$$

where

$$P_i = G_{ij}{J_1}^\beta \partial_\beta Y^j + iB_{ij}\partial_1 Y^j, \quad P_\mu = G_{\mu\nu}{J_1}^\beta \partial_\beta X^\nu. \tag{2.5}$$

are the momentum and

$${J_\alpha}^\beta = \frac{1}{\sqrt{g}} g_{\alpha\gamma}\epsilon^{\gamma\beta} \tag{2.6}$$

is the complex structure ($\alpha, \beta = 1, 2$) on the worldsheet. Substitute the conjugate momentum, we obtain

$$\frac{k_{ij}}{Y^{\beta-\alpha}} {J_1}^\alpha \partial_\alpha Y^i {J_1}^\beta \partial_\beta Y^j + h_{\mu\nu} {J_1}^\alpha \partial_\alpha X^\mu {J_1}^\beta \partial_\beta X^\nu = \frac{k_{ij}}{Y^{\beta-\alpha}} \partial_1 Y^i \partial_1 Y^j + h_{\mu\nu}\partial_1 X^\mu \partial_1 X^\nu \tag{2.7}$$

near $Y = 0$.

One like to know how this equation put constraint on the boundary variables of the theory. To do this we need the boundary conditions for the string coordinates. Suppose that the Wilson loop is parametrized by $(x^\mu(\sigma_1), y^i(\sigma_1))$ and choose the world-sheet coordinates such that the boundary is located at $\sigma_2 = 0$. First we have the Dirichlet boundary condition for the coordinates

$$X^\mu(\sigma_1, 0) = x^\mu(\sigma_1). \tag{2.8}$$

For the remaining coordinates $Y^i(\sigma_1, \sigma_2)$, due to the presence of the B-field, we propose the mixed boundary condition

$$J_1^\alpha \partial_\alpha Y^k(\sigma_1, 0) + iB^k{}_l \partial_1 Y^l(\sigma_1, 0) = E^k{}_l\, \dot{y}^l(\sigma_1), \tag{2.9}$$

where $E^k{}_l$ is some invertible matrix which can depend on Y, θ^i. Its form will be determined later.

For now, focus on the first term on the RHS of (2.7). For a string which terminates at the boundary, it is $Y^i(\sigma_1, 0) = 0$. This would imply also $\partial_1 Y^i(\sigma_1, 0) = 0$. If $\beta - \alpha \leq 0$, then we can get rid of this term immediately. If $\beta - \alpha > 0$, then this term indeterminate. To proceed, we consider a limiting process of letting $Y \to 0$. One can get rid of this term if [2] $\partial_1 Y^i = o(Y^{\frac{\beta-\alpha}{2}})$. As in the $AdS_5 \times S^5$ case, the term $h_{\mu\nu} J_1{}^\alpha \partial_\alpha X^\mu J_1{}^\alpha \partial_\alpha X^\nu$ on the LHS of (2.7) has to vanish near a smooth boundary since otherwise the determinant of the induced metric will blow up and this will cost an infinite area. Therefore we arrive at the condition

$$h_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \frac{1}{Y^{\beta-\alpha}} k_{ij} J_1{}^\alpha \partial_\alpha Y^i J_1{}^\beta \partial_\beta Y^j \tag{2.10}$$

for a worldsheet which terminates on the boundary $Y = 0$. In order for the condition to make sense, one need $J_1{}^\alpha \partial_\alpha Y^i$ to be of the order of $Y^{\frac{\beta-\alpha}{2}}$.

[2]We use the symbol $f = o(g)$ to mean $\lim f/g = 0$, i.e. f tends to infinity slower than g or f tends to zero faster than g. We also use $f = O(g)$ to mean $\lim f/g = k$, $0 \leq k < \infty$. i.e. f tends to infinity not faster than g or f tends to zero not slower than g or f tends to infinity not faster than g.

Before analyzing further the boundary condition, let us turn to an analysis of the divergence in the worldsheet action I and its Legendre transform

$$\tilde{I} = I - \oint d\sigma_1 P_i Y^i. \tag{2.11}$$

As in the $AdS_5 \times S^5$ case, the area A may pick up a divergent contribution from the boundary. This can be seen by writing the metric in the form

$$G_{ij} dY^i dY^j = \frac{k_{ij}\theta^i\theta^j}{Y^\beta} dY^2 + \frac{1}{Y^{\beta-2}} k_{ij} d\theta^i d\theta^j + \frac{2}{Y^{\beta-1}} k_{ij}\theta^i d\theta^j dY + \cdots , \tag{2.12}$$

where $\cdots$ denotes terms coming from the subleading expansion terms in the metric (2.2). Near the boundary, A picks up the dominant contribution

$$\int dY\, d\sigma_1 \frac{\sqrt{k_{ij}\theta^i\theta^j}}{Y^{\frac{\alpha+\beta}{2}}} \sqrt{h_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} + \cdots . \tag{2.13}$$

Since the metric is singular at $Y = 0$, we introduce a regulator $Y = \epsilon$ and evaluate the regularized action for $Y \geq \epsilon$. The divergent part of the area is

$$A = \frac{c}{\epsilon^{(\alpha+\beta)/2-1}} \int d\sigma_1 \sqrt{k_{ij}\theta^i\theta^j} \sqrt{h_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} + \cdots , \tag{2.14}$$

where $c^{-1} := (\alpha + \beta)/2 - 1$ and $\cdots$ denotes possible subleading divergent terms. The B-field coupling can be written as

$$-i \int B_{ij} \partial_1 Y^i \partial_2 Y^j = -i \int \partial_2 (B_{ij} \partial_1 Y^i Y^j) + i \int \partial_2 (B_{ij} \partial_1 Y^i) Y^j. \tag{2.15}$$

With the cutoff $Y = \epsilon$, the first term on the RHS contributes the boundary term

$$\oint d\sigma_1 i B_{ij} Y^i \partial_1 Y^j \big|_{Y=\epsilon}, \tag{2.16}$$

which cancels against the B-dependent term from the Legendre transform

$$P_i Y^i = G_{ij} Y^i J_1{}^\alpha \partial_\alpha Y^j + i B_{ij} Y^i \partial_1 Y^j. \tag{2.17}$$

Therefore we can write

$$\tilde{I} = \tilde{I}_A + \tilde{I}_B, \tag{2.18}$$

where

$$\tilde{I}_A := A - \oint d\sigma_1\; G_{ij} Y^i J_1{}^\alpha \partial_\alpha Y^j, \tag{2.19}$$

$$\tilde{I}_B := i \int d^2\sigma \; \partial_2(B_{ij}\partial_1 Y^i)Y^j, \tag{2.20}$$

are the Legendre transform modified contributions of the area and B-coupling term. There is a reason we group the terms in this way. Note that the term $G_{ij}Y^i J_1{}^\alpha \partial_\alpha Y^j$ is of the order of $1/Y^{\frac{\alpha+\beta}{2}-1}$ and is of precisely the same order of divergence as in A. Note also that A has a dependence in $J_1{}^\alpha \partial_\alpha Y^j$ due to (2.10). Thus it is in principle possible to cancel the divergence in A using the term $\oint G_{ij}Y^i J_1{}^\alpha \partial_\alpha Y^j$. On the other hand, the term $\tilde{I}_B$ depends on $\partial_1 Y^i$. This dependence is different from the other terms. Thus the B-field contribution, if divergent, corresponds to a new divergence with a different type of functional dependence on the variables of the theory.

Let us consider a B-field such that

$$B_{ij}\partial_1 Y^i = o\left(\frac{1}{Y^{\frac{\alpha+\beta}{2}}}\right). \tag{2.21}$$

This implies that the divergence in $\tilde{I}_B$ will be subleading compared to $\tilde{I}_A$. This condition also implies that the second term on the LHS of (2.9) behaves asymptotically as

$$iB^k{}_l \partial_1 Y^l = o(Y^{\frac{\beta-\alpha}{2}}). \tag{2.22}$$

Since $J_1{}^\alpha \partial_\alpha Y^k$ is the order of $Y^{\frac{\beta-\alpha}{2}}$, one can drop the B-term in (2.9). It is convenient to define $E^k{}_l = Y^{\frac{\beta-\alpha}{2}} \Lambda^k{}_l$ and the boundary condition (2.9) can be written as

$$J_1^\alpha \partial_\alpha Y^k(\sigma_1, 0) = Y^{\frac{\beta-\alpha}{2}} \Lambda^k{}_l \, \dot{y}^l(\sigma_1). \tag{2.23}$$

The Hamilton-Jacobi equation (2.10) becomes

$$h_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = k_{ij}\Lambda^i{}_m \Lambda^j{}_n \dot{y}^m \dot{y}^n. \tag{2.24}$$

This condition will play a key role in the cancelation of the divergences in $\tilde{I}_A$. To see this, note that

$$G_{ij}Y^i J_1{}^\alpha \partial_\alpha Y^j = \frac{1}{Y^{\beta-1}} k_{ij}\theta^i\theta^j J_1{}^\alpha \partial_\alpha Y + \frac{1}{Y^{\beta-2}} k_{ij} J_1{}^\alpha \theta^i \partial_\alpha \theta^j + \cdots, \tag{2.25}$$

where $\cdots$ denotes the subleading contribution from the asymptotic expansion of the metric (2.2). This is to be compared with the leading divergence $\sqrt{k_{ij}\theta^i\theta^j}\, \cdot$

$\sqrt{h_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}/Y^{\frac{\alpha+\beta}{2}-1}$ in A, which, using (2.10), can be written as follows:

$$\frac{\sqrt{k_{ij}\theta^i\theta^j}}{Y^{\beta-1}}\sqrt{(J_1{}^\alpha\partial_\alpha Y)^2 k_{ij}\theta^i\theta^j + 2YJ_1{}^\alpha\partial_\alpha Y J_1{}^\beta k_{ij}\theta^i\partial_\beta\theta^j + Y^2 J_1{}^\alpha J_1{}^\beta k_{ij}\partial_\alpha\theta^i\partial_\beta\theta^j}. \tag{2.26}$$

Obviously (2.25) and (2.26) cannot match in general. Doing so will require an extra constraint among the derivatives of θ^i and Y, which, first of all, is not obvious it is in consistent with the relation (2.10). Moreover this relation does not have any obvious physical interpretation in field theory. On the other hand there is a particularly simple set of conditions which guarantee that (2.25) and (2.26) are equal, namely,

$$k_{ij}\theta^i = \theta^j, \tag{2.27}$$

$$\beta - \alpha < 2. \tag{2.28}$$

In fact the first condition implies immediately $k_{ij}\theta^i\partial_\alpha\theta^j = 0$ and hence the vanishing of the second term in (2.25) and (2.26); while the second condition says that the last term in (2.26) is subleading compared to the first term. As a result of (2.21), (2.27) and (2.28), we can write

$$G_{ij}Y^iJ_1{}^\alpha\partial_\alpha Y^j = \frac{1}{Y^{\beta-1}}J_1{}^\alpha\partial_\alpha Y + \cdots = \frac{1}{Y^{\beta-1}}\sqrt{k_{ij}J_1{}^\alpha\partial_\alpha Y^i J_1{}^\alpha\partial_\alpha Y^j} + \cdots \tag{2.29}$$

near $Y = 0$, and the Legendre transform contributes the singular terms

$$\oint d\sigma_1 P_i Y^i = \frac{1}{\epsilon^{(\alpha+\beta)/2-1}}\oint d\sigma_1\sqrt{k_{ij}\Lambda^i{}_m\Lambda^j{}_n\,\dot{y}^m\dot{y}^n} + \cdots, \tag{2.30}$$

where we have used (2.23). Therefore the leading divergence term in (2.14), (2.30) cancels if $c = 1$, i.e. if the leading divergence is linear:

$$\tilde{I}_A = \frac{1}{\epsilon}\oint\left(\sqrt{h_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} - \sqrt{k_{ij}\Lambda^i{}_m\Lambda^j{}_n\,\dot{y}^m\dot{y}^n}\right) + \cdots, \tag{2.31}$$

and if the Hamilton-Jacobi condition (2.24) holds. Here $\cdots$ denotes the subleading contribution from the asymptotic expansion of the metric (2.2). Whether there are further subleading singularity (like, for example, $1/\sqrt{\epsilon}$ or $\log\epsilon$ type) or not will depend on the specific details of the asymptotic form of the background metric. Note that since $\partial_1 Y^i$ is of order Y, the sufficient condition (2.21) for the $\tilde{I}_B$-term to be subleading divergent can be written as

$$B_{ij} = o\left(\frac{1}{Y^{\frac{\alpha+\beta}{2}+1}}\right). \tag{2.32}$$

On the other hand, if

$$B_{ij} = o\left(\frac{1}{Y^2}\right), \tag{2.33}$$

then the $\tilde{I}_B$-term is non-divergent.

Summarizing in a general supergravity background, the B-field coupling in the worldsheet action generically generates a divergence which cannot be canceled with the Legendre transform. A sufficient condition for the B-field contribution to be finite is (2.33). When there is no such divergence, the leading order divergence in the Wilson loop arises from the area and it can be canceled with the application of Legendre transform if the following conditions are satisfied:

1. supergravity background:

   - The supergravity metric takes the asymptotic form (2.2) near the boundary. Moreover

   $$\alpha + \beta = 4, \quad \beta - \alpha < 2. \tag{2.34}$$

   - The boundary metric $h_{\mu\nu}$ is independent of θ^i. The transverse part of the metric satisfies the boundary condition

   $$k_{ij}\theta^i = \theta^j. \tag{2.35}$$

   These conditions are conditions on the background and do not impose any extra constraint on the form of the Wilson loop variables.

2. string worldsheet:

   The boundary constraint (2.24) for the string worldsheet is satisfied.

In general, once the leading UV divergences are canceled, there may be further subleading singularity (like, for example, $1/\sqrt{\epsilon}$ or $\log \epsilon$ type). An extensive analysis of them will need information on the specific details of the asymptotic form of the background metric, the B-field and the dilaton. Generally we don't expect the subleading divergences can be canceled with the application of Legendre transform.

A special situation with no further subleading divergence is if the leading correction term in the asymptotic conditions (2.2) and (2.33) are of at least order Y. We will examine some examples of this kind later.

2.1.2 Comments: boundary constraint as loop constraint

Just as in the original $AdS_5 \times S^5$ case, one would like to interpret the boundary constraint (2.24) for the open string as a condition in the field theory. Since the Wilson loop is specified by the loop variables $\dot{x}^\mu$ and $\dot{y}^i$, and θ^i does not play any role, the loop constraint should not depend on θ^i. This means $h_{\mu\nu}$ should be independent of θ^i. For the same reason, one should choose $\Lambda^k{}_m$ such that $k_{kl}\Lambda^k{}_m\Lambda^l{}_n$ is independent of θ^i. Generally this can be achieved by taking $\Lambda^k{}_m$ of the form

$$\Lambda^k{}_m = \hat{\Lambda}^k{}_l M^l{}_m, \tag{2.36}$$

where $\hat{\Lambda}^k{}_l$ is the vielbein of the metric k_{kl} and $M^l{}_m$ is an invertible matrix which is independent of θ^i but can depends arbitrarily on parameters which have meaning both in supergravity and in the field theory (e.g. the 't Hooft coupling or parameters in the theory such as the β-deformation parameter in the Maldacena-Lunin duality). As a result, the condition (2.24) takes the form

$$h_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = a_{ij}\dot{y}^i\dot{y}^j, \qquad i,j = 1,\cdots,n \tag{2.37}$$

where we have defined $a_{ij} := M^n{}_i M^n{}_j$. In general the form of the matrix a_{ij} will be a function of the couplings of the theory and cannot be fixed from the supergravity analysis alone. In the original $\mathcal{N} = 4$ SYM case, the matrix a_{ij} is given by $a_{ij} = \delta_{ij}$ [13]. We have also computed the constraint for the $\mathcal{N} = 1$ β-deformed superconformal field theory and find $a_{ij} = \delta_{ij}$ up to λ^2 order in perturbation theory [24]. We emphasize that in general the constraint (2.37) has nothing to do with preservation of any supersymmetry. It is a pure coincidence that this loop constraint also implies a preservation of local Poincare supersymmetry in the $\mathcal{N} = 4$ SYM theory.

Let us make a consistency check on the boundary constraint (2.37). In the large N limit of gauge theory, Wilson loop satisfies a closed set of equations called the loop equation [60]. To further justify the supergravity procedure for the computation of the Wilson loop expectation value, one should check that the supergravity ansatz (1.27) satisfies the loop equation [60]. As in the $AdS_5 \times S^5$ case, although the leading linear divergence cancels out when the loop constraint (2.37) is satisfied, the loop variation does not commute with the constraint and so the linear divergence may gives a divergent contribution and violate the loop equation. We show this is not the case.

The loop derivative operator is given by

$$\hat{L} = \lim_{\eta \to 0} \oint ds \int_{s-\eta}^{s+\eta} ds' \left(\frac{\delta^2}{\delta x^\mu(s')\delta x_\mu(s)} - a^{ij} \frac{\delta^2}{\delta y^i(s')\delta y^j(s)} \right). \tag{2.38}$$

That this definition is correct can be confirmed by checking that $\hat{L}\langle W \rangle = 0$ in field theory for the Wilson loop operator (1.22). As usual the loop regulator η has to be taken much smaller than the UV cutoff scale ϵ in order to extract the equation of motion terms. Now acting on the supergravity ansatz (1.27) with the loop operator, we get the leading term in large λ,

$$\lambda \lim_{\eta \to 0} \oint ds \int_{s-\eta}^{s+\eta} ds' \left(\frac{\delta \tilde{I}_A}{\delta x^\mu(s')} \frac{\delta \tilde{I}_A}{\delta x_\mu(s)} - \frac{\delta \tilde{I}_A}{\delta y^i(s')} \frac{\delta \tilde{I}_A}{\delta y_i(s)} \right). \tag{2.39}$$

Let us now extract the divergent contribution from $\tilde{I}_A$ in (2.31). Given the condition (2.37), we can choose a parametrization such that $h_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = a_{ij}\dot{y}^i\dot{y}^j = 1$ and get

$$\hat{L}\langle W \rangle = \frac{\lambda\eta}{\epsilon^2} \oint ds \left(h_{\mu\nu}\ddot{x}_\mu \ddot{x}^\nu - a_{ij}\ddot{y}^i\ddot{y}^j) \right). \tag{2.40}$$

For a smooth loop the terms in the integral are finite. Therefore by taking η going to zero faster than ϵ^2, we find

$$\hat{L}\langle W \rangle = 0 \tag{2.41}$$

and the loop equation is satisfied.

2.2 General solution to the conditions on SUGRA background and examples

2.2.1 General solution to the metric condition

The condition (2.27) on the metric may look a little restrictive at first sight. We show now that it is in fact satisfied by a general class of metric of the form

$$ds^2 = H_1(Y)dT^2 + H_2(Y)d\vec{X}^2 + F(Y)dY^2 + g_{ij}d\theta^i d\theta^j, \tag{2.42}$$

where θ^i, $i, j = 1, \cdots, n$ are the coordinates of the $n-1$ dimensional space X_{n-1}; and the metric g_{ij} is a function of Y^i, e.g. as in the Klebanov-Strassler metric [61]. The metric can be thought as a warped product of the boundary spacetime $(T, \vec{X})$ and the transverse space (Y, θ^i).

Defining $Y^i = Y\theta^i$ and making the coordinate transformation we get

$$g_{ij}d\theta^i d\theta^j = \frac{1}{Y^2}(g_{kl} + g_{ij}\theta^i\theta^j\theta^k\theta^l - g_{il}\theta^i\theta^k - g_{ki}\theta^i\theta^l)dY^l dY^k. \tag{2.43}$$

So our metric become

$$ds^2 = H_1(Y)dT^2 + H_2(Y)d\vec{X}^2 + G_{ij}dY^i dY^j, \tag{2.44}$$

where

$$G_{ij} := F(Y)\theta^i\theta^j + \frac{1}{Y^2}A_{ij}, \tag{2.45}$$

and

$$A_{ij} := g_{ij} + g_{kl}\theta^k\theta^l\theta^i\theta^j - g_{il}\theta^l\theta^j - g_{jl}\theta^l\theta^i. \tag{2.46}$$

The matrix A_{ij} satisfies the following identity,

$$A_{ij}\theta^j = 0, \tag{2.47}$$

and so

$$G_{ij}Y^j = F(Y)Y^i \tag{2.48}$$

Note that (2.48) is of the form of (2.27). Therefore if F behaves as

$$F(Y) = \frac{1}{Y^\beta}, \quad Y \to 0, \tag{2.49}$$

near the boundary, then the condition (2.27) is satisfied. Therefore if also $\alpha + \beta = 4$ and $\beta - \alpha < 2$, then the metric conditions are satisfied.

It is easy to give example where the condition (2.27) is not satisfied. For example, if we have started with a metric with an additional cross-terms $dYd\theta^i$

$$ds^2 = H_1(Y)dT^2 + H_2(Y)d\vec{X}^2 + F(Y)dY^2 + K_i(Y)dYd\theta^i + g_{ij}d\theta^i d\theta^j, \tag{2.50}$$

then under the same coordinate transformation, the additional term takes the form

$$K_i(Y)dYd\theta^i = \frac{1}{Y}\Big(\frac{1}{2}(\theta^k K_l + \theta^l K_k) - (K_i\theta^i)\theta^k\theta^l\Big)dY^k dY^l := \frac{1}{Y}\xi_{kl}dY^k dY^l. \tag{2.51}$$

ξ_{kl} satisfies the following identities

$$\xi_{ij}\theta^j = \frac{1}{2}(K_i - (K_l\theta^l)\theta^i), \qquad \xi_{ij}\theta^i\theta^j = 0, \qquad \xi_{ij}\theta^i\partial\theta^j = \frac{1}{2}K_l\partial\theta^l \tag{2.52}$$

Denote the whole metric as $G_{ij} := H_{ij} + Y^{-1}\xi_{ij}$, where H_{ij} is given by the RHS of (2.45). It is

$$G_{ij}\theta^j = F(Y)\theta^i + \frac{1}{2Y}(K_i - (K_l\theta^l)\theta^i). \tag{2.53}$$

Since the right hand side is generally not proportional to θ^i, the condition (2.27) is no longer satisfied. Note that the cross-terms in (2.50) may be eliminated with a shift of $\theta^i \to \theta^i + a_i(Y)$. However the new θ's will not satisfy the condition $(\theta^i)^2 = 1$ anymore. This is another way to see that the metric conditions are not satisfied.

2.2.2 Examples

Here we examine some backgrounds with known dual field theories, to which our analysis can be applied.

- **Background with $AdS_5 \times X^5$ metric**

This is a standard example. The metric of the space can be written as

$$ds^2 = U^2 \sum_{\mu=0}^{3} dX^\mu dX^\mu + \frac{dU^2}{U^2} + dX_5^2. \tag{2.54}$$

where X^5 is an internal compact space. In this case $\alpha = 2 = \beta$ and the condition (2.34) is satisfied. The linear divergence in A is canceled by the Legendre transform and $\tilde{I}_A$ is finite. Some explicit examples are, $X^5 = S^5$, $\tilde{S}^5$, $\tilde{S}^5_{\gamma_1,\gamma_2,\gamma_3}$, $T^{1,1}$, $Y^{p,q}$, $L^{p,q,r}$, etc., where respectively these spaces are the 5-sphere for the original Maldacena AdS/CFT correspondence, the β-deformed 5-sphere for the Lunin-Maldacena β-deformation [22], the multi-parameter β-deformed sphere, and the Sasaki-Einstein spaces [100–103]. The boundary condition for the string minimal surface is

$$J_1^\alpha \partial_\alpha Y^k(\sigma_1, 0) = \hat{\Lambda}^k{}_m M^m{}_l \dot{y}^l(\sigma_1). \tag{2.55}$$

It is easy to see that $\tilde{I}_B$ is finite for these cases. In the $AdS_5 \times S^5$ case or in the duality with Sasaki-Einstein spaces, there is simply no B-field. In the β-deformation or the multi-parameters β-deformation, the B-field is of the form

$$B = \frac{1}{2} B_{ab} d\phi^a d\phi^b, \tag{2.56}$$

where $\sum(\mu_a)^2 = 1$, ϕ^a $(a = 1, 2, 3)$ are the azimuth angles defined by (3.6) and B_{ab} is a function of μ_a given from (3.20). This form of the B-field respects the symmetries of the β-deformed sphere and we will take it to be the B-field where the string is coupled to. In general one may get a different answer by using a different gauge equivalent B-field. This is similar to the situation discussed in [59] where an open D3-brane is employed to compute the expectation value of Wilson loop in higher representation. There the answer is shown to depend on the gauge choice of the RR 4-form potential C_4 which appears in the Wess-Zumino coupling. A symmetry argument was used to suggest the natural form of the C_4 to be used.

Obviously the B-term in the worldsheet action is finite. For the piece $B_{ij} Y^i \partial_1 Y^j$ in the Legendre transform, since B_{ij} is of order $1/Y^2$, this term is potentially linear divergent. However this does not happen since, as we will show in the next chapter using the relations (3.20), a B-field of the form (2.56) satisfies the condition

$$B_{ij} Y^i = 0 \tag{2.57}$$

exactly [24].

As a result, the piece $B_{ij} Y^i \partial_1 Y^j$ in the Legendre transform is zero. Therefore, there is no divergence associated with the B-field. This can also be checked using (2.20). For example the contributions from B_{12}, B_{15} to $\partial_2(B_{ij}\partial_1 Y^i) Y^j$ is of the form $\sim \frac{Y_4 (Y_2)^2}{Y^4} \partial_1 Y_1 \partial_2 \frac{Y_5}{Y_2}$. This is finite as $Y \to 0$ and so $\tilde{I}_B$ is free from any divergence. Also since there is no subleading correction terms to the metric and the B-field, there is no subleading divergence at all. The Wilson loop is finite.

We remark that the background $AdS_5 \times \tilde{S}^5_{\gamma_1,\gamma_2,\gamma_3}$ for the multi-parameters β-deformation is not supersymmetric, but the Wilson loop expectation value is finite. This clearly shows that supersymmetry or the satisfaction of the BPS condition for the loop is not what is required for the finiteness of Wilson loop expectation value.

- **Supergravity background with asymptotically $AdS_5 \times X^5$ metric**

The first kind of example is given by a finite temperature deformation of any of the metric above. For example for $\mathcal{N}=4$ at finite temperature, the metric is

$$ds^2 = U^2\Big(-(1-\frac{U_T^4}{U^4})dt^2 + (dX^i)^2\Big) + \Big(1-\frac{U_T^4}{U^4}\Big)^{-1}\frac{dU^2}{U^2} + d\Omega_5^2 \tag{2.58}$$

Asymptotically, the metric behaves identically to that of the $AdS_5 \times S^5$ background. So the cancelation of the infinity occurs with the same boundary conditions as in the $AdS_5 \times S^5$ case. Putting a finite temperature deforms the asymptotic form of the metric with power-like terms and this does not introduce any additional subleading singularity.

- **Sakai-Sugimoto QCD model**

The background consists of a dilaton, a RR 3-form potential and the metric [62]

$$\begin{aligned} ds^2 &= \left(\frac{U}{R}\right)^{3/2}(\eta_{\mu\nu}dX^\mu dX^\nu + f(U)dz^2) + \left(\frac{R}{U}\right)^{3/2}\left(\frac{dU^2}{f(U)} + U^2 d\Omega_4^2\right), \\ e^\phi &= g_s\left(\frac{U}{R}\right)^{3/4}, \qquad f(U) = 1 - \frac{U_{KK}^3}{U^3}. \end{aligned} \tag{2.59}$$

Here X^μ ($\mu = 0,1,2,3$) is the spacetime. $z = X^5$ is periodic and describes the compact direction of the D4-brane. $U > U_{KK}$ corresponds to the radial direction transverse to the D4-brane. With the coordinate transformation $Y = R^2/U$, the metric near the boundary $U = \infty$ reads

$$ds^2 = \left(\frac{R}{Y}\right)^{3/2}(\eta_{\mu\nu}dX^\mu dX^\nu + dz^2) + \left(\frac{R}{Y}\right)^{5/2}(dY^2 + Y^2 d\Omega_4^2). \tag{2.60}$$

In this case $\alpha = 3/2, \beta = 5/2$ and the condition (2.34) is satisfied. The leading UV divergence is a linear one and it can be canceled with a choice of the boundary condition for the string minimal surface

$$J_1^\alpha \partial_\alpha Y^k(\sigma_1, 0) = Y^{1/2} M^k{}_l\, \dot{y}^l(\sigma_1). \tag{2.61}$$

The vielbein is trivial since $k_{ij} = \delta_{ij}$ ($i,j = 1,\cdots,5$) for the boundary metric. Including the contribution of the pion field φ_0, we propose the following form of the Wilson loop operator for the Sakai-Sugimoto QCD model,

$$W[C] = \frac{1}{N}\mathrm{Tr}\, P\exp\left(\oint_C d\tau(iA_\mu\dot{x}^\mu + i\varphi_0\dot{z} + \varphi_i\dot{y}^i)\right), \tag{2.62}$$

and the constraint is

$$\dot{x}_\mu{}^2 = \dot{y}_i{}^2 - \dot{z}^2. \tag{2.63}$$

Moreover since the subleading correction terms to the metric is power-like, therefore there is no further subleading UV divergences.

- **Klebanov-Strassler background**

Another example is the Klebanov-Strassler background [61] which describes a warped deformed conifold. In this case the asymptotic behavior of the metric is different from the power ansatz (2.2). However it is not difficult to repeat our analysis above.

The background has a constant dilaton, a RR 2-form, and the metric and B-field

$$\begin{aligned} ds^2 &= h^{-1/2}m^2 dx_m dx_m + h^{1/2}\frac{3^{1/3}}{2^{4/3}}K\left[\frac{1}{3K^3}(d\tau^2+(g_5)^2)+\cosh^2\frac{\tau}{2}[(g_3)^2+(g_4)^2]\right. \\ &\qquad \left. +\sinh^2\frac{\tau}{2}[(g_1)^2+(g_2)^2]\right], \quad (2.64) \\ B &= \frac{g_s M}{2}\left[f g_1\wedge g_2 + k g_3\wedge g_4\right], \quad (2.65) \end{aligned}$$

where g_i is a basis of invariant one-form on $T^{1,1}$

$$\begin{aligned} &g_1 = \frac{1}{\sqrt{2}}(-s_1 d\phi_1 - c_\psi s_2 d\phi_2 + s_\psi d\theta_2), \qquad g_2 = \frac{1}{\sqrt{2}}(d\theta_1 - s_\psi s_2 d\phi_2 - c_\psi d\theta_2), \\ &g_3 = \frac{1}{\sqrt{2}}(-s_1 d\phi_1 + c_\psi s_2 d\phi_2 - s_\psi d\theta_2), \qquad g_4 = \frac{1}{\sqrt{2}}(d\theta_1 + s_\psi s_2 d\phi_2 + c_\psi d\theta_2), \\ &g_5 = d\psi + c_1 d\phi_1 + c_2 d\phi_2. \quad (2.66) \end{aligned}$$

The B-field respects the symmetries of $T^{1,1}$ and we will assume that this is the proper B-field where the string is coupled to. h, K, f and k are some functions of τ whose form can be found in [61]. For our purpose, we record their asymptotic form for large τ,

$$h = e^{-\frac{4\tau}{3}}(4\tau-1)+O(\tau^2 e^{-\frac{10\tau}{3}}), \qquad K = 2^{1/3}e^{-\tau/3}(1-\frac{4\tau}{3}e^{-2\tau})+O(e^{-\frac{2\tau}{3}}), \quad (2.67)$$

$$f \to \frac{\tau-1}{2} - \tau e^{-\tau} + O(\tau e^{-2\tau}), \qquad k \to \frac{\tau-1}{2} + \tau e^{-\tau} + O(\tau e^{-2\tau}). \quad (2.68)$$

In this limit, the metric becomes

$$ds^2 = h^{-1/2}(r)dx^2 + h^{1/2}(r)ds_6^2, \quad (2.69)$$

where the radial variable is defined by

$$r^3 = r_s^3 e^\tau \quad (2.70)$$

for some resolved scale r_s. The warp factor is

$$h = \frac{1}{r^4}\left(\log\frac{r}{r_s} - \frac{1}{4}\right) + o\left(\frac{1}{r^{10}}(\log\frac{r}{r_s})^2\right) \tag{2.71}$$

and ds_6^2 is the cone metric over $T^{1,1}$

$$ds_6^2 = dr^2 + r^2 ds_{T^{1,1}}^2. \tag{2.72}$$

The B-field behaves

$$B = O(\log\frac{r}{r_s})(s_1 d\theta_1 d\phi_1 - s_2 d\theta_2 d\phi_2). \tag{2.73}$$

Putting $Y = 1/r$, we have near the boundary $Y = 0$

$$G_{\mu\nu} = \frac{h_{\mu\nu}}{Y^2\sqrt{\log Y}}\left(1 + O(\frac{1}{\log Y})\right), \tag{2.74}$$

$$G_{ij} = k_{ij}\frac{\sqrt{\log Y}}{Y^2}\left(1 + O(\frac{1}{\log Y})\right), \tag{2.75}$$

and

$$B_{ij} = O(\frac{\log Y}{Y^2}). \tag{2.76}$$

Here $h_{\mu\nu} = \eta_{\mu\nu}$ and k_{ij} can be worked out using the metric of $T^{1,1}$. These details will not be important for us. Note that the metric (2.64) is of the form (2.42) and so it satisfies the condition (2.48).

The Hamilton-Jacobi equation (2.7) is replaced by

$$(\log Y)k_{ij}J_1{}^{\alpha}\partial_\alpha Y^i J_1{}^{\beta}\partial_\beta Y^j + h_{\mu\nu}J_1{}^{\alpha}\partial_\alpha X^\mu J_1{}^{\beta}\partial_\beta X^\nu = (\log Y)k_{ij}\partial_1 Y^i\partial_1 Y^j + h_{\mu\nu}\partial_1 X^\mu \partial_1 X^\nu. \tag{2.77}$$

The string boundary condition is given by the same Dirichlet condition (2.8) and mixed boundary condition (2.9). For a string terminating on the boundary, we have $Y^i(\sigma_1, 0) = 0$. To get rid of the first term on the RHS of (2.77), we require that $\partial_1 Y^i(\sigma_1, 0) = o(1/\sqrt{\log Y})$. This also implies that the B-term in the mixed boundary condition

$$iB^k{}_l\partial_1 Y^l = o(1). \tag{2.78}$$

The Hamilton-Jacobi equation in the limit $Y \to 0$ makes sense if $J_1{}^{\alpha}\partial_\alpha Y^i(\sigma_1, 0)$ is of the order of $1/\sqrt{\log Y}$. Therefore, we can drop the B-term in the mixed boundary condition (2.9) and write

$$J_1{}^{\alpha}\partial_\alpha Y^i(\sigma_1, 0) = \frac{1}{\sqrt{\log Y}}\Lambda^i{}_j\dot{y}^j(\sigma_1). \tag{2.79}$$

The Hamilton-Jacobi equation finally gives

$$h_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = k_{ij}\Lambda^{i}{}_{m}\Lambda^{j}{}_{n}\dot{y}^{m}\dot{y}^{n}. \tag{2.80}$$

Now we examine the structure of UV divergences. For the area part, it is easy to see that we get the same linear divergence (2.31) as before and so $\tilde{I}_A$ is finite if the loop condition (2.80) is satisfied. As for the B-field, since $\partial_2(B_{ij}\partial_1 Y^i)Y^j$ is of the order of $\log Y/Y$, therefore

$$\tilde{I}_B \sim (\log \epsilon)^2. \tag{2.81}$$

This is a new divergence which can not be canceled with the Legendre transform.

2.3 Discussions

In this chapter, we have analyzed of the structure of UV divergences in the Wilson loop from the supergravity point of view by including the effect of a non-trivial metric and a NSNS B-field. We find that in general there can be new divergences which cannot be canceled with the Legendre transform. We also find that when certain conditions are satisfied by the B-field and the metric, the leading UV divergence becomes a linear one and this can be canceled away by choosing the boundary condition of the string appropriately. In general there may still be divergences associated with the B-field, and if they do exist, there is no way to cancel them with the Legendre transform. This is similar to the result of [15] where analyzed the effect of a nontrivial dilaton on the structure of UV divergences in Wilson loop. We conclude that the Legendre transform is at best capable of canceling only linear UV divergences, but is incapable to canceling any subleading divergences which may be present, no matter whether it is due to the dilaton or the NSNS B-field.

Our analysis is performed on the supergravity side. It is an interesting question to check and confirm the form of the loop constraint (2.37) from the field theory perspective. To do this, one need to know the form of the Wilson loop operator that is dual to the supergravity computation. In the simplest case where the field theory has the same number of (adjoint) massless scalar with the dimension of the internal manifold, the natural candidate for the operator is a direct generalization of (1.22). However, the field theory may have different number of scalar fields in general. This is the case, for example, in the quiver theories that are dual to backgrounds with Sasaki-Einstein spaces [100–103]. There the form of the Wilson loop operator is unknown. In this example one may try to exponentiate a product of the bifundamental fields in order to construct the Wilson loop. But since scalar field has dimension one in four dimensions, one needs to compensate the dimension with another dimensional quantity. This is not completely clear what it might be in a conformal theory. It will be interesting to analyze this further and to construct the Wilson loop operator for these theories.

Since we already analyzed of the structure of UV divergences in the Wilson loop for a general gauge/gravity duality, would be very interesting to consider particular dualities, with less supersymmetries and see how these general features can be apply there. Moreover, the derivation of the Wilson loop operator in any theory it is important task on its own. These are some issues that we try to investigate in the β-deformed theories in the next chapter.

Chapter 3

Wilson Loop in β-deformed Theories

In this chapter we are looking at the Wilson loops in the Lunin-Maldacena correspondence. Aspects of the supergravity duals of Wilson loops in the β-deformed SYM theory has been studied before [37, 82]. The work of [37] is a generalization of the fact that the Wilson loops in the symmetric or antisymmetric representation in the original AdS/CFT correspondence can be described in terms of a single D3-brane or D5-brane with worldvolume RR flux. See [59, 64–69] for the 1/2 BPS case and [70] for the D3-brane dual for 1/4 BPS Wilson in symmetric representation. Moreover, analogous to the approach of [77], the supergravity description for certain half BPS Wilson loop has also been obtained [78–81].

However in the relevant works for the β deformed theory, the form of the field theory operators that are dual with the supergravity configurations has not been identified. We note that here the Wilson loop operator (1.22), (1.23) is non-BPS since the gauge bosons and the scalars are in different $\mathcal{N} = 1$ supersymmetry multiplets and so their supersymmetry variations cannot cancel out each other. Conformal supersymmetry also does not mix these multiplets. [1] One can check that even

[1] We note, however, that the Wilson loop operator (1.22) is half BPS if the curve is taken to be a lightlike line (possible in the Lorentzian case) and with $\dot{y}^i = 0$. This operator has no coupling to the scalar fields and is not sensitive to the deformation. In this chapter we focus in the case where the Wilson loop has coupling to the scalar fields since we are interested in the effects of the

by allowing general fermion coupling, it is not possible to construct a supersymmetric Wilson loop. It thus appears impossible to construct a Wilson loop operator which respects some of the $\mathcal{N} = 1$ superconformal symmetries of the β-deformed SYM.

Moreover as we saw before and we point out here again, although the Wilson loop operator (1.22), (1.23) is non-BPS [2], it shares a distinguished property of the locally BPS Wilson loop operator in the $\mathcal{N} = 4$ theory - namely, it has a finite vev. This is not true for a generic non-BPS Wilson loop. To distinguish it from a generic non BPS loop, we call the operator (1.22), (1.23) in the β deformed theories a *near BPS Wilson loop operator* (or maybe another appropriate name would be *like BPS Wilson loop operator*). An analogous example is the BMN operator in the $\mathcal{N} = 4$ SYM theory. The BMN operator is not a BPS operator, but it has a finite anomalous dimensions in a particular double scaling limit [83]. This operator is very interesting and have been studied extensively. We stress that the near BPS Wilson loop operator is not a deformation of a BPS one. The use of "near" is to emphasize that although it is not BPS, but it has finite expectation value just as a BPS Wilson loop operator in the $\mathcal{N} = 4$ theory does.

We propose that dual operators for the D-brane configurations in [37] are given by the *near BPS Wilson loop operators* (1.22), (1.23) whose path is a circle in the x-space and a point in the transverse space. When $\beta \approx 0$, an approximate half of the associated $\mathcal{N} = 4$ supersymmetry is preserved. And one may call this Wilson loop operator *near-half BPS*. We also consider the *near-1/4 BPS* case and construct the dual microscopic string description. The Wilson loop's expectation value is computed using the AdS/CFT correspondence and, as expected, it is finite. Unlike the near-1/2 BPS Wilson loops where the authors find that precisely the same undeformed ansatz has to be taken to construct the desired dual D-branes configurations, here we find that one has to employ a modified ansatz to construct the dual string minimal surface.

β-deformation. We thank Nadav Drukker for a discussion on this.

[2] non-BPS in the local sense. For simplicity, unless otherwise stated, we will omit "local" in the following. The meaning should be clear from the context.

The chapter organized as follows. In section 3.1, we review the Lunin-Maldacena background in its original form where the deformed sphere metric is written in the angular coordinate system. Since the 1/4 BPS Wilson loop necessarily involves a non-trivial coupling to the six real scalars field, for the purpose of using AdS/CFT, it is more convenient to re-express the deformed five-sphere metric and the B-field in terms of the embedding $\mathbf{R}^6$ coordinates. We then show that the metric satisfies the properties we expect according to our discussion in the previous chapter and we point out the remarkable property satisfied by the B-field, which used in chapter 2. In section 3.2 we claim, using some field theory arguments, that the form for the Wilson loop operator could remain undeformed in the β deformed theory. We also support our claims with the appendix A.1, where we try to derive the form of the Wilson loop in the large N limit using the phase factor associated with the infinitely massive quark obtained from the breaking $U(N+1) \to U(N) \times U(1)$. We finish, by giving in section 3.3 the dual string solution in the Lunin-Maldacena background of a near-1/4 BPS circular Wilson loop. Unlike the undeformed case where the string surface is confined on a S^2 in the five-sphere, the string now extends on a deformed $\tilde{S}^3$. The expectation value of the Wilson loop is computed and found to be undeformed. This could mean that the exact expectation value of the Wilson loop is given by the same matrix model as in the undeformed case.

A number of additional appendices are included. In appendix A.2, we collect some of the formula of the deformed metric expressed in the Cartesian coordinates. The Hamiltonian-Jacobi equation in the presence of B-field is also derived in Appendix A.3. Finally, we show that the 1-loop corrected scalar propagator and gauge boson propagator in the Feynman gauge remains equal. Using this result, we show that our near BPS Wilson loop operator is free from UV divergences up to order $(g^2N)^2$.

3.1 The Lunin-Maldacena Background

The type IIB supergravity solution that is dual to the β-deformation of $\mathcal{N}=4$ super Yang Mills was found in [22]. In the string frame it is:

$$ds^2 = R^2 \left[ds^2_{AdS_5} + \sum_i \left(d\mu_i^2 + G\mu_i^2 d\phi_i^2\right) + \hat{\gamma}^2 G\mu_1^2\mu_2^2\mu_3^2 \Big(\sum_i d\phi_i\Big)^2 \right], \tag{3.1a}$$

$$e^{2\phi} = g_s G, \tag{3.1b}$$

$$B = R^2\hat{\gamma}\ G\ (\mu_1^2\mu_2^2 d\phi_1 \wedge d\phi_2 + \mu_2^2\mu_3^2 d\phi_2 \wedge d\phi_3 + \mu_3^2\mu_1^2 d\phi_3 \wedge d\phi_1), \tag{3.1c}$$

$$C_2 = -4R^2\hat{\gamma}\ \omega_1 \wedge (d\phi_1 + d\phi_2 + d\phi_3), \tag{3.1d}$$

$$C_4 = \omega_4 + 4R^4 G\ \omega_1 \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3, \tag{3.1e}$$

where $R^4 = 4\pi g_s N$ (in units where $\alpha' = 1$),

$$G^{-1} = 1 + \hat{\gamma}^2(\mu_1^2\mu_2^2 + \mu_2^2\mu_3^2 + \mu_3^2\mu_1^2). \tag{3.2}$$

The parameter $\hat{\gamma}$ appearing in (3.1) is related to the deformation parameter β of the gauge theory by:

$$\hat{\gamma} = R^2\ \beta. \tag{3.3}$$

The definition of ω_1 and ω_4 can be found in [22].

The background has the $U(1)^3$ symmetry

$$\phi_k \to e^{i\delta_k}\phi_k, \quad \text{for arbitrary constant } \delta_k, \quad (k=1,2,3). \tag{3.4}$$

This is in correspondence with the $U(1)^3$ symmetry of the β-deformed super Yang-Mills theory which is also invariant under the same symmetry. Where, it's action on the scalar components is invariant under

$$\Phi_k \to e^{i\delta_k}\ \Phi_k, \quad \text{for arbitrary constants } \delta_k, \quad (k=1,2,3). \tag{3.5}$$

3.1.1 Properties of the deformed metric and B-field

It is convenient to introduce the Cartesian coordinates where the deformed $\tilde{S}^5$ is embedded

$$\begin{aligned} Y^1 &= Y\theta^1 = Y\mu_1\cos\phi_1, & Y^4 &= Y\theta^4 = Y\mu_1\sin\phi_1, \\ Y^2 &= Y\theta^2 = Y\mu_2\cos\phi_2, & Y^5 &= Y\theta^5 = Y\mu_2\sin\phi_2, \\ Y^3 &= Y\theta^3 = Y\mu_3\cos\phi_3, & Y^6 &= Y\theta^6 = Y\mu_3\sin\phi_3. \end{aligned} \tag{3.6}$$

Here $Y^2 = (Y^i)^2$ and $(\theta^i)^2 = 1$. With respect to this basis, the symmetry (3.4) is translated to

$$Y_1 + iY_4 \to e^{i\delta_1}(Y_1 + iY_4), \qquad Y_2 + iY_5 \to e^{i\delta_2}(Y_2 + iY_5), \qquad Y_3 + iY_6 \to e^{i\delta_3}(Y_3 + iY_6). \tag{3.7}$$

The metric (3.1a) becomes

$$ds^2 = \frac{R^2}{Y^2}\left(\sum_{\mu=0}^{3} dX^\mu dX^\mu + dY^2 + Y^2 d\tilde{\Omega}_5^2\right) = \frac{R^2}{Y^2}\left(\sum_{\mu=0}^{3} dX^\mu dX^\mu + \sum_{i=1}^{6} G_{ij} dY^i dY^j\right), \tag{3.8}$$

where G_{ij} is the embedding metric of the deformed $\tilde{S}^5$. The diagonal terms of the metric are

$$G_{ii} = \frac{1}{Y^2}(\cos^2\phi_i + GM_i \sin^2\phi_i), \quad G_{i+3\,i+3} = \frac{1}{Y^2}(\sin^2\phi_i + GM_i \cos^2\phi_i),\ i = 1,2,3 \tag{3.9}$$

where, for convenience, we have defined the new quantities

$$M_1 = 1 + \hat{\gamma}^2\mu_2^2\mu_3^2\,, \quad M_2 = 1 + \hat{\gamma}^2\mu_1^2\mu_3^2\,, \quad M_3 = 1 + \hat{\gamma}^2\mu_1^2\mu_2^2\,. \tag{3.10}$$

The non-diagonal elements are

$$\begin{aligned}
G_{12} &= \frac{\hat{\gamma}^2}{Y^2} G\mu_1\mu_2\mu_3^2 \sin\phi_1 \sin\phi_2, & G_{13} &= \frac{\hat{\gamma}^2}{Y^2} G\mu_1\mu_2^2\mu_3 \sin\phi_1 \sin\phi_3,\\
G_{15} &= -\frac{\hat{\gamma}^2}{Y^2} G\mu_1\mu_2\mu_3^2 \sin\phi_1 \cos\phi_2, & G_{16} &= -\frac{\hat{\gamma}^2}{Y^2} G\mu_1\mu_2^2\mu_3 \sin\phi_1 \cos\phi_3,\\
G_{23} &= \frac{\hat{\gamma}^2}{Y^2} G\mu_1^2\mu_2\mu_3 \sin\phi_2 \sin\phi_3, & G_{26} &= -\frac{\hat{\gamma}^2}{Y^2} G\mu_1^2\mu_2\mu_3 \sin\phi_2 \cos\phi_3.
\end{aligned} \tag{3.11}$$

The elements G_{45}, G_{46}, G_{24}, G_{34}, G_{56}, G_{35} differ respectively from G_{12}, G_{13}, G_{15}, G_{16}, G_{23}, G_{26} by switching all the cos and sin in each case. The remaining elements are

$$\begin{aligned}
G_{14} &= \frac{1}{2Y^2}(1 - GM_1)\sin 2\phi_1, \quad G_{25} = \frac{1}{2Y^2}(1 - GM_2)\sin 2\phi_2,\\
G_{36} &= \frac{1}{2Y^2}(1 - GM_3)\sin 2\phi_3.
\end{aligned} \tag{3.12}$$

In the above we have given the metric elements as a function of the angles. For convenience, we have also recorded in the appendix A.2 the expressions of the metric elements as a function of Y^i.

Even if as expected this deformed metric is not conformally flat, it displays some remarkable symmetries. One can check that the following identity is satisfied

$$Y^i G_{ij} Y^j = 1 \ , \tag{3.13}$$

which leads to

$$\theta^i g_{ij} \theta^j = 1 \ , \tag{3.14}$$

where we have defined

$$g_{ij} := Y^2 G_{ij} \ . \tag{3.15}$$

The g_{ij} is finite at the boundary as can be easily seen from (3.9), (3.11), (3.12). Another interesting property of the deformed metric is that

$$\theta^i (\partial_\alpha g_{ij}) \theta^j = 0 \ , \tag{3.16}$$

where ∂_α is an arbitrary derivative. Also we have

$$(\partial_\alpha \theta^i) g_{ij} \theta^j = 0 \ , \tag{3.17}$$

which follows immediately from (3.14), (3.16).

The B-field also satisfies an interesting identity. Writing the B-field as

$$B = R^2 \hat{\gamma} \ G \ (b_1 + b_2 + b_3) \ , \tag{3.18}$$

where

$$b_{\mathtt{m}} := \frac{1}{2} \epsilon_{\mathtt{mnk}} \mu_{\mathtt{n}}^2 \mu_{\mathtt{k}}^2 d\phi_{\mathtt{n}} \wedge d\phi_{\mathtt{k}}, \quad \mathtt{m}, \mathtt{n}, \mathtt{k} = 1, 2, 3. \tag{3.19}$$

It is

$$\begin{aligned}
b_3 &= Y^{-4}(Y^4 Y^5 dY^1 \wedge dY^2 + Y^1 Y^2 dY^4 \wedge dY^5 + Y^1 Y^5 dY^2 \wedge dY^4 - Y^2 Y^4 dY^1 \wedge dY^5), \\
b_2 &= -Y^{-4}(Y^4 Y^6 dY^1 \wedge dY^3 + Y^1 Y^3 dY^4 \wedge dY^6 + Y^1 Y^6 dY^3 \wedge dY^4 - Y^3 Y^4 dY^1 \wedge dY^6), \\
b_1 &= Y^{-4}(Y^5 Y^6 dY^2 \wedge dY^3 + Y^2 Y^3 dY^5 \wedge dY^6 + Y^2 Y^6 dY^3 \wedge dY^5 - Y^3 Y^5 dY^2 \wedge dY^6).
\end{aligned} \tag{3.20}$$

It is easy to check that the B-field gives the identity

$$B_{ik} \partial_\sigma Y^k Y^i = 0 \ . \tag{3.21}$$

In fact the stronger form

$$b_{\mathfrak{n}ik}\partial_\sigma Y^k Y^i = 0 \ , \tag{3.22}$$

holds for the individual pieces composing the B-field.

In the following section we use the results of the previous chapter, and the equations (3.14), (3.17), (3.21) of the metric and the B-field to study briefly the deformed boundary condition for the macroscopic string ending on the Wilson loop. Moreover we propose that the Wilson loop operator in the β deformed theories could remain undeformed.

3.2 Near-BPS Wilson Loop and Deformed Boundary Condition

We start out by recalling the arguments for the form of the Wilson loop operator (1.22) and the constraint (1.23) in the original undeformed $\mathcal{N} = 4$ case. Firstly, one can examine the unbroken supersymmetry on the Wilson loop operators [13,84,85]. The Wilson loop operator is locally supersymmetric if the constraint (1.23) is satisfied. A second way is from perturbation theory. One finds that the above constraint must be satisfied in order for the UV-divergence to cancel out in the expectation value of W. This is easy to check in the leading order in $g^2N := \lambda$ and can be extended to arbitrary higher orders in λ using arguments based on the present $SO(6)$ symmetry [13]. Another way to derive the Wilson loop operator is by decomposing the gauge group $U(N+1) \to U(N) \times U(1)$ in order to use the W-bosons, that appear from this breaking [11,13]. Finally, the constraint can also be understood from the dual supergravity point of view [13]. Imposing appropriate boundary conditions and using the Hamilton-Jacobi equation for the minimal surface, one can show for a smooth loop, that the minimal surface ends on the AdS boundary if and only if the loop variables obey the constraint $\dot{x}^2 = \dot{y}^2$. We remark that the first two methods work for any gauge group and any representation, while modifications will be needed in order to generalize the third and the fourth methods to other gauge group or higher representation.

In the β-deformed theory, as we explained in the introduction, it appears impossible to construct a supersymmetric Wilson loop. On the other hand, supergravity configurations have been constructed whose dual operators would have finite vev. We propose to study this form of the Wilson loop operator (1.22), (1.23) and that it provides the dual of the the D-brane configurations constructed in [37] [3].

We first give field theory arguments for the choice of this operator in the beta-deformed theories. First, as in the undeformed case, one may define the Wilson loop as the phase factor associated with the W-boson probe arising from the breaking $U(N+1) \to U(N) \times U(1)$. In appendix A.1, we calculate the deformed $\mathcal{N} = 4$ Lagrangian arising from this decomposition. The action looks quite complicated at finite N. However all the β-dependence drops out in the large N limit of the *classical* action and the resulting operator takes the form of (1.22), (1.23). We propose this form of the Wilson loop for any N.

Another field theory reason is that if ones tries to derive the constraint in the β-deformed theory using perturbation methods, the result at the leading order of 't Hooft coupling λ is the same as in the undeformed theory since the propagators of the β-deformed theory are not modified. Hence the UV pole cancels if the condition (1.23) is satisfied, as in the undeformed case. At higher orders of λ, the β-deformation breaks the $SO(6)$ invariance of the scalars and the simple argument of the undeformed case does not hold anymore. However one can check explicitly the gauge boson and scalar propagator remains equal up to order λ. As a result, the UV divergence cancels out explicitly up to order λ^2 if the constraint (1.23) holds. The details is presented in the appendix A.4. This could mean that the UV divergences cancel exactly in the β-deformed SYM theory. A better understanding of perturbative properties of the beta-deformed theory would give an answer to this problem.

This result is quite remarkable since although the $SO(6)$ symmetry is broken by the β-deformation, a $SO(6)$ invariant constraint is constructed. The same constraint is also obtained from the SUGRA analysis performed in the next subsections and

[3] In this case, the loop is taken to be a circle in the x-space and a point in the transverse space y^i.

give support to the validity of this constraint (1.23) and the form (1.22) of the Wilson loop operator.

We next turn to the supergravity picture for support of the form of the constraint (1.23) and the conjecture on the UV finiteness of the Wilson loop. Before we do this, a comment is in order. In order for the Wilson loop operator to respect the $U(1)^3$ symmetry (3.5) of the β-deformed SYM, one need to assign a corresponding rotation

$$y_1+iy_4 \to e^{i\delta_1}(y_1+iy_4), \quad y_2+iy_5 \to e^{i\delta_1}(y_2+iy_5), \quad y_3+iy_6 \to e^{i\delta_1}(y_3+iy_6), \quad (3.23)$$

to the loop variables y_i. Here we have used the identification of the scalar fields (A.4). The transformation properties (3.23) and (3.7) leads one to associate y_i with Y_i. This fact is important as, given a specific configuration of the loop variables y_i in the field theory, it tells which Y_i should be activated for the dual string configuration in supergravity. An example will be shown in next section.

Before we finish this section we make a comment on the Neumann boundary conditions. As we saw in chapter 2, for the β deformed case we have the Neumann boundary condition (2.55) which we write again here in terms of ${\Lambda^k}_m$

$$J_1^\alpha \partial_\alpha Y^k(\sigma_1, 0) = {\Lambda^k}_l \, \dot{y}^l(\sigma_1). \quad (3.24)$$

As we also saw there, that the Hamilton-Jacobi equation for the backgrounds discussed, which the beta deformed was a particular example, gives at the end of the relevant analysis a constraint

$$\dot{x}^2 = g_{kl} {\Lambda^k}_m {\Lambda^l}_n \, \dot{y}^m \dot{y}^n \ . \quad (3.25)$$

In particular for the Lunin-Maldacena theories, the constraint derived from supergravity agrees with the constraint (1.23) derived from field theory considerations of the condition if the matrix ${\Lambda^k}_i$ satisfies the condition

$$g_{kl} {\Lambda^k}_m {\Lambda^l}_n = \delta_{mn} \ . \quad (3.26)$$

This means that the *boundary condition matrix* ${\Lambda^k}_m$ is the vielbein of the deformed metric g_{kl}. We remark that in [37], the D-brane boundary condition in the β-deformed theory was obtained out using TsT transformation on the original undeformed boundary condition. It was easy in that case since only angles was involved.

In our case we still expect that one can perform a TsT-transformation on the angles to derive the modified boundary condition (3.24), (3.26), although it is less direct since the boundary condition is formulated in terms of the Cartesian coordinates while TsT transformations operates on the angles.

3.3 Near-1/4 BPS Wilson Loop

In the above, we have proposed that the D-brane configurations considered in [37] are dual to the near-1/2 BPS operators where the circular loop has a trivial dependence in the transverse space. Now we look at next non-trivial case where the loop involves a non-trivial rotation in the transverse space as well,

$$W[C] = \frac{1}{N}\mathrm{Tr}\, P\,\exp\left[\int d\tau\Big(iA_\mu \dot{x}^\mu(\tau) + |\dot{x}(\tau)|\varphi_i\theta^i(\tau)\Big)\right], \tag{3.27}$$

where the loop is a circular path of radius R_0 in space

$$x^1 = R_0\cos\tau, \quad x^2 = R_0\sin\tau, \tag{3.28}$$

and the coupling to the three scalars $\varphi_1, \varphi_2, \varphi_5$ is parametrized by

$$\theta^1 = \cos\theta_0, \quad \theta^2 = \sin\theta_0\cos\tau, \quad \theta^5 = \sin\theta_0\sin\tau, \tag{3.29}$$

with an arbitrary fixed θ_0. This operator in the undeformed theory is 1/2 BPS when $\theta_0 = 0$ and 1/4 BPS in general [59]. In this section we use the AdS/CFT correspondence to compute the value for the circular near BPS Wilson loop operator in the β-deformed SYM.

We use the following form for the (Euclidean) AdS_5 metric

$$ds^2 = du^2 + \cosh^2 u(d\rho^2 + \sinh^2\rho d\psi^2) + \sinh^2 u(d\chi^2 + \sin^2\chi d\phi^2). \tag{3.30}$$

For the deformed $\tilde{S}^5$ (3.1a), we parametrize the μ_i coordinates via

$$\mu_1 = \cos\theta, \quad \mu_2 = \sin\theta\cos\alpha, \quad \mu_3 = \sin\theta\sin\alpha \tag{3.31}$$

so that $\sum d\mu_i^2 = d\theta^2 + \sin^2\theta d\alpha^2$. For Euclidean space, the worldsheet coupling to the B-field get an extra factor of $-i$.

To find the dual string configuration, we note that

$$\begin{aligned}\theta^1 + i\theta^4 &= \cos\theta_0,\\ \theta^2 + i\theta^5 &= \sin\theta_0 e^{i\tau},\\ \theta^3 + i\theta^6 &= 0.\end{aligned} \tag{3.32}$$

Comparing with the definition (3.6) for θ^i, and using (3.31), this means the dual string configuration must satisfy $\phi_2 = \tau$, and $\theta = \theta_0$, $\phi_1 = \alpha = \phi_3 = 0$ at the boundary. Minimally, one wants to consider an ansatz involving only two angles ϕ_2 and θ. However due to the B-field, one can see easily that this is inconsistent. Let us therefore consider a motion on $R^2 \times \tilde{S}^3$ where $R^2 \subset AdS_5$ is parametrized by ψ and ρ, and the deformed 3-sphere is parametrized by the three angles θ, ϕ_1, ϕ_2 with $\alpha = \phi_3 = 0$. The Polyakov action for the Euclidean worldsheet (σ, τ) is

$$\begin{aligned} S &= \frac{\sqrt{\lambda}}{4\pi}\int d\sigma d\tau \Big[\rho'^2 + \dot{\rho}^2 + \sinh^2\rho(\psi'^2 + \dot{\psi}^2) + \theta'^2 + \dot{\theta}^2 + G\cos^2\theta(\phi_1'^2 + \dot{\phi}_1^2) \\ &\quad + G\sin^2\theta(\phi_2'^2 + \dot{\phi}_2^2) - 2i\hat{\gamma}G\sin^2\theta\cos^2\theta(\dot{\phi}_1\phi_2' - \phi_1'\dot{\phi}_2)\Big] \end{aligned} \tag{3.33}$$

where $'$ (resp. $\dot{}$) denotes ∂_σ (resp. ∂_τ) derivative. Due to the extra factor of $-i$ in the B-field coupling, a real configuration is possible only if one perform a Wick rotation $\phi_1 \to i\phi_1$. To match with the path specified by (3.28), (3.29), we look for solution of the form

$$u = 0, \quad \rho = \rho(\sigma), \quad \psi = \tau \tag{3.34}$$

$$\theta = \theta(\sigma), \quad \phi_1 = \phi_1(\sigma), \quad \phi_2 = \tau. \tag{3.35}$$

We remark that, compared to the solution [86] for the undeformed case, our ansatz has an additional angle ϕ_1 turned on. This is similar to the situation in the story of magnon. There the string configuration dual to the magnon was found [28] to expand from a motion on S^2 for the undeformed case to a motion on a deformed 3-sphere when the β-deformation is turned on. We also remark that the Wick rotation on ϕ_1 is natural and is consistent with a semi-classical interpretation of the AdS/CFT correspondence as a tunneling phenomena.

The classical equations of motion for our ansatz (3.34), (3.35) takes the form

$$
\begin{aligned}
\rho'' &= \cosh\rho \sinh\rho, & (3.36)\\
\theta'' &= \frac{1}{2}\partial_\theta(G\sin^2\theta) - \frac{1}{2}\partial_\theta(G\cos^2\theta)\ \phi_1'^2 + \partial_\theta(\hat{\gamma}G\sin^2\theta\ \cos^2\theta)\phi_1'\ , & (3.37)\\
0 &= \partial_\tau(-G\cos^2\theta\ \dot{\phi}_1 - \hat{\gamma}G\sin^2\theta\ \cos^2\theta\ \phi_2') + \partial_\sigma(-G\cos^2\theta\ \phi_1' + \hat{\gamma}G\sin^2\theta\ \cos^2\theta\ \dot{\phi}_2) & (3.38)\\
0 &= \partial_\tau(G\sin^2\theta\ \dot{\phi}_2 + \hat{\gamma}G\sin^2\theta\ \cos^2\theta\ \phi_1') + \partial_\sigma(-G\sin^2\theta\ \phi_2' - \hat{\gamma}G\sin^2\theta\ \cos^2\theta\ \dot{\phi}_1)\ . & (3.39)
\end{aligned}
$$

The equation (3.39) is satisfied trivially. Equation (3.38) gives

$$-G\cos^2\theta\phi_1' + \hat{\gamma}G\sin^2\theta\cos^2\theta = c_1. \tag{3.40}$$

For the surface to be closed, it must be possible to reach $\theta = 0$ (north pole) or π (south pole), and there the derivatives ϕ_1', ϕ_2' should be zero since no rotation is possible. Therefore $c_1 = 0$ and we have

$$\phi_1' = \hat{\gamma}\sin^2\theta. \tag{3.41}$$

Equation (3.37) then becomes

$$\theta'^2 = \sin^2\theta + c_2, \tag{3.42}$$

where c_2 is a constant. Notice how the G dependence disappears in the above calculations. Finally, we check also the Virasoro constraints, which reads

$$\rho'^2 - \sinh^2\rho + \theta'^2 - G\sin^2\theta - G\cos^2\theta\ \phi_1'^2 = 0,$$

which implies

$$-\rho'^2 + \sinh^2\rho = \theta'^2 - \sin^2\theta. \tag{3.43}$$

Again here notice that the G dependence disappears. To get a surface in correspondence to a single circle, we set $c_2 = 0$, and the final form of the equations of motion is

$$
\begin{aligned}
\rho'^2 &= \sinh^2\rho, & (3.44)\\
\theta'^2 &= \sin^2\theta. & (3.45)
\end{aligned}
$$

This give the solution

$$\sinh\rho = \frac{1}{\sinh\sigma}, \tag{3.46}$$

$$\sin\theta = \frac{1}{\cosh(\sigma_0 \pm \sigma)} \Leftrightarrow \cos\theta = \tanh(\sigma_0 \pm \sigma) \tag{3.47}$$

and

$$\phi_1 = \hat{\gamma}\big(\tanh(\sigma \pm \sigma_0) \mp \tanh(\sigma_0)\big). \tag{3.48}$$

To see how our solution behaves, consider the limits

$$\sigma \to 0 \quad \Rightarrow \quad \rho \to \infty, \quad \text{and} \quad \theta \to \theta_0, \quad \phi_1 \to 0, \tag{3.49}$$

$$\sigma \to \infty \quad \Rightarrow \quad \rho \to 0, \quad \text{and} \quad \theta \to 0 \text{ or } \pi. \tag{3.50}$$

Here $\cos\theta_0 = \tanh\sigma_0$. Depending on the sign in (3.47), the surface extends over the north or south pole of $\tilde{S}^5$.

Next we evaluate the action for this configuration. The bulk term is

$$S_{\text{bulk}} = \frac{\sqrt{\lambda}}{2\pi}\int d\sigma d\tau(\sinh^2\rho + \sin^2\theta), \tag{3.51}$$

from which we find

$$S_{\text{bulk}} = \sqrt{\lambda}(\cosh\rho_{\text{max}} \mp \cos\theta_0). \tag{3.52}$$

Here we have introduced a cutoff σ_{min} to regulate the boundary contribution, and ρ_{max} is the corresponding cutoff on ρ. The $\cosh\rho_{\text{max}}$ term will cancel with boundary term coming from the Legendre transformation as we have showed above. Hence, the final result is

$$S_{tot} = \mp\sqrt{\lambda}\cos\theta_0, \tag{3.53}$$

and

$$\langle W\rangle \sim \exp\big(\pm\sqrt{\lambda}\cos\theta_0\big), \tag{3.54}$$

where the sign is chosen to minimize the action. This is the same vev as the 1/4 BPS Wilson loop in the undeformed theory.

We note that in addition to this supergravity solution which involves 3 angles, one can also construct a solution which involves only the two angles

$$\theta = \theta(\sigma), \quad \alpha = \tau, \tag{3.55}$$

together with (3.34). This solution is exactly the same as the undeformed one given in [86] and gives rises to the same expectation value for the dual Wilson loop. It is straightforward to work out the Wilson loop operator that is dual to it. It is defined by the loop

$$\theta_1 = \cos\theta_0, \quad \theta_2 = \sin\theta_0 \cos\tau, \quad \theta_3 = \sin\theta_0 \sin\tau. \tag{3.56}$$

Due to a lack of $SO(6)$ invariance, the Wilson loop operator with the loop (3.56) is different from the one with the loop (3.29). It is quite amazing that they have the same expectation value.

To understand this result better. Let us first recall how the expectation value of the 1/2 BPS circular Wilson loop was computed in gauge theory [88, 89]. The circular loop is related to the straight line by a conformal transformation, one can therefore relate the circular Wilson loop to the expectation value of the Wilson straight line, which is one. The result is however non-trivial since under the conformal transformation, the gluon propagator is modified by a singular total derivative which gives non-zero contribution only when both ends of the propagator are located at the point which is conformally mapped to the infinity. It was conjectured by [88] that diagrams with internal vertexes cancel precisely and this is supported by a direct calculation at order g^4N^2. Assuming this is true, [89] showed that the sum of all the non-interacting diagrams can be written as a Hermitian matrix model

$$\langle W_R \rangle = \Big\langle \frac{1}{N}\mathrm{Tr}_R\big[e^M\big]\Big\rangle = \frac{1}{Z}\int \mathcal{D}M \frac{1}{N}\mathrm{Tr}_R\big[e^M\big] \exp\big(-\frac{2N}{\lambda}\mathrm{Tr}M^2\big). \tag{3.57}$$

This is exact to all order in λ and $1/N$ [89]. Explicit evaluation of the integral and hence the Wilson loop expectation value has been performed for loops in various representations [59, 64, 66, 67, 88, 89]. This argument has also been applied to the 1/4 BPS fundamental Wilson loop [86].

Now the β-deformed theory is exact conformal. So the above argument of conformal anomaly applies. The only thing one need to be sure is how interacting diagrams contribute. If they again sum up to zero, then there is no β-dependence left and one will get the same result as in the undeformed case. Our result of getting the same expectation value for the undeformed and the deformed Wilson loop operators suggests that the interacting diagrams again cancel exactly, at least in the large 't Hooft coupling limit. This is however not easy to prove from perturbation theory since one needs to identify terms with dependence on $\beta^2 N$ at each order of $1/N$. We believe a similar mechanism as in the undeformed case is at work. If this is the case, the exact expectation value of the circular Wilson loop in the β-deformed SYM will be given by the same matrix model as in the undeformed $\mathcal{N} = 4$ case. A better understanding of how this works in the undeformed case is necessary and will be very interesting.

For the same reason, we conjecture that the expectation value of the near-1/4 BPS Wilson loop in higher representations will also be unmodified. It will be interesting to construct the D3-brane and D5-brane dual to these Wilson loops in higher representations for the β-deformed theory and check this.

To summarize briefly, in this chapter we have proposed a definition of a near BPS Wilson loop operator in the β-deformed SYM theory. We conjectured that this operator has finite vev and provided supporting evidences both from field theory and from supergravity. Thus this operator is a natural candidate of a Wilson loop operator which admits a holographic description in the β-deformed AdS/CFT correspondence. We show, using the results of the chapter 2, that on the supergravity side, the finiteness of the vev of the Wilson loop implies the same constraint on the loop as is derived from the field theory analysis. That this is true relies on some remarkable properties satisfied by the metric and the B-field of the Lunin-Maldacena background, which classify these backgrounds to be special cases of the ones we examine in chapter 2. It will be interesting to be able to formulate and understand these symmetry properties in terms of the dual field theory language. Its origin is likely to be nonperturbative. This should provide us a better understanding of the mechanism responsible for the finiteness of the vev of the Wilson loop. Finally we also construct the string dual configuration for a near-1/4 BPS circular Wilson loop operator and its expectation value is computed using the AdS/CFT correspondence and found to be undeformed.

In the next chapter we move to a different gauge/gravity duality and start to investigate some semiclassical solutions of the Sasaki-Einstein backgrounds.

Chapter 4

Semi-classical Strings in Sasaki-Einstein Manifolds

In this chapter we investigate semi-classical string solutions in general $Y^{p,q}$ and $L^{p,q,r}$ manifolds. Work in this direction has been done for the very special case of $AdS_5 \times T^{1,1}$ examined in [107–109]. Moreover, a study for the case of BPS massless geodesics and their dual long BPS operators has been done in [111] for $Y^{p,q}$ manifolds and in [112] for $L^{p,q,r}$. Dual giant gravitons have been studied in [110] and recently giant magnons and spiky strings moving in a sector of $AdS_5 \times T^{1,1}$ have been examined in [113].

Here we mainly work on the gravity side and examine the motion of the string along some $U(1)$ directions in Sasaki-Einstein spaces which is localized at $\rho = 0$ in the AdS space. We will see that in some cases it is difficult to find acceptable string solutions, due to the constraints imposed on the Sasaki-Einstein parameters. For the solutions we find, we present the energy-spin relation. The energy expressed in the momenta, depends on the manifold considered i.e. on p, q, r. We also present an extensive discussion on point-like BPS string solutions. Notice that we do not examine the string dynamics in AdS_5 which are identical to the maximally supersymmetric case, since all the equations can be decoupled.

4.1 The backgrounds

4.1.1 $Y^{p,q}$ Metrics

The Sasaki–Einstein metrics $Y^{p,q}$ on $S^2 \times S^3$ can be presented in the following local form [100]:

$$\begin{aligned} ds^2 &= \frac{1-cy}{6}(d\theta^2 + \sin^2\theta d\phi^2) + \frac{1}{w(y)q(y)}dy^2 + \frac{q(y)}{9}(d\psi - \cos\theta d\phi)^2 \\ &+ w(y)\left[d\alpha + f(y)(d\psi - \cos\theta d\phi)\right]^2, \end{aligned} \tag{4.1}$$

or more compactly

$$ds^2 = ds^2(B) + w(y)[d\alpha + A]^2, \tag{4.2}$$

where

$$\begin{aligned} w(y) &= \frac{2(a-y^2)}{1-cy}, \\ q(y) &= \frac{a - 3y^2 + 2cy^3}{a-y^2}, \\ f(y) &= \frac{ac - 2y + y^2c}{6(a-y^2)}. \end{aligned} \tag{4.3}$$

For $c = 0$ the metric takes the local form of the standard homogeneous metric on $T^{1,1}$. Generally we can scale the constant c to 1 by a diffeomorphism, and this is what we do in the rest of the paper.

To make the space B a smooth complete compact manifold we should fix the coordinates appropriately [100]. The parameter a is restricted to the range

$$0 < a < 1 . \tag{4.4}$$

To make the base B_4 an axially squashed S^2 bundle over the round S^2 one can choose the ranges of the coordinates (θ, ϕ, y, ψ) to be $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, $y_1 \leq y \leq y_2$ and $0 \leq \psi \leq 2\pi$. The parameter ψ is the azimuthal coordinate on the axially squashed S^2 fibre and the round sphere S^2 parametrized by (θ, ϕ). Also, by choosing the above range for a, the following conditions of y are satisfied: $y^2 < a$, $w(y) > 0$ and $q(y) \geq 0$. The equation $q(y) = 0$ is cubic and has three real roots, one negative and two positive. Naming the negative root y_{q-} and the smallest positive root y_{q+} we must choose the range of the coordinate y to be

$$y_{q-} \leq y \leq y_{q+} , \tag{4.5}$$

with the boundaries corresponding to the south and north poles of the axially squashed S^2 fibre. Also, it is necessary to have p/q rational in order to have a complete manifold. Note that

$$y_{q+} - y_{q-} = \frac{3q}{2p} \equiv \xi \; , \tag{4.6}$$

where ξ is defined for later use. Therefore, if the roots y_{q+}, y_{q-} are rational we speak for quasi-regular Sasaki-Einstein manifolds with the property that the volume of these manifolds having a rational relation to the volume of the S^5. However the rationality of p/q can be achieved even in cases that the two roots are irrational which gives irregular Sasaki-Einstein metrics.

Using the expressions (B.1) presented in the appendix B.1, we can express y_{q-} in terms of ξ

$$y_{q-} = \frac{1}{2}(1 - \xi - \sqrt{1 - \frac{\xi^2}{3}}) \; , \tag{4.7}$$

where $0 < \xi < \sqrt{3}$. Since y_{q-} is the root of the cubic, a can be expressed in terms of ξ

$$a = \frac{1}{18}\left(9 - 3\sqrt{9 - 3\xi^2} + 4\xi^2\sqrt{9 - 3\xi^2}\right) \; , \tag{4.8}$$

and in order to ensure that y_{q+} is the smallest positive root we constrain ξ to the range $0 < \xi < 3/2$. If we prefer, we can express a in terms of p, q using (4.6)

$$a = \frac{1}{2} - \frac{p^2 - 3q^2}{4p^3}\sqrt{4p^2 - 3q^2} \; , \tag{4.9}$$

then the period of α is given by $2\pi l$ where

$$l = \frac{q}{3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}} \; . \tag{4.10}$$

4.1.2 $L^{p,q,r}$ Metrics

The metric of this manifold is [104]

$$ds^2_{p,q,r} \;=\; (d\xi + \sigma)^2 + ds^2_{[4]} \; , \tag{4.11}$$

where

$$ds^2_{[4]} \;=\; \frac{\rho^2}{4\Delta(x)}dx^2 + \frac{\rho^2}{h(\theta)}d\theta^2 + \frac{\Delta(x)}{\rho^2}\left(\frac{\sin^2\theta}{\alpha}d\phi + \frac{\cos^2\theta}{\beta}d\psi\right)^2 \tag{4.12}$$

$$+\frac{h(\theta)\sin^2\theta\cos^2\theta}{\rho^2}\left(\frac{\alpha - x}{\alpha}d\phi - \frac{\beta - x}{\beta}d\psi\right)^2 \tag{4.13}$$

and

$$\sigma = \frac{(\alpha - x)\sin^2\theta}{\alpha}d\phi + \frac{(\beta - x)\cos^2\theta}{\beta}d\psi \ , \tag{4.14}$$

$$\Delta(x) = x(\alpha - x)(\beta - x) - \mu \ , \tag{4.15}$$

$$\rho^2 = h(\theta) - x \ , \tag{4.16}$$

$$h(\theta) = \alpha\cos^2\theta + \beta\sin^2\theta \ . \tag{4.17}$$

Here p, q and r are relative positive coprime integers and $0 < p \leq q$, $0 < r < p + q$ and p, q are coprime to $s = p + q - r$. The metrics depends on two non-trivial parameters since α, β, μ are constants, and we can set one of them equal to a non-zero number by rescaling the other two and x. The function $\Delta(x)$ plays a similar role to the function $f(y)$ in the $Y^{p,q}$ manifold, so x should be restricted between the two lowest roots of $\Delta(x) = 0$, namely x_1 and x_2, where

$$x_1 < x < x_2 \ . \tag{4.18}$$

Moreover, in order to have a smooth geometry in 5 dimensions the parameters should satisfy $\alpha, \beta \geq x_2$ where $x_2 \geq x_1 \geq 0$, which imply the already presented inequalities for p, q, r. The constants appearing in the metric are related to the roots of $\Delta(x)$ as follows:

$$\mu = x_1x_2x_3, \qquad \alpha + \beta = x_1 + x_2 + x_3, \qquad \alpha\beta = x_1x_2 + x_1x_3 + x_2x_3, \tag{4.19}$$

where x_3 is the other root of $\Delta(x)$.

The metrics (4.1) of $Y^{p,q}$ can be derived as a special case of $L^{p,q,r}$ when $p+q = 2r$, which implies $\alpha = \beta$. The coordinate transformation is

$$\bar{\xi} = -2\alpha, \ \ \bar{\phi} = \frac{\psi + \phi}{2} + 3\alpha, \ \ \bar{\psi} = \frac{\psi - \phi}{2} + 3\alpha, \ \ \bar{\theta} = \frac{\theta}{2}, \ \ \bar{x} = \frac{(2y+1)\bar{\alpha}}{3} \ , \tag{4.20}$$

with μ related to a by

$$\bar{\mu} = \frac{4}{27}(1 - a)\bar{\alpha}^3 \ , \tag{4.21}$$

where we redefine the coordinates and the constants of $L^{p,q,r}$ using bars, in order to distinguish them from the ones of $Y^{p,q}$.

We also present the metric of AdS_5 in the Hopf coordinate system although we will be using the time element only

$$ds^2 = -\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho(d\beta_1^2 + \cos^2\beta_1 d\beta_2^2 + \sin^2\beta_1 d\beta_3^2) \ , \tag{4.22}$$

where the ranges of the angles are $0 \leq \beta_1 \leq \pi/2$, $0 \leq \beta_2, \beta_3 \leq 2\pi$.

4.2 String solutions in $Y^{p,q}$ background

4.2.1 Equations of motion and conserved quantities

In this section we present some spinning string solutions in the $Y^{p,q}$ manifold. We will fix the angle θ and hence we are allowing the string to move on a circle of the round sphere S^2 parametrized by the coordinate ϕ. On the squashed sphere the string can move on its azimuthal coordinate ψ, and sit at a constant value y_0 between the north and south poles. This value will be chosen carefully by solving the equations of motion. Finally, the string can move on the principle S^1 bundle over B parametrized by α. Notice that each of the directions that the string is allowed to spin has a $U(1)$ symmetry. As usual the global time is expressed through the world-sheet time as $t = \kappa\tau$, and the string is localized at the point $\rho = 0$. The Polyakov action in the conformal gauge is given by

$$\begin{aligned} S &= -\frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma\Big[-(-\dot{t}^2+t'^2)+\frac{1-y}{6}(-\dot{\theta}^2+\theta'^2)+\frac{1}{wq}(-\dot{y}^2+y'^2) \\ &\quad +(\frac{1-y}{6}s_\theta^2+\frac{q}{9}c_\theta^2+wf^2c_\theta^2)(-\dot{\phi}^2+\phi'^2)+(\frac{q}{9}+wf^2)(-\dot{\psi}^2+\psi'^2)+w(-\dot{\alpha}^2+\alpha'^2) \\ &\quad -2c_\theta(\frac{q}{9}+wf^2)(-\dot{\psi}\dot{\phi}+\psi'\phi')+2wf(-\dot{\alpha}\dot{\psi}+\alpha'\psi')-2wfc_\theta(-\dot{\alpha}\dot{\phi}+\alpha'\phi')\Big] \ . \end{aligned}$$

For convenience we do not write explicitly the dependence of y in the functions f, w and q. The classical equations of motion for constant θ and y take the form

$$\begin{aligned} &\frac{1-y}{6}(-\dot{\phi}^2+\phi'^2)s_{2\theta}+(\frac{q}{9}+wf^2)\big(s_{2\theta}(\dot{\phi}^2-\phi'^2)+2s_\theta(-\dot{\psi}\dot{\phi}+\psi'\phi')\big) \\ &\qquad +2wfs_\theta(-\dot{\alpha}\dot{\phi}+\alpha'\phi')=0 \ , \qquad (4.23) \\ &\frac{s_\theta^2}{6}(\dot{\phi}^2-\phi'^2)+(\frac{Q}{9}+A_1)\big(c_\theta^2(-\dot{\phi}^2+\phi'^2)-\dot{\psi}^2+\psi'^2-2c_\theta(-\dot{\psi}\dot{\phi}+\psi'\phi')\big) \\ &\qquad +W(-\dot{\alpha}^2+\alpha'^2)+2A_3(-\dot{\alpha}\dot{\psi}+\alpha'\psi'-c_\theta(-\dot{\alpha}\dot{\phi}+a'\phi'))=0 \ , \qquad (4.24) \end{aligned}$$

$$\partial_\beta[\gamma^{\beta\delta}\big(w\partial_\delta\alpha+wf(\partial_\delta\psi-c_\theta\partial_\delta\phi)\big)]=0 \ , \qquad (4.25)$$

$$\partial_\beta[\gamma^{\beta\delta}(\frac{1-y}{6}s_\theta^2\partial_\delta\phi+(\frac{q}{9}+wf^2)(c_\theta^2\partial_\delta\phi-c_\theta\partial_\delta\psi)-wfc_\theta\partial_\delta a)]=0 \ , \qquad (4.26)$$

$$\partial_\beta[\gamma^{\beta\delta}((\frac{q}{9}+wf^2)(\partial_\delta\psi-c_\theta\partial_\delta\phi)+wf\partial_\delta\alpha)]=0 \ , \qquad (4.27)$$

where we have used the conventions

$$A_1 \equiv \partial_y(wf^2), \quad A_2 \equiv \partial_y((wq)^{-1}), \quad A_3 \equiv \partial_y(wf), \quad Q \equiv \partial_y q, \quad W \equiv \partial_y w \quad (4.28)$$

and we have written the three last equations in a more compact form since for the ansatze we will choose they are satisfied trivially. In addition to the above equations, we get two more which come from the variation of the action with respect to the worldsheet metric. These are equivalent to the components of the energy-momentum tensor being set to zero. The Virasoro constraints read

$$\begin{aligned}
&\frac{1-y}{6}s_\theta^2\dot\phi\phi' + (\frac{q}{9}+wf^2)\big(c_\theta^2\dot\phi\phi' + \dot\psi\psi' - c_\theta(\dot\phi\psi' + \dot\psi\phi')\big) + w\dot\alpha\alpha' \\
&\qquad\qquad + wf\big(\dot\alpha\psi' + \dot\psi\alpha' - (\dot\alpha\phi' + \dot\phi\alpha')c_\theta\big) = 0, \qquad (4.29)\\
&-\kappa^2 + \frac{1-y}{6}s_\theta^2(\dot\phi^2 + \phi'^2) + w(\dot\alpha^2 + \alpha'^2) + 2wf\big(\dot\alpha\dot\psi + \alpha'\psi' - (\dot\alpha\dot\phi + \alpha'\phi')c_\theta\big) \\
&\qquad + (\frac{q}{9}+wf^2)\big(c_\theta^2(\dot\phi^2 + \phi'^2) + \dot\psi^2 + \psi'^2 - 2c_\theta(\dot\phi\dot\psi + \phi'\psi')\big) \\
&\qquad + \frac{1-y}{6}(\dot\theta^2 + \theta'^2) + \frac{1}{wq}(\dot y^2 + y'^2) = 0, \qquad (4.30)
\end{aligned}$$

where only in the last equation (4.30) we include the terms corresponding to a non-constant θ and y for later use in section 3.2.

The symmetry of $Y^{p,q}$ admits three conserved charges which are the angular momenta corresponding to strings rotating along the α, ϕ and ψ directions. Moreover, there exists one more conserved quantity, the classical energy, which is generated by the translational invariance along t. All of them are presented below:

$$\begin{aligned}
E &= \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma\,\kappa\,, \qquad (4.31)\\
J_a &= \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma\,(w\dot\alpha - wfc_\theta\dot\phi + wf\dot\psi)\,, \qquad (4.32)\\
J_\phi &= \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma\,\big(-wfc_\theta\dot\alpha + (\frac{1-y}{6}s_\theta^2 + \frac{q}{9}c_\theta^2 + wf^2c_\theta^2)\dot\phi \\
&\qquad - (\frac{q}{9}+wf^2)c_\theta\dot\psi\big)\,, \qquad (4.33)\\
J_\psi &= \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma\,\big(wf\dot\alpha - (\frac{q}{9}+wf^2)c_\theta\dot\phi + (\frac{q}{9}+wf^2)\dot\psi\big)\,. \qquad (4.34)
\end{aligned}$$

Let us also define the new quantities $J_{tot} = J_\alpha + J_\phi + J_\psi$, $\mathcal{E} = E/\sqrt{\lambda}$, $\mathcal{J}_i = J_i/\sqrt{\lambda}$ and $\mathcal{J}_{tot} = J_{tot}/\sqrt{\lambda}$ for later use. The conserved quantity corresponding to the total

$SU(2)$ angular momentum is

$$J^2 = J_\theta^2 + \frac{1}{s_\theta^2}(J_\phi + c_\theta J_\psi)^2 + J_\psi^2 \, . \tag{4.35}$$

Finally, for point-like stings localized at constant points on θ and y, the following identity holds

$$\sqrt{\lambda}\,\kappa^2 = \dot{\alpha} J_\alpha + \dot{\phi} J_\phi + \dot{\psi} J_\psi \, . \tag{4.36}$$

In the following we will choose an ansatz where the string is moving in $Y^{p,q}$ along the three angles α, ϕ, ψ and is at rest along all the other directions

$$\alpha = \omega_1 \tau + m_1 \sigma, \qquad \phi = \omega_2 \tau + m_2 \sigma, \qquad \psi = \omega_3 \tau + m_3 \sigma \ , \tag{4.37}$$

$$\theta = \theta_0 \ \text{ and } \ y = y_0, \tag{4.38}$$

where θ_0, y_0 are constants and their exact values should be chosen to be consistent with the solutions of the equations of motion and the Virasoro constraints. Notice also, that due to the periodicity condition in the global coordinates of the manifold on σ, the winding numbers have to be integers. For the linear dependence on τ, σ of (4.37), the equations of motion (4.25), (4.26) and (4.27) for α, ϕ and ψ respectively, are trivially satisfied.

4.2.2 Discussion on BPS solutions

In this section we discuss the BPS point-like solutions. The R-symmetry in the field theory is dual to the canonically defined Reeb Killing vector field K on the Sasaki-Einstein manifolds [102, 103], given by

$$K = 3\frac{\partial}{\partial \psi} - \frac{1}{2}\frac{\partial}{\partial \alpha} \tag{4.39}$$

and the R-charge is equal to

$$Q_R = 2J_\psi - \frac{1}{3}J_\alpha \, . \tag{4.40}$$

Now in order to express the Hamiltonian in terms of the momenta, we are initially considering a general situation, where all the parameters in the internal manifold are dependent on τ :

$$\theta = \theta(\tau), \qquad y = y(\tau), \qquad \alpha = \alpha(\tau), \qquad \phi = \phi(\tau), \qquad \psi = \psi(\tau), \tag{4.41}$$

and later we will focus on the string configurations mentioned in the previous section. The reason we are doing this, is to show how the general BPS solutions behave if we generalize our ansatz and activate simultaneously the motion on all angles. The process is equivalent to finding massless geodesics and is examined in [110, 111].

We start by expressing the energy in terms of the momenta. Now since we consider motion on the θ and y coordinates, we have the-non zero conjugate momenta

$$J_y = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma\,\frac{1}{w\,q}\dot{y}\ , \qquad J_\theta = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma\,\frac{1-y}{6}\dot{\theta}\ . \tag{4.42}$$

It is straightforward to substitute in the second Virasoro constraint the velocities in terms of their momenta and get

$$\kappa^2 = \frac{1}{\lambda}\Big(w\,q\,J_y^2 + \frac{6}{1-y}(J^2 - J_\psi^2) + \frac{1}{w}J_\alpha^2 + \frac{9}{q}(J_\psi - f\,J_\alpha)^2\Big)\ . \tag{4.43}$$

The energy of the string is given by (4.31), and is equal to the conformal dimension Δ of the dual operator, and to find the lower bound of it, we should express (4.43) in terms of the R-charge. Using (4.40), we obtain via the algebra in (4.43):

$$\Delta^2 = \big(\frac{3}{2}Q_R\big)^2 + \frac{1}{wq}(J_a + 3yQ_R)^2 + wqJ_y^2 + \frac{6}{1-y}(J^2 - J_\psi^2)\ . \tag{4.44}$$

Since $y_{q+} < 1$, which is the upper bound of y, and $J^2 \geq J_\psi^2$, all the terms in the above equation are positive, which leads to the inequality $\Delta \geq 3/2\,Q_R$. The solutions generated by the equality correspond to BPS operators, and in order to saturate the bound, all the following equations must be satisfied

$$J_y = 0, \qquad J_\theta = 0, \qquad J_\phi = -c_\theta J_\psi, \qquad J_\alpha = -3yQ_R. \tag{4.45}$$

The two first equations fix θ and y to unknown constants. The next two can be used to determine the relationship between α, ϕ, ψ using the constants y, θ. The situation now is getting closer to our initial configuration in the previous section where we considered y, θ as constant, with the difference that the τ dependence of the $U(1)$ angles is now unknown, and needs to be determined by the equations of motion and the Virasoro constraints. In order to find BPS solutions we need to solve the equations of motion (4.23-4.27) together with (4.45) and use (4.30) to calculate the energy. The first Virasoro constraint (4.29) is trivially satisfied.

It is more convenient to proceed by solving first the equations (4.25), (4.26) and (4.27) for constant y and θ, since their solutions are very special. As we said above, these equations are satisfied trivially for angles with linear dependence on τ, i.e. the one written in (4.37) where for the point-like case is equivalent to set all the m_i zero. The rest of the solutions we get constrain y to live on its maximum or minimum values, which means on the poles of the squashed sphere. More specifically the solutions are

$$\alpha = \omega_1 \tau, \qquad \phi = \omega_2 \tau, \qquad \psi = \omega_3 \tau \quad \text{or} \tag{4.46}$$

$$y = y_{q\pm}, \qquad \phi = \omega_2 \tau, \qquad \ddot{\alpha} = \frac{1-y}{6y}\ddot{\psi}. \tag{4.47}$$

However, when we take account of the boundary conditions in (4.47), we see that $\dot{\psi} = 0$, since we are at the poles of the squashed sphere. This fixes ψ to be constant and the other two $U(1)$ angles to have linear dependence on τ. This makes in a sense (4.47) a special case of (4.46). So the only way to satisfy the above equations for the non-constant angles is to have a linear dependence on τ, where the θ and y are not yet fixed to a specific value.

Bearing the above results in mind, we proceed by finding general solutions that satisfy the two first equations of motion (4.23), (4.24), together with the BPS equations (4.45) and present some solutions, starting with a solution which was also found in [111]

$$\dot{\phi} = 0, \qquad \dot{\alpha} = -\frac{\dot{\psi}}{6}. \tag{4.48}$$

We see that this solution is valid for any y that satisfies the inequality (4.5), since in this case, the last equation of (4.45) is satisfied trivially. Additionally, the third equation in (4.45) is satisfied also trivially and hence θ can take any constant value inside the region where it is defined. However notice that in order for the solution to satisfy (4.25-4.27), the α and ψ angles must have a linear dependence on τ, and hence

$$\alpha = -\frac{\omega_3}{6}\tau, \qquad \psi = \omega_3 \tau. \tag{4.49}$$

The energy for this solution is equal to $E = \sqrt{\lambda}|\dot{\psi}|/3$ and the conserved momenta in this case are

$$J_\alpha = -\frac{2}{3}\sqrt{\lambda} y \dot\psi, \qquad J_\phi = \frac{1}{9}\sqrt{\lambda}(y-1)c_\theta \dot\psi, \qquad J_\psi = -\frac{1}{9}\sqrt{\lambda}(y-1)\dot\psi \tag{4.50}$$

and are related each other by

$$J_\phi = -c_\theta\, J_\psi = \frac{1-y}{6y} c_\theta\, J_\alpha\ . \tag{4.51}$$

In this case the energy can be written as

$$E = \frac{3}{|(y-1)(c_\theta - 1) - 6y|}\, |J_{tot}|\ . \tag{4.52}$$

Notice that in the above relation the factor of proportionality is independent of a, and hence on the manifold considered.

Another set of possible solutions different than (4.49) are:

$$\theta = 0, \qquad y = 1 \pm \frac{\sqrt{1-a}}{\sqrt{3}}, \qquad \dot\alpha = \frac{\dot\phi - \dot\psi}{6}, \tag{4.53}$$

where again α, ϕ, ψ have to be linear with τ in order for the above expressions to satisfy (4.25), (4.26) and (4.27). The y solution is a special case of (4.55) for $\theta = 0$ and as we show in the next section, it is unfortunately always greater than y_{q+} except in the limit $y = y_{q+} = 1$. This is the case where $a = 1$ and we are not going to examine it any further. However, to be more accurate here, we have to set $\dot\phi = 0$, since for $\theta = 0$ we are on the pole of S^2 and there is no meaning in defining rotation along ϕ direction. Hence, the corresponding equation in (4.53) should be modified to $\dot\alpha = -\dot\psi/6$.

More interesting are the following solutions where y lives on its boundaries

$$y = y_{q\pm}, \qquad \theta = 0, \qquad \dot\phi = 0, \qquad \dot\alpha = -\frac{y+2}{6y}\dot\psi, \tag{4.54}$$

and the dependencies of the non-constant angles on τ are linear. This solution is acceptable since the inequality (4.5) is satisfied. However, by considering the boundary conditions, the solution become trivial, since we are on the poles of the squashed sphere where rotation along the ψ direction cannot be defined. Hence, we have to impose $\dot\psi = 0$ and then the whole string ansatz becomes static.

One could possibly find, other non-interesting solutions at the limits $a = 0,\ 1$, but it seems that there are no other BPS solutions than the ones presented above, which allow a to be at other points except zero and one. The solution (4.54) has as its main property to restrict y to the boundaries of (4.5) and to localise the string on the two three-submanifolds obtained by the initial manifold for $y = y_{q\pm}$, and denoted by $\Sigma_- \cong S^3/\mathcal{Z}_{p+q}, \Sigma_+ \cong S^3/\mathcal{Z}_{p+q}$. The cones over these Lens spaces are divisors of $C(Y^{p,q})$ and hence supersymmetric submanifolds [102]. This is because the induced volume form on Σ is equal to the four-form $\mathcal{J} \equiv J \wedge J/2$, where J here is the Kähler symplectic form. Hence, these cones are calibrated with respect to $\mathcal{J}$. However, by imposing the boundary conditions the string ansatz becomes static.

In the following sections, non-BPS point-like, as well as extended string solutions, will be examined.

4.2.3 One angle solution

In this section we examine the simplest case, where only the angle which parametrizes some $U(1)$ direction of the manifold in this coordinate system is turned on. In this case it is known that classical spinning string solutions that wrap around the circle do not exist due to diffeomorphism invariance, or equivalently because the first Virasoro constraint forces the metric element in the spinning direction to vanish[1], or make the string ansatz trivial. In $Y^{p,q}$ manifolds however, we just mention for completeness, that the diagonal metric elements in these three Killing vector directions can not vanish for y in the range (4.5) and hence the Virasoro constraints force the ansatz in that case to become static. Let us show briefly that the diagonal metric elements in the $U(1)$ directions can not vanish.

For the $g_{\alpha\alpha}$, this is obvious since it is equal to zero for $y = \pm\sqrt{a}$ and we already know that these values do not satisfy (4.5). For the $g_{\phi\phi}$ element, the situation is more complicated since it is equal to zero for

$$y_\pm = 1 \pm \frac{\sqrt{1 - ac_\theta^2}}{\sqrt{3}} \ . \tag{4.55}$$

Obviously the y_+ solution is discarded since is bigger than one, but also the y_- solution is outside the desirable area as can be seen in Figure 4.1. The last diagonal

[1]which conceptually does not look right since we consider motion along this direction only.

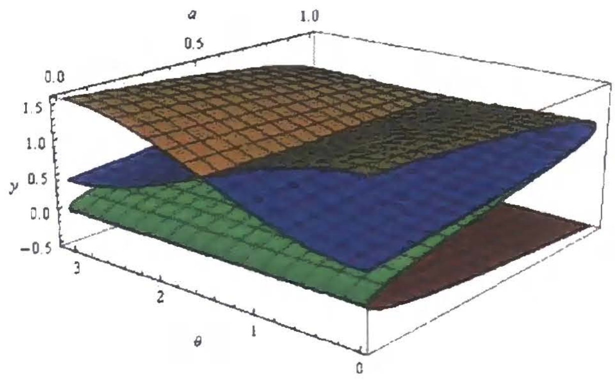

Figure 4.1: We plot $y_\pm = 1 \pm \sqrt{1 - a c_\theta^2}/\sqrt{3}$ together with the y_{q_+}, y_{q-} versus a, θ. The y_{q_-}, y_{q+} surfaces are colored red and green respectively, while y_-, y_+ colored blue and orange respectively. From the figure we notice that both $y_\pm$ are greater than y_{q+} for the whole range of a and θ.

metric element $g_{\psi\psi}$ is zero for $y_\pm = 1 \pm \sqrt{1-a}/\sqrt{3}$, where y_- is the lower bound of the previous solution and obviously this metric element cannot be zero too.

Let us now begin by looking for point-like string solutions. By allowing the string to move only along the α direction, and using the ansatz $\alpha = \omega_1 t$, the only non-trivially satisfied equation that we need to solve is (4.24), which takes the simple form:

$$\left(2 - \frac{2(a-1)}{(y-1)^2}\right)\omega_1^2 = 0 \;, \tag{4.56}$$

and has solutions

$$y_\pm = 1 \pm \sqrt{1-a} \;. \tag{4.57}$$

These solutions are plotted with $y_{q\pm}$ (Figure 4.2), and is obvious that only y_- is an acceptable solution, since $y_{q-} \le y_- \le y_{q+}$ for the whole range of a. Then, for $y = y_-$ the second Virasoro constraint (4.30) gives

$$\kappa^2 = w\,\omega_1^2 \Rightarrow \kappa^2 = 4\left(1 - \sqrt{1-a}\right)\omega_1^2 \;. \tag{4.58}$$

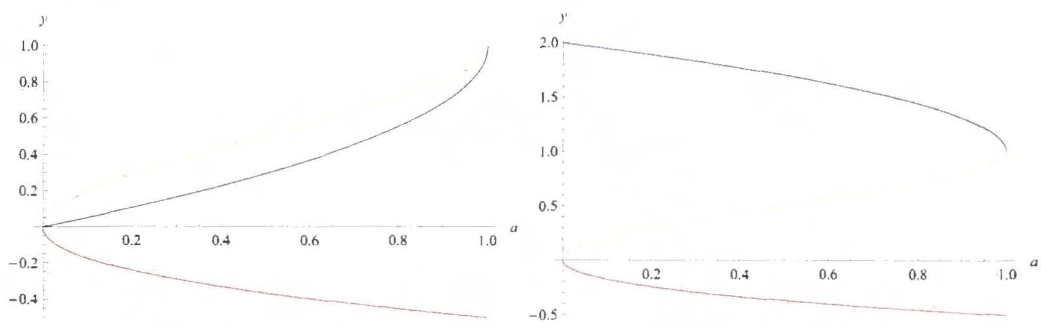

Figure 4.2: Plotting the roots y (blue) of (4.57) with y_{q-} (red), y_{q+} (green) versus a. This color mapping to the solution will be the same in the other plots too. In the first plot we see that the smaller solution y_- is between the two roots of $q(y) = 0$ and in the second one that the greater root y_+ is outside the allowed area since $y_+ > 1$.

Then using (4.31) - (4.34), we get for energy and the non zero conserved charges

$$E = \sqrt{\lambda}\sqrt{w\,\omega_1^2}\,, \tag{4.59}$$

$$J_\alpha = \sqrt{\lambda}\,w\,\omega_1\,. \tag{4.60}$$

Combining the above relations we end up with

$$E = \sqrt{\frac{1}{w}\,J_\alpha^2} \Rightarrow \quad E = \frac{\sqrt{\lambda}}{2}\sqrt{\frac{1}{1-\sqrt{1-a}}}\mathcal{J}_\alpha^2\,. \tag{4.61}$$

Note that the energy depends linearly on $\mathcal{J}_\alpha = \mathcal{J}_{tot}$ and depends on the manifold $Y^{p,q}$, since a is related to p, q by the equation (4.8), with the factor of proportionality being a monotonically decreasing function with respect to a. On the other hand, the function of the energy in terms of a, ω_1 is a monotonically increasing function with respect to a as can be seen from (4.58). As a final remark, we mention that all the calculations are independent of the angle θ, which can be chosen as any constant angle.

For completeness we now can consider a static string along the α direction with $\alpha = m_1\sigma$. The solution of the equation of motion remains the same, given by (4.57). The only difference is that all the momenta are now equal to zero and the energy is expressed as

$$\kappa^2 = w m_1^2 \Rightarrow E = 2\sqrt{\lambda}\sqrt{\left(1-\sqrt{1-a}\right)m_1^2}\,. \tag{4.62}$$

Following a similar procedure we choose a different angle ϕ with a similar ansatz and consider a point-like string motion with $\phi = \omega_2 t$. Then, the non-trivial equations are (4.23) and (4.24), which take the form

$$\frac{(a-1)s_{2\theta}}{18(y-1)}\omega_2^2 = 0 \,, \tag{4.63}$$

$$\frac{a-7-6(y-2)y+(a-1)c_{2\theta}}{36(y-1)^2}\omega_2^2 = 0 \,, \tag{4.64}$$

and do not give any new real solution, since the first equation is solved for $\theta = n\pi/2$, $n = (0,1,2)$ and the second one can not be solved for real y. The situation is similar when we choose the static string ansatz $\phi = m_2\sigma$, where the above equations remain the same with ω_2 replaced by m_2, and obviously again they do not give any new solutions.

Finally, for the third angle which parametrizes the remaining $U(1)$ direction, again consider the string ansatz $\psi = \omega_3 t$. The only non-trivial equation of motion is (4.24)

$$\frac{a-4-3(y-2)y}{18(y-1)^2}\omega_3^2 = 0 \,, \tag{4.65}$$

which has solutions $y = 1 \pm \sqrt{-1+a}/\sqrt{3}$ and are not real. The situation is similar for the ansatz $\psi = m_3\sigma$, since the equation (4.65) remains the same, with only difference the replacement of ω_3 by m_3.

Summarizing, if we restrict the string to rotate or wrap only along one $U(1)$ direction in $Y^{p,q}$ as we have done above, there is only one possible configuration. It is the point-like string moving along the fibre direction $\alpha = \omega_1\tau$. This is one spin solution, where the energy is proportional to $\mathcal{J}_\alpha$ and depends on the manifold $Y^{p,q}$ in a way that the factor of proportionality is a decreasing function of a. We expect that the solution (4.57) is not BPS. More specifically the Hamiltonian here is

$$H = \sqrt{\lambda}\frac{1}{2}w\omega_1^2 = 2\sqrt{\lambda}\left(1-\sqrt{1-a}\right)\omega_1^2 \,. \tag{4.66}$$

Hence the R-charge and the conformal dimension of the dual operator can be written as

$$Q_R = -\frac{\sqrt{\lambda}}{3}w\omega_1, \;\; \Delta^2 = (\frac{3}{2}Q_R)^2 + \lambda(4a-w)\omega_1^2 = (\frac{3}{2}Q_R)^2 + \frac{\sqrt{1-a}}{4(1-\sqrt{1-a})}J_\alpha^2 \,.$$

It is obvious that $\Delta > 3/2\, Q_R$ always, since the expression $(4a - w)$ is always positive, and become equal to zero only in the limits $a = 0,\ 1$. Hence, the solution is non-BPS.

4.2.4 The two Angle Solutions

Here we examine some solutions where the strings are moving along two $U(1)$ directions. We are looking for possible string solutions, with motion along the directions α, ψ, motivated by the fact that motion in these directions gives one BPS solution. Considering point-like strings and the simple ansatz[2]

$$\alpha = \omega_1 \tau, \qquad \psi = \omega_3 \tau, \tag{4.67}$$

we get only one non-trivial equation (4.24). This is solved by

$$\begin{aligned} y_{\pm} &= 1 \pm \frac{\sqrt{(1-a)(2\omega_1 - \omega_3)(6\omega_1 + \omega_3)}}{\sqrt{3}(2\omega_1 - \omega_3)}, \quad \text{or} \\ \omega_1 &= -\frac{\omega_3}{6}. \end{aligned} \tag{4.68}$$

We do not examine solutions in the limits $a = 0,\ 1$ and in the rest of the analysis we will ignore solutions that are valid only there. The solution $\omega_1 = -\omega_3/6$ is BPS and is examined in a previous section. The other solution for y, is acceptable in a region which will be specified. Notice, that for

$$\omega_1 = -\frac{y+2}{6y}\omega_3\ , \tag{4.69}$$

which is the solution derived in (4.54), the solution (4.68) becomes BPS and lives on $y = y_{q\pm}$, but by considering the boundary conditions becomes static.

For the general case, first of all, one must find the values for which the square root is real and these are for

$$\omega_3 > 0,\ \ \omega_1 > \frac{\omega_3}{2} \ \text{or}\ \omega_1 < -\frac{\omega_3}{6} \tag{4.70}$$

$$\omega_3 < 0,\ \ \omega_1 < \frac{\omega_3}{2} \ \text{or}\ \omega_1 > -\frac{\omega_3}{6}. \tag{4.71}$$

[2]Whenever we are writing an ansatz we suppose that all the parameters ω_i, m_i are non-zero.

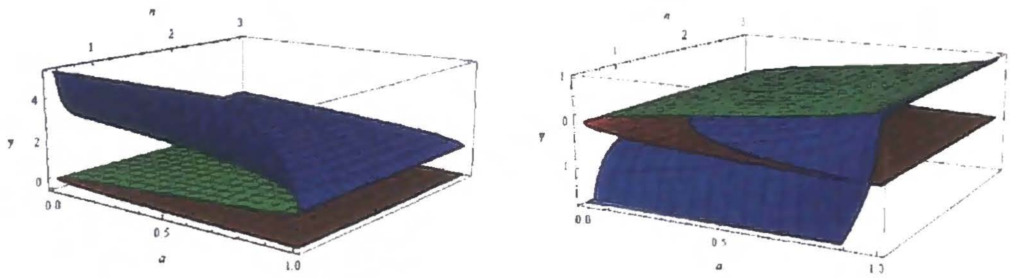

Figure 4.3: We plot $y_{\pm}$ versus n, a where $\omega_1 = n\,\omega_3$ with $n > 1/2$. In the first plot with y_+ we clearly see that there is no acceptable solution. In the second plot, the y_- give acceptable solutions.

Then we can see that indeed there exists solutions of y that are between y_{q-} and y_{q+}. Actually, the two solutions of (4.68) are equivalent depending on the sign of numbers ω_1, ω_3, i.e. if they are positive or negative. What one can say directly, is that the solution for y with RHS equal to 'one minus a positive quantity', is the one that could be acceptable in some intervals. In order keep the presentation simpler we choose ω_1, $\omega_3 > 0$. To give a visual picture of how the solutions behave we plot the surfaces for some random relation between ω_1, ω_3, say $\omega_1 = n\,\omega_3$ in Figure 4.3.

We see that y_- is an acceptable solution for

$$a > \frac{4(7+18n)}{(1+6n)^3}\,, \tag{4.72}$$

which also implies $n > 1/2$ in order for a to be smaller than 1.

For the general case, the equation (4.30) gives

$$\kappa^2 = 4\omega_1^2 - \frac{(6\omega_1+\omega_3)\sqrt{(1-a)(2\omega_1-\omega_3)(6\omega_1+\omega_3)}}{3\sqrt{3}}\,, \tag{4.73}$$

where we have substituted the solution y.

The conserved charges for our solution, given from (4.32), (4.33), (4.34) are

$$J_\alpha = \sqrt{\lambda}\Big(4\omega_1 + \frac{4(-3\omega_1+\omega_3)}{3}\sqrt{\frac{(1-a)(6\omega_1+\omega_3)}{3(2\omega_1-\omega_3)}}\Big)\,, \qquad (4.74)$$

$$J_\phi = -\sqrt{\lambda}\frac{\omega_3 c_\theta}{3}\sqrt{\frac{(1-a)(6\omega_1+\omega_3)}{3(2\omega_1-\omega_3)}}\,, \qquad (4.75)$$

$$J_\psi = \sqrt{\lambda}\frac{\omega_3}{3}\sqrt{\frac{(1-a)(6\omega_1+\omega_3)}{3(2\omega_1-\omega_3)}}\,, \qquad (4.76)$$

where (4.68) used. Notice the relation $J_\phi = -c_\theta J_\psi$. It seems that the energy depends on the momenta in a transcendental way. This is due to the complicated expression of y which depends on $\omega_{1,3}$, and enters in the momenta through the functions w, q, f. On the other hand, as we can see from (4.73), the energy is a monotonically increasing function with respect to a when ω_1, ω_3 are fixed. This behavior is similar to the one angle solution we found before, but it will be more interesting to have an energy-spin relation, which could be found by substituting in the second Virasoro constraint the $\omega_{1,3}$ with two conserved charges.

To finish, let us consider the example of $Y^{2,1}$. To find the corresponding value of a we can use (4.8) and find that

$$a = \frac{1}{4}\left(2 - \frac{\sqrt{13}}{8}\right) \simeq 0.387327\,. \qquad (4.77)$$

Substituting in (4.72), we see that indeed we can get solutions for y that are inside the desirable interval when $\omega_1 > 0.8568\,\omega_3$, or equivalently $n > 0.8568$. To simplify things even more we can choose $\theta = 0$, and solve for ω_1, ω_3 in terms of J_α, J_ψ to get the energy-spin relation. The energy-spin relation looks lengthy and complicated and seems to be completely transcendental, so we choose not to present it here.

Generalizing the ansatz by adding a σ dependence on the angle α and allowing the string to spin along this direction, we search for string solutions using

$$\alpha = \omega_1\tau + m_1\sigma, \qquad \psi = \omega_3\tau. \qquad (4.78)$$

The non-trivial equations that needs to be solved in this case are (4.24) and (4.29). One acceptable solution is

$$\omega_1 = \frac{\omega_3}{6}, \qquad y = a, \qquad a = \frac{2\omega_3^2}{9m_1^2+\omega_3^2}\,. \qquad (4.79)$$

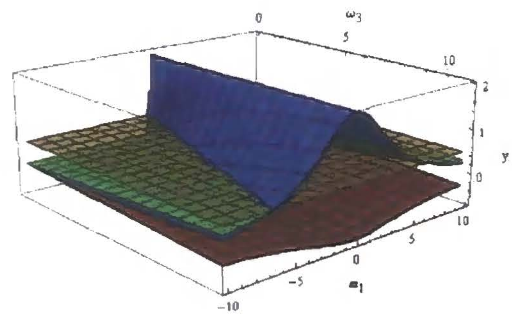

Figure 4.4: Plotting y, y_{q+}, y_{q-} versus ω_3, m_1. The transparent plane is at $a = 1/2$. When (4.80) satisfied, $a < 1/2$ and hence our solution lives in the area between the y_{q+}, y_{q-}. Also notice that for $a > 1$ there is no green surface, since the values of y_{q+} become complex, as expected.

By considering $a < 1$, and choosing $\omega_1, \omega_3 > 0$, we get $|m_1| > \omega_3/3$. To satisfy the condition (4.5), we have to restrict m_1 further as

$$|m_1| > \frac{\omega_3}{\sqrt{3}} \Rightarrow a < \frac{1}{2} \,. \tag{4.80}$$

Hence solutions that satisfy the above condition are acceptable as can be also seen clearly in Figure 4.4.

To calculate the energy we use (4.30) which gives

$$\kappa^2 = \frac{(15m_1^2 - \omega_3^2)\omega_3^2}{3\,(9m_1^2 + \omega_3^2)} \,, \tag{4.81}$$

and the corresponding charges are given by

$$J_\alpha = 0, \qquad J_\phi = \sqrt{\lambda}\frac{c_\theta\,(\omega_3^2 - 3m_1^2)}{3\,(9m_1^2 + \omega_3^2)}\omega_3, \qquad J_\psi = \sqrt{\lambda}\left(-\frac{1}{3} + \frac{4m_1^2}{9m_1^2 + \omega_3^2}\right)\omega_3. \tag{4.82}$$

Before we continue, we point out that one has to check if the function of a we get from (4.79) can give rational values to the expression $y_{q+} - y_{q-}$. This was not done in the previous cases since a was not dependent on ω or m, and hence it can take any appropriate value between zero and one, which would make the subtraction of the two roots of $q(y) = 0$ rational. In this case, one way to solve this problem is to equate the expression for a, given in (4.8), with our solution (4.79), and see if we can get rational solutions in the acceptable interval for ξ. One more reason to express ω_3

in terms of a is that is more preferable to choose a manifold $Y^{p,q}$, and from that to specify the allowed values for ω_i, m_i, instead of proceeding in the reverse direction. Using (4.8) and (4.79), we get for ω_3, m_3

$$\omega_3 = \pm\frac{3\sqrt{a}}{\sqrt{2-a}}m_1 = \pm\frac{3\sqrt{9+\sqrt{9-3\xi^2}\,(-3+4\xi^2)}}{\sqrt{27+(3-4\xi^2)\sqrt{9-3\xi^2}}}m_1 \equiv \pm D m_1\ , \tag{4.83}$$

where D is defined from the above equation. We are going examine the solutions with the plus sign which imply $m_1 > 0$. For $0 < \xi < 3/2$, the multiplicative factor satisfies $0 < D < 3$, and by imposing the constraint (4.80), we are limited in the interval

$$\xi < \frac{\sqrt{3}}{2} \Leftrightarrow a < \frac{1}{2}\ , \tag{4.84}$$

which also means that $D < \sqrt{3}$. The corresponding conserved charges in terms of a, m_1 are

$$J_\alpha = 0, \qquad J_\phi = -\sqrt{\lambda}\frac{(1-2a)\sqrt{a}c_\theta}{3\sqrt{2-a}}m_1, \qquad J_\psi = \sqrt{\lambda}\frac{(1-2a)\sqrt{a}}{3\sqrt{2-a}}m_1\ , \tag{4.85}$$

or using ω's

$$J_\phi = -\sqrt{\lambda}\frac{(1-2a)c_\theta}{9}\omega_3, \qquad J_\psi = \sqrt{\lambda}\frac{(1-2a)}{9}\omega_3\ . \tag{4.86}$$

The sum of the momenta is

$$J_{tot} = \frac{\sqrt{\alpha}(2\alpha-1)(c_\theta-1)}{3\sqrt{2-\alpha}}m_1\ . \tag{4.87}$$

Finally, the relation between the charges is

$$J_\phi = -c_\theta J_\psi = -\frac{c_\theta}{1-c_\theta}J_{tot}\ . \tag{4.88}$$

Notice, that the zeroth momentum J_α can follow from a more general solution such as $y = a$ and $\omega_1 = \omega_3/6$.

To find the energy, we use (4.30) to get

$$\kappa^2 = \frac{a(5-4a)}{2-a}m_1^2\ , \tag{4.89}$$

which can be written in terms of ω_3 by solving (4.83) for m_1. As before for a fixed ω_3, the energy is a monotonically increasing function with respect to a for $a < 1/2$. Using (4.85), we get the energy of our solution in terms of the momenta

$$E = \sqrt{\lambda}\sqrt{\frac{a(5-4a)}{2-a}m_1^2} = \frac{3\sqrt{5-4a}}{(1-2a)(1-c_\theta)}J_{tot}\ . \tag{4.90}$$

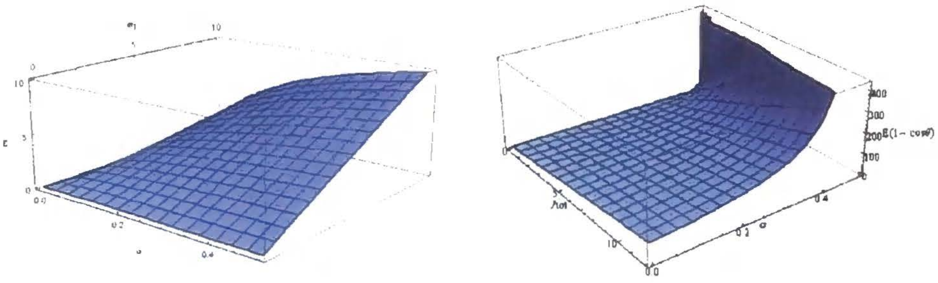

Figure 4.5: In the first plot is the energy versus m_1, a. In the second plot the expression $E \cdot (1 - c_\theta)$ is plotted versus J_{tot}, a, and one can notice the sharp rise of the energy as $a \to 1/2$. The plot range on energy is restricted intentionally to finite region in order to have a clear shape of the surface.

The angle θ is a constant, but in case we want to eliminate it, we can express c_θ in terms of the momenta from equation (4.88) and get

$$E = \frac{3\sqrt{5-4a}}{(1-2a)} J_\psi \ . \tag{4.91}$$

It is also interesting to insert the ξ parameters in the expressions we calculate using (4.83). Then the energy in terms of the momenta and the rational number ξ is

$$E = \frac{3\sqrt{3}}{(3-4\xi^2)(1-c_\theta)} \sqrt{\frac{27 + 2(3-4\xi^2)\sqrt{9-3\xi^2}}{3-\xi^2}} J_{tot} \ . \tag{4.92}$$

The equations (4.90) or (4.92) show that the energy is proportional to the momenta and depends on a. The exact dependence of the energy multiplied by $(1 - c_\theta)$, in order to avoid the dependence on the third parameter θ and to be able to plot a surface, is presented in Figure 4.5. We can see that the factor of proportionality between the energy and the momenta, is monotonically increasing with respect to a. However, the form of this factor does not seem to follow a specific pattern between the different string solutions. This occurs mainly because y depends on ω and m in a different way for each string solution. Also notice that the momenta are not taking continuous values but are quantized on a and m_1. Hence, the energy will not be a continuous function of the parameters, even if we plot it as continuous in order to show its behavior.

To finish the analysis for this case we mention that there are other solutions of (4.24) and (4.29), but these solutions are excluded since there is no a that satisfies the rationality constraint of $y_{q+} - y_{q-}$ and at the same time gives integer winding numbers.

4.3 String solutions in $L^{p,q,r}$ background

4.3.1 Equations of motion and conserved quantities

In this section we construct solutions for strings moving on the $L^{p,q,r}$ manifold. The configuration is chosen to be similar with the analysis of the $Y^{p,q}$ manifold. Firstly, we choose the string to sit at a constant angle θ. In this manifold, the role of the previous y coordinate is played by the coordinate x, and so we restrict the string to be localized at a constant point x_0, which has to respect the constraint (4.18). For some configurations the points θ and x that the string is sitting can be chosen arbitrary, but in most cases the equations of the system constrain at least one of them. As for the string dynamics, we are going to consider motion along some of the three $U(1)$ directions and try to find the energy-spin relation and how the energy is related on the properties of the general manifold.

Furthermore, we are not going to analyze the string dynamics in the AdS_5 since these are identical to the maximally supersymmetric case. To simplify things we are localizing the string also at $\rho = 0$ on AdS_5 and expressing the global time through the world-sheet time by $t = \kappa\tau$. Thus, the Polyakov action in the conformal gauge is given by

$$
\begin{aligned}
S \;=\; & -\frac{\sqrt{\lambda}}{4\pi}\int d\tau d\sigma\Big[-(-\dot{t}^2+t'^2)+(-\dot{\xi}^2+\xi'^2)+\frac{\rho^2}{4\Delta}(-\dot{x}^2+x'^2)+\\
&+\frac{\rho^2}{h}(-\dot{\theta}^2+\theta'^2)+\Big(\frac{(\alpha-x)^2}{\alpha^2}s_\theta^2+\frac{\Delta s_\theta^2+hc_\theta^2(\alpha-x)^2}{\rho^2\alpha^2}\Big)s_\theta^2(-\dot{\phi}^2+\phi'^2)+\\
&+\Big(\frac{(\beta-x)^2}{\beta^2}c_\theta^2+\frac{\Delta c_\theta^2+hs_\theta^2(\beta-x)^2}{\rho^2\beta^2}\Big)c_\theta^2(-\dot{\psi}^2+\psi'^2)+\\
&+2\Big((\alpha-x)(\beta-x)+\frac{\Delta-h(\alpha-x)(\beta-x)}{\rho^2}\Big)\frac{c_\theta^2 s_\theta^2}{\alpha\beta}(-\dot{\psi}\dot{\phi}+\psi'\phi')+\\
&+2\frac{\alpha-x}{\alpha}s_\theta^2(-\dot{\xi}\dot{\phi}+\xi'\phi')+2\frac{\beta-x}{\beta}c_\theta^2(-\dot{\xi}\dot{\psi}+\xi'\psi')\;.
\end{aligned}
$$

Again, we do not write the dependence of the functions on their arguments, in order to simplify the presentation. Also notice that α should not be confused with the same letter used in the case of $Y^{p,q}$ manifolds to name the angle. Before we start writing down the equations of motion, we define for convenience some new quantities. We identify the three expressions which are multiplied by $(-\dot{\phi}^2+\phi'^2)$, $(-\dot{\psi}^2+\psi'^2)$ and $(-\dot{\psi}\dot{\phi}+\psi'\phi')$, in the above action, which also correspond to metric elements, with the functions $d_1(x,\theta)\equiv g_{\phi\phi}$, $d_2(x,\theta)\equiv g_{\psi\psi}$ and $d_3(x,\theta)\equiv g_{\phi\psi}$ respectively. The partial derivatives of these functions are presented in Appendix B.2, and will be used later. The equations of motion for the θ, x are:

$$\begin{aligned}&\partial_\theta d_1(-\dot{\phi}^2+\phi'^2)+\partial_\theta d_2(-\dot{\psi}^2+\psi'^2)+2\partial_\theta d_3(-\dot{\psi}\dot{\phi}+\psi'\phi')\\&\qquad+2\frac{\alpha-x}{\alpha}s_{2\theta}(-\dot{\xi}\dot{\phi}+\xi'\phi')-2\frac{\beta-x}{\beta}s_{2\theta}(-\dot{\xi}\dot{\psi}+\xi'\psi')=0\qquad(4.93)\\&\partial_x d_1(-\dot{\phi}^2+\phi'^2)+\partial_x d_2(-\dot{\psi}^2+\psi'^2)+2\partial_x d_3(-\dot{\psi}\dot{\phi}+\psi'\phi')\\&\qquad-\frac{2}{\alpha}s_\theta^2(-\dot{\xi}\dot{\phi}+\xi'\phi')-\frac{2}{\beta}c_\theta^2(-\dot{\xi}\dot{\psi}+\xi'\psi')=0\qquad(4.94)\end{aligned}$$

where we have used the fact that the string is localized at two fixed points θ_0, x_0. The equations of motion for the three $U(1)$ directions ϕ, ψ, and ξ are

$$\partial_\kappa\big(\gamma^{\kappa\lambda}(d_1\partial_\lambda\phi+d_3\partial_\lambda\psi+2\frac{\alpha-x}{\alpha}s_\theta^2\partial_\lambda\xi)\big)=0\ ,\qquad(4.95)$$

$$\partial_\kappa\big(\gamma^{\kappa\lambda}(d_2\partial_\lambda\psi+d_3\partial_\lambda\phi+2\frac{\beta-x}{\beta}c_\theta^2\partial_\lambda\xi)\big)=0\ ,\qquad(4.96)$$

$$\partial_\kappa\big(\gamma^{\kappa\lambda}(2\frac{\alpha-x}{\alpha}s_\theta^2\partial_\lambda\phi+2\frac{\beta-x}{\beta}c_\theta^2\partial_\lambda\psi+\partial_\lambda\xi)\big)=0\ .\qquad(4.97)$$

Note that they are trivially satisfied for the linear ansatz we choose. Furthermore, the Virasoro constraints are given by

$$\begin{aligned}&\dot{\xi}\xi'+d_1\dot{\phi}\phi'+d_2\dot{\psi}\psi'+d_3(\dot{\psi}\phi'+\psi'\dot{\phi})+\frac{\alpha-x}{\alpha}s_\theta^2(\dot{\xi}\phi'+\xi'\dot{\phi})\\&\qquad+\frac{\beta-x}{\beta}c_\theta^2(\dot{\xi}\psi'+\xi'\dot{\psi})=0\qquad(4.98)\\&\kappa^2=\dot{\xi}^2+\xi'^2+d_1(\dot{\phi}^2+\phi'^2)+d_2(\dot{\psi}^2+\psi'^2)+2d_3(\dot{\psi}\dot{\phi}+\psi'\phi')\\&\qquad+2\frac{\alpha-x}{\alpha}s_\theta^2(\dot{\xi}\dot{\phi}+\xi'\phi')+2\frac{\beta-x}{\beta}c_\theta^2(\dot{\xi}\dot{\psi}+\xi'\psi')\qquad(4.99)\end{aligned}$$

Finally, the conserved charges associated to the three $U(1)$ isometries are

$$J_\xi = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma\,(\dot\xi + \frac{\alpha - x}{\alpha}s_\theta^2\dot\phi + \frac{\beta - x}{\beta}c_\theta^2\dot\psi)\,, \tag{4.100}$$

$$J_\phi = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma\,(\frac{\alpha - x}{\alpha}s_\theta^2\dot\xi + d_1\dot\phi + d_3\dot\psi)\,, \tag{4.101}$$

$$J_\psi = \frac{\sqrt{\lambda}}{2\pi}\int_0^{2\pi} d\sigma\,(\frac{\beta - x}{\beta}c_\theta^2\dot\xi + d_3\dot\phi + d_2\dot\psi)\,, \tag{4.102}$$

where the classical energy is given by (4.31). In the next section we use these equations to find some string solutions on $L^{p,q,r}$.

The ansatz for the string motion we consider have a linear dependence with τ and σ and are the following

$$\xi = \omega_1\tau + m_1\sigma, \qquad \phi = \omega_2\tau + m_2\sigma, \qquad \psi = \omega_3\tau + m_3\sigma \tag{4.103}$$

$$\theta = \theta_0 \quad \text{and} \quad x = x_0, \tag{4.104}$$

where θ_0, x_0 are constants and we are also setting $\mu = 1$.

4.3.2 One angle solution

It is straight-forward to see that in the coordinate system we chose for the metric, there are solutions for point-like strings moving on the direction ξ, since the metric element $g_{\xi\xi}$ is constant. As we also commented in a previous section, a general property of the Virasoro constraints is that they do not allow any extended spinning string solutions, where the strings are moving only along one $U(1)$ direction.

We start by considering the trivial case of $\xi = \omega_1\tau$, where all the equations of motion and the first Virasoro constraint are satisfied trivially. The conserved momenta are

$$J_\xi = \sqrt{\lambda}\omega_1, \quad J_\phi = \sqrt{\lambda}\frac{\alpha - x}{\alpha}s_\theta^2\omega_1, \quad J_\psi = \sqrt{\lambda}\frac{\beta - x}{\beta}c_\theta^2\omega_1. \tag{4.105}$$

The second Virasoro constraint gives the energy

$$E = \sqrt{\lambda}\sqrt{\omega_1^2} = |J_\xi| = \frac{2\alpha\beta}{4\alpha\beta - (\alpha+\beta)x - (\alpha-\beta)x\,c_{2\theta}}|J_{tot}|\,, \tag{4.106}$$

which is presented in terms of the momenta. We see that the energy is proportional to the total spin and the factor of proportionality depends on α, β, and hence on the manifold $L^{p,q,r}$ considered.

Now consider a point-like string rotating along ϕ direction with $\phi = \omega_2 \tau$. The first two equations of motion give

$$\partial_\theta d_1(x,\theta) = 0 \ , \qquad \partial_x d_1(x,\theta) = 0 \ . \tag{4.107}$$

In the region that α, β are defined, the only real solution is when $\theta = 0$, which makes the $g_{\phi\phi}$ element zero. As a final possibility, consider the string moving according to $\psi = \omega_3 \tau$, which has to satisfy the equations

$$\partial_\theta d_2(x,\theta) = 0 \ , \qquad \partial_x d_2(x,\theta) = 0 \ . \tag{4.108}$$

These have a real solution only for $\theta = \pi/2$ which makes the metric element in the direction of rotation equal to zero.

For completeness we mention that the corresponding static string ansatze $m_i\sigma$, gives an acceptable solution only for a wrapping around the α direction, and in this case all the conserved momenta are zero.

4.3.3 The two angle solutions

In this section we try to sketch a way of finding solutions of strings moving in two Killing vector directions simultaneously.

We choose to activate the angles ξ, ψ, and initially look for point-like string solutions

$$\xi = \omega_1 \tau, \qquad \psi = \omega_3 \tau. \tag{4.109}$$

The equations of motion reduce to

$$\partial_\theta d_2 \, \omega_3 - 2\frac{\beta - x}{\beta} s_{2\theta} \, \omega_1 = 0 \ , \qquad \partial_x d_2 \, \omega_3 - 2\frac{c_\theta^2}{\beta}\omega_1 = 0 \ , \tag{4.110}$$

where in order to check easier the inequality (4.18), and solve the above equations, it is convenient to give an appropriate value to angle θ. By choosing $\theta = \pi/4$ we get a solution

$$\alpha = \beta \pm \frac{3\omega_3}{\sqrt{2}\sqrt{-\beta\omega_3(2\omega_1 + \omega_3)}}, \qquad x = \frac{\alpha + 5\beta}{6} \ . \tag{4.111}$$

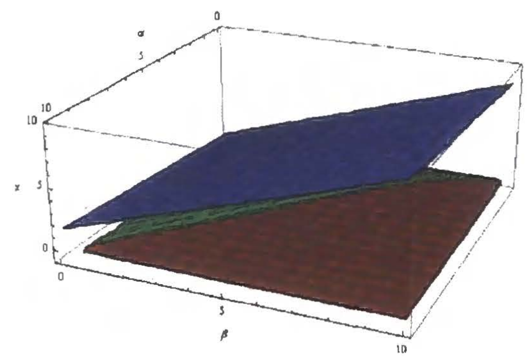

Figure 4.6: In the plot are presented the solutions of $\Delta(x)$, x_1 and x_2, together with x (4.111) versus the manifold parameters α, β. The solution x is plotted with blue and we see that there is no region such that x is between of x_1, x_2 where the manifold constraints are satisfied.

To make sure the solution is real we need to constrain ω_3 by

$$\omega_1 < 0 \quad \text{and} \quad 0 < \omega_3 < -2\omega_1 \quad \text{or} \quad \omega_1 > 0 \quad \text{and} \quad -2\omega_1 < \omega_3 < 0 \, . \tag{4.112}$$

Moreover, using the inequalities between x_2, α, and β we can easily see that $x_2 \leq (\alpha + 5\beta)/6$, which means that x can only be equal to x_2. This seems to be a general feature for these solutions, since even for $\theta = \pi/3$, the situation is similar. But for a general angle θ, it is more difficult to solve the equations of motion and identify this behavior. However, in the cases mentioned above, the solution x is always strictly greater than x_2, and hence not acceptable (Figure 4.6). There are also other solutions that could give acceptable answers, but need more extensive analysis.

To obtain an extended string configuration consider the ansatz

$$\alpha = \omega_1 \tau + m_1 \sigma, \quad \psi = \omega_3 \tau + m_3 \sigma, \tag{4.113}$$

which gives the following system of equations

$$\partial_\theta d_2(-\omega_3^2 + m_3^2) - 2\frac{\beta - x}{\beta} s_{2\theta}(-\omega_1\omega_3 + m_1 m_3) = 0 \, , \tag{4.114}$$

$$\partial_x d_2(-\omega_3^2 + m_3^2) - \frac{2}{\beta} c_\theta^2(-\omega_1\omega_3 + m_1 m_3) = 0 \, , \tag{4.115}$$

$$\omega_1 m_1 + d_2 \omega_3 m_3 + \frac{\beta - x}{\beta} c_\theta^2(\omega_1 m_3 + m_1 \omega_3) = 0 \, . \tag{4.116}$$

For $\omega_1 = m_1$ and $\omega_3 = m_3$ reduces to a single equation

$$1 + d_2\frac{\omega_3^2}{\omega_1^2} + 2\frac{\beta - x}{\beta}c_\theta^2\frac{\omega_3}{\omega_1} = 0 \tag{4.117}$$

and the simpler solution is for $\theta = \pi/4$,

$$\omega_1 = -\frac{\omega_3}{2}, \qquad x = \frac{-2 + \alpha\beta^2 + \beta^3}{2\beta^2}\,. \tag{4.118}$$

In order this solution to satisfy the inequalities between α, β, and x, is essential for the following inequality $|\alpha - \beta| < 2/\beta^2$ to be satisfied. However, one can see that the solution is not acceptable because it is always greater than x_2. It is also interesting to examine the more complicated solutions, and see if they can satisfy (4.18).

Since it is complicated to check analytically if the solutions which involve a general angle θ satisfy the manifold constraints, we choose not to present them here, and maybe leave the problem for further investigation in the future. However, we already gave the basic setup with all the equations, found some pointlike solutions and also show the method to follow to find new string solutions moving in the $U(1)$ directions of $L^{p,q,r}$ manifolds.

4.4 Discussions

In this chapter, we initially examine the string motion on $Y^{p,q}$ along the $U(1)$ directions. To accept the solutions of the equations of motion and the Virasoro constraints, we must make sure that they also satisfy the Sasaki-Einstein constraints in a way we described above. Due to the presence of all these constraints, the number of acceptable string solutions is limited drastically. Then we show that when the energy is expressed in terms of the conserved momenta, the factors multiplied with them depend on the manifold, i.e. on the parameter a, and they are monotonic functions with respect to it. Hence, the extrema of these functions occur for maximum or minimum values of a. Except this behavior, it seems that the dispersion relations depend on the parameter a in a transcendental way.

Furthermore, by looking at massless geodesics, we find that there is a unique BPS solution, which was already known. What we see here is that the string coordinates must depend linearly with time. Moreover, for this solution, the string can sit anywhere in the allowed y interval satisfying (4.5). An other point-like string solution was found which lives on the two supersymmetric three-submanifolds $S^3/\mathcal{Z}_{p+q}, S^3/\mathcal{Z}_{p+q}$, obtained by the initial manifold for $y = y_{q\pm}$. However, by considering the boundary conditions, the solution becomes static, and we argue that there are no other point-like BPS solutions in the analysis presented above.

One can work similarly in the cohomogeneity two manifolds $L^{p,q,r}$, finding some point-like and classical string solutions. Again, we expect that when the energy is expressed in terms of the conserved momenta, it does not take the same form uniformly over the family of manifolds $L^{p,q,r}$, and basically the discussion we presented for the $Y^{p,q}$ manifold remain similar with the $L^{p,q,r}$ manifold. However, in this case, due to the large number of parameters, it is more difficult to check analytically in full generality whether or not the solutions satisfy the manifold constraints. One can certainly try to find some more solutions for these manifolds.

Chapter 5

Concluding Remarks

In this chapter we discuss mostly some extensions of the work presented in the previous chapters and some recent developments.

Let us start this discussion with the Wilson loops in β deformed backgrounds. There are many indications, like the undeformed expectation value of the near $1/4$ BPS Wilson loop we considered, or the field theory analysis we presented, that at least in the large N the Wilson loop operator description is exactly the same with the undeformed $\mathcal{N}=4$ super Yang-Mills. This could be the case even in general, considering random N as we already have described. However, it is not impossible that for a generic N the Wilson loop operator will be the one in the undeformed case, multiplied by an exponential of β/N, which in the limit of large N reproduces the undeformed result. To get a final definite answer to this question, one has to find the Wilson loop operator with the method and the setup presented in Appendix A.1.

In gravity side would be interesting to try to check the supersymmetry of the near $1/4$ BPS Wilson loop configuration, using the Killing spinors and compare the behavior of the calculations with the ones already done in $\mathcal{N}=4$ super Yang-Mills [86]. Of course it is expected that no supersymmetry is preserved.

More interesting in our opinion, is to check in the gravity side whether the Wilson loop expectation value in the β deformed theories remain undeformed for a general smooth loop. It seems that the method that this can be done, is by examining the Polyakov on-shell action of the TsT deformed background, of a smooth Wilson loop that sits on the boundary of AdS. The area of the minimal surface that is produced it is likely to be undeformed. This analysis could be very demanding, so one can choose particular ansatze, other than the one considered above, of circular Wilson loops and check whether their areas of minimal surfaces are undeformed. If this happens, it is a strong evidence that in the general case the area of the minimal surface of the deformed background remains undeformed, but not a proof. To summarize, the problem is well defined: Take any minimal surface on $AdS_5 \times S^5$ with boundary on AdS and the desirable boundary conditions, deform the sphere, and consider again the minimal surface with the deformed boundary conditions. The question to be answered is whether the area of this minimal surface change, and if yes how depends on the deformation parameter. We have shown that at least in one case does not, and it seems that this could happen in more general cases too, but this is not proven yet.

From the work in chapter 2 it seems that the cancelation of the UV divergence and the Wilson loop constraint, are not related to the preserved supersymmetry of the theory. We saw several examples with theories with reduced supersymmetries, where the cancelation of the divergence with the use of Legendre transform and the Wilson loop constraint is possible. In most of these cases the Wilson loop constraint seems not to be possible to come from the supersymmetry preservation. One case, that it is definitely worth mentioning, is the case of the multi-β deformed theories, which do not preserve any of supersymmetries. In this theory the cancelation of the divergences with the use of the Wilson loop constraint and the Legendre transform is still accomplishable, and since there is no supersymmetry the cancelation can not depend on it. However, when one is using the β deformed backgrounds, should be aware that there could be some kind of hidden structure, that gives to these backgrounds some supersymmetric-like properties. This saying, is just a speculation and maybe not true, but we have to be cautious since several other results i.e. instantons, produced in the original theories, seems to carry on in the deformed theories, without affected crucially from this supersymmetry reduction. Of course, is more likely that these results are independent of the supersymmetry. In any case the conclusions for the UV divergences of the Wilson loops and their no relation to the supersymmetry are true, since we use several other backgrounds with reduced supersymmetries.

Furthermore, the fact that the Wilson loop expectation value for the case of the Sakai-Sugimoto model is finite can be used to produce several useful results in holographic models of QCD [115]. Moreover, recently it is examined the correspondence between the Wilson loops in $(p+1)$ dimensional super Yang-Mills and the minimal surface in the black p-brane background [116]. It seems that our analysis for the cancelation of the divergences with the Legendre transform, can be repeated there in a straightforward way. It is also straightforward to say that our results can be extended in different dimensions or similar spaces. For example one should expect that the Wilson loops in the $AdS_3 \times S^7$ will have divergence which canceled with the Legendre transform.

One very interesting case, is the Wilson loop operator in the 3-dimensional $\mathcal{N} = 6$ supersymmetric Chern-Simons theory [58], where recently the correspondence of Wilson loop has been analyzed [71–74] (see also [75] for related discussions). The ABJM theory has a $U(N) \times U(N)$ gauge and opposite levels k and $-k$. The matter fields are bifundamental scalar fields A_1, A_2 in the representation $(\mathbf{N}, \bar{\mathbf{N}})$ and anti-bifundamental fields B_1, B_2 in the representation $(\bar{\mathbf{N}}, \mathbf{N})$ and fermions. On the field theory side, a Wilson loop operator which couples to a certain bilinear combination of the bifundamental fields has been considered

$$W[C] = \frac{1}{N}\mathrm{Tr}\, P \exp\left[\oint_C d\tau \left(iA_\mu \dot{x}^\mu + \frac{2\pi}{k}|\dot{x}|M_I{}^J Y^I Y_J^\dagger\right)\right], \tag{5.1}$$

where $Y^I = (A_1, A_2, \bar{B}_1, \bar{B}_2)$ and the curve C is a straight line or a circle. For the special case where C is spacelike and $M = \mathrm{diag}(1, 1, -1, -1)$, the operator is 1/6 BPS. In this case the UV divergences of this operator canceled in the perturbation theory. It was also argued [72] that this 1/6 BPS Wilson loop operator describes a string smeared over a CP^1 in CP^3. The smeared string preserves a $SU(2) \times SU(2)$ subgroup of the $SU(4)$ isometry, which is precisely the amount of R-symmetry preserved by the operator (5.1) for this particular choice of M. As a smeared configuration, one would not expect to have a relation like (2.23) to relate the worldsheet boundary conditions with the couplings of the scalar fields in the Wilson loop. In general one may consider localized string in CP^3 and ask how it's boundary condition appears in the Wilson loop. We will consider a natural proposal in the following. However it turns out the correct operator has to be more complicated than this.

To describe the string theory on CP^3 (see for example, [76]), it is convenient to use the complex coordinates w^I

$$\sum_{I=1}^{4} w^I \bar{w}^I = 1, \tag{5.2}$$

subjected to the constraint

$$\sum_{I=1}^{4} (w^I \partial_\alpha \bar{w}^I - \bar{w}^I \partial_\alpha w^I) = 0, \quad \alpha = 1, 2. \tag{5.3}$$

This construction is a realization of the Hopf fibration since the first constraint describes a S^7 and the second constraint describes a $U(1)$ symmetry which reduces the embedding to CP^3. Using this description, one can think about the transverse space to the boundary spacetime $\mathbf{R}^3$ as described by the four coordinates $Z^I := Yw^I$ where Y is the radial coordinate of AdS_4. In terms of Z^I, we have $\sum_{I=1}^{4} Z^I \bar{Z}^I = Y^2$ and

$$\sum_{I=1}^{4} (Z^I \partial_\alpha \bar{Z}^I - \bar{Z}^I \partial_\alpha Z^I) = 0, \quad \alpha = 1, 2. \tag{5.4}$$

The string boundary condition is then given by the three Dirichlet conditions for the longitudinal coordinates and the eight Neumann boundary conditions

$$J_1^\alpha \partial_\alpha Z^I(\tau, 0) = \dot{z}^I(\tau), \quad I = 1, \cdots, 4. \tag{5.5}$$

Note that the boundary condition (5.5) is consistent with the constraint in (5.4) since $Z^I(\tau, 0) = 0$. In terms of real coordinates $Z^1 = Y^1 + iY^5, Z^2 = Y^2 + iY^6, Z^3 = Y^3 + iY^7, Z^4 = Y^4 + iY^8$, the embedding reads $\sum_{i=1}^{8} (Y^i)^2 = Y^2$ and

$$\sum_{I=1}^{4} (Y^I \partial_\alpha Y^{I+4} - Y^{I+4} \partial_\alpha Y^I) = 0. \tag{5.6}$$

The boundary condition reads

$$J_1^\alpha \partial_\alpha Y^i(\tau, 0) = \dot{y}^i, \quad i = 1, \cdots, 8, \tag{5.7}$$

where $z^1 = y^1 + iy^5, z^2 = y^2 + iy^6, z^3 = y^3 + iy^7, z^4 = y^4 + iy^8$.

To write down the Wilson loop, we note that due to the presence of the product gauge group, there are two independent Wilson loops one can write down. Let us concentrate for the moment on the first $U(N)$, one can form adjoint fields by multiplying the bi-fundamental fields in a certain order. It is natural to consider

$$W = \frac{1}{N}\mathrm{Tr}P\exp\left(\oint_C d\tau(iA_\mu\dot{x}^\mu + \dot{a}_{ab}A_a\bar{A}_b + \dot{b}_{ab}\bar{B}_aB_b)\right) \tag{5.8}$$

where C is a general spacelike curve. This operator is invariant under arbitrary reparametrization $\tau \to \tilde{\tau}$, including orientation reversing ones. Since scalar fields in three-dimensions is of dimension half, the variables a^{ab} and b^{ab} are of length dimension and therefore it make sense to try to identify them with the boundary variables z^I in (5.5). Since A_a (or B_a) is a doublet of $SU(2)_1$, $A_a\bar{A}_b$ (or $\bar{B}_aB_b$) contains a singlet and a triplet of $SU(2)_1$. Our proposal is to identify

$$\dot{a}_{ab} = \frac{2\sqrt{2}\pi}{k}\sum_{i=1}^{4}(\sigma^i)_{ab}\dot{y}^i, \quad \dot{b}_{ab} = \frac{2\sqrt{2}\pi}{k}\sum_{i=1}^{4}(\sigma^i)_{ab}\dot{y}^{i+4} \tag{5.9}$$

where $\sigma^i = (\tau^1, \tau^2, \tau^3, 1)$ and $\tau^{1,2,3}$ are the Pauli matrices. Note that the ABJM theory is manifestly invariant under $SU(2) \times SU(2)$ of the $SU(4)$ R-symmetry. Therefore (5.8) respects this symmetry if we assign (y^1, y^2, y^3) (respectively (y^5, y^6, y^7)) to be a triplet and y^4 (respectively y^8) to be a singlet $SU(2)_1$ (respectively $SU(2)_2$). For convenience, we have put a factor of $2\sqrt{2}\pi/k$ above since the propagator of the gauge bosons and the scalar field is different. This turns out to be a convenient normalization in perturbation theory. We remark that the identification (5.9) can also be written as

$$\dot{a}_{ab} + i\dot{b}_{ab} = \frac{2\sqrt{2}\pi}{k}\sum_{I=1}^{4}(\sigma^I)_{ab}\dot{z}^I \tag{5.10}$$

and our proposal for the Wilson loop operator that is dual to a string with the boundary condition (5.5) is

$$W = \frac{1}{N}\mathrm{Tr}P\exp\left[\oint_C d\tau\Big(iA_\mu\dot{x}^\mu + \frac{2\pi}{k}\sum_{I=1}^{4}\dot{z}^I\bar{R}^I + \dot{\bar{z}}^IR^I\Big)\right]. \tag{5.11}$$

Here R^I is the composite scalar $R^I := (\mathcal{A}^I + i\mathcal{B}^I)/\sqrt{2}$ where $\mathcal{A}^I := A_a(\sigma^I)_{ab}\bar{A}_b$, $\mathcal{B}^I := \bar{B}_a(\sigma^I)_{ab}B_b$.

By doing a perturbative computation as in, e.g. [72–74], one can show that the Wilson loop is in general linear divergent:

$$\sim \frac{N^2}{k^2\epsilon}\int d\tau_1(\dot{x}(\tau_1)^2-\dot{y}(\tau_1)^2). \tag{5.12}$$

Therefore the divergence cancels if the loop constraint

$$\dot{x}^2=\dot{y}^2 \tag{5.13}$$

is satisfied. The fact that we obtain precisely the same constraint as obtained from the Hamilton-Jacobi analysis provides some support that the ansatz (5.11) correctly encodes the boundary conditions of the dual open string. However this cannot be correct due to a mismatch. In fact, a half BPS string configuration which is localized at a point in CP^3 has been considered in [72–74]. One can show that there is no choice of $\dot{z}^I$ to make (5.11) half BPS. Even worse, it is easy to show, for the ansatz (5.1) which is coupled to a bilinear of scalars, there is no choice of the Hermitian matrix M so that there is 1/2 unbroken supersymmetry. Therefore the correct Wilson loop operator that is dual to localized string must be more complicated. The understanding of this will be very interesting.

Another more general direction to look at, is to try to apply our analysis to Wilson loops in higher representations. In these cases the suitable dual description is in terms of D3-branes or D5-branes [59, 64–70]. Presumably the correspondence will continue to hold for a more general class of gauge/gravity duality. It will be interesting to analyze the structure of the UV divergences there and to derive the corresponding boundary conditions for the corresponding D-brane description.

Finally, it is worthy to mention some recent developments in the Wilson loops especially in the study of 1/8-BPS supersymmetric Wilson loops in $\mathcal{N}=4$ super Yang-Mills theory and their string theory duals [117, 118]. The operators are defined for arbitrary contours on a two-sphere in space-time, and they were conjectured to be captured perturbatively by 2 dimensional bosonic Yang-Mills theory. In the AdS dual, they are described by pseudo-holomorphic string surfaces living on a certain submanifold of $AdS_5\times S^5$. It seems that it is important to try investigate more this conjecture. Moreover, the dual picture of these Wilson loops in higher representations is a very interesting problem and still not solved. Furthermore, it seems that the calibrations and the fact that the dual string worldsheets are pseudoholomorphic surfaces with respect to an almost complex structure J, is something that depend on the supersymmetry preserved. A very interesting question is what one would see for the pseudoholomorphic surfaces in the case of the backgrounds with less supersymmetries. It seems that in these backgrounds due to the lack of the existence of the $SO(6)$ the relevant analysis become more complicated. However, a good start would be to consider the β deformed backgrounds, where one hopes that their structure would simplify the analysis.

As far as concerns the semiclassical string solutions in Sasaki-Einstein manifolds, is true that the problem did not receive yet much attention in the literature. One can certainly try to find more solutions, and try to establish the energy spin relation. An interesting extension of this work could be to consider strings having σ dependence on one of the y, θ angles. By activating simultaneously one more $U(1)$ angle for the string's motion, the analysis should not be difficult. However, simultaneous string motion on more directions, could lead to systems of differential equations that might be difficult to solve. Also, worth looking at, is the effect of the β deformations on the string solutions on these manifolds. One can initially work with the point-like BPS solutions presented above, and try to derive the β deformed 'BPS condition'. It can be checked then if the BPS massless geodesics found above, remain undeformed. It must be possible to support these results accordingly in the dual beta deformed theory. Another interesting topic is to examine the string motion simultaneously in several $U(1)$ directions, and to analyze the energy-spin relations. Finally, it would be very good to identify the solutions found in this paper with the corresponding operators in field theory. Some of the above issues will be examined in a forthcoming publication [119].

Appendix A

A.1 Wilson loop from $U(N+1) \to U(N) \times U(1)$ breaking

For real β-deformation, the bosonic part of the Lagrangian of the β-deformed SYM theory is given by

$$\mathcal{L} = \mathrm{Tr}\left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D^\mu\bar{\Phi}^\alpha)(D_\mu\Phi_\alpha) - g^2[\Phi_\alpha, \Phi_\beta]_*[\bar{\Phi}^\alpha, \bar{\Phi}^\beta]_* + \frac{g^2}{2}[\Phi_\alpha, \bar{\Phi}^\alpha][\Phi_\beta, \bar{\Phi}^\beta]\right),$$

where Φ_α $(\alpha = 1, 2, 3)$ are the scalar components of the $\mathcal{N} = 1$ chiral superfield.

Next, let us break the gauge group $U(N+1) \to U(N) \times U(1)$ by turning non-zero vacuum expectation values for the scalar fields

$$\Phi_\alpha = \begin{pmatrix} 0_{N\times N} & 0 \\ 0 & M\Theta_\alpha \end{pmatrix}, \quad \alpha = 1, 2, 3\ . \tag{A.1}$$

Here Θ_α lies on a 5-sphere, $\Theta_\alpha\Theta^\alpha = 1$, corresponding to the direction of the symmetry breaking. Decomposing the fields as

$$\hat{A}_\mu = \begin{pmatrix} A_\mu & W_\mu \\ W_\mu^\dagger & a_\mu \end{pmatrix}, \qquad \hat{\Phi}_\alpha = \begin{pmatrix} \Phi_\alpha & W_\alpha \\ Y_\alpha & M\Theta_\alpha \end{pmatrix}, \tag{A.2}$$

we obtain the action in terms of W_α, Y_α :

$$\begin{aligned}
\hat{S} &= \frac{1}{4}F_{\mu\nu}^2 + (D_\mu\overline{\Phi}_\alpha)(D_\mu\Phi_\alpha) + \frac{1}{2}[\Phi_\alpha, \overline{\Phi}_\alpha][\Phi_\gamma, \overline{\Phi}_\gamma] + [\Phi_\alpha, \Phi_\gamma]_{\beta_{\alpha\gamma}}[\overline{\Phi}_\alpha, \overline{\Phi}_\gamma]_{\beta_{\alpha\gamma}} \\
&+\big((D_\mu - ia_\mu)W_\alpha^\dagger\big)\big((D_\mu + ia_\mu)Y_\alpha\big) + \big((D_\mu + ia_\mu)Y_\alpha^\dagger\big)\big((D_\mu - ia_\mu)W_\alpha\big) \\
&+\frac{1}{4}f_{\mu\nu}^2 + (\partial_\mu M\Theta_\alpha^\dagger)(\partial_\mu M\Theta_\alpha) \\
&-2Y_\alpha^\dagger\Big(\Phi_\gamma\Phi_\alpha e^{-2i\pi\beta_{\alpha\gamma}} - \Phi_\alpha\Phi_\gamma + \frac{1}{2}(\Phi_\alpha - M\Theta_\alpha)(\Phi_\gamma - M\Theta_\gamma) + M^2\Theta_\alpha\Theta_\gamma(e^{2i\pi\beta_{\alpha\gamma}} - 1)\Big)W_\gamma^\dagger \\
&-2Y_\alpha\Big(\overline{\Phi}_\gamma\overline{\Phi}_\alpha e^{-2i\pi\beta_{\alpha\gamma}} - \overline{\Phi}_\alpha\overline{\Phi}_\gamma + \frac{1}{2}(\overline{\Phi}_\alpha - M\overline{\Theta_\alpha})(\overline{\Phi}_\gamma - M\overline{\Theta}_\gamma) + M^2\overline{\Theta}_\alpha\overline{\Theta}_\gamma(e^{2i\pi\beta_{\alpha\gamma}} - 1)\Big)W_\gamma \\
&+Y_\alpha^\dagger\Big(\big(2(\overline{\Phi}_\kappa - M\overline{\Theta}_\kappa) **_{\beta_{\alpha k}}(\Phi_\kappa - M\Theta_\kappa) + [\Phi_k, \overline{\Phi}_\kappa]\big)\delta_{\alpha\gamma} - 2(\overline{\Phi}_\gamma - M\overline{\Theta}_\gamma) *_{\beta_{\alpha\gamma}} (\Phi_\alpha - M\Theta_\alpha) \\
&\qquad\qquad + (\Phi_\alpha - M\Theta_\alpha)(\overline{\Phi}_\gamma - M\overline{\Theta}_\gamma)\Big)W_\gamma \\
&+Y_\alpha\Big(\big(2(\Phi_\kappa - M\Theta_k) **_{\beta_{\alpha k}}(\overline{\Phi}_\kappa - M\overline{\Theta}_\kappa) + [\overline{\Phi}_\kappa, \Phi_\kappa]\big)\delta_{\alpha\gamma} - 2(\Phi_\gamma - M\Theta_\gamma) *_{\beta_{\alpha\gamma}} (\overline{\Phi}_\alpha - M\overline{\Theta}_\alpha) \\
&\qquad\qquad + (\overline{\Phi}_\alpha - M\overline{\Theta}_\alpha)(\Phi_\gamma - M\Theta_\gamma)\Big)W_\gamma^\dagger + \cdots , \qquad (A.3)
\end{aligned}$$

where we have defined

$$\begin{aligned}
(\overline{\Phi}_\kappa - M\overline{\Theta}_\kappa) **_{\beta_{\alpha\kappa}}(\Phi_\kappa - M\Theta_\kappa) &:= \overline{\Phi}_\kappa\Phi_\kappa + M^2\overline{\Theta}_\kappa\Theta_\kappa - \overline{\Phi}_\kappa M\Theta_\kappa e^{2i\pi\beta_{\alpha\kappa}} - M\overline{\Theta}_\kappa\Phi_\kappa e^{-2i\pi\beta_{\alpha\kappa}} , \\
(\overline{\Phi}_\gamma - M\overline{\Theta}_\gamma) *_{\beta_{\alpha\gamma}} (\Phi_\alpha - M\Theta_\alpha) &:= \overline{\Phi}_\gamma\Phi_\alpha e^{2i\pi\beta_{\alpha\gamma}} + M^2\overline{\Theta}_\gamma\Theta_\alpha e^{-2i\pi\beta_{\alpha\gamma}} - M\overline{\Theta}_\gamma\Phi_\alpha - \overline{\Phi}_\gamma M\Theta_\alpha.
\end{aligned}$$

In (A.3), $\cdots$ denotes terms of higher order (fourth) in the fields W, Y, and Tr over $U(N)$ is understood.

Next, we go to the real basis by introducing

$$\begin{aligned}
\Phi_1 &= \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_4) , \quad \Phi_2 = \frac{1}{\sqrt{2}}(\varphi_2 + i\varphi_5) , \quad \Phi_3 = \frac{1}{\sqrt{2}}(\varphi_3 + i\varphi_6) , &(A.4)\\
W_1 &= \frac{1}{\sqrt{2}}(w_1 + iw_4) , \quad W_2 = \frac{1}{\sqrt{2}}(w_2 + iw_5) , \quad W_3 = \frac{1}{\sqrt{2}}(w_3 + iw_6) &(A.5)
\end{aligned}$$

and similarly for Y_α and Θ_α. The terms $Y_\alpha^\dagger(\cdots)W_\gamma^\dagger$, $Y_\alpha(\cdots)W_\gamma$, $Y_\alpha(\cdots)W_\gamma^\dagger$ and $Y_\alpha^\dagger(\cdots)W_\gamma$ become

$$\begin{aligned}
&\sum_{i=1}^{6} w_i^\dagger\Big[\sum_{j=1}^{6} C_{jj} - C_{ii}^0\Big]w_i \\
&+ \sum_{ij=14,25,36} w_i^\dagger\Big[2\lambda_{ij} - C_{ij}^0 + 2i\sin 2\pi\beta\sum_{kl} s_{ikl}\varphi_k M\theta_l\Big]w_j + \ c.c. \qquad (A.6)\\
&+ \sum_{\substack{ij\neq 14,25,36 \\ i\neq j}} w_i^\dagger\Big[2\Lambda_{ij} - C_{ij}^0 - 2M^2\theta_i\theta_j(\cos 2\pi\beta - 1) + 2i\sum_{k,l=1}^{6} S_{iklj}\sin 2\pi\beta(\varphi_k\varphi_l - M^2\theta_k\theta_l)\Big]w_j \\
&\qquad + \ c.c.,
\end{aligned}$$

where we have defined

$$C_{ij} := (\varphi_i e^{i\pi\beta} - M\theta_i e^{-i\pi\beta})(\varphi_j e^{-i\pi\beta} - M\theta_j e^{i\pi\beta}) , \tag{A.7}$$

$$\Lambda_{ij} := \varphi_i \varphi_j - \varphi_j \varphi_i \cos 2\pi\beta , \tag{A.8}$$

$$\lambda_{ij} := [\varphi_i, \varphi_j] , \tag{A.9}$$

and $C^0_{ij} = C_{ij}(\beta = 0)$. The quantities s_{ijk}, S_{ijkm} are equal to ± 1 or zero, and their non-zero elements are shown below:

$$\begin{aligned} s_{ikl} &= 1 \quad \text{for } ikl = 125, 163, 241, 236, 314, 352, \quad \text{and} \quad s_{ikl} = -s_{ilk} , \\ S_{iklj} &= 1 \quad \text{for } iklj = 2451, 1245, 4512, 5124, 1643, 6431, 3164, 4316, \\ &\qquad 3562, 2356, 5623, 6235 . \end{aligned} \tag{A.10}$$

We have written our result in this form, so to be clear as much as possible the separation between the deformed and the undeformed part of the Lagrangian.

Following the derivation of [13], one can derive the form of the deformed Wilson loop. What is relevant is the eigenvalues of the mass matrix (A.6). In the undeformed case, the mass matrix has an eigenvalue which is 5-fold degenerated and a zero non-degenerate eigenvalue. The supersymmetric Wilson loop (1.22), (1.23) is derived from the (infinitely) massive quark probe. In the β-deformed case, the eigenvalues are generally deformed and degeneracy is lifted. However it is clear that the large N Wilson loop will be the same as in the undeformed case because there isn't any multiplicative factor depending on N in the mass matrix (A.6), therefore the classical Lagrangian is the same as the undeformed one in the large N limit (1.51).

For finite N, one will need to keep track of all the dependence of β in the Lagrangian (A.6). Due to the large amount of computational work, we were not able to work out the explicit expressions of the eigenvalues. However for the cases we have checked (for example by setting some of the ϕ_k and θ_k zero), it appears that there is always an eigenvalue which is equal to the undeformed one. It is the phase factor which is associated with this quark which gives rises to the Wilson loop (1.22), (1.23).

We remark that one may also utilize the star product (1.43) and use a star product path ordering to define the Wilson loop operator. Unlike the Wilson loop in the ordinary noncommutative geometry which is highly non-local [55], the closed Wilson loop operator is immediately local and there is no need to employ an open Wilson line. When one expands the exponent, will get higher and higher powers of the scalar fields and each of them is accompanied with a phase factor which depends on the charge configuration of the scalars. Since these phase factors becomes higher and higher power in β, in general one cannot drop the β-dependence even in the large N limit. This operator is not what one obtains from the probe analysis presented above. It is an interesting question whether this noncommutative Wilson loop also admits a nice holographic interpretation, and how.

A.2 The deformed metric in the Cartesian coordinate system

For convenience we collect and present the metric in the coordinate system (3.6) expressed in Y^i coordinates. Defining

$$\begin{aligned}
A_1 &= 1 + \hat{\gamma}^2 Y^{-4}(Y^{2^2} + Y^{5^2})(Y^{3^2} + Y^{6^2}),\\
A_2 &= 1 + \hat{\gamma}^2 Y^{-4}(Y^{1^2} + Y^{4^2})(Y^{3^2} + Y^{6^2}),\\
A_3 &= 1 + \hat{\gamma}^2 Y^{-4}(Y^{1^2} + Y^{4^2})(Y^{2^2} + Y^{5^2}),
\end{aligned} \tag{A.11}$$

the metric elements are:

$$\begin{aligned}
G_{11} &= Y^{-2}\frac{(Y^{1^2} + GY^{4^2}A_1)}{Y^{1^2} + Y^{4^2}}, & G_{44} &= Y^{-2}\frac{(Y^{4^2} + GY^{1^2}A_1)}{Y^{1^2} + Y^{4^2}},\\
G_{22} &= Y^{-2}\frac{(Y^{2^2} + GY^{5^2}A_2)}{Y^{2^2} + Y^{5^2}}, & G_{55} &= Y^{-2}\frac{(Y^{5^2} + GY^{2^2}A_2)}{Y^{2^2} + Y^{5^2}},\\
G_{33} &= Y^{-2}\frac{(Y^{3^2} + GY^{6^2}A_3)}{Y^{3^2} + Y^{6^2}}, & G_{44} &= Y^{-2}\frac{(Y^{6^2} + GY^{3^2}A_1)}{Y^{3^2} + Y^{6^2}},
\end{aligned} \tag{A.12}$$

$$\begin{aligned}
G_{12} &= 2Y^{-6}\hat{\gamma}^2 G(Y^{3^2} + Y^{6^2})Y^4Y^5, & G_{13} &= 2Y^{-6}\hat{\gamma}^2 G(Y^{2^2} + Y^{5^2})Y^4Y^6,\\
G_{15} &= -2Y^{-6}\hat{\gamma}^2 G(Y^{3^2} + Y^{6^2})Y^2Y^4, & G_{16} &= -2Y^{-6}\hat{\gamma}^2 G(Y^{2^2} + Y^{5^2})Y^4Y^3,\\
G_{23} &= 2Y^{-6}\hat{\gamma}^2 G(Y^{1^2} + Y^{4^2})Y^5Y^6, & G_{24} &= -2Y^{-6}\hat{\gamma}^2 G(Y^{3^2} + Y^{6^2})Y^1Y^5,\\
G_{26} &= -2Y^{-6}\hat{\gamma}^2 G(Y^{1^2} + Y^{4^2})Y^3Y^5, & G_{34} &= -2Y^{-6}\hat{\gamma}^2 G(Y^{2^2} + Y^{5^2})Y^1Y^6,\\
G_{35} &= -2Y^{-6}\hat{\gamma}^2 G(Y^{1^2} + Y^{4^2})Y^2Y^6, & G_{45} &= 2Y^{-6}\hat{\gamma}^2 G(Y^{3^2} + Y^{6^2})Y^1Y^2,\\
G_{46} &= 2Y^{-6}\hat{\gamma}^2 G(Y^{2^2} + Y^{5^2})Y^1Y^3, & G_{56} &= 2Y^{-6}\hat{\gamma}^2 G(Y^{1^2} + Y^{4^2})Y^2Y^3,
\end{aligned}$$

$$
\begin{aligned}
G_{14} &= 2Y^{-2}\frac{Y^1Y^4(1-GA_1)}{Y^{1^2}+Y^{4^2}}, \\
G_{25} &= 2Y^{-2}\frac{Y^2Y^5(1-GA_2)}{Y^{2^2}+Y^{5^2}}, \\
G_{36} &= 2Y^{-2}\frac{Y^3Y^6(1-GA_3)}{Y^{3^2}+Y^{6^2}}.
\end{aligned}
\tag{A.13}
$$

Substituting from (3.6) the coordinates we express the metric in angles, the diagonal terms are

$$
\begin{aligned}
G_{11} &= \frac{1}{Y^2}(\cos^2\phi_1 + G\sin^2\phi_1 M_1), & G_{44} &= \frac{1}{Y^2}(\sin^2\phi_1 + G\cos^2\phi_1 M_1), \\
G_{22} &= \frac{1}{Y^2}(\cos^2\phi_2 + G\sin^2\phi_2 M_2), & G_{55} &= \frac{1}{Y^2}(\sin^2\phi_2 + G\cos^2\phi_2 M_2), \\
G_{33} &= \frac{1}{Y^2}(\cos^2\phi_3 + G\sin^2\phi_3 M_3), & G_{66} &= \frac{1}{Y^2}(\sin^2\phi_3 + G\cos^2\phi_3 M_3)\ .
\end{aligned}
$$

The non-diagonal elements are

$$
\begin{aligned}
G_{12} &= \frac{1}{Y^2}\hat\gamma^2 G\mu_1\mu_2\mu_3^2\sin\phi_1\sin\phi_2, & G_{13} &= \frac{1}{Y^2}\hat\gamma^2 G\mu_1\mu_2^2\mu_3\sin\phi_1\sin\phi_3, \\
G_{15} &= -\frac{1}{Y^2}\hat\gamma^2 G\mu_1\mu_2\mu_3^2\sin\phi_1\cos\phi_2, & G_{16} &= -\frac{1}{Y^2}\hat\gamma^2 G\mu_1\mu_2^2\mu_3\sin\phi_1\cos\phi_3, \\
G_{23} &= \frac{1}{Y^2}\hat\gamma^2 G\mu_1^2\mu_2\mu_3\sin\phi_2\sin\phi_3, & G_{24} &= -\frac{1}{Y^2}\hat\gamma^2 G\mu_1\mu_2\mu_3^2\cos\phi_1\sin\phi_2, \\
G_{26} &= -\frac{1}{Y^2}\hat\gamma^2 G\mu_1^2\mu_2\mu_3\sin\phi_2\cos\phi_3, & G_{34} &= -\frac{1}{Y^2}\hat\gamma^2 G\mu_1\mu_2^2\mu_3\cos\phi_1\sin\phi_3, \\
G_{35} &= -\frac{1}{Y^2}\hat\gamma^2 G\mu_1^2\mu_2\mu_3\cos\phi_2\sin\phi_3, & G_{45} &= \frac{1}{Y^2}\hat\gamma^2 G\mu_1\mu_2\mu_3^2\cos\phi_1\cos\phi_2, \\
G_{46} &= \frac{1}{Y^2}\hat\gamma^2 G\mu_1\mu_2^2\mu_3\cos\phi_1\cos\phi_3, & G_{56} &= \frac{1}{Y^2}\hat\gamma^2 G\mu_1^2\mu_2\mu_3\cos\phi_2\cos\phi_3,
\end{aligned}
$$

and

$$
\begin{aligned}
G_{14} &= \frac{1}{2Y^2}\sin 2\phi_1(1-GM_1),\ G_{25} = \frac{1}{2Y^2}\sin 2\phi_2(1-GM_2), \\
G_{36} &= \frac{1}{2Y^2}\sin 2\phi_3(1-GM_3)\ .
\end{aligned}
\tag{A.14}
$$

Appendix A.3: Derivation of the Hamilton-Jacobi Equation

In this appendix we shortly derive the Hamilton-Jacobi (HJ) equation (2.4). Consider the action for the string

$$
S = \int d^2\sigma\big(\sqrt{\det g} - iB_{IJ}\partial_1 X^I\partial_2 X^J\big) \tag{A.15}
$$

where $g_{\alpha\beta} := G_{IJ}\partial_\alpha X^I \partial_\beta X^J$, $\alpha, \beta = 1, 2$. The conjugate momentum is

$$P_I = \frac{\delta S}{\delta(\partial_2 X^I)} = \frac{1}{\sqrt{g}} G_{IJ}(g_{11}\partial_2 X^J - g_{12}\partial_1 X^J) + iB_{IJ}\partial_1 X^J := \mathcal{P}_I + iB_{IJ}\partial_1 X^J, \tag{A.16}$$

where we have introduced $\mathcal{P}_I$ as defined above. This turns out to be a convenient variable for expressing the HJ equation. The Hamiltonian is

$$H = P_I \partial_2 X^I - L = \mathcal{P}_I \partial_2 X^I - \sqrt{g}. \tag{A.17}$$

Eliminate $\partial_2 X^I$ in terms of $\mathcal{P}_I$ and note that $\mathcal{P}_I \partial_1 X^I = 0$, we obtain

$$H = \frac{\sqrt{g}}{g_{11}}(G^{IJ}\mathcal{P}_I\mathcal{P}_J - g_{11}). \tag{A.18}$$

And we obtain the HJ equation $H = 0$,

$$G^{IJ}\mathcal{P}_I\mathcal{P}_J = G_{IJ}\partial_1 X^I \partial_1 X^J. \tag{A.19}$$

This is the form of HJ equation we used in the main text of the thesis.

A.4 Cancelation of UV divergences up to order $(g^2N)^2$

We first demonstrate that that the scalar propagator and the gauge boson propagator in the Feynman gauge remains equal up to first order in g^2N. The simplest way to show this is to use superspace Feynman graphs. In terms of superfields, the Lagrangian for the β-deformed SYM theory is

$$\begin{aligned} L &= \int d^2\theta d^2\overline{\theta}\, \mathrm{Tr}(e^{-gV}\overline{\Phi}_i e^{gV}\Phi_i) + \frac{1}{2g^2}\int d^2\theta\, \mathrm{Tr} W^\alpha W_\alpha + \text{c.c.} \qquad \text{(A.20)} \\ &+ ih\int d^2\theta\, \mathrm{Tr}(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2) + ih^*\int d^2\overline{\theta}\, \mathrm{Tr}(e^{i\pi\beta}\overline{\Phi}_1\overline{\Phi}_2\overline{\Phi}_3 - e^{-i\pi\beta}\overline{\Phi}_1\overline{\Phi}_3\overline{\Phi}_2). \end{aligned}$$

Using $f_{abc} := -i\mathrm{Tr}(T_a[T_b, T_c])$, $d_{abc} := \mathrm{Tr}(T_a\{T_b, T_c\})$, the superpotential can be written as

$$-h(f_{abc}\cos\pi\beta + d_{abc}\sin\pi\beta)\int d^2\theta\, \Phi_1^a\Phi_2^b\Phi_3^c + \text{c.c.}\,. \tag{A.21}$$

The relation between h and g is obtained from the requirement of superconformal invariance, which gives up to two-loop order [56, 57],

$$|h|^2\big(C_2\cos^2\pi\beta + D_2\sin^2\pi\beta\big) = Ng^2. \tag{A.22}$$

Figure A.1: 1-loop contribution to scalar propagator

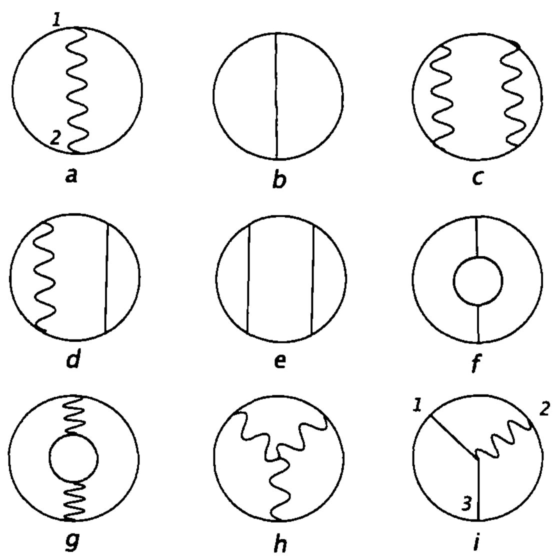

Figure A.2: Feynman diagrams of leading and next-to-leading orders

Here $f_{abc}f_{a'bc} = \delta_{aa'}C_2$, $d_{abc}d_{a'bc} = \delta_{aa'}D_2$ and $\mathrm{Tr}(T_aT_{a'}) = \delta_{aa'}/2$. Now the 1-loop correction to the scalar propagator is contained in the diagrams in figure 1. It is obvious that the graph (b) is independent of β. For the graph (a), it has a interaction vertex proportional to $|h|^2(f_{abc}f_{a'bc}\cos^2\pi\beta + d_{abc}d_{a'bc}\sin^2\pi\beta)$. Using the superconformal invariance condition (A.22), this is equal to $g^2N\delta_{aa'}$ and is independent of β. Thus the one loop contribution to the scalar propagator is independent of β. It is obvious that the one loop contribution to the gauge boson propagator is also independent of β. Using the result of [88], we conclude that the scalar propagator and the gauge boson propagator remains equal up to first order in g^2N.

Using this result, it is easy to see that the Wilson loop operator (1.22) is free from UV divergence up to order $(g^2N)^2$ if the constraint (1.23) is satisfied. The proof is a slight adaption of the computation of [85]. At leading and next-to-leading orders, we have the Feynman diagrams given in figure 2. The linear divergences in diagrams (a-g) got canceled out immediately due to the equality of the 1-loop corrected scalar and gauge boson propagators. As for the diagrams (h) and (i), we have

$$\begin{aligned}(h)+(i) &= 2(g^2N)^2\int d^4x \oint ds_1ds_2ds_3\theta_c(s_1,s_2,s_3)\cdot \\ &\quad \cdot(D_{xx_1}\partial_\lambda D_{xx_2}-\partial_\lambda D_{xx_1}D_{xx_2})D_{xx_3}\dot{x}_3^\lambda\cdot(\dot{x}_1^\mu\dot{x}_2^\nu\delta_{\mu\nu}-\dot{y}_1^i\dot{y}_2^i) \qquad (A.23)\end{aligned}$$

The contribution to (A.23) from the region $s_1\sim s_2\sim s_3$ is linear divergent for a generic loop,

$$(h)+(i)\sim\oint ds_1\frac{1}{\epsilon}(\dot{x}_1^2-\dot{y}_1^2+\epsilon). \qquad (A.24)$$

However when the constraint (1.23) is satisfied, the contribution is finite. Thus we conclude that the Wilson loop operator (1.22) has a expectation value that is free from UV divergence up to order $(g^2N)^2$ when the constraint is satisfied.

We also remark that, due to the equality of the propagators, the Wilson loop operator with the constraint [85]

$$\dot{y}^i=M^i_\mu\dot{x}^\mu,\quad M^i_\mu M^i_\nu=\delta_{\mu\nu} \qquad (A.25)$$

has expectation value 1 up to order $(g^2N)^2$. This Wilson loop possibly has an exact expectation value 1 just as in the $\mathcal{N}=4$ theory.

Appendix B

B.1 Some formulas of Sasaki-Einstein manifolds

The three roots of cubic satisfy

$$y_{q+} + y_{q-} + y_3 = 3/2, \quad y_{q+}y_{q-} + y_{q+}y_3 + y_{q-}y_3 = 0, \quad 2y_{q+}y_{q-}y_3 = -a \qquad \text{(B.1)}$$

and also can be expressed in terms of p, q

$$y_{q\pm} = \frac{1}{4p}(2p \pm 3q - \sqrt{4p^2 - 3q^2}), \quad y_3 = \frac{1}{4p}(2p + 2\sqrt{4p^2 - 3q^2}) \qquad \text{(B.2)}$$

and the period of a (4.10), can be rewritten in a more compact form

$$l = -\frac{q}{4p^2 y_{q+} y_{q-}}, \qquad \text{(B.3)}$$

which is always positive since y_{q-} is negative. The volume $Y^{p,q}$ is given by

$$Vol(Y^{p,q}) = \frac{q(2p + \sqrt{4p^2 - 3q^2})l\pi^3}{3p^2} \qquad \text{(B.4)}$$

and is bounded by

$$Vol(T^{1,1}/Z_p) > Vol(Y^{p,q}) > Vol(S^5/Z_2 \times Z_p). \qquad \text{(B.5)}$$

B.2 Definition of functions used in $L^{p,q,r}$ string solutions

In order to make the presentation shorter we define the following functions as

$$d_1(x,\theta) \equiv \left(\frac{(\alpha-x)^2}{\alpha^2}s_\theta^2 + \frac{\Delta s_\theta^2 + hc_\theta^2(\alpha-x)^2}{\rho^2\alpha^2}\right)s_\theta^2\,, \tag{B.6}$$

$$d_2(x,\theta) \equiv \left(\frac{(\beta-x)^2}{\beta^2}c_\theta^2 + \frac{\Delta c_\theta^2 + hs_\theta^2(\beta-x)^2}{\rho^2\beta^2}\right)c_\theta^2\,, \tag{B.7}$$

$$d_3(x,\theta) \equiv \left((\alpha-x)(\beta-x) + \frac{\Delta - h(\alpha-x)(\beta-x)}{\rho^2}\right)\frac{c_\theta^2 s_\theta^2}{\alpha\beta}\,, \tag{B.8}$$

which have the following partial derivatives. With respect to θ:

$$\partial_\theta d_1(x,\theta) = \frac{s_{2\theta}}{\alpha^2(\alpha-\beta)}\left(\mu + \alpha(\alpha-\beta)(\alpha-x) - \frac{4\mu(\alpha-x)^2}{(\alpha+\beta-2x+(\alpha-\beta)c_{2\theta})^2}\right)\,,$$

$$\partial_\theta d_2(x,\theta) = \frac{s_{2\theta}}{\beta^2(\alpha-\beta)}\left(\mu + \beta(\beta-\alpha)(\beta-x) - \frac{4\mu(\beta-x)^2}{(\alpha+\beta-2x+(\alpha-\beta)c_{2\theta})^2}\right)\,,$$

$$\partial_\theta d_3(x,\theta) = \frac{\mu s_{2\theta}\left((x-\alpha c_\theta^2)c_\theta^2 - (x-\beta s_\theta^2)s_\theta^2\right)}{\alpha\beta\left(-x+\alpha c_\theta^2+\beta s_\theta^2\right)^2}$$

and with respect to x:

$$\begin{aligned}\partial_x d_1(x,\theta) = &-\frac{s_\theta^2}{8\alpha^2\left(-x+\alpha c_\theta^2+bs_\theta^2\right)^2}\Big(4\mu + \alpha\left(3\alpha^2+2\alpha\beta+3\beta^2-8(\alpha+\beta)x+8x^2\right) + \\ &+4(-\mu+\alpha(\alpha-\beta)(\alpha+\beta-2x))c_{2\theta} + \alpha(\alpha-\beta)^2c_{4\theta}\Big)\,,\end{aligned} \tag{B.9}$$

$$\begin{aligned}\partial_x d_2(x,\theta) = &-\frac{c_\theta^2}{8\beta^2\left(-x+\alpha c_\theta^2+\beta s_\theta^2\right)^2}\Big(4\mu + \beta\left(3\alpha^2+2\alpha\beta+3\beta^2-8(\alpha+\beta)x+8x^2\right) + \\ &+4(\mu+(\alpha-\beta)\beta(\alpha+\beta-2x))c_{2\theta} + (\alpha-\beta)^2\beta c_{4\theta}\Big)\,,\end{aligned} \tag{B.10}$$

$$\partial_x d_3(x,\theta) = -\frac{\mu s_{2\theta}^2}{4\alpha\beta\left(-x+\alpha c_\theta^2+\beta s_\theta^2\right)^2}\,. \tag{B.11}$$

We use the above expressions to write the equations of motion and the Virasoro constraints in a more compact form.

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