STRUCTURALISM IN DIFFERENTIAL EQUATIONS

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> There are many things one would say about [the Dedekind cut corresponding to an irrational number] such as that it is a set of infinitely many things . . . that one would certainly be most reluctant to impose as a burden on the number itself. (Dedekind letter to Weber 24 January 1888, quoted by Stein (1988, p. 248).)
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> The field of differential equations has never been transformed in a profound way by the intrusion of structuralist methods. (Abstract for the session *The Limits of Mathematical Structuralism: a practice-oriented analysis* 17th International Congress on Logic, Methodology and Philosophy of Science and Technology, Buenos Aires.)

1. Introduction

Whether the field of differential equations has been transformed by structuralist methods depends on what is meant by "structuralist." Carter (2023, p. 214) describes narrower and broader scopes for structuralism:

> mathematical activities (reasoning or introducing new entities) rely not only on relations internal to the considered structures; equally important—as I will show—are "global" relations, the relations being set up between different structures or mathematical fields.

Section 2 associates narrow scope structuralism with Benacerraf (1965) *What numbers could not be*, and broad scope with Dedekind (1872) *Continuity and Irrational Numbers.* Sections 3–6 describe how *functional analysis* parallels Dedekind, and how it is central in differential equations teaching and research. Section 7 describes the current practice of "definition up to isomorphism." While practice in functional analysis certainly does not determine a full metaphysics of mathematical existence, Section 8 explains in what way and for what reason the practice is "structuralist." Benacerraf says any attempt to specify uniquely what numbers *are* "miss(es) the point of what arithmetic, at least, is all about" (1965, p. 69). This paper argues that any similar attempt for the spaces of functional analysis misses the the point of that subject.

1.1. **Sources.** We adopt three paradigms for the mathematics: a calculus textbook *Differential Equations and Linear Algebra*, Strang (2015); advanced undergrad lectures on *Functional Analysis* by Stein and Shakarchi (2011); and a research survey, Lemarié-Rieusset (2024) *The Navier-Stokes Problem in the 21st Century.* Typical current mathematics in content, all three are unusually up to date in outlook and unusually informative on history.

Strang (2015) grew from an inspired reorganization of MIT's differential equations course for engineering students. Stein and Shakarchi (2011) is one of four volumes from Elias Stein's radically re-conceived analysis sequence at Princeton. These celebrated lectures emphasize how problems in the inchoate 19th century insights of Charles Fourier led to current methods (Fefferman et al., 2012; Wikipedia contributors, 2023b). The title of Lemarié-Rieusset (2024) declares its focus on the latest methods for one famous equation: Navier-Stokes.

1.2. **A timeline of structural methods for differential equations.** An outline of the events creating this mathematics shows structural methods are no abstract alternative to concrete calculation. They are calculating tools.

- 1820: Fourier solves important differential equations by using "functions" that violate the (later) set theoretic definition of function. The most familiar today is the Dirac delta function $\delta(t)$.
- 1927: Dirac uses $\delta(t)$ and other "improper functions" in Quantum Mechanics, noting they are not functions by the usual mathematical definition (Dirac, 1930, p. 60ff.).
- 1934: Leray extends Fourier by deep use of topology, creating the modern approach to Navier-Stokes and many other differential equations.
- 1936: Sobolev generalizes the use of *topological vector spaces* (Babich, 2009).
- 1944: Schwartz organizes all this in a *theory of distributions* (Barany, 2018).

Tao (2008a,b) puts the current state of the art very concisely.

Today the Dirac delta $\delta(t)$ and related "functions" occur as calculating tools in standard second-year calculus and engineering math textbooks, with examples of their use but no precise definition. More or less rigorous versions of all the topics in this list are standard upper-level undergraduate pure and applied math.

2. Two scopes for structuralism

Structuralist philosophy of mathematics pursues "the image of mathematics as revolving around the concept of structure" (Corry, 2004, p. 337). This image is more or less true to different areas of mathematics at different times. The contrast between narrow and broad scope structuralism already occurs in Benacceraf's *What numbers could not be* which opens by quoting R.M. Martin:

> [T]he mathematician focuses primarily upon mathematical structure ... seeing how one structure is "modelled" in another, or in exhibiting some new structure and showing how it relates to previously studied ones.... (Martin quoted by Benacerraf 1965, p. 47)

This is broad scope structuralism. When it refers to entirely isomorphism invariant means we will call it "Dedekind structuralism." We could as well associate it with Emmy Noether, or category theory, or many others. To be clear, Noether not only produced mathematics, she taught a structuralist conception of how it should be produced. One of her great students wrote of "Noether's principle: base all of algebra so far as possible on consideration of isomorphisms" (Krull, 1935, p. 4).

Benacerraf refocuses the question in a narrower way:

> [In] an abstract structure [...] the 'elements' of the structure have no properties other than those relating them to other 'elements' of the same structure. (1965, p. 70)

Call this "Benacerraf structuralism." This image of structural relations holding only among the elements of one structure is faithful to some current mathematics:

(1) Strictly elementary arithmetic. (But not even introductory number theory.)
(2) Axiomatic geometry. (But not analytic or differential geometry.)
(3) Some model theory.

It is not true to much undergraduate mathematics, let alone research.

2.1. **Dedekind's continuum, and *functional analysis*.** Dedekind defined the real numbers by beginning with the rational numbers, and saying irrational real numbers *correspond to* cuts on the rational numbers:

> Whenever, then, we have to do with a cut (A_1, A_2) produced by no rational number, we create a new, an irrational number α, which we regard as completely defined by this cut (A_1, A_2); we shall say that the number α corresponds to this cut, or that it produces this cut. (Dedekind, 1872, p. 15)

The irrational number α is assigned no properties except what follows from being greater than any rational number in A_1 and less than any in A_2. Dedekind explicitly refuses to say irrational numbers *are* cuts, because that would assign them irrelevant set theoretic properties. But his whole understanding of irrational numbers—like every analyst's then or now—rests on knowing which rational numbers are less than a given irrational α, and which are greater. So Dedekind's real numbers do not have only properties "relating them to each other." He defines them in relation to *the antecedently assumed rational numbers* $\mathbb{Q}$, which form an indispensable structure themselves. Dedekind defines all his number systems, from the natural numbers $\mathbb{N}$ through $\mathbb{Q}$ and $\mathbb{R}$ only up to isomorphism, but each in relation to the ones before.[1] So Dedekind is structuralist, but not Benacerraf structuralist.

Of course Dedekind also knew—like every analyst then or now—irrational numbers can be specified by Cauchy sequences of rational numbers. This follows from his definition of irrationals as corresponding to cuts. Dedekind just refuses to say a real number *is* a cut on the rational numbers, or *is* an equivalence class of Cauchy sequences of rational numbers. Cuts and sequences are equally indispensable to working with real numbers and neither has a claim to be what real numbers *are*. Mathematicians today often prefer an explicitly isomorphism invariant higher-order definition: Let the real numbers $\mathbb{R}$ be any *complete ordered field*.

These three treatments of the real numbers are closely analogous to the structural methods of *functional analysis*.

3. A remarkable, slightly illegal function

> Now we meet a remarkable function $\delta(t)$. This "delta function" is everywhere zero, except at the instant $t = 0$. In that one moment it gives a unit input.... This "impulse" is by no means an ordinary function. (Strang (2015, p. 23).)

Since Fourier's 1822 *Analytic Theory of Heat*, a central tool for solving differential equations has been the *Dirac delta function* $\delta(t)$. Fourier did not use the symbol δ but worked with this integral expression for a function of the variable t:

$$\int_{q=0}^{q=\infty} \cos(q \cdot t)\, dq.$$

[1]Ferreirós (2007, Ch. VII), see also Ferreirós and Reck (2020). Reck (2023) reviews other interpretations including by Dedekind's "philosophical critics."

Fourier's critics were more right than wrong when they called this expression nonsense.[2] But Fourier applied standard rules of calculus just as if this integral did mean something. He got impressive, independently verifiable solutions to difficult differential equations. Today nearly no one ever sees this ill-defined integral.[3]

Textbooks now introduce $\delta(t)$ as a function with $\delta(t) = 0$ for $t \neq 0$, and $\delta(0)$ so high that the area under the graph is 1. They refer to $\delta(t)$ as "impulse input" and warn it is *not* an ordinary function. Strang (2015, p. 98) calls it "slightly illegal." No function in the set theoretic sense has the required properties.

Textbooks give one step further precision by an integral equation implicit in Fourier's work. For all functions $g : \mathbb{R} \to \mathbb{R}$:

$$\int_{t=-\infty}^{t=\infty} g(t) \cdot \delta(t)\, dt \;=\; g(0) \tag{1}$$

This has successfully taught math, physics, and engineering students to use $\delta(t)$. But this "integral sign" $\int$ cannot mean the familiar Riemann (or less familiar Lebesgue) integral. With no definition of this $\int$, students just gain intuition from examples using Equation 1. Filling it out rigorously is a good bit of work which is done by Stein and Shakarchi (2011, p. 100f.) for example.

An alternative approach motivates $\delta(t)$ by infinite sequences of curves like the sequence begun in Fig. 1. These are *normal* or *Gaussian* curves with mean 0, and successively smaller standard deviation. So the area under each curve is 1, and they eventually become vanishingly small everywhere but $t = 0$.

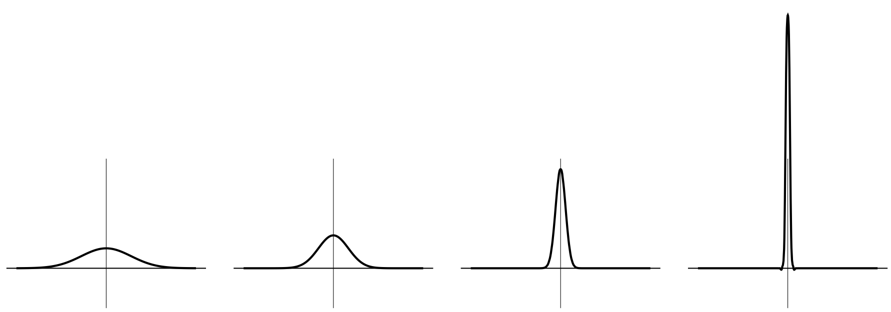

FIGURE 1. Smooth functions approaching $\delta(t)$.

Intuitively, think of these curves as approaching or *converging to* the graph of $\delta(t)$. But geometrically they converge to the x-axis plus a vertical line up the y-axis, and that is not the graph of a function. It takes a good bit of work to spell out the correct, relevant sense of *convergence* using *topological vector spaces.* But then this can be made a rigorous definition of $\delta(t)$. See Stein and Shakarchi (2011, p. 146 Ex. 4).

[2]See (Lützen, 1982, p. 113) and many references to Fourier in (Kline, 1972).

[3]Related well-defined integrals show $\delta(t)$ is the Fourier transform of 1 (Strang, 2015, Example 5, p. 441).

3.1. **Remarkable *weak* derivatives.**

> This was the great talent of Schwartz: to give a simple idea that works. (Bourbaki member Pierre Cartier 2021)

These methods would not work if they did not well match relevant intuitions, including intuitions of the calculus. One key to using the Dirac delta function is that this "function" has a derivative, written $\delta'(t)$.

Certainly $\delta'(t)$ is not a derivative of $\delta(t)$ in the sense of limits of *difference quotients* the way Calculus I classes define derivatives. It cannot be that, since $\delta(0)$ has no specifiable value to begin with. Rather, $\delta'(t)$ is a derivative in a symbolic sense as some (but not all) familiar rules of calculus apply to it. It is introduced by the same means we just used for $\delta(t)$: It is motivated verbally, it has a suggestive integral equation, and smooth curves can approach $\delta'(t)$. All three ways are made rigorous by topological vector space methods the same as for $\delta(t)$. See Section 5.

Putting it in words, $\delta'(t) = 0$ for $t \neq 0$. This makes perfect sense since $\delta(t)$ is constant when $t \neq 0$. But let t approach 0 from the negative side. From its value of 0 for $t < 0$, $\delta'(0)$ first rockets up to infinity, then down to negative infinity, and then returns to 0. All this action happens over the single point $t = 0$. Clearly this is not possible for any function from $\mathbb{R}$ to $\mathbb{R}$ as defined in set theory. Since it cannot be a set theoretic function, but it follows (many of) the calculating rules for a derivative, it is called a *weak derivative* of $\delta(t)$.

To visualize $\delta'(t)$, picture the infinitely high and narrow limit of smooth curves as in Figure 2. These smooth curves are the derivatives of normal curves. As $\delta(t)$

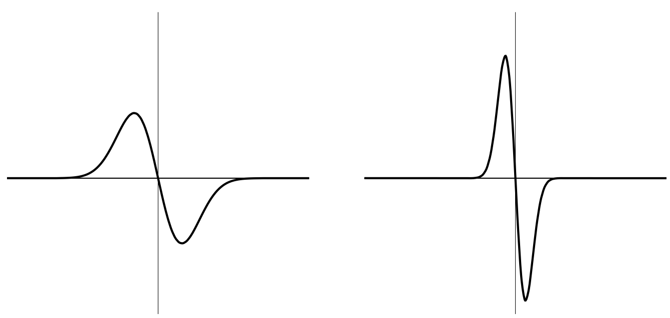

FIGURE 2. Smooth functions approaching $\delta'(t)$.

is a kind of limit of ever higher narrower normal curves, so $\delta'(t)$ is that kind of first high then low narrow limit of their derivatives.

The integral equation for $\delta'(t)$ says, for all functions $g:\mathbb{R}\to\mathbb{R}$ with a well defined derivative $g'(t)$:

$$\int_{-\infty}^{\infty} g(t) \cdot \delta'(t)\, dt \;=\; -g'(0) \tag{2}$$

Any reference on $\delta(t)$ will explain the negative sign in Equation 2. In short, this makes (many of) the usual rules of calculus work in this broader context.

4. Numerical methods using weak derivatives

Given a differential equation $F(u) = 0$ for a function u, we may want numerical estimates of values $u(a), u(b), u(c), \ldots$ at specified points $a, b, c, \ldots$.

(1) Practical applications always need specific calculated values $u(a), u(b), u(c)\ldots$.
(2) If exact solutions for u are unknown, numerical approximations may be a good way to explore the problem.

Numerical methods are extremely important, extensively developed, and often extremely reliable; but the pitfalls and the general theory are extremely intricate. See discussion by Sterrett (2023).

Numerical methods for differential equations often use weak derivatives in the sense of our Section 3.1 rather than classical derivatives (Evans, 2010, p. 8 and much passim). The widely stated reason for this in the literature is that numerical approximations normally are patched together from individually smooth pieces, but with "kinks" where different pieces join. They are not classically differentiable at the "kinks," but do have weak derivatives. Section 6 returns briefly to this.

5. Function spaces

Rigorous versions of generalized functions like $\delta(t)$ and $\delta'(t)$ are due to Sobolev and Schwartz using *function spaces*.[4] Tao (2008b, p. 210) says the elements of function spaces "are functions." But in that same series of articles Tao states he uses "function" in a wider sense than functions as defined in set theory (2008a, p. 185). The elements of function spaces most often *are not* functions in the set theoretic sense. They are always intuitively *like* functions defined in set theory, the way $\delta(t)$ and $\delta'(t)$ are intuitively *like* set theoretic functions from $\mathbb{R}$ to $\mathbb{R}$. And all function spaces are *structurally related* to spaces of set theoretic functions. We can see an example:

There is a function space called $C_c^\infty(\mathbb{R})$ containing those set theoretically defined functions $f: \mathbb{R} \to \mathbb{R}$ which are infinitely differentiable and have $f(x) = 0$ for all x outside some finite interval. It carries a topology we will not define.

The point for us is that $C_c^\infty(\mathbb{R})$ has a *dual space* $\mathcal{D}'(\mathbb{R})$ whose elements are definable in basically three ways:

(1) Continuous linear functions from the space $C_c^\infty(\mathbb{R})$ to $\mathbb{R}$.
   (a) This officially defines $\mathcal{D}'(\mathbb{R})$ for Stein and Shakarchi (2011, p. 100).
   (b) Compare defining $\mathbb{R}$ by Dedekind cuts on the rational numbers.

(2) Equivalence classes of suitable sequences of functions in $C_c^\infty(\mathbb{R})$.
   (a) Stein and Shakarchi (2011, p. 146 Ex. 4) shows this can define $\mathcal{D}'(\mathbb{R})$.
   (b) Compare defining $\mathbb{R}$ by equivalence classes of Cauchy sequences of rational numbers.

(3) Up to isomorphism by more abstract properties.
   (a) Compare defining $\mathbb{R}$ as a complete ordered field.

Elements of $\mathcal{D}'(\mathbb{R})$ are called *Schwartz distributions* on $\mathbb{R}$ no matter which definition of $\mathcal{D}'(\mathbb{R})$ is used. This is often shortened to just *distributions*. The Dirac delta function $\delta(t)$ can be precisely defined as the Schwartz distribution on $\mathbb{R}$ which

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[4] Andrei Rodin points out the early history is poorly known and may go back to Nikolai Gunter in Saint Petersburg in 1916. That would be valuable to know but current practice traces to Sobolev and Schwartz.

satisfies Equation 1. Authors often use distributions without specifying which definition they mean. The definitions of $\mathcal{D}'(\mathbb{R})$ agree up to isomorphism so they all work alike exactly as the definitions of $\mathbb{R}$ all work alike. Section 7 returns to this.

The space of Schwartz distributions $\mathcal{D}'(\mathbb{R})$ is one of many, many different *function spaces* used in *functional analysis*. These are not Benacerraf structures, since the elements of one function space are not only related to the elements of that space.

They are Dedekind structures: They are defined, in practice, up to isomorphism. And the elements of a function space are defined by relations to each other, and to the elements of a few other related structures. In our example, the elements of $\mathcal{D}'(\mathbb{R})$ are related to the real numbers $\mathbb{R}$ and to elements of the more basic function space $C_c^\infty(\mathbb{R})$.

This barely touches the surface of current functional analysis. But be assured the more advanced reaches are no less structural than this!

5.1. **Aside on Dedekind cuts as order-preserving functions.** Some readers may enjoy a fuller account of Item 1b above, comparing Dedekind cuts to continuous linear functions. The point is:

> A Dedekind cut gives the same information as a continuous order-preserving function from $\mathbb{Q}$ to the ordered set $\{0, 1\}$ (with $0 \leq 1$, and topology making $\{0\}$ open and $\{1\}$ closed).

This rarely comes up outside textbooks on order-theory. But it is not hard.

Partitioning $\mathbb{Q}$ into a lower part A_1 and an upper part A_2 is just the same as giving an order preserving function $\mathbb{Q} \to \{0, 1\}$ mapping A_1 to 0 and A_2 to 1. Dedekind (1872, p. 13) notes each irrational number corresponds to one such partition of $\mathbb{Q}$, while each rational number q corresponds to two since q might be the greatest element of A_1, or the least element of A_2. We can remove this ambiguity by requiring that part A_1 of a Dedekind cut must have no greatest element. In other words A_1 must be open in $\mathbb{Q}$. And this is exactly the same as requiring $\mathbb{Q} \to \{0, 1\}$ to be continuous for this topology on $\{0, 1\}$.

So the real numbers $\mathbb{R}$ can be defined (up to isomorphism) as continuous order preserving functions $\mathbb{Q} \to \{0, 1\}$. Schwartz distributions commonly are defined (up to isomorphism) as continuous linear functions $C_c^\infty(\mathbb{R}) \to \mathbb{R}$.

6. What Fourier's "functions" do for differential equations

Generalized functions u solving some differential equation are studied numerically as described in Section 4, or when set theoretically defined solutions do not exist or are not yet known. The Navier-Stokes equation has been studied this way for 90 years. As another use, a differential equation may depend on some input function. Then the general solutions for arbitrary inputs may be well organized around the special solution with the Dirac delta $\delta(t)$ as input.

6.1. **Leray's weak solutions to Navier-Stokes.** The Navier-Stokes equation expresses Newton's law $F = ma$ for the flow of viscous fluids.[5] Its widespread use in physics and engineering relies on coarse approximations and ad hoc corrections because the math is so hard (Patton, 2023; Sterrett, 2023).

[5]Introductions emphasizing pure mathematics are: Fefferman (2008), Lemarié-Rieusset (2024, opening chapters), McLarty (2023), Wikipedia contributors (2023a). The huge engineering and physics literature on Navier-Stokes is beyond the scope of this paper.

The equation posits three forces on a flowing fluid: viscous drag within the flow, fluid pressure, and an external force such as gravity. The 2-dimensional case models a fluid layer of negligible depth and it has been completely solved (Fefferman, 2008). For 3-dimensional flow, you can win a $\$1,000,000$ Clay Millennium Prize without finding a single solution just by settling the existence of smooth solutions.

> Prove or refute: The 3-dimensional Navier-Stokes equation has a *smooth* global solution for every smooth initial condition (see details in Fefferman (2000)).

Here a smooth global solution means a function in the set theoretic sense, with well-defined derivatives at every point meeting the equation. Current work on this is heavily based on Leray's 1934 result: The 3-dimensional Navier-Stokes equation has a global solution *in Fourier's sense* for every smooth initial condition.

These are called *weak solutions*. A weak solution could be a function in the set theoretic sense. Or it could be a *generalized function* which verifies many rules of calculus in a symbolic way while not being a function set theoretically (or at least not current known to be one).

Leray (1934) took advantage of two facts:

(1) "Functions" in Fourier's sense include all smooth functions, even all continuous functions, but there are far more functions in Fourier's sense (as there are more real numbers than rational).[6]
(2) The key point: Spaces of (what I have called) Fourier's "functions" have good topological properties that the related spaces of set theoretically defined functions lack. This is precisely analogous to the continuous real line $\mathbb{R}$ supporting techniques of calculus that do not work for the discontinuous rational line $\mathbb{Q}$.

Leray found a nice kind of approximate solutions to the 3-dimensional Navier-Stokes equation, and gave an innovative topological argument showing suitable sequences of these approximations converge to weak solutions.[7]

6.2. Fundamental solutions.

> The most important solution to a linear differential equation [is] the *fundamental solution*. In engineering [it is called] the *impulse response*. (Strang, 2015, p. 78).

Strang expands on fundamental solutions throughout his book. For a theoretical introduction see Stein and Shakarchi (2011, p. 125–34). Lemarié-Rieusset (2024, p. 715) notes they are central to his discussion of classical solutions to Navier-Stokes.

Suppose a savings account pays 3% yearly interest compounded continuously. Let $y(t)$ be the amount in that account at time t measured in years. Then the derivative $y'(t)$ is the rate of change of the balance at time t and it is a sum of two terms:

$$y'(t) = 0.03 \cdot y(t) + f(t). \tag{3}$$

[6] Leray's solutions are *measurable functions* and are often treated as distributions as a convenient more general context. See the introduction to Chapter 5 of Lemarié-Rieusset (2024).

[7] The proof is non-constructive. While Leray's solutions have good properties beyond being weak solutions, striking work by Albritton et al. (2022) confirmed the expectation that they are not unique.

Here $0.03 \cdot y(t)$ is the interest on the balance of $y(t)$, and $f(t)$ is the rate of deposits or withdrawals made at time t. Mathematically f is called the *input function.*

The fundamental solution to Equation 3 is just the solution with $\delta(t)$ as input. That means there are no deposits or withdrawals at any time except $t = 0$ when the balance $y(0)$ instantaneously jumps to 1. Then the balance for $t \geq 0$ grows exponentially at the rate of interest:

$$y' = 0.03 \cdot y(t) + \delta(t) \quad \text{with solution } \begin{cases} y(t) = 0, & \text{for } t < 0; \\ y(t) = e^{0.03 \cdot t}, & \text{for } t \geq 0. \end{cases} \tag{4}$$

To solve Equation 3 for any input $f(t)$, think of f as a "sum" of continuously many successive impulses where the value $f(t)$ is the magnitude of the impulse at time t. Then the solution with input f is the "sum" of continuously many successively shifted impulse solutions with these variously sized impulses.[8]

This method applies widely:

(1) The *fundamental solution* to any *linear differential equation with constant coefficients* and an input function, is the solution for input $\delta(t)$.

(2) Any *linear partial differential equation with constant coefficients* can be treated by a multi-variable analog of Equation 4 using a multi-variable version of $\delta(t)$.

The Navier-Stokes equation is not linear so it has no fundamental solution. But the method of fundamental solutions is so productive that a major part of Navier-Stokes research rests on fundamental solutions to related linear equations.

7. Working up to isomorphism

Dedekind had no such term as "structuralism." He expressed his view imagistically or philosophically, in terms of creating new objects, and Frege criticized Dedekind at length for this (Hallett, 2019). Still today mathematicians rarely discuss "structuralism." But now they have precise, standard techniques for defining structures "up to isomorphism" and working with them that way.

Today mathematical isomorphism is always sorted.[9] Two topological vector spaces might be isomorphic *as vector spaces*, but not in any topologically continuous way, so they are not isomorphic as *topological* vector spaces.

Section 5 said the three approaches to the real numbers "all work alike." Precisely, they all imply $\mathbb{R}$ is a *complete ordered field.* And there is only one complete ordered field, up to isomorphism of ordered fields.

A statement $\varphi(F)$ about ordered fields F is *invariant under isomorphism of ordered fields* if and only if $\varphi(F_1)$ agrees with $\varphi(F_2)$ whenever F_1 and F_2 are isomorphic as ordered fields. Intuitively such a statement just talks about the algebra and the order on F and not about any set theoretic construction. Two typical examples suggest why these are the mathematically important statements about an ordered field: a real number α has a square root if and only if $0 \leq \alpha$; and every upper bounded subset of $\mathbb{R}$ has a Least Upper Bound (LUB). These are isomorphism invariant as they refer only to ordered field properties.

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[8]Strang (2015, p. 78) gives a fully worked example. Then, because readers "may feel uncertain about working with delta functions," he gives three ways to verify the result.

[9]The sort can be clear from context. Group theory usually (not always) uses isomorphism of groups. More intricate contexts use multiple sorts of isomorphism and the sorts have to be specified.

Beginning analysis books like Tao (2016) often specify one set theoretic construction of the real numbers. But they teach students to discuss $\mathbb{R}$ in terms invariant under isomorphism of ordered fields. Then it becomes rigorously irrelevant whether $\mathbb{R}$ was defined by Cauchy sequences, or cuts, or simply as a complete ordered field. All those definitions imply exactly the same isomorphism invariant theorems. Notably, each of them implies real numbers *can be specified by* Dedekind cuts on $\mathbb{Q}$ *and can be specified by* Cauchy sequences on $\mathbb{Q}$. It is rigorously irrelevant to standard theorems of analysis what the real numbers *are* set theoretically.

Section 5 sketched three approaches to distributions. All are useful and often used. But usually none is taken to specify $\mathcal{D}'(\mathbb{R})$ uniquely. All are taken to define $\mathcal{D}'(\mathbb{R})$ uniquely *up to isomorphism of topological vector spaces extending* $C_c^\infty(\mathbb{R})$. Everything Lemarié-Rieusset (2024) says about distributions is invariant under those isomorphisms. So Lemarié-Rieusset never chooses one set theoretic construction of distributions. It would be rigorously irrelevant for him to do so.

8. What is structuralism and what is it good for?

Two philosophic questions stand out:[10]

(1) In what sense is "Dedekind structuralism" structural? Like Zermelo Fraenkel (ZF) set theory it defines some structures in terms of others, and Benacerraf (1965) took ZF definitions to typify non-structural methods.
(2) Is "definition up to isomorphism" conceptually rigorous? Or is it a fast and loose practice that "the philosophical logician ..., sensitive to matters of ontology" can correct? (Quoting Martin in (Benacerraf, 1965, p. 47).)

8.1. **Structuralism through relevant concepts.**

Dedekind structuralism, unlike ZF, describes structures only up to isomorphism and only by relation to other specifically relevant structures. A structuralist account defines distributions by relation to real numbers and to (set theoretically well defined) differentiable functions. These relations are used in all calculations with distributions. And they are explicitly *relations*. They do not say what distributions *are*.

No statement in our three paradigm sources places numbers or distributions in the transfinite cumulative hierarchy that uniquely identifies each ZF set. Dedekind structuralism admits no question of uniquely identifying the elements of any structure. The kind of sets that Benacerraf (1965) says numbers cannot be, our argument says distributions also cannot be.

8.2. **Epistemology: trusting these concepts.**

> Socrates: [There are] people you would not care to trust (*pisteuō*) claiming they are good practitioners, if they cannot show some example of their skill—some work well carried out—once and many times. (Plato, *Laches* 185e–186a, at `www.perseus.tufts.edu`)

It is a testimony to mathematical progress that, where Frege and Russell found Dedekind's idea of "new creations" wrong, Martin only says 1960s structural mathematics leaves philosophers wanting to know more:[11]

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[10] Thanks to an anonymous referee for posing these sharply.
[11] On Frege and Russell see Hallett (2019); Heis (2020).

> The philosophical logician.... will not be satisfied with being told merely that such and such entities exhibit such and such a mathematical structure. He will wish to inquire more deeply into what these entities are.... he will wish to ask whether the entity dealt with is sui generis or whether it is in some sense reducible to (or constructible in terms of) other, perhaps more fundamental entities. (Martin quoted by Benacerraf 1965, p. 47)

Martin was wrong if he thought structuralist mathematics neglects set theoretic constructions. Section 5 gave two set theoretic constructions for distributions from Stein and Shakarchi (2011). But Stein and Shakarchi do not offer either construction as *ontology*. They use both. They take the construction by linear functions as definitive (item 1a of our Section 5). But this makes no difference after their Chapter 3. The construction by sequences of curves is definitive for Lighthill (2008, p. 10f.).

Lemarié-Rieusset (2024) uses distributions without knowing or caring whether the reader defines distributions by linear functions, or sequences of smooth functions, or any other definition. Those constructions all define the space of distributions *up to isomorphism*. The theorems in (Lemarié-Rieusset, 2024)—and essentially all books on differential equations on that level—are isomorphism invariant. Those books use distributions rigorously without choosing between the constructions.

The mathematicians are right from the viewpoint of fruitfulness, logical rigor, and conceptual coherence. Philosophers could valuably tease out the social versus individual epistemology (De Toffoli, 2023). How do structural methods help individuals? How do they help organize and coordinate the community? But in plain fact we have two centuries of fruitful, successful, rough and ready use of "generalized functions" like $\delta(t)$ from Fourier to Dirac and on, made rigorous for the past 80 years by the structural function space methods of Sobolev and Schwartz. That record exhibits both the heuristic value and the epistemic reliability of structural methods, tested from many pure and applied perspectives.

9. Conclusion

It is a long road, both in the history of mathematics and in today's undergrad math curriculum, from calculus through current progress on Navier-Stokes. Our sources show extensive structural work well carried out, meeting Socrates' demand for trusting good practitioners. Not only are the theorems true. The vast work of conceiving, stating, proving, communicating, and applying them is well carried out. A philosopher like Martin (as quoted by Benacerraf) is free to ask what distributions *are* specifically, not just *up to isomorphism*. But the question is rigorously irrelevant to our paradigm sources Strang (2015); Stein and Shakarchi (2011); Lemarié-Rieusset (2024). It misses the point of what functional analysis is all about. Without settling all philosophical questions about "structuralism," existing mathematical practice does show philosophers can trust the epistemology and ontology of current, working, Dedekind-structural, functional analysis.

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