Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.14, No.1, 2024

Exponentiated Generalized Exponential Lomax Distribution with Its Associated Properties and Application

ADEYEMI-GIDADO, S. A AKOMOLAFE, A.A.
Federal University of Technology, Akure, Nigeria
aaakomolafe@futa.edu.ng

Abstract
In this research, a five parameter Exponentiated Generalized Exponential lomax (EGEL) distribution was derived from Exponentiated Generalized Family (EGF) of distribution. we consider certain results characterizing the generalization of the derived Distribution through their distribution functions and asymptotic properties. The resulting Exponentiated Generalized Exponential lomax was defined and some of the conventional properties like moment generating function, survival rate function, hazard rate function, , median, quantile, mean deviation, median deviation, skewness, kurtosis, Renyi and entropy were investigated. The estimation of the model parameters was performed using maximum likelihood estimation method. The usefulness and flexibility of the new model is illustrated using real data. The distribution was found to generalize some known distributions thereby providing a great flexibility in modeling symmetric, heavy tailed, skewed and bimodal distributions, the use of the new lifetime distribution was illustrated using failure time life data.
Keywords: Reversed Hazard function, Moment, Kurtosis, Odd function, skewness, and quantile function.
DOI: 10.7176/MTM/14-1-01
Publication date: April 30th 2024

1. Introduction

The Lomax distribution is one of the well-known distributions that is used to fit heavily tailed data. It is also referred to as a Pareto Type II distribution that has been shifted, that its support begins at zero. The Lomax (or Pareto II) distribution has widely applied in so many fields such as medical, wealth inequality and income, engineering reliability and lifetime modeling actuarial science, and biological sciences, and size of cities. The quality of statistical analysis depends heavily on the underlying probability distribution. Because of this, considerable e®ort over the years has been expended in the development of large classes of probability distributions along with relevant statistical methodologies. In fact, the statistics literature is ¯lled with hundreds of continuous univariate distributions but a real data set following the classical distributions are more often the exception rather than the reality. Since there is a clear need for extended forms of these distributions a significant progress has been made towards the generalization of some well-known distributions and their successful application to problems in areas such as engineering, economics and biomedical sciences, among others.In a context of lifetime, the Lomax distribution belongs to the family of decreasing failure rate Chahkandi and Ganjali, (2009) and arises as a limiting distribution of residual lifetimes at great age Balkema and de Hann, (1974). Different generalizations of Lomax distribution have been studied such as the exponentiated Lomax, by Abdul-Moniem and Abdel-Hameed (2012), Marshall-Olkin extended Lomax defined by Ghitany et al. (2007), McDonald Lomax investigated by Lemonte and Cordeiro (2013),Gamma Lomax introduced by Cordeiro et al. (2015), the Weibull Lomax distribution studied by Tahir et al. (2015), the Transmuted Weibull Lomax distribution given by Afify et al. (2015), On Five-Parameter Lomax Distribution studied by M. E. Mead (2016) and most recently, a new generalization of lomax distribution with increasing, decreasing and constant hazard rate by Oguntunde et al (2017). Modifying existing classical distribution to fit diverse nature of real life data using established generator or family of distribution in recent terms have been studied in literature. In this study, the Exponentiated generalize family of distribution proposed by Cordeiro et al (2013) is used to modify the existing Exponentiated lomax distribution.

2. METHODOLOGY

Derivation and Development of Exponentiated Generalized Exponential Lomax

The Exponentiated Lomax distribution density and cumulative distribution function are presented in equation (1) and (2) as:

$$g(x) = \theta\lambda(1 + \lambda x)^{-(\theta+1)} \quad (1)$$

$$G(x) = 1 - (1 + \lambda x)^{-\theta} \quad (2)$$

while equation (3) and (4) are the probability density function (pdf) and cumulative distribution function (cdf) of Exponentiated generalized family proposed by Cordeiro et al (2013)

$$f(x) = \alpha\beta\{1 - G(x)\}^{\alpha-1}[1 - \{1 - G(x)\}^{\alpha}]^{\beta-1}g(x) \quad (3)$$

$$F(x) = [1 - \{1 - G(x)\}^{\alpha}]^{\beta} \quad (4)$$

The propose Exponentiated Generalized Exponentiated Lomax (EGEL) distribution pdf is

$$f(x) = \alpha\beta k\theta\lambda(1+\lambda x)^{-(\theta+1)}\left[1-(1+\lambda x)^{-\theta}\right]^{k-1}\left(1-\left[1-(1+\lambda x)^{-\theta}\right]^{k}\right)^{\alpha-1} \times \left(1-\left(1-\left[1-(1+\lambda x)^{-\theta}\right]^{k}\right)^{\alpha}\right)^{\beta-1} \quad (5)$$

and the cdf as:

$$F(x) = \left(1-\left(1-\left[1-(1+\lambda x)^{-\theta}\right]^{k}\right)^{\alpha}\right)^{\beta} \quad (6)$$

while the Hazard function, Survival function, Reverse Hazard function and odd function are presented in equations (7),(8)(9) and (10) respectively as explained and shown below

2.1 Hazard Function

Hazard function is represented in mathematical form as

$$h(x) = \frac{\alpha\beta k\theta\lambda(1+\lambda x)^{-(\theta+1)}\left[1-(1+\lambda x)^{-\theta}\right]^{k-1}\left(1-\left[1-(1+\lambda x)^{-\theta}\right]^{k}\right)^{\alpha-1} * \left(1-\left(1-\left[1-(1+\lambda x)^{-\theta}\right]^{k}\right)^{\alpha}\right)^{\beta-1}}{1-\left(1-\left(1-\left[1-(1+\lambda x)^{-\theta}\right]^{k}\right)^{\alpha}\right)^{\beta}} \quad (7)$$

2.2 The Survival Function

$$s(x) = 1-\left(1-\left(1-\left[1-(1+\lambda x)^{-\theta}\right]^{k}\right)^{\alpha}\right)^{\beta} \quad (8)$$

2.3 The Reversed Hazard Function

$$r(x) = \frac{\alpha\beta k\theta\lambda(1+\lambda x)^{-(\theta+1)}\left[1-(1+\lambda x)^{-\theta}\right]^{k-1}\left(1-\left[1-(1+\lambda x)^{-\theta}\right]^{k}\right)^{\alpha-1}\left(1-\left(1-\left[1-(1+\lambda x)^{-\theta}\right]^{k}\right)^{\alpha}\right)^{\beta-1}}{\left(1-\left(1-\left[1-(1+\lambda x)^{-\theta}\right]^{k}\right)^{\alpha}\right)^{\beta}} \quad (9)$$

2.4 The Odds Function.

The Odd function can be shown mathematically as

$$O(x) = \frac{\left(1-\left(1-\left[1-(1+\lambda x)^{-\theta}\right]^{k}\right)^{\alpha}\right)^{\beta}}{1-(1-(1-[1-(1+\lambda x)^{-\theta}]^{k})^{\alpha})^{\beta}} \quad (10)$$

2.5 Special cases of EGEL distribution

- The Lomax (LD) distribution is a special case of EGEE when $\alpha = \beta = k = 1$.
- For $\alpha = \beta = 1$ the EGEL gives an Exponentiated Lomax (EL) distribution
- When $k = 1$ the EGL gives a member of Exponentiated Generalized Family which is Exponentiated Generalized Lomax (EGL) distribution.

Table 1:Summary of EGEL and Sub-Models

Distribution	$\boldsymbol{\beta}$	$\boldsymbol{\alpha}$	$\boldsymbol{k}$	$\boldsymbol{\theta}$	$\boldsymbol{\lambda}$
EGELD	β	α	k	θ	$\boldsymbol{\lambda}$
EGLD	β	α	1	θ	$\boldsymbol{\lambda}$
EL	1	1	K	θ	$\boldsymbol{\lambda}$
LD	1	1	1	θ	$\boldsymbol{\lambda}$

Table 1 presents sub-model of the proposed EGEL distribution. The presence of all the unknown parameter gives the proposed distribution, the absence of a shape parameter $\boldsymbol{k}$ gives the Exponetiated Generalized Lomax (EGL) distribution, the absence of two shape parameters β and $\boldsymbol{\alpha}$ leads to Exponetiated Lomax (EL) distribution while the absence of all the three shape parameters $\boldsymbol{k}$,β and $\boldsymbol{\alpha}$ gives a Lomax distribution(LD),

2.6 Quantile function

$$x = \frac{1-\left(\left[1-\left(1-\left(1-(p)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}\right)^{\frac{1}{k}}\right]^{-\frac{1}{\theta}}\right)}{\lambda} \quad (11)$$

2.7 Median

Fig. 1: Probability of Density Function of EGEE And Its Sub-Models

$$x = \frac{1-\left(\left[1-\left(1-\left(1-(0.5)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}\right)^{\frac{1}{k}}\right]^{-\frac{1}{\theta}}\right)}{\lambda} \quad (12)$$

2.8 General Moment

$$\mu^{r} = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(-1)^{i}\binom{kj-1}{i} jw_{j}k\theta\lambda^{-r}\, B(r+1,(1+i)\theta-r) \quad (13)$$

2.9 Order Statistics

$$\frac{\propto \boldsymbol{\beta}}{\boldsymbol{B(i,n-i+1)}}\boldsymbol{k\theta\lambda} * \sum_{q=0}^{n-i}\sum_{p=0}^{\infty}\sum_{l=o}^{\infty}\sum_{d=0}^{\infty}(-1)^{p+q+l+d}\frac{\Gamma(n-i+1)\Gamma(\beta(q+i)\Gamma\alpha(p+1)\Gamma(k(l+1))}{\Gamma(n-i+q+1)\Gamma(\beta(q+i)-p)\Gamma(\alpha(p+1)-l)\Gamma(k(l+1)-d)d!\,q!\,p!\,l!}(1+\lambda x)^{-\theta d} \quad (14)$$

3.0 Analysis and Results

In this section the proposed EGEL distribution is used to fit two real life data sets and the estimates obtained are used to compare the fitted values of some existing Lomax generated distribution proposed by M. E. Mead (2016).

3.1 First data set

The first real life data set adopted was used by M. E. Mead (2016). The data set in table 2 represents the relief period of twenty patients receiving an analgesic which was generated by Gross and Clark (1975).

Table 2: Twenty Patients Relief Period

1.1	**1.4**	**1.3**	**1.7**	**1.9**	**1.8**	**1.6**	**2.2**	**1.7**	**2.7**
4.1	**1.8**	**1.5**	**1.2**	**1.4**	**3.0**	**1.7**	**2.3**	**1.6**	**2.0**

Source: M. E. Mead (2016).

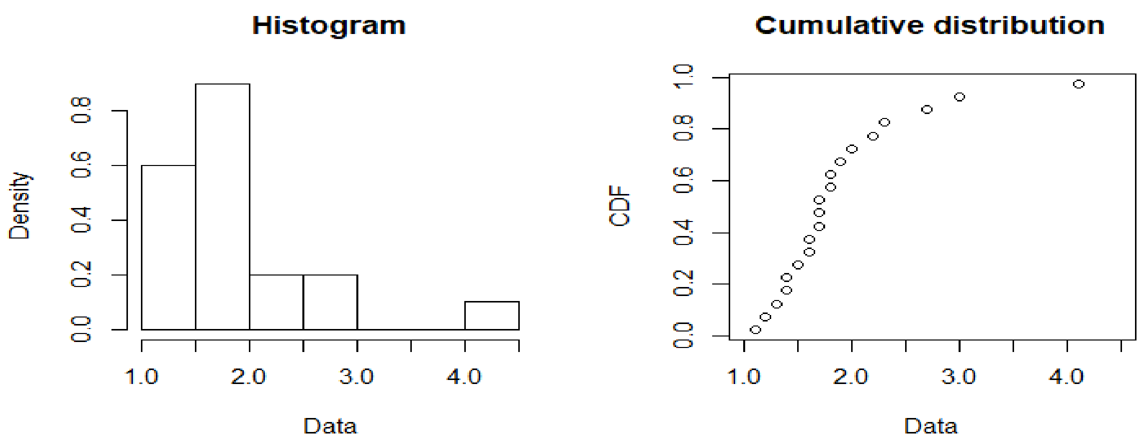

Fig. 1: Histogram and Cummulative Plot of Analgesic Data

The histogram in figure 2 describes the right skewed nature of the Analgesic data

Table 3: Summary Statistics of Analgesic Data

Min	Max	Median	S.D	Mean	Skewness	Kurtosis
1.100	4.100	1.700	0.70412	1.900	1.7197	2.9241

The statistical summary on table 3 shows that, the Analgesic data is under-disperse, non-symmetry and right skewed.

The correlation matrix of the parameters of EGEL distribution estimated from Analgesic data is:

	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\theta}$	$\boldsymbol{k}$	$\boldsymbol{\lambda}$
$\boldsymbol{\alpha}$	1.0000	0.0982	-0.9694	-0.1084	0.8895.
$\boldsymbol{\beta}$	0.0982	1.0000	-0.1929	-0.4042	0.2773
$\boldsymbol{\theta}$	-0.9694	-0.1929	1.0000	-0.0177	-0.9722
$\boldsymbol{k}$	-0.1084	-0.4042	-0.0177	1.0000	0.1685
$\boldsymbol{\lambda}$	0.8895	0.2773	-0.9722	0.1685	1.0000

Correlation matrix above shows the relationship between combine sets of parameters. The off-diagonal entries coefficients with values 0.0982 , -0.9694, -0.1084and 0.8895 shows the relationship between parameters $(\alpha,\beta),(\alpha,\theta)(\alpha\ k)$and $(\ \alpha,\lambda)$ respectively, the values -0.1929,-0.4042and 0.2773 depicts the relationship of β parameter with $\theta, k\ and\ \lambda$, while θ relationship with $k\ and\ \lambda$ shows a negative relationship with values -0.0177 and -0.9722 respectively. The final entry of 0.1685 indicates a positive relationship between parameters $K and\ \lambda$.

Table 4: Maximum Likelihood Estimate of Parameters Standard Errors in Parenthesis, Anderson-Darling($\boldsymbol{A^*}$)and Cramer-von Mises ($\boldsymbol{W^*}$) statistic

Models	**Estimates**					$\boldsymbol{-2l}$	$\boldsymbol{W^*}$	$\boldsymbol{A^*}$
EGEL ($\boldsymbol{\hat{\alpha},\hat{\beta},\hat{k},\hat{\theta}\hat{\lambda}}$)	1.2591	10.0258	6.2085	12.1344	0.6256	31.1886	0.03113	0.1834
	(2.8333)	(17.8975)	(22.8082)	(22.3150)	(2.4305)			
BEL($\hat{a},\hat{b},\hat{\beta},\hat{\theta}\hat{\lambda}$)	12.5749	2.2163	8.9554	3.4561	1.3121	31.5890	0.0413	0.2348
	(22.3500)	(6.3030)	(15.4360)	(8.6350)	(3.9370)			
BL($\hat{a},\hat{b},\hat{\theta}\hat{\lambda}$)	41.07035	1.9286	5.7740	0.4289		32.2190	0.0495	0.2888
	(41.2740)	(2.3480)	(9.0860)	(0.7340)				
EL($\hat{\beta},\hat{\theta}\hat{\lambda}$)	59.4578	14.3611	0.2059			31.7096	0.0432	0.2479
	(65.6460)	(20.4000)	(0.3860)					
McL($\hat{a},\hat{b},\hat{\beta},\hat{\theta}\hat{\lambda}$)	14.1723	16.4803	8.4082	2.7554	0.6094	34.5610	0.0897	0.5272
	(10.7050)	(37.3160)	(14.1580)	(6.6540)	(1.9920)			
TWL($\hat{a},\hat{b},\hat{\beta},\hat{\theta}\hat{\lambda}$)	8.6188	6.2149	0.2479	0.2255	0.6966	37.8040	0.1319	0.7954
	(42.8320)	(4.5010)	(0.6660)	(0.2020)	(0.3380)			
WL($\hat{a},\hat{b},\hat{\beta},\hat{\theta}$)	14.7394	5.5854	0.2633	0.2191		39.2610	0.1485	0.9065
	(64.6700)	(3.8398)	(0.6730)	(0.1840)				
GL($\hat{a},\hat{\beta},\hat{\theta}$)	26.5061	25.3134	0.9907			33.2200	0.0666	0.3911
	(24.4554)	(8.8660)	(1.6420)					

The reported values in table 4 shows the estimated values of the unknown parameters for the proposed distribution and the compared existing Beta Exponentiated Lomax (BEL), Beta Lomax (BL), Exponentiated Lomax (EL), McDonald Lomax (McL), the Transmuted Weibull Lomax (TWL), Weibull Lomax (WL), and Gamma Lomax (GL) distributions.

The -2log-likelihood (-2l) values obtained as31.1886, 31.5890, 32.2190, 31.7096, 34.5610, 37.8040, 39.2610 and 33.2200 for EGEL, BEL, BL, EL, McL, TWL, WL and GL distributions respectively with Cramer-Von Mises Test (W*) values of0.03113, 0.0413, 0.0495, 0.0432, 0.0897, 0.1319, 0.1485 and 0.0666. The Anderson-Darling(A^*) indicates values of 0.1834, 0.2348, 0.2888, 0.2479, 0.5272, 0.7954, 0.9065 and 0.3911 for all the distributions.

The goodness of fit statistics values of 31.1886, 0.03113 and 0.1834 for -2log-likelihood (-2l), Cramer-Von Mises Test (W*) and Anderson-Darling(A^*) respectively, reported the least values for the proposed EGEL distribution which shows that the proposed distribution fits the Analgesic data better than the existing distributions.

4.0 Conclusion

The Exponentiated Generalized Exponential Lomax distribution with five parameters (four shapes and one scale parameters) is derived from Exponentiated family of distribution. Some statistical properties ofthe distribution such as general moment, quantile function, survival function, hazard function, reverse hazard function, odd function, median and order statistics were derived. Some distribution such as Exponentiated Lomax, Exponentiated Generalized and Lomax were found to be special cases of the proposed distribution. The application of real life data to the proposed distribution shows it can serve as an alternative and better distribution to some existing distribution.

References

Abdul-Moniem, I. B., and Abdel-Hameed, H. F. (2012). A lifetime distribution with decreasing failure rate. *International Journal of Mathematical Education*, *33*(5), 1-7.

Akomolafe A. A., Maradesa A. (2017): The Beta-Halfnormal Distribution and its properties. *International Journal of Advanced Research and Publications* (IJARP). 1(4), 17-22.

Akomolafe, A. A. and Yussuf, T. O. (2018). Dirichilet- Multinomial Model: its mixture and Application using Bayesian Approach. *Rep Opinion* 10(2) 1-15. http://www.sciencepub.net/report doi: 10:7517/marsroj100518.03.

Chahkandi, M., and Ganjali, M. (2009). On some lifetime distributions with decreasing failure rate.*Comput Statist Data Analysis*, **53**, 4433-4440 Balkema and de Hann, 1974

Cordeiro, G. M., Ortega, E. M., and da Cunha, D. C. (2013). The exponentiated generalized class of distributions. *Journal of Data Science*, *11*(1), 1-27.

da Silva, R. V., Gomes-Silva, F., Ramos, M. W. A., andCordeiro, G. M. (2015). A new extended gammageneralized model. *International Journal of Pure and Applied Mathematics*, *100*(2), 309-335.

Ghitany, M. E., Al-Awadhi, F. A., andAlkhalfan, L. (2007). Marshall–Olkin extended Lomax distribution and its application to censored data. *Communications in Statistics—Theory and Methods*, *36*(10), 1855-1866.

Lemonte, A.J., Barreto-Souza, W. and Cordeiro, G.M. (2013). The exponentiated Kumaraswamy distribution and its log-transform. *Brazilian Journal of probability and Statistics,*27(1), 31-53

Maradesa,.A. and Akomolafe, A.A. (2017): Dirichilet Model: its consistency and Tracking pattern using Buying Behaviour Data. *Srilanka Journal of Applied Statistics.* 18(2) 89-103.

Mead, M. E. (2016). On five-parameter Lomax distribution: properties and applications. *Pakistan Journal of Statistics and Operation Research*, 185-199.

Oguntunde, P. E., Khaleel, M. A., Ahmed, M. T., Adejumo, A. O., andOdetunmibi, O. A.(2017). A New Generalization of the Lomax Distribution with Increasing, Decreasing, and Constant Failure Rate. *Modelling and Simulation in Engineering*, *2017*.

Oseghale, O.I. and Akomolafe A.A**.** (2017): Performance rating of the Kumaraswamy Transmuted Weibull Distribution: An Analytic Approach. *American Journal of Mathematics and Statistics* 7(3), 125-135.

Tahir, M. H., Cordeiro, G. M., Alizadeh, M., Mansoor, M., Zubair, M., and Hamedani, G. G. (2015). The odd generalized exponential family of distributions with applications. *Journal of Statistical Distributions and Applications, 2:1, 1-28.*

Yousof, H. M., Afify, A. Z., Alizadeh, M., Butt, N. S., andHamedani, G. (2015). The transmutedexponentiated generalized-G family of distributions. *Pakistan Journal of Statistics and Operation Research*.