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Prospect theory and asset allocation☆

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ABSTRACT

We study the asset allocation of an investor with prospect theory (PT) preferences. First, we solve analytically the two-asset problem of the PT investor for one risk-free and one risky asset and find that the reference return and the level of risk aversion or risk seeking (diminishing sensitivity) affect differently less ambitious and more ambitious investors: the less ambitious investor decreases her exposure to the risky asset when increasing her reference return or the level of diminishing sensitivity, while the more ambitious investor increases her exposure to the risky asset when increasing her reference return or the level of diminishing sensitivity. However, both less and more ambitious investors decrease their exposures to the risky asset when increasing their degrees of loss aversion. In a comprehensive sensitivity analysis, we investigate how different aspects of the PT investor's preferences contribute to her risk taking, performance and happiness. We observe, for instance, that the investor's happiness decreases with her increasing level of ambition. Second, we perform simulations to examine concrete solutions of the theoretical two-asset problem for different types of the PT investor and for different characteristics of the risky asset and find that the assumption of skewness, as opposed to symmetry, changes the optimal investment in the risky asset. Third, we empirically investigate the performance of a PT portfolio when diversifying among a stock market index, a government bond and gold, in Europe and the US. We focus on investors with PT preferences under different scenarios regarding the reference return and the degree of loss aversion and compare their portfolio performance with the performance of investors under mean–variance (MV), linear loss averse and CVaR preferences. We find that, in the US, PT portfolios significantly outperform MV portfolios (in terms of returns) in most cases.

1. Introduction

The mainstream expected utility model cannot explain many aspects of financial market characteristics. An alternative that has been proposed to describe investors' behavior under risk is prospect theory (Kahneman & Tversky, 1979; Tversky & Kahneman, 1992). This theory can explain many of the anomalies observed in asset returns including the equity premium puzzle (An et al., 2020; Barberis et al., 2001, 2021; Benartzi & Thaler, 1995). Experiments by Kahneman and Tversky found that the utility function does not depend on the absolute level of terminal wealth, as is hypothesized with the expected utility model, but depends on the change in the level of wealth. In addition, the probabilities assigned to the utility of the outcomes are expressed within a weighted function. These weight functions allow one to capture the tendency of people to underreact when faced with large probability events and overreact when faced with small probability events.[1] Kahneman and Tversky proposed a utility function that is defined over terminal wealth in relation to some reference level, such as the status quo, i.e., investors have reference-dependent preferences. In addition, they found that investors exhibit loss aversion, meaning that their disutility of a loss is greater than their utility of a gain of the same magnitude, i.e., investors are more sensitive when they experience a loss in financial wealth than when they experience a gain. Even when there is no commonly accepted measure of loss aversion in the literature and there are many alternatives introduced (Abdellaoui et al., 2007), the main characteristic seems to be that the utility function is steeper in the domain of losses than in the domain of gains. The simplest form of such loss aversion is linear loss aversion, where the marginal utility of gains and losses is fixed. The optimal asset allocation decision under linear loss aversion has been studied widely, see, for example, Best and Grauer (2016), Best et al. (2014), Fortin and Hlouskova (2011), Grauer (2013), and Siegmann and Lucas (2005). Further, Kahnemann and Tversky found that investors prefer

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[1] This is referred to in the literature as cumulative prospect theory. However, in our study we do not use weight functions (subjective probabilities).

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risk aversion options when they are confronted with gains, while they are more willing to select risk seeking options when confronted with losses. This behavior can be captured by an S-shaped utility function. We generalize linear loss aversion, as studied in Fortin and Hlouskova (2011), to S-shaped loss aversion and study optimal asset allocation in this setup, both theoretically and empirically.

In order to explain stock market anomalies (Barberis et al., 2021) introduce a new model of asset prices in which investors evaluate risk according to prospect theory. This model incorporates all elements of prospect theory, accounts for investors' prior gains and losses, and makes quantitative predictions about an asset's average return based on empirical estimates of the asset's return volatility, return skewness, and past capital gain. With this model, average asset returns can thus be predicted under prospect theory preferences, as with the capital asset pricing model under mean–variance preferences. In our study, however, we consider the asset allocation problem of a given sufficiently loss averse prospect theory investor, which is different from equilibrium asset pricing models.

Other studies examine the investor's asset allocation problem using the continuous-time framework and the martingale method (under the assumption that the market is complete) to solve the S-shaped utility maximization problem. For example, Berkelaar et al. (2004) derive the optimal investment strategies for two prospect theory utility functions. Chen et al. (2017) examine S-shaped preferences with a minimum performance constraint and inflation risk. He and Kou (2018) investigate the S-shaped utility maximization under a minimum guarantee. Dong and Zheng (2019) include short-selling and portfolio insurance constraints in the model, and Dong and Zheng (2020) impose trading and Value-at-Risk constraints; both apply the dual control method to solve the corresponding constrained optimization problem. In this framework, however, the solutions for the optimal wealth and trading strategy are not given directly, as in our discrete one-period setup. Choi et al. (2022) solve an intertemporal consumption investment problem, considering disutility from changing consumption levels. This is closely related to loss aversion towards a consumption change when the previous consumption level is the reference consumption and, in our setup, translates to loss aversion towards an investment change when the previous portfolio return is the reference return. The investment behavior in Choi et al. (2022) exhibits a U-shaped relationship between the share of the risky assets in wealth and actual wealth.

In this paper we focus on a single-period model, which allows us to examine certain properties of the optimal portfolio in more detail and, in some sense, is in line with myopic investors who prefer immediate gratification over long-term objectives. Multi-period models of an investor with prospect theory preferences are developed in Barberis and Huang (2009), He and Zhou (2014) and Shi et al. (2015), but only for a piecewise linear value function, not for the general S-shaped value function. De Giorgi and Legg (2012) extend the approach of Barberis and Huang (2009) to a convex–concave value function, however, examples are again presented only for the piecewise linear value function. In a comprehensive sensitivity analysis, we investigate how different aspects of the prospect theory investor's preferences contribute to her risk taking, performance and happiness. More precisely we examine the following properties: (i) the dependence of the optimal solution (investment in the risky asset) on the degree of loss aversion, the level of ambition,[2] and the degree of diminishing sensitivity; (ii) the behavioral pattern of the first three moments of the portfolio return with respect to the changing degree of the investor's loss aversion, level of ambition and diminishing sensitivity; (iii) the sensitivity of the investor's "happiness/satisfaction" with respect to the level of ambition, where we observe that happiness decreases with an increasing level of ambition; and (iv) the investor's expected gains and losses, where we find different risk attitudes for different diminishing sensitivity parameters. We derive an explicit solution when the risky asset return follows a discrete Bernoulli distribution and a semi-analytical solution when it follows a general continuous distribution. The shortage of one-period models is that the reference return is assumed to be exogenous, while it can be assumed to be endogenous in a multi-period model, depending, e.g., on the portfolio return of the previous period. However, an exogenous reference return allows us to analyze the dependence of our results and performance measures with respect to the investor's ambition. Hlouskova et al. (2019) examine the consumption–investment decision of a prospect theory household in a two-period model with an endogenous second period reference consumption (under the assumption of the risky asset return following a Bernoulli distribution, where the investor draws her utility from gains and losses in intertemporal consumption). In this setup the second period reference consumption is determined as a convex combination of the previous (first) period reference consumption and previous period consumption, i.e., the reference level increases after prior "gains" and decreases after prior "losses". This, of course, cannot be done in a one-period model.[3] In our empirical analysis we try to address this by considering dynamic scenarios which update the reference level according to the portfolio return gained in the previous period. What is not investigated in Hlouskova et al. (2017, 2019), is the analysis of prospect theory portfolio properties under more general distributions, which is one of the focuses in this study.

As the S-shaped prospect theory utility function is not concave and the problem cannot be easily transformed to a sufficiently smooth higher dimensional concave problem, grid search algorithms (as in the empirical part of our paper) or alternative special algorithms are necessary to solve the problem. More recent empirical studies propose numerical portfolio optimization techniques which maximize the prospect theory utility implied by the empirical distribution of asset returns. Harris and Mazibas (2022), for example, solve the one-period multi-asset model in two steps, first smoothing the utility (as it is non-differentiable) employing the cubic spline smoothing approach and then using a multistart scatter search heuristic approach; Luxenberg et al. (2022) provide several practical methods to maximize prospect theory utility for the multi-asset case, which exploit the special structure of the prospect theory maximization problem; Barro et al. (2020) use the partial swarm optimization approach; Cui et al. (2022) propose a method based on the alternating direction method of multipliers; and Grishina et al. (2017), employ intelligent algorithms. Also older empirical studies use numerical algorithms to solve the maximization problem for more risky assets. De Giorgi and Hens (2009), for example, consider data of private clients and measure the clients' added value from holding prospect theory portfolio as compared to a mean–variance asset allocation; they find considerable monetary gains. Hens and Mayer (2014) perform a similar analysis using eight indices of different asset classes and, again, find that prospect theory investors are better off than mean–variance investors. In the empirical part of our study we also take the mean–variance portfolio as the main benchmark when assessing the performance of prospect theory portfolios. Grishina et al. (2017) compute prospect theory portfolios composed of up to 225 stocks, taking the stock index (in which the individual stocks are included) as the reference point, and find that prospect theory portfolios perform better than index tracking models in bullish markets, and worse in bearish markets. We also compute prospect theory portfolios empirically, for three assets (stock, bond, gold), in the European and the US markets and compare these portfolios with other benchmark portfolios, in particular with the traditional mean–variance portfolios, in terms of different performance measures. Finally,

[2] We show analytically the V-shaped dependence of risk taking with respect to the reference return for the general continuous distribution, which, to our knowledge, was not tackled in the literature in such detail before.

[3] However, there are no large differences in the sensitivity analyses in the cases when the second period reference level is exogenous or endogenous, see Hlouskova et al. (2017, 2019).

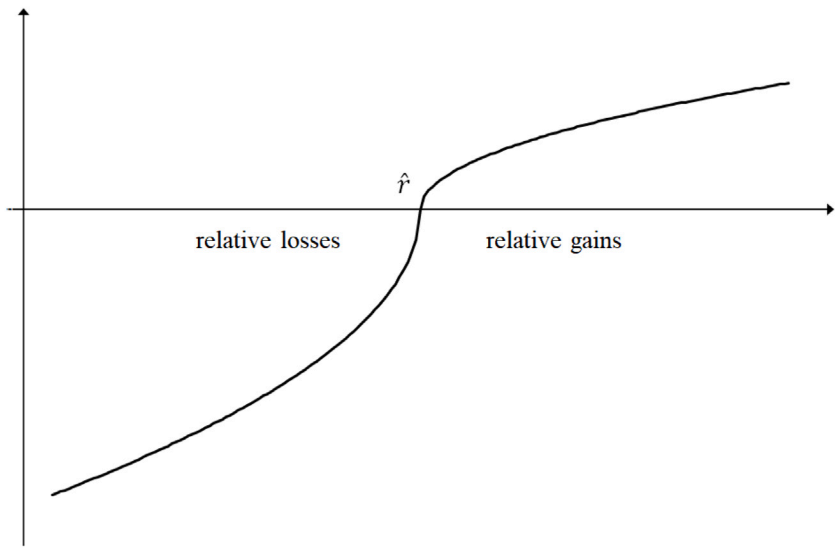

Fig. 1. Value function of S-shaped prospect theory investor.

Pirvu and Schulze (2012) consider a one-period multi-asset model of cumulative prospect theory investment under the assumption that asset returns follow a multivariate elliptical (symmetric) distribution. Their main result is that agents invest in the risk-free asset and in the mean–variance portfolio, where the mean–variance portfolio is the same for all investors, but the participation differs. In our study we consider one risk-free and one risky asset and assume a general continuous distribution, which enables us to analyze the effects of asymmetric returns.

The remaining paper is organized as follows. In Section 2 we explore the two-asset problem of an S-shaped sufficiently loss averse prospect theory investor, with a risky and a risk-free asset, and derive properties of the optimal weight of the risky asset under the assumptions of Bernoulli and (generally) continuously distributed returns, both for the case when the reference point is equal to the risk-free rate and for the case when it is not. In Section 3 we perform simulations for a prospect theory investor with different characteristics, with a focus on the effect of (negatively) skewed returns of the risky asset. In Section 4 we implement different trading strategies of the prospect theory investor, who reallocates her portfolio on a monthly basis, and study the performance of the resulting optimal portfolios with respect to different performance measures that are based solely on portfolio returns (mean, median, realized returns), on risk-adjusted returns (Omega measure, Sharpe ratio, Sortino ratio, conditional value-at-risk) or on risk (volatility, downside volatility). We also compare the performance of the prospect theory portfolios with the performance of risk neutral, linear loss averse, conditional value-at-risk and, in particular, with the performance of traditional mean–variance portfolios. Section 5 provides a summary of the results and concludes.

2. Portfolio optimization under prospect theory preferences: analytical solution

We consider a sufficiently loss averse investor characterized by the following S-shaped prospect theory value function of portfolio return R

$$v(R) = \begin{cases} \frac{(R-\hat{r})^{1-\gamma}}{1-\gamma}, & R > \hat{r} \\ -\lambda \frac{(\hat{r}-R)^{1-\gamma}}{1-\gamma}, & R \le \hat{r} \end{cases} = \frac{1}{1-\gamma}\left[|\hat{r} - R|^{1-\gamma} - (1+\lambda)\left([\hat{r}-R]^{+}\right)^{1-\gamma} \right] \quad (1)$$

where $\hat{r} \in \mathbb{R}$ is the reference return with respect to which relative gains and losses are coded, $\gamma \in (0,1)$ is a parameter determining the curvature of the utility function for relative gains and losses (diminishing sensitivity parameter),[4] and $[t]^+$ denotes the maximum of 0 and t. We assume net returns but the whole analysis holds also under the assumption of gross returns.[5] The parameter $\lambda > 1$ is the penalty parameter, which captures the degree of loss aversion making thus utility steeper in the loss domain ($R < \hat{r}$) than in the gain domain ($R > \hat{r}$). The investor's reduction in utility arising from a loss is greater (in absolute terms) than the marginal utility from a gain or, in other words, the investor is more sensitive when experiencing a loss than when experiencing a gain of the same size. Investors also display risk aversion in the domain of gains (the value function is concave for $R > \hat{r}$) but become risk lovers when they deal with losses (the value function is convex for $R < \hat{r}$). See Fig. 1 for a graphical illustration of the value function, which is non-differentiable at the reference return.

We study the optimal asset allocation behavior of an investor with S-shaped prospect theory preferences. This behavior depends on the reference return $\hat{r}$ and, in particular, on whether this reference return is below, equal to, or above the risk-free interest rate, r^0. The position of the investor's reference return with respect to the risk-free rate is determined exogenously by the investor's incentive, e.g., investors with their reference return being below the risk-free rate ($\hat{r} < r^0$) can be viewed as less ambitious investors while investors with their reference return being above the risk-free rate ($\hat{r} > r^0$) can be viewed as more ambitious investors.

Investors maximize their expected utility of returns

$$\max_{\mathbf{x}} \left\{ \mathbb{E}\left(v(\mathbf{r}'\mathbf{x})\right) \middle| \mathbf{A}\mathbf{x} \le \mathbf{b} \right\} \quad (2)$$

where $\mathbf{x} = (x_1, \dots, x_n)'$, with x_i denoting the proportion of wealth invested in asset i,[6] $i = 1, \dots, n$, and $\mathbf{r}$ is the n–dimensional random vector of net returns, subject to the usual asset constraints $\mathbf{A}\mathbf{x} \le \mathbf{b}$, with $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Note that in general the proportion invested in a given asset may be negative, due to short-selling, or larger than one.

[4] We assume that $\gamma \in (0,1)$ in order to be consistent with the experimental findings of Tversky and Kahneman (1992). Booij and van de Kuilen (2009) find the γ parameter to be (statistically) significantly less than unity.

[5] Without loss of generality we assume that the initial wealth (W_0) is unity, as otherwise the reference wealth becomes $W_0\hat{r}$ which gives the value function $W_0^{1-\gamma}v(r)$ and does not change the solution when considering only $v(r)$.

[6] Throughout this paper, prime (′) is used to denote matrix transposition and any unprimed vector is a column vector.

To better understand the attitude with respect to risk of an investor with prospect theory preferences, we consider a simple two-asset world, where one asset is risk-free and the other is risky, and analyze what proportion of wealth is invested in the risky asset.[7] Let r^0 be the certain (deterministic) return of the risk-free asset and let r be the (stochastic) return of the risky asset. Then the portfolio return is $R = R(x) = xr+(1-x)r^0 = r^0+(r-r^0)x$, where x is the proportion of wealth invested in the risky asset, and the maximization problem of the investor with prospect theory preferences is

$$\max_x \left\{ \mathbb{E}\left(v\left(\left(R(x)\right)\right)\right) = \mathbb{E}\left(v\left(r^0 + \left(r - r^0\right)x\right)\right) \mid x \in \mathbb{R}\right\} \tag{3}$$

with value function $v(\cdot)$ given by (1). As will be seen later, the results will be sensitive to the position of the reference return with respect to the risk-free rate.

In the following we present optimal solutions and their properties for the following three cases, when the investor: (i) is modest in setting her return goals, i.e., $\hat{r} < r^0$ (less ambitious investor), (ii) stays out of the market, i.e., $\hat{r} = r^0$, and (iii) is more ambitious in setting her goals, i.e., $\hat{r} > r^0$ (more ambitious investor). We assume first that the risky asset return is Bernoulli distributed (discrete distribution) and later on that it is (generally) continuously distributed.

2.1. The risky asset return follows the Bernoulli distribution

First we assume, for the sake of simplicity and because in this case we can show a number of results analytically, that the return of the risky asset follows the Bernoulli distribution. We assume two states of nature: a good state of nature which yields return r_g such that $r_g > r^0$ and which occurs with probability p, and a bad state of nature which yields return r_b such that $r_b < r^0$ and which occurs with probability $1-p$, i.e., $r_b < r^0 < r_g$.[8] In the good state of nature the portfolio yields return $R_g(x) = r^0 + (r_g - r^0)x$ with probability p and in the bad state of nature it yields return $R_b(x) = r^0 - (r^0 - r_b)x$ with probability $1-p$. Thus, based on (1) and (3), the expected prospect theory utility (value function) of the two-asset portfolio including the risk-free asset and the Bernoulli distributed risky asset is the following continuous function

$$\mathbb{E}\left(v\left(R(x)\right)\right) = \begin{cases} \frac{1}{1-\gamma}\left[p\left(R_g(x)-\hat{r}\right)^{1-\gamma} + (1-p)(R_b(x)-\hat{r})^{1-\gamma}\right], & R_g(x) \geq \hat{r},\ R_b(x) \geq \hat{r} \\ \frac{1}{1-\gamma}\left[p(R_g(x)-\hat{r})^{1-\gamma} - \lambda(1-p)(\hat{r}-R_b(x))^{1-\gamma}\right], & R_g(x) \geq \hat{r},\ R_b(x) \leq \hat{r} \\ \frac{1}{1-\gamma}\left[-\lambda p(\hat{r}-R_g(x))^{1-\gamma} + (1-p)(R_b(x)-\hat{r})^{1-\gamma}\right], & R_g(x) \leq \hat{r},\ R_b(x) \geq \hat{r} \\ -\lambda\frac{1}{1-\gamma}\left[p(\hat{r}-R_g(x))^{1-\gamma} + (1-p)(\hat{r}-R_b(x))^{1-\gamma}\right], & R_g(x) \leq \hat{r},\ R_b(x) \leq \hat{r} \end{cases} \tag{4}$$

To proceed with the analysis, we define the following threshold

$$K_\gamma = \frac{(1-p)(r^0-r_b)^{1-\gamma}}{p(r_g-r^0)^{1-\gamma}} \tag{5}$$

$1/K_\gamma$ shows similar features as the Omega measure (see Shadwick & Keating, 2002), which reflects the ratio of the upside potential of the risky asset return relative to its downside potential with respect to the risk-free rate. It coincides with the Omega measure of the risky asset for $\gamma = 0$.

The next proposition presents the analytical solution of the prospect theory investor with preferences described by (1) and (3), who is less ambitious and who is also sufficiently loss averse.

Proposition 2.1. *Let $\mathbb{E}(r) > r^0$, $\hat{r} < r^0$, and $\lambda > \max\left\{K_\gamma, 1/K_\gamma\right\}$, where K_γ is defined by (5). Then there exists the only solution x^* of (3) such that*

$$x^* = \frac{\left(1-K_0^{1/\gamma}\right)(r^0-\hat{r})}{r^0-r_b+K_0^{1/\gamma}(r_g-r^0)} > 0 \tag{6}$$

Proof. See Appendix A. □

Note that solution (6) does not depend on the degree of loss aversion, λ, but a sufficiently large degree of loss aversion is needed to guarantee the monotonic properties of the prospect theory utility function in its certain domains and thus the uniqueness of the solution. In addition, the optimal portfolio return exceeds the reference return, i.e., $R(x^*) > \hat{r}$. See the proof of Proposition 2.1 in Appendix A for more details.

Note that if $\tilde{v}(\cdot)$ is the power utility, namely $\tilde{v}(y) = \frac{y^{1-\gamma}}{1-\gamma}$, then the solution of the maximization of the expected power utility under the Bernoulli distributed risky asset as specified above, namely $\max\{\mathbb{E}(\tilde{v}(R(x))) \mid x \in \mathbb{R}\}$, is $\tilde{x}^* = \frac{\left(1-K_0^{1/\gamma}\right)r^0}{r^0-r_b+K_0^{1/\gamma}(r_g-r^0)}$, and thus for $\hat{r} < r^0$ we have $x^* < \tilde{x}^*$ when $\hat{r} > 0$.[9] The proportion invested in the risky asset is thus smaller for the less ambitious prospect theory investor than for the investor characterized by power utility, as long as the reference return is positive. Based on (6) we can formulate the following corollary, which formally states that when the investor is less ambitious, i.e., when the reference return is below the risk-free rate, then the investor is not sensitive to the degree of loss aversion (λ) and she becomes more conservative with an increasing reference return and curvature parameter, i.e., her investment in the risky asset decreases with an increasing level of ambition and diminishing sensitivity γ.

Corollary 2.1. *Let the assumptions of Proposition 2.1 be satisfied. Then the optimal solution of (3), x^*, has the following properties*

$$\frac{dx^*}{d\lambda} = 0$$

$$\frac{dx^*}{d\hat{r}} = -\frac{\left(1-K_0^{1/\gamma}\right)}{r^0-r_b+K_0^{1/\gamma}(r_g-r^0)} < 0 \tag{7}$$

and

$$\frac{dx^*}{d\gamma} = \frac{(r^0-\hat{r})(r_g-r_b)\,K_0^{1/\gamma}\ln K_0}{\left[\gamma\left(r^0-r_b+K_0^{1/\gamma}(r_g-r^0)\right)\right]^2} < 0 \tag{8}$$

The following proposition states that when the reference return coincides with the risk-free rate ($\hat{r} = r^0$) the prospect theory investor stays out of the market ($x^* = 0$), i.e., everything is invested in the risk-free asset.[10]

Proposition 2.2. *Let $\hat{r} = r^0$ and $\lambda \geq \max\left\{K_\gamma, 1/K_\gamma\right\}$, where K_γ is defined by (5). Then the solution of (3) is $x^* = 0$.*

Proof. See Appendix A. □

[7] Another motivation for looking at this problem is the following. When Tobin's separation theorem holds, the investment decision problem can be simplified to deciding which proportion to invest in the safe asset and which to invest in some risky portfolio. Levy et al. (2004) have shown that Tobin's separation principle holds, under certain assumptions, also for the Tversky and Kahneman's prospect theory utility.

[8] This assumption rules out any arbitrage between the risky and risk-free assets.

[9] Note in addition that $x^* \geq \tilde{x}^*$ when $\hat{r} \leq 0$.

[10] This is also the case for the linear loss averse investor ($\gamma = 0$), see Fortin and Hlouskova (2011), but not for the loss averse investor with quadratic shortfall, where the optimal investment in the risky asset is strictly positive, see Fortin and Hlouskova (2015).

Table 1
Summary of optimal solutions under S-shaped prospect theory and linear loss aversion.

Assumptions on $\hat{r}$	Additional assumptions	Solutions		
$\hat{r} < r^0$	$\mathbb{E}(r) > r^0$	$0 <$	$x^* <$	$x^*_{\text{LLA}} = \frac{r^0-\hat{r}}{r^0-r_b}$
$\hat{r} = r^0$			$x^* =$	$x^*_{\text{LLA}} = 0$
$\hat{r} > r^0$	$p > \bar{p}$	$x^*_p =$	$x^* >$	$x^*_{\text{LLA}} = \frac{\hat{r}-r^0}{r_g-r^0} > 0$
$\hat{r} > r^0$	$p < \bar{p}$	$x^*_n =$	$x^* < 0 <$	$x^*_{\text{LLA}} = \frac{\hat{r}-r^0}{r_g-r^0}$

We assume that $\lambda > \max\{K_\gamma, 1/K_\gamma, 1/K_0\}$. Note that x^* is given by (6) when $\hat{r} < r^0$; x^*_p is given by (9), x^*_n by (10), $\bar{p}$ by (11) and x^*_{LLA} is derived in Fortin and Hlouskova (2011).

Note that this is the only case when the portfolio of the prospect theory investor coincides with the portfolio of the mean–variance investor.[11] From the proof of Proposition 2.2 it is easy to see that for the less loss averse investor the following holds: For $1 < \lambda < 1/K_\gamma$ the investor would benefit from an ever increasing long position, i.e., $x^* = +\infty$,[12] while for $1 < \lambda < K_\gamma$ the investor would benefit from an ever increasing short position, i.e., $x^* = -\infty$.

Before proceeding further let us introduce the following notation

$$x^*_p = \frac{\hat{r} - r^0}{r_g - r^0} \times \frac{\lambda^{1/\gamma} + \left(\frac{1}{K_0}\right)^{1/\gamma}}{\lambda^{1/\gamma} - \left(\frac{1}{K_\gamma}\right)^{1/\gamma}} \tag{9}$$

$$x^*_n = -\frac{\hat{r} - r^0}{r^0 - r_b} \times \frac{\lambda^{1/\gamma} + K_0^{1/\gamma}}{\lambda^{1/\gamma} - K_\gamma^{1/\gamma}} \tag{10}$$

$$\bar{p} = \frac{\left(r^0 - r_b\right)^{1-\gamma}}{\left(r^0 - r_b\right)^{1-\gamma} + \left(r_g - r^0\right)^{1-\gamma}} \tag{11}$$

Note that for the more ambitious investor, i.e., $\hat{r} > r^0$, is $x^*_p > 0$ when $\lambda > 1/K_\gamma$ and $x^*_n < 0$ when $\lambda > K_\gamma$.

The following proposition presents the analytical solution of the investor with prospect theory preferences, see (1) and (3), who is more ambitious and who is also sufficiently loss averse.

Proposition 2.3. *Let $\hat{r} > r^0$ and $\lambda > \max\left\{K_\gamma, 1/K_\gamma\right\}$, where K_γ is defined by (5). Then there exists the solution x^* of (3) such that*

$$x^* \begin{cases} = x^*_p > 0, & \text{for } \; p > \bar{p} \\ = x^*_n < 0, & \text{for } \; p < \bar{p} \\ \in \left\{x^*_p, x^*_n\right\} & \text{for } \; p = \bar{p} \end{cases} \tag{12}$$

which is unique for $p \neq \bar{p}$.

Proof. See Appendix A. □

Proposition 2.3 implies that the more ambitious sufficiently loss averse investor purchases the risky asset when the probability of the good state to occur ($r = r_g$) is sufficiently large. In this case the portfolio return exceeds the reference return if $r = r_g$ ($R(x^*) > \hat{r}$), while it is below the reference return if $r = r_b$ ($R(x^*) < \hat{r}$). However, if the probability of the bad state to occur ($r = r_b$) is sufficiently large, the investor takes a short position in the stock market.[13] In this case the portfolio return exceeds the reference return if $r = r_b$ ($R(x^*) > \hat{r}$), while it is below the reference return if $r = r_g$ ($R(x^*) < \hat{r}$).

Based on (12) we can formulate the following corollary, which presents the sensitivity of the risky investment of a more ambitious investor ($\hat{r} > r^0$) with respect to her degree of loss aversion, her reference return and her diminishing sensitivity (risk aversion/risk seeking parameter γ).

Corollary 2.2. *Let the assumptions of Proposition 2.3 be satisfied. Then the optimal solution has the following properties:*

$$\frac{dx^*}{d\lambda} = \begin{cases} -\frac{1}{\gamma}\frac{r_g - r_b}{r_g - r^0} \times \frac{\lambda^{1/\gamma - 1}}{K_0^{1/\gamma}\left[\lambda^{1/\gamma} - \left(\frac{1}{K_\gamma}\right)^{1/\gamma}\right]^2} < 0, & \text{for } p > \bar{p} \\ \frac{1}{\gamma}\frac{r_g - r_b}{r^0 - r_b} \times \frac{K_0^{1/\gamma}\lambda^{1/\gamma - 1}}{\left[\lambda^{1/\gamma} - K_\gamma^{1/\gamma}\right]^2} > 0, & \text{for } p < \bar{p} \end{cases} \tag{13}$$

$$\frac{dx^*}{d\hat{r}} = \begin{cases} \frac{1}{r_g - r^0} \times \frac{\lambda^{1/\gamma} + \left(\frac{1}{K_0}\right)^{1/\gamma}}{\lambda^{1/\gamma} - \left(\frac{1}{K_\gamma}\right)^{1/\gamma}} > 0, & \text{for } p > \bar{p} \\ -\frac{1}{r^0 - r_b} \times \frac{\lambda^{1/\gamma} + K_0^{1/\gamma}}{\lambda^{1/\gamma} - K_\gamma^{1/\gamma}} < 0, & \text{for } p < \bar{p} \end{cases} \tag{14}$$

and

$$\frac{dx^*}{d\gamma} = \begin{cases} \frac{(\hat{r} - r^0)(r_g - r_b)}{\left[\gamma(r_g - r^0)\left(\lambda^{1/\gamma} - \left(\frac{1}{K_\gamma}\right)^{1/\gamma}\right)\right]^2}\left(\frac{\lambda}{K_0}\right)^{1/\gamma}\left(\ln\lambda - \ln\frac{1}{K_0}\right) > 0, & \\ & \text{for } p > \bar{p} \text{ and } \lambda > \frac{1}{K_0} \\ -\frac{(\hat{r} - r^0)(r_g - r_b)}{\left[\gamma(r^0 - r_b)\left(\lambda^{1/\gamma} - K_\gamma^{1/\gamma}\right)\right]^2}\left(\lambda K_0\right)^{1/\gamma}\left(\ln\lambda - \ln K_0\right) < 0, & \\ & \text{for } p < \bar{p} \text{ and } \lambda > K_0 \end{cases} \tag{15}$$

Corollary 2.2 implies the following findings regarding the comparative statics for the more ambitious investor. For a sufficiently large probability of the good state to occur, the risk taking (i.e., the investment in the risky asset, x^*) decreases with an increasing degree of loss aversion, while it increases with an increasing level of ambition and an increasing degree of diminishing sensitivity. However, when the probability of the good state to occur is sufficiently small then the risk taking increases (i.e., the short position decreases) with an increasing loss aversion, while it decreases (i.e., the short position increases) with an increasing level of ambition and an increasing degree of diminishing sensitivity. These findings together with the findings in Corollary 2.1, namely (7), imply that risk taking, x^*, as a function of the reference return, $\hat{r}$, is non-differentiable at $\hat{r} = r^0$, which implies the V-shaped relation between x^* and $\hat{r}$ for sufficiently large p ($p > \bar{p}$).

Table 1 summarizes and contrasts the optimal investments into the risky asset of the linear loss averse (LLA) investor, who is defined by value function (1) with $\gamma = 0$, and the prospect theory (PT) investor, who is defined by value function (1) with $\gamma \in (0, 1)$.[14] Note that the investment in the risky asset of the less ambitious LLA investor exceeds the investment in the risky asset of the less ambitious PT investor. The opposite holds for the more ambitious investor when the probability of the good state to occur is sufficiently large, i.e., when $p > \bar{p}$. If $p < \bar{p}$

[11] As the variance of the portfolio return $R(x) = xr + (1-x)r^0$ is $x^2\text{Var}(r)$, the minimum variance of the mean–variance portfolio is reached at $x = 0$.

[12] An infinite long position is also observed in Ang et al. (2005), when calibrating a binomial tree to the US stock market and working with the original Kahneman and Tversky (1979) specification.

[13] Note that $K_\gamma < 1$ if and only if $p > \bar{p}$ and $K_\gamma = 1$ if and only if $p = \bar{p}$.

[14] For more details regarding the LLA investor, see Fortin and Hlouskova (2011).

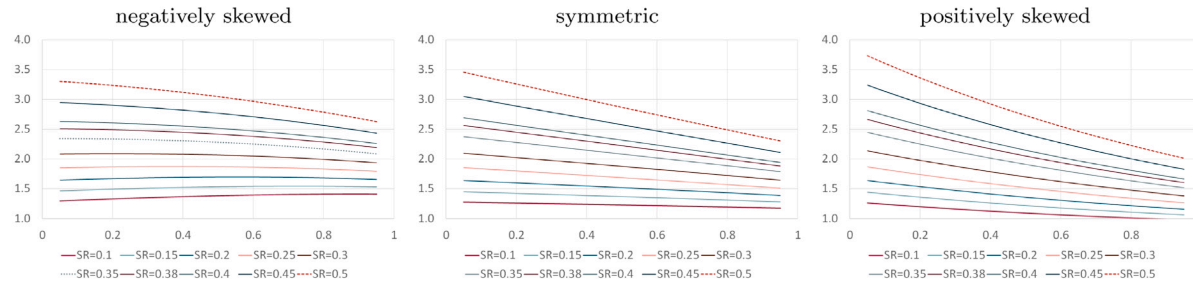

Fig. 2. $1/K_\gamma$ with respect to γ for different Sharpe ratios.
The results are shown for symmetric, negatively skewed and positively skewed distributions of the risky asset return, where the parameters of the risky asset return and the risk-free rate are in line with the EU data (stock market and risk-free rate), see Table B.2. Note that only for negatively skewed returns can the line be upward sloping, however, it is downward sloping for a sufficiently large Sharpe ratio (SR); in the example shown, for $SR \geq 0.35$.

then the PT investor takes a short position and thus the risk taking of the LLA investor again exceeds the risk taking of the PT investor. Note in addition that only when the reference return coincides with the risk-free rate is the investment in the risky asset the same for both investors, namely staying out of the market. Risk taking for the LLA investor is always positive when the reference return does not coincide with the risk-free rate.

Propositions 2.1, 2.3 and their proofs imply the following form of the indirect utility function, i.e., the value of the expected value function at its maximum.

Corollary 2.3. *The investor's level of satisfaction/happiness can be expressed as follows*

$$\mathbb{E}(v(R(x^*))) = \begin{cases} p(r^0-\hat{r})^{1-\gamma}\left(1+K_\gamma^{1/\gamma}\right)^\gamma\left(\frac{r_g-r_b}{r^0-r_b}\right)^{1-\gamma}, \\ \quad \text{for } \hat{r}<r^0,\ \lambda>\max\left\{K_\gamma, 1/K_\gamma\right\} \\ -(1-p)(\hat{r}-r^0)^{1-\gamma}\left[\lambda^{1/\gamma}-\left(\frac{1}{K_\gamma}\right)^{1/\gamma}\right]^\gamma\left(\frac{r_g-r_b}{r_g-r^0}\right)^{1-\gamma}, \\ \quad \text{for } \hat{r}>r^0,\ p\geq\bar{p},\ \lambda>1/K_\gamma \\ -p(\hat{r}-r^0)^{1-\gamma}\left(\lambda^{1/\gamma}-K_\gamma^{1/\gamma}\right)^\gamma\left(\frac{r_g-r_b}{r^0-r_b}\right)^{1-\gamma}, \\ \quad \text{for } \hat{r}>r^0,\ p\leq\bar{p},\ \lambda>K_\gamma \end{cases} \tag{16}$$

This implies that for $\lambda > \max\left\{K_\gamma, 1/K_\gamma\right\}$ is $\frac{d\mathbb{E}(v(R(x^*)))}{d\hat{r}} < 0$, i.e., happiness decreases with an increasing level of ambition and thus upward comparison seems to spoil the happiness. The same is true in our numerical example when the risky asset return is continuously distributed, see Fig. 7. Note in addition that $\frac{d\mathbb{E}(v(R(x^*)))}{d\lambda} = 0$ for $\hat{r} < r^0$, $\frac{d\mathbb{E}(v(R(x^*)))}{d\lambda} < 0$ for $\hat{r} > r^0$, i.e., for more ambitious investors happiness decreases with an increasing level of loss aversion.

2.2. The risky asset is continuously distributed

Now we assume that the risky asset return r is continuously distributed with probability density function $f(\cdot)$ and cumulative distribution function $F(\cdot)$, such that $\mathbb{E}(r^2) = \int_{-\infty}^{+\infty} r^2 f(r)dr < +\infty$, i.e., the variance exists, and $F(c) < 1$ for any $c \in \mathbb{R}$. These assumptions are met by a large number of distributions including the normal, the skew normal, the student t with degrees of freedom strictly larger than two, the Gamma, etc.

Let R be a continuous random variable describing the stochastic portfolio return and $f_R(\cdot)$ be its probability density function. Then based on (1) we can define the expected prospect theory utility of return R as

$$\mathbb{E}(v(R)) = \frac{1}{1-\gamma}\left(-\lambda\int_{-\infty}^{\hat{r}}(\hat{r}-z)^{1-\gamma}f_R(z)dz + \int_{\hat{r}}^{+\infty}(z-\hat{r})^{1-\gamma}f_R(z)dz\right) \tag{17}$$

and thus based on (17), the expected prospect theory utility function of portfolio return $R = R(x)$ is

$$\mathbb{E}(v(R(x))) = \begin{cases} \frac{(-x)^{1-\gamma}}{1-\gamma}\left[\int_{-\infty}^{z(x)}(z(x)-r)^{1-\gamma}f(r)dr - \lambda\int_{z(x)}^{+\infty}(r-z(x))^{1-\gamma}f(r)dr\right], & x<0 \\ \frac{1}{1-\gamma}\left(r^0-\hat{r}\right)^{1-\gamma}, & x=0 \text{ and } \hat{r}\leq r^0 \\ -\frac{\lambda}{1-\gamma}\left(\hat{r}-r^0\right)^{1-\gamma}, & x=0 \text{ and } \hat{r}>r^0 \\ \frac{x^{1-\gamma}}{1-\gamma}\left[-\lambda\int_{-\infty}^{z(x)}(z(x)-r)^{1-\gamma}f(r)dr + \int_{z(x)}^{+\infty}(r-z(x))^{1-\gamma}f(r)dr\right], & x>0 \end{cases} \tag{18}$$

where $z(x) = \frac{\hat{r}-r^0}{x} + r^0$. It is easy to see that $\mathbb{E}(v(R(x)))$ is continuous in x, also for $x = 0$.

The problem we want to solve is

$$\max_x \{\, \mathbb{E}(v(R(x))) \mid x \in \mathbb{R} \,\} \tag{19}$$

The case when $\hat{r} = r^0$ is already solved in the literature (see Bernard & Ghossoub, 2010, or He & Zhou, 2011) and, as in the discrete case, the PT investor stays out of the market. This holds also for the mean–variance investor. We summarize it in the following proposition, where

$$K_\gamma = \frac{\int_{-\infty}^{r^0}(r^0-r)^{1-\gamma}f(r)dr}{\int_{r^0}^{+\infty}(r-r^0)^{1-\gamma}f(r)dr} \tag{20}$$

K_γ plays an important role as it, together with $1/K_\gamma$, represents the lower bound of loss aversion (λ), which rules out infinite investment in the risky asset ($x^* = +\infty$) if $\lambda > 1/K_\gamma$, or infinite short-selling ($x^* = -\infty$) if $\lambda > K_\gamma$.[15] Thus, when $\lambda > 1/K_\gamma$ the investor's loss aversion is strong enough to offset the attractiveness of investing in the risky asset indicated by $1/K_\gamma$, while when $\lambda > K_\gamma$ loss aversion is sufficient to offset the attractiveness of short selling the risky asset indicated by K_γ. As mentioned in Section 2.1, $1/K_0$ coincides with the Omega measure of the risky asset and thus shows similar features. Harris and Mazibas (2022) refer to $1/K_\gamma$ as the hope-fear ratio, as it presents optimism (about the portfolio return exceeding a benchmark) relative to pessimism (about the portfolio return being below a benchmark), while Bernard and Ghossoub (2010) refer to $1/K_\gamma$ as the generalized Omega measure or the CPT-ratio.[16] Based on the assumption of the

[15] Note that $1/K_\gamma$ can be also written as $1/K_\gamma = \frac{\mathbb{E}\left((r-r^0)^{1-\gamma}|r\geq r^0\right)\times P(r\geq r^0)}{\mathbb{E}\left((r^0-r)^{1-\gamma}|r\leq r^0\right)\times P(r\leq r^0)}$, where $P(\cdot)$ is the probability implied by the cumulative distribution function of the risky asset return.

[16] Harris and Mazibas (2022) propose the discrete version of $1/K_\gamma$ as an alternative behavioral objective function, where instead of the risky asset return r they use the portfolio return. In addition they use subjective probabilities.

risky asset's distribution, namely $\mathbb{E}(r^2) = \int_{-\infty}^{+\infty} r^2 f(r)dr < +\infty$, it follows that K_γ is finite, see Lemma A.1 in Appendix A.

In order to better understand the dynamics of $1/K_\gamma$, we compute $1/K_\gamma$ for $\gamma \in (0,1)$ under the assumption of both symmetric and skewed distributions that we present in more detail in Section 3. Our findings suggest that for the risky asset return following a symmetric or a positively skewed distribution is $1/K_\gamma$ a decreasing function of γ, while for a negatively skewed distribution the results are mixed, see Fig. 2.[17] However, there seems to be the following pattern for negatively skewed returns: for a sufficiently large Sharpe ratio of the risky asset is $1/K_\gamma$ a decreasing function of γ, while for a sufficiently small Sharpe ratio is $1/K_\gamma$ increasing (at least for a sufficiently small diminishing sensitivity parameter γ). Note that stocks and stock indices are usually negatively skewed.

Proposition 2.4. *Let $\hat{r} = r^0$ and $\lambda > \max\left\{K_\gamma, \frac{1}{K_\gamma}\right\}$ where K_γ is given by (20). Then $x^* = 0$ is the solution of problem (19).*

Proof. See Appendix A. □

For $\hat{r} \neq r^0$ we can find the solution only in its implicit semi-analytical form but we can still perform comparative statics and show how the solution depends on the degree of loss aversion λ and on the reference return $\hat{r}$.

The following propositions present sufficient conditions for the existence of a global maximum or at least local maxima of problem (3).

$$\lambda \int_{-\infty}^{z(x^*)} \frac{r-r^0}{[z(x^*)-r]^\gamma} f(r)dr + \int_{z(x^*)}^{+\infty} \frac{r-r^0}{[r-z(x^*)]^\gamma} f(r)dr = 0 \tag{21}$$

where $z(x^*) = \frac{\hat{r}-r^0}{x^*} + r^0$. In more detail, the proposition below states that the optimal investment in the risky asset of a sufficiently loss averse less ambitious investor is strictly positive. To ease the exposition we introduce the following notation

$$\hat{K}_\gamma = \max_c \left\{ K_\gamma(c) = \frac{\int_{-\infty}^{c}(c-r)^{1-\gamma} f(r)dr - (c-r^0)^{1-\gamma}}{\int_c^{+\infty}(r-c)^{1-\gamma} f(r)dr} \,\middle|\, c \geq r^0 \right\} \tag{22}$$

Proposition 2.5. *Let $\mathbb{E}(r) > r^0$, $\hat{r} < r^0$, and $\lambda > \max\{\hat{K}_\gamma, 1/K_\gamma\}$, where K_γ is given by (20) and $\hat{K}_\gamma$ is given by (22). Then there exists a finite positive global maximum of problem (19), i.e., $x^* > 0$, which satisfies Eq. (21).*

Proof. See Appendix A. □

Note that $K_\gamma(r^0) = K_\gamma$ and thus from the assumption of Proposition 2.5 it follows that $\lambda > K_\gamma$. Lemma A.1 (see Appendix A) shows that $\hat{K}_\gamma < +\infty$, i.e., $\hat{K}_\gamma$ is bounded from above and thus there exist such λs that the assumption $\lambda > \max\{\hat{K}_\gamma, 1/K_\gamma\}$ of Proposition 2.5 is satisfied.

The following proposition presents the results again in the implicit form, for both less and more ambitious investors. It proves the existence of at least one positive local maximum and also shows that any local (and thus also global) maximum is finite, which is guaranteed by the assumption that the investor is sufficiently loss averse. If this is not the case, i.e., if the investor is less loss averse, then she holds an infinite long or short position in the risky asset. Namely, for $1 < \lambda < 1/K_\gamma$ the investors holds an infinite long position, while for $1 < \lambda < K_\gamma$ the investors holds an infinite short position.

[17] An analogous pattern can be observed in the case of the Bernoulli distribution. Namely, $1/K_\gamma$ is an increasing function of γ when $r^0 > \frac{r_g+r_b}{2}$, which together with $\mathbb{E}(r) > r^0$ implies that $p > 0.5$, i.e., when the Bernoulli distribution is negatively skewed, and thus $1/K_\gamma$ can be increasing only for a negatively skewed Bernoulli distribution.

Proposition 2.6. *Let $\mathbb{E}(r) > r^0$, $\hat{r} \neq r^0$, and $\lambda > \max\{K_\gamma, 1/K_\gamma\}$, where K_γ is given by (20). Then there exists at least one local maximum of problem (19) such that $x^* > 0$ and any positive local maximum $x^* > 0$ satisfies (21). If there exists also a local maximum such that $x^* < 0$ then it satisfies the following equation*

$$\int_{-\infty}^{z(x^*)} \frac{r-r^0}{[z(x^*)-r]^\gamma} f(r)dr + \lambda \int_{z(x^*)}^{+\infty} \frac{r-r^0}{[r-z(x^*)]^\gamma} f(r)dr = 0 \tag{23}$$

where $z(x^*) = \frac{\hat{r}-r^0}{x^*} + r^0$. Finally, any global maximum is finite, i.e. $-\infty < x^* < +\infty$.

Proof. See Appendix A. □

The next proposition presents the sensitivity analysis of risky investment for both less and more ambitious investors with respect to their degrees of loss aversion and reference rates.

Proposition 2.7. *Let $x^* \neq 0$ be the optimal solution of (19), $\hat{r} \neq r^0$, let*

$$\lim_{r\to\pm\infty} |r|^{2-\gamma} f(r) = 0 \tag{24}$$

and let x^* satisfies (21) if $x^* > 0$ and let x^* satisfies (23) if $x^* < 0$. Then x^* has the following properties

$$\frac{d|x^*|}{d\lambda} < 0 \tag{25}$$

and

$$\frac{d|x^*|}{d\hat{r}} = \left\{ \begin{array}{ll} <0, & \text{if } \hat{r} < r^0 \\ >0, & \text{if } \hat{r} > r^0 \end{array} \right\} \tag{26}$$

Proof. See Appendix A. □

Proposition 2.7 implies that any positive solution (investment in the risky asset) of (19) satisfying (21) decreases with an increasing degree of loss aversion, and it also decreases with an increasing reference return for the less ambitious investor, while it increases with an increasing reference return for the more ambitious investor. The reference return thus plays an important role in asset allocation as the investor's risk attitude changes differently with respect to the reference return, depending on whether $\hat{r}$ is smaller or larger than the risk-free rate. The proof of Proposition 2.7, namely (A.20) and (A.21), shows that

$$\frac{dx^*}{d\hat{r}} = \frac{x^*}{\hat{r}-r^0} \tag{27}$$

This implies that x^* is proportional to $\hat{r}-r^0$, i.e., x^* is bi-linear with respect to $\hat{r}$ and at $\hat{r} = r^0$ it is non-differentiable.[18] Thus, the proportion invested in the risky asset, if it is positive, shows a V-shaped pattern with respect to the reference return. The less ambitious investor ($\hat{r} < r^0$) prefers to accept less risk and move away from the stock market when increasing her level of ambition, due to her risk aversion in the domain of gains, while the more ambitious investor ($\hat{r} > r^0$) puts more money into the risky asset when increasing her level of ambition, due to her risk seeking behavior in the domain of losses. The opposite holds for any negative solution (short-position in the risky asset) of (19) satisfying (23), in which case we observe an inverse V-shaped pattern of risk taking with respect to the reference return. However, a similar argument for this behavior applies: the less ambitious investor decreases her short position when increasing $\hat{r}$ as she is more risk averse

[18] See (A.22) and the proof of Proposition 2.7 for more details. More precisely, for $\hat{r} < r^0$ is $x^* = \frac{r^0-\hat{r}}{K}$, where $K > 0$ satisfies (A.23) for the positive solution and $K < 0$ satisfies (A.25) for a local negative solution, while for $\hat{r} > r^0$ is $x^* = \frac{\hat{r}-r^0}{K}$, where $K > 0$ satisfies (A.24) for the positive solution and $K < 0$ satisfies (A.26) if a local negative solution exists. In all these cases is K a function of λ, γ, r^0 and the distribution of the risky asset return, but not of $\hat{r}$. This property can be seen also in the case of the Bernoulli distribution.

Table 2
Summary of optimal solutions and sensitivities under S-shaped prospect theory and linear loss aversion.

Conditions on λ	Conditions on $\hat{r}$	Bernoulli	Continuous
S-shaped prospect theory			
		$\mathbf{x}^*$	$\mathbf{x}^*$
$\lambda > \max\{K_\gamma, \hat{K}_\gamma, 1/K_\gamma\}$	$\hat{r} < r^0$	> 0	> 0
$\lambda > 1/K_\gamma,\ p > \bar{p}$	$\hat{r} > r^0$	> 0	–
$\lambda > K_\gamma,\ p < \bar{p}$	$\hat{r} > r^0$	< 0	–
$\lambda > \max\{K_\gamma, 1/K_\gamma\}$, (21) is satisfied	$\hat{r} \neq r^0$	–	> 0, local
$\lambda > \max\{K_\gamma, 1/K_\gamma\}$, (23) is satisfied	$\hat{r} \neq r^0$	–	< 0, local
$\lambda > \max\{K_\gamma, 1/K_\gamma\}$	$\hat{r} = r^0$	$= 0$	$= 0$
		$\mathbf{d}\vert\mathbf{x}^*\vert/\mathbf{d}\lambda$	$\mathbf{d}\vert\mathbf{x}^*\vert/\mathbf{d}\lambda$
$\lambda > \max\{K_\gamma, \hat{K}_\gamma, 1/K_\gamma\}$	$\hat{r} < r^0$	$= 0$	$< 0^*$
$\lambda > \max\{K_\gamma, 1/K_\gamma\}$	$\hat{r} > r^0$	< 0	–
$\lambda > \max\{K_\gamma, 1/K_\gamma\}$, (21) or (23) is satisfied	$\hat{r} \neq r^0$	–	$< 0^*$
	$\hat{r} = r^0$	$= 0$	$= 0$
		$\mathbf{d}\vert\mathbf{x}^*\vert/\mathbf{d}\hat{\mathbf{r}}$	$\mathbf{d}\vert\mathbf{x}^*\vert/\mathbf{d}\hat{\mathbf{r}}$
$\lambda > \max\{K_\gamma, \hat{K}_\gamma, 1/K_\gamma\}$	$\hat{r} < r^0$	< 0	$< 0^*$
$\lambda > \max\{K_\gamma, 1/K_\gamma\}$	$\hat{r} > r^0$	> 0	–
$\lambda > \max\{K_\gamma, 1/K_\gamma\}$, (21) or (23) is satisfied	$\hat{r} > r^0$	–	$> 0^*$
	$\hat{r} = r^0$	$= 0$	$= 0$
Linear loss aversion			
		$\mathbf{x}^*$	$\mathbf{x}^*$
$\lambda > 1/K_0$	$\hat{r} \neq r^0$	> 0	> 0
	$\hat{r} = r^0$	$= 0$	$= 0$
		$\mathbf{dx}^*/\mathbf{d}\lambda$	$\mathbf{dx}^*/\mathbf{d}\lambda$
$\lambda > 1/K_0$	$\hat{r} \neq r^0$	$= 0$	< 0
	$\hat{r} = r^0$	$= 0$	$= 0$
		$\mathbf{dx}^*/\mathbf{d}\hat{\mathbf{r}}$	$\mathbf{dx}^*/\mathbf{d}\hat{\mathbf{r}}$
$\lambda > 1/K_0$	$\hat{r} < r^0$	< 0	< 0
	$\hat{r} > r^0$	> 0	> 0
	$\hat{r} = r^0$	$= 0$	$= 0$

We assume that $\mathbb{E}(r) > r^0$ and $\lambda > 1$. Note that K_γ for the Bernoulli case is defined by (5) while for the continuous case it is defined by (20). In addition, $\hat{K}_\gamma$ is defined by (22) and $\bar{p}$ by (11). 0^* denotes the cases when an additional assumption is required, namely $\lim_{r\to\pm\infty} |r|^{2-\gamma} f(r) = 0$. For more details regarding the linear loss averse investor see Fortin and Hlouskova (2011).

in the domain of gains, while the more ambitious investor increases her short position when increasing $\hat{r}$ as she is more risk seeking in the domain of losses. Finally, with an increasing degree of loss aversion the short position decreases.

A complete summary of the results including comparative statics with respect to loss aversion (λ) and the level of ambition ($\hat{r}$) for both the prospect theory investor and the linear loss averse investor[19] can be found in Table 2.

2.3. Additional portfolio properties

In this section we present additional portfolio's properties, such as the sensitivities of the first three moments of the portfolio return as well as its expected excess return with respect to the reference return, $\mathbb{E}(r) - \hat{r}$, to changes of the degree of loss aversion, λ, the level of ambition, $\hat{r}$, and diminishing sensitivity, γ. We analyze also the portfolio performance, such as the Sharpe ratio, the Sortino ratio, downside volatility, and the Omega measure, with respect to the same parameters $(\lambda, \hat{r}, \gamma)$. We show these properties analytically, under the assumption that the risky asset is continuously distributed and that $\mathbb{E}(r^2)$ is finite, i.e., the variance exists, as stated above.

Note that $\mathbb{E}(R(x^*)) = r^0 + \left(\mathbb{E}(r) - r^0\right) x^*$. Thus, $\frac{d\mathbb{E}(R(x^*))}{d\lambda} = \left(\mathbb{E}(r) - r^0\right) \frac{dx^*}{d\lambda}$, and as $\mathbb{E}(r) > r^0$ then

$$\operatorname{sgn}\left(\frac{d\mathbb{E}(R(x^*))}{d\lambda}\right) = \operatorname{sgn}\left(\frac{dx^*}{d\lambda}\right) \tag{28}$$

I.e., the sign of the change of the expected portfolio return, $\mathbb{E}(R(x^*))$, with respect to λ, and also with respect to $\hat{r}$ and γ, is the same as the sign of the change of risk taking, x^*, with respect to these parameters.[20] Thus, based on Proposition 2.7 it follows that, e.g., for $x^* > 0$ is the expected portfolio return decreasing with an increasing loss aversion and has a V-shaped pattern with respect to the reference return level, i.e., the expected portfolio return decreases with an increasing reference return for $\hat{r} < r^0$ and it increases with an increasing reference return for $\hat{r} > r^0$. Similar findings hold when $x^* < 0$.

The expected excess return with respect to the reference return, or expected gain/loss, has the form

$$\mathbb{E}(R(x^*)) - \hat{r} = r^0 - \hat{r} + (\mathbb{E}(r) - r^0)x^* \tag{29}$$

We refer to it as expected gain, when $\mathbb{E}(r) > \hat{r}$, or expected loss, when $\mathbb{E}(r) < \hat{r}$. Let us focus on $x^* > 0$. Then it follows from $\mathbb{E}(r) > r^0$ that

$$\mathbb{E}(R(x^*)) - \hat{r} \begin{cases} < 0 & \text{if } x^* < \frac{\hat{r}-r^0}{\mathbb{E}(r)-r^0} \text{ (expected loss)} \\ > 0 & \text{if } x^* > \frac{\hat{r}-r^0}{\mathbb{E}(r)-r^0} \text{ (expected gain)} \end{cases} \tag{30}$$

This implies that an expected loss can only occur for more ambitious investors. On the other hand, for less ambitious investors only an expected gain can occur (as $x^* > 0$). Finally, more ambitious investors can achieve an expected gain only for a sufficiently large x^*, namely when $x^* > \frac{\hat{r}-r^0}{\mathbb{E}(r)-r^0}$. Thus, the positioning of the reference return with respect to the risk-free rate is crucial for the perception of gains or losses.

Note that (27) and (29) imply that the expected gain/loss has the following property with respect to the reference return, for fixed λ and γ

$\frac{d}{d\hat{r}}\left(\mathbb{E}(R(x^*)) - \hat{r}\right)$

[19] For more details regarding the LLA investor, see again Fortin and Hlouskova (2011).

[20] Note in addition that for $x^* > 0$ is $\mathbb{E}(R(x^*)) > r^0$ and for $x^* < 0$ is $\mathbb{E}(R(x^*)) < r^0$.

$$= -1 + \left(\mathbb{E}(r) - r^0\right)\frac{dx^*}{d\hat{r}} \begin{cases} < 0, & \text{if } \hat{r} < r^0 \quad \text{(expected gain) or} \\ & \text{if } \hat{r} > r^0 \text{ and } x^* < \frac{\hat{r}-r^0}{\mathbb{E}(r)-r^0} \quad \text{(expected loss)} \\ > 0, & \text{if } \hat{r} > r^0 \text{ and } x^* > \frac{\hat{r}-r^0}{\mathbb{E}(r)-r^0} \quad \text{(expected gain)} \end{cases} \tag{31}$$

Thus, for less ambitious investors is the expected gain a decreasing function of $\hat{r}$, while for more ambitious investors is the expected gain an increasing function of $\hat{r}$, if $x^* > \frac{\hat{r}-r^0}{\mathbb{E}(r)-r^0}$, and the expected gain/loss a decreasing function of $\hat{r}$, if $x^* < \frac{\hat{r}-r^0}{\mathbb{E}(r)-r^0}$.

Sensitivity analysis of higher moments of the portfolio return and of performance measures

Regarding the sensitivity analysis of the variance of the portfolio return,[21] we observe that the variance responds, in terms of its sign, to the changes in the parameters (λ, $\hat{r}$ and γ) in the same way as risk taking x^* does. As $Var(R(x)) = Var\left(r^0 + (r - r^0)x\right) = x^2 Var(r)$ we have $\frac{dVar(R(x^*))}{d\lambda} = 2x^* Var(r)\frac{dx^*}{d\lambda}$, which implies that

$$\text{sgn}\left(\frac{dVar(R(x^*))}{d\lambda}\right) = \begin{cases} \text{sgn}\left(\frac{dx^*}{d\lambda}\right) & \text{for } x^* > 0 \\ -\text{sgn}\left(\frac{dx^*}{d\lambda}\right) & \text{for } x^* < 0 \end{cases} \tag{32}$$

The same holds for $\frac{dVar(R(x^*))}{d\hat{r}}$ and $\frac{dVar(R(x^*))}{d\gamma}$. Thus, e.g., for $x^* > 0$ is the portfolio variance decreasing with an increasing level of loss aversion, decreasing with an increasing reference return (or diminishing sensitivity), if investors are less ambitious, while the portfolio variance is increasing with an increasing reference return (or diminishing sensitivity) if investors are more ambitious.

When looking at the sensitivity analysis of the third moment of the portfolio return we observe that it responds, in terms of the sign, to the changes in the parameters (λ, $\hat{r}$ and γ) in the same way as the risk taking x^* for a positively skewed return of the risky asset and in the opposite way for a negatively skewed return. As $\mathbb{E}\left(R(x) - \mathbb{E}(R(x))\right)^3 = x^3\mathbb{E}\left(r - \mathbb{E}(r)\right)^3$, we have $\frac{d\mathbb{E}\left(R(x^*)-\mathbb{E}(R(x^*))\right)^3}{d\lambda} = 3(x^*)^2\mathbb{E}\left(r - \mathbb{E}(r)\right)^3\frac{dx^*}{d\lambda}$, which implies that

$$\text{sgn}\left(\frac{d\mathbb{E}\left(R(x^*) - \mathbb{E}(R(x^*))\right)^3}{d\lambda}\right) = \begin{cases} \text{sgn}\left(\frac{dx^*}{d\lambda}\right) & \text{for } \mathbb{E}\left(r - \mathbb{E}(r)\right)^3 > 0 \\ -\text{sgn}\left(\frac{dx^*}{d\lambda}\right) & \text{for } \mathbb{E}\left(r - \mathbb{E}(r)\right)^3 < 0 \end{cases} \tag{33}$$

The same again applies to sensitivities with respect to $\hat{r}$ and γ. Thus, if for instance the return of the risky asset is negatively skewed, i.e., $\mathbb{E}\left(r - \mathbb{E}(r)\right)^3 < 0$, then the third moment of the portfolio return increases with increasing degree of loss aversion, as investment in the risky asset decreases with increasing λ. The same dynamics occur for an increasing reference return and diminishing sensitivity for less ambitious investors, while the third moment of the portfolio return decreases with an increasing $\hat{r}$ and γ for more ambitious investors. Considering the third standardized moment of the portfolio return we have $\mathbb{E}\left(\frac{R(x)-\mathbb{E}(R(x))}{\sqrt{Var(R(x))}}\right)^3 = \text{sgn}(x)\cdot\mathbb{E}\left(\frac{r-\mathbb{E}(r)}{\sqrt{Var(r)}}\right)^3$, which is independent of any parameters (λ, $\hat{r}$ and γ).

When analyzing risk-adjusted performance measures of the portfolio return, such as the Sharpe ratio (SR) or the Sortino ratio, we can see that they are independent of loss aversion, the reference return and diminishing sensitivity. E.g., in the case of the Sharpe ratio

$$SR(R(x)) = \frac{\mathbb{E}(R(x) - r^0)}{\sqrt{Var(R(x) - r^0)}} = \frac{x\left(\mathbb{E}(r) - r^0\right)}{|x|\sqrt{Var(r - r^0)}} = \begin{cases} SR(r), & \text{for } x > 0 \\ -SR(r), & \text{for } x < 0 \end{cases} \tag{34}$$

[21] The same applies to the downside variance.

and the Sharpe ratio of the risky asset return, $SR(r)$, does not depend on the above mentioned parameters. As will be seen later in the empirical part in Section 4, this is not the case when more risky assets are included in the portfolio.

Finally, it is straightforward to see that when the risky asset return follows the Bernoulli distribution then

$$\text{Omega}(R(x)) = \begin{cases} 1/K_0 & \text{for } x > 0 \\ K_0 & \text{for } x < 0 \end{cases} \tag{35}$$

where K_0 is given by (5) for $\gamma = 0$. Thus also the Omega measure does not depend on the parameters of the model.[22]

3. Simulations

In this section we perform different simulation exercises to present the results for our theoretical two-asset problem of a prospect theory investor with different values of the reference return, loss aversion, diminishing sensitivity and different characteristics of the financial market, when $x^* > 0$. In particular, we examine whether the assumption of an asymmetric distribution of the risky asset return has major effects on the results. We consider the normal and the skew normal distributions to describe symmetric and asymmetric distributions of the risky asset return, respectively. We focus on negative skewness as this type of skewness is mainly observed in stock markets. For the risky asset return we consider a mean of 13.6%, a standard deviation of 14.9% and, for the negatively skewed distribution, a shape parameter of −3.1, resulting in a skewness of −0.68. The risk-free rate is 3.5%. These values are in line with the observed statistics of the US (stock) market in the period January 1983 to December 2020 (see Table B.2), which we consider in the empirical application. In some examples we fix the degree of loss aversion at 2.25, which is the value suggested by experimental results, see, e.g., Tversky and Kahneman (1992). In the simulations we numerically compute the optimal weight of the risky asset, x^*, which is derived theoretically in Propositions 2.2, 2.5, and 2.6, using the densities of the normal and skew normal distributions and generating numerical integral solutions.[23]

First we present risk taking, x^*, as a function of the diminishing sensitivity parameter, γ, see Fig. 3.[24] We observe that less ambitious investors ($\hat{r} < r^0$, left graph in Fig. 3) are becoming more conservative (i.e., decrease the investment in the risky asset) with an increasing degree of diminishing sensitivity, while more ambitious investors ($\hat{r} > r^0$, right graph in Fig. 3) are becoming less conservative (i.e., increase the investment in the risky asset) with an increasing degree of diminishing sensitivity. Thus, the investor's risk attitude is different for less and more ambitious investors. This can be explained as follows: (30) implies that less ambitious investors experience only expected gains, $\mathbb{E}(R(x^*)) > \hat{r}$, and as in the domain of gains investors become more risk averse with an increasing γ, they invest less in the risky asset. However, more ambitious investors experience also expected losses, $\mathbb{E}(R(x^*)) < \hat{r}$, and

[22] At least for this particular distribution of the risky asset returns. This feature we have observed also in our simulations when assuming that the risky asset return followed normal and skew normal distributions.

[23] Alternatively, we simulate from the normal and skew normal distributions and compute expectations, where we use the method introduced in Henze (1986) to simulate from the skew normal. The number of simulations is 10^6. The two solution methods basically deliver the same solutions.

[24] Let us remind ourselves that a larger γ parameter in the value function implies a smaller sensitivity with respect to the return of the risky asset that is far away from the reference return, $r > \hat{r}$ (higher degree of risk aversion), but a higher sensitivity for the risky asset return that is closer to $\hat{r}$. Similarly, we observe a smaller sensitivity with respect to the return of the risky asset that is much smaller than the reference return, $r < \hat{r}$ (higher degree of risk seeking), but a higher sensitivity for the risky asset return that is closer to $\hat{r}$.

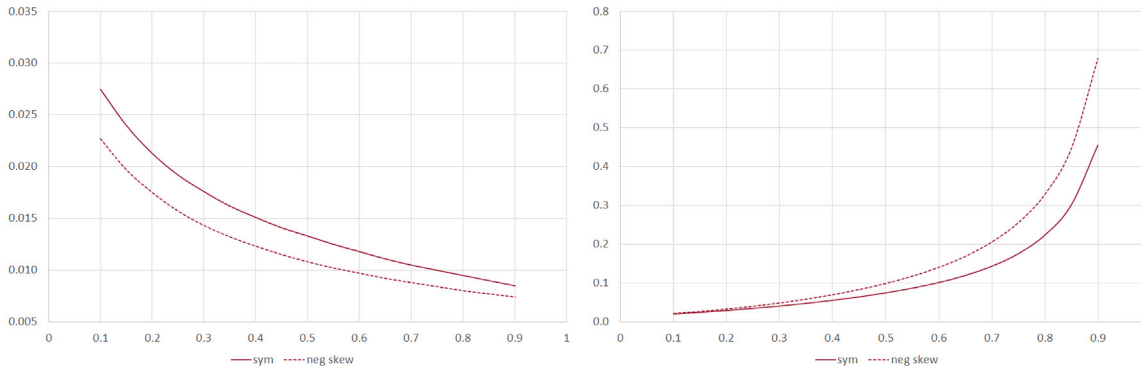

Fig. 3. Risk taking, x^*, as a function of diminishing sensitivity, γ.
The horizontal axis shows the diminishing sensitivity parameter. In the left graph the reference level is smaller than the risk-free rate, in the right graph the reference level is larger than the risk-free rate. The results are shown for both symmetric and negatively skewed distributions, where the parameters of the risky asset return and the risk-free rate ($r^0 = 3.5\%$) are in line with the US data, see Table B.2, $\lambda = 2.25$, $\hat{r} = 1.2\% < r^0$ (left graph), and $\hat{r} = 4.3\% > r^0$ (right graph).

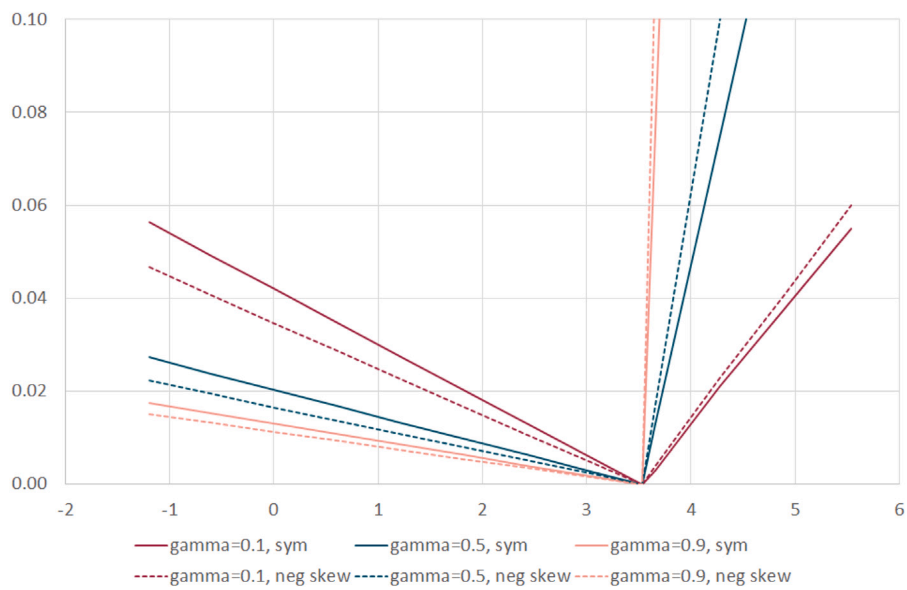

Fig. 4. Risk taking, x^*, as a function of the reference return, $\hat{r}$, for $\gamma = 0.1, 0.5, 0.9$.
The horizontal axis shows the reference return. The results are shown for both symmetric and negatively skewed distributions, where the parameters of the risky asset return and the risk-free rate ($r^0 = 3.5\%$) are in line with the US data, see Table B.2, and $\lambda = 2.25$.

in the domain of losses they are more risk seeking, and thus they invest more in the risky asset with increasing γ.[25]

Fig. 4 presents risk taking as a function of the reference return, $\hat{r}$, which is V-shaped and which is shown analytically in Proposition 2.7 for a general continuous distribution of the risky asset. Fig. 5 presents risk taking as a function of loss aversion, λ, for both less ambitious and more ambitious investors. In both cases risk taking decreases with an increasing degree of loss aversion, which is in line with Proposition 2.7.

Based on Figs. 3, 4 and 5 we observe the following common result related to comparing risk taking implied by the symmetric distribution of the risky asset return, x^*_{sym}, and risk taking implied by the negatively skewed distribution of the risky asset return, x^*_{negsk}. For less ambitious investors risk taking implied by the negatively skewed distribution is smaller than risk taking implied by the symmetric distribution ($x^*_{negsk} < x^*_{sym}$ for $\hat{r} < r^0$), while for more ambitious investors risk taking implied by the negatively skewed distribution is larger than risk taking implied by the symmetric distribution ($x^*_{negsk} > x^*_{sym}$ for $\hat{r} > r^0$), given all other parameters are fixed. Thus, if one falsely assumes symmetric returns when in fact they are negatively skewed, one overestimates the investment in the risky asset for less ambitious investors; and underestimates investment in the risky asset for more ambitious investors. The positive skewness of the risky asset return has the opposite effect on risk taking.[26]

Fig. 6 presents the expected gain/loss functions for investors with $\gamma = 0.1, 0.5, 0.9$, for a symmetric and a negatively skewed distribution of the risky asset return. The graph additionally shows risk taking, x^*, and the ratio $\frac{\hat{r}-r^0}{\mathbb{E}(r)-r^0}$, as the relation of x^* to this ratio determines whether the more ambitious investor incurs expected gains or losses, see (30). The more ambitious investor with $\gamma = 0.9$ experiences expected gains, as $x^* > \frac{\hat{r}-r^0}{\mathbb{E}(r)-r^0}$, while the more ambitious investor with $\gamma = 0.1$ experiences expected losses, as $x^* < \frac{\hat{r}-r^0}{\mathbb{E}(r)-r^0}$. For the more ambitious investor with $\gamma = 0.5$ the expected gain/loss function is slightly negative

[25] This is analytically shown for the case when the risky asset return follows the Bernoulli distribution (see Corollaries 2.1 and 2.2). It seems to be equally true when the risky return is generally continuously distributed, as suggested by the numerical solutions of risk taking as a function of γ.

[26] Namely, $x^*_{negsk} < x^*_{sym} < x^*_{possk}$ for $\hat{r} < r^0$, while $x^*_{possk} < x^*_{sym} < x^*_{negsk}$ for $\hat{r} > r^0$, for fixed λ, $\hat{r}$ and γ, where x^*_{possk} is risk taking when the return of the risky asset is positively skewed.

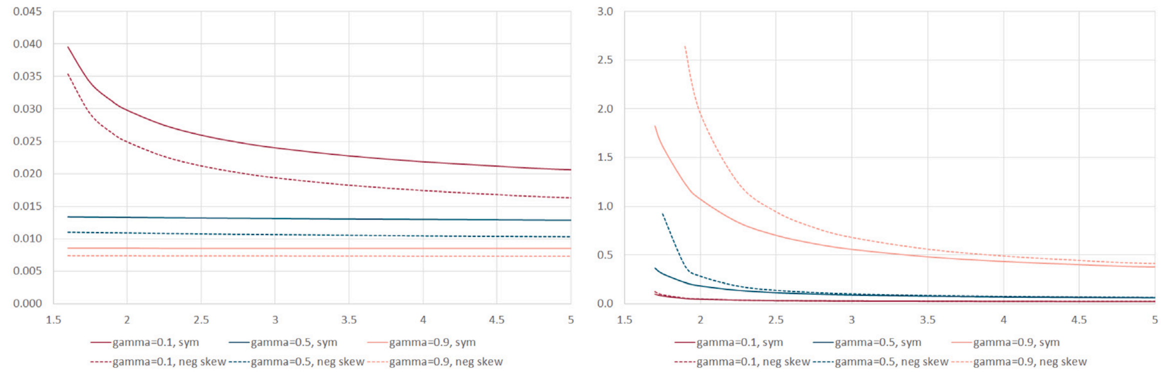

Fig. 5. Risk taking, x^*, as a function of loss aversion, λ, for $\gamma = 0.1, 0.5, 0.9$.
The horizontal axis shows the level of loss aversion. In the left graph the reference level is smaller than the risk-free rate, in the right graph the reference level is larger than the risk-free rate. The results are shown for both symmetric and negatively skewed distributions, where $r^0 = 3.5\%$ (in line with the US data, see Table B.2), $\lambda = 2.25$, $\hat{r} = 1.2\% < r^0$ (left graph), and $\hat{r} = 4.9\% > r^0$ (right graph).

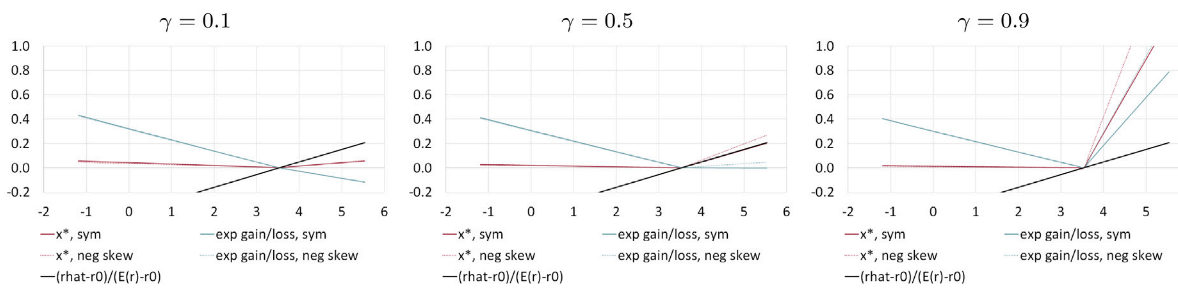

Fig. 6. Risk taking, x^*, and expected portfolio gain/loss, $\mathbb{E}(R(x^*)) - \hat{r}$, with respect to the reference return.
The vertical axis shows risk taking, x^*, and the expected excess portfolio return with respect to the reference return, $\mathbb{E}(R(x^*)) - \hat{r}$, which we call expected portfolio gain if it is positive, $\mathbb{E}(R(x^*)) > \hat{r}$, and expected portfolio loss if it is negative, $\mathbb{E}(R(x^*)) < \hat{r}$. The horizontal axis shows the reference return. The results are shown for both symmetric and negatively skewed distributions, where the parameters of the risky asset return and the risk-free rate ($r^0 = 3.5\%$) are in line with the US data, see Table B.2, and $\lambda = 2.25$.

for symmetric returns and it is positive for negatively skewed returns. The negative skewness of returns thus turns expected losses to expected gains in this example. So the value of the diminishing sensitivity parameter as well as the skewness/symmetry of the risky asset return are crucial for incurring expected gains or losses, when the investor is more ambitious. The less ambitious investor always achieves expected gains, see again (30).

In our example, the expected gain decreases with an increasing reference return $\hat{r}$ for less ambitious investors for any γ, see (31), while for more ambitious investors the expected loss increases (with increasing $\hat{r}$) for $\gamma = 0.1$, as $x^* < \frac{\hat{r}-r^0}{\mathbb{E}(r)-r^0}$, and the expected gain increases for $\gamma = 0.9$ as $x^* > \frac{\hat{r}-r^0}{\mathbb{E}(r)-r^0}$. For $\gamma = 0.5$ the situation is different for symmetric and negatively skewed returns. Namely, more ambitious investors increase their expected *losses* with an increasing reference return if the risky asset return is symmetric, as $x^*_{sym} < \frac{\hat{r}-r^0}{\mathbb{E}(r)-r^0}$, while they increase their expected *gains* if the risky asset return is negatively skewed, as $x^*_{negsk} > \frac{\hat{r}-r^0}{\mathbb{E}(r)-r^0}$, see (31). Thus, in this context, the negative skewness of the risky asset return seems to improve the investor's performance in terms of the expected gain/loss measure.

In the following we consider our concrete example to analyze the dynamics of the indirect utility function (the expected value function at its maximum which expresses the investor's level of satisfaction or happiness) with respect to the investor's level of ambition represented by the reference return $\hat{r}$, see Fig. 7. As in the case of the discrete Bernoulli distribution is the investor's level of happiness decreasing with an increasing reference return. We can see that the shape of indirect utility (happiness) as a function of the reference return mirrors, in some sense, the value function (where the reference return is replaced by the risk-free rate r^0). Hence the indirect utility function is decreasing with an increasing reference return, is concave in the domain of less ambitious investors, and is convex in the domain of more ambitious investors. This implies that the sensitivity of happiness with respect to the reference return increases with an increasing reference return for less ambitious investors (i.e., when $\hat{r}$ is moving towards r^0) and decreases with an increasing reference return for more ambitious investors (i.e., when $\hat{r}$ is moving away from r^0). Thus, investors with reference returns around the risk-free rate are the most sensitive ones with respect to changes of their levels of ambition. In addition, the investor's happiness for more ambitious investors is larger when the risky asset return is negatively skewed than when it is symmetric.[27] In summary, higher ambition decreases happiness, but less so for negatively skewed risky asset returns.

4. Empirical application

In this section we illustrate the investment problem of a prospect theory (PT) investor. Investors typically include safe assets (bonds), risky assets (stocks) and additional assets, which are only weakly related to the other assets (e.g., commodities), in their portfolios. The simplest such type of portfolio includes a bond, a stock and a commodity, and this is what we consider. We compare the results implied by PT preferences with the results implied by other types of preferences, in particular, the mean–variance (MV) approach.[28] We also examine how different characteristics of the PT investor (loss aversion, diminishing sensitivity, reference level) affect the optimal portfolio.[29]

27 The opposite holds, to a lesser extent, for less ambitious investors.

28 Throughout this section we refer to the mean–variance (MV) portfolio as the portfolio with the minimum variance of the portfolio return.

29 For example, does the expected return of the portfolio, its volatility, and downside volatility decrease with an increasing degree of loss aversion, as found theoretically for the two-asset case?

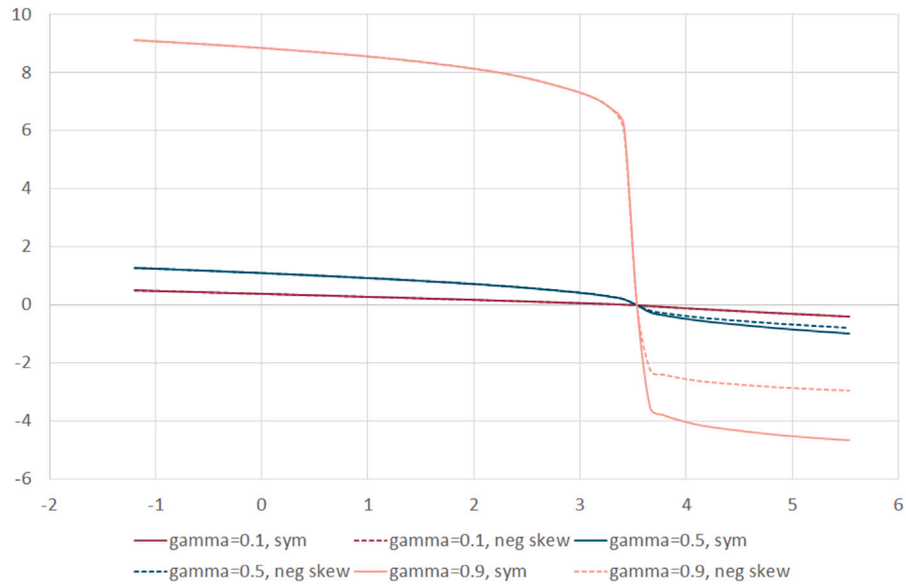

Fig. 7. Expected utility at its maximum (indirect utility function, happiness, investor satisfaction) for $\gamma = 0.1, 0.5, 0.9$.
The horizontal axis shows the reference return. The results are shown for both symmetric and negatively skewed distributions, where the parameters of the risky asset return and the risk-free rate ($r^0 = 3.5\%$) are in line with the US data, see Table B.2, and $\lambda = 2.25$.

We separately consider a European and a US investor in order to acknowledge geographical differences. The investor re-optimizes her portfolio each month and we also consider dynamic scenarios in which the reference return and the loss aversion parameter are updated based on the previous period portfolio performance.

4.1. Different types of prospect theory investors

We investigate the performance of an optimal asset portfolio constructed by a PT investor. We study the *benchmark scenario*, where the penalty parameter is constant and the reference return is equal to zero percent, and three modified versions of the benchmark scenario. The first modification uses again a constant penalty parameter and the risk-free interest rate as the reference point (*risk-free scenario*), while the remaining two dynamic modifications employ time-changing versions of the penalty parameter, which depend on previous gains and losses, while the reference return is either zero, ($\hat{r}_t = 0$), the risk-free interest rate ($\hat{r}_t = r^0$) or the portfolio return of the previous period ($\hat{r}_t = R_{t-1}$).[30] So, we consider two constant and two dynamic scenarios with respect to the penalty parameter. The first dynamic scenario describes the usual conservative loss averse investor, who becomes even more loss averse after losses (*conservative scenario*), while the second dynamic scenario describes a more aggressive non-conventional risk seeking investor, who becomes less loss averse after losses and accepts further risk and gambles which offer a chance to break even (*aggressive scenario*). Our conservative and aggressive (break-even) scenarios are modified versions of the scenarios suggested by Barberis and Huang (2001) and Zhang and Semmler (2009), respectively.[31]

Thus, the value function adjusted for the time-changing penalty parameter and a certain reference return is

$$v(R_t) = \begin{cases} \frac{(R_t - \hat{r}_t)^{1-\gamma}}{1-\gamma}, & R_t \geq \hat{r}_t \\ -\lambda_t \frac{(\hat{r}_t - R_t)^{1-\gamma}}{1-\gamma}, & R_t < \hat{r}_t \end{cases}$$

The *conservative scenario* is modeled as follows. If the investor has experienced gains then her penalty parameter is equal to the prespecified λ while, on the other hand, if the investor has experienced losses then she increases her degree of loss aversion with respect to the prespecified level, i.e.,

$$\lambda_t = \begin{cases} \lambda, & R_{t-1} \geq R_{t-2} \text{ (gains, } \lambda \text{ at prespecified level)} \\ \lambda + \left(z_{t-1} - 1\right), & R_{t-1} < R_{t-2} \text{ (losses, } \lambda \text{ increases with respect} \\ & \text{to prespecified level)} \end{cases} \tag{36}$$

where $z_t = \frac{1+R_{t-1}}{1+R_t} \geq 0$[32] and $\lambda_t \geq \lambda$. See the left plot in Fig. 8, where the dashed line represents the value function of a conservative investor, when loss aversion increases after prior losses.

The *aggressive scenario* is based on the idea that sometimes both private and institutional investors become more risk seeking after losses in order to make up for previous losses. In other words, even if they have experienced a loss in the previous period, investors may be ready to incur further risks and accept gambles which offer them a chance to break even. So in this case losses imply a decreasing loss aversion due to the investor's increased risk seeking. The gain, on the other hand, is treated as in the conservative scenario, i.e., the degree of loss aversion is equal to the prespecified level. The time-changing penalty parameter

[30] This idea is developed in Choi et al. (2022) and it was also used in Hlouskova et al. (2019). Using the last period's information (in our case, the last period's portfolio return) as a reference point is also supported by Baucells et al. (2011), who find in experiments about updating reference levels that the first and the last prices in a sequence of information are most important.

[31] Note that although this analysis covers many empirical aspects of the problem a more thorough analysis is required to shed light on all details and we will deal with this in our future research. For example, as we focus mainly on loss aversion, the dynamic scenarios in our study update only the investor's loss aversion, and only after prior losses. They do not update the reference return and do not apply dynamic updates of both the reference return and loss aversion after prior gains. So we cannot study here the house money effect, for instance, when people tend to take on increased risk subsequent to a successful investment experience, see Thaler and Johnson (1990).

[32] Note that in the case of gains, i.e., $R_{t-1} \geq R_{t-2}$, is $z_{t-1} \leq 1$ and in the case of losses, i.e., $R_{t-1} < R_{t-2}$, is $z_{t-1} > 1$.

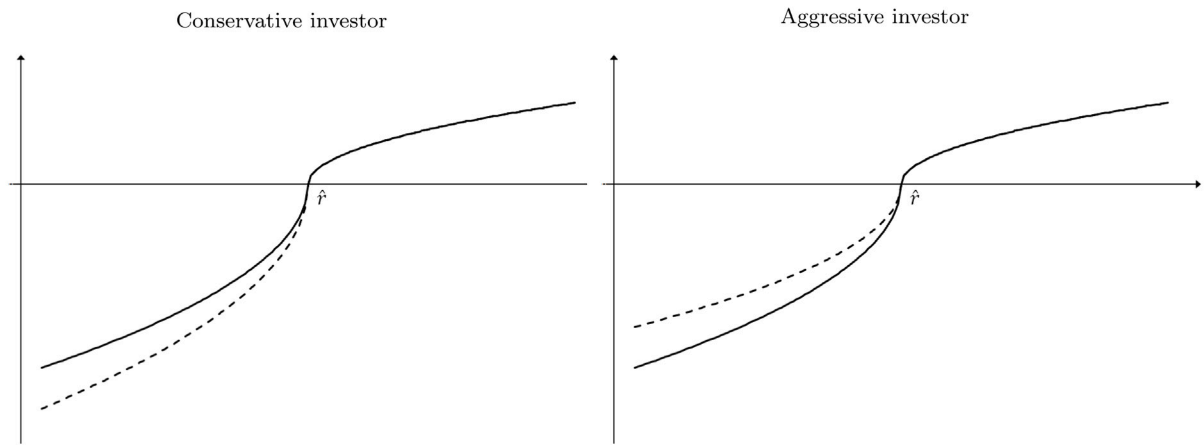

Fig. 8. Value function of conservative and aggressive investors.
The value function after prior gains is plotted as a solid line, the value function after prior losses as a dashed line; $\hat{r}$ denotes the reference return.

is then

$$\lambda_t = \begin{cases} \lambda, & R_{t-1} \geq R_{t-2} \text{ (gains, } \lambda \text{ at prespecified level)} \\ \lambda + \left(\frac{1}{z_{t-1}} - 1\right), & R_{t-1} < R_{t-2} \text{ (losses, } \lambda \text{ decreases with respect to prespecified level)} \end{cases} \tag{37}$$

where $\lambda_t \leq \lambda$. With the current lambda adjustment a sufficient condition for $\lambda_t \geq 1$ is $\lambda \geq 2$.[33] See the right plot in Fig. 8, where the dashed line represents the value function of an aggressive investor, when loss aversion decreases after prior losses.

We use different values of λ in all scenarios to allow for different degrees of loss aversion. Specifically, we let the penalty parameter be equal to 1.5, 2, 2.25, 2.5 and 3.[34] In addition we account for the following values of the risk aversion/risk seeking – or diminishing sensitivity – parameter, namely $\gamma = 0$, 0.1, 0.5 and 0.9.[35] Note that an investor with $\gamma = 0$ represents the linear loss averse (LLA) investor and an investor with $\gamma = 0$ and $\lambda = 1$ corresponds to the risk neutral investor. The latter is thus a special case of LLA investors.

4.2. Data

In the empirical analysis we consider two geographical markets, the European (EU) and the US markets, where the euro area represents the EU market. We consider three different assets among which the investor may select: a stock market index, a 10-year government bond[36] and gold.[37] We solve problem (2) numerically by approximating the expectation of the value function of the portfolio return by its empirical counterpart, using the empirical distribution

$$\max_x \left\{ \frac{1}{S(1-\gamma)} \sum_{s=1}^{S} \left[\left([r_s' x - \hat{r}]^+ \right)^{1-\gamma} - \lambda \left([\hat{r} - r_s' x]^+ \right)^{1-\gamma} \right] \mid \sum_{i=1}^{3} x_i = 1, \ x \geq 0 \right\} \tag{38}$$

where $x = (x_1, x_2, x_3)'$ and its elements denote the proportions of wealth invested in the stock market, the 10-year government bond and gold, $r_s = (r_{s1}, r_{s2}, r_{s3})'$ is the vector of corresponding observed net returns at state s, $s = 1, \ldots, S$, and $[t]^+$ is maximum of 0 and t. We solve (38) numerically by applying the grid search method.[38] Returns are computed as $r_t = 100\,(P_t/P_{t-1} - 1)$, where P_t is the monthly closing price at time t and we consider EU and US investors who completely hedge their respective currency risk.[39] All prices are extracted from Refinitiv Datastream from January 1983 to December 2020. The stock market indices for the EU and the US are broad indices and are calculated by Datastream. Table B.1 in Appendix B provides the data description and the data sources, and Table B.2 in Appendix B reports the summary statistics of all asset returns including correlations. In general, the stock index exhibits a comparatively high risk and return, the government bond shows a much lower risk and return, and gold exhibits a relatively high risk but also a small return. The correlations between the assets are small, which suggest that – at least in mean–variance portfolios – all three assets should be included in the optimal portfolio at non-negligible rates for diversification reasons (provided the volatilities of these asset are not too large and the given correlations also prevail in subperiods).

The investor is assumed to re-optimize her portfolio once a month considering an optimization sample of 36 months, i.e., $S = 36$. This yields an out-of-sample evaluation period from January 1986 until December 2020. For the EU and US prospect theory investors we report optimization results for different scenarios and for different values of loss aversion and diminishing sensitivity, as described above. In addition we present results for the LLA investors (including risk-neutral investors) and for the benchmark investors, i.e, the mean–variance investor and the conditional value-at-risk investor. In particular, we present descriptive statistics including mean, standard deviation, downside volatility, conditional value at risk, and various risk-adjusted performance measures of the optimal portfolio returns as well as the average optimal portfolio weights. Risk-adjusted performance measures

[33] Note that this assumption might be violated in our empirical applications only for $\lambda = 1.5$, which we handle in our code such that if λ_t happens to be below one then we impose $\lambda_t = 1$.

[34] Note that the value $\lambda = 2.25$ is the one initially estimated by Kahneman and Tversky (see Tversky & Kahneman, 1992). Chapman et al. (2018) provide a median estimate of $\lambda = 1.99$ for the US in lab experiments. Finally, Kahneman (2011) finds a range for λ of 1.5 to 2.5, doing several experiments.

[35] Among these values, 0.1 is closest to the one determined by Kahneman and Tversky in their lab experiments, which is $\gamma = 0.12$, and we sometimes call the PT investor with $\gamma = 0.1$ the "typical" PT investor.

[36] We consider the German 10-year government bond as a proxy, because euro area government bonds do not exist and we want to use an investable asset not an artificial aggregate.

[37] Note that gold has a low correlation with stocks and bonds, and hence including gold in the portfolio provides a natural hedge, and it is also (slightly) positively skewed, see Table B.2. Gold is also considered a safe haven in turbulent times.

[38] The whole procedure is implemented in MATLAB R2021a.

[39] Prices in the EU market are quoted in, or transformed to, Euro; prices in the US market are quoted in US dollar.

include the Omega measure and the Sharpe and Sortino ratios. All presented numbers are annualized.

4.3. Results: portfolio performance and asset allocation

The performance results of all investors for all scenarios under consideration are presented in Appendix C, Tables C.1–C.6.[40] There are many different results and our discussion will focus on selected aspects.

Prospect theory versus mean–variance

Our empirical results suggest that PT investment leads to clearly higher means of portfolio returns, but also to much higher risk (measured by the volatility of portfolio returns), than traditional MV investment, for all types of PT investors[41] and for both the EU and US markets. If we consider risk-adjusted performance measures like the Omega measure, the Sharpe ratio and the Sortino ratio, however, then the geographical market seems to matter. In this case MV investment outperforms PT investment in the EU, while PT investment outperforms MV investment in the US in most cases. Considering the benchmark scenario in PT investment with $\gamma = 0.5$, for example, we observe an average Sharpe ratio of portfolio returns (over the different degrees of loss aversion) of 49.9 (59.7) for PT investment versus 62.7 (57.4) for MV investment in the EU (US) market. This is probably driven by the larger weight of bonds in MV portfolios in the EU (81%) than in the US (65%), a less volatile EU than US bond, and a larger mean of the US than the EU stock. See Table 4, Fig. 9 and Tables C.1 and C.2.

In the US, PT portfolios perform significantly better than MV portfolios in terms of their mean returns[42] in the majority of cases, at the 5% significance level, see Table 3. For a PT investor with $\gamma = 0.1$ this is true for all scenarios and for degrees of loss aversion of up to 2.25, for a PT investor with $\gamma = 0.5$ it is true nearly all the time, and for a PT investor with $\gamma = 0.9$ it is true for all portfolios except the ones generated by scenarios with a zero reference return. In the EU, however, PT portfolios hardly ever significantly outperform MV portfolios, except for three cases, namely for an investor with $\gamma = 0.9$ and $\lambda = 3$, where the reference return is equal to zero (including the benchmark scenario as well as the conservative and aggressive scenarios). This difference between the portfolio performance implied by various types of PT investors in the EU and US is probably partly due to the higher returns of stock markets in the US than in the EU. In general the differences between PT and MV investment depend on the type of performance measure used for comparison, and the different results can be explained with the different preferences of PT and MV investors. While PT investors maximize the portfolio return under certain conditions for the deviations from a given reference return, MV investors minimize the portfolio variance.

In addition, the PT investor shows a clearly different investment behavior from the MV investor with respect to implied asset weights. Fig. 10 shows the optimal portfolio weights of stocks, bonds and gold for a PT investor (characterized by $\lambda = 2.25$, $\gamma = 0.5$ and $\hat{r} = 0$) and for an MV investor from January 1986 to December 2020.[43] While the PT investor holds large proportions of stocks during certain time periods and her asset weights may sometimes vary strongly over short periods, the MV investor holds rather large proportions of bonds throughout the total period and her asset weights do not change a lot over short periods.[44] Note that the PT investor substantially increases her investment in gold during (and after) the global financial crisis 2007–2009, which is not observed to the same extent for the MV investor. This behavior of the PT investor is thus more in line with the safe haven behavior observed in crisis periods, considering gold as the safe asset. See again Tables C.1 and C.2.

We can explain the "smooth" investment behavior of the MV investor, dominated by bonds, by her strong preference for a small portfolio risk, which clearly favors bonds whose risk is constantly small and much lower than that of stocks and gold. The PT investor seems to be driven more by the different levels of stock returns resulting, sometimes, in large changes in stock weights.[45] The large difference in investment behavior would be reduced when applying transaction costs. However, we did not impose transaction costs in order to see investment patterns solely driven by different preferences.

In general, investors with PT preferences seem to be more willing to take risks than MV investors, as their preferences are designed to maximize positive deviations from their reference return, minimize negative deviations from the reference return and they also exhibit risk taking behavior in the domain of losses. The MV investors considered in this study, however, minimize the variance of their portfolio returns, which is quite different from targeting returns. Nonetheless, some studies show that under certain conditions, in particular when asset returns are normally, i.e., symmetrically, distributed, the performance of PT and MV portfolios does not deviate too much, see, e.g., Hens and Mayer (2014).[46] However, this is not the case in our study.[47] The quite substantial difference in asset allocations between MV and PT investors, we observe, could thus be caused by asset returns being not symmetrically distributed. Note that, for more ambitious investors, we observe a better performance in terms of the expected gain/loss in our theoretical two-asset world, if the risky asset return is negatively skewed (as opposed to being symmetric), see Section 3.

Reference return updates

Our empirical results suggest that for a given level of diminishing sensitivity and a given degree of loss aversion the reference return seems to be a crucial factor in determining the performance of PT investment. This is in line with our theoretical findings in the two-asset case that the optimal solution of a prospect theory investor depends on her reference level and that expected gains/losses are determined

[40] Additional tables for all possible PT scenarios, all reference returns and all levels of diminishing sensitivity, for the EU and the US, are available upon request. In total we have 48 results tables, of which we present six selected ones in the appendix.

[41] By all types of PT investors we mean investors with respect to all different values (under consideration) of $\hat{r}$, λ and γ, as well as both types of scenarios (constant and dynamic).

[42] In order to find out whether the means of PT portfolio returns (r_t^{PT}) are significantly larger than the means of MV portfolio returns (r_t^{MV}), we estimate coefficient c in the regression $r_t^{PT} - r_t^{MV} = c + \varepsilon_t$, and test whether c is significantly different from zero. We use the HAC adjustment and thus this is equivalent to the Diebold–Mariano test, which is performed in EViews 12.

[43] Note that this particular example shows an investment behavior very similar to other types of PT investors, including conservative and aggressive types. So the following discussion holds overall for any PT investor.

[44] These different weight patterns across PT and MV investors are also suggested by the means and standard deviations of the portfolio weights presented in the bottom parts of Tables C.1–C.6. In particular the high standard deviation of the weights implied by the PT investor vis-a-vis the much lower standard deviation of the weights implied by the MV investor reflects the non-smooth investment behavior of PT investors versus the smooth investment behavior of MV investors. For stocks and bonds, the standard deviation of the weight of the MV investor is often only a third, or even a forth, of the standard deviation of the weight of the PT investor.

[45] This is supported by the high correlations between PT stock weights and stock returns over rolling windows of three years (the optimization period).

[46] Fortin and Hlouskova (2011) show analytically that, under mild assumptions, the optimal portfolio for linear loss averse and mean–variance investors is identical when the risky asset return is normally distributed.

[47] The only case in our theoretical part, when MV investment coincides with PT investment, is when the reference return coincides with the risk-free rate and thus both investors stay out of the market.

Table 3
Diebold–Mariano test for prospect theory returns being larger than mean–variance returns. The table shows annualized estimated coefficients $\hat{c}$ from the regression $r_t^{PT} - r_t^{MV} = c + \varepsilon_t$, estimated for the period January 1986 to December 2020. R is the portfolio return from the previous period.

Scenario:	Benchmark	Risk-free	Conservative			Aggressive		
Reference return:	$\hat{r} = 0$	$\hat{r} = r^0$	$\hat{r} = 0$	$\hat{r} = r^0$	$\hat{r} = R$	$\hat{r} = 0$	$\hat{r} = r^0$	$\hat{r} = R$
EU								
$\gamma = 0.1$								
$\lambda = 1.5$	2.25	2.54	2.01	2.38	3.70	2.31	2.49	3.91
$\lambda = 2$	2.02	1.89	1.80	1.85	1.93	1.82	1.78	1.92
$\lambda = 2.25$	1.35	1.51	1.50	1.59	1.57	1.33	1.57	1.35
$\lambda = 2.5$	1.20	1.35	1.28	1.37	0.95	1.23	1.33	1.08
$\lambda = 3$	0.79	0.74	0.76	0.70	1.06	0.79	0.71	1.07
$\gamma = 0.5$								
$\lambda = 1.5$	1.55	1.78	1.69	1.72	1.18	1.68	1.69	1.75
$\lambda = 2$	1.77	1.45	1.82	1.59	0.26	1.83	1.70	1.43
$\lambda = 2.25$	0.97	1.43	0.95	1.49	0.46	1.01	1.38	0.23
$\lambda = 2.5$	1.57	1.72	1.56	1.79	0.60	1.33	1.73	0.02
$\lambda = 3$	1.32	1.38	1.53	1.43	0.38	1.31	1.40	0.33
$\gamma = 0.9$								
$\lambda = 1.5$	1.77	2.16	2.12	1.91	1.94	2.00	2.16	0.79
$\lambda = 2$	1.28	1.90	1.37	2.15	0.16	1.47	1.99	1.09
$\lambda = 2.25$	1.90	1.72	1.82	1.75	−0.93	1.79	2.05	0.92
$\lambda = 2.5$	1.79	2.07	1.80	1.98	1.29	1.55	2.05	2.30
$\lambda = 3$	2.06*	1.63	2.29**	1.80	0.84	1.99*	1.53	1.13
US								
$\gamma = 0.1$								
$\lambda = 1.5$	5.01**	5.27**	4.70**	5.03**	6.27**	5.20**	5.31**	6.25**
$\lambda = 2$	2.94**	3.03**	2.88*	3.36**	4.20**	2.94**	3.10**	4.02**
$\lambda = 2.25$	2.50*	2.94**	2.45*	2.72*	3.44**	2.54*	2.91**	3.59**
$\lambda = 2.5$	1.82	2.18	1.73	2.08	2.71*	1.81	2.30*	2.59
$\lambda = 3$	1.41	1.56	1.31	1.52	2.48*	1.39	1.63	2.34
$\gamma = 0.5$								
$\lambda = 1.5$	2.92*	3.84**	3.03*	3.88**	2.68	3.16*	3.91**	4.23**
$\lambda = 2$	3.30**	3.04*	3.21**	3.21**	4.03**	3.24**	3.07**	4.00**
$\lambda = 2.25$	2.77**	3.21**	2.66*	3.10**	4.10**	2.68*	3.04**	4.29**
$\lambda = 2.5$	2.53*	3.23**	2.51*	3.18**	4.01**	2.35*	3.08**	4.66**
$\lambda = 3$	1.82	2.82**	1.77	2.83**	2.90*	2.10*	2.83**	3.21*
$\gamma = 0.9$								
$\lambda = 1.5$	2.46	4.28**	2.47	4.10**	3.62*	2.48	4.38**	3.39*
$\lambda = 2$	1.98	4.34**	1.97	4.27**	3.44*	1.93	4.36**	2.02
$\lambda = 2.25$	1.65	3.96**	1.77	4.12**	4.28**	1.58	4.09**	3.41*
$\lambda = 2.5$	1.80	4.08**	1.80	4.15**	4.05**	1.71	4.20**	3.26*
$\lambda = 3$	1.54	2.83*	1.39	2.89**	2.23	1.65	2.87*	3.89**

* indicates rejection of the null hypothesis $c = 0$ at the 10% level, using the HAC adjustment.
** indicates rejection of the null hypothesis $c = 0$ at the 5% level, using the HAC adjustment.

by the position of the reference return with respect to the risk-free rate. So the reference return seems to be more of a game changer than whether the investor follows a constant or a conservative or an aggressive strategy. However, this is an observation based on the specific (limited) scenarios we consider, see Footnote 31. For the reference return being equal to zero or the risk-free interest rate, the constant, conservative and aggressive scenarios yield very similar performance results, for both the EU and the US markets; see Fig. 9, which shows Omega measures for different PT investors with $\gamma = 0.5$.[48] Only when the reference return is equal to the portfolio return of the previous period is the situation different. In this case the difference between the performance of conservative and aggressive investors is generally more pronounced. In the US market the Omegas implied by aggressive investment are larger than those implied by conservative investment for all degrees of loss aversion, while in the EU market the Omegas implied by aggressive investment are larger for smaller degrees of loss aversion and smaller for higher degrees of loss aversion.[49] The similarities for zero and risk-free reference returns (across constant and dynamic scenarios) are mainly due to portfolio returns being rather close to the reference return combined with the small differences in the respective penalty parameters across the scenarios.[50] In the case of the reference return being equal to the portfolio return of the previous period, the differences between the corresponding penalty parameters across conservative and aggressive scenarios are also small, but the portfolio returns are much further away from the reference returns, and thus the resulting differences in performance measures are more pronounced.

Note finally, that the property shown in the theoretical part for the case with two assets, namely, that the expected return of portfolio, its volatility and downside volatility decrease with an increasing degree of

48 Similar observations apply for other performance measures.

49 The investors' characteristics for which the aggressive scenarios lead to a better performance are, however, not similar across the three levels of diminishing sensitivity.

50 Across all different types of investors (with respect to different γ, λ and $\hat{r}$) the loss aversion parameter is at most by 0.33 larger than the prespecified value in the conservative scenarios, and it is at most by 0.25 smaller than the prespecified value in the aggressive scenarios. Mostly, however, the deviations from the prespecified λ are much smaller. In at least 90% of the deviations upwards (conservative scenario) the deviations are smaller than 0.08 (for a given investor), and in at least 90% of the deviations downwards (aggressive scenario) the absolute deviations are smaller than 0.07 (for a given investor).

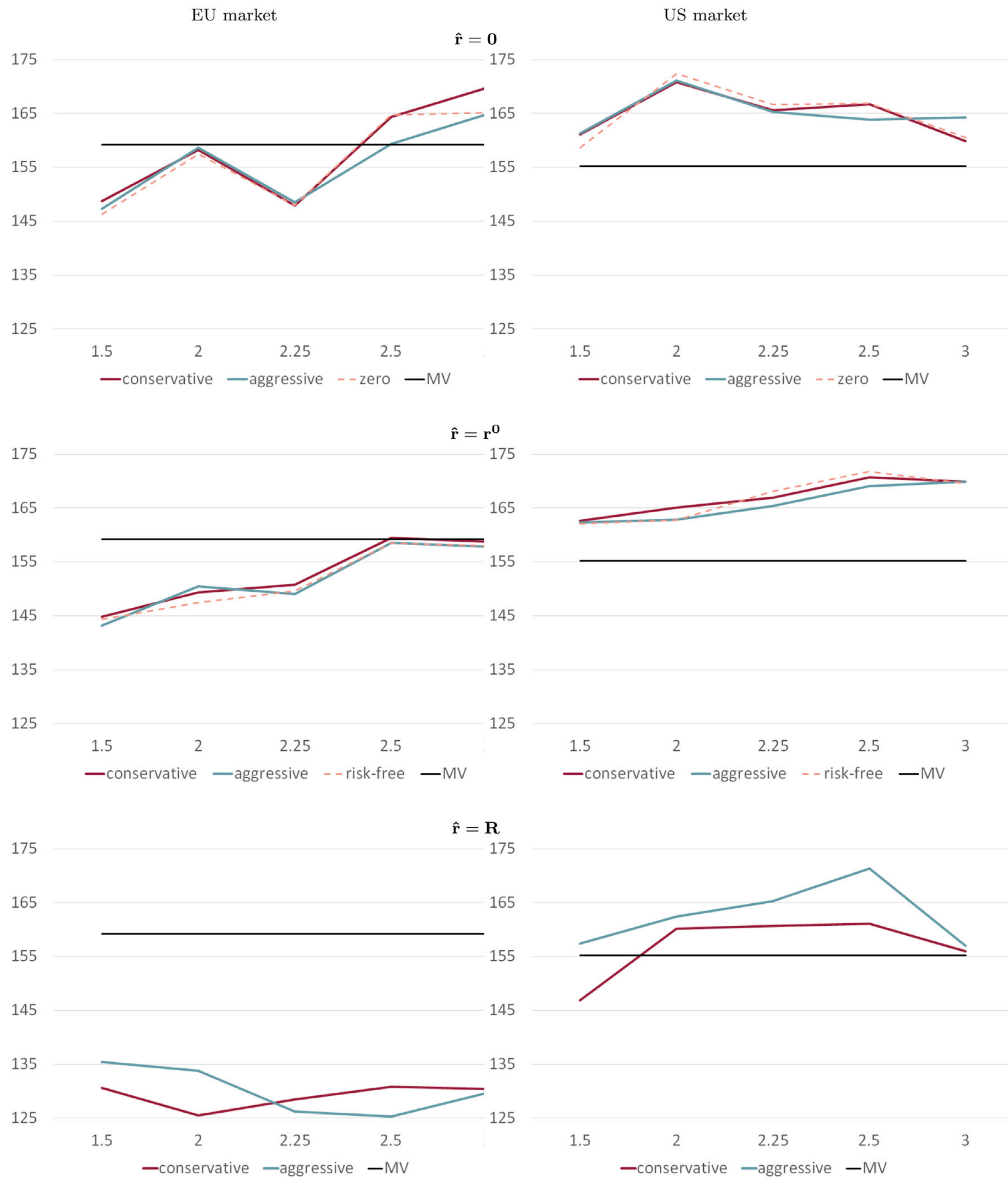

Fig. 9. Conservative versus aggressive prospect theory investing: Omega measure.
The graphs show the Omega measures of portfolio returns implied by conservative and aggressive prospect theory investing ($\gamma = 0.5$) for different degrees of loss aversion (shown on the x-axis) and for a given reference return (top row: $\hat{r} = 0$, middle row: $\hat{r} = r^0$, bottom row: $\hat{r} = R$) in the EU market (left) and the US market (right).

loss aversion holds, in some sense, also for the case with three assets. We observe this property for $\gamma = 0.1$ (see Footnote 35) and to a lesser extent also for $\gamma = 0.5, 0.9$. See Proposition 2.7, Eqs. (25), (28), (32), and Tables C.1–C.6.

EU versus US

When comparing prospect theory investment in the EU and the US we notice that the means of portfolio returns are clearly larger in the US than in the EU (by roughly two to three percentage points, for a given PT investor), and also risk-adjusted performance measures are mostly larger in the US, see Tables 4 and 5. If one targets risk measures like the volatility of portfolio returns, however, then the EU seems to be the market to go for, as volatilities of PT portfolios are generally smaller in the EU than in the US, where for higher levels of loss aversion (all other things equal) the difference in volatility is usually more pronounced. See Tables C.1–C.6.

With respect to the best performance we observe that investors show a different behavior in the EU and US markets. The best risk-adjusted

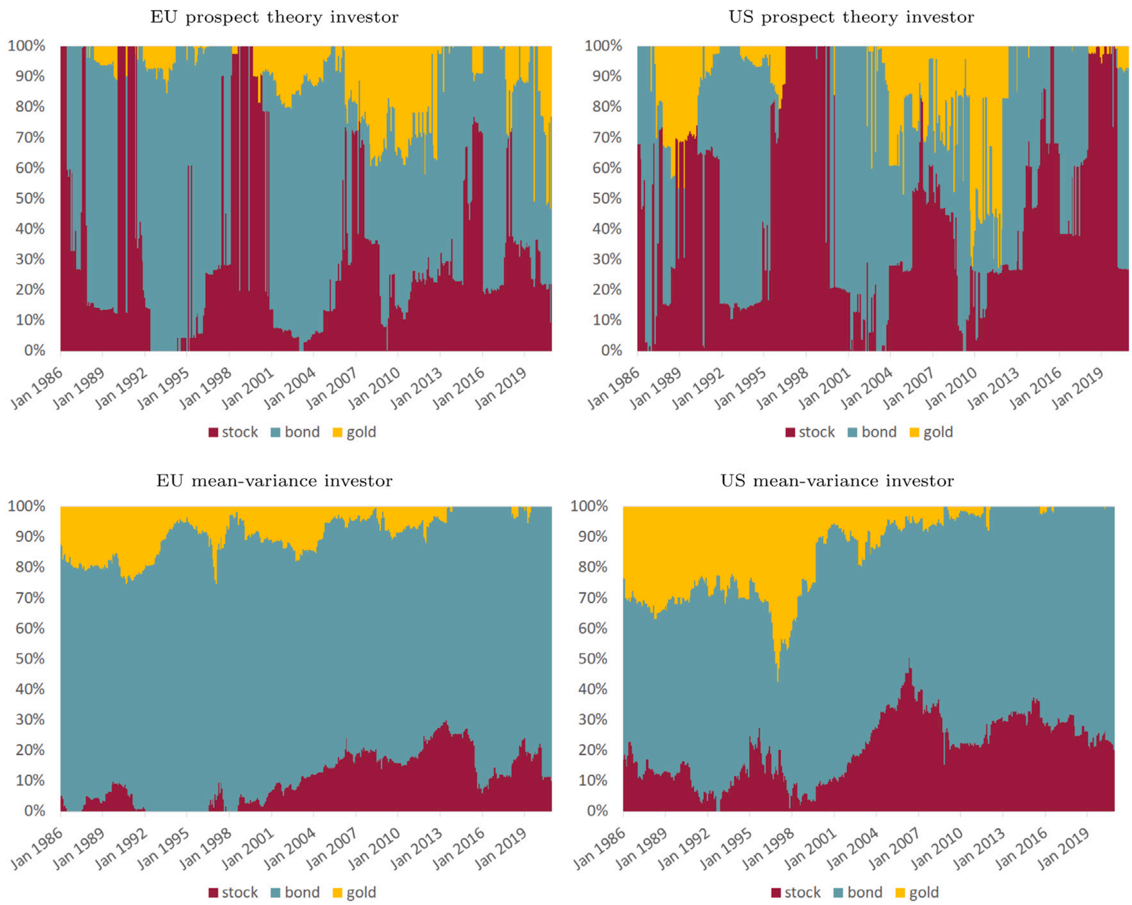

Fig. 10. Optimal portfolio weights for prospect theory and mean–variance investors.
The graph shows optimal asset weights for investors in the EU (left) and in the US (right) markets. The prospect theory investor is characterized by $\lambda = 2.25$, $\gamma = 0.5$ and a zero reference return.

Table 4
Best performing scenarios and models with respect to risk-adjusted performance measures (Omega measure, CVaR, Sharpe ratio, Sortino ratio) for the EU and the US markets. R is the portfolio return from the previous period.

	Omega	Scenario	Model	CVaR	Scenario	Model
EU						
$\gamma = 0.1$	167.86/159.23		risk neutral/MV	−27.23		MV
$\gamma = 0.5$	169.79	Conservative	$\lambda = 3,\ \hat{r} = 0$	−27.23		MV
$\gamma = 0.9$	180.76	Conservative	$\lambda = 3,\ \hat{r} = 0$	−27.23		MV
US						
$\gamma = 0.1$	173.28	Conservative	$\lambda = 1.5,\ \hat{r} = R$	−31.16		MV
$\gamma = 0.5$	172.36	Benchmark	$\lambda = 2,\ \hat{r} = 0$	−31.16		MV
$\gamma = 0.9$	181.59	Aggressive	$\lambda = 2,\ \hat{r} = r^0$	−31.16		MV
	Sharpe	Scenario	Model	Sortino	Scenario	Model
EU						
$\gamma = 0.1$	63.16/62.74		Risk neutral/MV	98.64/97.7		Risk neutral/MV
$\gamma = 0.5$	63.16/62.74		Risk neutral/MV	98.64/97.7		Risk neutral/MV
$\gamma = 0.9$	69.82	Conservative	$\lambda = 3,\ \hat{r} = 0$	116.87	Conservative	$\lambda = 3,\ \hat{r} = r^0$
US						
$\gamma = 0.1$	68.70	Conservative	$\lambda = 1.5,\ \hat{r} = R$	103.06	Conservative	$\lambda = 1.5,\ \hat{r} = R$
$\gamma = 0.5$	67.57	Aggressive	$\lambda = 2.5,\ \hat{r} = R$	106.28	Aggressive	$\lambda = 2.5,\ \hat{r} = R$
$\gamma = 0.9$	71.71	Aggressive	$\lambda = 2,\ \hat{r} = r^0$	109.87	Aggressive	$\lambda = 2,\ \hat{r} = r^0$

performance[51] in the EU market is implied by the conservative PT investor with the largest diminishing sensitivity ($\gamma = 0.9$), the largest degree of loss aversion ($\lambda = 3$) and a zero reference return, see Table 4. On the other hand, the best risk-adjusted performance in the US market is implied by the aggressive PT investor with the largest diminishing sensitivity ($\gamma = 0.9$), $\lambda = 2$, and the risk-free reference return, see again Table 4. Though in both markets the largest risk-adjusted performance is implied by a PT investor with the largest diminishing sensitivity,

[51] What follows is equally true for the Omega measure, the Sharpe ratio and the Sortino ratio, for both the EU and the US.

Table 5
Best performing scenarios and models with respect to the mean and median for the EU and the US markets. Presented values for means and medians are annual averages in percent. For comparison, the mean–variance investor shows a mean of 6.17 (EU) and 6.95 (US), and a median of 7.32 (EU) and 6.17 (US), respectively. R is the portfolio return from the previous period.

	Mean	Scenario	Model	Median	Scenario	Model
EU						
$\gamma = 0.1$	10.30	Aggressive	$\lambda = 1.5,\ \hat{r} = R$	10.62	Aggressive	$\lambda = 1.5,\ \hat{r} = R$
$\gamma = 0.5$	8.11	Aggressive	$\lambda = 2,\quad \hat{r} = 0$	9.49	Aggressive	$\lambda = 1.5,\ \hat{r} = r^0$
$\gamma = 0.9$	8.61	Aggressive	$\lambda = 2.5,\ \hat{r} = R$	9.53	Risk-free	$\lambda = 1.5,\ \hat{r} = r^0$
US						
$\gamma = 0.1$	13.62	Conservative	$\lambda = 1.5,\ \hat{r} = R$	18.86	Aggressive	$\lambda = 1.5,\ \hat{r} = R$
$\gamma = 0.5$	11.91	Aggressive	$\lambda = 2.5,\ \hat{r} = R$	13.57	Aggressive	$\lambda = 1.5,\ \hat{r} = R$
$\gamma = 0.9$	11.61	Aggressive	$\lambda = 1.5,\ \hat{r} = r^0$	14.30	Aggressive	$\lambda = 2.5,\ \hat{r} = R$

in the EU the conservative strategy seems to beat all other strategies, while in the US the aggressive strategy provides the best risk-adjusted performance. Note that the best EU investor is more loss averse than the best US investor, and the reference return of the best EU investor is most of the time zero, while it is equal to the risk-free rate in the case of the best US investor. Note also that the best EU investor allocates on average 29%, 60% and 11% to the stock, bond and gold markets, while the best US investor allocates on average 50%, 35% and 15% to the stock, bond and gold markets. Thus, while the bond market is the main focus for the PT investor in the EU, it is the stock market for the PT investor in the US. See Tables C.3 and C.6.

The smallest volatility and smallest downside volatility are implied by the MV model in both the EU and the US. This does not come as a surprise, as the mean–variance model explicitly targets the minimization of the variance, which is closely related to these risk measures. Note that volatility and downside volatility decrease with increasing λ (when all other model parameters are fixed), which is in line with the theoretical results in Section 2.3. Note in addition that CVaR mostly increases with increasing λ (when all other model parameters are fixed).

Finally, our results suggest that for higher degrees of loss aversion ($\lambda = 2, 2.25, 2.5, 3$) PT investors mostly outperform LLA investors in terms of the mean and risk-adjusted performance measures, in both the EU and the US.

In a nutshell, the best typical PT investor (i.e., $\gamma = 0.1$, see Footnote 35) in the US market is the conservative one who targets her portfolio return to the portfolio return from the previous period, is not too much loss averse ($\lambda = 1.5$), and outperforms other benchmark investors such as MV and risk neutral. This is also the investor with the largest mean of portfolio returns, see Table 5. However, in the EU market, the best typical PT investor is outperformed by both MV and risk neutral investors.

5. Conclusion

In this paper we investigate the behavior of an S-shaped prospect theory investor. In the theoretical part we derive the analytical closed form solution for a two-asset portfolio consisting of one risk-free asset and one risky asset with returns following the Bernoulli distribution. When the risky asset return follows a general continuous distribution then the solution can be expressed in a semi-analytical way. In a comprehensive sensitivity analysis, we investigate how different aspects of the prospect theory investor's preferences contribute to her risk-taking, performance, and happiness. For both assumptions with respect to the distribution of the risky asset return we observe similar findings. Namely, the reference return, in particular its position relative to the risk-free interest rate, plays an important role: the less ambitious investor decreases her exposure to the risky asset when increasing her reference return, while the more ambitious investor increases her exposure to the risky asset when increasing her reference return. More precisely, we observe a V-shaped relationship between risk taking and the level of ambition when risk taking is positive, and an inverse V-shaped relationship when risk taking is negative. However, both less and more ambitious investors decrease their exposures to the risky asset when increasing their degrees of loss aversion. When the reference return coincides with the risk-free rate then the investor invests everything into the risk-free asset. In addition, we observe that the investor's level of happiness (or satisfaction) decreases with an increasing reference return (level of ambition). We also analyze the investor's expected gains and losses and observe different risk attitudes with respect to her level of diminishing sensitivity. Namely, for less ambitious investors with any degree of diminishing sensitivity, the expected gain decreases with an increasing reference return, while more ambitious investors with a smaller degree of diminishing sensitivity experience an increasing expected *loss* with an increasing reference return and more ambitious investors with a larger degree of diminishing sensitivity experience an increasing expected *gain* with an increasing reference return.

In the simulation part we investigate the theoretical results for a prospect theory investor with different values of the reference return, loss aversion, diminishing sensitivity and for different characteristics of the financial market. In particular, we examine whether the assumption of negative skewness of the risky asset return has major effects on the results. We find that risk taking implied by a negatively skewed distribution of the risky asset return is smaller than the risk taking implied by a symmetric distribution of the risky asset return for less ambitious investors, while risk taking implied by a negatively skewed distribution is larger than the risk taking implied by a symmetric distribution for more ambitious investors.

In the empirical part we investigate the performance of optimal asset portfolios implied by PT preferences. We study two scenarios with a constant penalty parameter, where the reference return is either equal to zero or equal to the risk-free interest rate, and two dynamic scenarios, where the penalty parameter is time-changing conditional on previous gains and losses and the reference return is either zero, the risk-free interest rate or the portfolio return of the previous period. In one of the two dynamic scenarios the PT investor becomes more loss averse after losses (conservative scenario), while in the other the PT investor becomes less loss averse after losses (aggressive scenario). We consider different levels of diminishing sensitivity and different degrees of loss aversion. The investor selects among three risky assets, a stock market index, a government bond and gold, and she operates either in the EU or in the US market. We use various performance measures, including risk-adjusted measures like the Omega measure and the Sharpe and Sortino ratios. In addition to PT portfolios, we examine optimal portfolios implied by linear loss averse, risk neutral, conditional value-at-risk and, in particular, traditional mean–variance (MV) preferences.

There are many different findings. First, PT investment leads to clearly higher means of portfolio returns, but also to much higher risk, than MV investment, for all types of PT investors and for both the EU and US markets. We actually find that, in the US market, returns of PT portfolios are significantly larger than returns of MV portfolios for almost all types of PT investors, while this is hardly the case in the EU. If we consider risk-adjusted performance, PT investment (mostly) outperforms MV investment in the US, while MV investment outperforms PT investment in the EU. If we consider the variance, downside volatility or the conditional value-at-risk, MV investment outperforms PT investment. These differences between the performance of PT and MV portfolios are due to the different preferences of PT and MV investors. While PT investors maximize the portfolio return under certain conditions for the deviations from a given reference return, MV investors minimize the portfolio variance. In addition we find that the performance results of optimal portfolios in the conservative and aggressive scenarios are very similar, for a given reference return of either zero or the risk-free rate, in both the EU and the US markets. The situation is different when the reference return is the portfolio return of the previous period. In this case the differences between conservative and aggressive scenarios are more pronounced, which is mainly due to the rather large difference between the portfolio return and the reference return. Rather unexpectedly, the risk neutral investor performs quite well empirically. In the EU market she mostly performs better than any PT investor in terms of the mean and risk-adjusted measures, while in the US the PT investor usually performs best. Linear loss averse investors generally perform worse than PT investors in terms of the mean and risk-adjusted measures, for larger degrees of loss aversion, in both the EU and the US.

Note that although our empirical analysis covers many aspects of the prospect theory asset allocation problem, a more thorough analysis is required to shed light on all details. For example, as we focus on loss aversion, the dynamic investors in our study update mainly their loss aversion, and only after prior losses. They do not update the reference return and do not apply dynamic updates of both the reference return and loss aversion after both prior gains and losses. These effects will be explored in future research.

The single-period model allows us to focus on a myopic loss averse investor, who prefers immediate gratification over her investment choices rather than considering long-term consequences. Considering one risky and one risk-free asset allows us to analyze certain portfolio properties in a straightforward way, when assuming a general continuous distribution of the risky asset. In the future it would be interesting to see how these properties change when the investor chooses among more risky assets, how these properties are affected by certain, in particular non-symmetric, joint distribution of the risky assets returns, and under what conditions a global (unique) solution exists. In addition, we would like to explore the PT investment problem in a multi-period model in order to see whether and in which direction the main results are affected.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Proofs

Proof of Proposition 2.1. To calculate the expected value of $v(R(x))$ with respect to the Bernoulli distribution, we evaluate four options following from (4): (i) $R_b(x) > \hat{r}$ and $R_g(x) > \hat{r}$, (ii) $R_b(x) \le \hat{r}$ and $R_g(x) > \hat{r}$, (iii) $R_b(x) > \hat{r}$ and $R_g(x) \le \hat{r}$, and (iv) $R_b(x) \le \hat{r}$ and $R_g(x) \le \hat{r}$.

Case (i) can occur only when $x \in \left[\frac{\hat{r}-r^0}{r_g-r^0}, \frac{r^0-\hat{r}}{r^0-r_b}\right]$, case (ii) occurs when $x \ge \max\left\{\frac{r^0-\hat{r}}{r^0-r_b}, \frac{\hat{r}-r^0}{r_g-r^0}\right\}$, case (iii) occurs when $x \le \min\left\{\frac{r^0-\hat{r}}{r^0-r_b}, \frac{\hat{r}-r^0}{r_g-r^0}\right\}$, and case (iv) occurs when $x \in \left[\frac{r^0-\hat{r}}{r^0-r_b}, \frac{\hat{r}-r^0}{r_g-r^0}\right]$. Note that for $\hat{r} < r^0$, which is the assumption of this proposition, is case (iv) infeasible. Based on these we thus solve the following three maximization problems

$$\left.\begin{array}{rl} \max: & \mathbb{E}(v(R(x))) = \frac{1}{1-\gamma}\left[(1-p)(r^0-\hat{r}-(r^0-r_b)x)^{1-\gamma}\right. \\ & \left.+\, p(r^0-\hat{r}+(r_g-r^0)x)^{1-\gamma}\right] \\ \text{such that}: & -\frac{r^0-\hat{r}}{r_g-r^0} \le x \le \frac{r^0-\hat{r}}{r^0-r_b} \end{array}\right\}\text{(i)}$$

$$\left.\begin{array}{rl} \max: & \mathbb{E}(v(R(x))) = \frac{1}{1-\gamma}\left[-\lambda(1-p)(\hat{r}-r^0+(r^0-r_b)x)^{1-\gamma}\right. \\ & \left.+\, p(r^0-\hat{r}+(r_g-r^0)x)^{1-\gamma}\right] \\ \text{such that}: & x \ge \frac{r^0-\hat{r}}{r^0-r_b} \end{array}\right\}\text{(ii)}$$

$$\left.\begin{array}{rl} \max: & \mathbb{E}(v(R(x))) = \frac{1}{1-\gamma}\left[(1-p)(r^0-\hat{r}-(r^0-r_b)x)^{1-\gamma}\right. \\ & \left.-\, \lambda p(\hat{r}-r^0-(r_g-r^0)x)^{1-\gamma}\right] \\ \text{such that}: & x \le -\frac{r^0-\hat{r}}{r_g-r^0} \end{array}\right\}\text{(iii)}$$

The idea of the proof is to show now that (i) is a concave problem with an unique maximum and as the objective function of (iii) is increasing at its domain and the objective function of (ii) is decreasing at its domain, and as $\mathbb{E}(v(R(x)))$ is continuous function, then the maximum of problem (i) coincides with the maximum of (3).

By differentiating the objective function of problem (i) we obtain

$$\frac{d}{dx}\mathbb{E}(v(R(x))) = -(1-p)[r^0-\hat{r}-(r^0-r_b)x]^{-\gamma}(r^0-r_b)+p[r^0-\hat{r}+(r_g-r^0)x]^{-\gamma}(r_g-r^0)$$

and

$$\begin{aligned}\frac{d^2}{dx^2}\mathbb{E}(v(R(x))) &= -\gamma(1-p)[r^0-\hat{r}-(r^0-r_b)x]^{-1-\gamma}(r^0-r_b)^2 \\ &\quad -\gamma p[r^0-\hat{r}+(r_g-r^0)x]^{-1-\gamma}(r_g-r^0)^2 < 0\end{aligned}$$

which implies that (i) is a concave programming problem and thus the maximum satisfies the following first order conditions

$$\frac{d}{dx}\mathbb{E}(v(R(x))) = -(1-p)[r^0-\hat{r}-(r^0-r_b)x]^{-\gamma}(r^0-r_b)+p[r^0-\hat{r}+(r_g-r^0)x]^{-\gamma}(r_g-r^0) = 0$$

Thus,

$$(p(r_g-r^0))^{1/\gamma}\left[r^0-\hat{r}-(r^0-r_b)x\right] = ((1-p)(r^0-r_b))^{1/\gamma}\left[r^0-\hat{r}+(r_g-r^0)x\right]$$

which implies

$$\begin{aligned} x &= \frac{(p(r_g-r^0))^{1/\gamma}-((1-p)(r^0-r_b))^{1/\gamma}}{\left[(p(r_g-r^0))^{1/\gamma}(r^0-r_b)+((1-p)(r^0-r_b))^{1/\gamma}(r_g-r^0)\right]}(r^0-\hat{r}) \\ &= \frac{\left(1-K_0^{1/\gamma}\right)(r^0-\hat{r})}{r^0-r_b+K_0^{1/\gamma}(r_g-r^0)} \end{aligned} \tag{A.1}$$

and this coincides with (6). Note that the conditions of the proposition imply that $0 < K_0 < 1$ (following from $r_b < r^0 < r_g$ and $\mathbb{E}(r) > r^0$) and thus $x^* > 0$.

Problem (iii) is increasing at its domain if

$$\frac{d}{dx}\mathbb{E}(v(R(x))) = -(1-p)[r^0-\hat{r}-(r^0-r_b)x]^{-\gamma}(r^0-r_b)+\lambda p[\hat{r}-r^0-(r_g-r^0)x]^{-\gamma}(r_g-r^0) > 0$$

which is guaranteed if

$$\lambda > \frac{(1-p)\left[\hat{r}-r^0-(r_g-r^0)x\right]^{\gamma}(r^0-r_b)}{p\left[r^0-\hat{r}-(r^0-r_b)x\right]^{\gamma}(r_g-r^0)} = \left(\frac{\frac{\hat{r}-r^0}{r_g-r^0}-x}{\frac{r^0-\hat{r}}{r^0-r_b}-x}\right)^{\gamma} K_\gamma \tag{A.2}$$

It follows from the assumptions of the theorem that $\lambda > K_\gamma$ and as $\frac{\frac{\hat{r}-r^0}{r_g-r^0}-x}{\frac{r^0-\hat{r}}{r^0-r_b}-x} < 1$ then (A.2) is satisfied and thus the objective function of (iii) is increasing.

Problem (ii) is decreasing at its domain if

$$\frac{d}{dx}\mathbb{E}(v(R(x))) = -\lambda(1-p)[\hat{r}-r^0+(r^0-r_b)x]^{-\gamma}(r^0-r_b)+p[r^0-\hat{r}+(r_g-r^0)x]^{-\gamma}(r_g-r^0) < 0$$

which is guaranteed if

$$\lambda > \frac{p(\hat{r} - r^0 + (r^0 - r_b)x)^\gamma (r_g - r^0)}{(1-p)(r^0 - \hat{r} + (r_g - r^0)x)^\gamma (r^0 - r_b)} = \left(\frac{\frac{\hat{r}-r^0}{r^0-r_b} + x}{\frac{r^0-\hat{r}}{r_g-r^0} + x} \right)^\gamma \frac{1}{K_\gamma} \tag{A.3}$$

It follows from assumptions of the theorem that $\lambda > \frac{1}{K_\gamma}$ and as $\frac{\frac{\hat{r}-r^0}{r^0-r_b}+x}{\frac{r^0-\hat{r}}{r_g-r^0}+x} < 1$ then (A.3) is satisfied and thus the objective function of (ii) is decreasing. This finishes the proof. □

Proof of Proposition 2.2. Note that solving (3) boils down to solving problems (ii) and (iii) for $r^0 = \hat{r}$ as in cases (i) and (iv) is $x = 0$ the optimal solution. As the objective function of (iii) is increasing for $x \le 0$ and the objective function of (ii) is decreasing for $x \ge 0$ then this implies that zero is the solution of (3). This finishes the proof. □

Proof of Proposition 2.3. Based on (4) we consider the following four cases: (i) $R_b(x) > \hat{r}$ and $R_g(x) > \hat{r}$, (ii) $R_b(x) \le \hat{r}$ and $R_g(x) > \hat{r}$, (iii) $R_b(x) > \hat{r}$ and $R_g(x) \le \hat{r}$, and (iv) $R_b(x) \le \hat{r}$ and $R_g(x) \le \hat{r}$. Case (i) can occur only when $x \in \left[\frac{\hat{r}-r^0}{r_g-r^0}, -\frac{\hat{r}-r^0}{r^0-r_b}\right]$, case (ii) occurs when $x \ge \frac{\hat{r}-r^0}{r_g-r^0}$, case (iii) occurs when $x \le -\frac{\hat{r}-r^0}{r^0-r_b}$, and case (iv) occurs when $x \in \left[-\frac{\hat{r}-r^0}{r^0-r_b}, \frac{\hat{r}-r^0}{r_g-r^0}\right]$. Note that for $\hat{r} > r^0$, which is the assumption of this proposition, is case (i) infeasible. Based on these we thus solve the following three maximization problems

$$\left.\begin{array}{ll} \max: & \mathbb{E}(v(R(x))) = \frac{1}{1-\gamma}\left[-\lambda(1-p)(\hat{r} - r^0 + (r^0 - r_b)x)^{1-\gamma} \right. \\ & \left. + \, p(r^0 - \hat{r} + (r_g - r^0)x)^{1-\gamma}\right] \\ \text{such that}: & x \ge \frac{\hat{r}-r^0}{r_g-r^0} \end{array}\right\} \text{(ii)}$$

$$\left.\begin{array}{ll} \max: & \mathbb{E}(v(R(x))) = \frac{1}{1-\gamma}\left[(1-p)(r^0 - \hat{r} - (r^0 - r_b)x)^{1-\gamma} \right. \\ & \left. - \, \lambda p(\hat{r} - r^0 - (r_g - r^0)x)^{1-\gamma}\right] \\ \text{such that}: & x \le \frac{r^0-\hat{r}}{r^0-r_b} \end{array}\right\} \text{(iii)}$$

$$\left.\begin{array}{ll} \max: & \mathbb{E}(v(R(x))) = \frac{-\lambda}{1-\gamma}\left[(1-p)(\hat{r} - r^0 + (r^0 - r_b)x)^{1-\gamma} \right. \\ & \left. + \, p(\hat{r} - r^0 - (r_g - r^0)x)^{1-\gamma}\right] \\ \text{such that}: & -\frac{\hat{r}-r^0}{r^0-r_b} \le x \le \frac{\hat{r}-r^0}{r_g-r^0} \end{array}\right\} \text{(iv)}$$

By differentiating (iv) we obtain

$$\frac{d}{dx}\mathbb{E}(v(R(x))) = -\lambda\left[(1-p)[\hat{r} - r^0 + (r^0 - r_b)x]^{-\gamma}(r^0 - r_b) \right. \\ \left. - \, p[\hat{r} - r^0 - (r_g - r^0)x]^{-\gamma}(r_g - r^0)\right]$$

and

$$\frac{1}{\gamma}\frac{d^2}{dx^2}\mathbb{E}(v(R(x))) = \lambda(1-p)[\hat{r} - r^0 + (r^0 - r_b)x]^{-1-\gamma}(r^0 - r_b)^2 \\ + \lambda p[\hat{r} - r^0 - (r_g - r^0)x]^{-1-\gamma}(r_g - r^0)^2 > 0 \tag{A.4}$$

which implies that (iv) is a convex programming problem and thus the maximum is reached in one of the corner points, i.e., $-\frac{\hat{r}-r^0}{r^0-r_b}$ or $\frac{\hat{r}-r^0}{r_g-r^0}$.

The first order conditions (FOC) for (iii) are

$$\frac{d}{dx}\mathbb{E}(v(R(x))) = -(1-p)[r^0-\hat{r}-(r^0-r_b)x]^{-\gamma}(r^0-r_b) + \lambda p[\hat{r}-r^0-(r_g-r^0)x]^{-\gamma}(r_g-r^0) = 0$$

It can be shown that x_n^*, given by (10), satisfies the FOC and that (iii) is concave, i.e.,

$$\frac{1}{\gamma}\frac{d^2}{dx^2}\mathbb{E}(v(R(x))) = \\ -(1-p)[r^0 - \hat{r} - (r^0 - r_b)x]^{-1-\gamma}(r^0 - r_b)^2 \\ + \lambda p[\hat{r} - r^0 - (r_g - r^0)x]^{-1-\gamma}(r_g - r^0)^2 \\ < 0$$

for $\lambda > K_\gamma$ and $x > x_L \equiv \frac{\left[1+\left(\frac{\lambda}{K_{-1}}\right)^{1/(1+\gamma)}\right](\hat{r}-r^0)}{r_g-r^0-\left(\frac{\lambda}{K_{-1}}\right)^{1/(1+\gamma)}(r^0-r_b)} = -\frac{\hat{r}-r^0}{r^0-r_b} \times \frac{\left(\lambda^{\frac{1}{1+\gamma}} + K_{-1}^{\frac{1}{1+\gamma}}\right)}{\left(\lambda^{\frac{1}{1+\gamma}} - K_\gamma^{\frac{1}{1+\gamma}}\right)}$ and that $x_L < x_n^* < \frac{r^0-\hat{r}}{r^0-r_b}$ for $\lambda > K_\gamma$. Note in addition that (iii) is a convex problem for $x < x_L$ and that $\lim_{x\to-\infty}\mathbb{E}(v(R(x))) = -\infty$. This shows that x_n^* is the only maximum of (iii).

The first order conditions (FOC) for (ii) are

$$\frac{d}{dx}\mathbb{E}(v(R(x))) = -\lambda(1-p)[\hat{r} - r^0 + (r^0 - r_b)x]^{-\gamma}(r^0 - r_b) \\ + p[r^0 - \hat{r} + (r_g - r^0)x]^{-\gamma}(r_g - r^0) = 0$$

It can be shown that x_p^*, given by (9), satisfies the FOC and that (ii) is concave, i.e.,

$$\frac{1}{\gamma}\frac{d^2}{dx^2}\mathbb{E}(v(R(x))) = \\ \lambda(1-p)[\hat{r} - r^0 + (r^0 - r_b)x]^{-1-\gamma}(r^0 - r_b)^2 \\ -p[r^0 - \hat{r} + (r_g - r^0)x]^{-1-\gamma}(r_g - r^0)^2 \\ < 0$$

for $\lambda > 1/K_\gamma$ and $x < x_U \equiv \frac{\left[1+\left(\frac{1}{\lambda K_{-1}}\right)^{1/(1+\gamma)}\right](\hat{r}-r^0)}{r_g-r^0-\left(\frac{1}{\lambda K_{-1}}\right)^{1/(1+\gamma)}(r^0-r_b)} = \frac{\hat{r}-r^0}{r_g-r^0} \times \frac{\left(\lambda^{\frac{1}{1+\gamma}} + \left(\frac{1}{K_{-1}}\right)^{\frac{1}{1+\gamma}}\right)}{\left(\lambda^{\frac{1}{1+\gamma}} - \left(\frac{1}{K_\gamma}\right)^{\frac{1}{1+\gamma}}\right)}$ and that $\frac{\hat{r}-r^0}{r_g-r^0} < x_p^* < x_U$ for $\lambda > 1/K_\gamma$. Note in addition that (ii) is a convex problem for $x > x_U$ and that $\lim_{x\to-\infty}\mathbb{E}(v(R(x))) = -\infty$. This shows that x_p^* is the only maximum of (ii) and thus the global maximum of (3) is achieved at $x^* = argmax\{\mathbb{E}(v(R(x_p^*))), \mathbb{E}(v(R(x_n^*)))\}$. Next we analyze the case when $\mathbb{E}(v(R(x_p^*))) > \mathbb{E}(v(R(x_n^*)))$. It can be shown that

$$\mathbb{E}(v(R(x_p^*))) = -\frac{1-p}{1-\gamma}\left[\frac{(r_g - r_b)(\hat{r} - r^0)}{r_g - r^0}\right]^{1-\gamma}\left[\lambda^{1/\gamma} - \left(\frac{1}{K_\gamma}\right)^{1/\gamma}\right]^\gamma \\ < 0 \quad \text{as} \quad \lambda > \frac{1}{K_\gamma}$$

and

$$\mathbb{E}(v(R(x_n^*))) = -\frac{1-p}{1-\gamma}\left[\frac{(r_g - r_b)(\hat{r} - r^0)}{r_g - r^0}\right]^{1-\gamma}\left[\left(\frac{\lambda}{K_\gamma}\right)^{1/\gamma} - 1\right]^\gamma \\ < 0 \quad \text{as} \quad \lambda > K_\gamma$$

and thus showing $\mathbb{E}(v(R(x_p^*))) > \mathbb{E}(v(R(x_n^*)))$ boils down to proving that

$$\lambda^{1/\gamma} - \left(\frac{1}{K_\gamma}\right)^{1/\gamma} < \left(\frac{\lambda}{K_\gamma}\right)^{1/\gamma} - 1$$

or

$$\lambda^{1/\gamma}\left(1 - K_\gamma^{1/\gamma}\right) > K_\gamma^{1/\gamma} - 1$$

where the last inequality is implied by $K_\gamma < 1$ which follows from $p > \bar{p}$. The other cases follow directly. This concludes the proof. □

Proof of Proposition 2.4. Note that the derivative of $\mathbb{E}(v(R(x)))$ with respect to x when $\hat{r} = r^0$ is

$$\frac{d}{dx}\mathbb{E}(v(R(x))) = \\ \left\{\begin{array}{ll} (-x)^{-\gamma}\left[\lambda\int_{r^0}^{+\infty}(r - r^0)^{1-\gamma}f(r)dr - \int_{-\infty}^{r^0}(r^0 - r)^{1-\gamma}f(r)dr\right], & x < 0 \\ x^{-\gamma}\left[\int_{r^0}^{+\infty}(r - r^0)^{1-\gamma}f(r)dr - \lambda\int_{-\infty}^{r^0}(r^0 - r)^{1-\gamma}f(r)dr\right], & x > 0 \end{array}\right\} \tag{A.5}$$

and thus $\frac{d}{dx}\mathbb{E}(v(R(x))) > 0$ for $x < 0$ and $\lambda > K_\gamma$, and $\frac{d}{dx}\mathbb{E}(v(R(x))) < 0$ for $x > 0$ and $\lambda > 1/K_\gamma$. $\mathbb{E}(v(R(x)))$ is continuous for $x = 0$ as

$$\lim_{x\to 0^+}\mathbb{E}(v(R(x))) = \lim_{x\to 0^-}\mathbb{E}(v(R(x))) = 0 = \mathbb{E}(v(R(0)))$$

which follows from (18) and the assumption $\hat{r} = r^0$. Then, based on this and (A.5) is the maximum of (19) reached at zero when $\lambda > \max\left\{K_\gamma, \frac{1}{K_\gamma}\right\}$. This finishes the proof. □

Proof of Proposition 2.5. The expected prospect theory utility function (18) is continuous as

$$\lim_{x\to 0^+}\mathbb{E}(v(R(x))) = \lim_{x\to 0^-}\mathbb{E}(v(R(x))) = \frac{(r^0-\hat{r})^{1-\gamma}}{1-\gamma} = \mathbb{E}(v(R(0))) \tag{A.6}$$

which follows from (18) and the assumption $\hat{r} < r^0$. Note that $\mathbb{E}(v(R(x))) < \mathbb{E}(v(R(0)))$ for $x < 0$ if

$$\begin{aligned}(1-\gamma)\mathbb{E}(v(R(x))) &= (-x)^{1-\gamma}\int_{-\infty}^{z(x)}[z(x)-r]^{1-\gamma}f(r)dr\\ &\quad -(-x)^{1-\gamma}\lambda\int_{z(x)}^{+\infty}[r-z(x)]^{1-\gamma}f(r)dr\\ &< (r^0-\hat{r})^{1-\gamma}\end{aligned}$$

where $z(x) = \frac{r^0-\hat{r}}{-x} + r^0$. Thus

$$\lambda > \frac{\int_{-\infty}^{z(x)}[z(x)-r]^{1-\gamma}f(r)dr - \left[z(x)-r^0\right]^{1-\gamma}}{\int_{z(x)}^{+\infty}[r-z(x)]^{1-\gamma}f(r)dr} = K_\gamma z(x) \tag{A.7}$$

as $\int_{z(x)}^{+\infty}[r-z(x)]^{1-\gamma}f(r)dr > 0$.[52] The assumption of the theorem, namely $\lambda > \hat{K}_\gamma$ and the definition of $\hat{K}_\gamma$, see (22), imply that (A.7) holds and thus $\mathbb{E}(v(R(x))) < \mathbb{E}(v(R(0)))$ for $x < 0$.

Note that based on Leibniz integral rule the derivative of $\mathbb{E}(v(R(x)))$ with respect to x is

$$\frac{d}{dx}\mathbb{E}(v(R(x))) = \left\{\begin{array}{ll}\frac{1}{(-x)^\gamma}\int_{-\infty}^{z(x)}\frac{r-r^0}{[z(x)-r]^\gamma}f(r)dr + \frac{\lambda}{x^\gamma}\int_{z(x)}^{+\infty}\frac{r-r^0}{[r-z(x)]^\gamma}f(r)dr, & x<0\\ \frac{\mathbb{E}(r)-r^0}{(r^0-\hat{r})^\gamma} > 0, \quad x=0, \quad \hat{r}<r^0 & \\ \lambda\frac{\mathbb{E}(r)-r^0}{(\hat{r}-r^0)^\gamma} > 0, \quad x=0, \quad \hat{r}>r^0 & \\ \frac{\lambda}{x^\gamma}\int_{-\infty}^{z(x)}\frac{r-r^0}{[z(x)-r]^\gamma}f(r)dr + \frac{1}{x^\gamma}\int_{z(x)}^{+\infty}\frac{r-r^0}{[r-z(x)]^\gamma}f(r)dr, & x>0\end{array}\right\} \tag{A.8}$$

where $z(x) = \frac{r^0-\hat{r}}{-x} + r^0$. This follows from $\mathbb{E}(r) > r^0$ and

$$\begin{aligned}\lim_{x\to 0^+}\frac{d}{dx}\mathbb{E}(v(R(x))) = \lim_{x\to 0^-}\frac{d}{dx}\mathbb{E}(v(R(x))) &= \int_{-\infty}^{+\infty}\frac{r-r^0}{(r^0-\hat{r})^\gamma}f(r)dr\\ &= \frac{\mathbb{E}(r)-r^0}{(r^0-\hat{r})^\gamma} > 0\end{aligned} \tag{A.9}$$

for $\hat{r} < r^0$ and from

$$\begin{aligned}\lim_{x\to 0^+}\frac{d}{dx}\mathbb{E}(v(R(x))) &= \lim_{x\to 0^-}\frac{d}{dx}\mathbb{E}(v(R(x)))\\ &= \lambda\int_{-\infty}^{+\infty}\frac{r-r^0}{(\hat{r}-r^0)^\gamma}f(r)dr = \lambda\frac{\mathbb{E}(r)-r^0}{(\hat{r}-r^0)^\gamma} > 0\end{aligned} \tag{A.10}$$

for $\hat{r} > r^0$. Thus, for any $\hat{r} \neq r^0$ is $\mathbb{E}(v(R(x)))$ increasing in zero. Note in addition that based on (18) we obtain the following

$$\begin{aligned}&\lim_{x\to+\infty}\mathbb{E}(v(R(x)))\\ &= +\infty\times\left[-\lambda\int_{-\infty}^{r^0}\left(r^0-r\right)^{1-\gamma}f(r)dr + \int_{r^0}^{+\infty}\left(r-r^0\right)^{1-\gamma}f(r)dr\right] = -\infty\end{aligned} \tag{A.11}$$

as $\lambda > 1/K_\gamma$, i.e., the expression in the brackets is negative.[53]

[52] Let $\phi(c) = \int_c^{+\infty}(r-c)^{1-\gamma}f(r)dr$ and let us assume that there exists $c_0 \in \mathbb{R}$ such that $\phi(c_0) = 0$. Then $f(r) = 0$ for any $r \geq c_0$ and $F(c_0) = \int_{-\infty}^{c_0}f(r)dr = \int_{-\infty}^{+\infty}f(r)dr = 1$, which is a contradiction to the assumption of the distribution of the risky asset's return. Thus, $\phi(c) > 0$ for any $c \in \mathbb{R}$.

[53] Note that this property holds also for $\hat{r} > r^0$.

Thus, based on the fact that $\mathbb{E}(v(R(x))) < \mathbb{E}(v(R(0)))$ for $x < 0$, $\mathbb{E}(v(R(x)))$ being continuous (also at zero, see (A.6)), increasing at zero, see (A.8), for $x = 0$, and achieving $-\infty$ in infinity, see (A.11), it follows then that the solution x^* of (19) is positive and such that the first order conditions are satisfied, i.e., $\frac{d}{dx}\mathbb{E}(v(R(x^*))) = 0$ for $x^* > 0$, which coincides with (21). This finishes the proof. □

Lemma A.1. *Let $\mathbb{E}(r) > r^0$. Then the function $K_\gamma : [r^0, +\infty) \to \mathbb{R}$*

$$K_\gamma(c) = \frac{\int_{-\infty}^{c}(c-r)^{1-\gamma}f(r)dr - (c-r^0)^{1-\gamma}}{\int_c^{\infty}(r-c)^{1-\gamma}f(r)dr}$$

is bounded from above; i.e., there exists a constant $M \geq 0$ such that $K_\gamma(c) \leq M$ for any $c \in [r^0, +\infty)$.

Proof. Note that $c \in \mathbb{R}$ and that the denominator of $K_\gamma(c)$, namely $\int_c^{+\infty}(r-c)^{1-\gamma}f(r)dr$, is strictly positive and that both integrals $\int_{-\infty}^{c}(c-r)^{1-\gamma}f(r)dr$ and $\int_c^{+\infty}(r-c)^{1-\gamma}f(r)dr$ are nonnegative and finite for any $c \in \mathbb{R}$. The latter is a consequence of the assumption $\mathbb{E}(|r|) = \int_{-\infty}^{+\infty}|r|f(r)dr < +\infty$ and the inequality $|r-c|^{1-\gamma} \leq C_\gamma(1+|c|+|r|)$ for any $r, c \in \mathbb{R}$ where $C_\gamma > 0$ is a constant.

Let $c \geq r^0$ be fixed. The function $H(r) \equiv (c-r)^{1-\gamma}$ is concave on the set $(-\infty, c)$. We remind ourselves Jensen's inequality

$$\int_{-\infty}^{c}H(r)g(r)dr \leq H\left(\int_{-\infty}^{c}rg(r)dr\right)$$

where $g(r) = f(r)/F(c) \geq 0$ is such that $\int_{-\infty}^{c}g(r)dr = (1/F(c))\int_{-\infty}^{c}f(r)dr = 1$. Therefore

$$\begin{aligned}\int_{-\infty}^{c}(c-r)^{1-\gamma}f(r)dr &\leq F(c)\left(c-\int_{-\infty}^{c}rg(r)dr\right)^{1-\gamma}\\ &= (F(c))^\gamma\left(\int_{-\infty}^{c}(c-r)f(r)dr\right)^{1-\gamma}\\ &\leq \left(\int_{-\infty}^{c}(c-r)f(r)dr\right)^{1-\gamma}\end{aligned} \tag{A.12}$$

For $c \geq 0$ we have

$$\int_{-\infty}^{c}(c-r)f(r)dr-(c-r^0) = c(F(c)-1)+r^0-\int_{-\infty}^{c}rf(r)dr \leq r^0-\int_{-\infty}^{c}rf(r)dr \tag{A.13}$$

Since $\mathbb{E}(r) > r^0$, there exists $c_* \geq \max\{0, r^0\}$ such that $\int_{-\infty}^{c}rf(r)dr > r^0$ for any $c \geq c_*$. This and (A.13) imply that for any $c \geq c_*$

$$\int_{-\infty}^{c}(c-r)f(r)dr < c-r^0$$

which gives, together with (A.12), the following

$$\int_{-\infty}^{c}(c-r)^{1-\gamma}f(r)dr-(c-r^0)^{1-\gamma} < 0 \quad \text{for any } c \geq c_*$$

Since the denominator of $K_\gamma(c)$ is strictly positive the function $K_\gamma(c)$ is continuous on a compact interval $[r^0, c_*]$. Hence it attains its maximum $M \geq 0$ on $[r^0, c_*]$. As $K_\gamma(c) < 0$ for $c \geq c_*$ the lemma follows. □

Proof of Proposition 2.6. The following holds based on (18) and assumption $\lambda > K_\gamma$

$$\begin{aligned}&\lim_{x\to-\infty}\mathbb{E}(v(R(x)))\\ &= +\infty\times\left[\int_{-\infty}^{r^0}\left(r^0-r\right)^{1-\gamma}f(r)dr - \lambda\int_{r^0}^{+\infty}\left(r-r^0\right)^{1-\gamma}f(r)dr\right] = -\infty\end{aligned} \tag{A.14}$$

The same holds also when x reaches $+\infty$, see (A.11) and assumption $\lambda > 1/K_\gamma$. Then, it follows based on this, the continuity of $\mathbb{E}(v(R(x)))$ and the fact that $\mathbb{E}(v(R(x)))$ is increasing at zero, see (A.9) and (A.10), that there is at least one local maximum x^* of problem (19) such that $x^* > 0$ and (21) is satisfied. In addition, continuity of $\mathbb{E}(v(R(x)))$ and

(A.14) imply that if there is any local maxima of problem (19), x^*, such that $x^* < 0$, then the first order conditions hold and (23) is satisfied. Finally, continuity of $\mathbb{E}(v(R(x)))$, (A.11) and (A.14) imply that any global maxima is finite. This concludes the proof. □

Proof of Proposition 2.7. The proof is based on implicit function differentiation and Eq. (21). Let $x^* > 0$ be a solution of

$$\frac{d}{dx}\mathbb{E}(v(R(x))) = 0$$

and let it be fixed for all the following analysis. Thus, the first order conditions are satisfied and

$$G(\lambda,\hat{r},x) \equiv \lambda \int_{-\infty}^{z(x)} \frac{r-r^0}{[z(x)-r]^\gamma} f(r)dr + \int_{z(x)}^{+\infty} \frac{r-r^0}{[r-z(x)]^\gamma} f(r)dr = 0 \quad \text{(A.15)}$$

where $z(x) = r^0 + \frac{\hat{r}-r^0}{x}$. Then

$$\frac{dx}{d\lambda} = -\frac{\frac{dG}{d\lambda}}{\frac{dG}{dx}} \quad \text{and} \quad \frac{dx}{d\hat{r}} = -\frac{\frac{dG}{d\hat{r}}}{\frac{dG}{dx}} \quad \text{(A.16)}$$

where $\hat{r}$ is fixed in the first case and λ is fixed in the second case. Note that $\frac{G(\lambda,\hat{r},x)}{x^\gamma} = \frac{d}{dx}\mathbb{E}(v(R(x)))$ and as x^* is the point of local maximum then $\frac{d^2\mathbb{E}(v(R(x)))}{dx^2} < 0$ for $x = x^*$ which implies then that $\frac{dG(\lambda,\hat{r},x)}{dx} < 0$. Namely

$$\begin{aligned}\frac{d^2\mathbb{E}(v(R(x)))}{dx^2} &= \frac{d}{dx}(x^{-\gamma}G(\lambda,\hat{r},x)) = -x^{-1-\gamma}\gamma G(\lambda,\hat{r},x) + x^{-\gamma}\frac{dG(\lambda,\hat{r},x)}{dx}\\ &= -\frac{\gamma}{x}\frac{d}{dx}\mathbb{E}(v(R(x))) + x^{-\gamma}\frac{dG(\lambda,\hat{r},x)}{dx} = x^{-\gamma}\frac{dG(\lambda,\hat{r},x)}{dx} < 0\end{aligned}$$

Thus, based on this and (A.16), the sign of $\frac{dx}{d\lambda}$ coincides with the sign of $\frac{dG(\lambda,\hat{r},x)}{d\lambda}$. (A.15) implies that

$$\frac{dG(\lambda,\hat{r},x)}{d\lambda} = \int_{-\infty}^{z(x)} \frac{r-r^0}{[z(x)-r]^\gamma} f(r)dr \quad \text{(A.17)}$$

For $\hat{r} < r^0$ and $x > 0$ is $z(x) < r^0$ and thus based on (A.17) is $\frac{dG(\lambda,\hat{r},x)}{d\lambda} < 0$. On the other hand, for $\hat{r} > r^0$ and $x > 0$ is $z(x) > r^0$ and thus $\int_{z(x)}^{+\infty} \frac{r-r^0}{[r-z(x)]^\gamma} f(r)dr > 0$. This and Eq. (A.15) imply that $\lambda \int_{-\infty}^{z(x)} \frac{r-r^0}{[z(x)-r]^\gamma} f(r)dr < 0$ and thus $\frac{dG(\lambda,\hat{r},x)}{d\lambda} < 0$. This and (A.16) imply that $\frac{dx}{d\lambda}|_{x=x^*} < 0$.

Note that

$$\begin{aligned}\int \frac{r-r^0}{[r-z(x)]^\gamma}dr &= \frac{[r-z(x)]^{2-\gamma}}{2-\gamma} + \frac{[r-z(x)]^{1-\gamma}[z(x)-r^0]}{1-\gamma}\\ \int \frac{r^0-r}{[z(x)-r]^\gamma}dr &= -\frac{[z(x)-r]^{2-\gamma}}{2-\gamma} + \frac{[z(x)-r]^{1-\gamma}[z(x)-r^0]}{1-\gamma}\end{aligned}$$

Using this and condition (24), when applying integration by parts on (A.15), gives

$$\begin{aligned}G(\lambda,\hat{r},x) &= -\int_{z(x)}^{+\infty}\left(\frac{[r-z(x)]^{2-\gamma}}{2-\gamma} + \frac{[r-z(x)]^{1-\gamma}[z(x)-r^0]}{1-\gamma}\right)\frac{df(r)}{dr}dr\\ &\quad + \lambda\int_{-\infty}^{z(x)}\left(-\frac{[z(x)-r]^{2-\gamma}}{2-\gamma} + \frac{[z(x)-r]^{1-\gamma}[z(x)-r^0]}{1-\gamma}\right)\frac{df(r)}{dr}dr\\ &= 0 \end{aligned} \quad \text{(A.18)}$$

As $\frac{dz(x)}{dx} = -\frac{\hat{r}-r^0}{(x^*)^2}$ and $\frac{dz(x)}{d\hat{r}} = \frac{1}{x^*}$, Equation $G(\lambda,\hat{r},x) = 0$ given by (A.18) implies

$$\left.\begin{aligned}\frac{dG}{dx}|_{x=x^*} &= \frac{dz(x)}{dx}\Big[\lambda\int_{-\infty}^{z(x)}\left(\tfrac{\gamma}{1-\gamma}[z(x)-r]^{1-\gamma} + \tfrac{z(x)-r^0}{[z(x)-r]^\gamma}\right)\tfrac{df(r)}{dr}dr\\ &\quad -\int_{z(x)}^{+\infty}\left(\tfrac{\gamma}{1-\gamma}[r-z(x)]^{1-\gamma} - \tfrac{z(x)-r^0}{[r-z(x)]^\gamma}\right)\tfrac{df(r)}{dr}dr\Big]\\ \frac{dG}{d\hat{r}}|_{x=x^*} &= \frac{dz(x)}{d\hat{r}}\Big[\lambda\int_{-\infty}^{z(x)}\left(\tfrac{\gamma}{1-\gamma}[z(x)-r]^{1-\gamma} + \tfrac{z(x)-r^0}{[z(x)-r]^\gamma}\right)\tfrac{df(r)}{dr}dr\\ &\quad -\int_{z(x)}^{+\infty}\left(\tfrac{\gamma}{1-\gamma}[r-z(x)]^{1-\gamma} - \tfrac{z(x)-r^0}{[r-z(x)]^\gamma}\right)\tfrac{df(r)}{dr}dr\Big]\end{aligned}\right\} \quad \text{(A.19)}$$

This and (A.16) imply that $\frac{dx}{d\hat{r}} = -\frac{\frac{dz(x)}{\hat{r}}}{\frac{dz(x)}{dx}} = \frac{x}{\hat{r}-r^0}$ and thus

$$\frac{dx^*}{d\hat{r}} = \frac{x^*}{\hat{r}-r^0}\begin{cases} <0, & \text{if } \hat{r} < r^0\\ >0, & \text{if } \hat{r} > r^0\end{cases} \quad \text{(A.20)}$$

In a similar matter it can be shown that for $x^* < 0$ is $\frac{dx^*}{d\lambda} > 0$ and

$$\frac{dx^*}{d\hat{r}} = \frac{x^*}{\hat{r}-r^0}\begin{cases} >0, & \text{if } \hat{r} < r^0\\ <0, & \text{if } \hat{r} > r^0\end{cases} \quad \text{(A.21)}$$

Note that both (A.20) and (A.21) are first order linear differential equations, which then imply (piece-vice linear) dependence of risk taking with respect to $\hat{r}$ when $x^* > 0$ and an inverse V-shaped dependence when $x^* < 0$.

$$x^* = \begin{cases} \frac{r^0-\hat{r}}{K_1(\lambda,r^0,\gamma)} > 0, & \text{if } x^* \text{ satisfies (21) and } \hat{r} < r^0\\ & \text{where } K_1(\lambda,r^0,\gamma) > 0\\ \frac{\hat{r}-r^0}{K_2(\lambda,r^0,\gamma)} > 0, & \text{if } x^* \text{ satisfies (21) and } \hat{r} > r^0\\ & \text{where } K_2(\lambda,r^0,\gamma) > 0\\ \frac{r^0-\hat{r}}{K_3(\lambda,r^0,\gamma)} < 0, & \text{if } x^* \text{ satisfies (23) and } \hat{r} < r^0\\ & \text{where } K_3(\lambda,r^0,\gamma) < 0\\ \frac{\hat{r}-r^0}{K_4(\lambda,r^0,\gamma)} < 0, & \text{if } x^* \text{ satisfies (23) and } \hat{r} > r^0\\ & \text{where } K_4(\lambda,r^0,\gamma) < 0\end{cases} \quad \text{(A.22)}$$

where K_1, K_2, K_3, K_4 are functions of λ, r^0, γ, and the distribution of the risky asset return does not depend on the reference return $\hat{r}$. This implies the V-shaped (piecewise linear) dependence of risk taking with respect to $\hat{r}$ when $x^* > 0$ and an inverse V-shaped dependence when $x^* < 0$.

Note that based on (A.22), Proposition 2.5 can be re-formulated such that for $\hat{r} < r^0$ the global maximum is obtained for $x^* = \frac{r^0-\hat{r}}{K(\lambda,r^0,\gamma)} > 0$, where $K = K(\lambda,r^0,\gamma) > 0$ satisfies

$$\lambda\int_{-\infty}^{r^0-K}\frac{r-r^0}{\left(r^0-K-r\right)^\gamma}f(r)dr + \int_{r^0-K}^{+\infty}\frac{r-r^0}{\left(r-r^0+K\right)^\gamma}f(r)dr = 0 \quad \text{(A.23)}$$

Then the statement of Proposition 2.6 can be re-formulated such that for $\hat{r} < r^0$ the local positive maximum is obtained for $x^* = \frac{r^0-\hat{r}}{K(\lambda,r^0,\gamma)} > 0$, where $K = K(\lambda,r^0,\gamma) > 0$ satisfies (A.23) while for $\hat{r} > r^0$ the local positive maximum is obtained for $x^* = \frac{\hat{r}-r^0}{K(\lambda,r^0,\gamma)} > 0$, where $K = K(\lambda,r^0,\gamma) > 0$ satisfies

$$\lambda\int_{-\infty}^{r^0+K}\frac{r-r^0}{\left(r^0+K-r\right)^\gamma}f(r)dr + \int_{r^0+K}^{+\infty}\frac{r-r^0}{\left(r-r^0-K\right)^\gamma}f(r)dr = 0 \quad \text{(A.24)}$$

If there exists also a local negative maximum $x^* < 0$, then for $\hat{r} < r^0$ it is of the form $x^* = \frac{r^0-\hat{r}}{K(\lambda,r^0,\gamma)} < 0$, where $K = K(\lambda,r^0,\gamma) < 0$ satisfies

$$\int_{-\infty}^{r^0-K}\frac{r-r^0}{\left(r^0-K-r\right)^\gamma}f(r)dr + \lambda\int_{r^0-K}^{+\infty}\frac{r-r^0}{\left(r-r^0+K\right)^\gamma}f(r)dr = 0 \quad \text{(A.25)}$$

and for $\hat{r} > r^0$ is $x^* = \frac{\hat{r}-r^0}{K(\lambda,r^0,\gamma)} < 0$, where $K = K(\lambda,r^0,\gamma) < 0$ satisfies

$$\int_{-\infty}^{r^0+K}\frac{r-r^0}{\left(r^0+K-r\right)^\gamma}f(r)dr + \lambda\int_{r^0+K}^{+\infty}\frac{r-r^0}{\left(r-r^0-K\right)^\gamma}f(r)dr = 0 \quad \text{(A.26)}$$

This concludes the proof. □

Appendix B. Data description and summary statistics

See Tables B.1 and B.2

Appendix C. Empirical results

See Tables C.1–C.6

Table B.1
Data description and sources.

Abbr	Variable	Unit	Note	Source	Code	Start
EU markets						
Stock	EU stock market ind	Index		Ref DS	TOTMKEM(RI)~E	1983:1
Bond	German gov bond ind	Index	Total ret ind, 10 yrs	Ref DS	BMBD10Y(RI)	1983:1
Gold	Gold bullion LBM	EUR	A.M. official fixing	Ref DS: ICE	GOLDBLN(OF)~E	1983:1
Risk-free	One-month LIBOR	Percent		Ref DS: ICE	BBDEM1M	1986:1
US markets						
Stock	US stock market ind	Index		Ref DS	TOTMKUS(RI)	1983:1
Bond	US gov bond ind	Index	Total ret ind, 10 yrs	Ref DS	BMUS10Y(RI)	1983:1
Gold	Gold bullion LBM	USD	A.M. official fixing	Ref DS: ICE	GOLDBLN(OF)	1983:1
Risk-free	One-month LIBOR	Percent		Ref DS: ICE	BBUSD1M	1986:1

Gov = government, ind = index, ret = return, yrs = years, Ref DS = Refinitiv Datastream, EUR = Euro, USD = US dollar, LBM = London Bullion Market, ICE = ICE Benchmark Administration Ltd. The frequency of all data is monthly, where monthly values are end-of-month values for a given month. Returns are calculated as $100\,(P_t/P_{t-1}-1)$, where P_t is the price of the index observed in month t, and are quoted in percent. Note that the risk-free rate is used in the evaluation process, not in the optimization process. It is only available from January 1986, which is why we start with the asset data in January 1983, granting us an optimization period of three years. Assets in Europe are quoted in Euro, assets in the US are quoted in US dollar. Prices for gold are quoted per troy ounce.

Table B.2
Summary statistics for EU and US markets.

	EU markets				US markets			
	Stock	Bond	Gold	Risk-free	Stock	Bond	Gold	Risk-free
Summary statistics of one-month returns (in percent p.a.)								
Mean	12.28	6.62	4.77	3.04	13.61	7.36	5.13	3.50
Std.dev.	16.80	5.44	15.85	0.79	14.87	7.50	15.62	0.77
Skewness	−0.61	−0.18	0.22	0.57	−0.68	0.24	0.13	0.27
Kurtosis	1.89	0.11	1.34	−0.39	2.31	0.95	1.61	−1.13
VaR	−60.92	−22.29	−56.98	−0.42	−58.08	−28.95	−53.76	0.17
CVaR	−76.81	−28.13	−69.38	−0.50	−71.08	−38.44	−68.32	0.16
Minimum	−93.99	−49.47	−89.62	−0.59	−93.40	−58.63	−91.77	0.14
Maximum	546.04	90.58	653.20	9.88	376.67	209.02	731.56	10.06
Percentiles (in percent p.a.)								
5	−60.92	−22.29	−56.98	−0.42	−58.08	−28.95	−53.76	0.17
10	−43.88	−17.59	−44.99	−0.40	−41.24	−21.81	−43.82	0.20
25	−17.40	−6.51	−23.90	0.35	−15.64	−8.46	−24.90	0.66
50	18.58	8.56	1.92	3.18	18.54	6.07	−0.39	3.19
75	57.11	21.18	39.20	4.57	52.22	24.19	42.22	5.68
90	104.72	32.64	101.53	7.93	102.38	46.48	104.64	7.13
95	149.39	40.55	157.25	9.06	134.23	65.99	157.16	8.31
Correlations across EU and US assets								
Stock, EU								
Bond, EU	−0.07							
Gold, EU	−0.08	0.06						
Risk-free, EU	−0.04	0.07	−0.06					
Stock, US	0.74	−0.09	−0.13	−0.03				
Bond, US	−0.17	0.67	−0.02	0.10	−0.04			
Gold, US	−0.13	0.05	0.79	−0.05	−0.06	0.06		
Risk-free, US	0.02	−0.01	−0.07	0.72	0.02	0.07	−0.07	

Statistics are calculated on the basis of monthly returns and then annualized using discrete compounding, for the period January 1983 to December 2020 (for the period January 1986 to December 2020 in case of the risk-free rate). The annual standard deviation is computed as $\sigma_{pa} = \sqrt{12}\sigma_{pm}$. Skewness and kurtosis are not adjusted. The risk-free rate, originally given in percent p.a., is first converted to percent per month using discrete compounding and then the statistics are computed similarly to the other data.

Table C.1
Portfolio performance of PT portfolios in the EU: benchmark scenario, $\hat{r} = 0$, $\gamma = 0.5$.

	MV	CVaR	Risk neutral	Linear loss aversion λ 1.5	2	2.25	2.5	3	S-shaped prospect theory, $\gamma = 0.5$ λ 1.5	2	2.25	2.5	3
Performance of one-month returns (in percent p.a.)													
Mean	6.17	5.75	12.92	9.42	8.36	7.69	7.36	7.03	7.81	8.04	7.19	7.84	7.57
Omega	159.23	143.34	167.86	150.45	151.48	149.66	148.38	149.69	146.31	157.46	147.98	164.78	165.16
Sharpe ratio	62.74	46.93	63.16	45.99	46.14	44.15	42.63	44.06	41.74	50.63	41.82	57.01	58.13
Sortino ratio	97.70	72.35	98.64	70.31	68.96	65.16	62.46	61.82	59.84	78.42	60.14	90.83	91.60
Additional descriptive statistics (in percent p.a.)													
Median	7.32	6.10	14.93	9.51	8.99	8.44	8.74	9.02	9.11	8.34	7.81	7.74	7.86
Volatility	4.79	5.54	15.19	13.47	11.21	10.23	9.83	8.79	11.09	9.58	9.63	8.16	7.54
Down. vol.	2.69	3.16	9.35	8.48	7.17	6.61	6.40	5.97	7.38	5.82	6.38	4.77	4.45
CVaR	−27.23	−31.10	−69.37	−66.91	−59.46	−56.03	−54.62	−51.37	−61.23	−52.28	−53.48	−44.46	−42.27
Skewness	−0.34	−0.10	−0.31	−0.27	−0.56	−0.71	−0.78	−1.53	−0.83	−0.07	−1.05	0.00	−0.11
Kurtosis	3.59	4.10	5.19	6.86	10.15	12.95	14.71	19.12	11.00	9.17	15.82	10.71	10.96
Realized returns (in percent p.a.)													
Last 10 years	5.67	5.18	7.19	4.52	4.15	4.50	4.51	5.03	4.44	5.19	4.94	5.73	5.99
Last 5 years	3.49	3.47	7.18	1.12	2.24	2.57	2.37	3.52	3.04	2.04	2.39	4.06	4.72
Last 3 years	3.23	3.50	7.83	1.26	2.62	2.84	2.21	3.56	3.77	2.50	2.90	4.35	4.57
Last year	1.13	−3.83	6.70	1.54	−2.41	−2.96	−3.95	−0.44	5.52	−0.79	−0.69	4.54	4.72
Mean portfolio weights (in percent)													
Stock	10.55	14.66	66.19	48.11	38.54	34.13	31.41	27.84	41.47	35.20	32.85	28.73	25.07
Bond	80.65	70.40	11.43	31.28	47.28	53.33	56.53	60.86	45.82	54.30	56.88	61.17	65.10
Gold	8.80	14.95	22.38	20.61	14.18	12.54	12.06	11.29	12.70	10.50	10.27	10.10	9.83
Standard deviation of portfolio weights													
Stock	8.51	8.57	47.31	42.57	36.20	32.03	29.62	26.03	35.64	31.51	29.95	25.98	21.28
Bond	7.42	16.49	31.82	38.66	35.41	32.54	30.91	27.89	35.98	31.51	29.99	26.93	22.73
Gold	7.10	12.71	41.68	35.66	22.75	18.67	16.54	13.65	18.27	12.80	11.92	11.03	10.05

The table reports statistics of monthly reallocated optimal portfolio returns based on an optimization period of 36 months as well as the average and standard deviation of the optimal asset weights. The table also reports results for portfolios implied by mean–variance (MV), conditional value-at-risk (CVaR), risk neutral, and linear loss averse investors. The evaluation period covers January 1986 to December 2020. Statistics are calculated on the basis of monthly returns and then annualized assuming discrete compounding. The annual standard deviation is computed as $\sigma_{pa} = \sqrt{12}\sigma_{pm}$. The Omega measure and the Sortino ratio use the risk-free interest rate as target return.

Table C.2
Portfolio performance of PT portfolios in the US: benchmark scenario, $\hat{r} = 0$, $\gamma = 0.5$.

	MV	CVaR	Risk neutral	Linear loss aversion λ 1.5	2	2.25	2.5	3	S-shaped prospect theory, $\gamma = 0.5$ λ 1.5	2	2.25	2.5	3
Performance of one-month returns (in percent p.a.)													
Mean	6.95	7.35	13.34	12.34	10.08	9.35	8.62	8.46	10.06	10.46	9.90	9.64	8.89
Omega	155.26	159.99	163.86	167.92	158.80	155.92	152.46	155.95	158.69	172.36	166.71	166.93	160.51
Sharpe ratio	57.37	58.99	62.23	63.43	55.15	53.36	50.51	53.19	54.56	65.46	61.21	61.06	56.38
Sortino ratio	91.75	93.98	92.15	94.49	78.09	75.44	71.24	77.06	78.40	99.15	91.38	91.46	84.18
Additional descriptive statistics (in percent p.a.)													
Median	6.17	6.91	18.72	14.24	11.46	10.57	10.60	10.04	10.69	9.91	9.69	9.34	8.45
Volatility	5.77	6.28	15.33	13.50	11.59	10.66	9.83	9.04	11.66	10.31	10.14	9.76	9.27
Down. vol.	3.14	3.49	9.90	8.65	7.78	7.13	6.55	5.82	7.72	6.40	6.38	6.10	5.79
CVaR	−31.16	−33.82	−71.48	−66.52	−62.15	−59.59	−56.00	−51.24	−63.19	−55.62	−55.97	−53.93	−51.32
Skewness	−0.02	−0.07	−0.68	−0.78	−1.30	−1.16	−1.12	−0.93	−1.03	−0.69	−0.73	−0.75	−0.70
Kurtosis	4.54	6.10	5.20	6.92	9.93	8.40	8.76	8.79	7.97	6.83	6.98	7.52	7.74
Realized returns (in percent p.a.)													
Last 10 Years	7.72	7.64	9.21	10.24	6.84	6.01	6.06	6.26	6.79	6.81	6.78	6.91	6.95
Last 5 Years	8.00	7.41	8.90	9.79	6.75	6.70	6.40	6.82	5.14	7.43	7.72	8.04	7.86
Last 3 Years	9.43	9.14	4.50	8.29	6.83	7.34	6.31	7.19	3.74	7.77	7.79	8.88	8.86
Last Year	15.05	14.66	−7.97	4.01	6.44	7.70	11.76	14.08	−9.99	4.27	5.03	8.04	11.14
Mean portfolio weights (in percent)													
Stock	20.74	25.78	65.48	58.99	49.04	45.89	42.21	38.63	50.92	46.34	45.95	43.47	41.47
Bond	64.64	55.68	6.43	17.51	34.52	38.67	43.72	48.52	34.06	40.48	41.79	45.35	48.38
Gold	14.62	18.54	28.10	23.51	16.44	15.44	14.08	12.84	15.02	13.18	12.26	11.18	10.15
Standard deviation of portfolio weights													
Stock	10.49	13.46	47.54	41.39	35.08	32.53	29.84	27.02	33.99	32.18	31.57	30.32	28.81
Bond	10.46	16.18	24.53	30.12	31.74	31.62	30.70	28.92	33.51	32.64	32.23	31.42	29.72
Gold	14.20	15.77	44.95	35.11	22.64	20.09	17.29	14.89	22.62	18.65	16.88	15.45	13.61

The table reports statistics of monthly reallocated optimal portfolio returns based on an optimization period of 36 months as well as the average and standard deviation of the optimal asset weights. The table also reports results for portfolios implied by mean–variance (MV), conditional value-at-risk (CVaR), risk neutral, and linear loss averse investors. The evaluation period covers January 1986 to December 2020. Statistics are calculated on the basis of monthly returns and then annualized assuming discrete compounding. The annual standard deviation is computed as $\sigma_{pa} = \sqrt{12}\sigma_{pm}$. The Omega measure and the Sortino ratio use the risk-free interest rate as target return.

Table C.3
Portfolio performance of PT portfolios in the EU: conservative scenario, $\hat{r} = 0$, $\gamma = 0.9$.

	MV	CVaR	Risk neutral	Linear loss aversion					S-shaped prospect theory, $\gamma = 0.9$				
				λ 1.5	2	2.25	2.5	3	λ 1.5	2	2.25	2.5	3
Performance of one-month returns (in percent p.a.)													
Mean	6.17	5.75	12.92	9.20	8.29	7.73	7.30	7.00	8.41	7.62	8.10	8.08	8.59
Omega	159.23	143.34	167.86	148.83	151.48	150.25	148.25	149.33	167.98	159.76	169.07	170.64	180.76
Sharpe ratio	62.74	46.93	63.16	44.76	46.08	44.57	42.69	43.78	60.04	54.74	61.86	63.01	69.82
Sortino ratio	97.70	72.35	98.64	67.84	68.92	65.82	61.93	61.24	97.31	85.46	100.05	103.25	116.87
Additional descriptive statistics (in percent p.a.)													
Median	7.32	6.10	14.93	9.43	8.99	8.57	8.74	9.02	7.54	7.47	8.19	8.01	8.10
Volatility	4.79	5.54	15.19	13.36	11.07	10.22	9.70	8.76	8.66	8.11	7.92	7.74	7.70
Down. vol.	2.69	3.16	9.35	8.48	7.08	6.60	6.37	5.96	4.99	4.85	4.55	4.38	4.26
CVaR	−27.23	−31.10	−69.37	−66.97	−58.74	−56.03	−54.58	−51.37	−46.04	−45.33	−42.81	−41.48	−40.60
Skewness	−0.34	−0.10	−0.31	−0.31	−0.56	−0.71	−0.91	−1.56	−0.08	−0.29	−0.13	−0.07	−0.03
Kurtosis	3.59	4.10	5.19	6.96	10.44	12.99	15.14	19.33	8.15	8.79	8.86	9.07	9.20
Realized returns (in percent p.a.)													
Last 10 Years	5.67	5.18	7.19	4.08	4.24	4.56	4.55	5.06	6.01	4.73	5.20	5.11	5.80
Last 5 Years	3.49	3.47	7.18	0.59	2.41	2.57	2.37	3.58	3.55	3.00	2.42	2.38	4.35
Last 3 Years	3.23	3.50	7.83	0.46	2.74	2.84	2.21	3.66	3.21	1.30	2.73	2.20	4.30
Last Year	1.13	−3.83	6.70	−0.03	−2.41	−2.96	−3.95	−0.08	0.92	−1.55	−0.82	1.01	1.40
Mean portfolio weights (in percent)													
Stock	10.55	14.66	66.19	48.01	37.96	34.05	31.08	27.79	33.24	31.56	31.21	29.57	28.57
Bond	80.65	70.40	11.43	32.05	47.93	53.47	56.83	60.89	55.53	57.97	58.24	59.49	60.45
Gold	8.80	14.95	22.38	19.94	14.11	12.47	12.09	11.32	11.23	10.47	10.55	10.94	10.98
Standard deviation of portfolio weights													
Stock	8.51	8.57	47.31	42.50	35.64	32.01	29.24	26.00	29.19	27.34	26.62	25.53	24.56
Bond	7.42	16.49	31.82	38.72	35.13	32.54	30.65	27.89	29.80	27.84	27.51	27.12	26.55
Gold	7.10	12.71	41.68	34.71	22.61	18.59	16.49	13.64	12.10	10.26	9.88	10.13	10.01

The table reports statistics of monthly reallocated optimal portfolio returns based on an optimization period of 36 months as well as the average and standard deviation of the optimal asset weights. The table also reports results for portfolios implied by mean–variance (MV), conditional value-at-risk (CVaR), risk neutral, and linear loss averse investors. The evaluation period covers January 1986 to December 2020. Statistics are calculated on the basis of monthly returns and then annualized assuming discrete compounding. The annual standard deviation is computed as $\sigma_{pa} = \sqrt{12}\sigma_{pm}$. The Omega measure and the Sortino ratio use the risk-free interest rate as target return.

Table C.4
Portfolio performance of PT portfolios in the EU: aggressive scenario, $\hat{r}$ is the portfolio return from the previous period, $\gamma = 0.1$.

	MV	CVaR	Risk neutral	Linear loss aversion					S-shaped prospect theory, $\gamma = 0.1$				
				λ 1.5	2	2.25	2.5	3	λ 1.5	2	2.25	2.5	3
Performance of one-month returns (in percent p.a.)													
Mean	6.17	5.75	12.92	10.64	8.80	7.98	7.69	7.67	10.30	8.20	7.60	7.32	7.31
Omega	159.23	143.34	167.86	154.49	145.96	142.39	142.32	145.74	151.07	139.91	137.58	137.06	140.61
Sharpe ratio	62.74	46.93	63.16	50.99	43.32	40.26	39.63	41.73	48.29	38.68	36.27	35.50	37.89
Sortino ratio	97.70	72.35	98.64	78.31	65.27	59.44	58.59	61.50	73.82	57.16	53.06	52.47	55.44
Additional descriptive statistics (in percent p.a.)													
Median	7.32	6.10	14.93	11.38	9.30	9.39	8.48	7.73	10.62	9.40	8.41	7.70	7.77
Volatility	4.79	5.54	15.19	14.47	12.91	11.91	11.38	10.76	14.60	12.95	12.21	11.71	10.94
Down. vol.	2.69	3.16	9.35	9.06	8.23	7.74	7.37	6.98	9.18	8.40	8.00	7.58	7.15
CVaR	−27.23	−31.10	−69.37	−68.48	−65.40	−62.51	−60.50	−58.07	−68.75	−65.93	−64.24	−61.48	−59.05
Skewness	−0.34	−0.10	−0.31	−0.26	−0.35	−0.53	−0.56	−0.67	−0.24	−0.44	−0.52	−0.48	−0.64
Kurtosis	3.59	4.10	5.19	5.92	7.18	8.56	9.64	11.29	5.79	7.13	8.15	8.88	10.73
Realized returns (in percent p.a.)													
Last 10 Years	5.67	5.18	7.19	5.24	4.34	3.70	4.10	4.31	4.43	3.74	2.84	2.56	3.17
Last 5 Years	3.49	3.47	7.18	3.30	1.42	1.09	1.29	1.24	2.29	1.09	−0.22	−0.47	−0.33
Last 3 Years	3.23	3.50	7.83	3.46	1.49	1.67	1.86	1.13	3.42	2.13	0.64	−0.48	−0.53
Last Year	1.13	−3.83	6.70	1.64	−2.24	−2.67	−1.70	−3.18	−1.18	−0.52	−2.98	−1.23	−7.07
Mean portfolio weights (in percent)													
Stock	10.55	14.66	66.19	58.53	49.56	45.64	42.82	39.17	58.74	51.07	47.21	44.01	40.19
Bond	80.65	70.40	11.43	19.91	33.51	40.12	43.65	47.76	20.21	32.41	38.51	42.19	47.01
Gold	8.80	14.95	22.38	21.56	16.93	14.24	13.53	13.07	21.05	16.53	14.28	13.81	12.79
Standard deviation of portfolio weights													
Stock	8.51	8.57	47.31	44.54	40.25	38.21	36.81	34.76	44.56	40.55	39.42	37.84	35.82
Bond	7.42	16.49	31.82	35.28	37.49	36.90	35.73	34.32	34.85	36.31	37.00	36.29	35.17
Gold	7.10	12.71	41.68	38.36	29.91	23.31	20.51	18.26	37.97	29.15	24.21	21.50	18.45

The table reports statistics of monthly reallocated optimal portfolio returns based on an optimization period of 36 months as well as the average and standard deviation of the optimal asset weights. The table also reports results for portfolios implied by mean–variance (MV), conditional value-at-risk (CVaR), risk neutral, and linear loss averse investors. The evaluation period covers January 1986 to December 2020. Statistics are calculated on the basis of monthly returns and then annualized assuming discrete compounding. The annual standard deviation is computed as $\sigma_{pa} = \sqrt{12}\sigma_{pm}$. The Omega measure and the Sortino ratio use the risk-free interest rate as target return.

Table C.5
Portfolio performance of PT portfolios in the US: conservative scenario, $\hat{r}$ is the portfolio return from the previous period, $\gamma = 0.1$.

	MV	CVaR	Risk neutral	Linear loss aversion λ 1.5	2	2.25	2.5	3	S-shaped prospect theory, $\gamma = 0.1$ λ 1.5	2	2.25	2.5	3
Performance of one-month returns (in percent p.a.)													
Mean	6.95	7.35	13.34	12.80	10.83	9.99	9.67	9.21	13.62	11.42	10.62	9.84	9.59
Omega	155.26	159.99	163.86	168.67	158.29	153.98	152.85	152.83	173.28	162.62	158.52	153.11	155.29
Sharpe ratio	57.37	58.99	62.23	64.53	55.06	51.99	51.27	52.14	68.70	58.49	55.32	51.39	54.33
Sortino ratio	91.75	93.98	92.15	96.39	80.40	75.33	73.80	76.80	103.06	85.94	79.56	73.87	79.66
Additional descriptive statistics (in percent p.a.)													
Median	6.17	6.91	18.72	15.84	12.83	11.98	11.80	10.62	18.32	13.61	13.00	11.54	11.37
Volatility	5.77	6.28	15.33	13.96	12.89	12.09	11.66	10.62	14.26	13.12	12.46	11.95	10.87
Down. vol.	3.14	3.49	9.90	8.93	8.42	7.92	7.68	6.76	9.08	8.52	8.25	7.89	6.97
CVaR	−31.16	−33.82	−71.48	−67.74	−65.21	−63.20	−62.07	−56.69	−68.40	−65.81	−64.42	−62.98	−58.76
Skewness	−0.02	−0.07	−0.68	−0.74	−0.88	−0.87	−0.90	−0.67	−0.73	−0.85	−1.06	−0.95	−0.69
Kurtosis	4.54	6.10	5.20	6.33	7.58	6.90	7.00	6.19	6.03	7.35	8.09	7.26	5.75
Realized returns (in percent p.a.)													
Last 10 Years	7.72	7.64	9.21	11.45	9.74	8.38	7.77	6.55	12.11	10.26	9.26	8.50	7.25
Last 5 Years	8.00	7.41	8.90	11.10	10.06	7.98	7.65	7.35	10.36	9.40	9.89	7.95	7.44
Last 3 Years	9.43	9.14	4.50	9.01	9.93	7.08	6.98	6.98	7.83	9.15	10.19	6.78	6.82
Last Year	15.05	14.66	−7.97	6.43	9.55	7.79	10.61	10.82	1.80	7.74	12.09	7.10	10.59
Mean portfolio weights (in percent)													
Stock	20.74	25.78	65.48	62.20	56.26	53.50	51.30	48.13	63.17	57.86	54.70	53.21	49.58
Bond	64.64	55.68	6.43	13.27	24.26	28.84	32.03	37.04	12.62	22.78	27.16	30.21	35.58
Gold	14.62	18.54	28.10	24.53	19.49	17.66	16.67	14.83	24.21	19.36	18.14	16.59	14.84
Standard deviation of portfolio weights													
Stock	10.49	13.46	47.54	43.36	39.25	38.20	36.98	34.86	43.78	39.64	38.95	37.79	36.50
Bond	10.46	16.18	24.53	29.23	30.23	30.65	30.87	30.82	28.43	30.85	31.88	31.47	31.87
Gold	14.20	15.77	44.95	37.77	28.15	24.45	21.95	18.59	39.02	28.95	25.75	22.53	18.63

The table reports statistics of monthly reallocated optimal portfolio returns based on an optimization period of 36 months as well as the average and standard deviation of the optimal asset weights. The table also reports results for portfolios implied by mean–variance (MV), conditional value-at-risk (CVaR), risk neutral, and linear loss averse investors. The evaluation period covers January 1986 to December 2020. Statistics are calculated on the basis of monthly returns and then annualized assuming discrete compounding. The annual standard deviation is computed as $\sigma_{pa} = \sqrt{12}\sigma_{pm}$. The Omega measure and the Sortino ratio use the risk-free interest rate as target return.

Table C.6
Portfolio performance of PT portfolios in the US: aggressive scenario, $\hat{r} = r^0$, $\gamma = 0.9$.

	MV	CVaR	Risk neutral	Linear loss aversion λ 1.5	2	2.25	2.5	3	S-shaped prospect theory, $\gamma = 0.9$ λ 1.5	2	2.25	2.5	3
Performance of one-month returns (in percent p.a.)													
Mean	6.95	7.35	13.34	12.59	10.07	9.69	8.90	8.40	11.61	11.59	11.31	11.42	10.00
Omega	155.26	159.99	163.86	169.14	158.63	159.29	155.41	154.70	177.74	181.59	178.08	181.25	166.21
Sharpe ratio	57.37	58.99	62.23	64.42	54.93	56.23	52.99	52.16	68.64	71.71	69.19	70.90	60.85
Sortino ratio	91.75	93.98	92.15	95.90	77.63	80.79	75.45	74.72	102.56	109.87	104.77	108.20	89.07
Additional descriptive statistics (in percent p.a.)													
Median	6.17	6.91	18.72	15.24	11.87	10.84	10.18	9.92	11.64	11.16	11.15	10.80	9.91
Volatility	5.77	6.28	15.33	13.65	11.60	10.69	9.88	9.11	11.46	10.95	10.95	10.84	10.36
Down. vol.	3.14	3.49	9.90	8.76	7.81	7.02	6.53	5.94	7.25	6.73	6.81	6.68	6.65
CVaR	−31.16	−33.82	−71.48	−67.17	−62.55	−59.44	−56.53	−52.42	−60.75	−58.00	−58.98	−58.35	−58.12
Skewness	−0.02	−0.07	−0.68	−0.78	−1.31	−1.02	−1.04	−1.01	−0.82	−0.64	−0.66	−0.64	−0.88
Kurtosis	4.54	6.10	5.20	6.72	9.91	7.56	8.18	8.94	7.42	6.95	6.84	6.96	7.08
Realized returns (in percent p.a.)													
Last 10 Years	7.72	7.64	9.21	10.73	6.96	6.18	6.34	6.41	7.95	8.07	7.59	7.78	6.04
Last 5 Years	8.00	7.41	8.90	10.22	6.92	6.80	6.87	7.11	8.94	9.33	8.06	8.42	4.96
Last 3 Years	9.43	9.14	4.50	8.99	7.15	7.44	7.08	7.66	9.26	9.70	9.08	9.13	3.80
Last Year	15.05	14.66	−7.97	5.74	6.11	7.35	12.35	15.97	8.26	9.22	9.23	9.24	−5.39
Mean portfolio weights (in percent)													
Stock	20.74	25.78	65.48	59.55	49.35	46.49	42.62	38.55	51.55	49.99	49.92	49.93	48.08
Bond	64.64	55.68	6.43	17.00	34.55	38.61	43.84	49.43	31.86	35.23	35.58	36.15	38.15
Gold	14.62	18.54	28.10	23.45	16.11	14.90	13.54	12.02	16.59	14.78	14.50	13.92	13.77
Standard deviation of portfolio weights													
Stock	10.49	13.46	47.54	41.64	35.13	33.23	30.54	27.69	30.48	30.47	30.33	30.32	29.50
Bond	10.46	16.18	24.53	30.67	32.38	32.24	31.26	29.39	27.29	27.58	27.47	27.67	27.71
Gold	14.20	15.77	44.95	35.56	22.46	19.80	17.16	14.55	21.21	19.01	18.44	17.66	16.92

The table reports statistics of monthly reallocated optimal portfolio returns based on an optimization period of 36 months as well as the average and standard deviation of the optimal asset weights. The table also reports results for portfolios implied by mean–variance (MV), conditional value-at-risk (CVaR), risk neutral, and linear loss averse investors. The evaluation period covers January 1986 to December 2020. Statistics are calculated on the basis of monthly returns and then annualized assuming discrete compounding. The annual standard deviation is computed as $\sigma_{pa} = \sqrt{12}\sigma_{pm}$. The Omega measure and the Sortino ratio use the risk-free interest rate as target return.

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