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Directed Search, Rationing and Wage Dispersion

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Abstract

This paper develops a microeconomic model of directed search, where firms are heterogeneous in the number of vacancies advertised, and wages affect workers' choices when both applying for jobs and accepting a job. An aggregate matching function is derived, which incorporates workers' preferences for firms. The aggregate level of matches is shown to be independent of the workers' preferences in the job acceptance stage. When firms' labor demands are heterogeneous, the matching market equilibrium outcome is suboptimal. Matching efficiency is, however, attained in equilibrium, when wages are employed as a rationing device. This results in wage dispersion, despite workers being homogeneous.

JEL Classification: J31, J64

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1 Introduction

Workers take into account many factors when they search for a job. In this paper I consider two of these, wages and the number of vacancies, and I develop a microeconomic model of directed search where workers' preferences of firms are explicitly modelled. An important contribution of the paper is to note and analyze the fundamental difference between the two factors that "direct" workers' searches. Specifically, the number of vacancies at a firm, which determines its probability of a job offer, affects a worker's choice of which jobs to apply for. However once job offers are received, this factor becomes irrelevant in the worker's choice of which job offer to accept. On the other hand, wages affect both the application choices and the acceptance choice of an applicant. In this sense, firms are *ex-post homogeneous* when they offer uniform wages, and *ex-post heterogeneous* when there is wage dispersion, where "ex-post" is in reference to the job-offer stage of a job-matching process. The first part of this paper asks the question of whether introduction of ex-post heterogeneity affects the matching outcome. It turns out that the aggregate level of matches is independent of the firms' ex-post heterogeneity. In other words, once job offers are made, the workers' preferences for the jobs do not affect the aggregate level of matches. The heterogeneity, however, affects the distribution of the matches. This implies that search friction in a job-matching market arises from coordination failures of the workers when they apply for jobs, and of the firms when they make job offers,[1] but no friction arises from the workers' actions after job offers are received. This is the first main result.

The second main result concerns efficient coordination in the matching market, defined as one that achieves the highest aggregate level of matches. When a worker chooses his optimal strategy of which jobs to apply for, he ignores the negative externality of his choice of firms on other applicants' probabilities of a job offer from those firms that he applies to. When firms offer heterogeneous numbers of vacancies, this leads to suboptimally high applications at firms offering higher number of vacancies. The argument is then that a wage policy can be employed to "redirect" the search, so that efficient coordination is attained in equilibrium. In this paper a simple example is given, where the socially optimal aggregate level of matches is attained with the use of heterogeneous wages. Intuitively, the wages reflect the social costs of job applications, and in the efficient equilibrium the externalities are internalized. This gives a matching efficiency argument for wage dispersion, even

[1] In Albrecht, Gautier and Vroman (2006), these are classified as the "urn-ball" and the "multiple-applications" frictions. See later for the definitions of these.

when workers are homogeneous. One policy implication is that if the objective of social planners is to minimize unemployment, then they should encourage *lower* wages at firms advertising higher numbers of vacancies. The result that wages can be used as an ex-ante allocation device to counter the coordination inefficiency caused by multiple vacancies, despite the fundamental difference between the two factors of wages and the number of vacancies, as described above, relies crucially on the independence result attained.

There has been some work in the microeconomic search literature modelling the role of wages in job searches. These have mainly investigated the trade-off between wage level and matching success, for example, to explain wage dispersion (e.g. Montogomery, 1991; Galenianos and Kircher, 2005). In this paper wages are used formally as a tool for ex-ante allocation of resources. To do this, a matching model that explicitly incorporates workers' preferences for higher wages is developed, and its equilibrium outcomes are investigated. Therefore it differs from the macroeconomic matching models, where in general the global equilibrium wage is determined as a result of bargaining after workers and firms are matched (e.g. Pissarides, 2000).

A related work is Moen (1997). In Moen's model the labor market is divided into submarkets, each with an exogenously assigned wage. Observing this, workers choose a submarket to join, within which matchings occur. This leads to wage dispersion in equilibrium as a result of a trade-off between the wage level and the expected duration of the unemployment period in the submarkets. However the model is partial, because while wages do play the role of allocating workers to the submarkets, once within the submarkets there are no worker preferences for the firms. Hence the matching mechanism itself, and therefore also the matching function, do not incorporate the wage's role as a rationing device.

The aggregate matching function derived here belongs to the family of multiple-application matching functions. Traditionally, matching functions were derived in an urn-ball set-up where the market contained firms consisting of a single vacancy, and workers made a single application (e.g. Pissarides, 1979; Blanchard and Diamond, 1994). Albrecht, Gautier and Vroman (2006) note that the advantage of allowing multiple applications is that, while the traditional models only capture what they term the "urn-ball friction", where some vacancies receive no applications while others receive more than one, the multiple-application models also capture the "multiple-application friction", where some workers receive multiple job offers while others receive none.[2]

[2]Julien, Kennes and King (2000), in which firms make a job offer to one worker after observing all applicants' reservation wages, capture the second friction but not the first.

More recently, models such as Albrecht et al. (2004) and Hori (2007) have derived multiple-application matching functions under random-matching setups. One disadvantage with multiple-application matching functions is that, for large markets, the computation becomes cumbersome very rapidly. For this reason a limiting expression is also derived here (as is done in Albrecht et al. and Hori for their functions), which reduces the computation time considerably.

There are in the literature other directed search models with multiple-applications, such as Albrecht, Gautier and Vroman (2006) and Galenianos and Kircher (2005, 2007).[3] These two in particular attain similar results as those presented here, but for very different reasons. In Galenianos and Kircher's first model, workers view their applications as a portfolio choice problem, and they are thus willing to apply for jobs offering different wage levels (i.e. "risk-diversify"). This incentivizes firms to post different wages, and in equilibrium every worker applies once to each distinct wage. In their second model, wage dispersion is driven by fundamentals where more productive firms post higher wages. However both of these models assume firms with a single vacancy; here, the multiple-vacancy element that allows heterogeneous labor demands drives the efficiency and the wage dispersion results.

Mortensen (2003) provides a comprehensive survey on the wage dispersion literature.[4] The explanations offered in the literature for the "frictional wage dispersion" (Hornstein, Krussell and Violante, 2006) include firms' productivity heterogeneity (Montgomery, 1991; Acemoglu and Shimer, 2000; Galenianos and Kircher, 2007) and the reservation wage heterogeneity of the workers (Albrecht and Axell, 1984). Those assuming homogeneous firms and workers rely on asymmetric information, with workers having only partial information regarding wages offered by firms (Burdett and Judd, 1983; Mortensen, 2003). This paper adds an alternative reason for wage dispersion to this list.

The paper is structured as follows. Section 2 derives the firm-level and aggregate matching functions for both finite and limiting cases, and obtains the independence result. Section 3 analyzes the equilibrium outcome and establishes wage dispersion for matching efficiency. Section 4 then gives the concluding remarks. Finally, an analysis for the single-application case is given in the appendix.

[3] Other models of directed search include Montgomery (1991), Acemoglu and Shimer (1999) and Burdett, Shi and Wright (2001).

[4] He estimates that "observable worker characteristics that are supposed to account for productivity differences typically explain no more than 30 percent of the variation in compensation across workers." (p.1)

2 Directed Search Matching Model

2.1 Job Matching Market Mechanism

The job market consists of J firms who offer one or more vacancies for each advertised job, and I homogeneous workers who apply to a multiple number of jobs. An advertised job $j = 1, ..., J$ is defined by its wage level w_j, uniform for all applicants, and the number of vacancies L_j. The jobs are otherwise identical.

The job market operates as follows. First firms announce their job characteristics, namely the wage levels $\mathbf{w} = (w_1, ..., w_J)'$ and the number of vacancies $\mathbf{L} = (L_1, ..., L_J)'$. Next, viewing these, workers $i = 1, ..., I$ select a firms and make job applications. It is assumed that workers can apply only once to a particular job, irrespective of the number of vacancies advertised for the job. The number of applications a per worker is assumed given. A worker's set of applied jobs is only known by the applicant himself. If the firms receive more applications than L_j, then they randomly select L_j candidates and make job offers; otherwise they offer jobs to all applicants. Finally, applicants with one or more job offers accept the job of their highest preference, or choose one randomly if they are indifferent between the most preferred firms. There is no wage renegotiation. Applicants with no job offer remain unemployed, and vacancies with no application or a rejected job offer remain unfilled.

I now formalize the job-matching market mechanism for the finite case where $I, J < \infty$. The limiting case $I, J \to \infty$ is considered in Section 2.3. To do this, the job-matching process is considered in three stages of job application, job offer and job acceptance.

Job Application

In the job application stage, the workers' problem is that of choosing between pure strategies $s \in \{1, ..., \sigma\}$ of selecting a jobs out of J. For example, $s = 1$ may be a strategy that applies to the first a firms $j = 1, 2, ..., a$. $\sigma = \binom{J}{a}$ is the total number of possible pure strategies. Given the number of workers n_s that choose each of the σ strategies, the number of applications received at each firm α_j can be calculated. More formally, first define

> *The* ***strategy matrix*** $\mathbf{S}$ *is a* $J \times \sigma$ *matrix representing the set of possible pure strategies of selecting* a *jobs out of* J*, where* $\sigma = \binom{J}{a}$.

Here each column of $\mathbf{S}$ is a permutation of selecting a out of J, where $S_{js} = 1$ if strategy s involves an application to job j, and is 0 otherwise. In the above example where in strategy 1 workers apply to the first a firms, the first column of $\mathbf{S}$ would contain a 1's followed by $J-a$ 0's. Next,

A ***strategy realization*** $\mathbf{n} = (n_1, ..., n_\sigma)'$ *is a* $\sigma \times 1$ *vector of the number of workers selecting each strategy. The set of all possible strategy realizations* $\mathbf{n}$ *when the number of applicants is* I is denoted Λ_I.

Clearly $\sum_{s=1}^{\sigma} n_s = I$. By assumption, firms do not know the strategies chosen by the workers, nor the resulting strategy realization $\mathbf{n}$. The number of possible realizations $\mathbf{n} \in \Lambda_I$ is given by $|\Lambda_I| = \lambda_I = \binom{I+\sigma-1}{I}$.[5] Finally,

An ***application outcome*** $\boldsymbol{\alpha}(\mathbf{n}) = (\alpha_1(\mathbf{n}), ..., \alpha_J(\mathbf{n}))'$ *is a* $J \times 1$ *vector of the number of applications at each job, resulting from a given realization* $\mathbf{n} \in \Lambda_I$.

Then $\sum_{j=1}^{J} \alpha_j(\mathbf{n}) = aI$. Given $\mathbf{n}$, $\boldsymbol{\alpha}(\mathbf{n})$ is calculated by,

$$\boldsymbol{\alpha}(\mathbf{n}) = \mathbf{S}\mathbf{n} \tag{1}$$

where the strategy matrix $\mathbf{S} : \mathbb{R}^\sigma \mapsto \mathbb{R}^J$ maps each strategy realization $\mathbf{n} \in \Lambda_I$ onto a unique application outcome $\boldsymbol{\alpha}(\mathbf{n})$.

As will be discussed in Section 3, in equilibrium applicants randomize between all pure strategies with probabilities $\boldsymbol{\theta} = (\theta_1, ..., \theta_\sigma)'$. Then the probability that a realization takes a particular $\mathbf{n} \in \Lambda_I$ is,

$$\phi_{\mathbf{n}}(\boldsymbol{\theta}) = \frac{I!}{\prod_{s=1}^{\sigma} n_s!} \prod_{t=1}^{\sigma} \theta_t^{n_t} \tag{2}$$

where $\sum_{\mathbf{n} \in \Lambda_I} \phi_{\mathbf{n}}(\boldsymbol{\theta}) = 1$. This is then also the probability that an application outcome takes a particular $\boldsymbol{\alpha}(\mathbf{n})$, given by equation (1). Now in this paper's model, there are two channels through which the applicants' preferences of the jobs is incorporated in the matching process. The choice of $\boldsymbol{\theta}$ is the first of these, where the applicants' preference for firms offering higher wages and higher numbers of vacancies can be reflected by larger θ_s for the strategies that apply to those firms. The second channel is the rule that determines a worker's choice of which job to accept if he receives more than one job offers. The latter is discussed later.

[5] This is the multiset coefficient $\left(\binom{\sigma}{I}\right)$, i.e. the number of multisets of cardinality I, with elements taken from a finite set of cardinality σ.

As an example, take a simple case $(I, J; a) = (3, 3; 2)$. There are $\sigma = \binom{3}{2} = 3$ possible pure strategies in this case, represented by the following 3×3 strategy matrix,

$$\mathbf{S} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \tag{3}$$

For example, the first column represents strategy 1, where workers apply to firms 1 and 2. There are $\lambda_3 = \binom{3+3-1}{3} = 10$ possible realizations for the three strategies that form the set Λ_3,

$$\Lambda_3 = \left\{ \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \right\}$$

For example, the first vector represents a strategy realization where all three applicants choose strategy 1. The corresponding application outcomes are then calculated using equation (1),

$$\boldsymbol{\alpha}(\mathbf{n}) \in \left\{ \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} \right\}$$

In the first application outcome, firms 1 and 2 receive three applications each, while firm 3 receives none. This results from the strategy realization $\mathbf{n} = (3, 0, 0)'$. Finally, when all applicants randomize between the 3 strategies with probabilities $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)'$, the probability of this particular strategy realization, and hence its corresponding application outcome occurring, is $\phi_{(3,0,0)'}(\boldsymbol{\theta}) = \frac{3!}{3!0!0!}\theta_1^3\theta_2^0\theta_3^0 = \theta_1^3$, using equation (2).

Job Offer

In the job offer stage of the job-matching process, given a realized $\mathbf{n} \in \Lambda_I$ with its corresponding $\boldsymbol{\alpha}(\mathbf{n})$, firm j chooses randomly L_j workers from the $\alpha_j(\mathbf{n})$ applications received if $\alpha_j(\mathbf{n}) > L_j$, or offers jobs to all applicants if $\alpha_j(\mathbf{n}) \leq L_j$. A worker's probability of a job offer from firm j is then,

$$p_j(\mathbf{n}) = \begin{cases} \min\left[\frac{L_j}{\alpha_j(\mathbf{n})}, 1\right] & , \text{ if } \alpha_j(\mathbf{n}) > 0 \\ 0 & , \text{ if } \alpha_j(\mathbf{n}) = 0 \end{cases} \tag{4}$$

Now irrespective of whether he applies to the firms, an applicant has 2^J different possible job offer outcomes from the J firms, defined as,

*A **job offer outcome** $\mathbf{h} = (h_1, ..., h_J)'$ is a $J \times 1$ vector representing the job offers received by an applicant, where $h_j = 1$ if firm j offers the applicant a job, and 0 otherwise. The set of all possible $\mathbf{h}$ is denoted H, where $|H| = 2^J$.*

Then given a realization $\mathbf{n} \in \Lambda_I$, the probability of $\mathbf{h} \in H$ occurring is,

$$p(\mathbf{h};\mathbf{n}) = \prod_{j=1}^{J} p_j^{h_j}(\mathbf{n})\left\{1 - p_j(\mathbf{n})\right\}^{1-h_j} \tag{5}$$

In the example with $J = 3$ firms, there are $2^3 = 8$ possible job offer outcomes for any applicant, given by,

$$H = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \tag{6}$$

In the first vector representation, the applicant receives no job offer from any of the three firms. The probability of this happening depends on the number of vacancies and the number of applications received at each firm. If, for example, $\mathbf{L} = (1, 1, 2)'$ and $\boldsymbol{\alpha}(\mathbf{n}) = (3, 3, 0)'$, the probabilities of a job offer at each firm are $p_1 = p_2 = \frac{1}{3}$ and $p_3 = 0$; and hence,

$$p\left((0,0,0)';(3,0,0)'\right) = \left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{3}\right)(1-0) = \frac{4}{9}$$

Job Acceptance

Finally in the job acceptance stage, each applicant is assumed to have an acceptance rule that is determined by the applicant's preference of the jobs ex-post of the job offers. This is represented by a set of job acceptance probabilities for each possible job offer outcome, as set out below,

An ***acceptance rule*** $\mathbf{R}_s(\mathbf{h}) = (R_{s1}(\mathbf{h}), ..., R_{sJ}(\mathbf{h}))'$ *is a* $J \times 1$ *vector of job acceptance probabilities, where* $R_{sj}(\mathbf{h})$ *is the probability that an applicant employing strategy* s *accepts the job offer from* j*, given that such a job offer is received in the realized job offer outcome* $\mathbf{h}$.

The rule must satisfy,

1. $R_{sj}(\mathbf{h}) = 0 \ \forall j$ such that $S_{js}h_j = 0$.
2. $\sum_{j=1}^{J} R_{sj}(\mathbf{h}) = \begin{cases} 0 \text{ if } \sum_{j=1}^{J} S_{js}h_j = 0 \\ 1 \text{ otherwise} \end{cases}$.

The first criterion states that an applicant assigns a zero-job-acceptance probability to all firms to which he does not apply to, or that he applies to, but does not receive a job offer. The second criterion is the adding-up constraint for the probabilities, except for the case when there are no job offers from all applications, in which case the sum equals 0. This acceptance rule provides the second channel of incorporating applicants' preferences of the jobs in this model of match.

Here I give two examples of $\mathbf{R}_s(\mathbf{h})$. The first is when firms are *ex-post homogeneous*; that is, workers are indifferent between all jobs once job offers are received. In this case,

$$R^{\dagger}_{sj}(\mathbf{h}) = \frac{S_{js}h_j}{\sum_{k=1}^{J} S_{ks}h_k} \tag{7}$$

Note $S_{js}h_j = 1$ for the firms that an applicant employing strategy s applies to (i.e. $S_{js} = 1$) and receives a job offer from (i.e. $h_j = 1$). The denominator is therefore the number of job offers received by a strategy s applicant, for a particular job offer outcome $\mathbf{h}$. This case occurs when all firms offer uniform wages. On the other hand, when they offer non-uniform wages, the firms are *ex-post heterogeneous*. Then if the applicants have a preference ordering $1 \succ 2 \succ ... \succ J$ for firms $j \in \{1, ..., J\}$, the acceptance rule is, this time,

$$R^{\dagger\dagger}_{sj}(\mathbf{h}) = S_{js}h_j \prod_{k=1}^{j-1} (1 - R_{sk}(\mathbf{h})) \tag{8}$$

Therefore the smallest indexed (i.e. the highest preferred) firm with $S_{js}h_j = 1$ would have $R^{\dagger\dagger}_{sj}(\mathbf{h}) = 1$, which makes $R^{\dagger\dagger}_{sj}(\mathbf{h})$ for all subsequent firms equal to 0. For the rest of the paper, the ex-post homogeneous and ex-post heterogeneous cases are denoted by the same superscripts "†" and "††".

Consider again the example of $(I, J; a) = (3, 3; 2)$. An applicant with strategy s has an acceptance rule $\mathbf{R}_s(\mathbf{h})$ defined on all possible job-offer outcomes $\mathbf{h} \in H$ in (6). Take a case where an applicant selects strategy 1 in the strategy matrix (3), where he applies to firms 1 and 2. Then his acceptance rule when firms are ex-post homogeneous is, as defined by equation (7),

$$\mathbf{R}^{\dagger}_1(\mathbf{h}) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} \right\}$$

For example, if the applicant receives a job offer from firms 1 and 3 (i.e. $\mathbf{h} = (1, 0, 1)'$, the sixth case in (6)), then he would accept the job offer from firm 1 with probability 1, while if he receives a job offer from all three ($\mathbf{h} = (1, 1, 1)'$, the eighth case) he would accept firms 1 and 2 with probabilities $\frac{1}{2}$. If, on the other hand, the firms are ex-post heterogeneous and an applicant had a preference ordering $1 \succ 2 \succ ... \succ J$ for the firms, the acceptance rule for the strategy 1 applicant given by equation (8) is,

$$\mathbf{R}_1^{\dagger\dagger}(\mathbf{h}) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Therefore firm 1, if it offers a job, would always have its offer accepted, while firm 2's offer would only be accepted when firm 1 does not offer a job.

2.2 Firm-Level and Aggregate Matching Functions

Under the above set-up, the firm-level and the aggregate job-matching functions are now derived. First, given a strategy realization $\mathbf{n} \in \mathbf{\Lambda}_I$ of the I applicants, let $z_{js}(\mathbf{n})$ be the probability that an applicant employing strategy s accepts a job offer from firm j, given that j offers the applicant a job. This is given by,

$$z_{js}(\mathbf{n}) = \sum_{\mathbf{h} \in H} \frac{p(\mathbf{h};\mathbf{n})}{p_j(\mathbf{n})} R_{sj}(\mathbf{h}) \tag{9}$$

This probability clearly depends on the applicants' ex-post preferences of the firms. For example, for the two cases considered above in equations (7) and (8), $z_{js}(\mathbf{n})$ is, respectively,

$$z_{js}^{\dagger}(\mathbf{n}) = S_{js} \sum_{i=0}^{a-1} \frac{(-1)^i}{(i+1)!} \underbrace{\sum_{k=1,k\neq j}^{J} \cdots \sum_{r=1,r\neq j,k,l,...}^{J}}_{i \text{ summations}} S_{ks}p_k(\mathbf{n})...S_{rs}p_r(\mathbf{n}) \tag{10}$$

$$z_{js}^{\dagger\dagger}(\mathbf{n}) = S_{js} \prod_{k=1}^{j-1} (1 - S_{ks}p_k(\mathbf{n})) \tag{11}$$

The proof of (10) is given in Appendix A. Equation (11) is the survival rate that an applicant with strategy s would not receive any job offers from firms of higher preference than j. Now given $z_{js}(\mathbf{n})$, the probability that there is a job match between firm j and a strategy s applicant is $z_{js}(\mathbf{n})p_j(\mathbf{n})$. Therefore, a firm's expected number of matches is calculated by the following *firm-level matching function*, when I workers make a applications to firms offering $\mathbf{L} = (L_1, ..., L_J)'$ vacancies,

$$m_j(I, \mathbf{L}; a) = \sum_{\mathbf{n} \in \Lambda_I} \phi_{\mathbf{n}}(\boldsymbol{\theta}) \sum_{s=1}^{\sigma} n_s z_{js}(\mathbf{n}) p_j(\mathbf{n}) \tag{12}$$

where again $\phi_{\mathbf{n}}(\boldsymbol{\theta})$ is the probability that the strategy realization is $\mathbf{n} \in \Lambda_I$ when applicants randomize between pure strategies with probabilities $\boldsymbol{\theta}$, given by equation (2). Section 3 analyzes the choice of $\boldsymbol{\theta}$; for now $\boldsymbol{\theta}$ is taken as given, and is suppressed for notational brevity in the argument of $m_j(.)$, or $m(.)$ and $\Psi(.)$ below. The job market's aggregate match level is then (12) summed over all firms $j = 1, ..., J$,

$$m(I, \mathbf{L}; a) = \sum_{j=1}^{J} m_j(I, \mathbf{L}; a) = \sum_{\mathbf{n} \in \Lambda_I} \phi_{\mathbf{n}}(\boldsymbol{\theta}) \sum_{s=1}^{\sigma} n_s \sum_{\mathbf{h} \in H} p(\mathbf{h}; \mathbf{n}) \sum_{j=1}^{J} R_{sj}(\mathbf{h}) \quad (13)$$

Now, recall the second criterion for $\mathbf{R}_s(\mathbf{h})$, which stated that when the job offer outcome is $\mathbf{h}$, $\sum_{j=1}^{J} R_{sj}(\mathbf{h}) = 0$ if no job offer is received from all applications, and is 1 otherwise. The term $\sum_{\mathbf{h} \in H} p(\mathbf{h}; \mathbf{n}) \sum_{j=1}^{J} R_{sj}(\mathbf{h})$ is therefore the probability that a strategy s applicant receives at least one job offer when the strategy realization is $\mathbf{n}$, which equals $1 - \prod_{j=1}^{J} (1 - S_{js} p_j(\mathbf{n}))$. Substituting this in (13) yields the *aggregate matching function*,

$$m(I, \mathbf{L}; a) = I \left[1 - \Psi(I, \mathbf{L}; a)\right] \quad (14)$$

where $\Psi(I, \mathbf{L}; a)$ is the probability that an applicant receives no job offer from all his applications,

$$\Psi(I, \mathbf{L}; a) = \sum_{\mathbf{n} \in \Lambda_I} \phi_{\mathbf{n}}(\boldsymbol{\theta}) \sum_{s=1}^{\sigma} \frac{n_s}{I} \prod_{j=1}^{J} (1 - S_{js} p_j(\mathbf{n})) \quad (15)$$

The level of unemployment is then given by $I\Psi(I, \mathbf{L}; a)$. Now crucially equations (14) and (15) are independent of $R_{sj}(\mathbf{h})$. Hence,

Proposition 1 (Independence) *The aggregate match level is independent of the applicants' ex-post preferences of the firms.*

As already explained, firms are ex-post heterogeneous when applicants have non-uniform preferences of the firms once job offers are received. The acceptance rule $\mathbf{R}_s(\mathbf{h})$ describes these preferences. The aggregate matching function's independence of $\mathbf{R}_s(\mathbf{h})$ implies that the *level* of aggregate match is already determined at the point when job offers are made, irrespective of which offer each worker chooses to accept. The acceptance rule then determines the *distribution* of the matches, given by equation (12), which depends on $\mathbf{R}_s(\mathbf{h})$.

This means that in the model of job match considered, search frictions are caused by coordination failures in the job-application and the job-offer stages of the job-matching process. In expression (13), these are the coordination failures when selecting $\mathbf{n} \in \Lambda_I$ and $\mathbf{h} \in H$ in the respective summations. Albrecht, Gautier and Vroman (2006) term these as the *urn-ball* and the *multiple-applications frictions.* The former refers to the coordination failure where some vacancies receive no application while others receive more than one, as captured in the original urn-ball models such as Pissarides (1979). The latter refers to the coordination failure where some workers receive multiple offers while others receive none. Proposition 1 states that the additional consideration of workers' ex-post preferences does not add to the job-matching market's search friction, but it affects the distributional outcome of the matches.

Note that the independence result does not imply that the levels of aggregate matches are the same when firms offer uniform or non-uniform wages. This is because the wages affect the applicants' ex-ante preferences of the firms, as well as their ex-post preferences. In the model, this leads to a different choice of $\boldsymbol{\theta}$, and thus a different $m(I, \mathbf{L}; a)$. The proposition states the equivalence of $m(I, \mathbf{L}; a)$ when $\boldsymbol{\theta}$ is the same, independent of the applicants' choice of $\mathbf{R}_s(\mathbf{h})$.

2.2.1 Example

To demonstrate this result, return to the example of $(I, J; a) = (3, 3; 2)$ with labor demands $\mathbf{L} = (1, 1, 2)'$. Take a case now where applicants $i = \{1, 2, 3\}$ choose respectively strategies $s = \{2, 3, 3\}$, where strategy 2 applies to firms 1 and 3 and strategy 3 applies to firms 2 and 3, as represented by the strategy matrix (3). The strategy realization is then $\mathbf{n} = (0, 1, 2)'$. This results in the application outcome $\boldsymbol{\alpha} = (1, 2, 3)'$, where firm 1 receives an application from applicant 1, firm 2 receives applications from applicants 2 and 3, and firm 3 receives applications from all three. For $\mathbf{L} = (1, 1, 2)'$, there are six possible outcomes when firms select their candidates, which from the point of view of the workers are,

$$\begin{bmatrix} 1,3 \\ 2,3 \\ - \end{bmatrix}, \begin{bmatrix} 1,3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2,3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1,3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1,3 \\ - \\ 2,3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2,3 \end{bmatrix} \tag{16}$$

Here, each outcome representation portrays the firms from which the workers receive a job offer. For example in the first outcome, applicant 1 receives job offers from firms 1 and 3. The number of aggregate matches is already determined at this point prior to the workers' decisions of which job to accept, which for each respective outcome in (16) are $\{2, 3, 3, 3, 2, 3\}$. Therefore which job offer each worker chooses to accept in the end has no effect on the number of aggregate matches. It does, however, affect the final distribution of the matches. For example, if the firms were ex-post heterogeneous and the applicants' acceptance rule was given by equation (8) (i.e. the preference ordering of $1 \succ 2 \succ 3$), then in the first outcome in (16) the two filled jobs will be at firms 1 and 2. If, on the other hand, the firms were ex-post homogeneous (i.e. $\mathbf{R}_s(\mathbf{h})$ given by equation (7)), then the possible distributions of the two matches among the three firms in the first outcome are $(1,1,0)'$ (i.e. firms 1 and 2 have one filled job each), $(1,0,1)'$, $(0,1,1)'$ and $(0,0,2)'$ (i.e. both filled vacancies are at firm 3).

2.2.2 Literature Comparison

As a matching function for directed search with multiple-vacancies firms and multiple-applications workers, (14) nests many of the existing matching functions in the literature. For example in Albrecht et al. (2004) and Hori (2007), workers making multiple applications are randomly matched to firms with a single vacancy. Then by setting $\theta_s = \frac{1}{\sigma}$ and $L_j = 1$ in equation (15), the workers' probability of no job offer is,

$$\Psi(I, J; a) = \frac{1}{\sigma^{I-1}} \sum_{\mathbf{n} \in \Lambda_I} \frac{(I-1)!}{(n_1 - 1)! \prod_{s=2}^{\sigma} n_s!} \prod_{j=1}^{a} (1 - p_j(\mathbf{n})) \tag{17}$$

where $p_j(\mathbf{n}) = \frac{1}{\alpha_j(\mathbf{n})}$ if $\alpha_j(\mathbf{n}) > 0$, and 0 otherwise.[6] Substituting this in (14) yields the matching function derived in Hori (2007). Hori further demonstrates that in taking the limit $I, J \to \infty$ and $\frac{J}{I} \to \mu < \infty$, equation (17) yields the same limiting result as that derived by Albrecht et al. (2004),

$$\Psi(I, J; a) = \left\{1 - \frac{\mu}{a}\left(1 - e^{-\frac{a}{\mu}}\right)\right\}^a \tag{18}$$

The limiting case of (15) is formally discussed in Section 2.3.

More traditional matching functions are derived in an urn-ball set-up where workers apply to one firm, and firms randomly select one candidate for their single vacancy. This is a special case of (17) when $a = 1$,

$$\begin{aligned}
\Psi(I, J; 1) &= \sum_{n_1=1}^{I} \binom{I-1}{n_1 - 1} \left(\frac{1}{J}\right)^{n_1 - 1} \left(1 - \frac{1}{J}\right)^{I - n_1} \left(1 - \frac{1}{n_1}\right) \\
&= 1 - \frac{J}{I}\left\{1 - \left(1 - \frac{1}{J}\right)^I\right\}
\end{aligned}$$

[6] This is derived using symmetry, by selecting strategy 1 that applies to the first a jobs as the representative applicant.

leading to the matching function derived in Pissarides (1979),

$$m(I,J;1)=J\left\{1-\left(1-\frac{1}{J}\right)^{I}\right\} \tag{19}$$

Petrongolo and Pissarides (2001) further note that for large J this becomes,

$$m(I,J;1)=J\left(1-e^{-\frac{I}{J}}\right) \tag{20}$$

which is simply equation (18) substituted in function (14) when $a=1$.[7] As already noted these functions capture the urn-ball friction of the job-matching market. Julien, Kennes and King (2000) on the other hand consider the case where firms choose one applicant to make one job offer, and workers with multiple offers auction their employment. In the derived model here, this is the case $L_j=1\ \forall j$ and $a=J$, which when applied to equation (17) yields,

$$m(I,J;J)=I\left\{1-\left(1-\frac{1}{I}\right)^{J}\right\} \tag{21}$$

This is the matching function derived by Julien, Kennes and King which captures the multiple-application friction. These are all random-matching models. In contrast in the literature, there have not been simple closed-form formulations of directed search matching. The derived function (14) is one such. Then for example, the directed search version of the Pissarides' (1979) single-vacancy, single-application function (19) can be derived by letting $L_j=1\ \forall j$ and $a=1$ in equation (15),

$$m(I,J;1)=J\left\{1-\frac{1}{J}\sum_{j=1}^{J}(1-\theta_j)^{I}\right\} \tag{22}$$

where $\boldsymbol{\theta}=\{\theta_1,...,\theta_J\}$ is the mixed-strategy probabilities of choosing firms $j=1,...,J$.

2.3 Limiting Case

Consider now the limit as $I,J\to\infty$ while $\frac{V}{I}\to\mu<\infty$, where $V=\sum_{j=1}^{J}L_j$. The number of vacancies L_j at each firm is assumed to remain relatively small,

[7] Blanchard and Diamond (1994) also uses this limiting form of matching function, with an additional exogenous parameter in the exponent representing the acceptable application probability.

as would be under competition. As argued by Albrecht, Gautier and Vroman (2003), the number of applications α_j at each firm can now be treated as being independent. In this case equation (15) becomes,

$$\lim_{I,J\to\infty} \Psi = \sum_{s=1}^{\sigma} \theta_s \prod_{j=1}^{J} \left\{ 1 - S_{js} \sum_{\alpha=1}^{I} \min \left[\frac{L_j}{\alpha}, 1 \right] \text{prob} \left[\alpha_j = \alpha | \; \alpha_j \geq 1 \right] \right\} \tag{23}$$

where, as the probability of any applicant applying to firm j is $\sum_{s=1}^{\sigma} S_{js}\theta_s$,

$$\text{prob} \left[\alpha_j = \alpha | \; \alpha_j \geq 1 \right] = \binom{I-1}{\alpha-1} \left(\sum_{s=1}^{\sigma} S_{js}\theta_s \right)^{\alpha-1} \left(1 - \sum_{s=1}^{\sigma} S_{js}\theta_s \right)^{I-\alpha}$$

Ignoring the minimum function in (23) in the limit,

$$\lim_{I,J\to\infty} \Psi = \sum_{s=1}^{\sigma} \theta_s \prod_{j=1}^{J} \left[1 - S_{js} \frac{L_j}{E\alpha_j} \left(1 - e^{-E\alpha_j} \right) \right] \tag{24}$$

where $E\alpha_j = I \sum_{s=1}^{\sigma} S_{js}\theta_s$ is the expected number of applicants at firm j given $\boldsymbol{\theta}$. Equation (18) is the symmetric case $L_j = 1$ and $\theta_s = \frac{1}{\sigma}$ of this. This approximation is useful, as in comparison to (15), equation (24) reduces the computation required for Ψ by a factor of λ_I, which for example for $(I, J; a) = (12, 6; 2)$ is $9,657,700$. Table 1 shows the simulated results for $(I, J; 2)$, which suggests that the approximated probability (24) rapidly converges to its true value (15).[8]

	$I = 3$	4	5	6	7	8	9	10	11	12
$J = 3$	0.253 0.322	0.242 0.274	0.243 0.262	0.237 0.246	0.227 0.231	0.234 0.238	0.227 0.230	0.220 0.221	0.227 0.229	0.221 0.223
4	–	0.270 0.322	0.259 0.285	0.254 0.269	0.247 0.254	0.244 0.247	0.249 0.254	0.243 0.245	0.233 0.232	0.242 0.244
5	–	–	0.281 0.322	0.270 0.292	0.264 0.276	0.257 0.261	0.256 0.258	0.247 0.244	0.253 0.253	0.246 0.243
6	–	–	–	0.288 0.322	0.278 0.296	0.272 0.281	0.267 0.272	0.262 0.262	0.260 0.259	0.257 0.255

< **Table 1**: $\Psi(I, \mathbf{L}; 2)$ computed using (15) (top) and (24) (bottom) >

[8]Here for each (I, J), L_j is set to be a decreasing step function to the nearest integer such that $\sum_{j=1}^{J} L_j = I$ (i.e. $\mu = 1$) and $L_J = 1$. For example for $(I, J) = (6, 3)$, $\mathbf{L} = (3, 2, 1)$. The mixed-strategy $\boldsymbol{\theta}$ is set using the formula $\theta_s = \sum_{j=1}^{J} S_{js} L_j \Big/ \sigma_1 I$, where $\sigma_1 = \sum_{s=1}^{\sigma} S_{js} = \binom{J-1}{a-1}$. Thus θ_s is higher for types that apply to firms with larger L_j.

3 Equilibrium Analysis

3.1 Factors Directing Search: Wages vs Number of Vacancies

Before proceeding to analyze the equilibrium outcomes of the directed search job-matching market established above, I note an important fundamental difference between wages and the number of vacancies as factors directing workers' searches. The number of vacancies at a firm, which given the number of applications at the firm determines its probability of a job offer, affects workers' ex-ante, but not ex-post, preferences of the firms. This is because once job offers are received, the number of vacancies is no longer a relevant factor in the worker's choice of a firm. On the other hand, wages affect both the ex-ante and ex-post preferences of the applicants. This seems to suggests that if, as shown below, heterogeneity in the number of vacancies offered at the firms leads to inefficiency in the matching outcome (where efficiency is measured by the level of successful aggregate matches), then wages may not be an optimal policy tool to use to resolve the inefficiency. This is because while counteracting the inefficiency effects of heterogeneous $\mathbf{L}$ in job application, such a wage policy could have an unintended effect on the applicants' choices in job acceptance. Crucially, however, Proposition 1 implies that this is not the case, so far as matching efficiency is concerned. This is now formally discussed.

3.2 Efficient Equilibrium

As Galenianos and Kircher (2007, p.18) state, in directed search "the standard notion of efficiency is that of constrained efficiency." For example in Montgomery (1991, p.173), a social planner is "constrained to operate within the existing institutional structure," and maximizes the expected output of the economy, which is calculated as the product of a firm's value of output and its probability of successfully filling the vacancy, aggregated over all firms. In this paper each matched job produces identical output. Therefore the outcome is constrained efficient, when the number of matches is maximized conditional on the matching frictions given I and $\mathbf{L}$. The efficiency concerned is thus that of "matching efficiency".[9] Intuitively, the planner's objective here is to minimize frictional unemployment.

[9] This differs from those models that investigate efficient level of labor market tightness, such as Moen (1997) and Albrecht, Gautier and Vroman (2006). A further discussion on the exogeneity assumption of $\mathbf{L}$ is given in Section 4.

The social planner thus chooses symmetric equilibrium strategy $\boldsymbol{\theta}^*$ that maximizes expression (14), or equivalently that minimizes (15), subject to the conditions $\sum_{s=1}^{\sigma}\theta_s^* = 1$ and $\boldsymbol{\theta}^* \geq \mathbf{0}$, given $(I, \mathbf{L}; a)$. In this section, for notational brevity, the arguments $(I, \mathbf{L}; a)$ are now suppressed. Then,

Proposition 2 (Efficient Coordination) *The job application coordination is efficient when applicants choose strategy probabilities* $\boldsymbol{\theta}^* \geq \mathbf{0}$, *such that,*

$$\bar{\Psi}_s(\boldsymbol{\theta}^*) - \Psi(\boldsymbol{\theta}^*) \geq 0 \text{ and } \theta_s^*\left[\bar{\Psi}_s(\boldsymbol{\theta}^*) - \Psi(\boldsymbol{\theta}^*)\right] = 0 \ \forall s = 1, ..., \sigma \tag{25}$$

where $\Psi(\boldsymbol{\theta})$ *is given by (15), and* $\bar{\Psi}_s(\boldsymbol{\theta})$ *is the average probability of no job offer when an applicant chooses strategy* s,

$$\bar{\Psi}_s(\boldsymbol{\theta}) = \sum_{\mathbf{n}\in\Lambda_{I-1}} \phi_{\mathbf{n}}(\boldsymbol{\theta}) \sum_{t=1}^{\sigma} \frac{n_t + \chi_t^s}{I} \prod_{j=1}^{J} (1 - S_{jt}p_j(\mathbf{n} + \boldsymbol{\chi}^s)) \tag{26}$$

and $\boldsymbol{\chi}^s$ *is a* $\sigma \times 1$ *vector with all elements* $\chi_t^s = 0$ *except* $\chi_s^s = 1$.

Proof. Minimize expression (15) with respect to $\boldsymbol{\theta}$, subject to the adding-up condition $\sum_{s=1}^{\sigma}\theta_s = 1$,

$$\min_{\boldsymbol{\theta},\eta} \ \pounds(\boldsymbol{\theta}, \eta) = \sum_{\mathbf{n}\in\Lambda_I} \phi_{\mathbf{n}}(\boldsymbol{\theta}) \sum_{s=1}^{\sigma} \frac{n_s}{I} \prod_{j=1}^{J} (1 - S_{js}p_j(\mathbf{n})) + \eta\left(1 - \sum_{s=1}^{\sigma}\theta_s\right) \tag{27}$$

where η is a Lagrangian multiplier and $\phi_{\mathbf{n}}(\boldsymbol{\theta})$ is given by equation (2). With the constraint $\boldsymbol{\theta}^* \geq \mathbf{0}$ the first-order conditions are,

$$I\bar{\Psi}_s(\boldsymbol{\theta}^*) - \eta \geq 0 \text{ and } \theta_s^*\left[I\bar{\Psi}_s(\boldsymbol{\theta}^*) - \eta\right] = 0 \ \forall s = 1, ..., \sigma \tag{28}$$

$$\text{where } \bar{\Psi}_s(\boldsymbol{\theta}) = \sum_{\mathbf{n}\in\Lambda_I} \frac{n_s}{I\theta_s}\phi_{\mathbf{n}}(\boldsymbol{\theta}) \sum_{t=1}^{\sigma} \frac{n_t}{I} \prod_{j=1}^{J} (1 - S_{jt}p_j(\mathbf{n})) \tag{29}$$

Equation (29) is the average probability of no job offer given that an applicant chooses strategy s, where $\frac{n_s}{I\theta_s}\phi_{\mathbf{n}}(\boldsymbol{\theta}) = \frac{(I-1)!}{(n_s-1)!\prod_{t=1,t\neq s}^{\sigma} n_t!} \prod_{u=1,u\neq s}^{\sigma} \theta_u^{n_u}\theta_s^{n_s-1}$ is the probability of the strategy realization $\mathbf{n} \in \Lambda_I$ for I workers, given that $n_s \geq 1$. This is equivalent to the probability $\phi_{\mathbf{n}}(\boldsymbol{\theta})$ of the strategy realization $\mathbf{n} \in \Lambda_{I-1}$ for $I-1$ workers. Rewriting equation (29) for $\mathbf{n} \in \Lambda_{I-1}$ is then (26). Condition (25) follows from (28) as $\Psi(\boldsymbol{\theta}) = \sum_{s=1}^{\sigma}\theta_s\bar{\Psi}_s(\boldsymbol{\theta}) = \frac{\eta}{I}$ at $\boldsymbol{\theta}^*$. Finally checking the second-order condition,

$$\frac{\partial^2\pounds}{\partial\theta_s^2} = \sum_{\mathbf{n}\in\Lambda_I}\left(\frac{n_s^2 - n_s}{\theta_s^2}\right)\phi_{\mathbf{n}}(\boldsymbol{\theta}) \sum_{t=1}^{\sigma} \frac{n_t}{I} \prod_{j=1}^{J} (1 - S_{jt}p_j(\mathbf{n})) \tag{30}$$

This is strictly greater than zero for $I > 1$, as $n_s^2 - n_s \geq 0$ for all non-negative integers n_s, and strictly so for $n_s \geq 2$. Hence $\pounds(\boldsymbol{\theta}, \eta)$ is a strictly convex function in $\boldsymbol{\theta}$, implying that the solution $\boldsymbol{\theta}^*$ is the global minimum. ■

Intuitively if $\bar{\Psi}_t < \bar{\Psi}_r$ for $t \neq r$, then decreasing the probability θ_r of choosing strategy r and increasing θ_t of choosing t, such that $d\theta_t = -d\theta_r > 0$, decreases the overall Ψ as,

$$d\Psi(\boldsymbol{\theta}) = \sum_{s=1}^{\sigma} \frac{\partial \Psi}{\partial \theta_s} d\theta_s = I\left(\bar{\Psi}_t - \bar{\Psi}_r\right) d\theta_t < 0$$

Therefore, higher coordination efficiency is achieved by shifting applicants out of r and into t until $\bar{\Psi}_t$ and $\bar{\Psi}_r$ equate, subject to the condition $\theta_r \geq 0$. Thus the socially optimal aggregate match level is achieved, when the average probability of no job offer if an applicant chooses strategy s with a positive probability equates for all s.

The result in Proposition 2 can be further refined for the case $a = 1$. The problem is then simplified to that of each worker choosing a strategy $j \in \{1, ..., J\}$ of applying to firm j. Appendix B gives a full analysis for this case. In reference to the socially optimal outcome $\boldsymbol{\theta}^*$, it is shown that, (i) $\theta_j^* > 0 \; \forall j$ (i.e. the socially optimal equilibrium is an interior solution), and (ii) if $L_j > L_k$, then $\theta_j^* > \theta_k^*$ (i.e. applicants always apply to higher labor demand firms with higher probabilities). These are not necessarily true for the case $a > 1$. Take for example the case $(I, J; a) = (5, 3; 2)$ with $\mathbf{L} = (3, 2, 1)'$, and the strategy matrix given in (3). The efficient equilibrium for this example is $\boldsymbol{\theta}^* = (0, 0.61, 0.39)'$ (i.e. it is socially optimal for workers not to choose strategy 1 that applies to the two firms with high labor demands). This is because for relatively large I, the probability of a job offer at firms with low labor demand becomes small, such that the optimization problem becomes that of finding efficient matching amongst the high labor demand firms. In the example then, the efficient allocation of the applicants at firms 1 and 2 is attained by using solely strategies 2 and 3, yielding a corner solution $\theta_1^* = 0$ for strategy 1.

3.3 Equilibrium with Uniform Wage

I now proceed to consider the symmetric Nash equilibrium of the job-matching market. In particular, I first consider the case where firms offer uniform wage. The number of vacancies $\mathbf{L}$ is, on the other hand, assumed to be non-uniform. Therefore the firms are ex-ante heterogeneous but ex-post homogeneous. An applicant's objective in a Nash equilibrium is to minimize his probability of no job offer, given the actions of others. Then,

Proposition 3 *When* $\mathbf{L}$ *is non-uniform but* $\mathbf{w}$ *is uniform, the symmetric Nash equilibrium* $\boldsymbol{\theta}^{\dagger}$ *is inefficient. Moreover for* $a=1$*, the Nash equilibrium exhibits over-crowding at the firm with the highest labor demand.*

Proof. In a job application game, given that all other applicants choose the mixed strategy $\boldsymbol{\theta}'$, an applicant chooses $\boldsymbol{\theta}$ to minimize the probability $\Psi(\boldsymbol{\theta};\boldsymbol{\theta}')$ of receiving no job offer given his choice of strategy, subject to the condition $\sum_{s=1}^{\sigma}\theta_s=1$ and $\boldsymbol{\theta}\geq\mathbf{0}$, where,

$$\Psi(\boldsymbol{\theta};\boldsymbol{\theta}')=\sum_{s=1}^{\sigma}\theta_s\sum_{\mathbf{n}\in\Lambda_{I-1}}\phi_{\mathbf{n}}(\boldsymbol{\theta}')\prod_{j=1}^{J}(1-S_{js}p_j(\mathbf{n}+\boldsymbol{\chi}^s)) \tag{31}$$

$\boldsymbol{\chi}^s$ is again a $\sigma\times 1$ vector with all elements $\chi_t^s=0$ except $\chi_s^s=1$. By the same argument as in Proposition 2, the symmetric solution $\boldsymbol{\theta}^{\dagger}$ when firms are ex-post homogeneous is given by

$$\Psi_s(\boldsymbol{\theta}^{\dagger})-\Psi(\boldsymbol{\theta}^{\dagger})\geq 0 \text{ and } \theta_s^{\dagger}\left[\Psi_s(\boldsymbol{\theta}^{\dagger})-\Psi(\boldsymbol{\theta}^{\dagger})\right]=0\ \forall s=1,...,\sigma \tag{32}$$

$$\text{where } \Psi_s(\boldsymbol{\theta})=\sum_{\mathbf{n}\in\Lambda_{I-1}}\phi_{\mathbf{n}}(\boldsymbol{\theta})\prod_{j=1}^{J}(1-S_{js}p_j(\mathbf{n}+\boldsymbol{\chi}^s)) \tag{33}$$

Equation (33) is the applicant's probability of no job offer given that he chooses strategy s, when all workers choose $\boldsymbol{\theta}$. Contrast this with $\bar{\Psi}_s(\boldsymbol{\theta})$ in (26), where $\bar{\Psi}_s(\boldsymbol{\theta})$ is the *average* over all applicants of the probability of no job offer given that the applicant chooses strategy s. It immediately follows then from a comparison with the condition (25), that the symmetric Nash equilibrium is inefficient. The proof for over-crowding at the firm with the highest labor demand for the case $a=1$ is given in Appendix B. ■

As with the efficient equilibrium, Appendix B further shows that when $a=1$, (i) the Nash equilibrium is also an interior solution, and (ii) that in the Nash equilibrium workers always apply to firms with higher labor demand with higher probabilities.

The reason for the matching inefficiency in the Nash equilibrium is that when an applicant chooses his optimal application strategy, he ignores the negative externality of his action on other applicants' probabilities of matching success. To see this, decompose $\bar{\Psi}_s(\boldsymbol{\theta})$ in (26) into two parts,

$$\bar{\Psi}_s(\boldsymbol{\theta})=\frac{1}{I}\Psi_s(\boldsymbol{\theta})+\frac{I-1}{I}\sum_{\mathbf{n}\in\Lambda_{I-1}}\phi_{\mathbf{n}}(\boldsymbol{\theta})\sum_{t=1}^{\sigma}\frac{n_t}{I-1}\prod_{j=1}^{J}(1-S_{jt}p_j(\mathbf{n}+\boldsymbol{\chi}^s)) \tag{34}$$

Now compare the second term with the average probability of no job offer in the expression (15) when there are $I-1$ applicants,

$$\Psi(I-1,\mathbf{L};a)=\sum_{\mathbf{n}\in\Lambda_{I-1}}\phi_{\mathbf{n}}\left(\boldsymbol{\theta}\right)\sum_{s=1}^{\sigma}\frac{n_s}{I-1}\prod_{j=1}^{J}\left(1-S_{js}p_j(\mathbf{n})\right) \tag{35}$$

Then, by joining in the labor market and choosing strategy s, an applicant adds his probability of no job offer to the average (the first term in equation (34)), but also makes the existing applicants' probabilities worse by increasing competition at those firms that he applies to (i.e. $p_j(\mathbf{n}+\boldsymbol{\chi}^s)\leq p_j(\mathbf{n})$ in equations (34) and (35)). Ignoring this negative externality, in Nash equilibrium, applicants over-apply to firms offering a higher number of vacancies.

In contrast, if the number of vacancies were also uniform,

Corollary 1 *For uniform labor demand, the Nash equilibrium outcome is efficient.*

Proof. By symmetry, $\theta_s^*=\theta_s^\dagger=\frac{1}{\sigma}\ \forall s=1,...,\sigma$ are the solutions in conditions (25) and (32) that make $\bar{\Psi}_s=\Psi_s=\Psi$. ■

In this case the firms are identical; and therefore the matching outcome is that of random search. The point of the analysis so far is then that when workers have strict preference ordering of the firms, such that they apply to some firms with higher intensity than others, in equilibrium there will be an inefficient aggregate level of matches due to the externality effect of the individual choices. Hence introduction of heterogeneity, be it in the levels of labor demand as here or otherwise, results in a suboptimal outcome. This agrees with the result attained by Galenianos and Kircher (2005). In their model firms with single vacancy offer non-uniform wages. Then the number of matches is maximized when workers apply to all firms with equal probability, but is lower when they apply to certain firms with higher intensity, which will be the case when workers have a preference for higher wages. The difference here is that in this section the heterogeneity is in the number of vacancies $\mathbf{L}$. Then the efficient coordination is attained by a non-uniform $\boldsymbol{\theta}^*$, defined by (25).

3.4 Equilibrium with Wage Dispersion

Now, consider the case where the firms offer both heterogeneous numbers of vacancies and non-uniform wages. If firms are otherwise identical, workers would always strictly prefer the firm offering the higher wage. For simplicity then, assume that the workers are risk-neutral and that their benefit from a job match with firm j is the wage level w_j. As an applicant's probability of matching with firm j when the realization outcome is $\mathbf{n} \in \Lambda_I$ is $z_{js}(\mathbf{n})p_j(\mathbf{n})$, where $z_{js}(\mathbf{n})$ was given in equation (9), the applicant's expected income when he chooses $\boldsymbol{\theta}$ while all other applicants choose $\boldsymbol{\theta}'$ is,

$$Ew(\boldsymbol{\theta}; \boldsymbol{\theta}') = \sum_{s=1}^{\sigma} \theta_s \sum_{\mathbf{n} \in \Lambda_{I-1}} \phi_{\mathbf{n}}(\boldsymbol{\theta}')\, z_{js}(\mathbf{n} + \boldsymbol{\chi}^s) p_j(\mathbf{n} + \boldsymbol{\chi}^s) w_j \tag{36}$$

In particular, if the wages are distinct and the firms are ordered in the descending order of the wages $w_1 > w_2 > ... > w_J$, then the workers' ex-post preferences are $1 \succ 2 \succ ... \succ J$ for which the expression for z_{js} was derived in (11). By the same argument as before then, the symmetric solution $\boldsymbol{\theta}^{\dagger\dagger}$ that maximizes this is given by,

$$Ew_s(\boldsymbol{\theta}^{\dagger\dagger}) - Ew(\boldsymbol{\theta}^{\dagger\dagger}) \leq 0 \text{ and } \theta_s^{\dagger\dagger}\left[Ew_s(\boldsymbol{\theta}^{\dagger\dagger}) - Ew(\boldsymbol{\theta}^{\dagger\dagger})\right] = 0 \;\forall s = 1, ..., \sigma \tag{37}$$

where $Ew_s(\boldsymbol{\theta})$ is an applicant's expected income given that he chooses strategy s,

$$Ew_s(\boldsymbol{\theta}) = \sum_{\mathbf{n} \in \Lambda_{I-1}} \phi_{\mathbf{n}}(\boldsymbol{\theta})\, z_{js}(\mathbf{n} + \boldsymbol{\chi}^s) p_j(\mathbf{n} + \boldsymbol{\chi}^s) w_j \tag{38}$$

The Nash equilibrium solution $\boldsymbol{\theta}^{\dagger\dagger}$ now depends on $\mathbf{w}$; the wage distribution $\mathbf{w}^{\dagger\dagger}$ that yields $\boldsymbol{\theta}^{\dagger\dagger} = \boldsymbol{\theta}^*$ attains efficient coordination in the job-matching market. One example was already given in Corollary 1: when labor demands $\mathbf{L}$ is uniform, then the uniform wage $w_1 = ... = w_J$ attains matching efficiency.

I now illustrate this using once again the simple example $(I, J; a) = (3, 3; 2)$. Let the labor demands this time be $\mathbf{L} = (1, 2, 2)'$. The strategy matrix $\mathbf{S}$ is again given in (3). Solving (25) and (32) yields the following efficient equilibrium and uniform wage Nash equilibrium solutions $\boldsymbol{\theta}^*$ and $\boldsymbol{\theta}^\dagger$, with their corresponding probabilities of no job offer $\Psi(\boldsymbol{\theta})$, aggregate match levels $m(I, \mathbf{L}; a)$ and the distributions of the matches $m_j(I, \mathbf{L}; a)$,

	$\boldsymbol{\theta}^*$ or $\boldsymbol{\theta}^\dagger$	$\Psi(\boldsymbol{\theta})$	$m(I, \mathbf{L}; a)$	$m_j(I, \mathbf{L}; a)$
Efficient eqm	$(0.25, 0.25, 0.50)'$	0.0417	2.875	$(0.536, 1.169, 1.169)'$
Nash eqm	$(0.18, 0.18, 0.64)'$	0.0456	2.863	$(0.465, 1.199, 1.199)'$

Thus, in this case, in the Nash equilibrium the applicants choose strategy 3 with a suboptimally high probability, leading to overcrowding at the two firms with higher labor demands.

Now, consider a wage distribution $\mathbf{w} = (w_1, 1, 1)$ where w_2 and w_3 are normalized to 1. Then $w_1^{\dagger\dagger}$ can be solved for which the solution $\boldsymbol{\theta}^{\dagger\dagger}$ to the condition (37) equals $\boldsymbol{\theta}^*$. This is found to be $w_1^{\dagger\dagger} = 1.21$, for which the corresponding results are,[10]

	$\boldsymbol{\theta}^{\dagger\dagger}$	$\Psi(\boldsymbol{\theta})$	$m(I, \mathbf{L}; a)$	$m_j(I, \mathbf{L}; a)$
Nash eqm	$(0.25, 0.25, 0.50)'$	0.0417	2.875	$(0.875, 1, 1)'$

With the use of heterogeneous wages as a rationing device, the matching efficiency is now attained in equilibrium. Note, however, that the distribution of the matches is shifted towards the firm offering the higher wage. This gives an example where, while introduction of one or more heterogeneities causes inefficient coordination in a matching market, a social planner may be able to design the market in such a way that the externality effects of the heterogeneities offset each other, increasing the overall matching efficiency. The above example is where wages are used to perfectly offset the externality effect of the heterogeneous labor demand. More specifically, by offering lower wages at the firms attracting suboptimally high number of applications, the externality associated with heterogeneous labor demands is fully internalized. Once again, given the difference between wages and the number of vacancies as factors directing search, as discussed in Section 3.1, this result relies crucially on the result established in Proposition 1 that the level of aggregate match is independent of the workers' preferences ex-post of job offers.[11]

As discussed in the introduction, some evidence of wage dispersion is surveyed in Mortensen (2003), who estimates that "observable worker characteristics that are supposed to account for productivity differences typically explain no more than 30 percent of the variation in compensation across workers" (p.1). Hornstein, Krussell and Violante (2006) also agrees with this estimation. Explanations offered for the remaining "frictional wage dispersion" include productivity heterogeneity of the firms, the reservation wage heterogeneity of the workers, and asymmetric information (i.e. workers have only partial information regarding wages offered). The above discussion suggests that, insofar

[10] In this case as firms 2 and 3 are identical, $z_{jt}(\mathbf{n})$ required is a hybrid of (10) and (11).

[11] In some sense this argument is analogous to that given with the Fundamental Theorems of Welfare Economics (FTWE). There, arguably, a social planner can simply force all agents to consume at the Pareto efficient outcome that is the most desirable to the planner, if such an outcome is known. Instead, the Second FTWE suggests that the planner can adjust the initial endowment through redistribution, and the desired Pareto efficient point is reached as a Walrasian equilibrium. Here, a planner can force all applicants to choose the efficient mixed strategy $\boldsymbol{\theta}^*$ in order to attain the maximum level of aggregate matches. The argument of this section is that, instead, the planner can use one of the tools - in this case the wages - as a rationing device so that the efficient outcome is achieved as a Nash equilibrium.

as wage allocation by social planners is allowed, there is a matching efficiency reason for having wage dispersion despite workers being homogeneous, when firms offer a heterogeneous number of vacancies.

4 Conclusions

Two main results are attained in the paper. First, in deriving the aggregate matching function that incorporates explicitly the workers' preferences for firms, it is shown that the aggregate level of matches is independent of the workers' preferences of jobs ex-post of job offers. However, the final distribution of the matches is affected by the preferences. Second, while heterogeneity in the number of vacancies at firms causes inefficiency in the job-matching outcome, it is argued that wages can be used as a rationing device to attain efficient coordination in equilibrium. This is despite the difference as factors directing search between wages and the number of vacancies, namely that the former affects workers' choices in both job application and job acceptance, while the latter only affects workers' job application. The second result relies crucially on the first. The results also give an alternative reason for wage dispersion.

There are short-comings in the model, which may be resolved with future research. For example, here it is assumed that social planners are able to implement the wage profile that leads to the efficient outcome. However, if wages are allowed to be determined endogenously by the firms, then efficient coordination would not be attained because firms with higher labor demand may raise wages to attract more applications, as predicted by Burdett, Shi and Wright (2001). The results of this paper suggests that this would further increase inefficiency. Similarly, in the paper the labor demands are assumed given exogenously. The situation envisaged here was, for example, where any employment contract has an exogenous probability of dissolution, and the firms are simply replacing their lost employees. Indeed the whole production side of the firm is (intentionally) absent. Endogenizing L_j will provide firms with a second rationing tool where they may strategically adjust L_j to attract more applicants. Allowing this would also permit a welfare analysis where one could investigate the efficiencies of the distribution of vacancies over firms, and the social efficiency of the equilibrium aggregate level of vacancies. The latter would be an extension to the constrained efficiency analyses of Moen (1997) and Albrecht, Gautier and Vroman (2006). Further, allowing firms to renegotiate wages post-application will provide the firms with an even larger strategy space. For example, depending on the realized number of applications, firms may wish to raise their wage offers in order to increase their matching probabilities. Knowing this possibility, firms may further strategically alter their ex-ante behavior. Finally, heterogeneity in jobs and workers may be introduced. The matching functions would then capture both market frictions induced by coordination failure and heterogeneity. The model here provides a concrete base for such extensions.

Appendix

A Ex-post Homogeneous Firms

For ex-post homogeneous firms, a strategy s worker who applies to j would accept j's offer with probability $\frac{1}{i+1}$ if the worker has i other offers. Then for a strategy realization $\mathbf{n}$, the probability of job acceptance by the strategy s worker is,

$$z_{js}^{\dagger}(\mathbf{n}) =$$

$$S_{js}\sum_{i=0}^{a-1}\frac{1}{(i+1)!}\underbrace{\sum_{k=1,k\neq j}^{J}\ldots\sum_{r=1,r\neq j,k,l,\ldots}^{J}}_{i\text{ summations}}S_{ks}p_k(\mathbf{n})\ldots S_{rs}p_r(\mathbf{n})\underbrace{\prod_{t=1,t\neq j,\ldots,r}^{J}\quad(1-S_{ts}p_t(\mathbf{n}))}_{J-1-i\text{ product sums}}$$

The term $\frac{1}{(i+1)!}$ reflects the $i!$ symmetries in the i summations, as well as the $\frac{1}{i+1}$ probability of job acceptance. Expanding the product-sum terms yields equation (10).

Consider now the following summation, noting that only a of S_{js}'s equal 1,

$$\begin{aligned}\sum_{j=1}^{J}z_{js}^{\dagger}(\mathbf{n})p_j(\mathbf{n}) &= \sum_{i=1}^{a}\frac{(-1)^{i-1}}{i!}\underbrace{\sum_{j=1}^{J}\sum_{k=1,k\neq j}^{J}\ldots\sum_{r=1,r\neq j,k,l,\ldots}^{J}}_{i\text{ summations}}S_{js}p_j(\mathbf{n})\ldots S_{rs}p_r(\mathbf{n}) \\ &= 1-\prod_{j=1}^{J}(1-S_{js}p_j(\mathbf{n}))\end{aligned}$$

Substituting this into $m_j(I,\mathbf{L};a)$ given by function (12), and summing up over $j=1,...,J$ also yields the aggregate matching function (14).

B Case $a = 1$ Analysis

When $a = 1$, the problem is simplified to that of each worker choosing one firm to apply to. The number of applicants at firm j is then $\alpha_j(\mathbf{n}) = n_j$.

First then, consider the efficient equilibrium established in Proposition 2 for this case. The average probability of no job offer when an applicant chooses firm j is now,

$$\bar{\Psi}_j(\boldsymbol{\theta}) = \sum_{n_1=0}^{N_{I-1,1}} \dots \sum_{n_{J-1}=0}^{N_{I-1,J-1}} \prod_{k=1}^{J} \binom{N_{I-1,k}}{n_k} \theta_k^{n_k} \sum_{h=1}^{J} \frac{n_h + \chi_h^j}{I} \left(1 - \left[\frac{L_h}{n_h + \chi_h^j}, 1\right]^-\right)$$

where $N_{I-1,j} = I - 1 - \sum_{l=1}^{j-1} n_l$, $n_J = N_{I-1,J}$, and $\chi_h^j = 1$ for $h = j$ and 0 otherwise. Here also, $[x, y]^- = \min[x, y]$. Using further the adding-up constraint $\sum_{j=0}^{I} \theta_j = 1$, this becomes,

$$\bar{\Psi}_j(\boldsymbol{\theta}) = 1 - \frac{1}{I} \sum_{k=1}^{J} \sum_{n_h=0}^{I-1} \binom{I-1}{n_h} \theta_h^{n_h} (1 - \theta_h)^{I-1-n_h} \left[L_h, n_h + \chi_h^j\right]^- \tag{39}$$

The second term is the average probability of a successful job application, calculated as the expected total number of job offers divided by the number of applicants I. This is so when $a = 1$ as in this case each job offer leads to a job match with certainty. Now, when a worker applies to a firm j, the total number of job offers is unchanged if the number of applications already received n_j is greater than or equal to L_j, while the number is increased by 1 if $n_j < L_j$. Hence the internal solution condition $\bar{\Psi}_j(\boldsymbol{\theta}^*) = \bar{\Psi}_k(\boldsymbol{\theta}^*)$ in (25) simplifies to,

$$\sum_{n_j=0}^{L_j-1} \binom{I-1}{n_j} \left(\theta_j^*\right)^{n_j} \left(1 - \theta_j^*\right)^{I-1-n_j} = \sum_{n_k=0}^{L_k-1} \binom{I-1}{n_k} \left(\theta_k^*\right)^{n_k} \left(1 - \theta_k^*\right)^{I-1-n_k} \tag{40}$$

That is, the probabilities that the number of applications already received is less than the number of advertised jobs are the same. The formal proof of this is given below.

For the uniform wage symmetric Nash equilibrium established in Proposition 3, again using the adding-up constraint $\sum_{j=0}^{I} \theta_j = 1$, the probability of no job offer for a strategy j applicant is now,

$$\Psi_j(\boldsymbol{\theta}) = 1 - \sum_{n_j=0}^{I-1} \binom{I-1}{n_j} \theta_j^{n_j} (1 - \theta_j)^{I-1-n_j} \left[\frac{L_j}{n_j + 1}, 1\right]^- \tag{41}$$

which is 1 minus the probability that the applicant's application is chosen by the firm.

Using these then, firstly,

Proposition 4 *For $a = 1$ and $L_j < I$ $\forall j$, both the efficient equilibrium and the uniform wage symmetric Nash equilibrium solutions $\boldsymbol{\theta}^*$ and $\boldsymbol{\theta}^\dagger$ are interior; that is, every firm receives a positive number of applications.*

Proof. For the efficient equilibrium solution to be interior in (25), I require that $\bar{\Psi}_j(\boldsymbol{\theta}) - \bar{\Psi}_k(\boldsymbol{\theta}) < 0$ whenever $\theta_j = 0$ and $\theta_k > 0$. Noting that in equation (39) the expressions for $\bar{\Psi}_j(\boldsymbol{\theta})$ and $\bar{\Psi}_k(\boldsymbol{\theta})$ differ only for $h = j$ and k in the h-summations,

$$\begin{aligned}
&\bar{\Psi}_j(\boldsymbol{\theta}) - \bar{\Psi}_k(\boldsymbol{\theta}) \\
= \;& \frac{1}{I}\sum_{n_j=0}^{I-1}\tbinom{I-1}{n_j}\theta_j^{n_j}(1-\theta_j)^{I-1-n_j}\left\{[L_j, n_j]^- - [L_j, n_j+1]^-\right\} \\
&+\frac{1}{I}\sum_{n_k=0}^{I-1}\tbinom{I-1}{n_k}\theta_k^{n_k}(1-\theta_k)^{I-1-n_k}\left\{[L_k, n_k+1]^- - [L_k, n_k]^-\right\} \\
= \;& -\frac{1}{I}\sum_{n_j=0}^{L_j-1}\tbinom{I-1}{n_j}\theta_j^{n_j}(1-\theta_j)^{I-1-n_j} + \frac{1}{I}\sum_{n_k=0}^{L_k-1}\tbinom{I-1}{n_k}\theta_k^{n_k}(1-\theta_k)^{I-1-n_k}
\end{aligned}$$

as $[L_j, n_j+1]^- - [L_j, n_j]^- = 1$ for $n_j < L_j$, and 0 for $n_j \geq L_j$. This formally proves the equality (40). Now when $\theta_j = 0$ the first term equals 1; and hence, at $\boldsymbol{\theta}$ such that $\theta_j = 0$ and $\theta_h > 0$,

$$\bar{\Psi}_j(\boldsymbol{\theta}) - \bar{\Psi}_k(\boldsymbol{\theta}) = -\frac{1}{I}\sum_{n_k=L_k}^{I-1}\tbinom{I-1}{n_k}(\theta_k)^{n_k}(1-\theta_k)^{I-1-n_k} < 0$$

as required.

Similarly, for the uniform wage Nash equilibrium, when $\theta_j = 0$, (41) is zero as either $\theta_j^{n_j} = 0$ when $n_j > 0$, or $\left[\frac{L_j}{n_j+1}, 1\right]^- = 1$ when $n_j = 0$. On the other hand, when $\theta_j > 0$, $L_j < I$ means that at least one term of $\left[\frac{L_j}{n_j+1}, 1\right]^-$ is less than 1, ensuring that $\Psi_j > 0$. Therefore, at $\boldsymbol{\theta}$ such that $\theta_j = 0$ and $\theta_k > 0$, $\Psi_j(\boldsymbol{\theta}) - \Psi_k(\boldsymbol{\theta}) < 0$, which does not satisfy (32). Thus, the Nash equilibrium solution must also be interior. ■

Next, regarding any two strategies j and k such that j applies to a higher labor demand firm than k,

Proposition 5 *For any j and k such that $1 \leq L_k < L_j < I$, in both the efficient equilibrium and the uniform wage symmetric Nash equilibrium the applicants apply to j with a higher probability than k; that is, $\theta_j^* > \theta_k^*$ and $\theta_j^\dagger > \theta_k^\dagger$.*

Proof. I prove these by contradiction. First, suppose that for $L_j > L_k$, $\theta_j^* \leq \theta_k^*$. As the solution must be interior, $\bar{\Psi}_j(\boldsymbol{\theta}^*) = \bar{\Psi}_k(\boldsymbol{\theta}^*)$. Now it follows from the equality (40) that the probabilities of the number of applications being more than the number of advertised jobs must also equate,

$$\sum_{n_j=L_j}^{I-1} \binom{I-1}{n_j} \left(\theta_j^*\right)^{n_j} \left(1-\theta_j^*\right)^{I-1-n_j} = \sum_{n_k=L_k}^{I-1} \binom{I-1}{n_k} \left(\theta_k^*\right)^{n_k} \left(1-\theta_k^*\right)^{I-1-n_k} \tag{42}$$

A couple of lines of algebra yields, for the following derivative,

$$\frac{\partial}{\partial \theta_k} \sum_{n_k=L_k}^{I-1} \binom{I-1}{n_k} \theta_k^{n_k} \left(1-\theta_k\right)^{I-1-n_k} = (I-1) \binom{I-2}{L_k-1} \theta_k^{L_k-1} \left(1-\theta_k\right)^{I-1-L_k} > 0$$

Therefore, in replacing θ_k^* with the weakly smaller θ_j^* in the right-hand side of (42),

$$\begin{aligned} \sum_{n_j=L_j}^{I-1} \binom{I-1}{n_j} \left(\theta_j^*\right)^{n_j} \left(1-\theta_j^*\right)^{I-1-n_j} &\geq \sum_{n_k=L_k}^{I-1} \binom{I-1}{n_k} \left(\theta_j^*\right)^{n_k} \left(1-\theta_j^*\right)^{I-1-n_k} \\ &> \sum_{n_k=L_j}^{I-1} \binom{I-1}{n_k} \left(\theta_j^*\right)^{n_k} \left(1-\theta_j^*\right)^{I-1-n_k} \end{aligned}$$

The second line follows as $L_j > L_k \geq 1$. This is a contradiction. Hence $\theta_j^* > \theta_k^*$.

Next suppose that for $L_j > L_k$, $\theta_j^\dagger \leq \theta_k^\dagger$. Again as the solution is always interior, $\Psi_j(\boldsymbol{\theta}^\dagger) = \Psi_k(\boldsymbol{\theta}^\dagger)$. This time in differentiating (41) with respect to θ_j,

$$\frac{\partial \Psi_j}{\partial \theta_j} = (I-1) \sum_{n_j=0}^{I-2} \binom{I-2}{n_j} \theta_j^{n_j} \left(1-\theta_j\right)^{I-n_j-2} \left(\left[\frac{L_j}{n_j+1}, 1\right]^- - \left[\frac{L_j}{n_j+2}, 1\right]^- \right) \tag{43}$$

This is strictly positive when $L_j < I$. Then in replacing $\theta_k^\dagger$ in $\Psi_k(\boldsymbol{\theta}^\dagger)$ with the weakly smaller $\theta_j^\dagger$,

$$\begin{aligned}\Psi_j(\boldsymbol{\theta}^\dagger) &\geq 1-\sum_{n_k=0}^{I-1}\tbinom{I-1}{n_k}(\theta_j^\dagger)^{n_k}(1-\theta_j^\dagger)^{I-1-n_k}\left[\frac{L_k}{n_k+1},1\right]^- \\ &> 1-\sum_{n_k=0}^{I-1}\tbinom{I-1}{n_k}(\theta_j^\dagger)^{n_k}(1-\theta_j^\dagger)^{I-1-n_k}\left[\frac{L_j}{n_k+1},1\right]^-\end{aligned}$$

which is a contradiction for $L_k < L_j < I$. Hence $\theta_j^\dagger > \theta_k^\dagger$. ■

Finally, let firm 1 be the firm with the highest labor demand. Then,

Proposition 6 *For firm* 1 *such that* $L_1 = \max\limits_{1\leq j\leq J}\{L_j\}$, $\theta_1^* < \theta_1^\dagger$.

Proof. I prove this by showing that at the efficient equilibrium $\boldsymbol{\theta}^*$, $\Psi_1(\boldsymbol{\theta}^*) < \Psi_j(\boldsymbol{\theta}^*)\ \forall j \neq 1$; hence θ_1 must be increased to reach the Nash equilibrium $\boldsymbol{\theta}^\dagger$. First rewrite equation (41) as,

$$\Psi_j(\boldsymbol{\theta}) = \sum_{n_j=L_j}^{I-1}\tbinom{I-1}{n_j}\theta_j^{n_j}\left(1-\theta_j\right)^{I-1-n_j} - \sum_{n_j=L_j}^{I-1}\tbinom{I-1}{n_j}\theta_j^{n_j}\left(1-\theta_j\right)^{I-1-n_j}\left(\frac{L_j}{n_j+1}\right)$$

Then, as at $\boldsymbol{\theta}^*$ equality (42) is true, $\Psi_1(\boldsymbol{\theta}^*) < \Psi_j(\boldsymbol{\theta}^*)$ if and only if,

$$\begin{aligned}&\sum_{n_1=L_1}^{I-1}\tbinom{I-1}{n_1}\left(\theta_1^*\right)^{n_1}\left(1-\theta_1^*\right)^{I-1-n_1}\left(\frac{L_1}{n_1+1}\right) \\ &> \sum_{n_j=L_j}^{I-1}\tbinom{I-1}{n_j}\left(\theta_j^*\right)^{n_j}\left(1-\theta_j^*\right)^{I-1-n_j}\left(\frac{L_j}{n_j+1}\right) \qquad (44)\end{aligned}$$

These terms represent the probabilities of a job offer at each firm, given that the total number of the applications n_k+1 is more than the number of advertised jobs L_k, $k \in \{1, j\}$. Now (42) ensures that at the efficient equilibrium, the cumulative probabilities of all outcomes n_k, such that $n_k+1 > L_k$, are the same at both firms. Moreover at each of these outcomes, the job offer probability $\frac{L_k}{n_k+1}$ is greater at firm 1 than at firm j; note in particular at $n_k = L_k$, $\frac{L_1}{L_1+1} > \frac{L_j}{L_j+1}$, and at $n_k = I-1$, $\frac{L_1}{I} > \frac{L_j}{I}$.[12] Therefore (44) is true, implying that $\Psi_1(\boldsymbol{\theta}^*) < \Psi_j(\boldsymbol{\theta}^*)$. Finally, we know from (43) that $\frac{\partial\Psi_k}{\partial\theta_k} > 0$ when

[12] The terms in (44) can be interpreted as the binomial tree pricing of options with payoffs $\frac{L_k}{n_k+1}$ if $n_k \geq L_k$, and 0 otherwise. The efficiency condition (42) ensures that the probability of the option being in-the-money at the maturity (i.e., the payoff being strictly greater than zero) is the same for both firms 1 and j, while $L_1 > L_j$ implies that the payoff curve for option 1 is always strictly above that for option j. Hence the value of the option (with zero discounting in this case) is unambiguously higher for the former than for the latter.

$\sum_{k=0}^{I} \theta_k = 1$. Hence θ_1 must be increased from θ_1^*, and θ_j decreased from θ_j^*, to attain $\boldsymbol{\theta}^\dagger$ such that $\Psi_1(\boldsymbol{\theta}^\dagger) = \Psi_j(\boldsymbol{\theta}^\dagger)$. This is true for all $j = 2, ..., J$, and hence $\theta_1^\dagger > \theta_1^*$ unambiguously. ■

There is, therefore, overcrowding at the firm with the highest labor demand in the uniform wage Nash equilibrium, compared with the efficient equilibrium. For example, when $(I, J; a) = (5, 3; 1)$ and $\mathbf{L} = (3, 2, 1)'$, the efficient equilibrium is calculated to be $\boldsymbol{\theta}^* = (0.554, 0.328, 0.119)'$, while the uniform wage Nash equilibrium turns out to be $\boldsymbol{\theta}^\dagger = (0.607, 0.316, 0.078)'$. This result agrees with both Propositions 5 and 6.

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