mathematics

Article

Discounting the Distant Future: What Do Historical Bond Prices Imply about the Long-Term Discount Rate?

J. Doyne Farmer [1,2], John Geanakoplos [2,3], Matteo G. Richiardi [4], Miquel Montero [5,6,*], Josep Perelló [5,6,*] and Jaume Masoliver [5,6,*]

[1] Institute for New Economic Thinking at the Oxford Martin School, University of Oxford, Oxford OX1 3UQ, UK; doyne.farmer@maths.ox.ac.uk
[2] Santa Fe Institute, Santa Fe, NM 87501, USA; john.geanakoplos@yale.edu
[3] Department of Economics, Yale University, New Haven, CT 06511, USA
[4] Institute for Social and Economic Research, University of Essex, Wivenhoe Park, Colchester CO4 3SQ, UK; matteo.richiardi@essex.ac.uk
[5] Departament de Física de la Matèria Condensada, Universitat de Barcelona, 08028 Barcelona, Spain
[6] Universitat de Barcelona Institute of Complex Systems (UBICS), Universitat de Barcelona, 08028 Barcelona, Spain
* Correspondence: miquel.montero@ub.edu (M.M.); josep.perello@ub.edu (J.P.); jaume.masoliver@ub.edu (J.M.)

Abstract: We present a thorough empirical study on real interest rates by also including risk aversion through the introduction of the market price of risk. From the viewpoint of complex systems science and its multidisciplinary approach, we use the theory of bond pricing to study the long-term discount rate to estimate the rate when taking historical US and UK data, and to further contribute to the discussion about the urgency of climate action in the context of environmental economics and stochastic methods. Century-long historical records of 3-month bonds, 10-year bonds, and inflation allow us to estimate real interest rates for the UK and the US. Real interest rates are negative about a third of the time and the real yield curves are inverted more than a third of the time, sometimes by substantial amounts. This rules out most of the standard bond-pricing models, which are designed for nominal rates that are assumed to be positive. We, therefore, use the Ornstein–Uhlenbeck model, which allows negative rates and gives a good match to inversions of the yield curve. We derive the discount function using the method of Fourier transforms and fit it to the historical data. The estimated long-term discount rate is 1.7% for the UK and 2.2% for the US. The value of 1.4% used by Stern is less than a standard deviation from our estimated long-run return rate for the UK, and less than two standard deviations of the estimated value for the US. All of this once more reinforces the need for immediate and substantial spending to combat climate change.

Keywords: Ornstein–Uhlenbeck; stochastic process; discount; climate; interest rates

MSC: 91B70

check for updates

Citation: Farmer, J.D.; Geanakoplos, J.; Richiardi, M.G.; Montero, M.; Perelló, J.; Masoliver, J. Discounting the Distant Future: What Do Historical Bond Prices Imply about the Long-Term Discount Rate? *Mathematics* **2024**, *12*, 645. https://doi.org/10.3390/math12050645

Academic Editor: Chuangyin Dang

Received: 2 January 2024
Revised: 6 February 2024
Accepted: 19 February 2024
Published: 22 February 2024

1. Introduction

China suffered direct economic losses of more than USD 42 billion during the first nine months of 2023 from natural disasters, including torrential rains, landslides, hailstorms, and typhoons according to government data. This figure was presented at the last COP28 summit during last November–December 2023 jointly with many others coming from different parts of the planet, as reported in recent posts from the World Economic Forum [1]. In Africa, disasters from 1970 to 2021 caused USD 43 billion in economic losses, according to the World Meteorological Organisation. Europe also reported a cost of around USD 562 billion in losses. For South America, the losses amounted to USD 115.2 billion and for North America, Central America, and the Caribbean it was USD 2 trillion. Meanwhile, the latest instalment of the US National Climate Assessment has concluded that extreme weather events currently cost the country USD 1 billion every three weeks and averaged USD 150 billion in damages each year between 2018 and 2022. All in all, extreme weather, climate- and water-related events caused almost USD 1.5 trillion of economic losses in the last decade according to the World Meteorological Organization. The figure is much higher compared to the USD 184 billion in losses reported during the 1970s.

These costs are expected to become even higher in the future, deeply influencing countries' GDP [2]. This might be a reason for each country to take immediate action, mitigate climate change, and avoid future larger damages. However, a careful evaluation of the future economic impact of global warming is required and future costs against present costs must be balanced under some economic assumptions and climate scenarios [2–4]. The most basic assumption considers an exponential with a constant discount rate. The choice of discount rate is part of current economic debates on the urgency of the response to global warming (see [2–10]). Most normative approaches attempt to derive the discount from axiomatic principles of justice, or from utility theory and assumptions about growth [11–13]. Discounting includes impatience, economic growth, and declining marginal utility and it can be embedded in the Ramsey formula [14–16]. This approach, however, makes it difficult to make quantitative estimates because the factors involved are difficult to measure empirically.

Here, we take a positive approach, assuming that discounting is equivalent to bond pricing. However, the time to maturity of bonds is not generally higher than 30 years, but climate change time horizons need the discount 100 years or even more into the future. Thus, inferring long maturity bond prices from empirical data on short maturity bonds is fundamental and long historical time series are needed for reliable statistical inference. These data are most often not available and it will be required to consider a reasonable model for real interest rates at different maturities.

There are two effects influencing long-term rates that must be considered. The first of these is risk aversion. Longer-term bonds bear greater risk than shorter ones, and higher risk should imply higher interest rates. The second effect is more subtle, and is due to the fact that interest rates are uncertain and highly persistent. In the environmental context, this effect was originally pointed out by Weitzman [17] and Gollier et al. [18] (see also [19]). However, the effect was also pointed out in a general context in [20], and it has been known much longer in the context of bond pricing [21]. The long-run rate is, thus, dominated by the lowest value, since asymptotically all the other discount factors will be negligible in comparison. To see this concretely, consider two possible rates, r_1 and r_2, with $r_1 > r_2$, and assume they have equal probability. Then, the average discount factor at time t in the future is $D(t) = (e^{-r_1 t} + e^{-r_2 t})/2$. Note that, since the sum of two exponentials is not an exponential, the discounting function is no longer an exponential. But for t which is sufficiently large, $D(t) \approx e^{-r_2 t}/2$, i.e., the discount function becomes approximately exponential with the lower interest rate. This illustrates that when interest rates are uncertain and persistent, lower interest rates tend to dominate.

We build here on earlier empirical work. Stochastic interest rates provided by Litterman et al. [22], Newell and Pizer [23], and Groom et al. [24] are more realistic processes than those provided by Weitzman and Gollier. Groom et al. [24] also noted that the drop in rates depends on uncertainty and persistence. All these authors calibrated the models suggested against long historical time series of 10-year bonds. We here add to these previous works in several ways. We study century-long records of the prices of 3-month bonds, 10-year bonds, and inflation, for both the US and the UK. Real interest rates are found to become negative more than one third of the time, often by substantial amounts. To our knowledge only a few authors have addressed this issue (e.g., Freeman et al. [25]), while the usual approach has been to ignore this fact, and instead has forced historical interest rates to be positive in order to be consistent with the proposed models.

We instead take the view that negative real interest rates are a fact that cannot be ignored. This leads us to choose the Ornstein–Uhlenbeck (OU) model, which is compatible with negative real interest rates. The model provides real interest rates that can be negative but they are constantly pulled back to a certain rate. The formula for the yield curve for the OU model has been derived in the finance literature using several approaches (see, for instance, [26–28], and the review in [29]). We derive this in a somewhat simpler way using Fourier transform methods. The previous work in environmental discounting cited above [24] was based on numerical simulations. In contrast, we take advantage of the existence of closed-form solutions, which allows us to better estimate the long-term discount rate. When the OU model is used in the finance literature, the mean reversion parameter is taken to be so strong that the nominal interest rate goes negative with a probability close to zero. In a different approach, Davidson et al. [30] examined a square root Ornstein–Uhlenbeck model which can never go negative and which is pulled down toward 0 as it is an absorbing state. Davidson et al. [30] solved the asymptotic (long) rate by using the Feynman–Kac functional, which is quite different from our approach.

The prevalence of negative real rates makes many of the standard nominal interest rate models inappropriate vehicles for studying real rates, and the OU becomes an obvious choice while preserving the highest possible simplicity. Other models might also be reasonable choices but they would, in any case, need to contain key features, such as mean reversion, random fluctuations, and a normal level of the process [27]. The paucity of data records available (hundreds of points) makes parameter estimation challenging. Among other possible candidates with the fewest number of parameters and which are analytically tractable with the methodology herein presented, it is possible to consider the shifted Feller model and the log-normal model as we have done elsewhere [31]. The shift avoids negative rates, but it becomes somewhat artificial and context dependent as it totally depends on the period analyzed [31]. Another alternative is to add more timescales into the time series and to consider a fast and a slow mean reversion with two different normal levels. This approach can be modeled with a two-dimensional OU process and enables moving forward the stationary limit beyond the data range available [32]. This sophistication already involves the addition of two more parameters, which are difficult to estimate due to the scarce data available [31,32].

Therefore, our main original contribution here regards properly taking risk aversion into account with the OU model. The works described above [23,24] were calibrated against a single maturity bond (10 years). Risk aversion is a well-accepted notion in bond pricing, and one would expect models that neglect it to underestimate long-term interest rates. We fix this by taking risk aversion into account and fit the resulting model to both 3-month and 10-year bond yields, which provides us with two points on the yield curve (one for very short maturities, thus capturing instantaneous interest rates, and another bringing out the long-term effects of the market price of risk). This approach can become more precise by taking additional bond maturities or even additional data sources to better account for risk aversion but, to avoid further sophistication in the whole methodology, we have discarded this option.

We here make the hypothesis that the long-term, risk-adjusted, bond discount rate is a proxy for measuring the urgency of climate action, the latter becoming higher as the former becomes lower, and vice versa. For this reason, we herein present a complete empirical study on real interest rates for the US and UK for which we have century-long records, and that are considered amongst the most economically stable countries as far as inflation and economic growth are concerned. Our focus is based on the interdisciplinary approach of complex systems developments. We have also presented elsewhere a similar study for other less stable countries, but with lower statistical reliability due to the scarcity of data and without considering risk aversion (i.e., the market price of risk) [31]. Several different theoretical aspects of discounting have recently been discussed by some of us, mainly regarding the effects on the discounting of extreme market situations [33] or the introduction of resettings in real interest rates [34] (see also the recent review [35]). These are, however, sophisticated models with a larger number of parameters, thus making model calibration more difficult. This is the reason why we here restrict the analysis to the OU process.

After Section 1, Section 2 presents the formal mathematical framework. It introduces the Ornstein–Uhlenbeck model and the addition of risk aversion in its simplest form for the discount function and the long-term interest rate. Section 3 focuses on the empirical estimates of the real interest rates by taking US and UK empirical data and provides an empirical analysis. It also includes the parameter estimation procedure for the OU model with the inclusion of the market price of risk. The main goal of the section is to estimate the discount function and the long-term interest rates. Their confidence intervals are also obtained. Section 4 offers a broader interpretation of the results obtained, focuses on some relevant aspects, such as the prevalence of negative rates, and points out some limitations of the current research, alongside consideration of future research directions. Section 5 summarizes the main results and highlights the practical relevance of the results obtained. Several technical details and descriptions are provided in three different appendices.

2. The Process of Discounting in Continuous Time

We now derive the form of the yield curve for the OU model. In the continuous limit the discount function is

$$D(t) = \mathbb{E}\left[\exp\left(-\int_{t_0}^{t} r(t')dt'\right)\right], \tag{1}$$

where t_0 is the initial time and the expectation $\mathbb{E}[\cdot]$ is an average over all possible real rate trajectories $r(t')$ $(t_0 \leq t' \leq t)$ up to time t. The mathematical expression is formally identical to the bond prices. The price $B(t_0|t)$ of a zero-coupon bond issued at time t_0 with unit payoff and time maturity t ($t \geq t_0$) reads

$$B(t_0|t) = \mathbb{E}\left[\exp\left(-\int_{t_0}^{t} n(t')dt'\right)\right], \tag{2}$$

which depends on the nominal rate $n(t)$.

2.1. The General Framework

Following the same strategy as for bonds, we compute $D(t)$ via a stochastic process model for the time evolution of $r(t)$. The simplest and most common assumption is to consider $r(t)$ as a diffusion process. The stochastic differential equation is

$$dr(t) = f(r)dt + g(r)dW(t), \tag{3}$$

where $f(r)$ gives the reason of the drift, while $g(r) > 0$ quantifies the noise intensity $dW(t) = \xi(t)dt$ (a Wiener process) characterized with

$$\mathbb{E}[\xi(t)] = 0, \qquad \mathbb{E}\left[\xi(t)\xi(t')\right] = \delta(t-t'), \tag{4}$$

and where $\delta(\cdot)$ is Dirac's delta function.

The discount function is, therefore, given by

$$D(t) = \mathbb{E}\left[e^{-x(t)}\right],$$

where the cumulative process $x(t)$ is defined as

$$x(t) = \int_{t_0}^{t} r(t')dt'. \tag{5}$$

Therefore, the discount function can be written in the following form:

$$D(t) = \int_{-\infty}^{\infty} dr \int_{-\infty}^{\infty} e^{-x} p(x, r, t|x_0, r_0, t_0)dx. \tag{6}$$

The function $p(x, r, t|x_0, r_0, t_0)$ corresponds to the probability density function (PDF) that depends on $(x(t), r(t))$. The measure p is often referred to as the *data-generating measure*. We note that when there are uncertainties in the value of certain parameters, which may appear in the estimation of the PDF from empirical data, then the average in Equation (6) has to be extended in order to include these additional uncertainties (see, e.g., [36]).

Equations (3) and (5) together refer to a bidimensional process defined in terms of a pair of stochastic differential equations

$$dx(t) = r(t)dt, \tag{7}$$

$$dr(t) = f(r) + g(r)dW(t). \tag{8}$$

These stochastic different equations imply that the joint density follows the Fokker–Planck equation (FPE)

$$\frac{\partial p}{\partial t} = -r\frac{\partial p}{\partial x} - \frac{\partial}{\partial r}[f(r)p] + \frac{1}{2}\frac{\partial^2}{\partial r^2}[g^2(r)p]. \tag{9}$$

The initial conditions of this equation are fixed and are

$$p(x, r, t_0|x_0, r_0, t_0) = \delta(x)\delta(r - r_0). \tag{10}$$

The process $r(t)$ is time-homogeneous because $f(r)$ and $g(r)$ do not depend on time explicitly. The process is, therefore, time-invariant and, without loss of generality, $t_0 = 0$.

2.2. The Ornstein–Uhlenbeck Process

We now make explicit choices for the functions f and g. For the reasons given in the introduction, we focus on the OU process, which is a standard model from finance that allows negative rates. In finance, the OU process was originally proposed by Vasicek [21] and is sometimes referred to as the Vasicek model. It is a diffusion process with linear drift and constant noise intensity

$$f(r) = -\alpha(r - m), \qquad g(r) = k. \tag{11}$$

The process is, thus, governed by Equation (3). Letting $r_0 = r(0)$ be the initial return, in the large time limit the probability density function $p(r, t|r_0)$ has mean m and variance $\sigma^2 = k^2/2\alpha$.

2.3. Adding Risk Aversion

The local expectation hypothesis (LEH) [37,38] assumes that prices can reasonably be modeled based on the data-generating measure p. This result is attributed to the fact that investors are assumed to be risk neutral. However, if investors are sensitive to risk, bonds are priced with an artificial probability density function, p^*, called either the *risk-neutral probability measure* or the *risk-correcting measure* (see Appendix B). The two measures p and p^* are related by the market price of risk (see below), which is the extra return per unit of risk that investors demand to bear risk.

Following a standard procedure for bond pricing [21,26,28,35], we take risk into account by adjusting the drift term according to

$$f(r) = -\alpha(r - m) + g(r)q(r), \tag{12}$$

where, in our case, $g(r) = k > 0$ and $q = q(r) \geq 0$ is the *market price of risk*, which, in general, may also depend on the current time, $q = q(r, t)$, but we here assume stationarity. The introduction of the market price of risk raises the effective interest rate, which in the context of bond pricing means that investors are compensated for taking increased risk. The most common assumption consists in taking $q(r, t)$ independent of r and t, that is

$$q(r,t) = q = \text{constant}.$$

In such a case, the adjusted drift can be rewritten as

$$f(r) = -\alpha(r - m^*), \tag{13}$$

where

$$m^* = m + \frac{qk}{\alpha}. \tag{14}$$

The result is that the effective mean interest rate m^* is increased relative to the historically observed interest rate m by a constant amount that depends on the volatility, the reversion rate, and the market price of risk.

For the case of a constant price of risk, the joint PDF is found by solving the Fokker–Planck Equation (9) , which now becomes (cf. Equation (13))

$$\frac{\partial p}{\partial t} = -r\frac{\partial p}{\partial x} + \alpha\frac{\partial}{\partial r}\left[(r - m^*)p\right] + \frac{1}{2}k^2\frac{\partial^2 p}{\partial r^2}, \tag{15}$$

with the initial condition provided in Equation (10). The equation problem can be addressed with the characteristic function (Fourier transform)

$$\tilde{p}(\omega_1, \omega_2, t|r_0) = \int_{-\infty}^{\infty} e^{-i\omega_1 x}dx \int_{-\infty}^{\infty} e^{-i\omega_2 r}p(x, r, t|r_0)dr. \tag{16}$$

Equation (15) can, therefore, take a simpler form:

$$\frac{\partial \tilde{p}}{\partial t} = (\omega_1 - \alpha\omega_2)\frac{\partial \tilde{p}}{\partial \omega_2} - \left(im^*\alpha\omega_2 + \frac{1}{2}k^2\omega_2^2\right)\tilde{p}, \tag{17}$$

with (cf. Equation (10))

$$\tilde{p}(\omega_1, \omega_2, 0|r_0) = e^{-i\omega_2 r_0}.$$

The solution is given by

$$\tilde{p}(\omega_1, \omega_2, t) = \exp\left\{-A(\omega_1, t)\omega_2^2 - B(\omega_1, t)\omega_2 - C(\omega_1, t)\right\}, \tag{18}$$

where the expressions for the functions $A(\omega_1, t)$, $B(\omega_1, t)$, and $C(\omega_1, t)$ are obtained in Appendix A. Comparing Equations (6) and (16), we obtain

$$D(t) = \tilde{p}(\omega_1 = -i, \omega_2 = 0, t). \tag{19}$$

Finally, it can be shown in Appendix A that $D(t) = \exp\{-C(-i, t)\}$, which is a tractable expression for the discount rate (cf. Equation (A10)).

We now make a change of notation. We have so far assumed that the discount function is computed for a time t in the future, starting at a fixed initial time $t_0 = 0$ with an initial interest rate r_0 for a time t in the future (let us remember that we have set $t_0 = 0$; had we kept $t_0 \neq 0$, then time t would have been replaced by $t - t_0$). However, in what follows, we will want to evaluate the discount starting from different initial times. Recalling that the whole process is invariant under time translations, we can, thus, perform the change of notation such that $t_0 \to t$, $r_0 \to r(t)$ and $t - t_0 \to \tau$. Note that, now t is the present time, $r(t) = r$ is the present interest rate and the maturity time is $t + \tau$; that is to say, τ is the "time to maturity". The discount function is then (cf. Equation (A10) of Appendix A)

$$\begin{aligned} \ln D(\tau) &= -\frac{r}{\alpha}\left(1 - e^{-\alpha\tau}\right) - m^*\left[\tau - \frac{1}{\alpha}\left(1 - e^{-\alpha\tau}\right)\right] \\ &+ \frac{k^2}{2\alpha^3}\left[\alpha\tau - 2\left(1 - e^{-\alpha\tau}\right) + \frac{1}{2}\left(1 - e^{-2\alpha\tau}\right)\right]. \end{aligned} \tag{20}$$

when the time to maturity τ is small, we can approximate $D(\tau)$ by expanding the exponentials to a first order using a Taylor series approximation, and this yields $\ln D(\tau) \approx r\tau$ ($\tau \to 0$), as expected.

In the limit $\tau \to \infty$, we see that

$$\ln D(\tau) \approx [-m^* + k^2/(2\alpha^2)]\tau$$

and the discount function becomes independent of the initial rate. It decays exponentially and can be written in the form

$$D(\tau) \simeq e^{-r_\infty \tau}, \tag{21}$$

where

$$r_\infty = m + \frac{qk}{\alpha} - \frac{k^2}{2\alpha^2}, \tag{22}$$

is the long-run rate.

In general, the long-run rate may be defined by

$$r_\infty = -\lim_{\tau \to \infty} \frac{\ln D(\tau)}{\tau}, \tag{23}$$

as long as the limit exists. Note that Equation (22) can be readily obtained after substituting Equation (20) into Equation (23).

From Equation (22), we see that the long-run discount rate depends on the historical rate m, but this is shifted by two terms. The first shift raises the long-run rate due to the market price of risk. The second shift lowers it by an amount given by the ratio of uncertainty (as measured by k) and persistence (as measured by α). We can trivially rewrite the equation above as

$$r_\infty = m + \frac{k}{\alpha}\left(q - \frac{k}{2\alpha}\right). \tag{24}$$

This makes it clear that whether or not the overall shift in the long-run discount rate is positive or negative depends on the size of the market price of risk in relation to the ratio of the volatility parameter and the reversion rate.

It is not surprising that the market price of risk raises the long-term rate, but it is not so obvious that uncertainty and persistence can lower it. Indeed for the OU process it can make it arbitrarily small. For any given mean interest rate m, by varying k and α, the long-run discount rate r_∞ can take on any value less than m, including negative values, while at the same time, the standard deviation σ can also be made to take on any arbitrary positive value. In particular, by choosing the appropriate (k, α), we can make r_∞ arbitrarily far below m and σ arbitrarily small. The probability that $r(t) < r_\infty$ can be arbitrarily small, even when $r_\infty \ll m$ (see Appendix A). Deducing the correct parameters (m, σ) of the stationary distribution of short-run interest rates does not determine r_∞ by itself; on the contrary, any $r_\infty < m$ is consistent with them. To infer r_∞ from the data, one must also tease out the mean reversion parameter α. Holding the long-run distribution (m, σ) constant, by raising the persistence parameter $1/\alpha$, it is possible to lower r_∞ to any desired level. Of course, this can always be offset by the market price of risk.

It is even possible for the long-run rate to be negative. This is due to the amplification of negative real interest rates $r(t)$. Computation of the discount function involves an average over exponentials, rather than the exponential of an average. As a result, periods where interest rates are negative are amplified, and can easily dominate periods where interest rates are large and positive, even if the negative rates are rarer and weaker. It does not take many such periods to substantially reduce the long-run interest rate.

To summarize, in the OU model, the long-run discounting rate can be much lower than the mean, and indeed can correspond to low interest rates that are rarely observed.

3. Empirical Results

3.1. Estimation of Real Interest Rates

We estimate real interest rates as nominal rates corrected by inflation, which we perform directly through the application of Fisher's equation by subtracting the realized inflation from the nominal interest rates (see Equation (26) below). This is not, however, the only way of obtaining real rates. Thus, for instance, Freeman et al. [25], among others, pursue an alternative procedure using co-integration methods to tease out real rates. Another way of obtaining real rates would be modeling nominal rates and inflation separately; that is, nominal rates by some positive random process (for example the Feller process as in the CIR model [39,40]) and the OU process for inflation, since the latter can assume positive and negative values. Such a procedure—which unfortunately enlarges the number of parameters to be estimated from the data—would also need to take into account possible correlations between bond prices and inflation. It is not clear a priori which procedure is better, and the direct procedure we follow has the virtue of being much simpler with a smaller number of free parameters to estimate.

We transform the annual rates into logarithmic rates and denote the resulting time series by $y(t|\tau)$ (with the maturity time τ equal to either 3 months or 10 years). Nominal rates $n(t)$ are then estimated by $n(t) \sim y(t|\tau)$ (see Appendix B for details). The inflation rate $i(t)$ is estimated through the Consumer Price Index (CPI) as

$$i(t) \sim \frac{1}{\tau} \ln\left[\frac{I(t+\tau)}{I(t)}\right], \tag{25}$$

where $I(t)$ is the aggregated inflation up to time t, and $\tau = 10$ years (cf. Appendix B). Finally, the real interest rate $r(t)$ is defined by Fisher's equation:

$$r(t) = n(t) - i(t). \tag{26}$$

We now estimate the OU model for real interest rates from historical data. To this end, we have collected long historical time series for both short- and long-run nominal interest rates, as well as inflation, for the United Kingdom and the United States. The properties of the data are summarized in Table 1. For each country, we have both three-month and ten-year interest rates, as well as an inflation index.

Table 1. Datasets used in our main analysis for the United Kingdom (UK) and the United States (US).

Country	Time Series	Frequency	From	To	# Records
UK	3-month Treasury bills	monthly	31 December 1900	31 December 2012	113
UK	10-year bonds	annual	31 December 1694	31 December 2012	309
UK	inflation index	annual	31 December 1694	31 December 2012	309
US	3-month Treasury bills	monthly	31 January 1920	30 October 2012	93
US	10-year bonds	annual	31 December 1820	30 October 2012	183
US	inflation index	annual	31 December 1820	30 October 2012	183

The recording frequency for each country is either annual or monthly. For ten-year government bonds (which pay out over a period of ten years), we smooth the inflation rates with a ten-year forward moving average, and subtract the annualized inflation index from the annualized nominal rate to compute the real interest rate. For the three-month bond rates, in contrast, we use the inflation adjustment for the corresponding year (since we do not have inflation adjustments at quarterly frequency). Figure 1 shows the nominal rates, inflation, and real interest rates for 3-month bonds for the UK and the US, and Figure 2 compares 3-month and 10-year real interest rates. This procedure assumes that people correctly forecast inflation; thus, in the absence of any knowledge of behavioral bias, we are assuming perfect rationality.

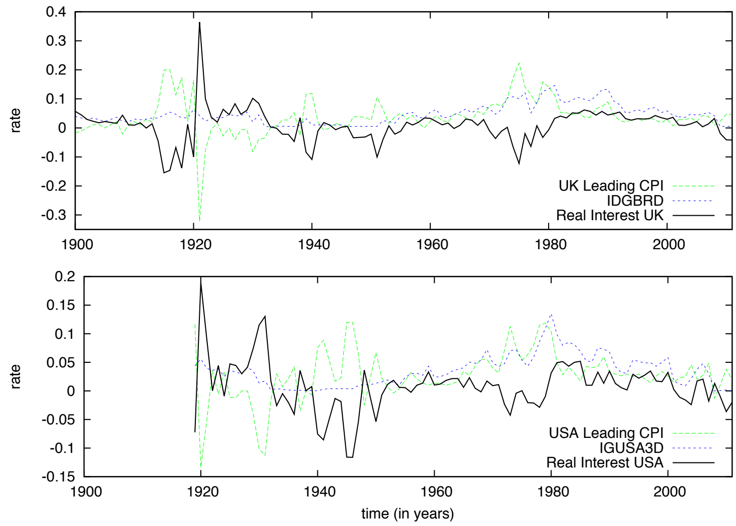

Figure 1. Time series for the three-month treasury bills for the UK (top) and US (bottom). The inflation index is shown in green (long dashes), the nominal interest rate in blue (short dashes), and the real interest rate is shown as a black solid line. Real interest rates display large fluctuations and negative rates are not uncommon.

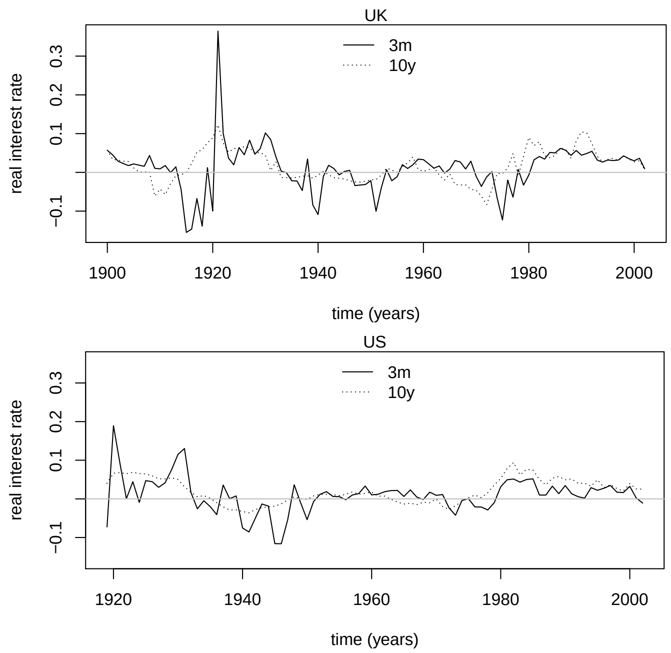

Figure 2. Comparison between three-month and ten-year real interest rates for the UK (top) and the US (bottom). The ten-year real interest rates are shown with dashed lines and the three-month real interest rates taken from Figure 1 are shown with solid lines. A horizontal line is drawn at zero to make it clear when the real rates are negative.

3.2. Empirical Properties of the Data

One of the most striking features of these time series is that real interest rates are often negative, in some cases by substantial amounts and for long periods of time (see Figures 1 and 2). This is evident looking at the histograms shown in Figure 3.

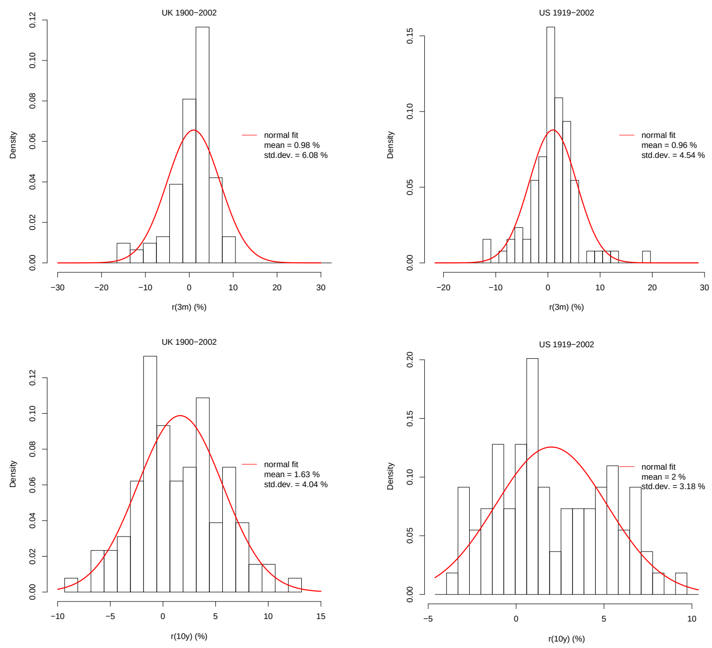

Figure 3. Histograms of 3-month (top) and 10-year (bottom) real interest rates for the UK and the US. The curve compares to a normal distribution. This illustrates that negative real interest rates are common and that the distribution of interest rates is heavy-tailed relative to a normal distribution.

Real 3-month interest rates for the UK are negative 32% of the time, and there are four distinct periods where they drop to nearly reach -10% and even below. The 10-year real interest rates for the UK are negative 38% of the time. For the US, the real 3-month interest rates are also negative 32% of the time, dropping below -10% during World War II, and the 10-year real interest rates are negative 30% of the time. Given that the real interest rates are negative about a third of the time, this makes it clear that models such as the log-normal process or the CIR model [39,40] that assume rates to be non-negative are far from being appropriate. We therefore confine our empirical work to the Ornstein–Uhlenbeck model.

From Figure 3, it is also evident that the distribution of interest rates is heavy-tailed. This is particularly true for the short-term interest rates. Heavy tails are apparent because of the excess in the center of the distribution but also because the observations in the tail exceed the normal distribution. However these deviations are not extreme and the OU process (which has a normal distribution) is at least a reasonable first approximation.

Another striking feature is that the yield curve is intermittently inverted; that is, the 10-year real interest rate is at intervals lower than the 3-month rate (see Figure 1). For the UK, the yield curve is inverted slightly more than 50% of the time, and for the US, it is inverted 32% of the time. Inversions of the yield curve are obviously important for understanding very long-term rates.

3.3. Parameter Estimation

The OU model with constant price of risk has four parameters to be estimated: m, k, α, and q. The 3-month rates are much less sensitive to the risk parameter q than the 10-year rates. Therefore, by making the approximation that the 3-month rate is equivalent to the instantaneous process, we can estimate m, k, and α from the 3-month rate time series alone (see below for the estimation of q). The parameters obtained in this way are shown in

Table 2 and the values displayed are based on the maximum likelihood estimators derived in Ref. [27] (see Appendix C).

To provide an appreciation for the robustness of the estimated parameters, we divide the time series into four blocks of equal size and evaluate the parameters m and k separately for each block, with the exception of the parameter α, which is always estimated using the complete dataset because the time series in each block are too short for reliable estimation of α. The quoted uncertainties in α are then simply the standard least square error computed when fitting an exponential autocorrelation function of the real interest time series. The maximum and minimum values obtained for each parameter are listed in Table 2. The variations are large, indicating some combination of autocorrelation, non-stationarity, and heavy tails.

We remind that m^* implicitly depends on the market price of risk q (cf. Equation (14)) and, as reported in Table 2, the value of m^* for a very short time window ($\tau = 3$ months) can be approximated by m. To estimate q, we, thus, make use of the $\tau = 10$-year real interest rate series as it is considered to be a large value of τ (much larger than three months). The averages of the 3-month and 10-year interest when considering Equation (20) give us two equations with two different values for m^*, where k and α are held fixed. And, finally, the market price of risk q can be obtained with Equation (14). For this estimation, we are also assuming in Equation (20) that $r(t) \simeq m$, i.e., that the mean historical interest rate is equal (or very close) to the current rate.

The estimates obtained in this manner are slightly distorted not only because the 3-month bond is sampled annually but also because we have treated it as though it were an instantaneous rate. We can estimate the size of this bias by simulating the instantaneous process, which we approximate as having daily frequency. The simulation procedure is standard. We, thus, generate the instantaneous process $r(t)$ by solving numerically the OU equation with the parameters α, m, and k given by the empirical values taken from the 3-month rates as given in Table 2 (see more details in Appendix C). We next create a surrogate time series for the 3-month real interest rate time series using Equation (20) with $\tau = 0.25$ year and the empirical $r = r(t)$ as the initial condition at each time t. We then mimic the procedure employed for the real data by estimating the parameters based on the surrogate 3-month rate, sampled at annual frequency. We can, thus, adjust the parameters of the instantaneous process to roughly match, on average, those observed for the real data (so that the estimated values based on the surrogate 3-month series match those of the observed 3-month series). The parameters with the bias corrected are given in Table 3 (see Appendix C for more details on the procedure we followed in order to correct the bias). The resulting shift in parameters is small, as can be seen by comparing Tables 2 and 3. The main difference is in the parameter α, which sets the timescale for the mean reversion; this changes by a little more than 10%. Our numerical experiments indicate that the main source of the bias is the annual sampling, and since parameter α sets the timescale for mean reversion, it is not surprising that it is affected by this.

Table 2. Raw estimates of parameters of the risk-free Ornstein–Uhlenbeck model for the United Kingdom (UK) and the United States (US). Results are based on the annually sampled time series of three-month real interest rates. We use annual units. The Min and Max columns correspond to the minimum and the maximum values of the parameters obtained by splitting the time series into four blocks of equal length and estimating the parameters separately in each block (except α, see main text). For better estimates, see Table 3.

Country	m^*	Min	Max	k	Min	Max	α	Min	Max
UK	0.88	−0.4	3.5	8.2	1.8	15.6	0.93	0.2	1.4
US	0.83	−1.2	2.2	5.7	2.5	10.5	0.74	0.3	1.3

Table 3. Parameters of the instantaneous Ornstein–Uhlenbeck process. We use the procedure described in the text. m and k are in percent.

Country	m	5%	95%	k	5%	95%	α	5%	95%	q	5%	95%
UK	0.84	−0.92	2.6	8.9	7.6	10.4	0.82	0.47	1.26	0.13	−0.04	0.34
US	0.83	−0.85	2.3	5.8	4.9	6.8	0.65	0.36	1.06	0.20	0.02	0.43

3.4. Contrasting The Model With Data

We now make some comparisons of the simulated OU model with real data (see Table 4). Similarly to the procedure for the 3-month rates, we create simulated 10-year rates using Equation (20), so that the rates are defined as $(1/\tau)\ln D(\tau)$ with $\tau = 10$ year, where $r = r(t)$ as the initial condition at each time t (recall that the maturity time is $t + \tau$). A comparison shows that the standard deviation of the 3-month rates matches reasonably well, but the standard deviation of the simulated rates is much lower than that of the 10-year rates. We correct this by adding IID normally distributed noise to match the standard deviation of the simulated series with real data. We also neglect the 10-year smoothing of the 10-year inflation data. The simulated result has a lower correlation between the 3-month and 10-year rates than the real data; for the UK, the correlations are 21% (simulated) vs. 39% real, and for the US, 24% (simulated) vs. 59% (real). However, the distributions for both 3-month and 10-years rates agree reasonably well. Figure 4 shows the simulated 3-month and 10-year interest rate time series, which should be compared to the real data shown in Figure 2. Not surprisingly, since the 3-month simulated rates are normally distributed, they lack the extreme values observed in the real data.

Table 4. A comparison of the percentage of the time real interest rates that are negative for both the UK and the US and for both the data and the model simulation. The simulation values are averaged over 1000 simulations.

Country	3-Month (Data)	3-Month (Model)	10-Year (Data)	10-Year (Model)
UK	32%	43%	38%	34%
US	32%	42%	30%	26%

The OU model does a good job of capturing the frequency of negative interest rates and yield curve inversions. Table 4 compares the frequency of negative interest rates for the real data and the simulation for both the 3-month and 10-year rates.

In Figure 5, we present a histogram of yield curve inversions for both the data and the model for the UK. The US histogram is qualitatively similar and we do not present it here. We use the difference between the 10-year real interest rate and the 3-month interest rate as our measure of inversion. The inversions of the data are somewhat more heavy-tailed than those of the model, but the agreement is surprisingly good. The real UK yield curve is inverted roughly 50% of the time and the simulated yield curves are inverted 46% of the time. Similarly, the real US yield curves are inverted 32% of the time and the simulated yield curves are inverted 41% of the time.

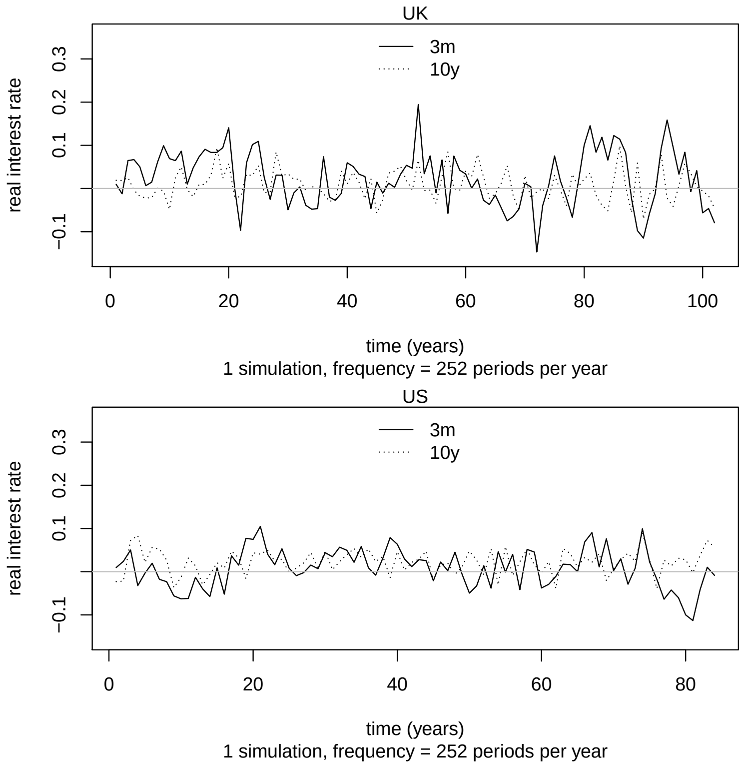

Figure 4. A simulation of the 3-month and 10-year interest rates for the UK (above) and the US (below) using the OU process. Compare with Figure 2.

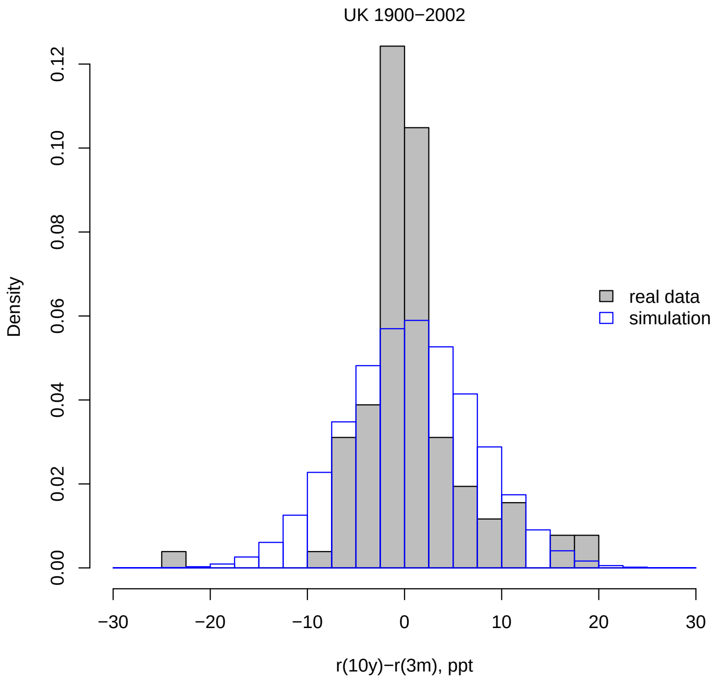

Figure 5. Histogram comparing yield curve inversions in the simulated vs. real data for the UK. We measure yield curve inversion based on the difference between the 10-year interest rate and the 3-month interest rate; positive values indicate a normal yield curve and negative values an inverted yield curve. Interest rates are measured in percent. The real data are shown in grey, the simulation in white. The real data are heavier-tailed, but the agreement is otherwise reasonably good.

3.5. Estimating Confidence Intervals

With such a short time series as we have, it is difficult to estimate confidence intervals by methods such as bootstrapping. This is particularly true for α, where the time series properties of the data matter, so that one would need to perform a block bootstrap, and, unfortunately, there are not many blocks of sufficient length. However, assuming that the model is well specified, we can at least compute error estimates which are consistent with our model. The width of the resulting confidence intervals can be regarded as lower bounds on the width of the confidence intervals, and provide a means of perceiving the magnitude of the estimating errors.

We repeatedly simulate the instantaneous process $r(t)$ using the parameters estimated from the data and generate 3-month and 10-year series as described above. In order to accurately mimic the constraints imposed by the data, we sample the simulated series at an annual frequency. We then apply the estimation procedure described above to estimate the four parameters. Doing this 1000 times allows us to compute the 5% and 95% quantiles for each parameter. The results are shown in Table 3.

The estimated discount functions, together with their confidence intervals, are shown in Figure 6. The uncertainty intervals are estimated by repeatedly simulating the instantaneous, 3-month and 10-year processes, as described above, applying the estimation procedure to the simulated data, and computing the discount function at each time interval. This is repeated 1000 times to estimate the 5% and 95% quantiles.

We are finally ready to present our key result. The long-term interest rate r_∞ is computed using Equation (22) based on the values in Table 3 and the final key result of this paper is presented in Table 5. The mean long-run rate is $r_\infty = 1.69\%$ for the UK and 2.21% for the US. Let us note that because of the scarcity of the data available, uncertainties are substantial, with standard deviations of roughly 0.45% in both cases.

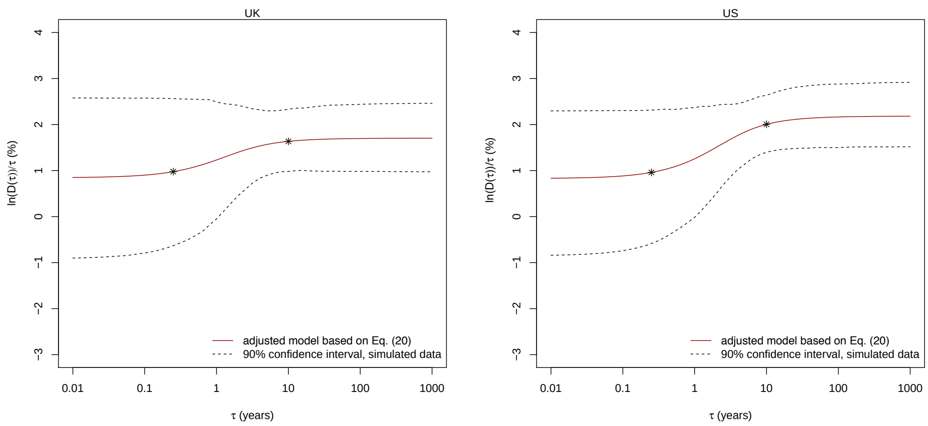

Figure 6. The estimated discount rate $\ln(D(\tau))/\tau$ (in%) for the UK (top) and US (bottom). The rates are shown as solid red lines (based on Equation (20)), plotted versus time to maturity τ (time is on a logarithmic scale). The dashed lines indicate the 5% and 95% quantiles based on the simulation procedure described in the text. These values are obtained with 1000 simulated series, each 84 years long, with 252 periods per year. Starred points are the empirical estimated values from UK and US historical time series.

Table 5. Long-term interest rate r_∞ for the United Kingdom (UK) and the United States (US). Measured in percent, as well as the 5% and 95% quantiles.

Country	r_∞	**5%**	**95%**
UK	1.69	0.76	2.63
US	2.21	1.35	3.07

4. Discussion

Climate change and climate action are widely studied from a variety of perspectives [41]. However, researchers in financial economics have only recently turned their attention to climate change, and climate finance is currently a quickly growing research field [42]. Finance academics participating in a recent survey have identified discount as a key topic to reduce climate risks [43]. With this paper, we have taken a multidisciplinary perspective from complex systems science and its related methods to study historical bond prices [44], with the aim to contribute to the need of new approaches to evaluate climate action urgency [45,46].

More specifically, we have wanted to highlight that long-term discount rates are not just a trivial matter of extrapolating mean interest rates, but rather that one must take several non-trivial factors into account. To begin with, because real interest rates are so often substantially negative, one should use a model that permits negative rates. This leads us to the Ornstein–Uhlenbeck model. While the presence of negative rates in this model may be viewed as a liability for describing nominal rates, for real rates, this becomes a virtue. Another factor that should be considered is the market price of risk, which tends to raise longer-term rates. Finally, one should properly take into account the uncertainty and persistence of interest rates, which tends to lower the long-term discount rate. The use of the OU model accommodates all of these factors. When we estimate the OU model and compare it with real data, we see good agreement on several essential properties, such as the frequencies of negative rates and yield curve inversions.

Our results indicate that the long-term interest rate used by Stern [2] is supported by historical data. His value of 1.4% is less than a standard deviation below the estimated long-term rate for the UK of 1.69%, and just under two standard deviations of the US long-term rate of 2.21%. More recent estimates by Stern indeed suggest a scenario with a value below 1% [45]. In contrast, higher long-run rates—see, for instance [6,7]—seem not to be supported, as they are well above the 95% confidence intervals of 2.63 for the UK and 3.07 for the US. Our estimates of 1.69% (UK) and 2.2% (US) are compatible with the rates recently estimated by Giglio et al. ([47–49]), which use data from UK housing markets during 2004–2013 and Singapore during 1995–2013 to estimate an annual discount rate of 2.6% for payments more than 100 years in the future.

The prevalence of negative rates and their role in the whole analysis merit an even deeper analysis than the one provided here so far. We have already explored the role of negative rates in a previous publication, which, however, did not incorporate the market price of the risk component [31]. By still considering the OU model as a valid model, one can then see that larger fluctuations (and, therefore, a higher probability to have negative rates) directly implies even lower long-term rates. This effect can indeed also be observed in Equation (22), where the term related to the noise amplitude, $k > 0$, has a negative sign. This effect is magnified in less stable countries compared to the UK and US [31]. The results from countries with a higher prevalence of negative rates and their even lower (and eventually negative) long-term interest rates further increase the urgency for taking action today rather than tomorrow. A more careful analysis would, however, need to be performed to enable a more quantitative discussion.

Our results could potentially also be improved in other ways, which, in turn, can lead to relevant future research avenues. One new research avenue would be to acquire more data. A broader exploration would enhance the applicability and robustness of the methods and the results being presented here. This could include data from more countries—as we undertook recently, but only for 10-year bonds and, therefore, unable to include risk aversion [31]. The 3-month data from other countries would allow us to incorporate the market price of risk into the analysis and we would be able to elaborate a more complete study, which indeed could include a rigorous discussion on the sensitivity of the different components that influence the discount function and the long-term interest rate r_∞.

One should, however, be aware that the methodology depends on long time series within which the risk premium evolves through time and within which many structural changes may occur. This certainly may affect the whole analysis, and in practical terms can deeply influence the results based on today's environment. These aspects should be included if we seek concrete applicability of the results obtained here. Along the same lines, the approach might not be valid for emerging markets and for some countries (climate change has different impacts on different countries), either because they may not have available the data being analysed here or because some countries/markets are far from stable.

In relation to the possibility of adding new data sources, it could also be of interest to include longer term bonds, which would certainly allow the inclusion of a more sophisticated market price of risk function. We have here limited the analysis to a constant q market price of risk independent of r and this is another limitation of the current work. Credit spreads data or supply/demand effects in bond markets, inflation expectations, and inflation-indexed bonds would be other possible datasets that could be used to provide a better estimation of the long-term interest ratio. The analysis would eventually need a more sophisticated methodology than the one presented in Section 3.

Another possible improvement would be to extend the model to better capture the non-stationarity and/or heavy-tailed behavior observed in the data [36,50]. In this regard, we suspect that had we been able to take the observed non-stationarity [45] and/or heavy tails into account, the mean values would have decreased because of the boosting of the uncertainty/persistence effect. This possibility is under investigation presently but we may anticipate that the result could suggest even lower discount rates [32].

At a broader level, both Stern and Stiglitz have recently provided new methods and models which could be further studied from a complex systems science perspective [46]. Stern [45] has indeed put the emphasis on incorporating a stronger multidisciplinary perspective to consider the social and behavioral dimensions [51–53]. The same author [2] also mentions a definition of social discounts that might include inputs from social dilemmas and predefined behavioral experiments [54].

5. Conclusions

Weitzman asserted that "the choice of an appropriate discount rate is one of the most critical problems in all of economics" [55]. We have shown that historical data indicate that the long-term discount rate is probably not very large. While the error bars remain large, a value of 2% or less seems plausible, corresponding to a present value of about 14% for a payment received 100 years in the future. The 2% is very close to the 1.4% from the Stern report [2], which has been criticised for being too low. The results presented here seem to support the Stern discount rate and this low value has important implications for many aspects of our societies (e.g., energy policy [56], insurances [57], and environmentally responsible investment decisions [58]). The higher values reported by other authors do not seem to be fully consistent with the historical data and the analysis performed here. The need for immediate and substantial spending to combat climate change is, thus, supported. However, our analysis still assumes a very simple model. The robustness and further applicability of the model should be explored using other data sources, with more sophisticated treatment of the market price of risk and consideration of stronger multidisciplinary perspectives. The current study has not covered a vast exploration of countries' datasets, the specific consideration of emerging markets, the more realistic consideration of non-stationarity, or the observed heavy-tailed behavior in the data, to name just a few of the limitations of the current research. However, behind all these additional considerations, there is an increase in uncertainty, which seems in any case to lead to even lower interest rates and, thus, to exacerbate the sense of urgency for climate action.

Author Contributions: Conceptualization, J.D.F., J.G., M.G.R., M.M., J.P. and J.M.; methodology, J.D.F., J.G., M.G.R., M.M., J.P. and J.M.; formal analysis, M.M., J.P. and J.M.; writing—original draft preparation, J.M.; writing—review and editing, all. All authors have read and agreed to the published version of the manuscript.

Funding: This work was partially supported by Ministerio de Ciencia e Innovación (MCIN, Spain), Agencia Estatal de Investigación (AEI) AEI/10.13039/501100011033 (grant number PID2019-106811GB-C33, M.M., J.P. and J.M.); by Ministerio de Ciencia e Innovación (MCIN, Spain), Agencia Estatal de Investigación (AEI) AEI/10.13039/501100011033 and by "ERDF A way of making Europe" (grant number PID2022-140757NB-I00, M.M., J.P. and J.M.); by Generalitat de Catalunya (Spain) through Complexity Lab Barcelona (grant numbers 2017 SGR 608 and 2021 SGR 00856; M.M., J.P. and J.M.).

Data Availability Statement: The raw data are available upon request.

Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

OU	Ornstein–Uhlenbeck
FPE	Fokker–Planck Equation
PDF	Probability Density Function
CIR	Cox–Ingersoll–Ross
CPI	Consumer Price Index
UK	United Kingdom
US	United States
IID	Independent and Identically Distributed

Appendix A. Discount Function for the Ornstein-Uhlenbeck Model

We have seen in the main text that when rates are described by the OU process, the joint characteristic function $\tilde{p}(\omega_1, \omega_2, t|r_0)$ obeys the first-order partial differential equation (cf. Equation (17))

$$\frac{\partial \tilde{p}}{\partial t} = (\omega_1 - \alpha\omega_2)\frac{\partial \tilde{p}}{\partial \omega_2} - \left(im\omega_2 + \frac{1}{2}k^2\omega_2^2\right)\tilde{p}, \tag{A1}$$

with initial condition ($t_0 = 0$)

$$\tilde{p}(\omega_1, \omega_2, 0|r_0) = e^{-i\omega_2 r_0}. \tag{A2}$$

Due to the linearity of the OU process and the Gaussian character of the input noise, we may look for a solution of the initial value problem (A1)–(A2) in the form of a Gaussian density:

$$\tilde{p}(\omega_1, \omega_2, t) = \exp\Big\{-A(\omega_1, t)\omega_2^2 - B(\omega_1, t)\omega_2 - C(\omega_1, t)\Big\}, \tag{A3}$$

where $A(\omega_1, t)$, $B(\omega_1, t)$, and $C(\omega_1, t)$ are unknown functions to be consistently determined. Substituting Equation (A3) into Equation (A1), identifying like powers in ω_2, and taking into account Equation (A2), we find that these functions satisfy the following set of differential equations:

$$\dot{A} = -2\alpha A - k^2/2, \qquad A(\omega_1, 0) = 0; \tag{A4}$$

$$\dot{B} = -\alpha B + 2\omega_1 A - im\alpha, \qquad B(\omega_1, 0) = ir_0; \tag{A5}$$

$$\dot{C} = \omega_1 B, \qquad C(\omega_1, 0) = 0. \tag{A6}$$

Equation (A4) is a first-order linear differential equation that can be readily solved giving

$$A(\omega_1, t) = \frac{k^2}{4\alpha}\left(1 - e^{-2\alpha t}\right), \tag{A7}$$

substituting this expression for $A(\omega_1, t)$ into Equation (A5) results in another first-order equation for $B(\omega_1, t)$, whose solution reads

$$B(\omega_1, t) = ir_0 e^{-\alpha t} + \frac{k^2 \omega_1}{2\alpha^2}\left(1 - 2e^{-\alpha t} + e^{-2\alpha t}\right) + im\left(1 - e^{-\alpha t}\right). \quad \text{(A8)}$$

Finally, the direct integration of Equation (A6) yields the expression for $C(\omega_1, t)$

$$\begin{aligned} C(\omega_1, t) = i\omega_1 r_0 \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right) &+ \frac{k^2 \omega_1^2}{2\alpha^3}\left[\alpha t - 2\left(1 - e^{-\alpha t}\right) + \frac{1}{2}\left(1 - e^{-2\alpha t}\right)\right] \\ &+ im\omega_1\left[t - \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right)\right]. \end{aligned} \quad \text{(A9)}$$

From Equation (19), we see that the effective discount is given by the characteristic function, $\tilde{p}(\omega_1, \omega_2, t|r_0)$, evaluated at the points $\omega_1 = -i$ and $\omega_2 = 0$. Thus, from Equations (A3) and (A9), we obtain

$$\begin{aligned} \ln D(t) = -\frac{r_0}{\alpha}\left(1 - e^{-\alpha t}\right) &+ \frac{k^2}{2\alpha^3}\left[\alpha t - 2\left(1 - e^{-\alpha t}\right) + \frac{1}{2}\left(1 - e^{-2\alpha t}\right)\right] \\ &- m\left[t - \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right)\right], \end{aligned} \quad \text{(A10)}$$

which, with the change of notation regarding time, as explained in the main text, agrees with Equation (20).

Knowing the exact form of the probability distribution of the joint process $(x(t), r(t))$—through its joint characteristic function (A3)—it is possible to study in detail some interesting properties of the rate $r(t)$, as we do next.

Appendix A.1. Negative Rates

As pointed out in the main text, a characteristic of the OU model is the possibility of attaining negative values. This probability is given by

$$P(r < 0, t|r_0) = \int_{-\infty}^{0} p(r, t|r_0) dr, \quad \text{(A11)}$$

where $p(r, t|r_0)$ is the probability density function of the rate process. This is given by the marginal density

$$p(r, t|r_0) = \int_{-\infty}^{\infty} p(x, r, t|r_0) dx,$$

and the characteristic function of the rate is related to the characteristic function of the bidimensional process $(x(t), r(t))$ by the simple relation

$$\tilde{p}(\omega_2, t|r_0) = \tilde{p}(\omega_1 = 0, \omega_2, t|r_0).$$

From Equation (A3) and Equations (A7)–(A9), we obtain

$$\tilde{p}(\omega_2, t|r_0) = \exp\left\{-\frac{k^2}{4\alpha}\left(1 - e^{-2\alpha t}\right)\omega_2^2 - i\left[r_0 e^{-\alpha t} + m\left(1 - e^{-\alpha t}\right)\right]\omega_2\right\},$$

which, after Fourier inversion, gives the Gaussian density

$$p(r, t|r_0) = \frac{(\alpha/k^2)^{1/2}}{\sqrt{\pi(1 - e^{-2\alpha t})}} \exp\left\{-\frac{(\alpha/k^2)[r - r_0 e^{-\alpha t} - m(1 - e^{-\alpha t})]^2}{1 - e^{-2\alpha t}}\right\}. \quad \text{(A12)}$$

The probability for $r(t)$ to be negative, Equation (A11), then reads

$$P(r<0,t|r_0)=\frac{1}{2}\mathrm{Erfc}\left(\frac{(\alpha/k^2)^{1/2}[r_0e^{-\alpha t}+m(1-e^{-\alpha t})]}{\sqrt{1-e^{-2\alpha t}}}\right), \tag{A13}$$

where $\mathrm{Erfc}(z)$ is the complementary error function [59],

$$\mathrm{Erfc}(z)=\frac{2}{\sqrt{\pi}}\int_z^\infty e^{-x^2}dx.$$

Note that, as time increases (in fact starting from $t>\alpha^{-1}$), the probability (A13) is well approximated by the stationary probability, defined as

$$P_s^{(-)}\equiv\lim_{t\to\infty}P(r<0,t|r_0).$$

That is

$$P_s^{(-)}=\frac{1}{2}\mathrm{Erfc}\left(m\sqrt{\alpha/k^2}\right). \tag{A14}$$

In terms of the dimensionless normal level μ and the dimensionless volatility κ defined by

$$\mu\equiv m/\alpha,\qquad \kappa\equiv k/\alpha^{3/2}, \tag{A15}$$

this probability reduces to

$$P_s^{(-)}=\frac{1}{2}\mathrm{Erfc}(\mu/\kappa). \tag{A16}$$

Let us now see the behavior of $P_s^{(-)}$ for the cases (i) $\mu<\kappa$ and (ii) $\mu>\kappa$.

(i) If the normal rate μ is smaller than the rate's volatility κ, we use the series expansion [59]

$$\mathrm{Erfc}(z)=1-\frac{2}{\sqrt{\pi}}z+O(z^2).$$

Hence,

$$P_s^{(-)}=\frac{1}{2}-\frac{1}{\sqrt{\pi}}(\mu/\kappa)+O(\mu^2/\kappa^2). \tag{A17}$$

For μ/κ sufficiently small, this probability approaches 1/2. In other words, the rates are positive or negative with almost equal probability. Note that this corresponds to the situation in which noise dominates over the mean. In the original units (cf. Equation (A15)), this case corresponds to the noise intensity k being larger than $m\alpha^{1/2}$.

(ii) When fluctuations around the normal level are smaller than the normal level itself, $\kappa<\mu$, we use the asymptotic approximation [59]

$$\mathrm{Erfc}(z)\sim\frac{e^{-z^2}}{\sqrt{\pi}z}\left[1+O\left(\frac{1}{z^2}\right)\right],$$

and

$$P_s^{(-)}\sim\frac{1}{2\sqrt{\pi}}\left(\frac{\kappa}{\mu}\right)e^{-\mu^2/\kappa^2}. \tag{A18}$$

Therefore, for mild fluctuations around the mean (that is, when $k\ll m\alpha^{1/2}$), the probability of negative rates is exponentially small.

Appendix A.2. Rates Below the Long-Run Rate

It is also interesting to know the probability that real rates $r(t)$ are below the long-run rate r_∞. This is given by

$$P_\infty(t)\equiv\mathrm{Prob}\{r(t)<r_\infty\}=\int_{-\infty}^{r_\infty}p(r,t|r_0)dr.$$

In the stationary regime, $t \to \infty$, we have

$$P_\infty = \int_{-\infty}^{r_\infty} p(r)dr, \tag{A19}$$

where $p(r)$ is the stationary PDF. For the OU model, $p(r)$ is obtained from Equation (A12) after taking the limit $t \to \infty$:

$$p(r) = \frac{1}{\sqrt{\pi}}\left(\frac{\alpha}{k^2}\right)^{1/2} e^{-\alpha(r-m)^2/k^2}. \tag{A20}$$

Substituting Equation (A20) into Equation (A19), we write

$$P_\infty = \frac{1}{\sqrt{\pi}}\left(\frac{\alpha}{k^2}\right)^{1/2} \int_{-\infty}^{r_\infty} e^{-\alpha(r-m)^2/k^2} dr = \frac{1}{\sqrt{\pi}}\left(\frac{\alpha}{k^2}\right)^{1/2} \int_{-r_\infty}^{\infty} e^{-\alpha(r+m)^2/k^2} dr,$$

or in terms of the complementary error function

$$P_\infty = \frac{1}{2}\mathrm{Erfc}\left[\frac{\sqrt{\alpha}}{k}(m - r_\infty)\right]. \tag{A21}$$

Using the asymptotic estimates of the complementary error function discussed above, we see that this probability is exponentially small if $(m - r_\infty) \to \infty$ with $\sqrt{\alpha}/k$ fixed, or if $\sqrt{\alpha}/k \to \infty$ with a fixed differential of rates $(m - r_\infty)$.

Appendix B. Real and Nominal Rates—The Market Price of Risk

Recall that the real rates are defined as the difference between the nominal rates and the inflation rates (cf. Equation (26)):

$$r(t) = n(t) - i(t).$$

We now discuss how to estimate $n(t)$ and $i(t)$ from empirical data .

Appendix B.1. Nominal Rates

Let $B(t|t+\tau)$ be the price at time t of a government bond maturing at time $t+\tau$ $(\tau \geq 0)$ with unit maturity, $B(t|t) = 1$. The instantaneous rate of return, $b(t|t+\tau)$, of this bond is defined as

$$b(t|t+\tau) \equiv \frac{1}{B(t|t+\tau)}\frac{dB(t|t+\tau)}{dt},$$

so that,

$$B(t|t+\tau) = \exp\left[-\int_t^{t+\tau} b(t|t')dt'\right]. \tag{A22}$$

It is also useful to define the "yield to maturity" $y(t|\tau)$ as

$$y(t|\tau) \equiv -\frac{1}{\tau}\ln B(t|t+\tau),$$

or, after using Equation (A22), as

$$y(t|\tau) = \frac{1}{\tau}\int_t^{t+\tau} b(t|t')dt'. \tag{A23}$$

This form of defining $y(t|\tau)$ has an interesting interpretation since it shows that the yield to maturity is the time average over the maturing period τ of the instantaneous rate of return.

Let us remark that the data at our disposal are not the historical values of $B(t|t+\tau)$ but the annual interest rates of the zero-coupon bond $\beta(t|\tau)$. In this case, we have

$$B(t|t+\tau) = \frac{1}{[1+\beta(t|\tau)]^\tau}, \tag{A24}$$

so that

$$y(t|\tau) = \ln[1+\beta(t|\tau)]. \tag{A25}$$

The spot or nominal rate $n(t)$ is defined as

$$n(t) \equiv \lim_{\tau\to 0} y(t|\tau), \tag{A26}$$

which, after substituting for Equation (A23), yields

$$n(t) = b(t|t). \tag{A27}$$

When dealing with empirical data, the nominal rates are, thus, estimated by the yield,

$$n(t) \sim y(t|\tau) = \ln[1+\beta(t|\tau)] \tag{A28}$$

and, attending to definition (A26), the shorter τ is, the better the estimation for $n(t)$.

Appendix B.2. Inflation Rates

$I(t)$ is defined as the aggregated inflation up to time t. The inflation rate $i(t)$ can then be estimated by the ex post mean inflation rate over a period of time τ, $i(t|\tau)$:

$$i(t|\tau) \equiv \frac{1}{\tau}\ln\frac{I(t+\tau)}{I(t)}, \tag{A29}$$

Also, the Consumer Price Index (CPI, $C(t)$) is related to the aggregated inflation

$$I(t+\tau) = I(t)\prod_{j=0}^{\tau-1}\big[1+C(t+j)\big]. \tag{A30}$$

Finally, the instantaneous rate of inflation $i(t)$ can be estimated by $i(t+\tau)$ and in terms of the CPI

$$i(t) \sim i(t+\tau) = \frac{1}{\tau}\sum_{j=0}^{\tau-1}\ln\big[1+C(t+j)\big] \tag{A31}$$

Appendix B.3. The Market Price of Risk

The concepts of risk-neutral probabilities and the market price of risk (MPR) were developed for bonds and nominal rates. They can, nonetheless, be extended formally to real rates despite practical difficulties, which arise because the real rates are not tradable and, thus, an empirical basis for constructing a risk-neutral measure is lacking.

Let us recall that the real rates $r(t)$ are estimated by the quantity $r(t|\tau)$:

$$r(t) \sim r(t|\tau) \equiv y(t|\tau) - i(t|\tau), \tag{A32}$$

where $y(t|\tau)$ is the yield to maturity τ for a zero-coupon bond $B(t|t+\tau)$ and $i(t|\tau)$ is the inflation rate over period τ.

From a theoretical point of view, the instantaneous real rate $r(t)$ is defined by

$$r(t) = \lim_{\tau\to 0} r(t|\tau).$$

This leads us to take the shortest possible yield, $y(t|\tau)$, at our disposal ($\tau = 3$ months) to construct a proxy of the real spot rate r.

Obviously the spot rate $r(t)$ is random, so is the quantity $r(t|\tau)$. We denote by μ and σ^2 the average and variance of $r(t|\tau)$, respectively. From empirical data these statistics are estimated by

$$\mu \sim \frac{1}{N}\sum_{t=1}^{N} r(t|\tau), \qquad \sigma^2 \sim \frac{1}{N}\sum_{t=1}^{N}\big[r(t|\tau) - \mu\big]^2,$$

where N is the number of samples. Note that, in the most general situation, $\mu = \mu(t, r|\tau)$ and $\sigma = \sigma(t, r|\tau)$ depend on the current time t, the rate r, and the maturing interval τ [21].

The risk premium is defined by the difference $\mu(t, r|\tau) - r$. Since this excess return depends on the maturity time, there can be arbitrage opportunities by buying and selling bonds at different maturities [21] (see also [60]). It can be shown that these arbitrage opportunities are ruled out as long as the Sharpe ratio of the excess return,

$$q(r, t) \equiv \frac{\mu(t, r|\tau) - r}{\sigma(t, r|\tau)}, \tag{A33}$$

is independent of the maturity time τ [21]. This ratio is called the market price of risk. It depends, in general, on the current time t and the spot rate r, although the most common and feasible assumption is that q is constant or, at most, a function of r (see main text).

Appendix C. Parameter Estimation and Uncertainties

Appendix C.1. Parameter Estimation

Let us recall that the OU model is defined by means of the linear stochastic differential equation:

$$dr(t) = -\alpha(r - m)dt + kdW(t)$$

whose solution is

$$r(t) = r(t_0)e^{-\alpha(t-t_0)} + m\Big[1 - e^{-\alpha(t-t_0)}\Big] + k\int_{t_0}^{t} e^{-\alpha(t-t')}dW(t'),$$

where t_0 is an arbitrary initial time. In what follows, we will assume that the process is in the stationary regime. That is to say, we assume the process started in the infinite past (i.e., $t_0 = -\infty$) and at the present time t all transient effects have faded away. Therefore,

$$r(t) = m + k\int_{-\infty}^{t} e^{-\alpha(t-t')}dW(t'). \tag{A34}$$

The parameter m is easily estimated by noting that since the Wiener process has a zero mean (stationary) rate

$$\mathbb{E}[r(t)] = m. \tag{A35}$$

The parameters α and k can be estimated in terms of the correlation function

$$K(t - t') = \mathbb{E}\big[(r(t) - m)(r(t') - m)\big].$$

From Equations (A34) and (A35), we write

$$K(t - t') = k^2 e^{-\alpha(t+t')}\int_{-\infty}^{t} e^{\alpha t_1}\int_{-\infty}^{t} e^{\alpha t_2}\mathbb{E}[dW(t_1)dW(t_2)].$$

Taking into account that [60]

$$\mathbb{E}[dW(t_1)dW(t_2)] = \delta(t_1 - t_2)dt_1dt_2,$$

where $\delta(\cdot)$ is the Dirac delta function, and performing the integration on t_2, we have

$$K(t-t') = k^2 e^{-\alpha(t+t')} \int_{-\infty}^{t} \Theta(t'-t_1) e^{2\alpha t_1} dt_1,$$

where $\Theta(\cdot)$ is the Heaviside step function. In the evaluation of the integral, we must take into account whether $t > t'$ or $t < t'$. It is a simple matter to see that for either case the final result is

$$K(t-t') = \frac{k^2}{2\alpha} e^{-\alpha|t-t'|}. \tag{A36}$$

Let us incidentally note that Equation (A36) proves that the correlation time of the OU process is given by α^{-1}. Indeed, the correlation time, τ_c, of any stationary random process with correlation function $K(\tau)$ is defined by the time integral of $K(\tau)/K(0)$. In our case,

$$\tau_c \equiv \frac{1}{K(0)} \int_0^{\infty} K(\tau) d\tau = \frac{1}{\alpha}. \tag{A37}$$

The empirical auto-correlation fit to an exponential (cf. Equation (A36)) allows to estimate α.

The parameter k can finally be obtained from the (empirical) standard deviation

$$\sigma^2 = \mathbb{E}\Big[(r(t)-m)^2\Big],$$

Hence,

$$k = \sigma\sqrt{2\alpha}. \tag{A38}$$

We estimate these quantities for the three-month interest rates using the maximum likelihood procedure given in [27].

Appendix C.2. Correcting for the Bias Using 3-Month Rates Sampled at Annual Frequency

Our parameter estimation process that corrects for the bias introduced by using the 3-month rate as an approximation to the instantaneous rate, and by sampling it at annual frequency, has the following steps:

1. Estimate parameters using the historical 3-month and 10-year data as described in the main text.
2. Simulate the instantaneous process (which we approximate as a daily process) using the parameters inferred in step (1) to generate a simulated time series $r(t)$ whose length matches that of the real data (roughly 100 years for the UK and 80 years for the US).
3. Construct simulated 3-month and 10-year time series based on Equation (20) with $\tau = 0.25$ and $\tau = 10$, using the time series for $r(t)$ from step 1 as the initial condition for each time t.
4. Estimate m, k, and α on the simulated 3-month series (sampled at annual frequency).
5. Repeat steps (2–4) for a 1000 times and compute the average value of each parameter under the estimation process of step (4). This yields systematic shifts in the parameters relative to those estimated on the historical data, making it clear that the estimation process is biased.
6. Correct for this bias by adjusting the parameters of the instantaneous process by the magnitude of the average shift, so that the estimation process for the simulated 3-month bond time series roughly matches the values estimated from the historical series.

References

1. World Economic Forum. This Is What the Climate Crisis Is Costing Economies Around the World. 29 November 2023. Available online: https://www.weforum.org/agenda/2023/11/climate-crisis-cost-global-economies/ (accessed on 6 February 2024).
2. Stern, N. *The Economics of Climate Change: The Stern Review*; Cambridge University Press: Cambridge, UK, 2006.
3. Dasgupta, P. *Human Well-Being and the Natural Environment*; Oxford University Press: Oxford, UK, 2004.
4. Nordhaus, W.D. *A Question of Balance*; Yale University Press: New Haven, CT, USA, 2008.

5. Arrow, K.J.; Cropper, M.L.; Gollier, C.; Groom, B.; Heal, G.M.; Newell, R.G.; Nordhaus, W. D.; Pindyck, R.S.; Pizer, W.A.; Portney, P.R.; et al. Determining benefits and costs for future generations. *Science* **2013**, *341*, 349–350. [CrossRef]
6. Nordhaus, W.D. The Stern Review on the economics of climate change. *J. Econ. Lit.* **2007**, *45*, 687–702. [CrossRef]
7. Nordhaus, W.D. Critical assumptions in the Stern Review on Climate Change. *Science* **2007**, *317*, 201–202. [CrossRef]
8. Dasgupta, P. *Comments on the Stern Review's Economics of Climate Change*; Cambridge University Press: Cambridge, UK, 2006.
9. Mendelsohn, R.O. A critique of the Stern report. *Regulation* **2006**, *29*, 42–46.
10. Weitzman, M.L. A review on the Stern Review on the economics of climate change. *J. Econ. Lit.* **2007**, *45*, 703–724. [CrossRef]
11. World Resources Institute. Better Growth, Better Climate: The New Climate Economy Report. The Synthesis Report. 2014. Available online: https://sustainabledevelopment.un.org/content/documents/1595TheNewClimateEconomyReport.pdf (accessed on 6 February 2024).
12. Stern, N. Ethics, Equity and the Economics of Climate Change. Paper 1: Science and Philosophy. *Econ. Philos.* **2014**, *30*, 397–444. [CrossRef]
13. Stern, N. Ethics, Equity and the Economics of Climate Change. Paper 2: Economics and Politics, *Econ. Philos.* **2014**, *30*, 445–501. [CrossRef]
14. Drupp, M.A.; Freeman, M.C.; Groom, B.; Nesje, F. Discounting disentangled. *Am. Econ. J. Econ. Policy* **2018**, *10*, 109–134. [CrossRef]
15. Ramsey, F.P. A mathematical theory of saving. *Econ. J.* **1928**, *38*, 543–559. [CrossRef]
16. Arrow, K.J.; Cropper, M.L.; Gollier, C.; Groom, B.; Heal, G.M.; Newell, R.G.; Nordhaus, W.D.; Pindyck, R.S.; Pizer, W.A.; Portney, P.R.; et al. How should benefits and costs be discounted in an intergenerational context? The views of an expert panel. *Tech. Rep. Resour. Future* **2012**, *2012*, 12–53. [CrossRef]
17. Weitzman, M.L. Why the far-distant future should be discounted at its lowest possible rate? *J. Environ. Econ. Manag.* **1998**, *36*, 201–208. [CrossRef]
18. Gollier, C.; Koundouri, P.; Pantelidis, T. Declining discount rates: Economic justifications and implications for the long-run policy. *Econ. Policy* **2008**, *23*, 757–795. [CrossRef]
19. Freeman, M.; Groom, B. How uncertain are we about the certainty-equivalent long-term social discount rate? *J. Environ. Econ. Manag.* **2016**, *79*, 152–168. [CrossRef]
20. Dybvig, P.; Ingersoll, J.; Ross, S. Long forward and zero-coupon rates can never fall. *J. Bus.* **1996**, *60*, 1–25. [CrossRef]
21. Vasicek, O. An equilibrium characterization of the terms structure, *J. Fin. Econ.* **1977**, *5*, 177–188. [CrossRef]
22. Litterman, R.; Scheinkman, J.; Weiss, L. Volatility and the yield curve. *J. Fixed Income* **1991**, *1*, 49–53. [CrossRef]
23. Newell, R.; Pizer, N. Discounting the Distant Future: How much do uncertain rates increase valuations? *J. Environ. Econ. Manag.* **2003**, *46*, 52–71. [CrossRef]
24. Groom, B.; Koundouri, P.; Panopoulou, P.; Pantelidis, T. Discounting distant future: How much selection affect the certainty equivalent rate. *J. Appl. Econom.* **2007**, *22*, 641–656. [CrossRef]
25. Freeman, M.C.; Groom, B.; Panopoulou, E.; Pantelidis, T. Declining discount rates and the fisher effect. Inflated past, discounted future? *J. Environ. Econ. Manag.* **2015**, *73*, 32–49. [CrossRef]
26. Duffie, D. *Dynamic Asset Pricing Theory*; Princeton University Press: Princeton, NJ, USA, 2001.
27. Brigo, D.; Mercurio, F. *Interest Rate Models—Theory and Practice*; Springer: Berlin, Germany, 2006.
28. Piazzesi, M. Affine term structure models. In *The Handbook of Financial Econometrics*; Sahala, Y. A., Hansen, L.P., Eds.; Elsevier: Amsterdam, The Netherlands, 2009.
29. Mamon, R. Three ways to solve bond prices in the Vasicek model. *J. Appl. Math. Decis. Sci.* **2004**, *8*, 1–14. [CrossRef]
30. Davidson, I.; Song, X.; Tippett, M. Time varying costs of capital and the expected present value of future cash flows. *Eur. J. Financ.* **2015**, *21*, 129–146. [CrossRef]
31. Perelló, J.; Montero, M.; Masoliver, J.; Farmer, J.D.; Geanakoplos, J. Statistical analysis and stochastic interest rate modeling for valuing the future with implications in climate change mitigation. *J. Stat. Mech.* **2020**, *2020*, 043210. [CrossRef]
32. Masoliver, J.; Montero, M.; Perelló, J. Valuing the future under random structural conditions: Non-stationary models for discounting. 2024, *in preparation*.
33. Masoliver, J.; Montero, M.; Perelló, J. Jump-Diffusion Models for Valuing the Future: Discounting under Extreme Situation. *Mathematics* **2021**, *9*, 1589. [CrossRef]
34. Montero,M.; Perelló, J.; Masoliver, J. Valuing the distant future under stochastic resettings: The effect on discounting. *J. Phys. A* **2022**, *55*, 464001. [CrossRef]
35. Masoliver, J.; Montero, M.; Perelló, J.; Farmer, J.D.; Geanakoplos, J. Valuing the Future and Discounting in Random Environments: A Review. *Entropy* **2022**, *24*, 496. [CrossRef] [PubMed]
36. Gollier, C. Discounting with fat-tailed economic growth. *J. Risk Uncertain.* **2008**, *37*, 171–186. [CrossRef]
37. Cox, J.C.; Ingersoll, J.E.; Ross, S.A. A re-examination of the traditional hypothesis about the term structure of interest rates. *J. Financ.* **1981**, *36*, 769–799.
38. Gilles, C.; Leroy, S.F. A note on the local expectation hypothesis. *J. Financ.* **1986**, *41*, 975–979.
39. Farmer, J.D.; Geanakoplos, J.; Masoliver, J.; Montero, M.; Perelló, J. Value of the future: Discounting in random environments. *Phys. Rev. E* **2015**, *91*, 052816. [CrossRef]
40. Cox, J.C.; Ingersoll, J.E.; Ross, S.A. A theory of the term structure of interest rates. *Econometrica* **1985**, *53*, 385–407. [CrossRef]

41. Soergel, B.; Kriegler, E.; Weindl, I.; Rauner, S.; Dirnaichner, A.; Ruhe, C.; Hofmann, M.; Bauer, N.; Bertram, C.; Bodirsky, B.; et al. A sustainable development pathway for climate action within the UN 2030 Agenda. *Nat. Clim. Change* **2021**, *11*, 656–664. [CrossRef]
42. Giglio, S.; Kelly, B.; Stroebel, J. Climate finance. *Annu. Rev. Financ. Econ.* **2021**, *13*, 15–36. [CrossRef]
43. Stroebel, J.; Wurgler, J. What do you think about climate finance? *J. Financ. Econ.* **2021**, *142*, 487–498. [CrossRef]
44. Bauer, M.D.; Rudebusch, G.D. The rising cost of climate change: Evidence from the bond market. *Rev. Econ. Stat.* **2023**, *105*, 1255–1270. [CrossRef]
45. Stern, N. A time for action on climate change and a time for change in economics. *Econ. J.* **2022**, *132*, 1259–1289. [CrossRef]
46. Stern, N.; Stiglitz, J.; Taylor, C. The economics of immense risk, urgent action and radical change: Towards new approaches to the economics of climate change. *J. Econ. Methodol.* **2021**, *29*, 181–216. [CrossRef]
47. Giglio, S.; Maggiori, M.; Stroebel, J. Very long-run discount rates. *Q. J. Econ.* **2015**, *130*, 1. [CrossRef]
48. Giglio, S.; Maggiori, M.; Stroebel, J. No-bubble conditions:model-free test in housing markets. *Econometrica* **2016**, *84*, 1047–1091. [CrossRef]
49. Giglio, S.; Maggiori, M.; Rao, K. Stroebel, J.; Weber, A. Climate change and long-run discount rates: Evidence from real estate. *Rev. Financ. Stud.* **2021**, *34*, 3527–3571. [CrossRef]
50. Sukono; Ibrahim, R.A.; Saputra, M.P.A.; Hidayat, Y.; Juahir, H.; Prihanto, I.G.; Halim, N.B.A. Modeling Multiple-Event Catastrophe Bond Prices Involving the Trigger Event Correlation, Interest, and Inflation Rates. *Mathematics* **2022**, *10*, 4685. [CrossRef]
51. Li, M.; Trencher, G.; Asuka, J. The clean energy claims of BP, Chevron, ExxonMobil and Shell: A mismatch between discourse, actions and investments. *PLoS ONE* **2022**, *17*, e0263596. [CrossRef]
52. Creutzig, F.; Kapmeier, F. Engage, don't preach: Active learning triggers climate action. *Energy Res. Soc. Sci.* **2020**, *70*, 101779. [CrossRef]
53. Kundzewicz, Z.W.; Matczak, P.; Otto, I.M.; Otto, P.E. From "atmosfear" to climate action. *Environ. Sci. Policy* **2020**, *105*, 75–83. [CrossRef]
54. Vicens, J.; Bueno-Guerra, N.; Gutiérrez-Roig, M.; Gracia-Lázaro, C.; Gómez-Gardeñes, J.; Perelló, J.; Sánchez, A.; Moreno, Y.; Duch, J. Resource heterogeneity leads to unjust effort distribution in climate change mitigation. *PLoS ONE* **2018**, *13*, e0204369. [CrossRef]
55. Weitzman, M.L. Gamma discounting. *Am. Econ. Rev.* **2001**, *91*, 260–271. [CrossRef]
56. Buła, R.; Foltyn-Zarychta, M. Declining Discount Rates for Energy Policy Investments in CEE EU Member Countries. *Energies* **2023**, *16*, 321. [CrossRef]
57. Suwanmalai, W.; Zaby, S. How Do Life Insurers Respond to a Prolonged Low Interest Rate Environment? A Literature Review. *Risks* **2022**, *10*, 155. [CrossRef]
58. Fregonara, E.; Ferrando, D.G. The Discount Rate in the Evaluation of Project Economic-Environmental Sustainability. *Sustainability* **2023**, *15*, 2467. [CrossRef]
59. Magnus, W.; Oberhettinger, F.; Soni, R.P. *Formulas and Theorems for the Special Functions of Mathematical Physics*; Springer: Berlin, Germany, 1966.
60. Masoliver, J. *Random Processes, First-Passage and Escape*; World Scientific: Singapore, 2018.